

# Thesis

Ben Agro

## 1 SDP Relaxation of Stereo Localization Problem

### 1.1 Re-write as QCQP

$$\mathbf{T}_{\text{cw}} = \operatorname{argmin}_{\mathbf{T}} \sum_k (\mathbf{y}_k - \mathbf{M} \frac{1}{z_k} \mathbf{T} \mathbf{p}_k)^T \mathbf{W}_k (\mathbf{y}_k - \mathbf{M} \frac{1}{z_k} \mathbf{T} \mathbf{p}_k), \quad (1)$$

$$\mathbf{T} \in SE(3), \quad (2)$$

$$z_k = \mathbf{a}^T \mathbf{T} \mathbf{p}_k, \quad (3)$$

where  $\mathbf{a}^T = [0 \ 0 \ 1 \ 0]$ . Let  $\mathbf{v}_k = \frac{1}{z_k} \mathbf{T} \mathbf{p}_k$ :

$$\mathbf{v}_k z_k = \mathbf{T} \mathbf{p}_k \quad (4)$$

$$\mathbf{v}_k \mathbf{a}^T \mathbf{T} \mathbf{p}_k = \mathbf{T} \mathbf{p}_k \quad (5)$$

$$(\mathbf{I} - \mathbf{v}_k \mathbf{a}^T) \mathbf{T} \mathbf{p}_k = \mathbf{0} \in \mathbb{R}^4 \quad (6)$$

Now we can re-write our optimization problem as

$$\mathbf{T}_{\text{cw}} = \operatorname{argmin}_{\mathbf{T}, \mathbf{v}_k} \sum_k (\mathbf{y}_k - \mathbf{M} \mathbf{v}_k)^T \mathbf{W}_k (\mathbf{y}_k - \mathbf{M} \mathbf{v}_k), \quad (7)$$

$$\mathbf{T} \in SE(3), \quad (8)$$

$$(\forall k) \quad (\mathbf{I} - \mathbf{v}_k \mathbf{a}^T) \mathbf{T} \mathbf{p}_k = \mathbf{0} \quad (9)$$

We want to write this in the standard, homogenous, form

$$\mathbf{x}^* = \operatorname{argmin}_{\mathbf{x}} \quad \mathbf{x}^T \mathbf{Q} \mathbf{x} \quad (10)$$

$$(\forall i) \quad \mathbf{x}^T \mathbf{A}_i \mathbf{x} = b_i. \quad (11)$$

### 1.1.1 Cost

We will start with the cost, adding the homogenization variable  $\omega_0$ :

$$\mathbf{T}_{\mathbf{cw}} = \operatorname{argmin}_{\mathbf{T}, \mathbf{v}_k, \omega_0} \sum_k (\omega_0 \mathbf{y}_k - \mathbf{M} \mathbf{v}_k)^T \mathbf{W}_k (\omega_0 \mathbf{y}_k - \mathbf{M} \mathbf{v}_k), \quad (12)$$

$$\mathbf{T} \in SE(3), \quad (13)$$

$$(\forall k) \quad (\mathbf{I} - \mathbf{v}_k \mathbf{a}^T) \mathbf{T} \mathbf{p}_k = \mathbf{0} \quad (14)$$

$$\omega_0^2 = 1. \quad (15)$$

Expanding the term in the sum:

$$(\omega_0 \mathbf{y}_k - \mathbf{M} \mathbf{v}_k)^T \mathbf{W}_k (\omega_0 \mathbf{y}_k - \mathbf{M} \mathbf{v}_k) = \quad (16)$$

$$\omega_0^2 \mathbf{y}_k^T \mathbf{W}_k \mathbf{y}_k - \omega_0 \mathbf{y}_k^T \mathbf{W}_k \mathbf{M} \mathbf{v}_k - \omega_0 \mathbf{v}_k^T \mathbf{M}^T \mathbf{W}_k \mathbf{y}_k + \mathbf{v}_k^T \mathbf{M}^T \mathbf{W}_k \mathbf{M} \mathbf{v}_k \quad (17)$$

$$= \begin{bmatrix} \mathbf{v}_k \\ \omega_0 \end{bmatrix}^T \begin{bmatrix} \mathbf{M}^T \mathbf{W}_k \mathbf{M} & -\mathbf{M}^T \mathbf{W}_k \mathbf{y}_k \\ -\mathbf{y}_k^T \mathbf{W}_k \mathbf{M} & \mathbf{y}_k^T \mathbf{W}_k \mathbf{y}_k \end{bmatrix} \begin{bmatrix} \mathbf{v}_k \\ \omega_0 \end{bmatrix}. \quad (18)$$

We can extend this to express the whole some in quadtratic form:

$$\sum_k (\omega_0 \mathbf{y}_k - \mathbf{M} \mathbf{v}_k)^T \mathbf{W}_k (\omega_0 \mathbf{y}_k - \mathbf{M} \mathbf{v}_k) = \quad (19)$$

$$\begin{bmatrix} \mathbf{v}_1 \\ \dots \\ \mathbf{v}_n \\ \omega_0 \end{bmatrix}^T \begin{bmatrix} \mathbf{M}^T \mathbf{W}_1 \mathbf{M} & 0 & \dots & 0 & -\mathbf{M}^T \mathbf{W}_1 \mathbf{y}_1 \\ 0 & \mathbf{M}^T \mathbf{W}_2 \mathbf{M} & \dots & 0 & -\mathbf{M}^T \mathbf{W}_2 \mathbf{y}_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \mathbf{M}^T \mathbf{W}_n \mathbf{M} & -\mathbf{M}^T \mathbf{W}_n \mathbf{y}_n \\ -\mathbf{y}_1^T \mathbf{W}_1 \mathbf{M} & -\mathbf{y}_2^T \mathbf{W}_2 \mathbf{M} & \dots & -\mathbf{y}_n^T \mathbf{W}_n \mathbf{M} & \sum_k \mathbf{y}_k^T \mathbf{W}_k \mathbf{y}_k \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \dots \\ \mathbf{v}_n \\ \omega_0 \end{bmatrix} \quad (20)$$

### 1.1.2 Constraints

Lets begin with the constraint  $\mathbf{T} \in SE(3)$ . Let

$$\mathbf{T} = \begin{bmatrix} \mathbf{C} & \mathbf{r} \\ \mathbf{0}^T & 1 \end{bmatrix}, \quad \mathbf{C} \in SO(3). \quad (21)$$

$$\mathbf{C}^T \mathbf{C} = \mathbf{I} \quad \text{and} \quad \det(\mathbf{C}) = 1. \quad (22)$$

We will drop the determinate constraint. Write

$$\mathbf{C} = [\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{c}_3] \quad (23)$$

$$\mathbf{C}^T \mathbf{C} = \begin{bmatrix} \mathbf{c}_1^T \\ \mathbf{c}_2^T \\ \mathbf{c}_3^T \end{bmatrix} [\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{c}_3] = \mathbf{I} \quad (24)$$

This implies 6 (due to symmetry we don't need all 9) quadratic constraints in the form:

$$\mathbf{c}_i^T \mathbf{c}_j = \delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}, \quad (25)$$

We can write these in matrix form. Let

$$\mathbf{x}_1 = \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \\ \mathbf{r} \end{bmatrix}. \quad (26)$$

Then the six constraints are

$$\mathbf{x}_1^T \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{x}_1 = 1, \quad (27)$$

$$\mathbf{x}_1^T \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{x}_1 = 1, \quad (28)$$

$$\mathbf{x}_1^T \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{x}_1 = 1, \quad (29)$$

$$\mathbf{x}_1^T \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{x}_1 = 1, \quad (30)$$

$$\mathbf{x}_1^T \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{x}_1 = 1, \quad (31)$$

$$\mathbf{x}_1^T \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{x}_1 = 1, \quad (32)$$

$$(33)$$

Next lets deal with the constraint  $(\forall k) \quad (\mathbf{I} - \mathbf{v}_k \mathbf{a}^T) \mathbf{T} \mathbf{p}_k = \mathbf{0}$ . Expand this and add our homogenization variable  $\omega_0$ :

$$\begin{bmatrix} \omega_0 & 0 & 0 & 0 \\ 0 & \omega_0 & 0 & 0 \\ 0 & 0 & \omega_0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 & \mathbf{r} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_{k1} \\ p_{k2} \\ p_{k3} \\ 1 \end{bmatrix} - \mathbf{v}_k \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 & \mathbf{r} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_{k1} \\ p_{k2} \\ p_{k3} \\ 1 \end{bmatrix} = \mathbf{0}. \quad (34)$$

Expanding further:

$$\begin{bmatrix} \omega_0 p_{k1} \mathbf{c}_1 + \omega_0 p_{k2} \mathbf{c}_2 + \omega_0 p_{k3} \mathbf{c}_3 + \omega_0 \mathbf{r} \\ 1 \end{bmatrix} - \mathbf{v}_k \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} p_{k1} \mathbf{c}_1 + p_{k2} \mathbf{c}_2 + p_{k3} \mathbf{c}_3 + \mathbf{r} \\ 1 \end{bmatrix} = \mathbf{0}. \quad (35)$$

Note that

$$\mathbf{C} = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}, \quad (36)$$

so we can expand the expression above:

$$\begin{bmatrix} \omega_0 p_{k1} \mathbf{c}_1 + \omega_0 p_{k2} \mathbf{c}_2 + \omega_0 p_{k3} \mathbf{c}_3 + \omega_0 \mathbf{r} \\ 1 \end{bmatrix} - \mathbf{v}_k (p_{k1} c_{31} + p_{k2} c_{32} + p_{k3} c_{33} + r_3) = \mathbf{0}. \quad (37)$$

Note that due from its definition,  $\mathbf{v}_k = [v_1 \ v_2 \ 1 \ v_4]^T$ . Because we want homogenous constraints, we substitute  $\mathbf{v}_k = [v_1 \ v_2 \ \omega_0 \ v_4]^T$  instead: Write this as four scalar equations:

$$p_{k1} \omega_0 c_{11} + p_{k2} \omega_0 c_{12} + p_{k3} \omega_0 c_{13} + \omega_0 r_1 - p_{k1} c_{31} v_{k1} - p_{k2} c_{32} v_{k1} - p_{k3} c_{33} v_{k1} - r_3 v_{k1} = 0, \quad (38)$$

$$p_{k1} \omega_0 c_{21} + p_{k2} \omega_0 c_{22} + p_{k3} \omega_0 c_{23} + \omega_0 r_1 - p_{k1} c_{31} v_{k2} - p_{k2} c_{32} v_{k2} - p_{k3} c_{33} v_{k2} - r_3 v_{k2} = 0, \quad (39)$$

$$p_{k1} \omega_0 c_{31} + p_{k2} \omega_0 c_{32} + p_{k3} \omega_0 c_{33} + \omega_0 r_1 - p_{k1} c_{31} \omega_0 - p_{k2} c_{32} \omega_0 - p_{k3} c_{33} \omega_0 - r_3 \omega_0 = 0, \quad (40)$$

$$1 - p_{k1} c_{31} v_{k4} - p_{k2} c_{32} v_{k4} - p_{k3} c_{33} v_{k4} - r_3 v_{k4} = 0. \quad (41)$$

$$\begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \\ \mathbf{r} \\ \mathbf{v}_k \\ \omega_0 \end{bmatrix}^T \left[ \begin{array}{ccccccccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -p_{k1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -p_{k2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -p_{k3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ p_{k1} & 0 & 0 & p_{k2} & 0 & 0 & p_{k3} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \\ \mathbf{r} \\ \mathbf{v}_k \\ \omega_0 \end{bmatrix} = 0, \quad (42)$$

5

(44)

(45)

Finally, the constraints for the homogenous variable is

(46)

(47)

Let

$$\mathbf{x} = \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \\ \mathbf{r} \\ \mathbf{v}_1 \\ \mathbf{v}_2 \\ \dots \\ \mathbf{v}_n \\ \omega_0 \end{bmatrix} \in \mathbb{R}^{(13+4n) \times 1}, \quad (48)$$

which includes all the decision variables. All of the  $12+4n+1$  constraints above can be formulated using  $\mathbf{x}$  by adding columns and rows of zero to the matrices above to obtain  $\mathbf{A}_i$ , to ignore the decision variables that are not used in the constraint. Any non-symmetric  $\mathbf{A}_i$  can be made symmetric and preserve the same constraint by doing

$$\mathbf{A}_{\text{new}_i} = \frac{1}{2}(\mathbf{A}_i + \mathbf{A}_i^T). \quad (49)$$

## 1.2 SDP Relaxation

Now that we have the QCQP in the homogenous form

$$\mathbf{x}^* = \operatorname{argmin}_{\mathbf{x}} \quad \mathbf{x}^T \mathbf{Q} \mathbf{x} \quad (50)$$

$$(\forall i) \quad \mathbf{x}^T \mathbf{A}_i \mathbf{x} = b_i, \quad (51)$$

we can easily turn it into an SDP:

$$\mathbf{x}^* = \operatorname{argmin}_{\mathbf{x}} \quad \operatorname{tr}(\mathbf{Q} \mathbf{x} \mathbf{x}^T), \quad (52)$$

$$\text{s.t.} \quad (\forall i) \quad \operatorname{tr}(\mathbf{A}_i \mathbf{x} \mathbf{x}^T) = b_i. \quad (53)$$

Let  $\mathbf{X} = \mathbf{x} \mathbf{x}^T \in \mathbb{R}^{(14+4n) \times (14+4n)}$ , then the above is equivalent to:

$$\mathbf{X}^* = \operatorname{argmin}_{\mathbf{X}} \quad \operatorname{tr}(\mathbf{Q} \mathbf{X}), \quad (54)$$

$$\text{s.t.} \quad (\forall i) \quad \operatorname{tr}(\mathbf{A}_i \mathbf{X}) = b_i, \quad (55)$$

$$\mathbf{X} = \mathbf{x} \mathbf{x}^T \quad (56)$$

$$\operatorname{rank}(\mathbf{X}) = 1 \quad (57)$$

We can relax these last two constraints to get an SDP:

$$\mathbf{X}^* = \operatorname{argmin}_{\mathbf{X}} \quad \operatorname{tr}(\mathbf{Q} \mathbf{X}), \quad (58)$$

$$\text{s.t.} \quad (\forall i) \quad \operatorname{tr}(\mathbf{A}_i \mathbf{X}) = b_i, \quad (59)$$

$$\mathbf{X} \succeq 0. \quad (60)$$

### 1.3 Redundant Constraints

Note that  $\mathbf{T}\mathbf{p}_k$  and  $\mathbf{v}_k$  differ by a constant factor, so we can add the constraint

$$\mathbf{v}_k(\mathbf{T}\mathbf{p}_k)^T = (\mathbf{T}\mathbf{p}_k)\mathbf{v}_k^T \quad \forall k \quad (61)$$

$$(62)$$

### 1.4 From 1D problem

Recall  $\mathbf{v}_n = \frac{1}{\mathbf{e}_3^T \mathbf{T}\mathbf{p}_n} \mathbf{T}\mathbf{p}_n$ .

$$\mathbf{e}_4^T(\mathbf{v}_i - \mathbf{v}_j) = \frac{\mathbf{e}_3^T \mathbf{T}\mathbf{p}_j - \mathbf{e}_3^T \mathbf{T}\mathbf{p}_i}{(\mathbf{e}_3^T \mathbf{T}\mathbf{p}_i)(\mathbf{e}_3^T \mathbf{T}\mathbf{p}_j)} \quad (63)$$

$$\mathbf{e}_4^T(\mathbf{v}_i - \mathbf{v}_j) = (\mathbf{e}_4^T \mathbf{v}_i)(\mathbf{e}_4^T \mathbf{v}_j)(\mathbf{e}_3^T \mathbf{T}\mathbf{p}_j - \mathbf{e}_3^T \mathbf{T}\mathbf{p}_i). \quad (64)$$

Introduce new variables  $q_{ij} = (\mathbf{e}_4^T \mathbf{v}_i)(\mathbf{e}_4^T \mathbf{v}_j) = \mathbf{u}_{i3}\mathbf{u}_{j3}$ . Then we add the quadratic constraints:

$$\mathbf{e}_4^T(\mathbf{v}_i - \mathbf{v}_j) = q_{ij}(\mathbf{e}_3^T \mathbf{T}\mathbf{p}_j - \mathbf{e}_3^T \mathbf{T}\mathbf{p}_i), \quad (65)$$

$$\omega q_{ij} = \mathbf{u}_{i3}\mathbf{u}_{j3} \quad (66)$$

## 2 1D problem

$$\mathcal{L} = \sum_{n=1}^N \left( y_n - \frac{1}{x - a_n} \right)^2 \quad (67)$$

Let  $z_n = \frac{1}{x - a_n}$ ,  $\omega = 1$ :

$$\mathcal{L} = \sum_n (\omega y_n - z_n)^2 \quad (68)$$

$$\text{subject to } z_n(x - a_n) = 1, \forall n \in 1, \dots, N \quad (69)$$

$$\omega^2 = 1 \quad (70)$$

### 2.1 Redundant Constraints

$$z_j - z_i = \frac{1}{x - a_j} - \frac{1}{x - a_i} \quad (71)$$

$$= \frac{a_j - a_i}{(x - a_j)(x - a_i)} \quad (72)$$

$$= z_j z_i (a_j - a_i) \quad (73)$$