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Abstract

- Linear viscoelastic models are not able to fit full strain rate uniaxial loading data for bone well [1]. Furthermore, biomechanics and the characterization of biological materials is rapidly evolving. As biological solids are impregnated with biological fluids which typically display non-Newtonian behaviour it stands to reason that many biological materials which are yet to be mechanically characterized could display non-linear viscoelastic behaviour. Hence, the existence of a flexible framework for non-linear viscoelasticity is an imperative in the field of biomechanics.
- A framework of non-linear viscoelasticity in an infinitesimal setting does not appear to exist in the literature and so one is developed here.
- A generalized Maxwell model is adopted without *a priori* assumption of the constitutive behaviour of the components in the Maxwell elements.
- The crux is to enforce that viscous and elastic stresses in the dashpot and spring, respectively, are equal. In an incremental (temporally discretized) setting this is achieved by constructing a residual function and applying local (integration point level) Newton-Raphson iterations to determine the viscous strain (that is, the strain of the dashpot) at the next iteration such that the viscous and elastic stresses are equal.
- This framework can be extended to finite strains and possesses some advantages over current finite strain non-linear viscoelastic models [2]. In particular, the model need not be formulated in terms of principal stretches.

Nomenclature

Number sets

$\mathbb{C}$ Complex numbers

$\mathbb{H}$ Quaternions

$\mathbb{R}$ Real numbers

Other symbols

ρ Friction index

V Constant volume

Physics constants

c Speed of light in a vacuum

G Gravitational constant

h Planck constant

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Instant strain

Constant strain rate

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3.5 Some models and their behaviour

We focus on the viscous component of the model assuming linear behaviour for the elastic component. However, linear behaviour of the elastic component is not a requirement for the framework presented in Section 3.4. We take inspiration from non-Newtonian models commonly used for biological fluids.

3.5.1 Linear viscoelasticity

3.5.2 Ostwald-de Waele model

3.5.3 Modified Ostwald-de Waele model

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4.1 Inverse FE to fit models??

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References

[1] T. J. Cloete, G. Paul, and E. B. Ismail. "Hopkinson bar techniques for the intermediate strain rate testing of bovine cortical bone". In: *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences* 372.2015 (2014). ISSN: 1364503X. DOI: 10.1098/rsta.2013.0210.

[2] Stefanie Reese and Sanjay Govindjee. "A theory of finite viscoelasticity and numerical aspects". In: *International Journal of Solids and Structures* 35.26-27 (1998), pp. 3455–3482. ISSN: 00207683. DOI: 10.1016/s0020-7683(97)00217-5.

A Other temporal approximations

B Why it's incorrect to simply insert a new viscosity at each time-step

C Some more detailed maths (maybe)

D Extension to finite strain