1 A framework for nonlinear generalized 1D rheological Maxwell networks

1.1 Kinematics and kinetics

Consider the network diagram illustrating the viscoelastic behaviour of a 1D constitutive model in Figure 1. The viscoelastic network

Figure 1:

The network diagram implies several axioms for the way in which the components interact:

1. The stress of sequential components is equal;
2. The stress of parallel components is added;
3. The strain of sequential components is added;
4. The strain of parallel components is equal.

Mathematically, these axioms assert that the following equations define the overall constitutive model and that each equation must be satisfied for the constitutive model to be self-consistent:

$$\sigma_i^s = \sigma_i^d = \sigma_i^e \,, \tag{1}$$

$$\sigma = \sigma^b + \sum_i^n \sigma_i^e \,, \tag{2}$$

$$\varepsilon_i^e = \varepsilon_i^d + \varepsilon_i^s \,, \tag{3}$$

$$\varepsilon = \varepsilon_i^e \,. \tag{4}$$

Note that, for the kinematic state of the network to be completely defined, that is, for the strain of each component to be defined ε, ε_i^d, ε_i^s, ε_i^e, $i = 1, ..., n$, one needs to define ε and ε_i^d. Then, the other strain values are defined by equations (3) and (4). Hence, we identify ε_i^d as a state variable.

$$\varepsilon = \varepsilon^d + \varepsilon^s \,. \tag{5}$$

$$\dot{\varepsilon} = \dot{\varepsilon}^d + \dot{\varepsilon}^s \,. \tag{6}$$

1.2 Thermodynamic considerations

Rate of work done on each Maxwell branch

$$\dot{w}_i^e = \sigma_i^e \dot{\varepsilon} = \underbrace{\sigma_i^s \left[\dot{\varepsilon} - \dot{\varepsilon}_i^d\right]}_{\dot{\Psi}_i^e} + \underbrace{\sigma_i^d \dot{\varepsilon}_i^d}_{\dot{\Phi}_i^e} \tag{7}$$

Axioms for the sub-constitutive models

1. $\sigma_i^s(\varepsilon_i^s)$, $\sigma_i^e(0) = 0$
2. $\sigma_i^d\left(\dot{\varepsilon}_i^d\right)$, $\sigma_i^d\left(0\right) = 0$

$$\frac{\mathrm{d}\Psi_i^e}{\mathrm{d}\varepsilon^s}\dot{\varepsilon}^s = \sigma_i^s\left[\dot{\varepsilon} - \dot{\varepsilon}_i^d\right] \Rightarrow \left[\frac{\mathrm{d}\Psi_i^e}{\mathrm{d}\varepsilon^s} - \sigma_i^s\right]\dot{\varepsilon}^s = 0 \Rightarrow \sigma_i^s = \frac{\mathrm{d}\Psi_i^e}{\mathrm{d}\varepsilon^s} \tag{8}$$

$$D = \sigma_i^e \dot{\varepsilon} - \dot{\Psi}_i^e \geq 0 \tag{9}$$

$$D = \sigma_i^d \dot{\varepsilon}_i^d = \dot{\Phi}_i^e \geq 0 \tag{10}$$

$$\dot{\Psi} = \dot{\Psi}^b + \sum_i^n \dot{\Psi}_i^e \tag{11}$$

1.3 Framework allowing for arbitrary choice of sub-constitutive models

Partition of the time domain

$$0 = t_0 < t_1 < ... < t_n < t_{n+1} = t \tag{12}$$

Define

$$\Delta t_{n+1} = t_{n+1} - t_n\,, \qquad t_{n+0.5} = \frac{1}{2}\left[t_{n+1} + t_n\right]\,. \tag{13}$$

$$\dot{\varepsilon}_{n+1}^d = \frac{1}{\Delta t_{n+1}}\left[\varepsilon_{n+1}^d - \varepsilon_n^d\right] \tag{14}$$

$$\dot{\varepsilon}_{n+1} = \frac{1}{\Delta t_{n+1}}\left[\varepsilon_{n+1} - \varepsilon_n\right] \tag{15}$$

$$\sigma^s(\varepsilon_{n+1} - \varepsilon_{n+1}^d) = \sigma^d\left(\dot{\varepsilon}_{n+1}^d\right) \tag{16}$$

$$r = \sigma^d\left(\dot{\varepsilon}_{n+1}^d\right) - \sigma^s(\varepsilon_{n+1} - \varepsilon_{n+1}^d) \tag{17}$$

$$L[r] = r_i + \frac{\mathrm{d}r}{\mathrm{d}\varepsilon^d_{n+1(i)}} \Delta\varepsilon^d_{n+1(i)} \tag{18}$$

$$\varepsilon^d_{n+1(i+1)} = \varepsilon^d_{n+1(i)} - \left[\frac{\mathrm{d}r}{\mathrm{d}\varepsilon^d_{n+1}}\right]^{-1}_i r_i \tag{19}$$

$$\frac{\mathrm{d}r}{\mathrm{d}\varepsilon^d_{n+1}} = \frac{\mathrm{d}\sigma^d}{\mathrm{d}\dot{\varepsilon}^d} \frac{\mathrm{d}}{\mathrm{d}\varepsilon^d_{n+1}} \left[\frac{1}{\Delta t_{n+1}} \left[\varepsilon^d_{n+1} - \varepsilon^d_n\right]\right] - \frac{\mathrm{d}\sigma^s}{\mathrm{d}\varepsilon^s} \frac{\mathrm{d}}{\mathrm{d}\varepsilon^d_{n+1}} \left[\varepsilon_{n+1} - \varepsilon^d_{n+1}\right] \tag{20}$$

$$\frac{\mathrm{d}r}{\mathrm{d}\varepsilon^d_{n+1}} = \frac{1}{\Delta t_{n+1}} \frac{\mathrm{d}\sigma^d}{\mathrm{d}\dot{\varepsilon}^d} + \frac{\mathrm{d}\sigma^s}{\mathrm{d}\varepsilon^s} \tag{21}$$

1.4 Some constitutive models

1.4.1 Linear models

$$\sigma^\infty = E^\infty \varepsilon \tag{22}$$

$$\sigma^s = E^s \left[\varepsilon - \varepsilon^d\right], \qquad \sigma^d = \eta \dot{\varepsilon}^d . \tag{23}$$

$\sigma^s = \sigma^d \Rightarrow$

$$E^s \left[\varepsilon - \varepsilon^d\right] = \eta \dot{\varepsilon}^d \tag{24}$$

$$\tau = \frac{\eta}{E^s} \tag{25}$$

$$\frac{1}{\tau}\varepsilon = \frac{1}{\tau}\varepsilon^d + \dot{\varepsilon}^d \tag{26}$$

Integrating factor:

$$\exp\left(\int \frac{1}{\tau}\,\mathrm{d}t\right) = \exp\left(\frac{t}{\tau}\right) \tag{27}$$

$$\frac{1}{\tau}\exp\left(\frac{t}{\tau}\right)\varepsilon = \frac{\mathrm{d}}{\mathrm{d}t}\left[\exp\left(\frac{t}{\tau}\right)\varepsilon^d\right] \tag{28}$$

$$\varepsilon^d = \int_{-\infty}^{t} \frac{1}{\tau}\exp\left(\frac{s-t}{\tau}\right)\varepsilon\,\mathrm{d}s \tag{29}$$

Integrate by parts ...