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# Fundamental Properties of Tent Spaces on Euclidean Spaces and Spaces of Homogeneous Type

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# *Abstract*

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**Fundamental Properties of Tent Spaces on Euclidean Spaces and Spaces of Homogeneous Type**

by Hedong Hou

This mémoire addresses fundamental properties of tent spaces. The definitions of tent spaces on Euclidean spaces were first proposed by R. Coifman, Y. Meyer, and E. Stein. They also elaborated several cardinal properties, such as independence of aperture, duality, atomic decomposition, and interpolation. The theory was further studied by numerous works, for instance, E. Russ for $T^{p,2}$ over spaces of homogeneous type with $p \in (0,1]$ and A. Amenta for tent spaces over doubling metric spaces. Our work first summarizes the above-mentioned works and extends their works in two aspects.

On Euclidean spaces itself, we discuss the independence of aperture for all the four types of tent spaces, and follow the work by P. Auscher to study the optimal constants of changing apertures in the full region $p, q \in (0, \infty)$.

On the other hand, we generalize the original definitions of tent spaces on Euclidean spaces to spaces of homogeneous type. We correspondingly demonstrate independence of aperture, duality, atomic decomposition, and interpolation theory of this novel space. The optimal constants of changing apertures are also clarified in this context.

We follow Coifman-Meyer-Stein to characterize Hardy spaces via tent spaces as an application. Furthermore, we summarize the work by H. Triebel to generalize such characterization to homogeneous Hardy-Soboev spaces, or even homogeneous Triebel-Lizorkin spaces via tent spaces.

Finally, we mention several further perspectives on this topic.

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To my family for their selfless unsurpassed supports. To all my friends for their friendship and efforts to relieve my stress, without whom, the work could have been finished long long ago.

To the Moon for her company night over night.

# Contents

# List of Figures

# Notations

| Notation | Definition |
|---|---|
| $\mathfrak{A} \lesssim \mathfrak{B}$ | $\mathfrak{A} \leq C\mathfrak{B}$ for some constant $C$ independent of $\mathfrak{A}, \mathfrak{B}$ |
| $\mathfrak{A} \lesssim_\rho \mathfrak{B}$ | $\mathfrak{A} \leq C_\rho\mathfrak{B}$ for some constant $C_\rho$ dependent on $\rho$, but independent of $\mathfrak{A}, \mathfrak{B}$ |
| $\mathfrak{A} \sim \mathfrak{B}$ or $\mathfrak{A} \simeq \mathfrak{B}$ | $\mathfrak{A} \lesssim \mathfrak{B}$ and $\mathfrak{B} \lesssim \mathfrak{A}$ |
| $\mathfrak{A} \sim_\rho \mathfrak{B}$ or $\mathfrak{A} \simeq_\rho \mathfrak{B}$ | $\mathfrak{A} \lesssim_\rho \mathfrak{B}$ and $\mathfrak{B} \lesssim_\rho \mathfrak{A}$ |
| $c(B)$ | center of a ball $B$ |
| $r(B)$ | radius of a ball $B$ |
| $\lambda B$ | for a ball $B = B(x, r)$, $\lambda B := B(x, \lambda r)$ |
| $\mathbb{1}_E$ | indicator function of a set $E$ |
| $\mathscr{L}$ | Lebesgue measure |
| $c_n$ | Lebesgue measure of the unit ball $B(0, 1)$ in $\mathbb{R}^n$ |
| $\\|\cdot\\|_p$ | $L^p$-norm for $p \in (0, \infty]$ |
| $p'$ | Hölder duality of $p \in [1, \infty]$, i.e., the real number satisfying $\frac{1}{p} + \frac{1}{p'} = 1$ |
| comp | a subscript as abbreviation of "compactly-supported" |
| cylin | a subscript as abbreviation of "cylinder-supported" |
| $\underline{\mathsf{N}}$ | the category of normed vector spaces, whose objects are normed vector spaces and whose morphisms are bounded linear maps between normed vector spaces |
| $X \hookrightarrow Y$ | the normed vector space $X$ is continuously embedded into the normed vector space $Y$ |
| $L(X, Y)$ | the vector space of bounded normed operators from a normed vector space $X$ to the other $Y$, equipped with the operator norm |
| $\underline{\mathsf{B}}$ | the category of Banach spaces, whose objects are Banach spaces and whose morphisms are bounded linear maps between Banach spaces. A sub-category of $\underline{\mathsf{N}}$ |
| $X^*$ | the Banach dual space of a Banach space $X$ |
| $\mathcal{S}$ | Schwartz space on $\mathbb{R}^n$, which consists of Schwartz functions and is canonically equipped with semi-norms $\{\\|\cdot\\|\}_{(\alpha,\beta)\in\mathbb{N}^n\times\mathbb{N}^n}$ as a Fréchet space [Hö3, Definition 7.1.2] |
| $\mathcal{S}'$ | the set of all tempered distributions, i.e., continuous linear functionals of $\mathcal{S}$ [Hö3, Definition 7.1.7] |
| $\widehat{f}$ | the Fourier transformation of the tempered distribution $f \in \mathcal{S}'$ |
| $\check{f}$ | the inverse Fourier transformation of the tempered distribution $f \in \mathcal{S}'$ |

# Chapter 1

# Definitions of tent spaces

In this chapter, we will provide the definitions of tent spaces on Euclidean spaces and spaces of homogeneous type. Furthermore, we shall prove that the definitions of tent spaces are independent of aperture. In this chapter, all the functions are valued in $\mathbb{C}$.

## 1.1 Tent spaces on Euclidean spaces

### 1.1.1 Geometry of cones and tents

Let $\alpha \in (0, \infty)$ be a fixed constant. Let $\Gamma_\alpha(x) \subset \mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0, \infty)$ be the *$\alpha$-cone* based on $x \in \mathbb{R}^n$, given by

$$\Gamma_\alpha(x) := \{(y,t) \in \mathbb{R}^{n+1}_+ : |y - x| < \alpha t\}.$$

The *aperture* of $\Gamma_\alpha(x)$ is $\alpha$. For any subset $F \subset \mathbb{R}^n$, set

$$\Gamma_\alpha(F) := \bigcup_{x \in F} \Gamma_\alpha(x).$$

Meanwhile, let

$$T_\alpha(F^c) := (\Gamma_\alpha(F))^c = \{(y,t) \in \mathbb{R}^{n+1}_+ : 0 < t \le \alpha^{-1} \operatorname{dist}(y, F)\}$$

be the *$\alpha$-tent* over $F^c$. The aperture of $T_\alpha(F^c)$ is $\alpha$. Set $T_1(F^c) := \widehat{F^c}$. Figure 1.1 and 1.2 show a cone and a tent, respectively.

Another two useful geometrical operators are shadowing and resource. Let $E$ be a subset of $\mathbb{R}^{n+1}_+$. The *$\alpha$-shadow* of $E$, $S_\alpha(E)$, is given by

$$S_\alpha(E) := \{x \in \mathbb{R}^n : \Gamma_\alpha(x) \cap E \ne \varnothing\}.$$

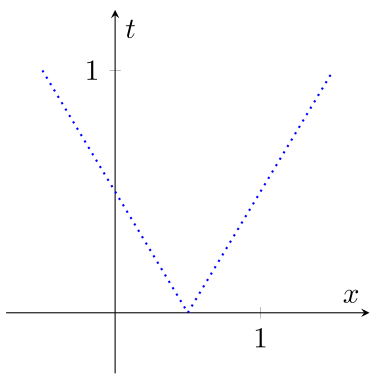


FIGURE 1.1: A cone $\Gamma(\frac{1}{2}) \subset \mathbb{R}^2_+$

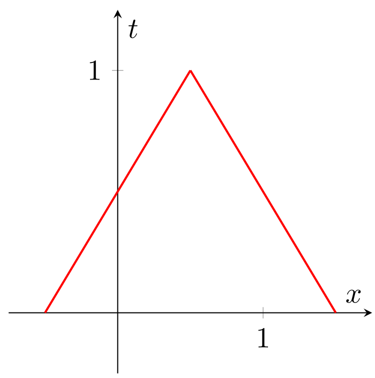


FIGURE 1.2: A tent $T(B(\frac{1}{2}, 1)) \subset \mathbb{R}^2_+$

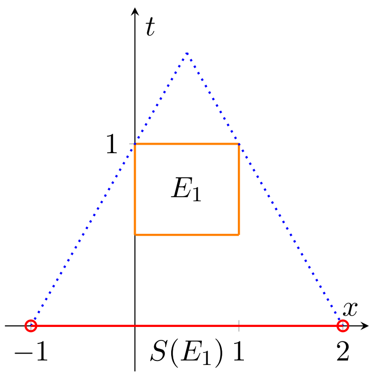


FIGURE 1.3: (1-)Shadow of $E_1$

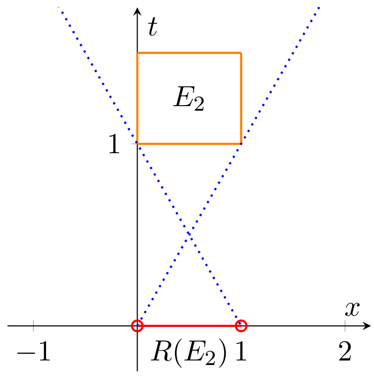


FIGURE 1.4: (1-)Resource of $E_2$

The $\alpha$-*resource* of $E$ is given by

$$R_\alpha(E) := \{x \in \mathbb{R}^n : E \subset \Gamma_\alpha(x)\}.$$

It is obvious that for any compact set $K \subset \mathbb{R}^{n+1}_+$, $S_\alpha(K)$ is open and $K \subset T_\alpha(S_\alpha(K))$. Figure 1.3 and 1.4 illustrate a shadow and a resource, respectively.

The following operators map "good" functions on $\mathbb{R}^{n+1}_+$ to functions on $\mathbb{R}^n$, which provide different methods to compress the information of functions on $\mathbb{R}^{n+1}_+$ to $\mathbb{R}^n$. Surprisingly, to some extent, these operators are equivalent [CMS85, Sec.6].

For $0 < q < \infty$, let $A_{q;\alpha}$ be the $(q;\alpha)$-*cone-averaging operator* on *measurable* functions on $\mathbb{R}^{n+1}_+$, given by

$$A_{q;\alpha}(f)(x) := \left( \int_{\Gamma_\alpha(x)} |f(y,t)|^q \frac{dy}{t^n} \frac{dt}{t} \right)^{\frac{1}{q}}.$$

Let $C_{q;\alpha}$ be the $(q;\alpha)$-*tent-maximal operator* on measurable functions, given by

$$C_{q;\alpha}(f)(x) := \sup_{\substack{B:\text{ open ball on } \mathbb{R}^n \\ x \in B}} \left( \frac{\alpha^n}{|B|} \int_{T_\alpha(B)} |f(y,t)|^q dy \frac{dt}{t} \right)^{\frac{1}{q}}.$$

We shall explain the reason for such normalization in the remark of Definition 1.3.

For $q = \infty$, $A_{\infty;\alpha}$ is identified as the *non-tangential maximal operator* on *continuous* functions as

$$A_{\infty;\alpha}(f)(x) := \sup_{(y,t) \in \Gamma_\alpha(x)} |f(y,t)|.$$

The first lemma establishes measurability.

**Lemma 1.1.** *Let $f : \mathbb{R}^{n+1}_+ \to \mathbb{C}$ be a measurable function. Then $A_{q;\alpha}(f)$ and $C_{q;\alpha}(f)$ are lower semi-continuous for any $q \in (0,\infty)$. If $f$ is further continuous, $A_{\infty;\alpha}(f)$ is also lower semi-continuous.*

*Proof.* The proof is based on shadowing. It suffices to verify that $E_1 := \{x \in \mathbb{R}^n : A_{q;\alpha}(f)(x) > \lambda\}$ and $E_2 := \{x \in \mathbb{R}^n : C_{q;\alpha}(f)(x) > \lambda\}$ are open for any $\lambda \geq 0$. Set that $\Gamma_\alpha(x) + h := \{(y,t) \in \mathbb{R}^{n+1}_+ : (y, t-h) \in \Gamma_\alpha(x)\}$ for any $h \geq 0$.

If $q \in (1,\infty)$, for any $x \in E_1$, note that

$$A_{q;\alpha}(f)(x) = \limsup_{h \to 0^+} \left( \int_{\Gamma_\alpha(x)+h} |f(y,t)|^q \frac{dy}{t^n} \frac{dt}{t} \right)^{\frac{1}{q}},$$

so there exists some $h_0 > 0$ such that RHS $> \lambda$. In particular, $\Gamma_\alpha(x) + h_0 \subset \Gamma_\alpha(z)$ for any $z \in B(x, \frac{\alpha}{2}h_0)$, so $E_1$ is hence open since $B(x, \frac{\alpha}{2}h_0) \subset E_1$. For any $x \in E_2$, we can directly pick some open ball $B$ such that $(\frac{1}{|B|}\int_{T_\alpha(B)} |f(y,t)|^q dy \frac{dt}{t})^{\frac{1}{q}} > \lambda$, so $B \subset E_2$, and $E_2$ is hence open.

If $q = \infty$, for any $x \in E_1$, there exists some $(y,t) \in \Gamma_\alpha(x)$ such that $|f(y,t)| > \lambda$. For any $z \in B(y, \alpha t)$, $(y,t) \in \Gamma_\alpha(z)$, i.e., $B(y, \alpha t) \subset S_\alpha(\{(y,t)\})$. Thus,

$$|f(y,t)| \leq \inf_{z \in B(y,\alpha t)} A_{\infty;\alpha}(f)(z), \tag{1.1}$$

so $B(y, \alpha t) \subset E_1$. Moreover, $x \in B(y, \alpha t)$ since $(y,t) \in \Gamma_\alpha(x)$. $\square$

Next lemma shows that $A_{q;\alpha}$ exactly behaves as an averaging operator, and the reason why we choose $\frac{dy}{t^n}\frac{dt}{t}$ as the measure attached to each cone $\Gamma_\alpha(x)$.

**Lemma 1.2** ($A$-averaging lemma)**.** *Let $\Phi : \mathbb{R}^{n+1}_+ \to [0,\infty)$ be a positive measurable function. Then*

$$\int_{\mathbb{R}^n} \left( \int_{\Gamma_\alpha(x)} \Phi(y,t) \frac{dy}{t^n} \frac{dt}{t} \right) dx = c_n \alpha^n \int_{\mathbb{R}^{n+1}_+} \Phi(y,t) dy \frac{dt}{t}.$$

*Proof.* Fubini theorem directly reads that

$$\text{LHS} = \int_{\mathbb{R}^{n+1}_+} \left( \int_{\mathbb{R}^n} \mathbb{1}_{\Gamma_\alpha(x)}(y,t) dx \right) \Phi(y,t) \frac{dy}{t^n} \frac{dt}{t} = \text{RHS},$$

since $(y,t) \in \Gamma_\alpha(x)$ if and only if $x \in B(y, \alpha t)$. $\square$

### 1.1.2 Definitions of tent spaces on Euclidean spaces

We generalize the original definitions of tent spaces in [CMS85]. The tent spaces consist of the following four types of function spaces on $\mathbb{R}^{n+1}_+$.

**Definition 1.3** (Tent spaces)**.** Let $p, q$ be two numbers in $(0, \infty]$.

1. For $(p,q) \in (0,\infty) \times (0,\infty)$, the *tent space* $T^{p,q}_{(\alpha)}$ consists of measurable functions $f$ on $\mathbb{R}^{n+1}_+$ so that $A_{q;\alpha}(f) \in L^p(\mathbb{R}^n)$, equipped with the quasi-norm
$$\|f\|_{T^{p,q}_{(\alpha)}} := \|A_{q;\alpha}(f)\|_{L^p(\mathbb{R}^n)}.$$

2. For $(p,\infty)$ with $p \in (0,\infty)$, the *tent space* $T^{p,\infty}_{(\alpha)}$ consists of *continuous* functions $f$ on $\mathbb{R}^{n+1}_+$ so that $A_{\infty;\alpha}(f) \in L^p(\mathbb{R}^n)$, equipped with the quasi-norm
$$\|f\|_{T^{p,\infty}_{(\alpha)}} := \|A_{\infty;\alpha}(f)\|_{L^p(\mathbb{R}^n)}.$$

3. For $(\infty, q)$ with $q \in (0,\infty)$, the *tent space* $T^{\infty,q}_{(\alpha)}$ consists of measurable functions $f$ on $\mathbb{R}^{n+1}_+$ so that $C_{q;\alpha}(f) \in L^\infty(\mathbb{R}^n)$, equipped with the quasi-norm
$$\|f\|_{T^{\infty,q}_{(\alpha)}} := \|C_{q;\alpha}(f)\|_{L^\infty(\mathbb{R}^n)}.$$

4. For $(\infty,\infty)$, set $T^{\infty,\infty}_{(\alpha)} =: L^\infty(\mathbb{R}^{n+1}_+)$ with the norm
$$\|f\|_{T^{\infty,\infty}_{(\alpha)}} := \|f\|_{L^\infty(\mathbb{R}^{n+1}_+)}.$$

Without special mention, the above four types will be correspondingly denoted as type $(p,q),(p,\infty),(\infty,q),(\infty,\infty)$, respectively, with $p,q\in(0,\infty)$.

We ignore the subscript of $\alpha$ or $q$ if $\alpha=1$ or $q=2$, respectively, for all the operators above mentioned. A tautological case is that when $q$ is ignored but $\alpha$ not, it is mandatory to write $A_{(\alpha)}$ or $C_{(\alpha)}$ for fear of misunderstandings.

*Remark.* There is another kind of definition for type $(p,\infty)$ by [Ame18, Definition 1.2] which reduces the continuity condition by measurability, and hence replacing supremum by essential supremum for $A_{\infty;\alpha}$. Such a case can be regarded as a variant of tent spaces on non-locally-compact spaces of homogeneous type in Sec.1.2.2.

*Remark.* Observe that the scaling $f(y,t)\mapsto\alpha^{-\frac{n}{q}}f(y,\frac{t}{\alpha})$ induces an isometry from $T^{p,q}_{(\alpha)}$ to $T^{p,q}$ for $p\in(0,\infty]$ and $q\in(0,\infty)$, which is ensured by the way of normalization in the definitions of $C_{q;\alpha}$.

### 1.1.3 Localization

The definitions of tent spaces combine the local regularity on each cone or tent and the global integrability on the bottom space $\mathbb{R}^n$. They provide with elaborate mixture but in company of complicated transitions. The following lemma reduces mixed integrals to the problem of convergence via a dense subspace of tent spaces by localization on compact sets. For convenience, a function space with subscript "comp" denotes its intersection with compactly supported functions.

**Lemma 1.4** (Localization lemma)**.** *For type $(p,q)$, $L^q_{\mathrm{comp}}(\mathbb{R}^{n+1}_+)$ is dense in $T^{p,q}_{(\alpha)}$. For type $(p,\infty)$, $C^0_{\mathrm{comp}}(\mathbb{R}^{n+1}_+)$ is dense in $T^{p,\infty}_{(\alpha)}$.*

*Proof.* First, fix a sequence of compact sets $K_m$ as an *exhaustion* of $\mathbb{R}^{n+1}_+$, i.e., $K_m\subset K_{m+1}$ for all $m\in\mathbb{N}$, and $\bigcup_{m\in\mathbb{N}}K_m=\mathbb{R}^{n+1}_+$.

Without loss of generality, assume that $K$ is of the form $\overline{B}(x_0,r)\times[a,b]$ for some open ball $B(x_0,r)\subset\mathbb{R}^n$ and $a,b\in(0,\infty)$. Pick $f\in T^{p,q}_{(\alpha)}$ and $g\in L^q(K)$. Note that $A_{q;\alpha}(g)$ is supported on $S_\alpha(K)=B(x_0,r+\alpha b)\subset\mathbb{R}^n$, which is a bounded subset.

For type $(p,q)$,

$$\begin{aligned}
\|g\|^p_{T^{p,q}_{(\alpha)}}&\le\int_{S_\alpha(K)}\left(\int_{\Gamma_\alpha(x)}|g(y,t)|^q\frac{dy}{t^n}\frac{dt}{t}\right)^{\frac{p}{q}}dx\\
&\le a^{-\frac{(n+1)p}{q}}\int_{S_\alpha(K)}\left(\int_K|g(y,t)|^q dydt\right)^{\frac{p}{q}}dx\\
&\le a^{-\frac{(n+1)p}{q}}|S_\alpha(K)|\|g\|^p_{L^q(K)}.
\end{aligned}$$

In turn, break $\overline{B}(x_0,r)$ by a finite open cover of balls of radius $\frac12\alpha a$, i.e., pick $x_1,\ldots,x_N$ such that $\overline{B}(x_0,r)\subset\bigcup_{j=1}^N B(x_j,\frac12\alpha a)$. Note that $B\left(x_j,\frac12\alpha a\right)\subset R_\alpha\left(B\left(x_j,\frac12\alpha a\right)\times[a,b]\right)$, which ensures the averaging estimate on each piece as

$$\begin{aligned}
\left(\int_{B(x_j,\frac12\alpha a)\times[a,b]}|f(y,t)|^q\frac{dy}{t^n}\frac{dt}{t}\right)^{\frac1q}&\le\left(⨍_{B(x_j,\frac12\alpha a)}\left(\int_{\Gamma_\alpha(x)}|f(y,t)|^q\frac{dy}{t^n}\frac{dt}{t}\right)^{\frac{p}{q}}dx\right)^{\frac1p}\\
&\lesssim_{n,p,\alpha,a}\left(\int_{B(x_j,\frac12\alpha a)}\left(A_{q;\alpha}(f)(x)\right)^p dx\right)^{\frac1p}\le\|f\|_{T^{p,q}_{(\alpha)}}.
\end{aligned}$$

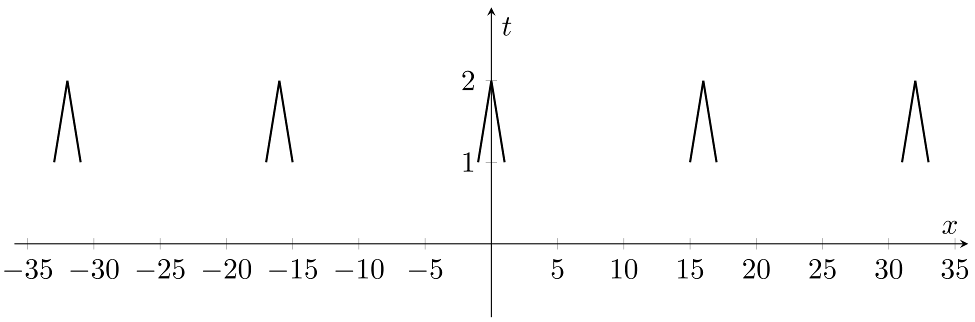


FIGURE 1.5: Counterexample for density of $L^2_{\text{comp}}(\mathbb{R}^2_+)$ in $T^\infty(\mathbb{R}^2_+)$

Glue these pieces up as

$$\begin{aligned}\|f\mathbb{1}_K\|_q &\le b^{\frac{n+1}{q}}\left(\int_K |f(y,t)|^q \frac{dy}{t^n}\frac{dt}{t}\right)^{\frac{1}{q}}\\ &\lesssim_{N,q} b^{\frac{n+1}{q}}\sum_{j=1}^N\left(\int_{B(x_j,\frac{1}{2}\alpha a)\times[a,b]} |f(y,t)|^q \frac{dy}{t^n}\frac{dt}{t}\right)^{\frac{1}{q}}\\ &\lesssim_{N,n,p,\alpha,a} b^{\frac{n+1}{q}}\|f\|_{T^{p,q}_{(\alpha)}}.\end{aligned}$$

The convergence follows immediately from Levy's theorem since $\{A_{q;\alpha}(f\mathbb{1}_{K_m})\}_{m\in\mathbb{N}}$ is an increasing sequence of positive functions, which pointwise converges to $A_{q;\alpha}(f)$.

For type $(p,\infty)$, similar technique gives that

$$\|g\|^p_{T^{p,\infty}_{(\alpha)}} \le \int_{S_\alpha(K)} (A_{\infty;\alpha}(g)(x))^p dx \le |S_\alpha(K)|\|g\|^p_\infty,$$

and

$$\sup_{(y,t)\in B(x_j,\frac{1}{2}\alpha a)\times[a,b]} |f(y,t)| \le \left(⨍_{B(x_j,\frac{1}{2}\alpha a)} (A_{\infty;\alpha}(f)(x))^p\, dx\right)^{\frac{1}{p}} \lesssim_{\alpha,a,n,p} \|f\|_{T^{\infty,q}_{(\alpha)}}.$$

So,

$$\|f\chi_K\|_\infty \le \max_{1\le j\le N} \sup_{(y,t)\in B(x_j,\frac{1}{2}\alpha a)\times[a,b]} |f(y,t)| \lesssim_{\alpha,a,n,p} \|f\|_{T^{\infty,q}_{(\alpha)}},$$

where $\chi_K$ is the mollification of $\mathbb{1}_K$, supported on $K$ and differing from $\mathbb{1}_K$ only on a small neighbourhood of $\partial K$. Note that $\{A_{\infty;\alpha}(f\chi_{K_m})\}_{m\in\mathbb{N}}$ is also an increasing sequence of positive functions, pointwise converging to $A_{\infty;\alpha}(f)$. □

*Remark.* Lemma 1.4 can not be generalized to the following two types. It is obvious for type $(\infty,\infty)$ considering $f\equiv 1$. For type $(\infty,q)$, consider the simplest case $\mathbb{R}^2_+$ with $q=2,\alpha=1$. Define $H_j^\pm = T(B(\pm 16j,1))\cap(\mathbb{R}\times[1,2])$ and $\phi := \sum_j \mathbb{1}_{H_j^\pm}$. We illustrate part of $\phi$ by Figure 1.5. For any open ball $B=B(x_0,r)$, if it touches only one of $H_j$'s, then $\left(\frac{1}{|B|}\int_{\widehat{B}}|\phi(y,t)|^2 dy\frac{dt}{t}\right)^{\frac{1}{2}} \le 1$. If $B$ touches $k$ of $H_j$'s for $k\ge 2$, then $r\ge 8(k-1)$, so $\left(\frac{1}{|B|}\int_{\widehat{B}}|\phi(y,t)|^2 dy\frac{dt}{t}\right)^{\frac{1}{2}} \le \frac{1}{2}$. Thus, $\|\phi\|_{T^\infty}\le 1$ but it always differ from any fixed compact-supported function on some sufficiently away $H_j^\pm$'s.

### 1.1.4 Completeness of tent spaces

**Proposition 1.5.** *Suppose that $p, q \in [1,\infty]$. Then $T^{p,q}_{(\alpha)}$ are Banach spaces.*

*Proof.* It suffices to check the completeness. The proof is based on localization. The proposition is trivial for the type $(\infty,\infty)$. Let $\{f_k\}_{k\in\mathbb{N}}$ be a Cauchy sequence in $T^{p,q}_{(\alpha)}$.

For type $(p,q)$, localization lemma, cf. Lemma 1.4 reads that $\{f_k \mathbb{1}_K\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^q(K)$ for any compact set $K \subset \mathbb{R}^{n+1}_+$, hence converging to some $f_K \in L^q(K)$. Pick $\{K_m\}_{m\in\mathbb{N}}$ as an exhaustion of $\mathbb{R}^{n+1}_+$. Define that $f = \lim_{m\to\infty} f_{K_m}$ pointwise, whose legality follows from identification, i.e., for any $i, j \in \mathbb{N}$ with $i > j$, $f_{K_i}\mathbb{1}_{K_j} = f_{K_j}$. For any $\epsilon > 0$, pick sufficiently large $m, k$ so that

$$\|f\|_{T^{p,q}_{(\alpha)}} \leq \|f\mathbb{1}_{K_m}\|_{T^{p,q}_{(\alpha)}} + \epsilon \lesssim \|f\|_{L^q(K_m)} + \epsilon \lesssim \|f_k\|_{L^q(K_m)} + \epsilon \lesssim \|f_k\|_{T^{p,q}_{(\alpha)}} + \epsilon < \infty,$$

so $f \in T^{p,q}_{(\alpha)}$. Moreover, for any $\epsilon > 0$, pick $N, m \in \mathbb{N}$ such that $\|f - f\mathbb{1}_{K_m}\|_{T^{p,q}_{(\alpha)}} < \epsilon$, $\|f_N - f_N\mathbb{1}_{K_m}\|_{T^{p,q}_{(\alpha)}} < \epsilon$, and $\|f_k - f_N\|_{T^{p,q}_{(\alpha)}} < \epsilon$ for any $k \geq N$. Thus, for any $k \geq N$,

$$\begin{aligned}
\|f - f_k\|_{T^{p,q}_{(\alpha)}} \leq{}& \|f - f\mathbb{1}_{K_m}\|_{T^{p,q}_{(\alpha)}} + \|(f - f_k)\mathbb{1}_{K_m}\|_{T^{p,q}_{(\alpha)}} + \|(f_k - f_N)\mathbb{1}_{K_m}\|_{T^{p,q}_{(\alpha)}} \\
&+ \|f_N\mathbb{1}_{K_m} - f_N\|_{T^{p,q}_{(\alpha)}} + \|f_N - f_k\|_{T^{p,q}_{(\alpha)}} \\
&< 4\epsilon + \|(f - f_k)\mathbb{1}_{K_m}\|_{T^{p,q}_{(\alpha)}}.
\end{aligned}$$

We conclude by taking limsup on both sides on $k$.

The proof for type $(p,\infty)$ is of the same. Extra continuity follows from the Banach property of $C^0(K)$ on each compact $K$.

For type $(\infty, q)$, for any open ball $B \subset \mathbb{R}^n$, $\{f_k\mathbb{1}_{T_\alpha(B)}\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $L^q(T_\alpha(B))$, hence converging to some $f_B \in L^q(T_\alpha(B))$. Exhausting $\mathbb{R}^{n+1}_+$ by $T_\alpha(B)$'s and verifying the identification via the same process, we obtain a measurable function $f$ such that $f\mathbb{1}_{T_\alpha(B)} = f_B$. For any ball $B$,

$$\left(\frac{\alpha^n}{|B|}\int_{T_\alpha(B)} |f(y,t)|^q dy\frac{dt}{t}\right)^{\frac{1}{q}} \leq \sup_{n\geq 1}\left(\frac{\alpha^n}{|B|}\int_{T_\alpha(B)} |f_n(y,t)|^q dy\frac{dt}{t}\right)^{\frac{1}{q}} \leq \sup_{n\geq 1}\|f_n\|_{T^{\infty,q}_{(\alpha)}},$$

so $f \in T^{\infty,q}_{(\alpha)}$. Note that

$$\begin{aligned}
&\sup_B\ \alpha^{\frac{n}{q}}|B|^{-\frac{1}{q}}\|f - f_k\|_{L^q(T_\alpha(B);dy\frac{dt}{t})} \\
&\leq \sup_B \alpha^{\frac{n}{q}}|B|^{-\frac{1}{q}}\left[\limsup_{\substack{m\to\infty\\ m\geq k}}\left(\|f - f_m\|_{L^q(T_\alpha(B);dy\frac{dt}{t})} + \|f_m - f_k\|_{L^q(T_\alpha(B);dy\frac{dt}{t})}\right)\right] \\
&\leq \sup_B \limsup_{\substack{m\to\infty\\ m\geq k}} \alpha^{\frac{n}{q}}|B|^{-\frac{1}{q}}\|f - f_m\|_{L^q(T_\alpha(B);dy\frac{dt}{t})} + \sup_{m\geq k}\|f_m - f_k\|_{T^{p,q}_{(\alpha)}} \\
&\leq \sup_{m\geq k}\|f_m - f_k\|_{T^{p,q}_{(\alpha)}}.
\end{aligned}$$

We conclude by Cauchy property. □

## 1.2 Tent spaces on spaces of homogeneous type

In this section, we shall generalize the work [Ame14] to define tent spaces on spaces of homogeneous type, a framework for unification of various typical geometrical objects in harmonic analysis. The completeness will also be discussed.

### 1.2.1 Geometry of spaces of homogeneous type

Let us first recall the definition and basic geometrical properties of spaces of homogeneous type. Let $X$ be a non-empty set.

**Definition 1.6** (Quasi-metric)**.** A function $d : X \times X \to [0, \infty)$ is called a *quasi-metric* on $X$ if for any $x, y, z \in X$,

1. $d(x, y) = d(y, x)$;
2. $d(x, y) = 0$ if and only if $x = y$;
3. there exists a constant $\kappa \geq 1$ such that $d(x, y) \leq \kappa(d(x, z) + d(z, y))$.

For any $x \in X$ and $r > 0$, the set $B(x, r) := \{y \in X : d(x, y) < r\}$ is called the *d-ball* or *ball* centred at $x$ with radius $r$ in $X$, and $\overline{B}(x, r) := \{y \in X : d(x, y) \leq r\}$.

**Definition 1.7** (Quasi-metric space)**.** Let $X$ be a non-empty set with a quasi-metric $d$. The space $(X, d)$ is called a *quasi-metric space* if $X$ is equipped with the topology induced by $d$, i.e., a subset $O \subset X$ is open if for any $x \in O$, there exists $r > 0$ such that $B(x, r) \subset O$.

A $d$-ball is not necessarily open if $\kappa > 1$.

**Definition 1.8** (Doubling, [DH09, Definition 1.1])**.** Let $(X, d)$ be a quasi-metric space. A positive $\sigma$-finite measure $\mu$ is called *doubling* if

1. it is defined on a $\sigma$-algebra containing all the Borel sets and $d$-balls;
2. there exists a constant $C > 0$ such that for any $x \in X$ and $r > 0$,

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)).$$

For convenience, we shall write $V(x, r) := \mu(B(x, r))$.

**Definition 1.9** (Space of homogeneous type)**.** Let $(X, d, \mu)$ be a measured quasi-metric space. It is called a *space of homogeneous type* if $\mu$ is non-zero and doubling.

There are many examples of spaces of homogeneous type.

**Example 1.10.**

1. $\mathbb{R}^n$ with Euclidean distance $d_2$ and Lebesgue measure $\mathscr{L}$.
2. An open subset $\Omega \subset \mathbb{R}^n$ of Lipschitz boundary, with $d_2$ and $\mathscr{L}$.
3. A compact Riemannian manifold with geodesic distance and Riemannian volume. It is a straightforward corollary of Bishop-Gromov volume comparison theorem [Pet16, Lemma 7.1.4].
4. Compact sub-Riemannian manifolds with generalized curvature-dimension inequality are spaces of homogeneous type [Bau+14, Theorem 6].

Several geometrical and measured properties of spaces of homogeneous type will be useful for the following deduction.

**Proposition 1.11.** *Let $(X, d, \mu)$ be a space of homogeneous type.*

1. *For any $\lambda > 0$, define that*

$$\rho(\lambda) := \sup_{B:\text{ball in } X} \frac{\mu(\lambda B)}{\mu(B)}.$$

*It holds that $\rho(\lambda) < \infty$.*

2. *It has* geometrical doubling property*, i.e., there exists $N \in \mathbb{N}$, such that for any $r > 0$ and $x_0 \in X$, we can always find $N$ points $\{x_j\}_{1\leq j\leq N} \subset \overline{B}(x_0, 2r)$ such that $\overline{B}(x_0, 2r) \subset \bigcup_{j=1}^N B(x_j, r)$.*

3. *For any ball $B$, it holds that $0 < \mu(B) < \infty$. Moreover, pick $0 < a \leq b < \infty$. Then, there are two constants $C_0, C_1 \in (0, \infty)$ such that $C_0 \leq \mu(B(y,t)) \leq C_1$ for any $y \in B(x,r)$ with $t \in [a,b]$.*

4. *It holds that $\mu(X) < \infty$ if and only if $X$ is bounded.*

*Proof.* See [CW71, Chapitre III §1]. □

### 1.2.2 Definitions of tent spaces on spaces of homogeneous type

All the geometrical operators can be defined on $X_+ := X \times (0, \infty)$ in analogy of those on $\mathbb{R}^{n+1}_+$, but just by changing Euclidean distances to quasi-metric distances. One thing we should emphasize is that, if $X$ is bounded, then for any set $O \supset X$,

$$T_\alpha(O) = (\Gamma_\alpha(O^c))^c = (\varnothing)^c = X_+. \tag{1.2}$$

Such case different from $\mathbb{R}^n$ should be treated more carefully.

The measure attached to each cone $\Gamma_\alpha(x)$, $\sigma_\alpha$, is correspondingly modified as

$$d\sigma_\alpha(y,t) = \frac{d\mu(y)}{V(y,\alpha t)} \frac{dt}{t}.$$

Such modification ensures the $A$-averaging, cf. Lemma 1.2, still holds as

$$\int_X \left( \int_{\Gamma_\alpha(x)} \Phi(y,t) \frac{d\mu(y)}{V(y,\alpha t)} \frac{dt}{t} \right) d\mu(x) = \int_{X_+} \Phi(y,t) d\mu(y) \frac{dt}{t}.$$

For $q \in (0, \infty)$, cone-averaging and tent-maximal operators on measurable functions are given by

$$A_{q;\alpha}(f)(x) := \left( \int_{\Gamma_\alpha(x)} |f(y,t)|^q \frac{d\mu(y)}{V(y,\alpha t)} \frac{dt}{t} \right)^{\frac{1}{q}},$$

and

$$C_{q;\alpha}(f)(x) := \sup_{\substack{B:d\text{-balls in } X \\ x \in B}} \left( \frac{1}{\mu(B)} \int_{T_\alpha(B)} |f(y,t)|^q d\mu(y) \frac{dt}{t} \right)^{\frac{1}{q}}.$$

The operators make sense due to finite measure of balls, cf. Proposition 1.11, and also map measurable functions to lower semi-continuous functions if it makes sense by the same proof of Lemma 1.1. The measurability is hence established. Tent spaces of type $(p,q)$, $(\infty, q)$, and $(\infty, \infty)$ are defined of the same as Definition 1.3. Note that the normalization here is to ensure that $f(y,t) \mapsto f(y, \frac{t}{\alpha})$ is an isometry of $T^{p,q}_{(\alpha)}$ to $T^{p,q}$ for $p \in (0, \infty]$ and $q \in (0, \infty)$.

For $q = \infty$, non-tangential maximal operators $A_{\infty;\alpha}$ are modified as

$$A_{\infty;\alpha}(f)(x) := \sigma_\alpha\text{-}\operatorname*{esssup}_{(y,t)\in\Gamma_\alpha(x)} |f(y,t)|.$$

It corresponds to the original on $\mathbb{R}^{n+1}_+$ since essential supremum for continuous functions acts of the same as supremum. We claim that, for a measurable function $f$, such modification still ensures that $A_{\infty;\alpha}(f)$ is lower semi-continuous. Indeed,

$$A_{\infty;\alpha}(x) = \lim_{h\to 0^+} \sigma_\alpha\text{-}\operatorname*{esssup}_{(y,t)\in\Gamma_\alpha(x)+h} |f(y,t)|.$$

Thus, if $A_{\infty;\alpha}(x) > \lambda$, we can pick a sufficiently small $h$ such that $\{(y,t) \in \Gamma_\alpha(x)+h : |f(y,t)| > \lambda\}$ takes positive measure for $\sigma_\alpha$, but it is contained in $B(x, \alpha h)$.

For $p \in (0,\infty)$, we define that the tent space $T^{p,\infty}_{(\alpha)}$ consists of

- *continuous* functions $f$ on $X_+$ so that $A_{\infty;\alpha}(f) \in L^p(X)$, if $X$ is locally compact;
- *measurable* functions $f$ on $X_+$ so that $A_{\infty;\alpha}(f) \in L^p(X)$, otherwise.

No matter of the case, the quasi-norm $\|\cdot\|_{T^{p,\infty}_{(\alpha)}}$ is always given by

$$\|f\|_{T^{p,\infty}_{(\alpha)}} := \|A_{\infty;\alpha}(f)\|_{L^p(\mathbb{R}^n)}.$$

The variant of localization, cf. Lemma 1.4, is more subtle. In general, it is achieved by cylinder-supported functions. By a *cylinder* in $X_+$, we denote a subset of the form $B(x_0,r)\times[a,b]$ for some $a,b > 0$ and some ball $B(x_0,r)$. For convenience, a function space with subscript "cylin" denotes its intersection with cylinder-supported functions.

**Lemma 1.12.** *1. For type $(p,q)$, $L^q_{\text{cylin}}(X_+)$ is dense in $T^{p,q}_{(\alpha)}$.*

*2. For type $(p,\infty)$, $C^0_{\text{cylin}}(X_+)$ is dense in $T^{p,\infty}_{(\alpha)}$ if $X$ is locally compact; $L^\infty_{\text{cylin}}(X_+)$ is dense in $T^{p,\infty}_{(\alpha)}$, otherwise.*

*Proof.* We just show several necessary modifications of the proof for Lemma 1.4. Fix $K$ as a cylinder of the form $\overline{B}(x_0,r)\times[a,b]$. Pick $g \in L^q(K)$ and $f \in T^{p,q}_{(\alpha)}$. Geometrical statements below have been summarized in Proposition 1.11.

For type $(p,q)$, $A_{q;\alpha}(g)$ is supported on $S_\alpha(K) \subset B(x_0, \kappa(\alpha b + r))$ that has finite measure. For any $(y,t) \in K$, $V(y,\alpha t)$ is uniformly bounded below by a strictly positive constant. It hence finalizes embedding $L^q(K)$ into $T^{p,q}_{(\alpha)}$. Cover $\overline{B}(x_0,r)$ by finite balls $\{B(x_j, \widetilde{r} := \frac{1}{2\kappa}\alpha a)\}_{1\le j\le N}$. For any $z \in B(x_j,\widetilde{r})$, $z \in B(y,\alpha a)$ for any $y \in B(x_j,\widetilde{r})$, i.e., $B(x_j,\widetilde{r}) \subset R_\alpha(B(x_j,\widetilde{a})\times[a,b])$. Thus,

$$\begin{aligned}\left(\int_{B(x_j,\widetilde{r})\times[a,b]} |f(y,t)|^q \frac{d\mu(y)}{V(y,\alpha t)}\frac{dt}{t}\right)^{\frac{1}{q}} &\le V(x_j,\widetilde{r})^{-\frac{1}{p}} \left(\int_{B(x_j,\widetilde{r})} (A_{q;\alpha}(f)(x))^p \, d\mu(x)\right)^{\frac{1}{p}} \\ &\le V(x_j,\widetilde{r})^{-\frac{1}{p}} \|f\|_{T^{p,q}_{(\alpha)}}.\end{aligned}$$

It implies that $f\mathbb{1}_K \in L^q(K)$ since $V(y,\alpha t)$ is upper bounded by a constant. Deduction for convergence by localization is of the same as Lemma 1.4.

For type $(p,\infty)$, it still holds that

$$\sigma_\alpha\text{-}\operatorname*{esssup}_{(y,t)\in E} |f(y,t)| \le \inf_{z\in R_\alpha(E)} A_{\infty;\alpha}(f)(z). \tag{1.3}$$

Thus, the proof is exactly of the same as that for Lemma 1.4. $\square$

The completeness, cf. Proposition 1.5, can also be established.

**Proposition 1.13.** *Suppose that $p, q \in [1, \infty]$. Then $T^{p,q}_{(\alpha)}$ are Banach spaces.*

*Proof.* The proof for types $(p, q), (\infty, \infty)$ is exactly of the same. For type $(p, \infty)$, if $X$ is locally compact, $C^0(K)$ is a Banach space for any cylinder $K$. Thus, the local limit of a Cauchy sequence in $T^{p,\infty}_{(\alpha)}$ is also continuous.

For type $(\infty, q)$, if $X$ is unbounded, i.e., $\mu(X) = \infty$, we can also exhaust $X$ by $\alpha$-tent of balls in $X$. The whole process is of the same as the proof of Proposition 1.5.

If $X$ bounded, fix $x_0 \in X$, pick sufficiently large $\tau$ so that $X \subset B(x_0, \tau)$. Eq.(1.2) reads that $T_\alpha(B(x_0, \tau)) = X_+$, so $\{f_n\}_{n\in\mathbb{N}}$ itself is a Cauchy sequence in $L^q(X_+; d\mu(y)\frac{dt}{t})$, hence converging to some $f \in L^q(X_+; d\mu(y)\frac{dt}{t})$. Note that

$$\left(\frac{1}{\mu(B)}\int_{T_\alpha(B)} |f(y,t)|^q d\mu(y)\frac{dt}{t}\right)^{\frac{1}{q}} \leq \sup_{n\geq 1} \|f_n\|_{T^{\infty,q}_{(\alpha)}} < \infty,$$

so $f \in T^{\infty,q}_{(\alpha)}$. We conclude again by Cauchy property as

$$\sup_B \ \mu(B)^{-\frac{1}{q}} \|f - f_k\|_{L^q(T_\alpha(B); d\mu(y)\frac{dt}{t})} \leq \sup_{m\geq k} \|f_m - f_k\|_{T^{p,q}_{(\alpha)}}.$$

via the same argument as the proof of Proposition 1.5. ☐

## 1.3 Independence of aperture

In this section, we shall prove that the definitions of tent spaces are independent of the aperture of cones or tents for all the four types. More precisely, the following theorem holds both on Euclidean spaces and spaces of homogeneous type.

**Theorem 1.14.** *For any $f$ as a measurable function on $\mathbb{R}^{n+1}_+$,*

$$\|f\|_{T^{p,q}_{(\beta)}} \sim \|f\|_{T^{p,q}_{(\alpha)}}$$

*for any real number $\alpha, \beta > 0$ and any type $(p, q)$.*

This theorem provides the freedom of changing apertures of cone-averaging or tent-maximal operators with a slight sacrifice on the controlling constant. Such property will be quite useful for covering. Nothing happens for type $(\infty, \infty)$ on any space.

### 1.3.1 On Euclidean spaces

We shall verify most of cases on Euclidean spaces in this subsection.

#### 1.3.1.1 Type $(p, q)$

We first make some simplification by spatially scaling. For any $\beta > 0$,

$$\|f\|_{T^{p,q}_{(\beta)}} \simeq_{\beta,n,q} \left(\int_{\mathbb{R}^n} \left(\int_{\Gamma(\frac{x}{\beta})} |f(\beta z, t)|^q \frac{dz}{t^n}\frac{dt}{t}\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}} \simeq_{\beta,n,p,q} \|f_\beta\|_{T^{p,q}}$$

where $f_\beta(x,t) = f(\beta x, t)$. If $\beta \leq 1$, we have that $\|f\|_{T^{p,q}_{(\beta)}} \sim \|f_\beta\|_{T^{p,q}}$, and $\|f_\beta\|_{T^{p,q}_{(\frac{1}{\beta})}} \sim \|f\|_{T^{p,q}}$ Thus, it suffices to prove that

$$\|f\|_{T^{p,q}_{(\alpha)}} \lesssim \|f\|_{T^{p,q}}. \tag{1.4}$$

for $\alpha > 1$ with $f \in T^{p,q}$. Indeed, if so, we can obtain that $\|f_\beta\|_{T^{p,q}} \sim \|f_\beta\|_{T^{p,q}_{(\frac{1}{\beta})}}$, hence $\|f\|_{T^{p,q}_{(\beta)}} \sim \|f\|_{T^{p,q}} \sim \|f\|_{T^{p,q}_{(\alpha)}}$. Other cases can also hence be established.

**Case** $p = q$ Fubini theorem, or $A$-averaging, reads that

$$\|f\|_{T^{p,p}_{(\alpha)}} = c_n^{\frac{1}{p}} \alpha^{\frac{n}{p}} \left( \int_{\mathbb{R}^{n+1}_+} |f(y,t)|^p dy \frac{dt}{t} \right)^{\frac{1}{p}} = \alpha^{\frac{n}{p}} \|f\|_{T^{p,p}}. \tag{1.5}$$

**Case** $p > q$ We have that

$$\|f\|_{T^{p,q}_{(\alpha)}} = \| \left(A_{q;\alpha}(f)\right)^q \|_{\frac{p}{q}}^{\frac{1}{q}} = \sup_{\substack{g \in L^r(\mathbb{R}^n;[0,\infty)) \\ \|g\|_r = 1}} \left( \int_{\mathbb{R}^n} \left(A_{q;\alpha}(f)(x)\right)^q g(x) dx \right)^{\frac{1}{q}},$$

where $r$ is the Hölder duality of $\frac{p}{q}$. Set that

$$M_\alpha(f)(x,t) = \fint_{B(x,\alpha t)} f(y) dy. \tag{1.6}$$

Let $\mathcal{M}$ be the uncentred Hardy-Littlewood maximal operator, so

$$\begin{aligned}
\int_{\mathbb{R}^n} \left(A_{q;\alpha}(f)(x)\right)^q g(x) dx &\simeq_{n,\alpha} \int_{\mathbb{R}^{n+1}_+} \left( \fint_{B(y,\alpha t)} g(x) dx \right) |f(y,t)|^q dy \frac{dt}{t} \\
&\simeq_n \int_{\mathbb{R}^n} \left( \int_{\Gamma(x)} |f(y,t)|^q M_\alpha(g)(y,t) \frac{dy}{t^n} \frac{dt}{t} \right) dx \\
&\leq \int_{\mathbb{R}^n} (A_q(f)(x))^q \mathcal{M}(g)(x) dx \lesssim \|f\|^q_{T^{p,q}}.
\end{aligned}$$

Such duality argument is quite useful for the case $p > q$ since the loss of Fubini theorem can be compensated by the duality.

**Case** $p < q$ Let us first observe what happens for $A$-averaging on a subset.

**Lemma 1.15.** *Let $F \subset \mathbb{R}^n$ be a measurable subset and $\Phi : \mathbb{R}^{n+1}_+ \to [0,\infty)$ a measurable non-negative function. Then,*

$$\int_F \left( \int_{\Gamma_\alpha(x)} \Phi(y,t) \frac{dy}{t^n} \frac{dt}{t} \right) dx \lesssim_{n,\alpha} \int_{\Gamma_\alpha(F)} \Phi(y,t) dy \frac{dt}{t}.$$

*Proof.* Fubini theorem reads that

$$\text{LHS} = \int_{\Gamma_\alpha(F)} \left( \int_F \mathbb{1}_{B(y,\alpha t)}(x) dx \right) \Phi(y,t) \frac{dy}{t^n} \frac{dt}{t} \lesssim_{n,\alpha} \text{RHS}.$$

∎

Note that the loss happens on the bound

$$|F \cap B(y, \alpha t)| \leq |B(y, \alpha t)|,$$

so the inverse requires that for any $(y,t) \in \Gamma_\alpha(F)$, $|B(y,\alpha t) \cap F|$ is always comparable to $t^n$. Such intuition enlightens the definition of *global density* [CMS85, Lemma 2].

**Definition 1.16** (Global density)**.** Let $F \subset \mathbb{R}^n$ be a measurable subset. Given $\gamma \in (0,1)$, we say that a point $x \in \mathbb{R}^n$ has *global $\gamma$-density* if

$$|F \cap B| \geq \gamma |B|,$$

for any ball $B$ containing $x$. The set $F^*_\gamma$ consists of all the points in $\mathbb{R}^n$ of global $\gamma$-density. Set $O := F^c$ and $O^*_\gamma := (F^*_\gamma)^c$.

Any point $x \in O^*_\gamma$ satisfies that

$$\sup_{x \in B} \frac{|B \cap O|}{|B|} > 1 - \gamma,$$

i.e.,

$$O^*_\gamma = \{x \in \mathbb{R}^n : \mathcal{M}(\mathbb{1}_O)(x) > 1 - \gamma\}. \tag{1.7}$$

*Remark.* Eq.(1.7) reads that $O^*_\gamma$ is open since $\mathcal{M}(f)$ is lower semi-continuous for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Furthermore, if $F$ is closed, any point in $F^*_\gamma$ is an accumulation point of $F$, so $F^*_\gamma \subset F$ and $O \subset O^*_\gamma$. But it is worthy of pointing out that the following deduction is independent of these topological properties and inclusion relation till the second case of Theorem 2.11, where we will mention again.

We can introduce the inverse version of Lemma 1.15.

**Lemma 1.17.** *For any $\alpha > 1$, there exists some $\gamma \in (0,1)$ so that for any subset $F \subset \mathbb{R}^n$ with finite-measure complement and any non-negative function $\Phi$,*

$$\int_{\Gamma_\alpha(F^*_\gamma)} \Phi(y,t) dy \frac{dt}{t} \lesssim_{n,\alpha,\gamma} \int_F \left( \int_{\Gamma(x)} \Phi(y,t) \frac{dy}{t^n} \frac{dt}{t} \right) dx.$$

*Proof.* For any $(y,t) \in \mathbb{R}^{n+1}_+$ such that $F^*_\gamma \cap B(y,\alpha t) \neq \varnothing$,

$$|F \cap B(y,t)| \geq |F \cap B(y,\alpha t)| - |B(y,\alpha t) \setminus B(y,t)| \geq c_n(\gamma \alpha^n - (\alpha^n - 1))t^n. \tag{1.8}$$

Pick $\gamma$ sufficiently close to 1 so that RHS is strictly positive. Then,

$$\begin{aligned}\int_{\Gamma_\alpha(F^*_\gamma)} \Phi(y,t) dy \frac{dt}{t} &\lesssim_{n,\alpha,\gamma} \int_{\Gamma_\alpha(F^*_\gamma)} \Phi(y,t) \left( \int_F \mathbb{1}_{B(y,t)}(x) dx \right) \frac{dy}{t^n} \frac{dt}{t} \\ &= \int_F \left( \int_{\Gamma_\alpha(F^*_\gamma)} \mathbb{1}_{\Gamma(x)}(y,t) \Phi(y,t) \frac{dy}{t^n} \frac{dt}{t} \right) dx \leq \text{RHS}\,.\end{aligned}$$

□

Lemma 1.15 gives *global* control of *mixed* integral, while Lemma 1.17 gives *mixed* control of *global* integration with the sacrifice of density. The combination of these two are called the *transition lemma* of tent spaces, as the generalization of $A$-averaging lemma, cf. Lemma 1.2.

Note that Lemma 1.17 actually reduce the aperture via raising density, so it is the time to prove the final case. Cavalieri's principle reads that

$$\|f\|_{T^{p,q}_{(\alpha)}}^p = p\int_0^\infty \lambda^{p-1}\left|\{x\in\mathbb{R}^n : A_{q;\alpha}(f)(x)>\lambda\}\right| d\lambda.$$

The following trick is a variant of "good $\lambda$-inequality" but adapted to the transition lemma. Let $F$ be the measurable subset $\{x\in\mathbb{R}^n : A_q(f)(x)\le\lambda\}$ and consider the decomposition as

$$|\{A_{q;\alpha}(f)(x)>\lambda\}| \le |\{x\in F^*_\gamma : A_{q;\alpha}(f)(x)>\lambda\}| + |O^*_\gamma|$$

for sufficiently large $\gamma$ chosen in Lemma 1.17 for $\alpha$. The first term is controlled as

$$\begin{aligned}
|\{x\in F^*_\gamma : A_{q;\alpha}(f)(x)>\lambda\}| &\le \frac{1}{\lambda^q}\int_{F^*_\gamma}(A_{q;\alpha}(f)(x))^q dx\\
&\le \frac{1}{\lambda^q}\int_{\Gamma_\alpha(F^*_\gamma)}|f(y,t)|^q dy\frac{dt}{t}\\
&\lesssim_{n,\alpha,\gamma} \frac{1}{\lambda^q}\int_F\left(\int_{\Gamma(x)}|f(y,t)|^q\frac{dy}{t^n}\frac{dt}{t}\right)dx\\
&= \frac{1}{\lambda^q}\int_F(A_q(f)(x))^q dx
\end{aligned}$$

by the transition lemma. Since $O$ has finite measure, Eq.(1.7) reads that

$$|O^*_\gamma| \lesssim_{n,\gamma} \frac{1}{1-\gamma}|O| = \frac{1}{1-\gamma}|\{x\in\mathbb{R}^n : A_q(f)(x)>\lambda\}|$$

by maximal theorem. We summarize that

$$\begin{aligned}
\|f\|_{T^{p,q}_{(\alpha)}}^p &\lesssim_{n,\alpha,\gamma} p\int_0^\infty \lambda^{p-1}\left[|\{A_q(f)>\lambda\}| + \frac{1}{\lambda^q}\int_{\{A_q(f)\le\lambda\}}(A_q(f)(x))^q dx\right]d\lambda.\\
&\lesssim_{p,q} \|A_q(f)\|_p^p + \int_{\{A_q(f)>0\}}\left(\int_{A_q(f)(x)}^\infty \lambda^{p-q-1}d\lambda\right)A_q(f)(x)^q dx \lesssim \|f\|_{T^{p,q}}^p.
\end{aligned}$$

Recall that we are working with the assumption that $p<q$ that ensures the legality.

#### 1.3.1.2 Type $(p,\infty)$ with $p>1$

For type $(p,\infty)$ with $p>1$, spatial scaling also reads that

$$\|f\|_{T^{p,\infty}_{(\beta)}} = \left(\int_{\mathbb{R}^n}\left(\sup_{(z,t)\in\Gamma(\frac{x}{\beta})}|f(\beta z,t)|\right)^p dx\right)^{\frac1p} \simeq_{\beta,n,p} \|f_\beta\|_{T^{p,\infty}}$$

for any $\beta>0$, so it also suffices to verify Eq.(1.4) for $\alpha>1$. Shadowing, cf. Eq.(1.1), reads that for any $(y,t)\in\Gamma_\alpha(x)$,

$$|f(y,t)| \le \inf_{z\in B(y,t)} A_\infty(f)(z) \le \alpha^n \fint_{B(y,\alpha t)} A_\infty(f)(z)dz \le \alpha^n\mathcal{M}(A_\infty f)(x).$$

Thus, $A_{\infty;\alpha}(f)(x) \leq \alpha^n \mathcal{M}(A_\infty f)(x)$, and

$$\|f\|_{T^{p,\infty}_{(\alpha)}} \leq \alpha^n \|\mathcal{M}(A_\infty f)\|_p \leq C(n,p)\alpha^n \|f\|_{T^{p,\infty}}, \tag{1.9}$$

where $C(n,p)$ is the controlling constant of strong type $(p,p)$ of uncentred maximal operator $\mathcal{M}$, especially independent of $\alpha$.

#### 1.3.1.3 Type $(\infty, q)$

Suppose that $0 < \beta < \alpha$. One side is straightforward that

$$\|f\|_{T^{\infty,q}_{(\alpha)}} \leq \left(\frac{\alpha}{\beta}\right)^{\frac{n}{q}} \|f\|_{T^{\infty,q}_{(\beta)}} \tag{1.10}$$

since $T_\alpha(B) \subset T_\beta(B)$ for any ball $B$. In turn, since $T_\beta(B) \subset T_\alpha(\frac{\alpha}{\beta}B)$,

$$\frac{\beta^n}{|B|}\int_{T_\beta(B)} |f(y,t)|^q dy \frac{dt}{t} \leq \frac{\alpha^n}{|\frac{\alpha}{\beta}B|}\int_{T_\alpha(\frac{\alpha}{\beta}B)} |f(y,t)|^q dy \frac{dt}{t} \leq \|f\|^q_{T^{\infty,q}_{(\alpha)}}.$$

By taking supremum of all the open balls, we conclude that

$$\|f\|_{T^{\infty,q}_{(\beta)}} \leq \|f\|_{T^{\infty,q}_{(\alpha)}}. \tag{1.11}$$

All the cases have been verified but type $(p,\infty)$ with $0 < p \leq 1$, while it will be discussed in Sec.2.3 via atomic decomposition. We shall also demonstrate the optimality of the controlling constants in Sec.3.5 via interpolation.

### 1.3.2 On spaces of homogeneous type

We shall prove Theorem 1.14 for tent spaces over spaces of homogeneous type.

**Type** $(p,q)$ First concern type $(p,q)$. For case $p = q$, $A$-averaging lemma directly reads that

$$\|f\|_{T^{p,p}_{(\alpha)}} = \|f\|_{T^{p,p}_{(\beta)}}. \tag{1.12}$$

However, spatial scaling fails if $X$ is bounded, and it does not necessarily hold that $A_{q;\beta}(f)(x) \leq A_{q;\alpha}(f)(x)$ even if $\alpha > \beta$ due to the averaging factor $V(y,\alpha t)^{-1}$. But it suffices to prove that $\|f\|_{T^{p,q}_{(\alpha)}} \lesssim \|f\|_{T^{p,q}_{(\beta)}}$ for any $\alpha, \beta > 0$ by symmetry.

The transition lemma still works. Lemma 1.15 directly holds as

$$\int_F \left(\int_{\Gamma_\alpha(x)} \Phi(y,t)\frac{d\mu(y)}{V(y,\alpha t)}\frac{dt}{t}\right) d\mu(x) \lesssim_{n,\alpha} \int_{\Gamma_\alpha(F)} \Phi(y,t) d\mu(y)\frac{dt}{t}. \tag{1.13}$$

Let $(y,t) \in X_+$ be such that $F^*_\gamma \cap B(y,\alpha t) \neq \varnothing$. If $\alpha \leq \beta$, it directly holds that

$$\mu(F \cap B(y,\beta t)) \geq \gamma\mu(B(y,\beta t))$$

since $F^*_\gamma \cap B(y,\alpha t) \subset F^*_\gamma \cap B(y,\beta t)$. If $\alpha > \beta$,

$$\begin{aligned}\mu(F \cap B(y,\beta t)) &\geq \mu(F \cap B(y,\alpha t)) - \mu(B(y,\alpha t) \setminus B(y,\beta t)) \\ &\geq \mu(B(y,\beta t)) - (1-\gamma)\mu(B(y,\alpha t)) \\ &\geq (1-(1-\gamma)\rho(\alpha/\beta))\,\mu(B(y,\beta t)).\end{aligned} \tag{1.14}$$

Pick $\gamma \in (0,1)$ sufficiently close to 1 such that RHS is strictly positive. Lemma (1.17) can hence be established as

$$\int_{\Gamma_\alpha(F^*_\gamma)} \Phi(y,t)d\mu(y)\frac{dt}{t} \lesssim_{\alpha,\beta,\gamma} \int_F \left(\int_{\Gamma_\beta(x)} \Phi(y,t)\frac{d\mu(y)}{V(y,\beta t)}\frac{dt}{t}\right) d\mu(x).$$

We just show some important modifications. Suppose that $f \in T^{p,q}_{(\alpha)}$. For case $p > q$, for any $g \in L^r(X)$ with $\|g\|_r = 1$ for $r = (\frac{p}{q})'$,

$$\int_X (A_{q;\alpha}(f)(x))^q\, g(x)d\mu(x) \lesssim_{\alpha,\beta} \|f\|^q_{T^{p,q}_{(\beta)}}\|\mathcal{M}(g)\|_r.$$

We conclude by the strong type of *uncentred* maximal operators on spaces of homogeneous type. For case $p < q$, define that $F = \{x \in X : A_{q;\beta}(f)(x) \le \lambda\}$, so

$$\mu\left(\{x \in F^*_\gamma : A_{q;\alpha}(f)(x) > \lambda\}\right) \lesssim_{\alpha,\beta,\gamma} \frac{1}{\lambda^q}\int_F (A_{q;\beta}(f)(x))^q d\mu(x)$$

by transition lemma. Write that $O = F^c$, and we conclude as

$$\mu(O^*_\gamma) \lesssim_{n,\gamma} \frac{1}{1-\gamma}\mu(O) = \frac{1}{1-\gamma}\mu\left(\{x \in \mathbb{R}^n : A_{q;\beta}(f)(x) > \lambda\}\right)$$

since maximal theorem also holds for uncentred Hardy-Littlewood maximal operator.

**Type** $(p,\infty)$ **with** $p > 1$ It still holds that $A_{\infty;\beta}(f)(x) \le A_{\infty;\alpha}(f)(x)$ if $\alpha > \beta$. In turn, set that $(\Gamma_\alpha|t_0,t_1)(x) := \{(y,t) \in \Gamma_\alpha(x) : t_0 \le t < t_1\}$. Let $k \in \mathbb{Z}$ be an integer. Cover $B(x, 2^{k+1}\alpha)$ by a sequence of balls $\{B^k_j = B(x^k_j, 2^{k-1}\beta)\}_{j\in I_k}$ satisfying that $x^k_j \in B(x, 2^{k+1}\alpha)$, $\#I_k < \infty$ and the bound of $\#I_k$ is independent of $k$. Write that $\Delta^{(k)}_j := (\Gamma_\alpha|2^k, 2^{k+1})(x) \cap (B^{(k)}_j \times [2^k, 2^{k+1}))$. Eq.(1.3) reads that

$$\begin{aligned}
\sigma_\alpha\text{-}\operatorname*{esssup}_{(y,t)\in\Delta^{(k)}_j} |f(y,t)| &\le \fint_{B^k_j} A_{\infty;\beta}(f)(z)d\mu(z) \\
&\le \rho(2(\alpha+\beta)/\beta)\fint_{B(x^k_j, 2^{k+1}(\alpha+\beta))} A_{\infty;\beta}(f)(z)d\mu(z) \\
&\le \rho(2(\alpha+\beta)/\beta)\mathcal{M}(A_{\infty;\beta}(f))(x),
\end{aligned}$$

since $B^k_j \subset R_\beta(B^k_j)$ and $d(x,x_j) \le 2^{k+1}\alpha$. We conclude that for any $x \in X$,

$$\sigma_\alpha\text{-}\operatorname*{esssup}_{(y,t)\in\Gamma_\alpha(x)} |f(y,t)| \le \rho(2(\alpha+\beta)/\beta)\mathcal{M}(A_{\infty;\beta}f)(x). \tag{1.15}$$

Therefore,

$$\|f\|_{T^{p,\infty}_{(\alpha)}} \lesssim_p \rho(\alpha/\beta)\|f\|_{T^{p,\infty}_{(\beta)}}, \quad \|f\|_{T^{\infty,q}_{(\beta)}} \le \|f\|_{T^{\infty,q}_{(\alpha)}}. \tag{1.16}$$

**Type** $(\infty, q)$ It follows that

$$\|f\|_{T^{\infty,q}_{(\alpha)}} \le \rho(\alpha/\beta)^{\frac{1}{q}}\|f\|_{T^{\infty,q}_{(\beta)}}, \quad \|f\|_{T^{\infty,q}_{(\beta)}} \le \|f\|_{T^{\infty,q}_{(\alpha)}} \tag{1.17}$$

for any $\alpha < \beta$. The proof is exactly of the same as that for Eq.(1.11).

We hence finish the proof.

We finalize this chapter by a review of previous proof for independent of aperture. Before that, we first mention a special property, so-called nice-intersection.

**Definition 1.18** (Nice-intersection)**.** A measured quasi-metric space $(X, d, \mu)$ is said to have *nice-intersection property*, if for any $\alpha, \beta > 0$, there exists a positive constant $c_{\alpha,\beta} > 0$ such that

$$\inf_{\substack{r>0 \\ x,y\in X; d(x,y)<\alpha r}} \frac{\mu(B(x,\alpha r) \cap B(y,\beta r))}{\mu(B(x,\alpha r))} \geq c_{\alpha,\beta} > 0.$$

Note that if a measured quasi-metric space has the nice-intersection property, then it must have doubling property. But the reverse is not true, even for metric connected spaces. See [Ame14, Section 2] for a counterexample.

There are different versions of proof for Theorem 1.14 in different contexts.

- The first proof was given by [CMS85, Proposition 4] for $p \in (0, \infty)$ and $q = 2$ in $\mathbb{R}^n$. They defined the global $\gamma$-density via centred maximal operators, which induced the proof to implicitly utilise the nice-intersection property of $\mathbb{R}^n$, cf. Eq.(1.8). Thus, their proof does not directly work in our context.

- There is sequential proof by [Aus11, Theorem 1.1] for $p \in (0, \infty]$ and $q = 2$ in $\mathbb{R}^n$ using the interpolation theory, moreover, in which the sharpness of the controlling constant has been also established with respect to $\alpha$ and $\beta$. We shall revisit and generalize this result in Sec.3.5.

- Another proof is given by [Ame14, Proposition 3.21] for $p, q \in (0, \infty)$ in *metric* doubling spaces via vector-valued method and raising-power trick. Both of these tools will be shown later in Sec.3.2 and Sec.2.2, respectively. This proof is independent of the nice-intersection property.

What we did above for type $(p, q)$ is to develop a new approach by adapting the original proof of [CMS85, Proposition 4] but modifying it with uncentred maximal operator and establishing Eq.(1.14) to avoid the nice-intersection property. Moreover, we have clarified the independence of aperture for all the possible types of tent spaces over spaces of homogeneous type.

# Chapter 2

# Duality and atomic decomposition of tent spaces

In this chapter, we will construct two fundamental weapons in harmonic analysis for tent spaces, duality and atomic decomposition, to deeply describe tent spaces and connections between them. Moreover, we shall show the elaborate and inextricable connections between these two weapons, as is shown in many other fields of harmonic analysis. Thanks to Theorem 1.14, we could concentrate on tent spaces with aperture 1 without special mention.

## 2.1 Duality

In this section, we first concern the duality of two typical types: type $(p,q)$ with $p,q \in [1,\infty)$ and type $(1,\infty)$. Let $(X,d,\mu)$ be a space of homogeneous type and $\Sigma(X)$ the smallest $\sigma$-algebra containing all the balls and Borel sets of $X$. Equip the space $X_+ = X \times (0,\infty)$ with the product $\sigma$-algebra $\Sigma(X) \otimes \mathfrak{B}(\mathbb{R}_+)$, where $\mathfrak{B}(\mathbb{R}_+)$ is the Borel $\sigma$-algebra of $\mathbb{R}_+$.

**Definition 2.1** (Carleson measure)**.** Let $\nu$ be a complex locally-finite measure defined on $\Sigma(X) \otimes \mathfrak{B}(\mathbb{R}_+)$ of $X_+$. We say that $\nu$ is a *Carleson measure* if there is a constant $C < +\infty$ such that

$$\frac{1}{\mu(B)} \int_{\widehat{B}} |d\nu(x,t)| \leq C < +\infty$$

for any ball $B \subset X$. Let $\mathfrak{M}_C$ be the collection of all the Carleson measures, equipped with the norm

$$\|\nu\|_C := \sup_{B\colon \text{ball in } X} \frac{1}{\mu(B)} \int_{\widehat{B}} |d\nu(x,t)|.$$

Note that if all the balls of $X$ are Borel sets, all the Carleson measures are Borel measures.

**Theorem 2.2** (Whitney covering lemma)**.** *There exists a constant $C > 1$, merely dependent on $X$, such that for any proper open subset $O \subsetneq X$, there is a* Whitney covering *of $O$, i.e., a collection of countably many balls $\{B_i\}_{i\in\mathbb{N}}$ satisfying that*

- $\Omega_\lambda = \bigcup_{i\in\mathbb{N}} B_i$*;*
- $CB_i \cap \Omega_\lambda^c \neq \varnothing$ *for any $i \in \mathbb{N}$;*
- *the* bounded overlapping property *holds, i.e., there exists a constant $C'$ only dependent on $X$ such that $\sum_i \mathbb{1}_{B_i} \leq C'$.*

*Proof.* See [CW71, Théorème 1.3, Chapitre III.1]. $\square$

### 2.1.1 On Euclidean spaces

#### 2.1.1.1 Type $(1,\infty)$

The following theorem identifies the dual space of $T^{1,\infty}$.

**Theorem 2.3.** *The dual space of $T^{1,\infty}$ can be identified as $\mathfrak{M}_C$ via the pairing*

$$(\nu, f) \mapsto \int_{\mathbb{R}^{n+1}_+} f(y,t) d\nu(y,t). \tag{2.1}$$

*for any $(\nu, f) \in \mathfrak{M}_C \times T^{1,\infty}$.*

*Proof.* First show that the integration makes sense. Cavalieri's principle reads that

$$\int_{\mathbb{R}^{n+1}_+} |f(y,t)||d\nu(y,t)| = \int_0^\infty |\nu| \left( \left\{ (y,t) \in \mathbb{R}^{n+1}_+ : |f(y,t)| > \lambda \right\} \right) d\lambda.$$

The shadowing trick, cf. Eq.(1.1) reads that

$$\left\{ (y,t) \in \mathbb{R}^{n+1}_+ : |f(y,t)| > \lambda \right\} \subset T\left( \{ x \in \mathbb{R}^n : A_\infty(f)(x) > \lambda \} \right) =: T(\Omega_\lambda). \tag{2.2}$$

Since $A_\infty(f) \in L^1(\mathbb{R})$, $\Omega$ is of finite measure, so Theorem 2.2 gives a Whitney covering of $\Omega_\lambda$, $\{B_i\}_{i\in\mathbb{N}}$. For any $(y,t) \in T(\Omega_\lambda)$, since $y \in \Omega_\lambda$, pick $B_i$ and $z$ such that $y \in B_i$ and $z \in CB_i \cap \Omega_\lambda^c$, so

$$\operatorname{dist}(y, \Omega_\lambda^c) \le d(y,z) \le (C+1) r(B_i) \le \operatorname{dist}(y, ((C+2)B_i)^c)$$

since $y \in B_i$. So,

$$T(\Omega_\lambda) \subset \bigcup_{i\in\mathbb{N}} T((C+2)B_i), \tag{2.3}$$

and hence,

$$\nu(T(\Omega_\lambda)) \le \sum_{i\in\mathbb{N}} \|\nu\|_C |(C+2)B_i| \le C'(C+2)^n \|\nu\|_C |\Omega_\lambda|$$

by bounded overlapping controlling constant $C'$. Thus,

$$\int_{\mathbb{R}^{n+1}_+} |f(y,t)||d\nu(y,t)| \lesssim_n \|\nu\|_C \int_0^\infty A_\infty(f)(x) dx = \|\nu\|_C \|f\|_{T^{1,\infty}}.$$

Each Carleson measure can hence define a linear functional on $T^{1,\infty}$ via Eq.(2.1).

Then, we prove the inverse. Let $l$ be a continuous linear functional of $T^{1,\infty}$, hence a continuous linear functional of $C_c^0(K) \subset C_c^0(\mathbb{R}^{n+1}_+)$ for any compact $K \subset \mathbb{R}^{n+1}_+$ by localization lemma, cf. Lemma 1.4. Riesz representation theorem [Rud87, Theorem 6.19] reads that there is a unique regular complex Borel measure $\nu_K$ supported on $K$ such that

$$l(f) = \int_K f d\nu_K \tag{2.4}$$

for any $f \in C_c^0(K)$. Same exhaustion and identification as in Lemma 1.4 give us a unique regular complex Borel measure $\nu$.

We verify that $\nu$ is a Carleson measure. Polar representation of $\nu$ [Rud87, Theorem 6.12] reads that there is a measurable function $h$ with $|h| = 1$ almost everywhere such that $|d\nu| = hd\nu$. Thus, for any ball $B \subset \mathbb{R}^n$,

$$\frac{1}{|B|}\int_{\widehat{B}} |d\nu| \le \frac{1}{|B|}\int_{\mathbb{R}^{n+1}_+} \varphi h d\nu = \frac{1}{|B|}(l, \varphi h) \lesssim \|l\|.$$

where $\varphi \in C^0(\mathbb{R}^{n+1}_+; [0,1])$, valued 1 on $\widehat{B}$ and supported on a small neighbourhood of $\widehat{B}$. The equality holds as $h\varphi \in T^{1,\infty}$ with $\|h\varphi\|_{T^{1,\infty}} \le 2|B|$. $\square$

#### 2.1.1.2 Type $(p, q)$

Note that the duality of $T^{1,1}$, i.e., $p, q = 1$ is trivial since $T^{1,1} = L^1(\mathbb{R}^{n+1}_+; dy\frac{dt}{t})$, whose dual space is $L^\infty(\mathbb{R}^{n+1}_+; dy\frac{dt}{t}) = T^{\infty,\infty}$. More cases are demonstrated by the following theorem.

**Theorem 2.4.** *For any $p \in [1,\infty), q \in (1,\infty)$, the dual space of $T^{p,q}$ can be identified as $T^{p',q'}$ via the pairing*

$$(g, f) \mapsto \int_{\mathbb{R}^{n+1}_+} g(y,t) f(y,t) dy \frac{dt}{t}. \tag{2.5}$$

*for any $(g, f) \in T^{p',q'} \times T^{p,q}$.*

The proof is divided into two cases: $p \in (1,\infty)$ and $p = 1$.

**Case** $p, q \in (1,\infty)$ We concern the case $p = q$ first, where the tent space $T^{p,q}$ is exactly $L^p\left(\mathbb{R}^{n+1}_+; dy\frac{dt}{t}\right)$ by $A$-averaging lemma, cf. Lemma 1.2. The dual space could hence be canonically identified as $L^{p'}\left(\mathbb{R}^{n+1}_+; dy\frac{dt}{t}\right) = T^{p',p'}$.

Assume that $p \neq q$. The pairing makes sense by $A$-averaging lemma as

$$\begin{aligned}\int_{\mathbb{R}^{n+1}_+} |g(y,t) f(y,t)| dy \frac{dt}{t} &\simeq_n \int_{\mathbb{R}^n} \left(\int_{\Gamma(x)} |g(y,t) f(y,t)| \frac{dy}{t^n}\frac{dt}{t}\right) dx \\ &\le \int_{\mathbb{R}^n} A_{q'}(g)(x) A_q(f)(x) dx \\ &\le \|A_{q'}(g)\|_{p'} \|A_q(f)\|_p = \|g\|_{T^{p',q'}} \|f\|_{T^{p,q}}.\end{aligned}$$

In turn, for any continuous linear functional $l \in (T^{p,q})^*$, localization lemma also gives us a $L^{q'}_{\text{loc}}$-function $g$ such that

$$l(f\mathbb{1}_K) = \int_K g(y,t) f(y,t) dy \frac{dt}{t} \tag{2.6}$$

for any $f \in T^{p,q}$ and compact set $K \subset \mathbb{R}^{n+1}_+$.

For $p' > q'$, i.e., $p < q$, duality argument reads that

$$\|g\mathbb{1}_K\|^{q'}_{T^{p',q'}} \simeq_n \sup_{\substack{\phi \in L^r \\ \|\phi\| = 1}} \int_K |(g\mathbb{1}_K)(y,t)|^{q'} M(\phi)(y,t) dy \frac{dt}{t},$$

where $r$ is the Hölder duality of $\frac{p'}{q'}$ and $M = M_1$ as defined in Eq.(1.6). Set that

$$f_\phi = \overline{g}^{\frac{q'}{2}} g^{\frac{q'}{2}-1} \mathbb{1}_K M(\phi),$$

so

$$\|g\mathbb{1}_K\|_{T^{p',q'}}^{q'} \simeq_n \sup_{\substack{\phi\in L^r\\ \|\phi\|=1}} l(f_\phi) \leq \|l\|\cdot \sup_{\substack{\phi\in L^r\\ \|\phi\|=1}} \|f_\phi\|_{T^{p,q}}.$$

Note that

$$\begin{aligned}\|f_\phi\|_{T^{p,q}} &= \left(\int_{\mathbb{R}^n}\left(\int_{\Gamma(x)} |(g\mathbb{1}_K)(y,t)|^{q(q'-1)}|M(\phi)(y,t)|^q \frac{dy}{t^n}\frac{dt}{t}\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}}\\ &\leq \left(\int_{\mathbb{R}^n} \mathcal{M}(\phi)(x)^p A_{q'}(g\mathbb{1}_K)(x)^{\frac{pq'}{q}} dx\right)^{\frac{1}{p}}\\ &\leq \|\mathcal{M}(\phi)\|_r \|A_{q'}(g\mathbb{1}_K)^{\frac{q'}{q}}\|_s \lesssim_n \|g\mathbb{1}_K\|_{T^{p',q'}}^{q'-1},\end{aligned}$$

where $\frac{1}{r}+\frac{1}{s}=\frac{1}{p}$. By Lemma 1.4, $\|g\mathbb{1}_K\|_{T^{p',q'}}<\infty$, so Levy theorem again reads that $g\in T^{p',q'}$ with

$$\|g\|_{T^{p',q'}} \lesssim_n \|l\|.$$

For $p'<q'$, note that $T^{p,q}=(T^{p',q'})^*$ by the above case, so to show that $(T^{p,q})^* = (T^{p',q'})^{**}\simeq T^{p',q'}$ with $p>q$, it suffices to verify that $T^{p,q}$ is reflexive for $p,q\in(1,\infty)$ and $p<q$. Since $T^{p,q}$ is a Banach space by Proposition 1.5, Banach-Alaoglu theorem reads that it is reflexive if and only if the unit ball of $T^{p,q}$ is weakly compact [Con07, Chapter V, Theorem 4.2]. Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of $T^{p,q}$-functions with $T^{p,q}$-norm at most 1. Pick an exhaustion of $\mathbb{R}^{n+1}_+$ by compact sets $\{K_m\}_{m\in\mathbb{N}}$. Since $L^q(K_m)$ is reflexive, Arzelà-Ascoli diagonal trick gives a subsequence $\{f_{k_j}\}_{j\in\mathbb{N}}$ weakly convergent in any $L^q(K_m)$. For any linear functional $l\in(T^{p,q})^*$, the above shows that it can be visualized as a $T^{p',q'}$-function $g$ via

$$l(f)=\int_{\mathbb{R}^{n+1}_+} g(y,t)f(y,t)dy\frac{dt}{t}=(g,f).$$

Then, for any $\epsilon>0$, pick $M\in\mathbb{N}$ such that $\|g-g\mathbb{1}_{K_M}\|_{T^{p',q'}}\leq\epsilon$, so

$$\begin{aligned}\left|l(f_{k_i})-l(f_{k_j})\right| = |(g,f_{k_i}-f_{k_j})| &\leq |(g-g\mathbb{1}_{K_M},f_{k_i}-f_{k_j})|+|(g\mathbb{1}_{K_M},f_{k_i}-f_{k_j})|\\ &\leq 2\epsilon+|(g\mathbb{1}_{K_M},(f_{k_i}\mathbb{1}_{K_M}-f_{k_j}\mathbb{1}_{K_M}))|.\end{aligned}$$

It indicates that $\{l(f_{k_j})\}_{j\in\mathbb{N}}$ is a Cauchy sequence, hence converges, so we conclude.

**Case** $p=1, q\in(1,\infty)$ The pairing makes sense via the "stopping-time" technique, which tames the behavior of $A$-functional by that of $C$-functional of $T^{\infty,q'}$-functions. For any $h>0$, the *truncated cone* by $h$ is given by

$$(\Gamma|h)(x):=\{(y,t)\in\Gamma(x): 0<t\leq h\}$$

if based on a point $x\in\mathbb{R}^n$, $(\Gamma|h)(F):=\bigcup_{x\in F}(\Gamma|h)(x)$, if based on a subset $F\subset\mathbb{R}^n$. The *truncated $A$-functional* by $h$ is given by

$$(A_q|h)(f)(x):=\left(\int_{(\Gamma|h)(x)}|f(y,t)|^q\frac{dy}{t^n}\frac{dt}{t}\right)^{\frac{1}{q}}.$$

Set $(\Gamma|\infty)(x):=\Gamma(x)$ and $(A_q|\infty)(f):=A_q(f)$ for coherence.

Let $f \in T^{1,q}, g \in T^{\infty,q'}$ be two non-zero fixed functions and $M$ a fixed constant to be determined later. The stopping-time $h(x)$ is given by

$$h(x) := \sup\{h > 0 : (A_{q'}|h)(g)(x) \leq MC_{q'}(g)(x)\}.$$

We claim that, for every ball $B$, tamed points $B^\top := \{x \in B : h(x) \geq r(B)\}$ also have $\gamma$-density, i.e.,

$$\inf_{B \subset \mathbb{R}^n} \frac{|B^\top|}{|B|} \geq \gamma$$

for some $\gamma \in (0,1)$. Indeed, note that $x \in B^\top$ if and only if $(A_{q'}|r(B))(g)(x) \leq MC_{q'}(g)(x)$, so Chebyshev-Markov inequality reads that

$$\begin{aligned}
\left(\inf_{x \in B} (MC_{q'}(g)(x))^{q'}\right) |(B^\top)^c| &\leq \int_B (A_{q'}|r(B))(g)(x)^{q'} dx \\
&\simeq_n \int_{(\Gamma|r(B))(B)} |g(y,t)|^{q'} dy \frac{dt}{t} \\
&\leq |3B| \inf_{x \in B} C_{q'}(g)(x)^{q'}.
\end{aligned}$$

The last inequality holds since $(\Gamma|r(B))(B) \subset T(3B)$. Note that $\inf_{x \in B} C_{q'}(g)(x) \in (0,\infty)$ since $g$ is a non-zero $T^{\infty,q'}$-function. Thus, we prove the claim via picking a sufficiently large $M$ but only dependent on the dimension $n$. Thus,

$$\begin{aligned}
\int_{\mathbb{R}^{n+1}_+} |g(y,t)f(y,t)| dy \frac{dt}{t} &\simeq_{\gamma,n} \int_{\mathbb{R}^{n+1}_+} |g(y,t)f(y,t)| \left(\int_{B(y,t)} \mathbb{1}_{B(y,t)^\top}(x) dx\right) \frac{dy}{t^n} \frac{dt}{t} \\
&\simeq_n \int_{\mathbb{R}^n} \left(\int_{\Gamma(x)} |g(y,t)f(y,t)| \mathbb{1}_{\{t \leq h(x)\}}(y,t) \frac{dy}{t^n} \frac{dt}{t}\right) dx \\
&\leq \int_{\mathbb{R}^n} (A_{q'}|h(x))(g)(x)(A_q|h(x))(f)(x) dx \\
&\leq M \int_{\mathbb{R}^n} C_{q'}(g)(x) A_q(f)(x) dx \leq M\|g\|_{T^{\infty,q'}} \|f\|_{T^{1,q}}.
\end{aligned}$$

In turn, for any $l \in (T^{1,q})^*$, localization gives an $L^{q'}_{\text{loc}}$-function $g$ such that

$$l(f\mathbb{1}_K) = \int_K g(y,t)f(y,t) dy dt$$

for any $f \in T^{1,q}$ and compact $K \subset \mathbb{R}^{n+1}_+$. For any $B$, duality argument gives that

$$\|g\mathbb{1}_{\widehat{B}}\|_{q'} = \sup_{\substack{\phi \in L^q \\ \|\phi\|_q = 1}} \int_{\mathbb{R}^{n+1}_+} (g\mathbb{1}_{\widehat{B}})(y,t)\phi(y,t) dy dt.$$

Note that

$$\|\phi\mathbb{1}_K\|_{T^{1,q}} = \int_B A_q(\phi\mathbb{1}_{\widehat{B}})(x) dx \lesssim_n |B|^{\frac{1}{q'}} \|\phi\|_q = |B|^{\frac{1}{q'}},$$

so an exhaustion of compact sets $\{K_m\}_{m \in \mathbb{N}}$ again gives the convergence as

$$\int_{\mathbb{R}^{n+1}_+} |(g\mathbb{1}_{\widehat{B}})\phi| = \limsup_{m \to +\infty} \int_{\widehat{B}} |g\phi\mathbb{1}_{K_m}| = \limsup_{m \to +\infty} l(\phi\mathbb{1}_{K_m}) \lesssim_n \|l\| |B|^{\frac{1}{q'}}.$$

We hence conclude that $g \in T^{\infty,q'}$ with

$$\|g\|_{T^{\infty,q'}} \lesssim_n \|l\|.$$

### 2.1.2 On spaces of homogeneous type

We generalize the above results to spaces of homogeneous type in this subsection.

**Theorem 2.5.** *Let $(X, d, \mu)$ be a locally compact space of homogeneous type with quasi-metric constant $\kappa$ and the property that all the balls are Borel sets. The dual space of $T^{1,\infty}$ can be identified as $\mathfrak{M}_C$ via the pairing*

$$(\nu, f) \mapsto \int_{X_+} f(y,t) d\nu(y,t). \tag{2.7}$$

*for any $(\nu, f) \in \mathfrak{M}_C \times T^{1,\infty}$.*

*Proof.* The proof of well-definedness of the pairing in Theorem 2.3 also works but with slight change on constants as

$$T(\Omega_\lambda) \subset \bigcup_{i \in \mathbb{N}} T(\kappa(C+2)B_i). \tag{2.8}$$

Indeed, pick $z \in CB_i \cap \Omega_\lambda^c$, for any $y \in B_i$,

$$\mathrm{dist}(y, \Omega_\lambda^c) \le \kappa(d(y, c(B_i)) + d(c(B_i), z)) \le \kappa(C+1)r(B_i) \le \mathrm{dist}(y, (\kappa(C+2)B_i)^c).$$

Note that Carleson measures are Borel measures in this case. We conclude since Riesz representation theorem holds for any locally compact Hausdorff space [Rud87, Theorem 6.19] and polar representation theorem works for any general complex-measured sets [Rud87, Theorem 6.12]. □

For type $(p, q)$, it also works.

**Theorem 2.6.** *For any $p \in [1, \infty), q \in (1, \infty)$, the dual space of $T^{p,q}$ can be identified as $T^{p',q'}$ via the pairing*

$$(g, f) \mapsto \int_{X_+} g(y,t) f(y,t) d\mu(y) \frac{dt}{t}. \tag{2.9}$$

*for any $(g, f) \in T^{p',q'} \times T^{p,q}$.*

*Proof.* There is no difficulty to generalize the proof of Theorem 2.4 since all the necessary ingredients have been already generalized such as $A$-averaging and localization lemma. □

## 2.2 Atomic decomposition

In this section, we describe the tent spaces of type $(p, q)$ with $p \in (0, 1], q \in [p, \infty)$ and type $(p, \infty)$ with $p \in (0, 1]$ via atomic decomposition. Before the theory, let us first prepare a trick, so-called the *raising-power trick*. For any $p, q \in (0, \infty)$, $\lambda \in (0, \infty)$, and $f \in T^{p,q}$ based on a space of homogeneous type $(X, d, \mu)$,

$$\|f\|_{T^{p,q}_{(\alpha)}} = \left( \int_X \left( \int_{\Gamma_\alpha(x)} |f(y,t)|^q \frac{d\mu(y)}{V(y, \alpha t)} \frac{dt}{t} \right)^{\frac{p}{q}} \right)^{\frac{1}{p}} = \||f|^{\frac{1}{\lambda}}\|^{\lambda}_{T^{\lambda p, \lambda q}_{(\alpha)}}. \tag{2.10}$$

Such a trick admits us to return back to $(1, \infty)$ for $p, q$ from a wider region.

### 2.2.1 On Euclidean spaces

We first deal with the case $T^{1,q}$ with $q \in [1, \infty]$.

**Definition 2.7** ($T^{1,q}$-atom)**.** Suppose that $q \in [1, \infty]$. A measurable function $a : \mathbb{R}^{n+1}_+ \to \mathbb{C}$ is called a $T^{1,q}$-*atom* if

- there exists a ball $B \subset \mathbb{R}^n$ such that $a$ is supported in $\widehat{B}$;
- $\|a\|_{L^q(\mathbb{R}^{n+1}_+; dy\frac{dt}{t})} \leq |B|^{-\frac{1}{q'}}$;
- $a$ is continuous if $q = \infty$.

Any ball satisfying all these conditions is called *associated to a*. Let $\mathscr{A}^{1,q}$ be the collection of all the $T^{1,q}$-atoms.

Note that $\mathscr{A}^{1,q}$ is uniformly bounded in $T^{1,q}$. Indeed,

$$\|a\|_{T^{1,q}} = \int_{\mathbb{R}^n} A_q(a)(x) dx \lesssim_{n,q} \|a\|_q |B|^{\frac{1}{q'}} \leq 1.$$

The following proposition in [Coi+93, Lemma III.1 and III.2] precisely describe the widely-believed heuristic principle that atomic decomposition and duality are actually different descriptions of the same thing on certain Banach spaces. In a normed vector space $F$, a subset is called *symmetric* if the condition $x \in V$ implies that $-x \in V$.

**Proposition 2.8.** *Let $V$ be a bounded symmetric subset of a normed vector space $F$. Suppose that for any $l \in F^*$, $\|l\|_{F^*}$ and $\sup_{x \in V}(l, x)$ are two equivalent norms. Then, any $x \in F$ can be written as*

$$x = \sum_{j=0}^{\infty} \lambda_j z_j$$

*for some $z_j \in V$ and $(\lambda_j)_{j \in \mathbb{N}} \in \ell^1(\mathbb{N})$.*

*Proof.* There exists some $\alpha > 0$ such that $\sup_{x \in V}(l, x) \geq \alpha \|l\|_{F^*}$. We claim that the ball $B(0, \alpha)$ is contained the closed convex closure of $V$, $\overline{\mathrm{Cl}(V)}$. Otherwise, pick $x_0$ as a counterexample. Hahn-Banach theorem reads that there exists some $l \in F^*$ with $\|l\|_{F^*} = 1$ such that

$$\sup_{x \in \overline{\mathrm{Cl}(V)}} (l, x) \leq (l, x_0) \leq \alpha,$$

which contradicts. Note that it suffices to prove the conclusion for $x \in F$ with $\|x\| < \alpha$ by normalization. Thus, $x \in B(0, \alpha) \subset \overline{\mathrm{Cl}(V)}$ and hence there exists $y_0 \in \mathrm{Cl}(V)$ such that $\|x - y_0\| < \frac{\alpha}{2}$. But $2(x - y_0) \in B(0, \alpha)$ and there exists $y_1 \in \mathrm{Cl}(V)$ such that $\|2(x - y_0) - y_1\| < \frac{\alpha}{2}$. Iteration hence gives a sequence $(y_j)_{j \in \mathbb{N}}$ such that

$$\left\| x - \sum_{j=0}^{N} \frac{1}{2^j} y_j \right\| < \frac{\alpha}{2^{N+1}},$$

so

$$x = \sum_{j=0}^{\infty} \frac{1}{2^j} y_j.$$

Note that $y_j \in \mathrm{Cl}(V)$, so it can be written as $y_j = \sum_{i=1}^{J_j} \eta_i z_i$ for some $z_i \in V$ with $\sum_i \eta_i = 1$. Since $V$ is bounded, we conclude by rearrangement. $\square$

We are ready to prove the atomic decomposition of tent spaces $T^{1,q}$.

**Proposition 2.9.** *For $q \in (1,\infty]$, any $T^{1,q}$-function $f$ can be written as the form*

$$f = \sum_{j=0}^{\infty} \lambda_j a_j$$

*with $a_j$'s as $T^{1,q}$-atoms, $\lambda_j \in \mathbb{C}$, and*

$$\|(\lambda_j)_j\|_{\ell^1(\mathbb{N})} = \sum_{j=0}^{\infty} |\lambda_j| \lesssim \|f\|_{T^{1,q}}.$$

*Proof.* It suffices to verify that $\|l\|_{(T^{1,q})^*}$ and $\sup_{x\in\mathscr{A}^{1,q}}(l,x)$ are equivalent norms for any $l \in (T^{1,q})^*$ and any $q \in (1,\infty]$ by Proposition 2.8.

For $q \in (1,\infty)$, Theorem 2.4 reads that for any $l \in (T^{1,q})^*$, there exists $g \in T^{\infty,q'}$ such that

$$\|l\| \simeq \|g\|_{T^{\infty,q'}} = \sup_B |B|^{-\frac{1}{q'}} \|g\|_{L^{q'}(\widehat{B};dy\frac{dt}{t})} = \sup_B \left( |B|^{-\frac{1}{q'}} \sup_{\phi}(\phi,g) \right) \lesssim \sup_{a\in\mathscr{A}^{1,q}} (l,a),$$

where $\phi$ is taken from $L^q(\widehat{B};dy\frac{dt}{t})$-functions with unit norm. The last inequality holds since $|B|^{-\frac{1}{q'}}\phi$ lies in $\mathscr{A}^{1,q}$.

For $q = \infty$, Theorem 2.3 shows that for any $l \in (T^{1,\infty})^*$, there is a Carleson measure $\nu$ such that

$$\|l\| \simeq \|\nu\|_C = \sup_B \frac{1}{|B|} \int_{\widehat{B}} |d\nu| = \sup_B \frac{1}{|B|} \int_{\mathbb{R}^{n+1}_+} h \mathbb{1}_{\widehat{B}} d\nu.$$

Set that $\varphi_B^{(\epsilon)} = ((h\mathbb{1}_{\widehat{B}}) * \chi_\epsilon)\mathbb{1}_{\mathbb{R}^{n+1}_+}$, where $\chi$ is the canonical mollifier and $\chi_\epsilon(x) = \epsilon^{-n}\chi(\frac{x}{\epsilon})$. Pick sufficiently small $\epsilon$ such that $\varphi_B^{(\epsilon)}$ is supported on $T(\eta B)$ with some fixed constant $\eta > 1$. Note that $\|\varphi_B^{(\epsilon)}\|_\infty \leq 1$ by Young's convolution inequality, so $|\eta B|^{-1}\varphi_B^{(\epsilon)} \in \mathscr{A}^{1,\infty}$. Thus, we conclude that

$$\|l\| \simeq \|\nu\|_C = \sup_B \left( \frac{1}{|B|} \limsup_{\epsilon\to 0^+} (\nu, \varphi_B^{(\epsilon)}) \right) \lesssim_{\eta,n} \sup_{a\in\mathscr{A}^{1,\infty}} (l,a).$$

□

The proof illustrates a typical approach from duality to atomic decomposition. However, the flaw is that for $p \in (0,1)$, neither we know the duality, nor the space is a Banach space. Therefore, the generalization relies on manually construction. Set that $[r,s] := \frac{1}{r} - \frac{1}{s}$.

**Definition 2.10** ($T^{p,q}$-atoms)**.** Suppose that $p \in (0,1]$ and $q \in [p,\infty]$. A measurable function $a : \mathbb{R}^{n+1}_+ \to \mathbb{C}$ is called a $T^{p,q}$-*atom* if

- there exists a ball $B \subset \mathbb{R}^n$ such that $a$ is supported in $\widehat{B}$;
- $\|a\|_{L^q(\mathbb{R}^{n+1}_+;dy\frac{dt}{t})} \leq |B|^{[q,p]}$;
- we further impose that $a$ is continuous, if $q = \infty$.

Any ball satisfying all these conditions is called *associated to $a$*. Let $\mathscr{A}^{p,q}$ be the collection of all the $T^{p,q}$-atoms.

*Remark.* Hölder's inequality does not hold for $p < 1$. But $\mathscr{A}^{p,q}$ is still uniformly bounded in $T^{p,q}$. The case for $q = \infty$ is straightforward since

$$\|a\|_{T^{p,\infty}} = \left(\int_B A_\infty(a)(x)^p\right)^{\frac{1}{p}} \leq 1.$$

The remaining follows by raising-power trick. Pick sufficiently large $\lambda$ such that $\lambda p > 1$ and $\lambda q > 1$. For any $a \in \mathscr{A}^{p,q}$,

$$\begin{aligned}\|a\|_{T^{p,q}} = \left\||a|^{\frac{1}{\lambda}}\right\|_{T^{\lambda p,\lambda q}}^{\lambda} &\leq \left\|A_{\lambda q}\left(|a|^{\frac{1}{\lambda}}\right)\right\|_{\lambda p}^{\lambda} \\ &\leq \left\|A_{\lambda q}\left(|a|^{\frac{1}{\lambda}}\right)\right\|_{\lambda q}^{\lambda} |B|^{\lambda[\lambda p,\lambda q]} \lesssim \|a\|_q \, |B|^{[p,q]} \leq 1.\end{aligned}$$

The condition $q \geq p$ is to ensure Hölder inequality for the second inequality.

**Theorem 2.11.** *Suppose that $p \in (0,1]$ and $q \in [p,\infty]$. Any $T^{p,q}$-function $f$ can be written as the form*

$$f = \sum_{j=0}^{\infty} \lambda_j a_j$$

*for some $a_j$'s as $T^{p,q}$-atoms, $\lambda_j \in \mathbb{C}$, and*

$$\|(\lambda_j)_{j\in\mathbb{N}}\|_{\ell^p(\mathbb{N})} = \left(\sum_{j=0}^{\infty} |\lambda_j|^p\right)^{\frac{1}{p}} \lesssim \|f\|_{T^{p,q}}.$$

The proof is divided into two cases for $q = \infty$ and $q \in [p,\infty)$.

**Case $q = \infty$** Let $f$ be a $T^{p,\infty}$-function with $p \in (0,1]$. Lemma 1.1 shows that

$$O_k := \{x \in \mathbb{R}^n : A_\infty(f)(x) > 2^k\}$$

is open for any $k \in \mathbb{Z}$. Chebyshev-Markov inequality reads that $O_k$ is of finite measure, hence not the whole space. Pick a Whitney covering of $O_k$, $\{B_j^{(k)}\}_{j\in I_k}$. Eq.(2.3) reads that $(B_j^{(k)} \times (0,\infty)) \cap \widehat{O}_k \subset T((C+2)B_j^{(k)})$. Let $\widetilde{O}_j^{(k)}$ be a small open neighbourhood of $(B_j^{(k)} \times (0,\infty)) \cap \widehat{O}_k$ but contained in $\operatorname{Int} T(2(C+2)B_j^{(k)})$. Set that

$$\Delta_j^{(k)} := \left(B_j^{(k)} \times (0,\infty)\right) \cap (\widetilde{O}_j^{(k)} \setminus \widehat{O}_{k+1}),$$

which is open. Pick a collection of continuous partition of unity $\{\phi_j^{(k)}\}_{j\in I_k}$ such that $\phi_j^{(k)}$ is supported on $\Delta_j^{(k)}$. It is obvious that

$$\{(y,t) \in \mathbb{R}^{n+1}_+ : f(y,t) \neq 0\} \subset \bigcup_k \left(\widehat{O}_k \setminus \widehat{O}_{k+1}\right) \subset \bigcup_{j,k} \Delta_j^{(k)}$$

by continuity and shadowing. Thus, the decomposition is natural as

$$f = \sum_{j,k} f\phi_j^{(k)} = \sum_{j,k} 2^{k+1}|2(C+2)B_j^{(k)}|^{\frac{1}{p}} a_{j,k} =: \sum_{j,k} \lambda_{j,k} a_{j,k},$$

with

$$a_{j,k} := 2^{-k-1}|2(C+2)B_j^{(k)}|^{-\frac{1}{p}}\phi_j^{(k)}f.$$

Note that $a_{j,k} \in \mathscr{A}^{p,\infty}$. Indeed, $a_{j,k}$ is supported on $T(2(C+2)B_j^{(k)})$, and for any $(y,t) \in \Delta_j^{(k)}$, $|f(y,t)| \le 2^{k+1}$, otherwise, $B(y,t) \subset O_{k+1}$ by shadowing and hence $(y,t) \in \widehat{O}_{k+1}$, which contradicts. So, $\|f\phi_j^{(k)}\|_\infty \le \|f\mathbb{1}_{\Delta_j^{(k)}}\|_\infty \le 2^{k+1}$. Finally,

$$\sum_{j,k} \lambda_{j,k}^p \lesssim_{n,p} \sum_{j,k} 2^{kp}|B_j^{(k)}| \lesssim_n \sum_{k\in\mathbb{Z}} 2^{kp}|O_k| \lesssim_{n,p} \|A_\infty(f)\|_p^p.$$

**Case** $q \in [p,\infty)$ Let us first remind that the topological properties and inclusion relation of global $\gamma$-density sets, cf. the remark for Definition 1.16, just start to be utilised since here.

Let $f$ be a $T^{p,q}$-function with $p \in (0,1]$, $\alpha \in (0,1)$ a fixed constant, and $\gamma \in (0,1)$ given by transition lemma, cf. Lemma 1.17 in terms of $\alpha$. Lemma 1.1 shows that $O_k := \{x \in \mathbb{R}^n : A_q(f)(x) > 2^k\}$ is open for any $k \in \mathbb{Z}$, so $O_k \subset (O_k)_\gamma^* =: U_k$ and $U_k$ is open. $O_k$ is of finite measure by Chebyshev-Markov inequality, so is $U_k$ by maximal theorem. Pick a Whitney covering of $U_k$, $\{B_j^{(k)}\}_{j\in I_k}$. Set that

$$\begin{aligned}\Delta_j^{(k)} &:= \left(B_j^{(k)} \times (0,\infty)\right) \cap (T_\alpha(U_k) \setminus T_\alpha(U_{k+1})) \\ &= \left(B_j^{(k)} \times (0,\infty)\right) \cap T_\alpha(U_k) \cap \Gamma_\alpha\left((F_{k+1})_\gamma^*\right),\end{aligned}$$

where $F_k := O_k^c$. We claim that $\bigcup_{j,k} \Delta_j^{(k)}$ forms an essential support of $f$. Indeed, $\bigcup_{j,k} \Delta_j^{(k)} = \bigcup_k (T_\alpha(U_k) \setminus T_\alpha(U_{k+1})) = \bigcup_k T_\alpha(U_k)$, whose complement is $\bigcap_k \Gamma_\alpha((F_k)_\gamma^*)$. Transition lemma reads that

$$\int_{\Gamma_\alpha((F_k)_\gamma^*)} |f(y,t)|^q dy\frac{dt}{t} \lesssim_{n,\alpha,\gamma} \int_{F_k} A_q(f)(x)^q dx.$$

Note that RHS tends to 0 since $F_k = \bigcap_{l\ge k} F_l$ and $A_q(f) = 0$ on $\bigcap_k F_k = (\bigcup_k O_k)^c$, so $f$ is 0 a.e. on $\bigcap_k \Gamma_\alpha((F_k)_\gamma^*)$.

Pick a collection of measurable partition of unity $\{\phi_j^{(k)}\}_{j\in I_k}$ such that $\phi_j^{(k)} = 0$ outside of $\Delta_j^{(k)}$, whose existence is ensured by the bounded overlapping property of Whitney covering. Thus,

$$\begin{aligned}\left\|f\phi_j^{(k)}\right\|_q^q &\le \int_{\Gamma_\alpha\left((F_{k+1})_\gamma^*\right)} \left|\mathbb{1}_{\Delta_j^{(k)}}(y)f(y,t)\right|^q dy\frac{dt}{t} \\ &\lesssim_{n,\alpha,\gamma} \int_{F_{k+1}} \left(\int_{\Gamma(x)} \left|\mathbb{1}_{\Delta_j^{(k)}}(y)f(y,t)\right|^q \frac{dy}{t^n}\frac{dt}{t}\right) dx \\ &\le \int_{\alpha^{-1}(C+2)B_j^{(k)}\cap F_{k+1}} A_q(f)(x)^q dx \le 2^{q(k+1)}|(C+2)B_j^{(k)}|.\end{aligned}$$

The third inequality holds by Eq.(2.3) as it reads that $\Delta_j^{(k)} \subset T(\alpha^{-1}(C+2)B_j^{(k)})$ since $T_\alpha((C+2)B_j^{(k)}) \subset T(\alpha^{-1}(C+2)B_j^{(k)})$. Therefore, we write that

$$f = \sum_{j,k} f\phi_j^{(k)} = \sum_{j,k} 2^{k+1}(\alpha(C+2))^{\frac{n}{q}}|B_j^{(k)}|^{\frac{1}{p}}a_{j,k} =: \sum_{j,k} \lambda_{j,k}a_{j,k},$$

with

$$a_{j,k} := 2^{-k-1}(\alpha(C+2))^{-\frac{n}{q}}|B_j^{(k)}|^{-\frac{1}{p}}\phi_j^{(k)}.$$

Computation directly shows that $a_{j,k} \in \mathscr{A}^{p,q}$ up to a uniform constant independent of $k$ and $j$. Moreover, observe that

$$\sum_{j,k} \lambda_{j,k}^p \lesssim_{n,p,q,\alpha} \sum_{k\in\mathbb{Z}} 2^{kp}|B_j^{(k)}| \lesssim_n \sum_{k\in\mathbb{Z}} 2^{kp}|O_k| \lesssim_{n,p} \|A_q(f)\|_p^p.$$

### 2.2.2 On spaces of homogeneous type

Let $(X,d,\mu)$ be a space of homogeneous type.

**Definition 2.12** ($T^{p,q}$-atoms)**.** Suppose that $p \in (0,1]$ and $q \in [p,\infty]$. A measurable function $a : X_+ \to \mathbb{C}$ is called a *$T^{p,q}$-atom* if

- there exists a ball $B \subset X$ such that $a$ is supported in $\widehat{B}$;
- $\|a\|_{L^q(X_+;d\mu\frac{dt}{t})} \leq \mu(B)^{[q,p]}$;
- It is further imposed that $a$ is continuous, if $q=\infty$ and $X$ is locally compact.

Any ball satisfying all these conditions is called *associated to $a$*. Let $\mathscr{A}^{p,q}$ be the collection of all the $T^{p,q}$-atoms.

Note that $\mathscr{A}^{p,q}$ is still bounded in $T^{p,q}$ by the same deduction as the remark for Definition 2.10. We state the following generalization of Theorem 2.11.

**Theorem 2.13.** *Suppose that $p \in (0,1]$ and $q \in [p,\infty]$. Then any $T^{p,q}$-function $f$ can be written as the form*

$$f = \sum_{j=0}^{\infty} \lambda_j a_j$$

*for some $a_j$'s as $T^{p,q}$-atoms, $\lambda_j \in \mathbb{C}$, and*

$$\|(\lambda_j)_{j\in\mathbb{N}}\|_{\ell^p(\mathbb{N})} = \left(\sum_{j=0}^{\infty} |\lambda_j|^p\right)^{\frac{1}{p}} \lesssim \|f\|_{T^{p,q}}.$$

The proof is also divided into two cases, $q \in [p,\infty)$ and $q=\infty$.

**Case** $q \in [p,\infty)$ The set $O_k := \{x \in X : A_q(f)(x) > 2^k\}$ is still open for any $k \in \mathbb{Z}$ since $A_q(f)$ is still lower semi-continuous. Set that $F_k := O_k^c$ and $U_k := (O_k)^*_\gamma$ that is open and of finite measure.

Suppose that $\mu(X)=\infty$. Note that $U_k \subsetneq X$ for any $k \in \mathbb{Z}$, thus, the proof is of the same as Theorem 2.11 with slight changes as

$$\left\|f\phi_j^{(k)}\right\|_q^q \lesssim 2^{q(k+1)}\mu(\alpha^{-1}\kappa(C+2)B_j^{(k)}) \tag{2.11}$$

by Eq.(2.8). The decomposition works as

$$f = \sum_{j,k} f\phi_j^{(k)} = \sum_{j,k} 2^{k+1}\rho(\alpha^{-1}\kappa(C+2))^{\frac{1}{q}}\mu(B_j^{(k)})^{\frac{1}{p}}a_{j,k} =: \sum_{j,k}\lambda_{j,k}a_{j,k},$$

with $a_{j,k} := 2^{-k-1}\rho(\alpha^{-1}\kappa(C+2))^{-\frac{1}{q}}\mu(B_j^{(k)})^{-\frac{1}{p}}\phi_j^{(k)}$.

Suppose that $\mu(X) < \infty$. Note that if $U_{k'} = X$ for some $k' \in \mathbb{Z}$, $U_r = X$ for any $r \leq k'$. Set $l = \max\{r \in \mathbb{Z} : U_r = X\} \in [-\infty, \infty)$, which differs from $+\infty$ since $A_q(f) \in L^p(X)$.

If $l = -\infty$, all the $U_k$'s differ from $X$, the construction is of the same as $\mu(X) = \infty$.

If $l \in (-\infty, +\infty)$, construct $\{\Delta_j^{(k)}\}_{j \in I_k}$ as follows.

- $I_k = \{0\}$ with $\Delta_0^{(k)} = \varnothing$, if $k < l$;
- $I_l = \{0\}$ with $\Delta_0^{(l)} = X_+ \setminus T_\alpha(U_{l+1}) = \Gamma_\alpha((F_{l+1})_\gamma^*)$;
- Pick a Whitney covering of $U_k$, $\{B_j^{(k)}\}_{j \in I_k}$, and define that $\Delta_j^{(k)} := (B_j^{(k)} \times (0,\infty)) \cap (T_\alpha(U_k) \setminus T_\alpha(U_{k+1})) = (B_j^{(k)} \times (0,\infty)) \cap T_\alpha(U_k) \cap \Gamma_\alpha((F_{k+1})_\gamma^*)$, if $k > l$.

Pick $\{\phi_j^{(k)}\}_{k \in \mathbb{Z}, j \in I_k}$ as a partition of unity of measurable functions, ensured by Theorem 2.2. For $k < l$, $\phi_0^{(k)} = 0$; For $k > l$, Eq.(2.11) still holds. For $k = l$,

$$\left\| f\phi_0^{(l)} \right\|_q \lesssim 2^{k+1} \mu(X)^{\frac{1}{q}}.$$

Thus, the composition is modified as

$$f = \sum_{\substack{k \geq l \\ j \in I_k}} f\phi_j^{(k)} = \sum_{\substack{k \geq l \\ j \in I_k}} \lambda_{j,k} a_{j,k},$$

where

- $a_{0,l} := 2^{-k-1} \mu(X)^{-\frac{1}{p}} f\phi_0^{(l)}$ and $\lambda_{0,l} := 2^{k+1} \mu(X)^{\frac{1}{p}}$;
- If $k > l$, set that $a_{j,k} := 2^{-k-1} \rho(\alpha^{-1}\kappa(C+2))^{-\frac{1}{q}} \mu(B_j^{(k)})^{-\frac{1}{p}} \phi_j^{(k)}$ and $\lambda_{j,k} := 2^{k+1} \rho(\alpha^{-1}\kappa(C+2))^{\frac{1}{q}} \mu(B_j^{(k)})^{\frac{1}{p}}$.

Direct computation suffices to conclude.

**Case $q = \infty$** If $X$ is locally compact, the proof is of the same as Theorem 2.11 with further discussion on whether $\mu(X) = \infty$. Note that the partition of unity of continuous functions can be satisfied by the local compactness [Tu11, Theorem 13.7].

If $X$ is not locally compact, set that $\alpha = \frac{1}{4}$, $O_k := \{x \in X : A_\infty(f)(x) > 2^k\}$ for $k \in \mathbb{Z}$, and we also claim that $\bigcup_k T_\alpha(O_k)$ is an essential support of $f$. Indeed, for any $m \in \mathbb{Z}$, pick $\{x_j\}_{j \in J_m}$ in $X$ so that $\{B(x_j, 2^{m-2})\}_{j \in J_m}$ covers $X$. The doubling property of $X$ reads that $J_m$ is at most countable. For any $j, m$ such that the set $\{(y,t) \in B(x_j, 2^{m-2}) \times [2^m, 2^{m+1}) : |f(y,t)| > 2^k\}$ takes non-zero measure with respect to $\sigma$ for some $k \in \mathbb{Z}$, Eq.(1.3) reads that $B(x_j, \frac{3}{4}2^k) \subset O_k$. Note that $B(x_j, 2^{m-2}) \times [2^m, 2^{m+1}) \subset T_{\frac{1}{4}}(B(x_j, \frac{3}{4}2^k)) \subset T_\alpha(O_k)$, which hence proves the claim. Observe that the claim holds for any $\alpha \in (0,1)$ by changing the radius of covering balls. The deduction also makes it clear that $\|f \mathbb{1}_{T_\alpha(O_{k+1})^c}\|_\infty \leq 2^{k+1}$.

Then we shall also discuss whether $\mu(X) = \infty$. If so, pick a Whitney covering of $O_k$, $\{B_j^{(k)}\}_{j \in I_k}$ and set that $\Delta_j^{(k)} := (B_j^{(k)} \times (0,\infty)) \cap (T_\alpha(O_k) \setminus T_\alpha(O_{k+1}))$ with a measurable partition of unity $\{\phi_j^{(k)}\}_{j \in I_k}$ such that $\phi_j^{(k)}$ is supported on $\Delta_j^{(k)}$. Thus, the decomposition works as

$$f = \sum_{j,k} f\phi_j^{(k)} = \sum_{j,k} 2^{k+1} \rho(\alpha^{-1}\kappa(C+2))^{\frac{1}{p}} \mu(B_j^{(k)})^{\frac{1}{p}} a_{j,k} =: \sum_{j,k} \lambda_{j,k} a_{j,k}$$

with

$$a_{j,k} := 2^{-k-1}\rho(\alpha^{-1}\kappa(C+2))^{-\frac{1}{p}}\mu(B_j^{(k)})^{-\frac{1}{p}}\phi_j^{(k)} f.$$

If not, pick $l := \max\{r \in \mathbb{Z} : O_r = X\}$ with the same construction. The decomposition finally turns out to be as

$$f = \sum_{\substack{k \geq l \\ j \in I_k}} f\phi_j^{(k)} = \sum_{\substack{k \geq l \\ j \in I_k}} \lambda_{j,k} a_{j,k},$$

where

- $a_{0,l} := 2^{-k-1}\mu(X)^{-\frac{1}{p}} f\phi_0^{(l)}$ and $\lambda_{0,l} := 2^{k+1}\mu(X)^{\frac{1}{p}}$;
- If $k > l$, set that $a_{j,k} := 2^{-k-1}\rho(\alpha^{-1}\kappa(C+2))^{-\frac{1}{q}}\mu(B_j^{(k)})^{-\frac{1}{p}}\phi_j^{(k)}$ and $\lambda_{j,k} := 2^{k+1}\rho(\alpha^{-1}\kappa(C+2))^{\frac{1}{q}}\mu(B_j^{(k)})^{\frac{1}{p}}$.

Let us finish this section by a remark of history of this problem and our work. Before that, let us recall the Whitney decomposition theorem.

**Theorem 2.14** (Whitney decomposition theorem)**.** *Let $O \subset \mathbb{R}^n$ be open. Then, there exists a collection of dyadic cubes $\{Q_i\}_{i \in I}$ such that:*

- *$Q_i$'s are mutually disjoint;*
- *$O = \bigcup_{i \in I} Q_i$;*
- *$\frac{1}{30}\operatorname{dist}(Q_i, O^c) \leq \operatorname{diam} Q_i \leq \frac{1}{10}\operatorname{dist}(Q_i, O^c)$.*

*Proof.* See [Aus12, Theorem 2.3.1]. □

Note that Whitney decomposition theorem can not be directly generalized to spaces of homogeneous type since dyadic grids do not directly work on spaces of homogeneous type. However, a highly non-trivial variant, so-called *Christ's variant*, of Whitney decomposition theorem on spaces of homogeneous type is given by [Chr90, Theorem 11] together with partition of unity [MS79, Lemma 2.16].

The followings are the previous works for Theorem 2.11 and 2.13 in different contexts.

- [CMS85, Proposition 2, Theorem 1(c)] used Whitney decomposition theorem, cf. Theorem 2.14, to prove the atomic decomposition for $T^{p,q}$ with $p \in (0,1]$ and $q \in (1,\infty]$ in $\mathbb{R}^n$. We shall mention that their deduction has a flaw that it is not convincing that $\bigcup_k \widehat{O}_k$ is an essential support of $f$ for if $q \neq \infty$. We fix this problem by introducing the aperture $\alpha$ and using transition lemma.
- [Rus07, Theorem 1.1] utilised Christ's variant to prove the atomic decomposition for $T^{p,2}$ with $p \in (0,1]$ over spaces of homogeneous type.
- [Ame18, Theorem 1.6] utilised Christ's variant to prove the atomic decomposition for $T^{p,q}$ with $p \in (0,1]$ and $q \in [p,\infty]$ over metric spaces of doubling property. In his settings, the continuity condition for type $(p,\infty)$ is reduced.

Nevertheless, we have shown that Whitney covering theorem suffices to conclude without introducing Christ's variant. What's more, our deduction is also compatible, if we also reduce the continuity condition for type $(p,\infty)$, by the proof for non-locally-compact spaces.

## 2.3 Application: independence of aperture revisited

In this section, we finalize the proof of independence of aperture for the definitions of tent spaces, cf. Theorem 1.14 via atomic decomposition. We first work on spaces of homogeneous type. For $p \in (0,1]$, $q \in [p,\infty]$ and $\alpha > 0$, the atomic decomposition on $T^{p,q}_{(\alpha)}$ for any $\alpha > 0$ and $p \in (0,1]$ can be similarly established.

**Definition 2.15** ($T^{p,q}_{(\alpha)}$**-atoms).** Suppose that $p \in (0,1]$ and $q \in [p,\infty]$. A measurable function $a : X_+ \to \mathbb{C}$ is called a $T^{p,q}_{(\alpha)}$*-atom* if

- there exists a ball $B \subset X$ such that $a$ is supported in $T_\alpha(B)$;
- $\|a\|_{L^q(X_+;d\mu\frac{dt}{t})} \leq \mu(B)^{[q,p]}$;
- $a$ is continuous if $q = \infty$ and $X$ is locally compact.

Any ball satisfying all these conditions is called *associated to* $a$. Let $\mathscr{A}^{p,q}_{(\alpha)}$ be the collection of all the $T^{p,q}_{(\alpha)}$-atoms, which is bounded in $T^{p,q}_{(\alpha)}$.

**Corollary 2.16.** *Suppose that* $p \in (0,1]$*,* $q \in [p,\infty]$*, and* $\alpha > 0$*. Then any* $T^{p,q}_{(\alpha)}$*-function* $f$ *can be written as the form*

$$f = \sum_{j=0}^{\infty} \lambda_j a_j$$

*for some* $a_j$*'s as* $T^{p,q}_{(\alpha)}$*-atoms,* $\lambda_j \in \mathbb{C}$*, and*

$$\|(\lambda_j)_{j\in\mathbb{N}}\|_{\ell^p(\mathbb{N})} = \left(\sum_{j=0}^{\infty} |\lambda_j|^p\right)^{\frac{1}{p}} \lesssim \|f\|_{T^{p,q}_{(\alpha)}}.$$

*Proof.* For $q \neq \infty$, it has been proved in Theorem 2.13 since the space is independent of aperture. For $q = \infty$, the same argument works without too much modification. $\square$

It hence suffices to conclude.

*Proof of Theorem 1.14.* Recall that the only remaining case is for type $(p,\infty)$ with $p \in (0,1]$. But we shall prove the case $p \in (0,1]$ and $q \in [p,\infty]$ as preparation for Sec.3.5. It still suffices to prove $\|f\|_{T^{p,\infty}_{(\alpha)}} \lesssim \|f\|_{T^{p,\infty}_{(\beta)}}$ for any $\alpha, \beta > 0$. For any $f \in T^{p,q}_{(\beta)}$, write its $T^{p,q}_{(\beta)}$-atomic decomposition form as $f = \sum_{j\in\mathbb{N}} \lambda_j a_j$. Pick $B_j$ as a ball associated to $a_j$ for any $j \in \mathbb{N}$.

Suppose that $\alpha > \beta$ first. Note that $\operatorname{supp} a_j \subset T_\beta(B_j) \subset T_\alpha(\frac{\alpha}{\beta}B_j)$ and $\mu(\frac{\alpha}{\beta}B_j) \leq \rho(\beta^{-1}\alpha)\mu(B_j)$. Thus,

$$\rho\left(\beta^{-1}\alpha\right)^{[q,p]} a_j \in \mathscr{A}^{p,q}_{(\alpha)}.$$

Therefore, the formula

$$f = \sum_{j\in\mathbb{N}} \left(\rho\left(\beta^{-1}\alpha\right)^{-[q,p]} \lambda_j\right) \left(\rho\left(\beta^{-1}\alpha\right)^{[q,p]} a_j\right)$$

works as an atomic decomposition of $f$ by $T^{p,q}_{(\alpha)}$-atoms, so $f \in T^{p,q}_{(\alpha)}$. Thus,

$$\|f\|_{T^{p,q}_{(\alpha)}} \leq \rho\left(\beta^{-1}\alpha\right)^{-[q,p]} \|(\lambda_j)_{j\in\mathbb{N}}\|_{\ell^p(\mathbb{N})} \lesssim \rho\left(\beta^{-1}\alpha\right)^{[p,q]} \|f\|_{T^{p,q}_{(\beta)}}. \tag{2.12}$$

If $\beta > \alpha$, $T_\beta(B_j) \subset T_\alpha(B_j)$, so $\mathscr{A}^{p,q}_{(\beta)} \subset \mathscr{A}^{p,q}_{(\alpha)}$. In particular, $f = \sum_{j\in\mathbb{N}} \lambda_j a_j$ directly performs as the atomic decomposition of $f$ by $T^{p,q}_{(\alpha)}$-atoms, so

$$\|f\|_{T^{p,q}_{(\alpha)}} \leq \|(\lambda_j)_{j\in\mathbb{N}}\|_{\ell^p(\mathbb{N})} \lesssim \|f\|_{T^{p,q}_{(\beta)}}. \tag{2.13}$$

It is worthy of pointing out that the controlling constant is independent of $\alpha, \beta$. □

*Remark.* The result also indicates the consistency of the normalization for $C_{q;\alpha}$. Without loss of generality, consider a simple case for $\alpha > \beta = 1$, for any $l \in (T^{1,q})^* = T^{\infty,q'}$,

$$\|l\|_{T^{\infty,q'}} \sim \|l\|_{(T^{1,q})^*} = \sup_{\|f\|_{T^{1,q}}\leq 1} (l,f) \gtrsim \sup_{\|f\|_{T^{1,q}_{(\alpha)}}\leq \alpha^{n/q'}} (l,f) = \alpha^{-\frac{n}{q'}}\|l\|_{(T^{1,q}_{(\alpha)})^*}$$

thanks to Eq.(2.12). It exactly corresponds to Eq.(1.11) and implies that the normalization of $C_{q;\alpha}$ ensures that there is no loss in terms of $\alpha$ in the identification $(T^{1,q}_{(\alpha)})^* = T^{\infty,q'}_{(\alpha)}$, which is decisive for the optimality of controlling constants of changing aperture to be discussed in Sec.3.5.

In the Euclidean settings, the normalization of $T^{p,q}_{(\alpha)}$-atoms is modified as

$$\|a\|_{L^q(\mathbb{R}^{n+1}_+;dy\frac{dt}{t})} \leq \alpha^{-\frac{n}{q}}|B|^{[q,p]}$$

to ensure that

$$\|a\|_{T^{p,q}_{(\alpha)}} \leq |B|^{[p,q]}\|A_{q;\alpha}(a)\|_q \leq 1.$$

Thus, the controlling constants of changing aperture are determined by interpolation as follows. If $\alpha > \beta$,

$$\|f\|_{T^{p,q}_{(\alpha)}} \lesssim \left(\frac{\alpha}{\beta}\right)^{\frac{n}{p}} \|f\|_{T^{p,q}_{(\beta)}}. \tag{2.14}$$

If $\alpha < \beta$,

$$\|f\|_{T^{p,q}_{(\alpha)}} \lesssim \left(\frac{\alpha}{\beta}\right)^{\frac{n}{q}} \|f\|_{T^{p,q}_{(\beta)}}. \tag{2.15}$$

## 2.4 Duality revisited

In this section, we get back to the duality from atomic decomposition to strengthen the viewpoint that duality and atomic decomposition are actually equivalent descriptions. We shall give new approaches of several results of duality in Sec.2.1.

For Theorem 2.3 on type $(1,\infty)$, we shall verify that the pairing, cf. Eq.(2.1), works via atomic decomposition. For any $T^{1,\infty}$-atom $a$, pick an associated ball $B$, so

$$|(\nu,a)| \leq \int_{\widehat{B}} |a(y,t)||d\nu(y,t)| \leq \frac{1}{|B|}\int_{\widehat{B}} |d\nu(y,t)| \leq \|\nu\|_C.$$

Thus, for any $T^{1,\infty}$-function $f$, pick an atomic decomposition as $f = \sum_j \lambda_j a_j$, so

$$|(\nu,f)| \leq \sum_j |\lambda_j||(\nu,a_j)| \lesssim \|\nu\|_C\|f\|_{T^{1,\infty}}.$$

The same deduction works for Theorem 2.5 on spaces of homogeneous type.

We can also verify the pairing of Theorem 2.4 on type $(1,q)$ with $q \in [1,\infty)$, cf. Eq.(2.5), works via atomic decomposition. For any $T^{1,q}$-atom $a$ with an associated ball $B$,

$$|(g,a)| \le \int_{\widehat{B}} |g(y,t)a(y,t)|dy\frac{dt}{t} \le \left(\int_{\widehat{B}} |g(y,t)|^{q'}dy\frac{dt}{t}\right)^{\frac{1}{q'}} \left(\int_{\widehat{B}} |a(y,t)|^{q}dy\frac{dt}{t}\right)^{\frac{1}{q}}$$
$$\le \left(\frac{1}{|B|}\int_{\widehat{B}} |g(y,t)|^{q'}dy\frac{dt}{t}\right)^{\frac{1}{q'}} \le \|g\|_{T^{\infty,q'}}.$$

for any $T^{\infty,q'}$-function $g$. For any $T^{1,q}$-function $f$, pick an atomic decomposition as $f = \sum_j \lambda_j a_j$. Therefore,

$$|(g,f)| \le \sum_j |\lambda_j||(g,a_j)| \lesssim \|g\|_{T^{\infty,q'}}\|f\|_{T^{1,q}}.$$

The same deduction works for Theorem 2.6 on spaces of homogeneous type.

Moreover, atomic decomposition will help reveal more on duality. For instance, we shall visualize the dual space of $T^{p,q}$ with $p \in (0,1], q \in [1,\infty)$ via a new kind of function spaces. We directly work on a space of homogeneous type $(X,d,\mu)$.

For $\sigma \ge 0$, the $(q;\alpha)$-*tent-maximal operator compensated by* $\sigma$, $C_{\sigma;q;\alpha}$ is given by

$$C_{\sigma;q;\alpha}(f)(x) = \sup_{x\in B} \frac{1}{\mu(B)^{\sigma}} \left(\frac{1}{\mu(B)}\int_{T_\alpha(B)} |f(y,t)|^q d\mu(y)\frac{dt}{t}\right)^{\frac{1}{q}}$$

if $q \in (0,\infty)$, and

$$C_{\sigma;\infty;\alpha}(f)(x) = \sup_{x\in B} \frac{1}{\mu(B)^{\sigma}} \operatorname*{esssup}_{(y,t)\in T_\alpha(B)} |f(y,t)|,$$

for any measurable function $f$ on $X_+$, where $B$ is taken from all the $d$-balls of $X$.

For $q \in (0,\infty]$, the space $T^{\infty,q}_{(\alpha),[\sigma]}$ consists of measurable functions $f$ such that $C_{\sigma;q;\alpha}(f) \in L^\infty(X)$, equipped with the quasi-norm

$$\|f\|_{T^{\infty,q}_{(\alpha),[\sigma]}} := \|C_{\sigma;q;\alpha}(f)\|_\infty.$$

Note that $T^{\infty,q}_{(\alpha),[0]} = T^{\infty,q}_{(\alpha)}$ for all the $q$'s, so we ignore $\sigma$ if $\sigma = 0$.

**Proposition 2.17.** *Let $(X,d,\mu)$ be a space of homogeneous type. Suppose that $p \in (0,1], q \in [1,\infty)$. Then the dual space of $T^{p,q}$ can be identified as $T^{\infty,q'}_{[[p,1]]}$ via the canonical pairing*

$$(g,f) \mapsto \int_{X_+} g(y,t)f(y,t)d\mu(y)\frac{dt}{t}.$$

*for any $(g,f) \in T^{\infty,q'}_{[[p,1]]} \times T^{p,q}$.*

*Proof.* We show first that the pairing makes sense. For any $T^{p,q}$-atom $a$, pick an associated ball $B$. For any $g \in T^{\infty,q'}_{[[p,1]]}$,

$$|(g,a)| \le \int_{T(B)} |a(y,t)g(y,t)|\, d\mu(y)\frac{dt}{t} \le \|a\|_q \mu(B)^{\frac{1}{q'}+[p,1]}\|g\|_{T^{\infty,q'}_{[[p,1]]}}$$
$$\le \mu(B)^{[q,p]+[p,q]}\|g\|_{T^{\infty,q'}_{[[p,1]]}} = \|g\|_{T^{\infty,q'}_{[[p,1]]}}.$$

Theorem 2.13 reads that any $T^{p,q}$-function $f$ has the atomic decomposition form as $f = \sum_j \lambda_j a_j$. Thus,

$$|(g,f)| \leq \sum_j |\lambda_j||(g,a_j)| \leq \|g\|_{T^{\infty,q'}_{[[p,1]]}} \|(\lambda_j)\|_{\ell^1} \leq \|g\|_{T^{\infty,q'}_{[[p,1]]}} \|(\lambda_j)\|_{\ell^p} \lesssim \|g\|_{T^{\infty,q'}_{[[p,1]]}} \|f\|_{T^{p,q}}.$$

The converse follows the conventional deduction. For any $l \in (T^{p,q})^*$, pick $g \in L^{q'}_{\mathrm{loc}}(X_+)$ so that $(l,f) = (g,f)$ for any $f \in T^{p,q}$ supported in some cylinder $K$. Duality argument again shows that for any $B \subset X$,

$$\begin{aligned}
\|g\mathbb{1}_{\widehat{B}}\|_{q'} = \sup_\phi \int_{\widehat{B}} g(y,t)\phi(y,t)d\mu(y)\frac{dt}{t} &= \sup_\phi \left(\lim_{m\to\infty} \int_{K_m} g(y,t)(\phi\mathbb{1}_{\widehat{B}})(y,t)d\mu(y)\frac{dt}{t}\right) \\
&= \sup_\phi \left(\lim_{m\to\infty} (l, \phi\mathbb{1}_{K_m})\right) \\
&\leq \sup_\phi \left(\|l\|\|\phi\mathbb{1}_{\widehat{B}}\|_{T^{p,q}}\right),
\end{aligned}$$

where $\phi$ is taken from $L^q(X_+; d\mu\frac{dt}{t})$ with unit norm, and $\{K_m\}_{m\in\mathbb{N}}$ is again an exhaustion of $X_+$. For any such $\phi$,

$$\|\phi\mathbb{1}_{\widehat{B}}\|_{T^{p,q}} = \left(\int_B |A_q(\phi)(x)|^p d\mu(x)\right)^{\frac{1}{p}} \leq \|A_q(\phi)\|_q \mu(B)^{[p,q]} \lesssim \mu(B)^{[p,q]}.$$

Thus, we conclude by observing that

$$\|g\|_{T^{\infty,q'}_{[[p,1]]}} = \sup_{B\subset X} \mu(B)^{-[p,1]-\frac{1}{q'}} \|g\mathbb{1}_{\widehat{B}}\|_{q'} \lesssim \|l\|.$$

□

Let us finalize this chapter by a full picture of duality and atomic decomposition of tent spaces over spaces of homogeneous type.

- For type $(\infty,\infty)$, the dual space of $T^{\infty,\infty}$ can be identified as $\mathrm{ba}(X_+, \Sigma, d\mu(y)\frac{dt}{t})$, the collection of all finitely-additive finite signed measures defined on $\Sigma$. Here $\Sigma$ is the $\sigma$-algebra of Lebesgue-measurable sets. See [DS88, Theorem IV.8.16].
- For type $(p,\infty)$:
  - For $p = 1$, we have shown both of duality and atomic decomposition in Theorem 2.5 and Theorem 2.13, respectively.
  - For $p \neq 1$, we have shown atomic decomposition in Theorem 2.13 for $p < 1$. If we directly extend Definition 2.12 to $T^{p,\infty}$-atoms, the same deduction will gives the case for $p > 1$. Duality for $p \in (1,\infty)$ and $p \in (0,1)$ has been shown in [AM88, Theorem 5.1 and Theorem 5.5].
- For type $(\infty, q)$, duality is open and atomic decomposition is not reasonable.
- For type $(p,q)$:
  - For $p \in (1,\infty)$:
    * For $q \in (1,\infty)$, duality has been provided in Theorem 2.6, and in general, atomic decomposition is not of concern.

    * For $q \in (0, 1]$, both of duality and atomic decomposition are open.

- For $p = 1$:
    * For $q \in [1, \infty)$, both of duality and atomic decomposition have been established in Proposition 2.17 and Theorem 2.13, respectively.
    * For $q \in (0, 1)$, both of duality and atomic decomposition are open.
- For $p \in (0, 1)$:
    * For $q \in [1, \infty)$, both of duality and atomic decomposition have been established in Proposition 2.17 and Theorem 2.13, respectively.
    * For $q \in [p, 1)$, only atomic decomposition has been developed in Theorem 2.13.
    * For $q \in (0, p)$, both of duality and atomic decomposition are open.

# Chapter 3

# Interpolation theory of tent spaces

In this chapter, the real and complex interpolation theory becomes the theme. Different from the original process in [CMS85, Sec.7], we adapt the vector-valued approach. We shall directly deal with tent spaces over spaces of homogeneous type in this chapter. Thanks to Theorem 1.14, it suffices to assume that $\alpha = 1$. Let $(X, d, \mu)$ be a space of homogeneous type. Equip $X_+ = X \times (0, \infty)$ with the measure $d\mu(y)\frac{dt}{t}$ as default, and write that

$$L^q_*(X_+) := L^q\left(X_+; \frac{d\mu(y)}{V(y,t)}\frac{dt}{t}\right).$$

All the functions discussed in this chapter are $\mathbb{C}$-valued without special mention.

## 3.1 Preliminaries for interpolation theory

We first introduce some fundamental notions of interpolation theory. Readers can refer to [BL76, Chapter 2] for more details.

### 3.1.1 Definition of interpolation spaces

Let $A_0, A_1$ be two normed vector spaces. They are said to be *comparable* if there is a Hausdorff topological vector spaces $A$ such that $A_0, A_1$ are subspaces of $A$. For two compatible normed vector spaces $A_0$ and $A_1$, the *intersection of* $A_0$ *and* $A_1$ is defined to be the space $A_0 \cap A_1$ equipped with the norm

$$\|a\|_{A_0 \cap A_1} = \max\{\|a\|_{A_0}, \|a\|_{A_1}\},$$

and the *sum of* $A_0$ *and* $A_1$ is defined to be the space $A_0 + A_1 = \{a \in A : \exists a_0 \in A_0, a_1 \in A_1 \text{ such that } a = a_0 + a_1\}$ equipped with the norm

$$\|a\|_{A_0+A_1} = \inf\{\|a_0\|_{A_0} + \|a_1\|_{A_1} : a = a_0 + a_1, a_0 \in A_0, a_1 \in A_1\}.$$

It is straightforward that both of them are still normed vector spaces, and they are complete if $A_0, A_1$ are complete [BL76, Lemma 2.3.1].

Given a sub-category $\mathscr{C}$ of the category of all the normed vector spaces $\underline{\mathsf{N}}$, we say that $\overline{A} := (A_0, A_1)$ is a *compatible couple* with respect to $\mathscr{C}$ if $A_0, A_1, A_0 + A_1, A_0 \cap A_1$ are objects in $\mathscr{C}$ and $A_0, A_1$ are compatible.

Let $\mathscr{C}_1$ be the category of compatible couples of $\mathscr{C}$, whose morphisms $T : \overline{A} \to \overline{B}$ consist of bounded linear operators from $A_0 + A_1$ to $B_0 + B_1$ such that $T_0 := T|_{A_0} : A_0 \to B_0, T_1 := T|_{A_0} : A_1 \to B_1$ are morphisms in $\mathscr{C}$. Note that the restriction of $T$ on any subspace $A$ of $A_0 + A_1$ makes sense directly since $T$ is well-defined on $A_0 + A_1$. We often write $T_A$ or just $T : A \to B$ instead of $T|_A$.

**Definition 3.1** (Interpolation spaces)**.** Let $\mathscr{C}$ be a given sub-category of $\underline{\mathsf{N}}$ and $\mathscr{C}_1$ the category of compatible couples of $\mathscr{C}$. For any $\overline{A} = (A_0, A_1), \overline{B} = (B_0, B_1) \in \mathscr{C}_1$ and $A, B \in \mathscr{C}$, the pairing $(A, B)$ is called *interpolation pairing* of $(\overline{A}, \overline{B})$, if

- $\Delta(\overline{A}) \hookrightarrow A \hookrightarrow \Sigma(\overline{A})$;
- $\Delta(\overline{B}) \hookrightarrow B \hookrightarrow \Sigma(\overline{B})$;
- $T : \overline{A} \to \overline{B}$ implies that $T : A \to B$ is a bounded linear operator.

We say that a space $A \in \mathscr{C}$ is an *interpolation space* of $\overline{A}$, or between $A_0$ and $A_1$, if the pairing $(A, A)$ is an interpolation pairing of $(\overline{A}, \overline{A})$.

Let $\theta \in [0, 1]$ be a fixed constant. We shall say that the interpolation pairing $(A, B)$ of $(\overline{A}, \overline{B})$ is *of exponent* $\theta$ if there is a constant $C$ such that for any $T : \overline{A} \to \overline{B}$ as morphisms in $\mathscr{C}_1$,

$$\|T\|_{A,B} \leq C\|T\|^{1-\theta}_{A_0,B_0}\|T\|^{\theta}_{A_1,B_1}.$$

Note that $C$ must be independent of $T$, but might be dependent on $\theta$. If $C = 1$, we shall say that it is *exact of exponent* $\theta$. Similarly, we say that an interpolation space $A$ of $\overline{A}$ is *(exact) of exponent* $\theta$ if $(A, A)$ is *(exact) of exponent* $\theta$.

### 3.1.2 Interpolation functors

To construct interpolation spaces, a practical approach is to construct functors. For instance, it is obvious that $A_0 \cap A_1$ and $A_0 + A_1$ are both interpolation spaces of $\overline{A} = (A_0, A_1)$. Actually they can be achieved by the *intersection functor* $\Delta : \mathscr{C}_1 \to \mathscr{C}$ and the *sum functor* $\Sigma : \mathscr{C}_1 \to \mathscr{C}$ given by

$$\Delta(\overline{A}) = A_0 \cap A_1, \quad \Sigma(\overline{A}) = A_0 + A_1.$$

Such an observation induces the following definition.

**Definition 3.2** (Interpolation functors)**.** Inherit the notations in Definition 3.1. A functor $F : \mathscr{C}_1 \to \mathscr{C}$ is called an *interpolation functor* or an *interpolation method* on $\mathscr{C}$, if

- for any $\overline{A}, \overline{B} \in \mathscr{C}_1$, $(F(\overline{A}), F(\overline{B}))$ is an interpolation pairing of $(\overline{A}, \overline{B})$;
- $F(T) = T_{F(\overline{A})}$ for any $T : \overline{A} \to \overline{B}$.

We say that the interpolation functor $F$ is *(exact) of exponent* $\theta$ if $(F(\overline{A}), F(\overline{B}))$ is exact of exponent $\theta$ with respect to $(\overline{A}, \overline{B})$.

In particular, we focus on a special but quite critical case. Let us recall notions of retraction, co-retraction, projections and complement spaces in the theory of Banach spaces first.

**Definition 3.3.** Let $\mathfrak{A}, \mathfrak{B}$ be two Banach spaces.

A bounded linear operator $R \in L(\mathfrak{A}, \mathfrak{B})$ is called a *retraction* if there exists a bounded linear operator $S \in L(\mathfrak{B}, \mathfrak{A})$ such that $R \circ S$ is the identity operator of $\mathfrak{B}$, where $S$ is hence called a *co-retraction* with respect to $R$.

A continuous linear operator on $\mathfrak{B}$, $P \in L(\mathfrak{B}, \mathfrak{B}) =: L(\mathfrak{B})$ is called a *projection* if $P^2 = P$. A subspace of a Banach space is called *complemented* if it is the range of some projection.

The following lemma shows the behaviors of interpolation functors with respect to retraction and complemented subspaces. Let $\underline{\mathsf{B}}$ be the category of Banach spaces and $\underline{\mathsf{B}}_1$ the category of compatible couples in $\underline{\mathsf{B}}$.

**Lemma 3.4.** *Let $\overline{A} = (A_0, A_1), \overline{B} = (B_0, B_1) \in \underline{\mathsf{B}}_1$ be two compatible couples of Banach spaces. The following statements hold:*

1. *Suppose that $R : \overline{A} \to \overline{B}, S : \overline{B} \to \overline{A}$ satisfy that $R_j : A_j \to B_j$ is retraction and $S_j : B_j \to A_j$ is the corresponding co-retraction, respectively, for $j = 0, 1$. Then, for any interpolation functor $F$, $S$ induces an isomorphism of $F(\overline{B})$ onto $SR(F(\overline{A}))$. Moreover, $SR(F(\overline{A}))$ is a complemented subspace of $F(\overline{A})$, whose associated projection is exactly $SR$.*

2. *Let $B$ be a complemented subspace of $\Sigma(\overline{A}) = A_0 + A_1$. Let $F$ be an interpolation functor. Then, $(A_0 \cap B, A_1 \cap B) \in \underline{\mathsf{B}}_1$ and*
$$F((A_0 \cap B, A_1 \cap B)) = F((A_0, A_1)) \cap B.$$

*Proof.* See [Tri78, Theorem in Sec.1.2.4 and Theorem 1 in Sec.1.17.1]. □

## 3.2 A vector-valued approach to tent spaces

This section provides us with a vector-valued approach to tent spaces. For a measured space $(Y, \nu)$, we write $L^p(Y, \nu; \mathfrak{B})$ as the $L^p$-space of $(Y, \nu)$ valued in a Banach space $\mathfrak{B}$ for $p \in (0, \infty]$. Sometimes, $Y$ or $\nu$ is omitted if clear. For basic properties of measurability and integration of Banach-valued functions, readers can refer to [Gra14, Sec.5.5.3].

Set that $p \in (0, \infty)$ and $q \in [1, \infty)$. The *embedding operator* $\iota : T^{p,q} \to L^p(X; L^q_*(X_+))$ is defined by
$$\iota(f)(x)(y,t) = \mathbb{1}_{\Gamma(x)}(y,t) f(y,t).$$
The measurability of $\iota(f)$ for any $T^{p,q}$-function $f$ is shown in [BC91, Lemma 2]. The only modification is to change compact sets approximation and dyadic cubes decomposition in Euclidean spaces as cylinder sets approximation and variant of dyadic cubes in spaces of homogeneous type given in [Chr90, Theorem 11]. It is obvious that $\iota$ is an isometry by definition.

The converse of the embedding operator is critical in the theory, shown by the following lemma. The idea was attributed by [HTV91, Theorem 2.1] and [Ber92, Theorem 1].

**Lemma 3.5.** *Suppose that $p, q \in (1, \infty)$. The operator $\pi$ mapping $L^p(X; L^q_*(X_+))$-functions to functions on $X_+$, given by*
$$\pi(F)(y,t) := \fint_{B(y,t)} F(x)(y,t) d\mu(x) = \frac{1}{\mu(B(y,t))} \int_{B(y,t)} F(x)(y,t) d\mu(x),$$
*is a bounded linear operator from $L^p(L^q_*(X_+))$ to $T^{p,q}$. Furthermore, $\pi$ is a retraction, $\iota$ is the corresponding co-retraction, and $T^{p,q}$ is hence a complemented subspace of $L^p(L^q_*(X_+))$ with respect to the projection $P := \iota \circ \pi$.*

*Proof.* It is trivial that the operator $\pi$ makes sense on locally integrable functions, so it makes sense on the whole range by approximation of linear combinations of simple functions. The measurability of $\pi(F)$ follows from [Mat99, Theorem 3.1].

For any $\phi \in T^{p',q'}$, note that $\iota(\phi) \in L^{p'}(X; L^{q'}_*(X_+))$, so Fubini's theorem for $\iota(\phi) \cdot F$ directly reads that

$$(\phi, \pi(F)) = (\iota(\phi), F).$$

Duality argument, cf. Theorem 2.6, reads that $\pi$ is hence bounded.

We conclude by observing that $\pi \circ \iota$ is the identity operator of $T^{p,q}$. Indeed, for any $T^{p,q}$-function $f$,

$$\pi \circ \iota(f)(y,t) = \fint_{B(y,t)} \iota(f)(x)(y,t) d\mu(x) = f(y,t) \fint_{B(y,t)} \mathbb{1}_{\Gamma(x)}(y,t) d\mu(x) = f(y,t),$$

so $\pi \circ \iota$ is the identity operator on $T^{p,q}$. □

The lemma is the most important weapon to simplify interpolation theory of tent spaces back to that of general $L^p$-spaces, which has been known quite well. Such observation significantly simplify our works in the following two chapters and generalizes the original results.

## 3.3 Complex interpolation theory

The complex interpolation theory was introduced by [Cal64] of Banach spaces, and there are some generalizations to quasi-Banach spaces such as [KM98, Theorem 3.4]. However, we only concentrate on that of Banach spaces in this chapter.

### 3.3.1 Definition of the complex interpolation method

Let $\overline{A} = (A_0, A_1)$ be a compatible couple of Banach spaces, i.e, $\overline{A} \in \underline{\mathsf{B}}_1$ which consist of compatible couples of the category of Banach spaces $\underline{\mathsf{B}}$. The vector space $\mathscr{F}(\overline{A})$ consists of functions $f : \mathbb{C} \to \Sigma(\overline{A})$ satisfying that

- $f$ is bounded and continuous on the strip $\{z \in \mathbb{C} : 0 \le \mathrm{Re}(z) \le 1\}$;
- $f$ is analytic on the open strip $\{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$;
- $t \mapsto f(j + it)$ decays to 0 as $|t| \to \infty$ for $j = 0, 1$.

Moreover, $\mathscr{F}(\overline{A})$ is equipped with the norm

$$\|f\|_{\mathscr{F}(\overline{A})} := \max \left\{ \sup_{t \in \mathbb{R}} \|f(it)\|_{A_0}, \sup_{t \in \mathbb{R}} \|f(1+it)\|_{A_1} \right\}.$$

The supremum makes sense by continuity and decaying property. It is easy to verify that $\mathscr{F}(\overline{A})$ is also a Banach space [BL76, Lemma 4.1.1].

The space $\overline{A}_{[\theta]}$ consists of $a \in \Sigma(\overline{A})$ such that $a = f(\theta)$ for some $f \in \mathscr{F}(\overline{A})$, equipped with the norm

$$\|a\|_{[\theta]} := \inf \left\{ \|f\|_{\mathscr{F}(A)} : f(\theta) = a, f \in \mathscr{F}(\overline{A}) \right\}.$$

It can be shown that $\overline{A}_{[\theta]}$ is a Banach space and an interpolation space of $\overline{A}$ exact of exponent $\theta$ [BL76, Theorem 4.1.2]. We summarize the above discussion by the following definition via functors.

**Definition 3.6** (Complex interpolation functor)**.** Let $\theta \in [0,1]$ be a fixed constant. The *complex interpolation functor* $C_\theta : \underline{\mathsf{B}}_1 \to \underline{\mathsf{B}}$ maps $\overline{A} \in \underline{\mathsf{B}}_1$ to $\overline{A}_{[\theta]}$, and the maps of morphisms are restriction by the definition of interpolation functors. It is exact of exponent $\theta$.

The following theorem shows complex interpolation of Banach-valued $L^p$-spaces.

**Theorem 3.7.** *Let $(Y, \nu)$ be a measured space and $A_0, A_1$ two Banach spaces. Suppose that $p_0, p_1 \in [1, \infty)$ and $\theta \in (0, 1)$. Then the identification holds that*

$$(L^{p_0}(Y, \nu; A_0), L^{p_1}(Y, \nu; A_1))_{[\theta]} = L^{p_\theta}(Y, \nu; (A_0, A_1)_{[\theta]})$$

*with*

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

*Proof.* See [BL76, Theorem 5.1.2]. □

We also prepare a theorem showing the behavior of complex interpolation functor with respect to duality.

**Theorem 3.8.** *Let $\overline{A} = (A_0, A_1) \in \underline{\mathsf{B}}_1$ be a compatible couple of Banach spaces. Suppose that $\Delta(\overline{A})$ is dense in both $A_0$ and $A_1$, and one of the spaces $A_0$ and $A_1$ is reflexive. Then*

$$(A_0, A_1)^*_{[\theta]} = (A_0^*, A_1^*)_{[\theta]}.$$

*Proof.* See [BL76, Theorem 4.5.1 and Corollary 4.5.2]. □

The following theorem is about reiteration, a critical weapon in the interpolation theory, in the context of complex interpolation. The following theorem is a simplified version by [Jan+84] of Wolff's original reiteration theorem [Wol82, Theorem 2].

**Theorem 3.9.** *Let $A_1, A_2, A_3, A_4$ be four Banach spaces which can be continuously embedded in some Hausdorff topological vector space. Suppose that $\theta, \eta, r_1, r_2$ are four constants such that $0 < \theta < \eta < 1$, $r_1 = \frac{\eta}{\theta}$ and $1 - r_2 = \frac{1-\eta}{1-\theta}$. If $(A_1, A_3)_{[r_1]} = A_2$ and $(A_2, A_4)_{[r_2]} = A_3$, then $(A_1, A_4)_{[\theta]} = A_2$ and $(A_1, A_4)_{[\eta]} = A_3$.*

*Proof.* See [Jan+84, Theorem 2]. □

### 3.3.2 Complex interpolation of tent spaces

We are ready to show the main theorem for complex interpolation theory of tent spaces.

**Theorem 3.10** (Complex interpolation)**.** *Let $(X, d, \mu)$ be a space of homogeneous type. Suppose that $p_0, p_1 \in [1, \infty]$ but excluding that $p_0 = p_1 = \infty$, $q_0, q_1 \in (1, \infty)$, and $\theta \in (0, 1)$. The identification holds that*

$$(T^{p_0,q_0}, T^{p_1,q_1})_{[\theta]} = T^{p_\theta,q_\theta},$$

*where*

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

The proof is divided into the following cases.

**Case** $p_0, p_1 \in (1, \infty)$ We first concern the typical case $p_0, p_1 \in (1, \infty)$. Theorem 3.7 reads that

$$\begin{aligned}(L^{p_0}(X, \mu; L^{q_0}_*(X_+)), L^{p_1}(X, \mu; L^{q_1}_*(X_+)))_{[\theta]} &= L^{p_\theta}(X, \mu; (L^{q_0}_*(X_+), L^{q_1}_*(X_+))_{[\theta]})]] \\ &= L^{p_\theta}(X, \mu; L^{q_\theta}_*(X_+)).\end{aligned}$$

Lemma 3.5 reads that the generalized map $\pi : (L^{p_0}(L_*^{q_0}), L^{p_1}(L_*^{q_1})) \to (T^{p_0,q_0}, T^{p_1,q_1})$ and $\iota : (T^{p_0,q_0}, T^{p_1,q_1}) \to (L^{p_0}(L_*^{q_0}), L^{p_1}(L_*^{q_1}))$ are bounded linear operators and satisfy the conditions in Lemma 3.4 as retractions and co-retractions, respectively. Thus, for the complex interpolation functor, we conclude that $\iota$ induces an isomorphism from $(T^{p_0,q_0}, T^{p_1,q_1})_{[\theta]}$ to $\iota \circ \pi(L^{p_\theta}(L_*^{q_\theta})) = T^{p_\theta,q_\theta}$ again by Lemma 3.5.

**Case** $p_0 = 1$ **and** $p_1 \in [1,\infty)$ Since $\iota : T^{p,q} \to L^p(L_*^q)$ is an isometry, the exactness of exponent $\theta$ of complex interpolation functor reads that

$$\begin{aligned}\|\iota(f)\|_{(L^{p_0}(L_*^{q_0}),L^{p_1}(L_*^{q_1}))_{[\theta]}} &\leq \|\iota\|^{1-\theta}_{L(T^{p_0,q_0},L^{p_0}(L_*^{q_0}))}\|\iota\|^{\theta}_{L(T^{p_1,q_1},L^{p_1}(L_*^{q_1}))}\|f\|_{(T^{p_0,q_0},T^{p_1,q_1})_{[\theta]}} \\ &= \|f\|_{(T^{p_0,q_0},T^{p_1,q_1})_{[\theta]}}.\end{aligned}$$

It follows that

$$(T^{p_0,q_0}, T^{p_1,q_1})_{[\theta]} \hookrightarrow T^{p_\theta,q_\theta}$$

since

$$\|f\|_{T^{p_\theta,q_\theta}} = \|\iota(f)\|_{L^{p_\theta}(L_*^{q_\theta})} = \|\iota(f)\|_{(L^{p_0}(L_*^{q_0}),L^{p_1}(L_*^{q_1}))_{[\theta]}}.$$

Note that the above deduction actually holds for any $p_0, p_1 \in [1,\infty)$. The mere difference is that we conclude the converse by the retraction $\pi$ for $p_0, p_1 \in (1,\infty)$. Now, we shall construct it manually by definition.

For any $f \in T^{p_\theta,q_\theta}$, define that $g = |f|^{\frac{1}{2}}$. Raising-power trick, cf. Eq.(2.10), reads that $g \in T^{2p_\theta,2q_\theta} = (T^{2p_0,2q_0}, T^{2p_1,2q_1})_{[\theta]}$. Thus, for any $\epsilon > 0$, there is some $\phi \in \mathscr{F}((T^{2p_0,2q_0}, T^{2p_1,2q_1})) =: \mathscr{F}'$ such that $\phi(\theta) = g$ and

$$\|\phi\|_{\mathscr{F}'} \leq \|g\|_{(T^{2p_0,2q_0},T^{2p_1,2q_1})_{[\theta]}} + \epsilon \lesssim \|g\|_{T^{2p_\theta,2q_\theta}} + \epsilon,$$

where the controlling constant is independent of $\phi$ and $g$. In turn, raising-power trick also reads that $\psi = \phi^2 \in \mathscr{F}((T^{p_0,q_0}, T^{p_1,q_1})) =: \mathscr{F}$ with $\psi(\theta) = \varphi(\theta)^2 = f$, so

$$\|f\|_{(T^{p_0,q_0},T^{p_1,q_1})_{[\theta]}} \leq \|\psi\|_{\mathscr{F}} \leq \|\phi\|^2_{\mathscr{F}'} \lesssim (\|g\|_{T^{2p_\theta,2q_\theta}} + \epsilon)^2 \lesssim \|f\|_{T^{p_\theta,q_\theta}} + \epsilon^2,$$

where the constant is independent of $f, g$ and $\epsilon$. We hence conclude the embedding

$$T^{p_\theta,q_\theta} \hookrightarrow (T^{p_0,q_0}, T^{p_1,q_1})_{[\theta]}$$

by taking $\epsilon$ to 0.

**Case** $p_0 \in (1,\infty), p_1 = \infty$ The duality theorem, cf. Theorem 2.6, reads that

$$(T^{p_0,q_0}, T^{\infty,q_1})_{[\theta]} = ((T^{p_0',q_0'})^*, (T^{1,q_1'})^*)_{[\theta]}.$$

Note that the the collection of linear combinations of cylinder-supported simple functions is dense both in $T^{p_0',q_0'}$ and $T^{1,q_1'}$, so Theorem 3.8 finally reads that

$$(T^{p_0,q_0}, T^{\infty,q_1})_{[\theta]} = (T^{p_0',q_0'}, T^{1,q_1'})^*_{[\theta]} = (T^{p_\theta',q_\theta'})^* = T^{p_\theta,q_\theta}. \tag{3.1}$$

The reason why $\theta \in (0,1)$ is to ensure that $p_\theta, q_\theta \in (1,\infty)$ for the last identification.

**Case** $p_0 = 1, p_1 = \infty$ For $\theta \in (0,1)$, pick some $r_1 \in (0,1)$ but sufficiently close to 1 such that $\eta = \frac{\theta}{r_1} \in (0,1)$. Write that $r_2 = 1 - \frac{1-\eta}{1-\theta} \in (0,1)$. Set that

$$A_1 = T^{1,q_0},\ A_2 = T^{p_\theta,q_\theta},\ A_3 = T^{p_\eta,q_\eta},\ A_4 = T^{\infty,q_1}.$$

The above discussion exactly reads that

$$(A_1, A_3)_{[r_1]} = A_2, \quad (A_2, A_4)_{[r_2]} = A_3,$$

since $p_\theta, p_\eta \in (1, \infty)$,

$$\frac{1-r_1}{1} + \frac{r_1}{p_\eta} = 1 - r_1 + r_1(1-\eta) = 1 - \theta = \frac{1}{p_\theta},$$
$$\frac{1-r_2}{p_\theta} = (1-r_2)(1-\theta) = 1 - \eta = \frac{1}{p_\eta},$$

and the same holds for $q_0, q_\theta, q_\eta, q_1$. Theorem 3.9 hence reads that

$$(T^{1,q_0}, T^{\infty,q_1})_{[\theta]} = T^{p_\theta,q_\theta}.$$

Let us finalize this section by a remark of some progress on complex interpolation theory of tent spaces. The relation

$$(T^{p_0,q_0}, T^{p_1,q_1})_{[\theta]} = T^{p_\theta,q_\theta},$$

has been proved true in the following contexts:

- $p_0, p_1 \in (0, \infty)$ and $q_0, q_1 \in (0, \infty)$ for $(\mathbb{R}^{n+1}_+, d_2, \mathscr{L})$, i.e., $\mathbb{R}^{n+1}_+$ with Euclidean distance $d_2$ and Lebesgue measure $\mathscr{L}$ in [HMM11, Lemma 8.23];
- $p_0, q_0 < \infty$ or $p_1, q_1 < \infty$ for $(\mathbb{R}^{n+1}_+, d_2, \mathscr{L})$ in [Hua16, Theorem 4.3].

Generalization of these results onto spaces of homogeneous type needs further work.

## 3.4 Real interpolation theory

### 3.4.1 Definition and fundamental properties of real interpolation method

For a compatible couple of normed vector spaces $\overline{A} = (A_0, A_1) \in \underline{\mathsf{N}}_1$, define that

$$K(t, a; \overline{A}) = \inf \{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1, a_0 \in A_0, a_1 \in A_1\}$$

as an equivalent norm on $\Sigma(\overline{A})$ for any $t > 0$‘. For $\theta \in (0, 1)$ and $q \in [1, \infty]$, define the space $\overline{A}_{\theta,q}$ consisting of $a \in \Sigma(\overline{A})$ such that

$$\|a\|_{\theta,q} := \left(\int_0^\infty (t^{-\theta} K(t, a; \overline{A}))^q \frac{dt}{t}\right)^{\frac{1}{q}} < \infty$$

if $q \in [1, \infty)$, and

$$\|a\|_{\theta,\infty} := \operatorname*{esssup}_{t \in (0,\infty)} |t^{-\theta} K(t, a; \overline{A})| < \infty$$

if $q = \infty$. The following facts have been fully discussed in [BL76, Sec.3.1] for $q \in [1, \infty]$, for which we omit the details:

- the legality of the conditions on the given region;
- $\| \cdot \|_{\theta,q}$ is a norm;
- $\overline{A}_{\theta,q}$ is an interpolation space of $\overline{A}$ exactly of exponent $\theta$.

Thus, it suffices to define an interpolation functor of $\underline{\mathsf{N}}_1$ to $\underline{\mathsf{N}}$.

**Definition 3.11** (Real interpolation functor)**.** Let $\theta \in (0,1), q \in [1,\infty]$ be two fixed constants. The *real interpolation functor* $K_{\theta,q} : \underline{\mathsf{N}}_1 \to \underline{\mathsf{N}}$ maps $\overline{A} \in \underline{\mathsf{N}}_1$ to $\overline{A}_{\theta,q}$, and the maps of morphisms are restriction by the definition of interpolation functors. It is exact of exponent $\theta$.

Similar to what we have done for complex interpolation, we also prepare a series of theorem to characterize behaviors of the real interpolation functor with respect to $L^p$-spaces, duality, and reiteration.

**Theorem 3.12.** *Let $(Y,\nu)$ be a measured space and $A$ a Banach space. Suppose that $1 \le p_0 < p_1 \le \infty$, $\theta \in (0,1)$, and $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Then the identification holds that*

$$(L^{p_0}(Y,\nu;A), L^{p_1}(Y,\nu;A))_{\theta,p_\theta} = L^{p_\theta}(Y,\nu;A).$$

*Proof.* See [BL76, Theorem 5.2.1]. □

**Theorem 3.13.** *Let $\overline{A} = (A_0, A_1) \in \underline{\mathsf{B}}_1$ be a compatible couple of Banach spaces. Suppose that $\Delta(\overline{A})$ is dense in both $A_0$ and $A_1$. Assume that $q \in [1,\infty)$ and $\theta \in (0,1)$. Then*

$$(A_0, A_1)^*_{\theta,q} = (A_0^*, A_1^*)_{\theta,q'}.$$

*Proof.* See [BL76, Theorem 3.7.1]. □

**Theorem 3.14.** *Inherit the notions in Theorem 3.9. Let $p,q \in [1,\infty]$ be two fixed constants. If $(A_1,A_3)_{r_1,p} = A_2$ and $(A_2,A_4)_{r_2,q} = A_3$, then $(A_1,A_4)_{\theta,p} = A_2$ and $(A_1,A_4)_{\eta,q} = A_3$.*

*Proof.* See [Jan+84, Theorem 1]. □

### 3.4.2 Real interpolation of tent spaces

The following is the baby version of real interpolation theory of tent spaces.

**Theorem 3.15** (Real interpolation)**.** *Let $(X,d,\mu)$ be a space of homogeneous type. Suppose that $1 < p_0 < p_1 < \infty$, $q \in (1,\infty)$, $\theta \in (0,1)$, and $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. The identification holds that*

$$(T^{p_0,q}, T^{p_1,q})_{\theta,p_\theta} = T^{p_\theta,q}. \tag{3.2}$$

*Proof.* It can be proved of the same argument as Theorem 3.10 since

$$(L^{p_0}(L^q_*), L^{p_1}(L^q_*))_{\theta,p_\theta} = L^{p_\theta}(L^q_*).$$

Again, $\pi$ and $\iota$ act as retraction and co-retraction, so we conclude by Lemma 3.4. □

We state the most conservative result, even in the context of Banach spaces since Eq.(3.2) is still true for $1 \le p_0 < p_1 \le \infty$. We give the sketch other than the proof here since there are too many tools not constructed yet. We start from the case $p_0 = 1$ and $p_1 \in (1,\infty)$. One side is clear that

$$(T^{1,q}, T^{p_1,q})_{\theta,p_\theta} \hookrightarrow T^{p_\theta,q} \tag{3.3}$$

by the same proof of Theorem 3.10 since real interpolation functor is also exact of exponent $\theta$. Duality theorem, cf. Theorem 3.13, reads that the embedding Eq.(3.3) is equivalent to the embedding

$$T^{p'_\theta,q'} \hookrightarrow (T^{\infty,q'}, T^{p'_1,q'})_{\theta,p'_\theta}. \tag{3.4}$$

Fundamental properties of real interpolation reads that

$$(T^{\infty,q'}, T^{p_1',q'})_{\theta,p_\theta'} = (T^{p_1',q'}, T^{\infty,q'})_{1-\theta,p_\theta'}.$$

Thus, it suffices to prove the embedding

$$(T^{p_0,q}, T^{\infty,q})_{\theta,p_\theta} \hookrightarrow T^{p_\theta,q} \tag{3.5}$$

for $p_0, q \in (1,\infty)$. Other than directly tedious construction in [CMS85, Theorem 4'], a simpler method to prove Eq.(3.5) is to construct tent spaces over Lorenz spaces and utilise factorisation of tent spaces. See [AM88, Sec.5] for details, which finalizes the proof for case $p_0 \in (1,\infty)$ and $p_1 = \infty$.

Duality theorem, cf. Theorem 3.13, in turn, proves the embedding, cf. Eq.(3.2), for $p_0 = 1$ and $p_1 \in (1,\infty)$.

The remaining case for $p_0 = 1$ and $p_1 = \infty$ follows from reiteration as we have shown in Theorem 3.10 ensured by Theorem 3.14.

We also finalize this section via remarks about the progress of real interpolation theory on tent spaces. Actually, the most fantastic part of the real interpolation theory lies on quasi-Banach spaces, or quasi-normed vector spaces, or even quasi-normed Abelian groups. In the context of tent spaces, it will give us the following extensions for instance.

- Eq.(3.2) still holds for $0 < p_0 < p_1 \leq \infty$ and $q \in (1,\infty)$. The above sketchy proof can be adapted without too much modification since the reiteration theorem, cf. Theorem 3.14 actually holds for quasi-Banach spaces $A_1, A_2, A_3, A_4$.
- For $q = \infty$, for any $p \in (0,\infty), \theta \in (0,1), r \in (0,\infty]$, set that $\frac{1}{p_\theta} = \frac{1-\theta}{p}$. Then, $(T^{p,\infty}, T^{\infty,\infty})_{\theta,r} = (T^{p,\infty}, L^\infty)_{\theta,r}$ can be shown via tent spaces over Lorenz spaces. See [AM88, Sec.5].

Generalization of these results on spaces of homogeneous type requires further work.

## 3.5 Application: independence of aperture revisited

In this section, we provide another proof of independence of aperture in the definitions of tent spaces, cf. Theorem 1.14, as an application of interpolation theory. The interpolation theory, cf. Theorem 3.10 and 3.15, can be imitated for $T^{p,q}_{(\alpha)}$ with the same argument, whose proof is be independent of changing aperture.

We accept the above proof for the following cases, whose proof is independent of changing aperture:

- Type $(\infty, q)$, cf. Eq.(1.17);
- Type $(p,q)$ with $p = q$, cf. Eq.(1.12), and $p \in (0,1], q \in [p,\infty)$, cf. (2.13);
- Type $(p,\infty)$ with $p > 1$, cf. Eq.(1.16), and $p \in (0,1]$, cf. (2.12).

Fig.3.1 illustrates the region, where, in particular, the red denotes the region proved by atomic decomposition.

We first deal with the case that $p, q \in (1,\infty)$ and $p \neq q$ via interpolation of the identification map $\mathcal{I} : T^{p,q}_{(\beta)} \to T^{p,q}_{(\alpha)}$ given by $\mathcal{I}(f) = f$. The idea is from [Aus11, Theorem 1.1].

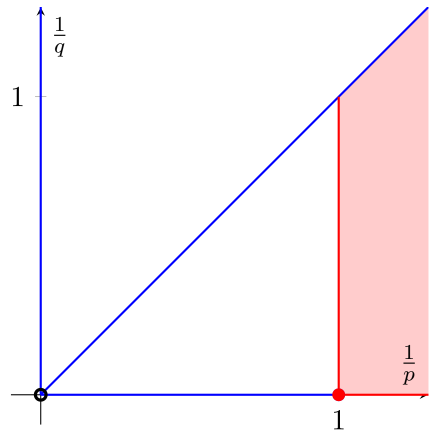


FIGURE 3.1: Region of proof of independence of aperture accepted

For $p > q$, Eq.(1.12) reads that $\mathcal{I} : T^{q,q}_{(\beta)} \to T^{q,q}_{(\alpha)}$ is of norm 1. Eq.(1.17) reads that the norm of $\mathcal{I} : T^{\infty,q}_{(\beta)} \to T^{\infty,q}_{(\alpha)}$ is controlled by 1 if $\alpha > \beta$ or by $\rho(\beta^{-1}\alpha)^{\frac{1}{q}}$ if $\alpha < \beta$. Pick $\theta = q[q,p]$. Theorem 3.10 reads that the norm of

$$\mathcal{I} : T^{p,q}_{(\beta)} = (T^{q,q}_{(\beta)}, T^{\infty,q}_{(\beta)})_{[\theta]} \to (T^{q,q}_{(\alpha)}, T^{\infty,q}_{(\alpha)})_{[\theta]} = T^{p,q}_{(\alpha)}$$

is controlled by 1 if $\alpha > \beta$ and by $\rho(\beta^{-1}\alpha)^{[q,p]}$ if $\alpha < \beta$ up to some constant, due to the exactness of exponent $\theta$ of complex interpolation.

For $p < q$, Eq.(2.12) reads that the norm of $\mathcal{I} : T^{1,q}_{(\beta)} \to T^{1,q}_{(\alpha)}$ is controlled by $\rho(\beta^{-1}\alpha)^{\frac{1}{q'}}$ if $\alpha > \beta$ or by 1 if $\alpha < \beta$, up to some constant. Pick $\theta = \frac{q'}{p'}$. Theorem 3.10 reads that the norm of

$$\mathcal{I} : T^{p,q}_{(\beta)} = (T^{1,q}_{(\beta)}, T^{q,q}_{(\beta)})_{[\theta]} \to (T^{1,q}_{(\alpha)}, T^{q,q}_{(\alpha)})_{[\theta]} = T^{p,q}_{(\alpha)}$$

is hence be controlled by $\rho(\beta^{-1}\alpha)^{[p,q]}$ if $\alpha > \beta$ and by 1 if $\alpha < \beta$ up to some constant, due to the exactness of exponent $\theta$ of complex interpolation. In summary, the norm of $\mathcal{I} : T^{p,q}_{(\beta)} \to: T^{p,q}_{(\alpha)}$ is controlled by

$$\rho(\beta^{-1}\alpha)^{|[p,q]|\delta(p,q;\alpha,\beta)},$$

where $\delta(p,q;\alpha,\beta)$ equals to 0, if $(p-q)(\alpha-\beta) > 0$, and equals to 1, otherwise.

The deduction can be extended to general $p, q \in (0,\infty)$ via raising-power trick. Let $f$ be a $T^{p,q}_{(\alpha)}$-function. Pick sufficiently large $\lambda > 1$ such that $\lambda p, \lambda q \in (1,\infty)$. Raising-power trick, cf. Eq.(2.10), reads that

$$\|f\|_{T^{p,q}_{(\alpha)}} = \||f|^{\frac{1}{\lambda}}\|^{\lambda}_{T^{\lambda p,\lambda q}_{(\alpha)}} \sim \||f|^{\frac{1}{\lambda}}\|^{\lambda}_{T^{\lambda p,\lambda q}_{(\beta)}} = \|f\|_{T^{p,q}_{(\beta)}}.$$

The controlling constant can be optimized by limit argument as

$$\max(p^{-1}, q^{-1})^2 \rho(\beta^{-1}\alpha)^{|[p,q]|\delta(p,q;\alpha,\beta)}.$$

Furthermore, in the context of $(\mathbb{R}^n, d_2, \mathscr{L})$, we could obtain similar results via the same argument but different control by Eq.(1.5), (1.10), (1.11), (2.14) and (2.15), due to different ways for normalization in Euclidean spaces and spaces of homogeneous type. For $q \in (1,\infty)$ and $p \in (0,\infty)$,

- If $\alpha > \beta$, the norm of $\mathcal{I} : T^{p,q}_{(\beta)} \to T^{p,q}_{(\alpha)}$ is controlled by $(\beta^{-1}\alpha)^{\frac{n}{q}}$ if $p > q$, or $(\beta^{-1}\alpha)^{\frac{n}{p}}$ if $p < q$, up to some constant;
- If $\alpha < \beta$, the norm of $\mathcal{I} : T^{p,q}_{(\beta)} \to T^{p,q}_{(\alpha)}$ is controlled by $(\beta^{-1}\alpha)^{\frac{n}{p}}$ if $p > q$, or $(\beta^{-1}\alpha)^{\frac{n}{q}}$ if $p < q$, up to some constant.

It hence generalizes the work [Aus11, Theorem 1.1] for $q = 2$. Moreover, we could also prove the optimality. It suffices to consider the simple case with $\alpha \gg \beta = 1$. Let $B$ denote the ball $B(0, 2)$. Define the function

$$a(y, t) := \mathbb{1}_B(y, t)\mathbb{1}_{[1,2]}(t),$$

whose $T^{p,q}$-norm is merely dependent on the dimension $n$, i.e., $\|a\|_{T^{p,q}} \sim_n 1$. But $A_{q;\alpha}(a)$ is supported on $B(0, 2\alpha)$ and takes the same value that is merely dependent on dimension $n$ on $B(0, \alpha-1)$ since $B(0, \alpha-1) \subset R_\alpha(\widehat{B} \cap (B \times [1, 2]))$. It implies that $\|a\|_{T^{p,q}_\alpha} \sim_n \alpha^{\frac{n}{p}}$. It shows the sharpness of controlling the embedding $\mathcal{I} : T^{p,q} \to T^{p,q}_{(\alpha)}$ if $p < q$, and the sharpness of controlling the embedding $\mathcal{I} : T^{p,q}_{(\alpha)} \to T^{p,q}$ if $p > q$.

In turn, scale $a$ as

$$\widetilde{a}(y, t) := a(y, \alpha t).$$

Note that $\|\widetilde{a}\|_{T^{p,q}_{(\alpha)}} = \alpha^{\frac{n}{q}} \|a\|_{T^{p,q}} \sim_n \alpha^{\frac{n}{q}}$. Note that $A_q(a)$ is supported on $B(0, 1 + \frac{1}{\alpha})$, and can be controlled by some constant merely depending on $n$ on $B(0, 1-\frac{1}{\alpha})$. Thus, it implies that $\|a\|_{T^{p,q}} \sim_n 1$, which indicates the sharpness of controlling the embedding $\mathcal{I} : T^{p,q} \to T^{p,q}_{(\alpha)}$ if $p > q$, , and the sharpness of controlling the embedding $\mathcal{I} : T^{p,q}_{(\alpha)} \to T^{p,q}$ if $p < q$.

# Chapter 4

# Application: characterization of Hardy spaces

In this chapter, we shall characterise Hardy spaces as an application of tent spaces to evince the close connections between tent spaces and other function spaces. In this chapter, we concentrate on Euclidean settings $(\mathbb{R}^n, d_2, \mathscr{L})$.

## 4.1 Preliminaries: Hardy spaces

Let us recall some fundamental concepts and results in Hardy spaces first.

**Definition 4.1** ($H^p$-atoms)**.** Suppose that $0 < p \leq 1$. An $L^q$-function $a$ is called an *$H^p$-atom* if there is some ball $B \subset \mathbb{R}^n$ such that:

1. $\operatorname{supp} a \subset B$;
2. $\|a\|_q \leq |B|^{[q,p]}$ with $1 < q \leq \infty$ if $p = 1$, or with $q = 1$ if $0 < p < 1$;
3. (*moment condition*) For any $\beta \in \mathbb{N}^n$ with $|\beta| \leq n[p, 1] = n(p^{-1} - 1)$,

$$\int_{\mathbb{R}^n} x^\beta a(x) dx = 0. \tag{4.1}$$

It is not obvious that $H^p$-atoms belong to $L^p(\mathbb{R}^n)$. See [SM93, Chapter III, §5.7] for the verification. For an $H^1$-atom $a$, we say that $a$ is a *$q$-$H^1$-atom* if $a \in L^q(\mathbb{R}^n)$ with $\|a\|_q \leq |B|^{[q,p]}$.

**Definition 4.2** ($H^p$-space)**.** For $0 < p \leq 1$, the space $H^p$ contains tempered distribution $f \in \mathcal{S}'$ that could be written as

$$f = \sum_{j \in \mathbb{N}} \lambda_j a_j$$

for some collection of $H^p$-atoms $\{a_j\}_{j \in \mathbb{N}}$ and an $\ell^p$-sequence $(\lambda_j)_{j \in \mathbb{N}}$. The space $H^p$ is equipped with the norm

$$\|f\|_{H^p} := \inf \left\{ \|(\lambda_j)_{j \in N}\|_{\ell^p} : \exists\ H^p\text{-atoms}\{a_j\}_{j \in \mathbb{N}} \text{ such that } f = \sum_{j \in \mathbb{N}} \lambda_j a_j \right\}.$$

For $1 < p < \infty$, the space $H^p$ is identified as $L^p$ with equivalent norms.

A well-known remark is that there is a method to uniformly define $H^p$-spaces for $0 < p < \infty$. See [SM93, Chapter III, Theorem 1, 2] for more details of Hardy spaces.

Another kind of essential elementary blocks of Hardy spaces is molecule, from which we have another characterization of Hardy spaces via decomposition.

**Definition 4.3** (Molecule)**.** Let $p \in (0,1]$ and $\epsilon > [p,1]$ be two given numbers. We say that an $L^2$-function $m$ is a $(p,\epsilon)$-*molecule* centred at $x_0$, if it satisfies the moment condition, cf. Eq.(4.1), for any $|\beta| \leq n[p,1]$, and that

$$\left(\int_{\mathbb{R}^n} |m(x)|^2 dx\right)\left(\int_{\mathbb{R}^n} |m(x)|^2 |x-x_0|^{n(1+2\epsilon)} dx\right)^{\frac{[p,2]}{[1,p]+\epsilon}} \leq 1. \tag{4.2}$$

The definition corresponds to the case $q=2, s=n[p,1]$ in the original definition by [TW80, Eq.(2.2)]. The following proposition states that $H^p(\mathbb{R}^n)$ can also be characterized by molecule decomposition.

**Theorem 4.4.** *A measurable function $f$ lies in $H^p(\mathbb{R}^n)$ if and only if there exist $\epsilon > [p,1]$ $p$-molecules $\{m_j\}_{j\in\mathbb{N}}$ and $(\lambda_j)_{j\in\mathbb{N}} \in \ell^p(\mathbb{N})$ such that*

$$f = \sum_{j=0}^{\infty} \lambda_j m_j,$$

*with*

$$\|(\lambda_j)_j\|_{\ell^p} \lesssim \|f\|_{H^p}.$$

*Proof.* See [TW80, Proposition 2.3 & Theorem 2.9]. □

## 4.2 Two critical operators

In this section, we shall provide two critical operators connecting tent spaces and Hardy spaces. One of them is the renowned square function, a typical operator in the theory of vector-valued singular integral, from where we commence. Let $\phi \in \mathcal{S}$ be a Schwartz function with mean-zero, i.e., $\int \phi = 0$.

### 4.2.1 Non-tangential square function

Define the operator $s_\phi$ as a so-called *non-tangential square function* by $\phi$, which maps functions on $\mathbb{R}^n$ to $L^2_*(\Gamma)$-valued functions on $\mathbb{R}^n$ if the formula

$$s_\phi(f)(x)(y,t) := \phi_t * f(x+y) = \int_{\mathbb{R}^n} \phi_t(x+y-z) f(z) dz \tag{4.3}$$

makes sense for any $(y,t) \in \Gamma$, where $\Gamma = \Gamma_1(0)$ is the standard cone, $L^2_*(\Gamma) := L^2(\Gamma, \frac{dy}{t^n}\frac{dt}{t})$, and $\phi_t(x) := \frac{1}{t^n}\phi\left(\frac{x}{t}\right)$.

We first identify this operator as mapping $L^2(\mathbb{R}^n)$-functions to $L^2(\mathbb{R}^n; L^2_*(\Gamma))$-functions. For any $f \in L^2(\mathbb{R}^n)$,

$$\begin{aligned} \|s_\phi(f)\|^2_{L^2(L^2_*(\Gamma))} &= \int_{\mathbb{R}^n}\left(\int_\Gamma |\phi_t * f(x+y)|^2 \frac{dy}{t^n}\frac{dt}{t}\right) dx \\ &= \int_{\mathbb{R}^n}\left(\int_{\Gamma(x)} |\phi_t * f(y)|^2 \frac{dy}{t^n}\frac{dt}{t}\right) dx \\ &\simeq_n \int_{\mathbb{R}^{n+1}_+} |\phi_t * f(y)|^2 dy \frac{dt}{t} = \int_{\mathbb{R}^{n+1}_+} |\widehat{\phi}(t\xi)\widehat{f}(\xi)|^2 d\xi \frac{dt}{t} \lesssim \|f\|_2^2. \end{aligned} \tag{4.4}$$

Note that the last inequality holds since

$$\mathfrak{c}_\phi := \sup_{\xi\in\mathbb{R}^n} \int_0^\infty |\widehat{\phi}(t\xi)|^2 \frac{dt}{t}$$

exists. Indeed, note that $\widehat{\phi} \in \mathcal{S}$ and $\widehat{\phi}(0) = \int \phi = 0$, so

$$|\widehat{\phi}(\xi) - \widehat{\phi}(0)| \lesssim_\phi |\xi|$$

for any $\xi \in \mathbb{R}^n$, and $|\eta_j \widehat{\phi}(\eta)| = |\widehat{\partial_j\phi}(\eta)| \leq \|\partial_j\phi\|_1$ for any $1 \leq j \leq n$, so

$$|\widehat{\phi}(\eta)| \lesssim_\phi |\eta|^{-1}$$

for any $\eta \neq 0$. Thus, for the non-trivial case that $\xi \neq 0$,

$$\int_0^\infty |\widehat{\phi}(t\xi)|^2 \frac{dt}{t} \lesssim_\phi |\xi|^2 \int_0^{|\xi|^{-1}} tdt + |\xi|^{-2} \int_{|\xi|^{-1}}^\infty \frac{dt}{t^3} = 1.$$

The following proposition extends the operator.

**Proposition 4.5.** *The operator $s_\phi$ given by Eq.(4.3) can be extended into a bounded linear operator in the following regions:*

1. *from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n; L^2_*(\Gamma))$ for $1 < p < \infty$;*
2. *from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n; L^2_*(\Gamma))$ for $p \leq 1$.*

Before the proof, recall that the Calderón-Zygmund theory also holds in the settings of Hilbert spaces [Aus12, Sec.7.6] or even Banach spaces [BCP62; dRT86].

*Proof.* It suffices to verify that the operator $s_\phi$ is a Hilbert-space valued Calderón-Zygmund operator. Define that

$$K(x,z)(y,t) := \phi_t(x - z + y).$$

We first verify that $K$ is a Calderón-Zygmund kernel of order 1. For any $x, z \in \mathbb{R}^n$ with $x \neq z$, $K(x,z)$ defines a linear operator from $\mathbb{C}$ to $L^2_*(\Gamma)$ by scalar multiplication as $\tau \mapsto \tau K(x,z)(y,t)$. Thus, the norm of the operator is exactly $L^2_*(\Gamma)$-norm of the corresponding image. Note that $K : \mathbb{R}^n \times \mathbb{R}^n \to L(\mathbb{R}^1; L^2_*(\mathbb{R}^{n+1}_+))$ is obviously continuous. Pick some constant $M > 0$ such that for $(y,t) \in \Gamma$ with $t \leq \frac{1}{2}|x - z|$,

$$|\phi_t(x - z - y)| \leq Mt|x - z - y|^{-n-1} \leq 2^{n+1}Mt|x - z|^{-n-1},$$

and for $t \geq \frac{1}{2}|x - z|$,

$$|\phi_t(x - z + y)| \leq Mt^{-n}.$$

Indeed, the former holds since $|x|^{n+1}|\phi(x)|$ can be bounded and $|y| \leq \frac{1}{2}|x - z|$ implies that $|x - z - y| \geq \frac{1}{2}|x - z|$. The latter holds since $\phi$ itself is bounded. Then, for any $x, z \in \mathbb{R}^n$ with $x \neq z$, the above discussion reads that

$$\begin{aligned}\|K(x,z)(y,t)\|^2_{L^2_*(\mathbb{R}^{n+1}_+)} &= \int_{(\Gamma|\frac{1}{2}|x-z|)(0)} |\phi_t(x - z + y)|^2 \frac{dy}{t^n}\frac{dt}{t} \\ &\quad + \int_{(\Gamma|\frac{1}{2}|x-z|)(0)^c} |\phi_t(x - z + y)|^2 \frac{dy}{t^n}\frac{dt}{t} \\ &\lesssim_n M^2|x - z|^{-2n}.\end{aligned}$$

The same argument for $\nabla_x K(x,z)(y,t) = t^{-1}(\nabla\phi)_t(x-z+y)$ reads that there also exists $M>0$ such that for $(y,t)\in\Gamma$,

$$|\nabla_x K(x,z)(y,t)| \le 2^{n+1}M|x-z|^{-n-1}$$

if $t\le \frac{1}{2}|x-z|$, and

$$|\nabla_x K(x,z)(y,t)| \le Mt^{-n-1}$$

if $t\ge \frac{1}{2}|x-z|$. Thus, it follows that

$$\|\nabla K(x,z)(y,t)\|_{L^2_*(\mathbb{R}^{n+1}_+)} \lesssim_n M|x-z|^{-n-1},$$

while the other side also follows by symmetry. Similarly, it also follows that

$$\|\nabla^\beta K(x,z)(y,t)\|_{L^2_*(\mathbb{R}^{n+1}_+)} \lesssim_n M|x-z|^{-n-|\beta|},$$

for any $\beta\in\mathbb{N}^n$ with $|\beta|\le n[p,1]+1$ if $p<1$.

The $L^2$-boundedness of $s_\phi$ has been shown in the above, and the association of $K$ and $s_\phi$ is obvious. □

**Corollary 4.6.** *The operator $S_\phi$ given by*

$$S_\phi(f)(y,t) := (\phi_t * f)(y) \tag{4.5}$$

*is bounded from $L^p(\mathbb{R}^n)$ to $T^{p,2}$ for $1<p<\infty$ and from $H^p(\mathbb{R}^n)$ to $T^{p,2}$ if $p\in(0,1]$.*

*Proof.* The translation gives an isometry as

$$\|s_\phi(f)(x)\|_{L^2_*(\Gamma)} = \left(\int_{\Gamma(x)} |\phi_t * f(y)|^2 \frac{dy}{t^n}\frac{dt}{t}\right)^{\frac{1}{2}} = A_2(S_\phi(f))(x).$$

Thus, we can achieve the extension by identification. □

### 4.2.2 Reverse direction

The reverse direction is more fantastic. Consider the operator $\pi_\phi$ mapping a measurable function on $\mathbb{R}^{n+1}_+$ to a function on $\mathbb{R}^n$, given by

$$\pi_\phi(f)(x) := \int_{\mathbb{R}^{n+1}_+} f(y,t)\phi_t(x-y)dy\frac{dt}{t} \tag{4.6}$$

if the formula makes sense. The following theorem gives the reverse.

**Theorem 4.7.** *The operator $\pi_\phi$ given by Eq.(4.6) can be extended into a bounded linear operator in the following regions:*

1. *from $T^{p,2}$ to $L^p(\mathbb{R}^n)$ for $1<p<\infty$;*
2. *from $T^{1,2}$ to $H^1(\mathbb{R}^n)$;*
3. *from $T^{\infty,2}$ to* BMO.

The proof is divided into the following cases in terms of valuation of $p$ in $T^{p,2}$.

**Case** $1 < p < \infty$ Duality argument shows that

$$\|\pi_\phi(f)\|_p = \sup_{\substack{\psi \in L^{p'} \\ \|\psi\|_{p'}=1}} \int_{\mathbb{R}^n} \pi_\phi(f)(x)\psi(x)dx.$$

Let $f$ be a compactly-supported $L^q$-function first so that Young's convolution theorem directly works as

$$\begin{aligned}
\int_{\mathbb{R}^n} \pi_\phi(f)(x)\psi(x)dx &= \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^{n+1}_+} f(y,t)\phi_t(x-y)dy\frac{dt}{t} \right) \psi(x)dx \\
&= \int_{\mathbb{R}^{n+1}_+} \left( \int_{\mathbb{R}^n} \phi_t(x-y)\psi(x)dx \right) f(y,t)dy\frac{dt}{t} \\
&= \int_{\mathbb{R}^{n+1}_+} (\widetilde{\phi}_t * \psi)(y)f(y,t)dy\frac{dt}{t},
\end{aligned}$$

where $\widetilde{\phi}(x) := \phi(-x)$. Corollary 4.6 hence reads that

$$\|\pi_\phi(f)\|_p \leq \sup_{\substack{\psi \in L^{p'} \\ \|\psi\|_{p'}=1}} \|S_{\widetilde{\phi}}(\psi)\|_{T^{p',2}} \|f\|_{T^{p,2}} \lesssim_{n,\phi} \|f\|_{T^{p,2}}, \tag{4.7}$$

where the first inequality follows from Eq.(2.5). Thus, it can be extended to the whole $T^{p,2}$ by density, cf. Lemma 1.4.

**Case** $p = 1$ We shall first observe the action of $\pi_\phi$ on a $T^{1,2}$-atom $a$. Let $B = B(x_0, r_0)$ be the ball associated to $a$. Pick $\gamma \gg 1$ as a fixed sufficiently large number. For $x \in (\gamma B)^c$,

$$\begin{aligned}
|\pi_\phi(a)(x)| &\leq \int_{\widehat{B}} |a(y,t)||\phi_t(x-y)|dy\frac{dt}{t} \\
&\leq M(1-\gamma^{-1})^{-n-s}|x-x_0|^{-n-s} \int_{\widehat{B}} |a(y,t)|t^{s-1}dydt \\
&\lesssim_{n,\phi,\gamma} |x-x_0|^{-n-s}r_0^s
\end{aligned} \tag{4.8}$$

for some $s > 1$ to be determined. The second inequality holds since there is some constant $M > 0$ such that

$$|\phi_t(x-y)| \leq Mt^s|x-y|^{-n-s}$$

by the Schwartz property of $\phi$, and $|x-y| \geq |x-x_0| - |x_0-y| \geq (1-\gamma^{-1})|x-x_0|$ since $|x_0-y| \leq r_0 \leq \gamma^{-1}|x-x_0|$. The last inequality holds since $t \leq r_0$ for $(y,t) \in \widehat{B}$, and

$$\int_{\widehat{B}} |a(y,t)|dydt \leq \left( r_0 \int_{\widehat{B}} |a(y,t)|^2 dy\frac{dt}{t} \right)^{\frac{1}{2}} \left( \int_{\widehat{B}} dydt \right)^{\frac{1}{2}} \lesssim_n r_0.$$

Thus,

$$\int_{(\gamma B)^c} |\pi_\phi(a)(x)|^2|x-x_0|^{n(1+2\epsilon)}dx \lesssim_{n,\gamma} r_0^{2s} \int_{(\gamma B)^c} |x-x_0|^{-n(1-2\epsilon)-2s}dx \lesssim_{n,\gamma,\epsilon} r_0^{2n\epsilon},$$

which shows that $s = n\epsilon + 1$ could ensure the convergence. On the other hand,

$$\|\pi_\phi(a)\|_2 \lesssim_n \|a\|_{T^{2,2}} \lesssim_n \|a\|_{L^2(\mathbb{R}^{n+1}_+;dy\frac{dt}{t})} \lesssim_n |B|^{-\frac{1}{2}} \tag{4.9}$$

by Eq.(4.7) and $A$-averaging. Thus,

$$\int_{\gamma B} |\pi_\phi(a)(x)|^2 |x - x_0|^{n(1+2\epsilon)} dx \lesssim_{\gamma,n,\epsilon} r_0^{n(1+2\epsilon)} \|\pi_\phi(a)\|_2^2 \lesssim_n r_0^{2n\epsilon}.$$

Eq.(4.2) turns to be as

$$\text{LHS} \lesssim_{n,\gamma,\epsilon} \|\pi_\phi(a)\|_2^2 (r_0^{2n\epsilon})^{(2\epsilon)^{-1}} \lesssim_n |B|^{-1} r_0^n \lesssim_n 1.$$

Thus, there is a constant $C$ independent of the atom $a$ such that $m = C^{-1}\pi_\phi(a)$ satisfies Eq.(4.2). Then, we conclude that $m$ is indeed a $p$-molecule for any $p > n$ as

$$\begin{aligned} \int_{\mathbb{R}^n} \pi_\phi(a)(x) dx &= \int_{\mathbb{R}^n} \left( \int_{\widehat{B}} a(y,t) \phi_t(x-y) dy \frac{dt}{t} \right) dx \\ &= \int_{\widehat{B}} \left( \int_{\mathbb{R}^n} \phi_t(x-y) dx \right) a(y,t) dy \frac{dt}{t} = 0. \end{aligned} \tag{4.10}$$

The second equality follows from Fubini's theorem, whose verification of legality is deferred to Appendix A. Therefore, for any $f \in T^{1,2}$, write its atomic decomposition form as $f = \sum_j \lambda_j a_j$. Then $\sum_j (C\lambda_j)(C^{-1}\pi_\phi(a_j))$ converges together with

$$\|(\lambda_j)_j\|_{\ell^1} \lesssim \|f\|_{T^{1,2}}.$$

Thus, the operator as

$$\pi_\phi(f) = \sum_j (C\lambda_j)(C^{-1}\pi_\phi(a_j))$$

is well-defined and gives the molecule decomposition, so $\pi_\phi(f) \in H^1(\mathbb{R}^n)$. Moreover, $\pi_\phi(f) \in H^1(\mathbb{R}^n)$ together with

$$\|\pi_\phi(f)\|_{H^1} \lesssim \|(\lambda_j)_j\|_{\ell^1} \lesssim \|f\|_{T^{1,2}}.$$

**Case $p = \infty$** It suffices to show that for any $g \in H^1(\mathbb{R}^n)$, the map

$$g \mapsto \int_{\mathbb{R}^n} \pi_\phi(f)(x) g(x) dx \tag{4.11}$$

gives a continuous linear functional. We first truncate and observe its behavior on atoms. For any $0 < \epsilon < R < \infty$ and $f \in T^{\infty,2}$, define that

$$(\pi_\phi|\epsilon, R)(f)(x) := \int_{\mathbb{R}^n \times (\epsilon,R)} f(y,t) \phi_t(x-y) dy \frac{dt}{t}.$$

Let $a$ be an $\infty$-$H^1$-atom and $B = B(x_0, r_0)$ as a ball associated to $a$. We claim that

$$((\pi_\phi|\epsilon, R)f, a) = \int_{\mathbb{R}^n} (\pi_\phi|\epsilon, R)(f)(x) a(x) dx = \int_{\mathbb{R}^n \times (\epsilon,R)} f(y,t) (\widetilde{\phi}_t * a)(y) dy \frac{dt}{t}. \tag{4.12}$$

Recall that $\widetilde{\phi}_t(x) = \phi_t(-x)$. Eq.(4.12) holds directly by Fubini theorem but the verification of legality of Fubini theorem is also deferred to Appendix A. Thus, Theorem

2.3 and Corollary 4.6 reads that $\widetilde{\phi}_t * a \in T^{1,2}$ and

$$|((\pi_\phi|\epsilon, R)f, a)| \lesssim \|f\|_{T^{\infty,2}} \|\widetilde{\phi}_t * a\|_{T^{1,2}} \lesssim \|f\|_{T^{\infty,2}} \|a\|_{H^1} \leq \|f\|_{T^{\infty,2}}.$$

It is worthy of pointing out particularly that the controlling constant is independent of $\epsilon$ and $R$. We can hence extend it as a continuous linear functional on $H^1(\mathbb{R}^n)$ as

$$((\pi_\phi|\epsilon, R)f, g) := \int_{\mathbb{R}^n} (\pi_\phi|\epsilon, R)(f)(x) g(x) dx$$

for any $g \in H^1(\mathbb{R}^n)$. Indeed, write $g = \sum_{j\in\mathbb{N}} \lambda_j a_j$ as atomic decomposition of $g$ by $\infty$-$H^1$-atoms. Then,

$$|((\pi_\phi|\epsilon, R)f, g)| \leq \sum_{j\in\mathbb{N}} |\lambda_j| |((\pi_\phi|\epsilon, R)f, a_j)| \lesssim \|f\|_{T^{\infty,2}} \sum_{j\in\mathbb{N}} |\lambda_j| \lesssim \|f\|_{T^{\infty,2}} \|g\|_{H^1}.$$

Note that the controlling constant is still independent of $\epsilon$ and $R$. Therefore, we define $\pi_\phi(f)$ as the limit of $\{(\pi_\phi|\epsilon, R)(f)\}_{0<\epsilon<R<\infty}$ as $\epsilon$ tends to 0 and $R$ tends to $\infty$ in the weak*-topology. Moreover, note that $(\pi_\phi|\epsilon, R)(f)$ belongs to BMO by the duality of $H^1$ and BMO [Aus12, Theorem 6.2.1]. Thus, $\pi_\phi(f)$, as the weak*-limit of $(\pi_\phi|\epsilon, R)(f)$, still lies in BMO. We hence finish the proof.

For general Hardy spaces, we can also extend the operator $\pi_\phi$ to some extents.

**Corollary 4.8.** *If $\phi$ further satisfies the moment condition, cf. Eq.(4.1) for any $|\beta| \leq n[p, 1]$. Then $\pi_\phi$ can be extended from $T^{p,2}$ to $H^p(\mathbb{R}^n)$ for $p \leq 1$.*

*Proof.* We mainly follow the proof of Theorem 4.7 for the case $p = 1$ but just show the corresponding modifications.

Let $a$ be a $T^{p,2}$-atom and $B$ the ball associated to $a$. The moment condition reads that, for any $|\beta| \leq n[p, 1]$,

$$\int_{\mathbb{R}^n} x^\beta \pi_\phi(a)(x) dx = \int_{\mathbb{R}^{n+1}_+} \left( \int_{\mathbb{R}^n} x^\beta \phi_t(x - y) dx \right) a(y, t) dy \frac{dt}{t} = 0, \tag{4.13}$$

since $\phi$ satisfies the moment condition. The verification of legality of Fubini theorem is also deferred to Appendix A. Since $a \in L^2(\mathbb{R}^{n+1}_+; dy\frac{dt}{t}) = T^{2,2}$, Theorem 4.7 reads that

$$\|\pi_\phi(a)\|_2 \lesssim \|a\|_2 \leq |B|^{-[p,2]}.$$

Pick $\gamma \gg 1$ sufficiently large. Correspondingly, Eq.(4.8) is modified as

$$|\pi_\phi(a)(x)| \lesssim |x - x_0|^{-n-s} r_0^s |B|^{[1,p]}$$

for any $x \in (\gamma B)^c$. Hence,

$$\int_{(\gamma B)^c} |\pi_\phi(a)(x)|^2 |x - x_0|^{n(1+2\epsilon)} dx \lesssim |B|^{2[1,p]} r_0^{2n\epsilon} \lesssim |B|^{2([1,p]+\epsilon)}.$$

Moreover,

$$\int_{\gamma B} |\pi_\phi(a)(x)|^2 |x - x_0|^{n(1+2\epsilon)} dx \lesssim r_0^{n(1+2\epsilon)} \|\pi_\phi(a)\|_2^2 \lesssim |B|^{2([1,p]+\epsilon)}.$$

It hence follows that, in Eq.(4.2) for $m = \pi_\phi(a)$,

$$\text{LHS} \leq |B|^{2[p,2]} \|\pi_\phi(a)\|_2^2 \lesssim 1.$$

Therefore, $\pi_\phi(a)$ is an $H^p$-molecule whose bound is independent of the atom, for which we extend $\pi_\phi$ to the whole $T^{p,2}$ by atomic decomposition, cf. Theorem 2.11, by the same argument as for Theorem 4.7 when $p=1$. □

## 4.3 Characterization of Hardy spaces

In this section, we shall utilise the above two operators to describe Hardy spaces $H^p$ via tent spaces $T^{p,2}$, where $p$ ranges among $(0,\infty)$ and we identify $H^p$ as $L^p$ for $p>1$. Let us start from a further condition on $\phi$.

**Definition 4.9** (Non-degenerate)**.** Suppose that $\phi \in \mathcal{S}$ with $\int \phi = 0$. We say that $\phi$ is *non-degenerate*, if there exists another function $\psi \in \mathcal{S}$ with $\int \psi = 0$ satisfying that

$$\int_0^\infty \widehat{\phi}(t\xi)\widehat{\psi}(t\xi)\frac{dt}{t} = 1. \tag{4.14}$$

Note that Eq.(4.14) is equivalent to the identity as

$$\int_0^\infty \phi_t * \psi_t \frac{dt}{t} = \delta$$

in the sense of distribution, where $\delta$ is the Dirac delta distribution.

*Remark.* An equivalent definition of non-degeneracy is that for any $\xi \neq 0$, there exists $t>0$ such that $\widehat{\phi}(t\xi) \neq 0$. See [SM93, Chapter IV, §6.19(a)].

The following theorem precisely shows how to describe Hardy spaces via tent spaces.

**Theorem 4.10.** *Suppose that $\phi$ is non-degenerate. Then, the operator $\pi_\phi$ defined in Eq.(4.6) is surjective of $T^{p,2}$ to $H^p$ for $p \in (0,\infty)$. More precisely, for any $f \in H^p$, there exists some $F \in T^{p,2}$ with*

$$\|F\|_{T^{p,2}} \simeq \|f\|_{H^p}$$

*such that $f = \pi_\phi(F)$.*

*Proof.* Let $\psi \in \mathcal{S}$ be the Schwartz function satisfying that $\int \psi = 0$ and Eq.(4.14) for $\phi$. Define that

$$F(y,t) = (\psi_t * f)(y) = S_\psi(f)(y,t).$$

Corollary 4.6 exactly reads that $F \in T^{p,2}$ since $f \in H^p$ and $\|F\|_{T^{p,2}} \lesssim \|f\|_{H^p}$ as required. We first verify the identity $f = \pi_\phi(F) = \pi_\phi(S_\psi)(f)$ for $f \in L^1 \cap H^p$ as

$$\begin{aligned}
\pi_\phi(F)(x) = \int_0^\infty \phi_t * F^t(x)\frac{dt}{t} &= \int_0^\infty (\phi_t * (\psi_t * f))(x)\frac{dt}{t} \\
&= \int_0^\infty ((\phi_t * \psi_t) * f)(x)\frac{dt}{t} \\
&= (\delta * f)(x) = f(x),
\end{aligned}$$

where the legality of the third equality is ensured by the fact that $\phi, \psi \in \mathcal{S}$ and $f \in L^1$, whose Fourier transformations are all well-defined. The fourth holds in the sense of distribution. It has been shown that $L^1 \cap H^p$ is dense in $H^p$, which is trivial if $1 < p < \infty$. For $p \leq 1$, refer to [SM93, Chapter III, §5.14]. Thus, we conclude the verification of the identity on the whole $H^p$ by density argument.

Recall that $\pi_\phi$ is bounded of $T^{p,2}$ to $H^p$, so if $f = \pi_\phi(F)$, it is natural that $\|f\|_{H^p} \lesssim \|F\|_{T^{p,2}}$. $\square$

Furthermore, via such a description, we can prove the complex and real interpolation of Hardy spaces.

**Corollary 4.11.** *Suppose that* $0 < p_0 < p_1 < \infty$ *and* $\theta \in (0,1)$ *be a given number. Then*

$$(H^{p_0}, H^{p_1})_{[\theta]} = H^{p_\theta},$$

*where* $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, *and*

$$(H^{p_0}, H^{p_1})_{\theta;p_\theta} = H^{p_\theta}.$$

*Proof.* Let $\phi, \psi \in \mathcal{S}$ be with $\int \phi = \int \psi = 0$. Then the operators $\pi_\phi : T^{p,2} \to H^p$ and $S_\psi : H^p \to T^{p,2}$ are retraction and co-retraction. We have obtained that $(T^{p_0,2}, T^{p_1,2})_\theta = T^{p_\theta}$ by the remark of Theorem 3.10 and $(T^{p_0,2}, T^{p_1,2})_{\theta,p_\theta} = T^{p_\theta}$ by the remark of Theorem 3.15. Lemma 3.4 directly reads the conclusion. $\square$

Let us end this chapter by further remarks on the generalized version of Theorem 4.10, characterizing homogeneous Hardy-Sobolev spaces via tent spaces.

We briefly state the construction of homogeneous Sobolev spaces first. Let $\mathcal{S}_0 \subset \mathcal{S}$ be the vector space consisting of Schwartz functions $f$ on $\mathbb{R}^n$ such that $\partial^\alpha \widehat{f}(0) = 0$ for any $\alpha \in \mathbb{N}^n$, equipped with the induced topology of $\mathcal{S}$. Let $\mathcal{S}_0'$ be the dual of $\mathcal{S}_0$, which can be identified as $\mathcal{S}'/\mathcal{P}$ with $\mathcal{P} = \mathbb{R}[x_1, \dots, x_n]$ as the space of all the polynomials on $\mathbb{R}^n$ [Tri83, Section 5.1.2]. For any $s \in \mathbb{R}$, let $I_s$ be the map of $\mathcal{S}_0$ to $\mathcal{S}_0$ as

$$I_s(f)(x) := (|\xi|^{-s}\widehat{f}(\xi))^\vee(x).$$

It is well-defined and can be extended to $\mathcal{S}_0'$ canonically [Tri83, Sec.5.2.3]. For any $u \in \mathcal{S}_0'$, $I_s u \in \mathcal{S}_0'$ is given by

$$(I_s u, f) := (u, I_s f)$$

for any $f \in \mathcal{S}_0$.

**Definition 4.12** (Homogeneous Sobolev spaces)**.** Suppose that $s \in \mathbb{R}$ and $1 < p < \infty$, the *homogeneous Sobolev space* $\dot{L}^p_s$ is defined as the image $I_s(L^p(\mathbb{R}^n))$, equipped with the norm

$$\|f\|_{\dot{L}^p_s} := \|I_{-s} f\|_p.$$

We start by a short but enlightening calculation. Set that the ring $R(x;a,b) = \{y \in \mathbb{R}^n : a < |y-x| < b\}$ and the closed ring $\overline{R}(x;a,b) = \overline{R(x;a,b)}$. Let $\varphi \in \mathcal{S}$ be a Schwartz function on $\mathbb{R}^n$ whose Fourier transformation $\widehat{\varphi}$ satisfies that:

- $\widehat{\varphi}(\xi) \in [0,1]$ for any $\xi \in \mathbb{R}^n$;
- $\widehat{\varphi}$ is supported on the closed ring $\overline{R}(0; \frac{1}{2}, 2)$;
- $\widehat{\varphi} \equiv 1$ on $\overline{R}(0; \frac{3}{4}, \frac{3}{2})$.

Recall the operator $S_\varphi$ defined by Eq.(4.5) as

$$S_\varphi(f)(x,t) = (\varphi_t * f)(x).$$

Define the *weight operator* on measurable functions on $\mathbb{R}^{n+1}_+$ as

$$(W^s f)(y,t) := t^s f(y,t) \tag{4.15}$$

for Lebesgue measure. We shall show that $f \in \dot{L}^2_s$ if and only if $W^{-s}S_\varphi(f) \in T^{2,2}$ by equivalent norms. Indeed,

$$\begin{aligned}\|f\|^2_{\dot{L}^2_s} = \|I_{-s}f\|^2_2 &= \int_{\mathbb{R}^n} |\xi|^{2s}|\widehat{f}(\xi)|^2 d\xi \\ &\simeq \int_{\mathbb{R}^n} \left(\int_0^\infty t^{-2s}|\widehat{\varphi}(t\xi)|^2 \frac{dt}{t}\right)|\widehat{f}(\xi)|^2 d\xi \\ &= \int_{\mathbb{R}^{n+1}_+} t^{-2s}|S_\varphi(f)(y,t)| dy \frac{dt}{t} \simeq \|W^{-s}(S_\varphi(f))\|^2_{T^{2,2}}.\end{aligned} \tag{4.16}$$

We can extend this intuition to all the $p$'s, but for that, we need discretization. Set that $\varphi_j(x) := 2^{jn}\varphi(2^jx)$, equal to $\varphi_{2^{-j}}(x)$ by the original definition of $\varphi_t$. Without loss of generality, assume that $\sum_{j\in\mathbb{Z}}\widehat{\varphi_j}(\xi) = 1$ for any $\xi \in \mathbb{R}^n \setminus \{0\}$ by normalization. Let $\Delta_j$ be the Littlewood-Paley operator given by $\Delta_j(f) := \varphi_j * f$.

**Definition 4.13** (Homogeneous Hardy-Sobolev spaces)**.** Suppose that $p \in (0,\infty)$ and $s \in \mathbb{R}$. The *homogeneous Hardy-Sobolev space* $\dot{H}^p_s$ consists of $f \in \mathcal{S}_0'$ such that the following quasi-norm is finite:

$$\|f\|_{\dot{H}^p_s} := \left\| \left\| (2^{js}\Delta_j(f)(x))_{j\in\mathbb{Z}} \right\|_{\ell^2(\mathbb{Z})} \right\|_{L^p(\mathbb{R}^n_x)}.$$

A remark is that when $p \in (1,\infty)$, the space $\dot{H}^p_s$ can be identified as the homogeneous Sobolev space $\dot{L}^p_s$ [Tri83, Theorem 1, Sec.5.2.3]. When $s = 0$, the space $\dot{H}^p_0$ can be identified as Hardy spaces $H^p$ for $0 < p < \infty$ [Tri83, Theorem, Sec.5.2.4].

The following theorem describes the homogeneous Hardy-Sobolev spaces via tent spaces. In particular, we can reduce the condition on $\varphi$ given above in Eq.(4.16).

**Theorem 4.14.** *Suppose that $p \in (0,\infty)$ and $s \in \mathbb{R}$. Let $\varphi \in \mathcal{S}_0$ be such that $\widehat{\varphi}(\xi) > 0$ on $\overline{R}(0;\frac{1}{2};2)$. Then, for any $f \in \mathcal{S}_0'$, the quasi-norms are equivalent as*

$$\|f\|_{\dot{H}^p_s} \simeq \|W^{-s}(S_\varphi f)\|_{T^{p,2}},$$

*where $S_\varphi$ is given by Eq.(4.5) and $W^{-s}$ is given by Eq.(4.15).*

*Proof.* See [AA18, Corollary 2.50]. □

A seminal generalization of homogeneous Hardy-Sobolev spaces is Triebel-Lizorkin spaces.

**Definition 4.15** (Triebel-Lizorkin spaces)**.** Suppose that $p, q \in (0,\infty)$ and $s \in \mathbb{R}$. The *Triebel-Lizorkin space* $\dot{F}^{p,q}_s$ consists of $f \in \mathcal{S}_0'$ such that the quasi-norm

$$\|f\|_{\dot{F}^{p,q}_s} := \left\| \left\| (2^{js}\Delta_j(f)(x))_{j\in\mathbb{Z}} \right\|_{\ell^q(\mathbb{Z})} \right\|_{L^p(\mathbb{R}^n_x)}$$

is finite.

It is easy to observe that $\dot{F}^{p,2}_s$ is exactly the Hardy-Sobolev space $\dot{H}^p_s$.

**Theorem 4.16.** *Suppose that $p, q \in (0,\infty)$ and $s \in \mathbb{R}$. Let $\varphi \in \mathcal{S}_0$ be such that $\widehat{\varphi}(\xi) > 0$ on $\overline{R}(0;\frac{1}{2};2)$. Then, for any $f \in \mathcal{S}_0'$, the quasi-norms are equivalent as*

$$\|f\|_{\dot{F}^{p,q}_s} \simeq \|W^{-s}(S_\varphi f)\|_{T^{p,q}},$$

*where $S_\varphi$ is given by Eq.(4.5) and $W^{-s}$ is given by Eq.(4.15).*

*Proof.* See [AA18, Theorem 2.57]. □

# Chapter 5

# Further remarks

In the final chapter, we provide a series of further works.

1. **Relations between cone-averaging operators and tent-maximal operators**. The following is actually a surprising proposition that reveals the equivalence of cone-averaging operators $A$ and tent-maximal operators $C$.

   **Proposition 5.1.** *If $0 < p < \infty$, then*

   $$\|A_q(f)\|_p \lesssim_{p,q} \|C_q(f)\|_p. \tag{5.1}$$

   *Furthermore, if $0 < q < p \leq \infty$, then the converse is true as*

   $$\|C_q(f)\|_p \lesssim_{p,q} \|A_q(f)\|_p. \tag{5.2}$$

   This proposition actually reveals the reason why we introduced tent-maximal operators to define the type $(\infty, q)$.

   - The original proof is given by [CMS85, Theorem 3] for $q = 2$ in $\mathbb{R}^n$. Moreover, they gave concrete examples to show that the above ranges could not be improved [CMS85, Remark (a) of Theorem 3].
   - [Ame14] further generalized Eq.(5.2) to *metric* spaces satisfying Hardy-Littlewood condition, i.e., the uncentred Hardy-Littlewood maximal operator is of strong type $(p, p)$ for any $p > 1$, and Eq.(5.1) to *metric* doubling spaces.

   We propose that Proposition 5.1 also holds for tent spaces over spaces of homogeneous type by modifying the original proof in [CMS85, Theorem 3] without showing details due to limited time and the size of this mémoire.

2. **Generalization of characterization of Hardy spaces on spaces of homogeneous type**. Note that the proof of Theorem 4.10 mainly depends on the Calderón-Zygmund operator theory, which also holds on spaces of homogeneous type. See [CW71, Théorème 2.4, Chapitre 3] for instance. Therefore, the theory follows without too much effort if we can find appropriate $\phi, \psi$ with non-degeneracy condition, cf. Eq.(4.14) in the sense of spaces of homogeneous type, such that $\pi_\phi$ and $S_\psi$ are still Calderón-Zygmund operators. But it is still open.

# Appendix A

# Complementary proof

In this chapter, we give some complementary proof to several statements. We inherit all the assumptions and notations with respect to the corresponding equations, respectively. We first verify Eq.(4.10), in particular for the second equality, and Eq.(4.13).

*Proof of Eq.(4.10).* It suffices to verify the integrability of

$$\int_{\mathbb{R}^n\times\widehat{B}} |a(y,t)\phi_t(x-y)|dxdy\frac{dt}{t}.$$

It follows by separating into two parts by picking sufficiently large $\gamma \gg 1$. One side is clear that

$$\begin{aligned}\int_{(\gamma B)\times\widehat{B}} |a(y,t)\phi_t(x-y)|dxdy\frac{dt}{t} &\lesssim \int_{\widehat{B}}\left(\int_{\gamma B} dx\right) t^{-n}|a(y,t)|dy\frac{dt}{t}\\ &\lesssim \int_{\widehat{B}} |a(y,t)|dy\frac{dt}{t} < \infty.\end{aligned}$$

The other also works as

$$\int_{(\gamma B)^c\times\widehat{B}} |a(y,t)\phi_t(x-y)|dxdy\frac{dt}{t} \lesssim \int_{(\gamma B)^c} |x-x_0|^{-n-1}dx < \infty$$

thanks to estimate in Eq.(4.8). □

*Proof of Eq.(4.13).* It suffices to verify the integrability of

$$\int_{\mathbb{R}^n\times\widehat{B}} |x|^k|a(y,t)\phi_t(x-y)|dxdy\frac{dt}{t}$$

for any $k \le n[p,1]$. Pick $\gamma \gg 1$ such that $B(0, r_0 = r(B)) \subset \gamma B$, so

$$\int_{(\gamma B)\times\widehat{B}} |x|^k|a(y,t)\phi_t(x-y)|dxdy\frac{dt}{t} \lesssim \int_{(\gamma B)\times\widehat{B}} |a(y,t)\phi_t(x-y)|dxdy\frac{dt}{t} < \infty.$$

On the other hand,

$$\begin{aligned}&\int_{(\gamma B)^c\times\widehat{B}} |x|^k|a(y,t)\phi_t(x-y)|dxdy\frac{dt}{t}\\ &\quad\lesssim |x_0|^k\left(\int_{(\gamma B)^c} |x-x_0|^{-n-s}dx\right)\left(\int_{\widehat{B}} |a(y,t)|t^{s-1}dydt\right)\\ &\quad+\left(\int_{(\gamma B)^c} |x-x_0|^{-n+k-s}dx\right)\left(\int_{\widehat{B}} |a(y,t)|t^{s-1}dydt\right) < \infty\end{aligned}$$

by estimate in Eq.(4.8), and inequalities as $|x|^k \lesssim_k |x-x_0|^k + |x_0|^k$ and $|\phi_t(x-y)| \le t^s|x-y|^{-n-s}$ for any $s>0$. $\square$

Then consider the legality of Eq.(4.12).

*Proof of Eq.(4.12).* We need verify the integrability of

$$\int_{\mathbb{R}^n\times(\epsilon,R)\times B} |f(y,t)\phi_t(x-y)a(x)|dy\frac{dt}{t}dx,$$

for which, by Theorem 2.3, it suffices to show that

$$(y,t)\mapsto \mathbb{1}_{(\epsilon,R)}(t)\int_B |\phi_t(z-y)a(z)|dz =: \mathbb{1}_{(\epsilon,R)}(t)h(y,t)$$

is a $T^{1,2}$-function. Note that

$$h(y,t)^2 \le \left(\int_B |\phi_t(z-y)|^2dz\right)\left(\int_B |a(z)|^2dz\right) \le |B|^{-1}\int_B |\phi_t(z-y)|^2dz.$$

Thus, it suffices to show that

$$\mathfrak{V} := \int_{\mathbb{R}^n}\left(\int_{(\Gamma|\epsilon,R)(x)}\left(\int_B |\phi_t(z-y)|^2dz\right)\frac{dy}{t^n}\frac{dt}{t}\right)^{\frac{1}{2}} dx < \infty$$

We still separate into two parts by a sufficiently large $\gamma \gg \frac{2(R+r_0)}{r_0}+10$.

For any $x\in(\gamma B)^c$, $(y,t)\in(\Gamma|\epsilon,R)(x)$, and $z\in B$,

$$|y-z| \ge |y-c_0| + |c_0-z| \ge |x-c_0| - R - r_0 \ge \frac{1}{2}|x-c_0|$$

In particular, since $x\in(10B)^c$, $|x-c_0|\ge 10r_0$, so $(\Gamma|\epsilon,R)(x)\cap 2B=\varnothing$. Thus,

$$|\phi_t(z-y)| \le t|z-y|^{-n-1} \lesssim_n t|x-c_0|^{-n-1}$$

in this region. In summary, it follows that

$$\begin{aligned}\mathfrak{V}_1 &:= \int_{(\gamma B)^c}\left(\int_{(\Gamma|\epsilon,R)(x)}\left(\int_B |\phi_t(z-y)|^2dz\right)\frac{dy}{t^n}\frac{dt}{t}\right)^{\frac{1}{2}} dx \\ &\lesssim \int_{(\gamma B)^c} |x-c_0|^{-n-1}\left(\int_{(\Gamma|\epsilon,R)(x)} t^{-n+1}dydt\right) dx < \infty.\end{aligned}$$

On the other hand, for $y\in\gamma B, z\in B$, $|\phi_t(z-y)|\lesssim t^{-n}$. Thus,

$$\begin{aligned}\mathfrak{V}_2 &:= \int_{\gamma B}\left(\int_{(\Gamma|\epsilon,R)(x)}\left(\int_B |\phi_t(z-y)|^2dz\right)\frac{dy}{t^n}\frac{dt}{t}\right)^{\frac{1}{2}} dx \\ &\lesssim \int_{\gamma B}\left(\int_\epsilon^R t^{-2n-1}dt\right)^{\frac{1}{2}} dx < \infty.\end{aligned}$$

$\square$

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