An Introduction to the Dynamics on the Circle

To the Great Henri Poincaré

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Abstract

This paper gives a quick review from basic notions of dynamical systems to Poincaré's rotation number theory, which categorizes all the preserving-orientation homeomorphisms of the circle.

Contents

1 Introduction

Dynamical system is widely believed to be derived from Poincaré's studies on celestial mechanics, more precisely, the famous n-body problem.

Newton's gravitation theory reads that movements of n bodies in $\mathbb{R}^3$ with masses $m_1, \dots, m_n$ are determined by the differential equation

$$m_i \frac{d^2\mathbf{x}_i}{dt^2} = -G \sum_{j \neq i} m_i m_j \frac{\mathbf{x}_i - \mathbf{x}_j}{\|\mathbf{x}_i - \mathbf{x}_j\|^3}.$$

where $\mathbf{x}_i \in \mathbb{R}^3$ is the position of the i-th body and G is the gravitation constant. For $n = 2$, the system is completely determined via "mass center". A corollary is that every planet moves approximately in an ellipse, if perturbations of others are ignored.

However, things go weird considering the case $n = 3$. There are many types of motions, such as homothetic one, homographic one, and relative equilibriums. Mathematicians and physicists were hence divided into two sides. One struggled to find intelligent methods to quantitatively analyse it, while the other turned into numerical computation experiments.

Poincaré first pointed out, in his celebrated series of papers, *Les Méthodes nouvelles de la mécanique céleste* [Poi99], and *Leçons de mécanique céleste* [Poi10], that, instead of seeking solutions, we should concentrate more on the asymptotic behaviors of these motions. His works bursted out plentiful studies on "integrability" of this system, which was finally proved to be non-integrable.

However, the birth of dynamical system couldn't be regarded as a triumph of numerical computations. They cannot answer the following problems:

- May the earth be ejected from the solar system or collide with the sun, i.e., what happens when the time goes to ∞?
- If we slightly perturb the initial conditions, such as an accidental collision with a planet, on which level the behavior of the solutions will be changed, slightly or strongly differently?

Obviously, we could only obtain answers in a finite scope, i.e., finite-time evolution and finite possibilities taken into consideration, from numerical simulations. That is why we are here.

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2 Preliminaries

2.1 Notions

Let X be a metric space, $T : X \to X$ a continuous map. The iteration of T defines an action of a semi-group $\mathbb{N}$ on X as $T^n(x) = T \circ \cdots \circ T(x)$ for n times. It can be generalized to the group $\mathbb{Z}$ if T is a homeomorphism, as $T^{-n}(x) = (T^{-1})^n(x)$. The *orbit* of a point $x \in X$ denotes the set

$$O^+(x) = \{T^n(x) \in X : n \in \mathbb{N}\},$$

or

$$O(x) = \{T^n(x) \in X : n \in \mathbb{Z}\}$$

if T is a homeomorphism, where we call $O^+(x)$ as the *positive orbit*. The simplest orbit comes from *fixed points*,

$$\mathrm{Fix}(T) := \{x \in X : T(x) = x\},$$

and then *periodic points*,

$$\mathrm{Per}(T) := \{x \in X : \exists q \in \mathbb{N}^* \ T^q(x) = x\}.$$

The minimal $q \in \mathbb{N}^*$ satisfying that $T^q(x) = x$ is called the *period* of x.

2.2 Premier examples: rotations

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the torus with coordinate $\widehat{x} := x + \mathbb{Z}$, $R_{\widehat{a}} : \mathbb{T} \to \mathbb{T}$ the rotation by angle $\widehat{x}$, i.e., $R_{\widehat{a}}(\widehat{x}) = \widehat{x} + \widehat{a} = \widehat{x+a}$.

Let us introduce a fundamental concept describing orbits compared to the space, defined by Birkhoff [Bir12].

Definition 2.1. The transformation $T : X \to X$ is called *positively minimal*, if for any $x \in X$, $O^+(x)$ is dense. If T is not invertible, we can simplify it as *minimal*.

If T is invertible, the system is called *minimal* if for any $x \in X$, $O(x)$ is dense.

The behaviour of rotations can be completely classified by its rationality.

Proposition 1.

1. *If $\widehat{a} \in \mathbb{Q}/\mathbb{Z}$, all the orbits are periodic. More precisely, if $a = \frac{p}{q}$ is the simplest representative of $\widehat{a}$, i.e., $p \in \mathbb{Z}, q \in \mathbb{N}^*$ with $p \wedge q = 1$, then all the points are periodic, of period q.*

2. *If $\widehat{a} \in (\mathbb{R} \setminus \mathbb{Q})/\mathbb{Z}$, then the system $(\mathbb{T}, R_{\widehat{a}})$ is minimal.*

3 Homeomorphisms and diffeomorphisms of the circle

An important tool to describe behaviours of general continuous maps on $\mathbb{S}^1 = \mathbb{T}$ is degree. Note that $\mathbb{R}$ is the universal covering space of $\mathbb{T}$ with the canonical projection $\pi : \mathbb{R} \to \mathbb{T}$, so any continuous map F of $\mathbb{T}$ can be lifted as a continuous map f of $\mathbb{R}$. Different lifts of the same map differ with an integer. Note that the map $f(x+1) - f(x)$ takes value at a fixed value, independent of the lift f, hence a nature of F. It is called the *degree* of F.

For fear of falling into more discussion on intrinsic properties of geometric objects, we choose to study better maps first, such as homeomorphisms. Let $F : \mathbb{T} \to \mathbb{T}$ be a homeomorphism of T, whose degree is obviously ± 1. The former type is called to *preserve* the orientation, while the latter *reverses* the orientation. Let $\mathrm{Homeo}_+(\mathbb{T})$ be the group consisting of preserving-orientation homeomorphisms of $\mathbb{T}$, equipped with the operation, composition. Let $D^0(\mathbb{T})$ be the group consisting of all the lifts of elements in $\mathrm{Homeo}_+(\mathbb{T})$. Without special mention, we concentrate on preserving-orientation homeomorphisms in the following.

3.1 Rotation number

Heuristically, preserving-orientation homeomorphisms behave just like rotations but maybe with different velocity. Enlightened by studies of rotations, Poincaré established the fundamental concept, *rotation number*, which approximates the homeomorphism by a certain rotation.

Proposition 2 (Approximation to the preserving-orientation homeomorphism)**.** *Let $f \in D^0(\mathbb{T})$. There exists $\rho \in \mathbb{R}$ such that for any $x \in \mathbb{R}, k \in \mathbb{Z}$, it holds that*

$$-1 < f^k(x) - x - k\rho < 1.$$

Such ρ is unique, hence called the rotation number of f, denoted by $\rho(f)$.

Proof. The critical observation is that for any $x, y \in \mathbb{R}$, $-1 < \varphi(y) - \varphi(x) < 1$, where $\varphi = f - \mathrm{id}_{\mathbb{R}}$. It is based on the fact that φ is 1-periodic, so we can fold x, y as $x \leq y < x + 1$ and conclude by increasing.

The following estimation comes from studying $m_k = \min_{x\in\mathbb{R}} f^k(x) - x$, $M_k = \max_{x\in\mathbb{R}} f^k(x) - x$, two kinds of sub-additive sequences. □

Corollary 2.1. *The rotation number $\rho(f)$ can hence be given as*

$$\rho = \lim_{n\to\infty} \frac{f^n(x)}{n}$$

for any $x \in \mathbb{R}$.

Remark.

1. The rotation number of $T_a : \mathbb{R} \to \mathbb{R}$ as $T_a(x) = x + a$ is a, just as expected.
2. For any $f \in D^0(\mathbb{T})$ and $p, q \in \mathbb{Z}$, $\rho(f + p) = \rho(f) + p, \rho(f^q) = q\rho(f)$.

There is hence an induced map $\rho : \mathrm{Homeo}_+(\mathbb{T}) \to \mathbb{T}$ as $\rho(F) = \rho(f)$ for any lift f of F.

3.2 Dynamics of homeomorphisms with rational rotation number

We start by the simplest case, $\rho(f) = 0$. Stereotypes of such homeomorphisms are illustrated by Fig.1 and 2.

Lemma 3. *Let $f \in D^0(\mathbb{T})$. Then $\rho(f) = 0$ if and only if f has a fixed point.*

Proof. The other side is given by intermediate value theorem. □

To better comprehend such systems, we introduce the following concepts describing the eventual behaviours of an orbit.

Definition 3.1. Let (X, T) be a topological dynamical system. For any $x \in X$, the set consisting of accumulation points of $O^+(x)$ is called the ω*-limit set* of x.

If T is invertible, the set consisting of accumulation points of $\{T^{-n}(x) : n \in \mathbb{N}\}$ is called the α*-limit set* of x.

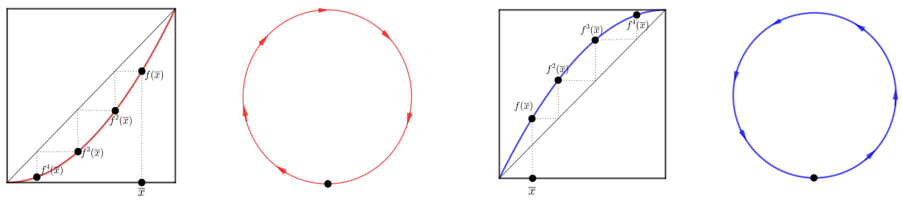

Figure 1: Stereotypes of homeomorphisms with one fixed point [San18]

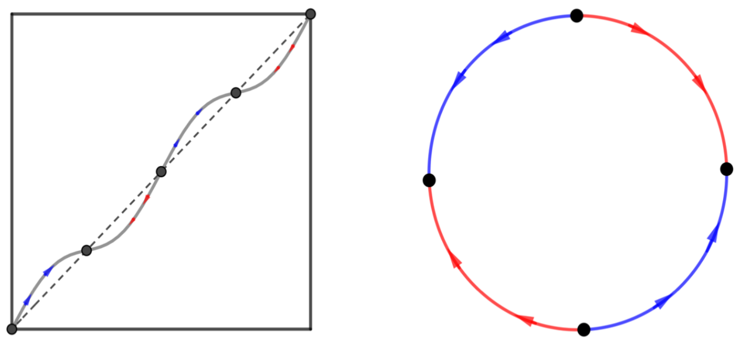

Figure 2: An example of homeomorphisms with two or more fixed points [San18]

It is obvious that
$$\omega(x) = \bigcap_{n \geq 0} \overline{O^+(T^n(x))},$$
and
$$\alpha(x) = \bigcap_{n \geq 0} \overline{\{T^{-m}(x) : m \geq n\}}.$$

Proposition 4 (Case for $\rho(F) = 0$)**.** *Let F be in* $\mathrm{Homeo}_+(\mathbb{T})$*. Then $\rho(F) = 0$ if and only if F has a fixed point. Moreover, if so, for any $\widehat{x} \in \mathbb{T}$, $\alpha(\widehat{x}), \omega(\widehat{x}) \subset$* $\mathrm{Fix}(F)$*.*

Proof. Evident. □

Proposition 5 (Case for $\rho(F) \in \mathbb{Q}/\mathbb{Z}$)**.** *Let F be in* $\mathrm{Homeo}_+(\mathbb{T})$*. Then $\rho(F) \in \mathbb{Q}/\mathbb{Z}$ if and only if F has a periodic point. If so, write that $\rho(F) = \frac{p}{q} + \mathbb{Z}$ with the simplest form $\frac{p}{q}$. The followings hold.*

1. *There is a periodic point of period q.*
2. *All the periodic points are of period q.*
3. *For any $\widehat{x} \in \mathbb{T}$, $\omega(\widehat{x}), \alpha(\widehat{x})$ are all periodic orbits.*

Proof. Observe F^q. □

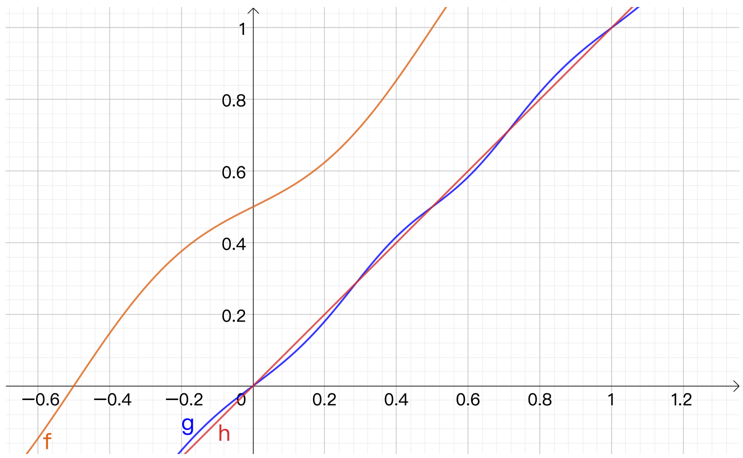

Figure 3: A homeomorphism of rotation number $\frac{1}{2}$

A non-trivial example is as follows.

Example 6. *Let $F : \mathbb{T} \to \mathbb{T}$ be the projection of $f \in D^0(\mathbb{T})$ defined as*

$$f(x) = x + \frac{1}{2} - \frac{1}{4\pi}\sin(2\pi x).$$

It is illustrated by Fig.3. It is of rotation number $\frac{1}{2}$. A clear periodic orbit is $\{0, \frac{1}{2}\}$.

3.3 Dynamics of homeomorphisms of irrational rotation number

Homeomorphisms with irrational rotation number behave completely differently. We introduce the concepts for conjugated dynamical systems first.

Definition 3.2. Let (X, T), (Y, S) be two dynamical systems. A *semi-conjugation* of X to Y denotes a surjective continuous map $H : X \to Y$ satisfying that $H \circ T = S \circ H$, i.e., the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{T} & X \\ \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle H} \\ Y & \xrightarrow{T} & Y \end{array}$$

If H is furthermore bijective, we say H is a *conjugation* and two systems are *conjugated*.

Conjugation designates an equivalence relation of dynamical systems for simplification and categorization. What's more, all the invariants in dynamical system must be preserved in this equivalence relation.

Theorem 7. *Let $F \in \mathrm{Homeo}_+(\mathbb{T})$ be with irrational rotation number. There exists a semi-conjugation $H : (\mathbb{T}, F) \to (\mathbb{T}, R_{\rho(F)})$ of degree 1, which can be lifted by an increasing continuous map $h : \mathbb{R} \to \mathbb{R}$.*

Proof. Pick $x \in \mathbb{R}$, and define h on $\{f^q(x) + p \in \mathbb{R} : (q,p) \in \mathbb{Z}^2\}$ as

$$h(f^q(x) + p) = q\rho + p.$$

Note that h is increasing with dense image, so we can extend it on $\mathbb{R}$ into a continuous map. □

Full description requires the concept of non-wandering points, which came from Birkhoff [Bir27].

Definition 3.3. Let (X, T) be a topological dynamical system. A point $x \in X$ is said to be *non-wandering* if for any its neighbourhood U, there exists some $n \in \mathbb{N}^*$ such that $f^n(U) \cap U \neq \varnothing$. The collection of non-wandering points is denoted by $\Omega(T)$.

Remark. The set $\Omega(T)$ is closed and invariant by T, i.e., $T(\Omega(T)) \subset \Omega(T)$. It is non-empty if X is compact by Birkhoff recurrence theorem.

Theorem 8 (Categorization of homeomorphisms with irrational rotation number)**.** *Let $F \in \mathrm{Homeo}_+(\mathbb{T})$ be with irrational rotation number. The smallest element X in the set*

$$\mathfrak{S} := \{S \subset \mathbb{T} : S \text{ is non-empty, closed and invariant by } F\}$$

corresponds with $\Omega(F)$. Moreover, two branches emerge as follows.

- *If $X = \mathbb{T}$, then $(\mathbb{T}, F)$ is conjugated to $(\mathbb{T}, R_{\rho(F)})$.*
- *If $X \subsetneq \mathbb{T}$, then any connected component of $\mathbb{T} \setminus X$ is a wandering domain, and X is a Cantor set.*

Corollary 8.1. *For any $x \in \mathbb{T}$, it always holds that $\alpha(x) = \omega(x) = X$.*

Instead of our process, Poincaré first established the last theorem, which exactly in turn enlightened him to define "rotation number" as above.

References

[Bir12] George D. Birkhoff. Quelques théorèmes sur le mouvement des systèmes dynamiques. *Bulletin de la Société mathématiques de France*, 40:305–323, 1912.

[Bir27] G.D. Birkhoff. *Dynamical Systems.* American Mathematical Society / Providence, Estados Unidos. American Mathematical Society, 1927.

[Poi99] Henri Poincaré. *Les Méthodes nouvelles de la mécanique céleste.* Gauthier-Villars, Paris, 1892, 1893, 1899.

[Poi10] Henri Poincaré. *Leçons de mécanique céleste*. Gauthier-Villars, Paris, 1905, 1907, 1910.

[San18] Diego A. S. Sanhueza. Homeomorphisms on the circle, 2018. http://www.im.ufrj.br/~gelfert/cursos/2019-1-SisDin/Homeomorphisms%20on%20the%20circle.pdf.