Project 5

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1 Prove the following lemma by PMI. If $u, v \in \Sigma^*$, then $(uv)^R = v^R u^R$.

Basis:
If $|v|$ and $|u|$ are 0, $v \in \Sigma^*$,
then $u = \lambda$
s.t.

$$\begin{aligned}
(uv)^R &= (\lambda v)^R \\
&= (v)^R \\
&= v^R(\lambda)^R \\
&= v^R \lambda^R \\
&= v^R u^R
\end{aligned}$$

Inductive Step:
Let $|u| = n > 0$
assume $u = aw$, where w is a string, $|w| = n - 1$ and $a \in \Sigma$
Then,

$$\begin{aligned}
(uv)^R &= (awv)^R \\
&= (awv)^R \\
&= v^R(aw)^R \\
&= v^R w^R (a)^R \\
&= v^R w^R a^R \lambda^R \\
&= v^R w^R a^R \lambda v^R w^R a^R \\
&= v^R (wa)^R \\
&= v^R u^R
\end{aligned}$$

We have now proven that $(uv)^R = v^R u^R$ through mathematical induction.

2 Using induction on the length of the string, and the lemma you proved in problem 1, prove that $(w^R)^R = w$ for all strings $w \in \Sigma^*$.

Let $w = uv$

$$
\begin{aligned}
((w)^R)^R &= ((uv)^R)^R \\
&= (v^R u^R)^R \\
&= u^R (v^R)^R \\
&= u^R v^R \\
&= (uv)^R \text{ [From Q1 Lemma]} \\
&= vu \\
&= w
\end{aligned}
$$

We have now proven that $((w)^R)^R = w$ through mathematical induction.

3 Let J be the set of palindromes over $\{a, b\}$.

3.1 Give a formal definition of J using set notation.

$J = \{w | w \in \{a \cup b\}^* \text{ s.t. } w^R = w\}\}$

3.2 Using the pumping lemma for regular languages, show that J is not regular.

The pumping lemma can be used to determine if J is a regular language by determining if there is a number P, the pumping length, where if s is any string in L of length at least P, then s may be divided into 3 parts, $S = xyz$ such that:
1. for each $i \geq 0$, $xy^iz \in J$
2. $|y| = 0$
3. $|xy| \leq P$

In this case, due to J being the set of palindromic strings formed from $\{a, b\}$, we can define $S = a^p b^p a^p$.

Through the first criteria for the modified version of s used in the pumping lemma, $\forall i > 0, xy^iz \in J$, we can deduce the following.

If $S = a^h a^i a^{p-h-i} b^p a^p \in J$, then $S' = a^h a^i a^i a^{p-h-i} b^p a^p \in J$ which is equivalent to $a^{i+p} b^p a^p \in J$.

However, when defining this, we specified that $i > 0$, which is contradicted by this statement.
Therefore:
Since $i \neq 0$, $S'^R \neq S'$, so $S' \notin J$.
This means that S is not a regular language.

4 Let $L = \{ww | w \in \{a, b\}*\}$. Using the pumping lemma for regular languages, show that L is not regular.

Lets assume that L is regular:
Consider the constant k ($k \geq 1$)
Consider a string $s = a^m b^m b^m a^m$ where $m \geq 1$
Now consider decomposition's for s.
They will be of the form $s = xy$, where $y = a^j$ where $1 \leq j \leq k$.
Now by pumping lemma, $kv^i k$ belongs to L for all $i \geq 0$.
Consider $i = 0$, then we have a string in the form $a^l b^m a^m b^m$ where $l < m$.
Since the number of a's on the left side of the string is less than a's on the right side of the string, they can never be represented as ww, where $w \in ab^*$.
This **violates** our assumption that L is regular.
So by **contradiction**, we have shown that L is not regular.