Thesis

Ben Agro

1 SDP Relaxation of Stereo Localization Problem

1.1 Re-write as QCQP

$$\mathbf{T_{cw}} = \text{argmin}_{\mathbf{T}} \sum_k (\mathbf{y}_k - \mathbf{M}\frac{1}{z_k}\mathbf{T}\mathbf{p}_k)^T \mathbf{W}_k (\mathbf{y}_k - \mathbf{M}\frac{1}{z_k}\mathbf{T}\mathbf{p}_k), \tag{1}$$

$$\mathbf{T} \in SE(3), \tag{2}$$

$$z_k = \mathbf{a}^T \mathbf{T}\mathbf{p}_k, \tag{3}$$

where $\mathbf{a}^T = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$. Let $\mathbf{v}_k = \frac{1}{z_k}\mathbf{T}\mathbf{p}_k$:

$$\mathbf{v}_k z_k = \mathbf{T}\mathbf{p}_k \tag{4}$$

$$\mathbf{v}_k \mathbf{a}^T \mathbf{T}\mathbf{p}_k = \mathbf{T}\mathbf{p}_k \tag{5}$$

$$(\mathbf{I} - \mathbf{v}_k \mathbf{a}^T)\mathbf{T}\mathbf{p}_k = \mathbf{0} \in \mathbb{R}^4 \tag{6}$$

Now we can re-write our optimization problem as

$$\mathbf{T_{cw}} = \text{argmin}_{\mathbf{T},\mathbf{v}_k} \sum_k (\mathbf{y}_k - \mathbf{M}\mathbf{v}_k)^T \mathbf{W}_k (\mathbf{y}_k - \mathbf{M}\mathbf{v}_k), \tag{7}$$

$$\mathbf{T} \in SE(3), \tag{8}$$

$$(\forall k) \quad (\mathbf{I} - \mathbf{v}_k \mathbf{a}^T)\mathbf{T}\mathbf{p}_k = \mathbf{0} \tag{9}$$

We want to write this in the standard, homogenous, form

$$\mathbf{x}^* = \text{argmin}_{\mathbf{x}} \quad \mathbf{x}^T \mathbf{Q} \mathbf{x} \tag{10}$$

$$(\forall i) \quad \mathbf{x}^T \mathbf{A}_i \mathbf{x} = b_i. \tag{11}$$

1.1.1 Cost

We will start with the cost, adding the homogenization variable ω_0:

$$\mathbf{T_{cw}} = \text{argmin}_{\mathbf{T},\mathbf{v}_k,\omega_0} \sum_k (\omega_0 \mathbf{y}_k - \mathbf{M}\mathbf{v}_k)^T \mathbf{W}_k (\omega_0 \mathbf{y}_k - \mathbf{M}\mathbf{v}_k), \tag{12}$$

$$\mathbf{T} \in SE(3), \tag{13}$$

$$(\forall k) \quad (\mathbf{I} - \mathbf{v}_k \mathbf{a}^T)\mathbf{T}\mathbf{p}_k = \mathbf{0} \tag{14}$$

$$\omega_0^2 = 1. \tag{15}$$

Expanding the term in the sum:

$$(\omega_0 \mathbf{y}_k - \mathbf{M}\mathbf{v}_k)^T \mathbf{W}_k (\omega_0 \mathbf{y}_k - \mathbf{M}\mathbf{v}_k) = \tag{16}$$

$$\omega_0^2 \mathbf{y}_k^T \mathbf{W}_k \mathbf{y_k} - \omega_0 \mathbf{y}_k^T \mathbf{W}_k \mathbf{M}\mathbf{v}_k - \omega_0 \mathbf{v}_k^T \mathbf{M}^T \mathbf{W}_k \mathbf{y}_k + \mathbf{v}_k^T \mathbf{M}^T \mathbf{W}_k \mathbf{M}\mathbf{v}_k \tag{17}$$

$$= \begin{bmatrix} \mathbf{v}_k \\ \omega_0 \end{bmatrix}^T \begin{bmatrix} \mathbf{M}^T \mathbf{W}_k \mathbf{M} & -\mathbf{M}^T \mathbf{W}_k \mathbf{y}_k \\ -\mathbf{y}_k^T \mathbf{W}_k \mathbf{M} & \mathbf{y}^T \mathbf{W}_k \mathbf{y} \end{bmatrix} \begin{bmatrix} \mathbf{v}_k \\ \omega_0 \end{bmatrix}. \tag{18}$$

We can extend this to express the whole some in quadtratic form:

$$\sum_k (\omega_0 \mathbf{y}_k - \mathbf{M}\mathbf{v}_k)^T \mathbf{W}_k (\omega_0 \mathbf{y}_k - \mathbf{M}\mathbf{v}_k) = \tag{19}$$

$$\begin{bmatrix} \mathbf{v}_1 \\ \dots \\ \mathbf{v}_n \\ \omega_0 \end{bmatrix}^T \begin{bmatrix} \mathbf{M}^T \mathbf{W}_1 \mathbf{M} & 0 & \dots & 0 & -\mathbf{M}^T \mathbf{W}_1 \mathbf{y}_1 \\ 0 & \mathbf{M}^T \mathbf{W}_2 \mathbf{M} & \dots & 0 & -\mathbf{M}^T \mathbf{W}_2 \mathbf{y}_2 \\ \dots & & & & \\ 0 & 0 & \dots & \mathbf{M}^T \mathbf{W}_n \mathbf{M} & -\mathbf{M}^T \mathbf{W}_n \mathbf{y}_n \\ -\mathbf{y}_1^T \mathbf{W}_1 \mathbf{M} & -\mathbf{y}_2^T \mathbf{W}_2^T \mathbf{M} & \dots & -\mathbf{y}_n^T \mathbf{W}_n^T \mathbf{M} & \sum_k \mathbf{y}_k^T \mathbf{W}_k \mathbf{y}_k \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \dots \\ \mathbf{v}_n \\ \omega_0 \end{bmatrix} \tag{20}$$

1.1.2 Constraints

Lets begin with the constraint $\mathbf{T} \in SE(3)$. Let

$$\mathbf{T} = \begin{bmatrix} \mathbf{C} & \mathbf{r} \\ \mathbf{0}^T & 1 \end{bmatrix}, \quad \mathbf{C} \in SO(3). \tag{21}$$

$$\mathbf{C}^T \mathbf{C} = \mathbf{I} \quad \text{and} \quad \det(\mathbf{C}) = 1. \tag{22}$$

We will drop the determinate constraint. Write

$$\mathbf{C} = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 \end{bmatrix} \tag{23}$$

$$\mathbf{C}^T \mathbf{C} = \begin{bmatrix} \mathbf{c}_1^T \\ \mathbf{c}_2^T \\ \mathbf{c}_3^T \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 \end{bmatrix} = \mathbf{I} \tag{24}$$

This implies 6 (due to symmetry we don't need all 9) quadtratic constraints in the form:

$$\mathbf{c}_i^T \mathbf{c}_j = \delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}, \tag{25}$$

We can write these in matrix form. Let

$$\mathbf{x}_1 = \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \\ \mathbf{r} \end{bmatrix}. \tag{26}$$

Then the six constraints are

$$\mathbf{x}_1^T \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{x}_1 = 1, \tag{27}$$

$$\mathbf{x}_1^T \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{x}_1 = 1, \tag{28}$$

$$\mathbf{x}_1^T \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{x}_1 = 1, \tag{29}$$

$$\mathbf{x}_1^T \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{x}_1 = 1, \tag{30}$$

$$\mathbf{x}_1^T \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{x}_1 = 1, \tag{31}$$

$$\mathbf{x}_1^T \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{x}_1 = 1, \tag{32}$$

$$\tag{33}$$

Next lets deal with the constraint $(\forall k) \quad (\mathbf{I} - \mathbf{v}_k\mathbf{a}^T)\mathbf{T}\mathbf{p}_k = \mathbf{0}$. Expand this and add our homogenization variable ω_0:

$$\begin{bmatrix} \omega_0 & 0 & 0 & 0 \\ 0 & \omega_0 & 0 & 0 \\ 0 & 0 & \omega_0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 & \mathbf{r} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_{k1} \\ p_{k2} \\ p_{k3} \\ 1 \end{bmatrix} - \mathbf{v}_k \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 & \mathbf{r} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_{k1} \\ p_{k2} \\ p_{k3} \\ 1 \end{bmatrix} = \mathbf{0}. \tag{34}$$

Expanding further:

$$\begin{bmatrix} \omega_0 p_{k1}\mathbf{c}_1 + \omega_0 p_{k2}\mathbf{c}_2 + \omega_0 p_{k3}\mathbf{c}_3 + \omega_0\mathbf{r} \\ 1 \end{bmatrix} - \mathbf{v}_k \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} p_{k1}\mathbf{c}_1 + p_{k2}\mathbf{c}_2 + p_{k3}\mathbf{c}_3 + \mathbf{r} \\ 1 \end{bmatrix} = \mathbf{0}. \tag{35}$$

Note that

$$\mathbf{C} = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}, \tag{36}$$

so we can expand the expression above:

$$\begin{bmatrix} \omega_0 p_{k1}\mathbf{c}_1 + \omega_0 p_{k2}\mathbf{c}_2 + \omega_0 p_{k3}\mathbf{c}_3 + \omega_0\mathbf{r} \\ 1 \end{bmatrix} - \mathbf{v}_k(p_{k1}c_{31} + p_{k2}c_{32} + p_{k3}c_{33} + r_3) = \mathbf{0}. \tag{37}$$

Note that due from its definition, $\mathbf{v}_k = \begin{bmatrix} v_1 & v_2 & 1 & v_4 \end{bmatrix}^T$. Because we want homogenous constraints, we substitute $\mathbf{v}_k = \begin{bmatrix} v_1 & v_2 & \omega_0 & v_4 \end{bmatrix}^T$ instead: Write this as four scalar equations:

$$p_{k1}\omega_0 c_{11} + p_{k2}\omega_0 c_{12} + p_{k3}\omega_0 c_{13} + \omega_0 r_1 - p_{k1}c_{31}v_{k1} - p_{k2}c_{32}v_{k1} - p_{k3}c_{33}v_{k1} - r_3 v_{k1} = 0, \tag{38}$$

$$p_{k1}\omega_0 c_{21} + p_{k2}\omega_0 c_{22} + p_{k3}\omega_0 c_{23} + \omega_0 r_1 - p_{k1}c_{31}v_{k2} - p_{k2}c_{32}v_{k2} - p_{k3}c_{33}v_{k2} - r_3 v_{k2} = 0, \tag{39}$$

$$p_{k1}\omega_0 c_{31} + p_{k2}\omega_0 c_{32} + p_{k3}\omega_0 c_{33} + \omega_0 r_1 - p_{k1}c_{31}\omega_0 - p_{k2}c_{32}\omega_0 - p_{k3}c_{33}\omega_0 - r_3\omega_0 = 0, \tag{40}$$

$$1 - p_{k1}c_{31}v_{k4} - p_{k2}c_{32}v_{k4} - p_{k3}c_{33}v_{k4} - r_3 v_{k4} = 0. \tag{41}$$

Now we can turn these into four matrix form:

$$
\begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \\ \mathbf{r} \\ \mathbf{v}_k \\ \omega_0 \end{bmatrix}^T
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -p_{k1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -p_{k2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -p_{k3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p_{k1} & 0 & 0 & p_{k2} & 0 & 0 & p_{k3} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \\ \mathbf{r} \\ \mathbf{v}_k \\ \omega_0 \end{bmatrix} = 0, \tag{42}
$$

$$
\begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \\ \mathbf{r} \\ \mathbf{v}_k \\ \omega_0 \end{bmatrix}^T
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -p_{k1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -p_{k2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -p_{k3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & p_{k1} & 0 & 0 & p_{k2} & 0 & 0 & p_{k3} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \\ \mathbf{r} \\ \mathbf{v}_k \\ \omega_0 \end{bmatrix} = 0, \tag{43}
$$

$$
\begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \\ \mathbf{r} \\ \mathbf{v}_k \\ \omega_0 \end{bmatrix}^T
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -p_{k1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -p_{k2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -p_{k3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & p_{k1} & 0 & 0 & p_{k2} & 0 & 0 & p_{k3} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \\ \mathbf{r} \\ \mathbf{v}_k \\ \omega_0 \end{bmatrix} = 0,
\tag{44}
$$

$$
\begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \\ \mathbf{r} \\ \mathbf{v}_k \\ \omega_0 \end{bmatrix}^T
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_{k1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_{k2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_{k3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \\ \mathbf{r} \\ \mathbf{v}_k \\ \omega_0 \end{bmatrix} = 1,
\tag{45}
$$

Finally, the constraints for the homogenous variable is

$$
\omega_0 \begin{bmatrix} 1 \end{bmatrix} \omega_0 = 1, \tag{46}
$$

$$
\tag{47}
$$

Let

$$\mathbf{x} = \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \\ \mathbf{r} \\ \mathbf{v}_1 \\ \mathbf{v}_2 \\ \dots \\ \mathbf{v}_n \\ \omega_0 \end{bmatrix} \in \mathbb{R}^{(13+4n)\times 1}, \tag{48}$$

which includes all the decision variables. All of the $12+4n+1$ constraints above can be formulated using $\mathbf{x}$ by adding columns and rows of zero to the matrices above to obtain $\mathbf{A}_i$, to ignore the decision variables that are not used in the constraint. Any non-symmetric $\mathbf{A}_i$ can be made symmetric and preserve the same constraint by doing

$$\mathbf{A}_{\text{new}_i} = \frac{1}{2}(\mathbf{A}_i + \mathbf{A}_i^T). \tag{49}$$

1.2 SDP Relaxation

Now that we have the QCQP in the homogenous form

$$\mathbf{x}^* = \text{argmin}_{\mathbf{x}} \quad \mathbf{x}^T\mathbf{Q}\mathbf{x} \tag{50}$$

$$(\forall i) \quad \mathbf{x}^T\mathbf{A}_i\mathbf{x} = b_i, \tag{51}$$

we can easily turn it into an SDP:

$$\mathbf{x}^* = \text{argmin}_{\mathbf{x}} \quad \text{tr}(\mathbf{Q}\mathbf{x}\mathbf{x}^T), \tag{52}$$

$$\text{s.t} \quad (\forall i) \quad \text{tr}(\mathbf{A}_i\mathbf{x}\mathbf{x}^T) = b_i. \tag{53}$$

Let $\mathbf{X} = \mathbf{x}\mathbf{x}^T \in \mathbb{R}^{(14+4n)\times(14+4n)}$, then the above is equivalent to:

$$\mathbf{X}^* = \text{argmin}_{\mathbf{X}} \quad \text{tr}(\mathbf{Q}\mathbf{X}), \tag{54}$$

$$\text{s.t} \quad (\forall i) \quad \text{tr}(\mathbf{A}_i\mathbf{X}) = b_i, \tag{55}$$

$$\mathbf{X} = \mathbf{x}\mathbf{x}^T \tag{56}$$

$$\text{rank}(\mathbf{X}) = 1 \tag{57}$$

We can relax these last two constraints to get an SDP:

$$\mathbf{X}^* = \text{argmin}_{\mathbf{X}} \quad \text{tr}(\mathbf{Q}\mathbf{X}), \tag{58}$$

$$\text{s.t} \quad (\forall i) \quad \text{tr}(\mathbf{A}_i\mathbf{X}) = b_i, \tag{59}$$

$$\mathbf{X} \succeq 0. \tag{60}$$

1.3 Redundant Constraints

Note that $\mathbf{Tp}_k$ and $\mathbf{v}_k$ differ by a constant factor, so we can add the constraint

$$\mathbf{v}_k(\mathbf{Tp}_k)^T = (\mathbf{Tp}_k)\mathbf{v}_k^T \quad \forall k \tag{61}$$
$$\tag{62}$$

2 1D problem

$$\mathcal{L} = \sum_{n=1}^{N} \left(y_n - \frac{1}{x - a_n} \right)^2 \tag{63}$$

Let $z_n = \frac{1}{x - a_n}$, $\omega = 1$:

$$\mathcal{L} = \sum_n (\omega y_n - z_n)^2 \tag{64}$$
$$\text{subject to } z_n(x - a_n) = 1, \forall n \in 1, \ldots, N \tag{65}$$
$$\omega^2 = 1 \tag{66}$$

2.1 Redundant Constraints

$$z_j - z_i = \frac{1}{x - a_j} - \frac{1}{x - a_i} \tag{67}$$
$$= \frac{a_j - a_i}{(x - a_j)(x - a_i)} \tag{68}$$
$$= z_j z_i (a_j - a_i) \tag{69}$$