Thesis

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1 SDP Relaxation of Stereo Localization Problem

1.1 Re-write as QCQP

$$\mathbf{T_{cw}} = \text{argmin}_{\mathbf{T}} \sum_k (\mathbf{y}_k - \mathbf{M}\frac{1}{z_k}\mathbf{T}\mathbf{p}_k)^T \mathbf{W}_k (\mathbf{y}_k - \mathbf{M}\frac{1}{z_k}\mathbf{T}\mathbf{p}_k), \tag{1}$$

$$\mathbf{T} \in SE(3), \tag{2}$$

$$z_k = \mathbf{e}_3^T \mathbf{T}\mathbf{p}_k, \tag{3}$$

where $\mathbf{e}_3^T = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$. Let $\mathbf{v}_k = \frac{1}{z_k}\mathbf{T}\mathbf{p}_k$:

$$\mathbf{v}_k z_k = \mathbf{T}\mathbf{p}_k \tag{4}$$

$$\mathbf{v}_k \mathbf{e}_3^T \mathbf{T}\mathbf{p}_k = \mathbf{T}\mathbf{p}_k \tag{5}$$

$$(\mathbf{I} - \mathbf{v}_k \mathbf{e}_3^T)\mathbf{T}\mathbf{p}_k = \mathbf{0} \in \mathbb{R}^4 \tag{6}$$

Now we can re-write our optimization problem as

$$\mathbf{T_{cw}} = \text{argmin}_{\mathbf{T},\mathbf{v}_k} \sum_k (\mathbf{y}_k - \mathbf{M}\mathbf{v}_k)^T \mathbf{W}_k (\mathbf{y}_k - \mathbf{M}\mathbf{v}_k), \tag{7}$$

$$\mathbf{T} \in SE(3), \tag{8}$$

$$(\forall k) \quad (\mathbf{I} - \mathbf{v}_k \mathbf{e}_3^T)\mathbf{T}\mathbf{p}_k = \mathbf{0} \tag{9}$$

We want to write this in the standard, homogenous, form

$$\mathbf{x}^* = \text{argmin}_{\mathbf{x}} \quad \mathbf{x}^T \mathbf{Q} \mathbf{x} \tag{10}$$

$$(\forall i) \quad \mathbf{x}^T \mathbf{A}_i \mathbf{x} = b_i. \tag{11}$$

1.1.1 Cost

We will start with the cost, adding the homogenization variable ω:

$$\mathbf{T_{cw}} = \text{argmin}_{\mathbf{T},\mathbf{v}_k,\omega} \sum_k (\omega \mathbf{y}_k - \mathbf{M}\mathbf{v}_k)^T \mathbf{W}_k (\omega \mathbf{y}_k - \mathbf{M}\mathbf{v}_k), \tag{12}$$

$$\mathbf{T} \in SE(3), \tag{13}$$

$$(\forall k) \quad (\mathbf{I} - \mathbf{v}_k \mathbf{a}^T)\mathbf{T}\mathbf{p}_k = \mathbf{0} \tag{14}$$

$$\omega^2 = 1. \tag{15}$$

We denote

$$\mathbf{v}_k = \frac{\mathbf{T}\mathbf{p}_k}{\mathbf{e}_3^T \mathbf{T}\mathbf{p}_k} = \begin{bmatrix} v_{k1} \\ v_{k2} \\ 1 \\ v_{k4} \end{bmatrix} = \begin{bmatrix} v_{k1} \\ v_{k2} \\ \omega \\ v_{k4} \end{bmatrix}. \tag{16}$$

We add the homogenization variable so the cost and constraints remain quadratic.

Expanding the term in the sum:

$$(\omega \mathbf{y}_k - \mathbf{M}\mathbf{v}_k)^T \mathbf{W}_k (\omega \mathbf{y}_k - \mathbf{M}\mathbf{v}_k) = \tag{17}$$

$$\omega^2 \mathbf{y}_k^T \mathbf{W}_k \mathbf{y_k} - \omega \mathbf{y}_k^T \mathbf{W}_k \mathbf{M}\mathbf{v}_k - \omega \mathbf{v}_k^T \mathbf{M}^T \mathbf{W}_k \mathbf{y}_k + \mathbf{v}_k^T \mathbf{M}^T \mathbf{W}_k \mathbf{M}\mathbf{v}_k \tag{18}$$

Let

$$\mathbf{E} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_4 \end{bmatrix} \in \mathbb{R}^{4\times 3}, \tag{19}$$

and

$$\mathbf{u}_k = \begin{bmatrix} v_{k1} \\ v_{k2} \\ v_{k4} \end{bmatrix}, \tag{20}$$

where $\mathbf{e}_i \in \mathbb{R}^{4\times 1}$ is all zeros except for the i^{th} entry which is 1. We can then re-write $\mathbf{v}_k$:

$$\mathbf{v}_k = \mathbf{E}\mathbf{u}_k + \mathbf{e}_3 \omega, \tag{21}$$

and use this to expand equation 18:

$$\omega^2 \mathbf{y}_k^T \mathbf{W}_k \mathbf{y}_k \tag{22}$$

$$- \omega \mathbf{y}_k^T \mathbf{W}_k \mathbf{M}\mathbf{E}\mathbf{u}_k - \mathbf{y}_k^T \mathbf{W}_k \mathbf{M}\mathbf{e}_3^T \omega^2 \tag{23}$$

$$- \omega \mathbf{u}_k^T \mathbf{E}^T \mathbf{M}^T \mathbf{W}_k \mathbf{y}_k - \mathbf{e}_3^T \mathbf{M}^T \mathbf{W}_k \mathbf{y}_k \omega^2 \tag{24}$$

$$+ \mathbf{u}_k \mathbf{E}^T \mathbf{M}^T \mathbf{W}_k \mathbf{M}\mathbf{E}\mathbf{u}_k + \omega \mathbf{e}_3^T \mathbf{M}^T \mathbf{W}_k \mathbf{M}\mathbf{E}\mathbf{u}_k + \mathbf{u}_k^T \mathbf{E}^T \mathbf{M}^T \mathbf{W}_k \mathbf{M}\mathbf{e}_3 \omega + \mathbf{e}_3^T \mathbf{M}^T \mathbf{W}_k \mathbf{M}\mathbf{e}_3 \omega^T. \tag{25}$$

Now we can write the cost in matrix form:

$$\mathbf{x}_2^T \begin{bmatrix} \mathbf{H}_1 & 0 & \dots & 0 & \mathbf{g}_1 \\ 0 & \mathbf{H}_2 & \dots & 0 & \mathbf{g}_2 \\ \dots & & & & \\ 0 & 0 & \dots & \mathbf{H}_N & \mathbf{g}_N \\ \mathbf{g}_1^T & \mathbf{g}_2^T & \dots & \mathbf{g}_N^T & \mathbf{\Omega} \end{bmatrix} \mathbf{x}_2 \tag{26}$$

where

$$\mathbf{x}_2 = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \dots \\ \mathbf{u}_N \\ \omega \end{bmatrix}, \tag{27}$$

$$\mathbf{\Omega} = \sum_k \left(\mathbf{y}_k^T \mathbf{W}_k \mathbf{y}_k - \mathbf{y}_k^T \mathbf{W}_k \mathbf{M} \mathbf{e}_3 - \mathbf{e}_3^T \mathbf{M}^T \mathbf{W}_k \mathbf{y}_k + \mathbf{e}_3^T \mathbf{M}^T \mathbf{W}_k \mathbf{M} \mathbf{e}_3 \right) \in \mathbb{R}, \tag{28}$$

$$\mathbf{g}_k = -\mathbf{E}^T \mathbf{M}^T \mathbf{W}_k \mathbf{y}_k + \mathbf{E}^T \mathbf{M}^T \mathbf{W}_k \mathbf{M} \mathbf{e}_3 \in \mathbb{R}^{3\times 1}, \tag{29}$$

$$\mathbf{H}_k = \mathbf{E}^T \mathbf{M}^T \mathbf{W}_k \mathbf{M} \mathbf{E} \in \mathbb{R}^{3\times 3} \tag{30}$$

1.1.2 Constraints

Lets begin with the constraint $\mathbf{T} \in SE(3)$. Let

$$\mathbf{T} = \begin{bmatrix} \mathbf{C} & \mathbf{r} \\ \mathbf{0}^T & 1 \end{bmatrix}, \quad \mathbf{C} \in SO(3). \tag{31}$$

$$\mathbf{C}^T \mathbf{C} = \mathbf{I} \quad \text{and} \quad \det(\mathbf{C}) = 1. \tag{32}$$

We will drop the determinate constraint. Write

$$\mathbf{C} = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 \end{bmatrix} \tag{33}$$

$$\mathbf{C}^T \mathbf{C} = \begin{bmatrix} \mathbf{c}_1^T \\ \mathbf{c}_2^T \\ \mathbf{c}_3^T \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 \end{bmatrix} = \mathbf{I} \tag{34}$$

This implies 6 (due to symmetry we don't need all 9) quadtratic constraints in the form:

$$\mathbf{c}_i^T \mathbf{c}_j = \delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}, \tag{35}$$

We can write these in matrix form. Let

$$\mathbf{x}_1 = \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \\ \mathbf{r} \end{bmatrix}. \tag{36}$$

Then the six constraints are

$$\mathbf{x}_1^T \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{x}_1 = 1, \tag{37}$$

$$\mathbf{x}_1^T \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{x}_1 = 1, \tag{38}$$

$$\mathbf{x}_1^T \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{x}_1 = 1, \tag{39}$$

$$\mathbf{x}_1^T \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{x}_1 = 0, \tag{40}$$

$$\mathbf{x}_1^T \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{x}_1 = 0, \tag{41}$$

$$\mathbf{x}_1^T \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{x}_1 = 0, \tag{42}$$

$$\tag{43}$$

Next lets deal with the constraint $(\forall k) \quad (\mathbf{I} - \mathbf{v}_k \mathbf{e}_3^T)\mathbf{T}\mathbf{p}_k = \mathbf{0}$. Expand this and add our homogenization variable ω:

$$(\mathbf{I} - \mathbf{v}_k \mathbf{e}_3^T)\mathbf{T}\mathbf{p}_k = \mathbf{0} \quad (44)$$

$$\left(\begin{bmatrix} \omega & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} v_{k1} \\ v_{k2} \\ \omega \\ v_{k4} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \right) \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 & \mathbf{r} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_{k1} \\ p_{k2} \\ p_{k3} \\ 1 \end{bmatrix} = \mathbf{0} \quad (45)$$

$$\begin{bmatrix} \omega & 0 & -v_{k1} & 0 \\ 0 & \omega & -v_{k2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -v_{k4} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 p_{k1} + \mathbf{c}_2 p_{k2} + \mathbf{c}_3 p_{k3} + \mathbf{r} \\ 1 \end{bmatrix} = \mathbf{0} \quad (46)$$

$$(47)$$

This last equation yields four three equality constraints:

$$\omega \mathbf{e}_1^T \mathbf{c}_1 p_{k1} + \omega \mathbf{e}_1^T \mathbf{c}_2 p_{k2} + \omega \mathbf{e}_1^T \mathbf{c}_3 p_{k3} + \omega \mathbf{e}_1^T \mathbf{r} - v_{k1} \mathbf{e}_3^T \mathbf{c}_1 p_{k1} - v_{k1} \mathbf{e}_3^T \mathbf{c}_2 p_{k2} - v_{k1} \mathbf{e}_3^T \mathbf{c}_3 p_{k3} - v_{k1} \mathbf{e}_3^T \mathbf{r} = 0, \quad (48)$$

$$\omega \mathbf{e}_2^T \mathbf{c}_1 p_{k1} + \omega \mathbf{e}_2^T \mathbf{c}_2 p_{k2} + \omega \mathbf{e}_2^T \mathbf{c}_3 p_{k3} + \omega \mathbf{e}_2^T \mathbf{r} - v_{k2} \mathbf{e}_3^T \mathbf{c}_1 p_{k1} - v_{k2} \mathbf{e}_3^T \mathbf{c}_2 p_{k2} - v_{k2} \mathbf{e}_3^T \mathbf{c}_3 p_{k3} - v_{k2} \mathbf{e}_3^T \mathbf{r} = 0, \quad (49)$$

$$v_{k4} \mathbf{e}_3^T \mathbf{c}_1 p_{k1} + v_{k4} \mathbf{e}_3^T \mathbf{c}_2 p_{k2} + v_{k4} \mathbf{e}_3^T \mathbf{c}_3 p_{k3} + v_{k4} \mathbf{e}_3^T \mathbf{r} = 1, \quad (50)$$

$$(51)$$

where we have slightly abused notation, because in the above equation $\mathbf{e}_i \in \mathbb{R}^{3\times 1}$. We can write this in matrix form:

$$\begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \\ \mathbf{r} \\ \mathbf{u}_k \\ \omega \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -p_{k1}\mathbf{e}_1\mathbf{e}_3^T & -p_{k2}\mathbf{e}_1\mathbf{e}_3^T & -p_{k3}\mathbf{e}_1\mathbf{e}_3^T & -\mathbf{e}_1\mathbf{e}_3^T & 0 & 0 \\ p_{k1}\mathbf{e}_1^T & p_{k2}\mathbf{e}_1^T & p_{k3}\mathbf{e}_1^T & \mathbf{e}_1^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \\ \mathbf{r} \\ \mathbf{u}_k \\ \omega \end{bmatrix} = 0 \quad (52)$$

$$\begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \\ \mathbf{r} \\ \mathbf{u}_k \\ \omega \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -p_{k1}\mathbf{e}_2\mathbf{e}_3^T & -p_{k2}\mathbf{e}_2\mathbf{e}_3^T & -p_{k3}\mathbf{e}_2\mathbf{e}_3^T & -\mathbf{e}_2\mathbf{e}_3^T & 0 & 0 \\ p_{k1}\mathbf{e}_2^T & p_{k2}\mathbf{e}_2^T & p_{k3}\mathbf{e}_2^T & \mathbf{e}_2^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \\ \mathbf{r} \\ \mathbf{u}_k \\ \omega \end{bmatrix} = 0 \quad (53)$$

$$\begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \\ \mathbf{r} \\ \mathbf{u}_k \\ \omega \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ p_{k1}\mathbf{e}_3\mathbf{e}_3^T & p_{k2}\mathbf{e}_3\mathbf{e}_3^T & p_{k3}\mathbf{e}_3\mathbf{e}_3^T & \mathbf{e}_3\mathbf{e}_3^T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \\ \mathbf{r} \\ \mathbf{u}_k \\ \omega \end{bmatrix} = 1 \quad (54)$$

1.2 SDP Relaxation

Now that we have the QCQP in the homogenous form

$$\mathbf{x}^* = \operatorname{argmin}_{\mathbf{x}} \quad \mathbf{x}^T \mathbf{Q} \mathbf{x} \tag{55}$$

$$(\forall i) \quad \mathbf{x}^T \mathbf{A}_i \mathbf{x} = b_i, \tag{56}$$

we can easily turn it into an SDP:

$$\mathbf{x}^* = \operatorname{argmin}_{\mathbf{x}} \quad \operatorname{tr}(\mathbf{Q}\mathbf{x}\mathbf{x}^T), \tag{57}$$

$$\text{s.t} \quad (\forall i) \quad \operatorname{tr}(\mathbf{A}_i \mathbf{x}\mathbf{x}^T) = b_i. \tag{58}$$

Let $\mathbf{X} = \mathbf{x}\mathbf{x}^T \in \mathbb{R}^{(14+4n)\times(14+4n)}$, then the above is equivalent to:

$$\mathbf{X}^* = \operatorname{argmin}_{\mathbf{X}} \quad \operatorname{tr}(\mathbf{Q}\mathbf{X}), \tag{59}$$

$$\text{s.t} \quad (\forall i) \quad \operatorname{tr}(\mathbf{A}_i \mathbf{X}) = b_i, \tag{60}$$

$$\mathbf{X} = \mathbf{x}\mathbf{x}^T \tag{61}$$

$$\operatorname{rank}(\mathbf{X}) = 1 \tag{62}$$

We can relax these last two constraints to get an SDP:

$$\mathbf{X}^* = \operatorname{argmin}_{\mathbf{X}} \quad \operatorname{tr}(\mathbf{Q}\mathbf{X}), \tag{63}$$

$$\text{s.t} \quad (\forall i) \quad \operatorname{tr}(\mathbf{A}_i \mathbf{X}) = b_i, \tag{64}$$

$$\mathbf{X} \succeq 0. \tag{65}$$

2 Redundant Constraints

2.1 Parallel Constraints

Note that because $\mathbf{v}_k = \frac{1}{z_k}\mathbf{T}\mathbf{p}_k$, $\mathbf{v}_k$ and $\mathbf{T}\mathbf{p}_k$ are parallel. Further, for any parallel column vectors $\mathbf{u}, \mathbf{w}$, $\exists \alpha \in \mathbb{R}$ such that

$$\mathbf{u} = \alpha\mathbf{w} \implies \mathbf{u}\mathbf{w}^T = \alpha\mathbf{w}\mathbf{w}^T = \mathbf{w}\mathbf{u}^T. \tag{66}$$

Thus, we can right:

$$\mathbf{v}_k(\mathbf{T}\mathbf{p}_k)^T = (\mathbf{T}\mathbf{p}_k)\mathbf{v}_k^T \in \mathbb{R}^{4\times 4}. \tag{67}$$

We want to re-write these 16 scalar equality constraints in the form $\mathbf{x}^T\mathbf{A}\mathbf{x} = b$ for some A, b. Note that

$$\mathbf{v}_k = \begin{bmatrix} 0 & \dots & 0 & \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_4 & 0 & \dots 0 & \mathbf{e}_3 \end{bmatrix} \mathbf{x} = \mathbf{B}_k\mathbf{x}, \tag{68}$$

and

$$\mathbf{T}\mathbf{p}_k = \begin{bmatrix} p_{k1}\mathbf{c}_1 + p_{k2}\mathbf{c}_2 + p_{k3}\mathbf{c}_3 + \mathbf{r} \\ 1 \end{bmatrix} \tag{69}$$

$$= \begin{bmatrix} p_{k1}\mathbf{I} & p_{k2}\mathbf{I} & p_{k3}\mathbf{I} & \mathbf{I} & \mathbf{0} & \dots & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & 1 \end{bmatrix} \mathbf{x} = \mathbf{C}_k\mathbf{x}. \tag{70}$$

Now the i^{th}, j^{th} constraint is

$$\mathbf{e}_i^T\mathbf{v}_k(\mathbf{T}\mathbf{p}_k)^T\mathbf{e}_j = \mathbf{e}_i^T(\mathbf{T}\mathbf{p}_k)\mathbf{v}_k^T\mathbf{e}_j, \forall i, j \in 1, \dots, 4 \tag{71}$$

$$\implies \mathbf{e}_i^T\mathbf{B}_k\mathbf{x}\mathbf{x}^T\mathbf{C}_k^T\mathbf{e}_j = \mathbf{e}_i^T\mathbf{C}_k\mathbf{x}\mathbf{x}^T\mathbf{B}_k^T\mathbf{e}_j. \tag{72}$$

Because $\mathbf{x}^T\mathbf{C}_k^T\mathbf{e}_j$ and $\mathbf{e}_i^T\mathbf{B}_k\mathbf{x}$ are scalars, they commute:

$$\mathbf{x}^T\mathbf{C}_k^T\mathbf{e}_j\mathbf{e}_i^T\mathbf{B}_k\mathbf{x} - \mathbf{x}^T\mathbf{B}_k^T\mathbf{e}_j\mathbf{e}_i^T\mathbf{C}_k\mathbf{x} = 0 \tag{73}$$

$$\implies \mathbf{x}^T(\mathbf{C}_k^T\mathbf{e}_j\mathbf{e}_i^T\mathbf{B}_k - \mathbf{B}_k^T\mathbf{e}_j\mathbf{e}_i^T\mathbf{C}_k)\mathbf{x} = 0 \tag{74}$$

which is of the desired form. Note that when $i = j$, $(\mathbf{C}_k^T\mathbf{e}_j\mathbf{e}_i^T\mathbf{B}_k - \mathbf{B}_k^T\mathbf{e}_j\mathbf{e}_i^T\mathbf{C}_k)$ is skew symmetric, so equation 74 is trivially satisfied. Therefore, we will add $(16-4)K = 12K$ constraints to the SDP.

3 Iterative SDP Method