

Thesis

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1 SDP Relaxation of Stereo Localization Problem

1.1 Re-write as QCQP

$$\mathbf{T}_{\text{cw}} = \operatorname{argmin}_{\mathbf{T}} \sum_k (\mathbf{y}_k - \mathbf{M} \frac{1}{z_k} \mathbf{T} \mathbf{p}_k)^T \mathbf{W}_k (\mathbf{y}_k - \mathbf{M} \frac{1}{z_k} \mathbf{T} \mathbf{p}_k), \quad (1)$$

$$\mathbf{T} \in SE(3), \quad (2)$$

$$z_k = \mathbf{e}_3^T \mathbf{T} \mathbf{p}_k, \quad (3)$$

where $\mathbf{e}_3^T = [0 \ 0 \ 1 \ 0]$. Let $\mathbf{v}_k = \frac{1}{z_k} \mathbf{T} \mathbf{p}_k$:

$$\mathbf{v}_k z_k = \mathbf{T} \mathbf{p}_k \quad (4)$$

$$\mathbf{v}_k \mathbf{e}_3^T \mathbf{T} \mathbf{p}_k = \mathbf{T} \mathbf{p}_k \quad (5)$$

$$(\mathbf{I} - \mathbf{v}_k \mathbf{e}_3^T) \mathbf{T} \mathbf{p}_k = \mathbf{0} \in \mathbb{R}^4 \quad (6)$$

Now we can re-write our optimization problem as

$$\mathbf{T}_{\text{cw}} = \operatorname{argmin}_{\mathbf{T}, \mathbf{v}_k} \sum_k (\mathbf{y}_k - \mathbf{M} \mathbf{v}_k)^T \mathbf{W}_k (\mathbf{y}_k - \mathbf{M} \mathbf{v}_k), \quad (7)$$

$$\mathbf{T} \in SE(3), \quad (8)$$

$$(\forall k) \quad (\mathbf{I} - \mathbf{v}_k \mathbf{e}_3^T) \mathbf{T} \mathbf{p}_k = \mathbf{0} \quad (9)$$

We want to write this in the standard, homogenous, form

$$\mathbf{x}^* = \operatorname{argmin}_{\mathbf{x}} \quad \mathbf{x}^T \mathbf{Q} \mathbf{x} \quad (10)$$

$$(\forall i) \quad \mathbf{x}^T \mathbf{A}_i \mathbf{x} = b_i. \quad (11)$$

1.1.1 Cost

We will start with the cost, adding the homogenization variable ω :

$$\mathbf{T}_{\mathbf{c}\mathbf{w}} = \operatorname{argmin}_{\mathbf{T}, \mathbf{v}_k, \omega} \sum_k (\omega \mathbf{y}_k - \mathbf{M} \mathbf{v}_k)^T \mathbf{W}_k (\omega \mathbf{y}_k - \mathbf{M} \mathbf{v}_k), \quad (12)$$

$$\mathbf{T} \in SE(3), \quad (13)$$

$$(\forall k) \quad (\mathbf{I} - \mathbf{v}_k \mathbf{a}^T) \mathbf{T} \mathbf{p}_k = \mathbf{0} \quad (14)$$

$$\omega^2 = 1. \quad (15)$$

We denote

$$\mathbf{v}_k = \frac{\mathbf{T} \mathbf{p}_k}{\mathbf{e}_3^T \mathbf{T} \mathbf{p}_k} = \begin{bmatrix} v_{k1} \\ v_{k2} \\ 1 \\ v_{k4} \end{bmatrix} = \begin{bmatrix} v_{k1} \\ v_{k2} \\ \omega \\ v_{k4} \end{bmatrix}. \quad (16)$$

We add the homogenization variable so the cost and constraints remain quadratic.

Expanding the term in the sum:

$$(\omega \mathbf{y}_k - \mathbf{M} \mathbf{v}_k)^T \mathbf{W}_k (\omega \mathbf{y}_k - \mathbf{M} \mathbf{v}_k) = \quad (17)$$

$$\omega^2 \mathbf{y}_k^T \mathbf{W}_k \mathbf{y}_k - \omega \mathbf{y}_k^T \mathbf{W}_k \mathbf{M} \mathbf{v}_k - \omega \mathbf{v}_k^T \mathbf{M}^T \mathbf{W}_k \mathbf{y}_k + \mathbf{v}_k^T \mathbf{M}^T \mathbf{W}_k \mathbf{M} \mathbf{v}_k \quad (18)$$

Let

$$\mathbf{E} = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_4] \in \mathbb{R}^{4 \times 3}, \quad (19)$$

and

$$\mathbf{u}_k = \begin{bmatrix} v_{k1} \\ v_{k2} \\ v_{k4} \end{bmatrix}, \quad (20)$$

where $\mathbf{e}_i \in \mathbb{R}^{4 \times 1}$ is all zeros except for the i^{th} entry which is 1. We can then re-write \mathbf{v}_k :

$$\mathbf{v}_k = \mathbf{E} \mathbf{u}_k + \mathbf{e}_3 \omega, \quad (21)$$

and use this to expand equation 18:

$$\omega^2 \mathbf{y}_k^T \mathbf{W}_k \mathbf{y}_k \quad (22)$$

$$- \omega \mathbf{y}_k^T \mathbf{W}_k \mathbf{M} \mathbf{E} \mathbf{u}_k - \mathbf{y}_k^T \mathbf{W}_k \mathbf{M} \mathbf{e}_3 \omega^2 \quad (23)$$

$$- \omega \mathbf{u}_k^T \mathbf{E}^T \mathbf{M}^T \mathbf{W}_k \mathbf{y}_k - \mathbf{e}_3^T \mathbf{M}^T \mathbf{W}_k \mathbf{y}_k \omega^2 \quad (24)$$

$$+ \mathbf{u}_k^T \mathbf{E}^T \mathbf{M}^T \mathbf{W}_k \mathbf{M} \mathbf{E} \mathbf{u}_k + \omega \mathbf{e}_3^T \mathbf{M}^T \mathbf{W}_k \mathbf{M} \mathbf{E} \mathbf{u}_k + \mathbf{u}_k^T \mathbf{E}^T \mathbf{M}^T \mathbf{W}_k \mathbf{M} \mathbf{e}_3 \omega + \mathbf{e}_3^T \mathbf{M}^T \mathbf{W}_k \mathbf{M} \mathbf{e}_3 \omega^2. \quad (25)$$

Now we can write the cost in matrix form:

$$\mathbf{x}_2^T \begin{bmatrix} \mathbf{H}_1 & 0 & \dots & 0 & \mathbf{g}_1 \\ 0 & \mathbf{H}_2 & \dots & 0 & \mathbf{g}_2 \\ \dots & & & & \\ 0 & 0 & \dots & \mathbf{H}_N & \mathbf{g}_N \\ \mathbf{g}_1^T & \mathbf{g}_2^T & \dots & \mathbf{g}_N^T & \boldsymbol{\Omega} \end{bmatrix} \mathbf{x}_2 \quad (26)$$

where

$$\mathbf{x}_2 = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \dots \\ \mathbf{u}_N \\ \omega \end{bmatrix}, \quad (27)$$

$$\boldsymbol{\Omega} = \sum_k (\mathbf{y}_k^T \mathbf{W}_k \mathbf{y}_k - \mathbf{y}_k^T \mathbf{W}_k \mathbf{M} \mathbf{e}_3 - \mathbf{e}_3^T \mathbf{M}^T \mathbf{W}_k \mathbf{y}_k + \mathbf{e}_3^T \mathbf{M}^T \mathbf{W}_k \mathbf{M} \mathbf{e}_3) \in \mathbb{R}, \quad (28)$$

$$\mathbf{g}_k = -\mathbf{E}^T \mathbf{M}^T \mathbf{W}_k \mathbf{y}_k + \mathbf{E}^T \mathbf{M}^T \mathbf{W}_k \mathbf{M} \mathbf{e}_3 \in \mathbb{R}^{3 \times 1}, \quad (29)$$

$$\mathbf{H}_k = \mathbf{E}^T \mathbf{M}^T \mathbf{W}_k \mathbf{M} \mathbf{E} \in \mathbb{R}^{3 \times 3} \quad (30)$$

1.1.2 Constraints

Lets begin with the constraint $\mathbf{T} \in SE(3)$. Let

$$\mathbf{T} = \begin{bmatrix} \mathbf{C} & \mathbf{r} \\ \mathbf{0}^T & 1 \end{bmatrix}, \quad \mathbf{C} \in SO(3). \quad (31)$$

$$\mathbf{C}^T \mathbf{C} = \mathbf{I} \quad \text{and} \quad \det(\mathbf{C}) = 1. \quad (32)$$

We will drop the determinate constraint. Write

$$\mathbf{C} = [\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{c}_3] \quad (33)$$

$$\mathbf{C}^T \mathbf{C} = \begin{bmatrix} \mathbf{c}_1^T \\ \mathbf{c}_2^T \\ \mathbf{c}_3^T \end{bmatrix} [\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{c}_3] = \mathbf{I} \quad (34)$$

This implies 6 (due to symmetry we don't need all 9) quadratic constraints in the form:

$$\mathbf{c}_i^T \mathbf{c}_j = \delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}, \quad (35)$$

We can write these in matrix form. Let

$$\mathbf{x}_1 = \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \\ \mathbf{r} \end{bmatrix}. \quad (36)$$

Then the six constraints are

$$\mathbf{x}_1^T \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{x}_1 = 1, \quad (37)$$

$$\mathbf{x}_1^T \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{x}_1 = 1, \quad (38)$$

$$\mathbf{x}_1^T \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{x}_1 = 1, \quad (39)$$

$$\mathbf{x}_1^T \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{x}_1 = 1, \quad (40)$$

$$\mathbf{x}_1^T \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{x}_1 = 1, \quad (41)$$

$$\mathbf{x}_1^T \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{x}_1 = 1, \quad (42)$$

$$(43)$$

Next lets deal with the constraint $(\forall k) \quad (\mathbf{I} - \mathbf{v}_k \mathbf{e}_3^T) \mathbf{T} \mathbf{p}_k = \mathbf{0}$. Expand this

and add our homogenization variable ω :

$$(\mathbf{I} - \mathbf{v}_k \mathbf{e}_3^T) \mathbf{T} \mathbf{p}_k = \mathbf{0} \quad (44)$$

$$\left(\begin{bmatrix} \omega & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} v_{k1} \\ v_{k2} \\ \omega \\ v_{k4} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \right) \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 & \mathbf{r} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_{k1} \\ p_{k2} \\ p_{k3} \\ 1 \end{bmatrix} = \mathbf{0} \quad (45)$$

$$\begin{bmatrix} \omega & 0 & -v_{k1} & 0 \\ 0 & \omega & -v_{k2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -v_{k4} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 p_{k1} + \mathbf{c}_2 p_{k2} + \mathbf{c}_3 p_{k3} + \mathbf{r} \\ 1 \end{bmatrix} = \mathbf{0} \quad (46)$$

$$(47)$$

This last equation yields four three equality constraints:

$$\omega \mathbf{e}_1^T \mathbf{c}_1 p_{k1} + \omega \mathbf{e}_1^T \mathbf{c}_2 p_{k2} + \omega \mathbf{e}_1^T \mathbf{c}_3 p_{k3} + \omega \mathbf{e}_1^T \mathbf{r} - v_{k1} \mathbf{e}_3^T \mathbf{c}_1 p_{k1} - v_{k1} \mathbf{e}_3^T \mathbf{c}_2 p_{k2} - v_{k1} \mathbf{e}_3^T \mathbf{c}_3 p_{k3} - v_{k1} \mathbf{e}_3^T \mathbf{r} = 0, \quad (48)$$

$$\omega \mathbf{e}_2^T \mathbf{c}_1 p_{k1} + \omega \mathbf{e}_2^T \mathbf{c}_2 p_{k2} + \omega \mathbf{e}_2^T \mathbf{c}_3 p_{k3} + \omega \mathbf{e}_2^T \mathbf{r} - v_{k2} \mathbf{e}_3^T \mathbf{c}_1 p_{k1} - v_{k2} \mathbf{e}_3^T \mathbf{c}_2 p_{k2} - v_{k2} \mathbf{e}_3^T \mathbf{c}_3 p_{k3} - v_{k2} \mathbf{e}_3^T \mathbf{r} = 0, \quad (49)$$

$$v_{k4} \mathbf{e}_3^T \mathbf{c}_1 p_{k1} + v_{k4} \mathbf{e}_3^T \mathbf{c}_2 p_{k2} + v_{k4} \mathbf{e}_3^T \mathbf{c}_3 p_{k3} + v_{k4} \mathbf{e}_3^T \mathbf{r} = 1, \quad (50)$$

$$(51)$$

where we have slightly abused notation, because in the above equation $\mathbf{e}_i \in \mathbb{R}^{3 \times 1}$. We can write this in matrix form:

$$\begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \\ \mathbf{r} \\ \mathbf{u}_k \\ \omega \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -p_{k1} \mathbf{e}_1 \mathbf{e}_3^T & -p_{k2} \mathbf{e}_1 \mathbf{e}_3^T & -p_{k3} \mathbf{e}_1 \mathbf{e}_3^T & -\mathbf{e}_1 \mathbf{e}_3^T & 0 & 0 \\ p_{k1} \mathbf{e}_1^T & p_{k2} \mathbf{e}_1^T & p_{k3} \mathbf{e}_1^T & \mathbf{e}_1^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \\ \mathbf{r} \\ \mathbf{u}_k \\ \omega \end{bmatrix} = 0 \quad (52)$$

$$\begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \\ \mathbf{r} \\ \mathbf{u}_k \\ \omega \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -p_{k1} \mathbf{e}_2 \mathbf{e}_3^T & -p_{k2} \mathbf{e}_2 \mathbf{e}_3^T & -p_{k3} \mathbf{e}_2 \mathbf{e}_3^T & -\mathbf{e}_2 \mathbf{e}_3^T & 0 & 0 \\ p_{k1} \mathbf{e}_2^T & p_{k2} \mathbf{e}_2^T & p_{k3} \mathbf{e}_2^T & \mathbf{e}_2^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \\ \mathbf{r} \\ \mathbf{u}_k \\ \omega \end{bmatrix} = 0 \quad (53)$$

$$\begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \\ \mathbf{r} \\ \mathbf{u}_k \\ \omega \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ p_{k1} \mathbf{e}_3 \mathbf{e}_3^T & p_{k2} \mathbf{e}_3 \mathbf{e}_3^T & p_{k3} \mathbf{e}_3 \mathbf{e}_3^T & \mathbf{e}_3 \mathbf{e}_3^T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \\ \mathbf{r} \\ \mathbf{u}_k \\ \omega \end{bmatrix} = 1 \quad (54)$$

1.2 SDP Relaxation

Now that we have the QCQP in the homogenous form

$$\mathbf{x}^* = \operatorname{argmin}_{\mathbf{x}} \quad \mathbf{x}^T \mathbf{Q} \mathbf{x} \quad (55)$$

$$(\forall i) \quad \mathbf{x}^T \mathbf{A}_i \mathbf{x} = b_i, \quad (56)$$

we can easily turn it into an SDP:

$$\mathbf{x}^* = \operatorname{argmin}_{\mathbf{x}} \quad \operatorname{tr}(\mathbf{Q} \mathbf{x} \mathbf{x}^T), \quad (57)$$

$$\text{s.t.} \quad (\forall i) \quad \operatorname{tr}(\mathbf{A}_i \mathbf{x} \mathbf{x}^T) = b_i. \quad (58)$$

Let $\mathbf{X} = \mathbf{x} \mathbf{x}^T \in \mathbb{R}^{(14+4n) \times (14+4n)}$, then the above is equivalent to:

$$\mathbf{X}^* = \operatorname{argmin}_{\mathbf{X}} \quad \operatorname{tr}(\mathbf{Q} \mathbf{X}), \quad (59)$$

$$\text{s.t.} \quad (\forall i) \quad \operatorname{tr}(\mathbf{A}_i \mathbf{X}) = b_i, \quad (60)$$

$$\mathbf{X} = \mathbf{x} \mathbf{x}^T \quad (61)$$

$$\operatorname{rank}(\mathbf{X}) = 1 \quad (62)$$

We can relax these last two constraints to get an SDP:

$$\mathbf{X}^* = \operatorname{argmin}_{\mathbf{X}} \quad \operatorname{tr}(\mathbf{Q} \mathbf{X}), \quad (63)$$

$$\text{s.t.} \quad (\forall i) \quad \operatorname{tr}(\mathbf{A}_i \mathbf{X}) = b_i, \quad (64)$$

$$\mathbf{X} \succeq 0. \quad (65)$$