Abstract

A numerical framework for modelling viscoelastic solids with varying viscosity, henceforth referred to as nonlinear viscoelastic solids, is proposed. Such models can predict a larger set of material responses than linear viscoelastic models. In particular, nonlinear viscoelastic models are useful for capturing the behaviour of biological materials impregnated with non-Newtonian fluids. For example, linear viscoelastic models are unable to accurately capture the stress response of bone over a large range of strain rates; nonlinear viscoelastic models remedy this. In addition to presenting the modelling framework, several examples of nonlinear models are presented and fitted to data from axial loading of bovine cortical bone test specimens at different rates. The framework is introduced in a small strain 1D setting and extended to a finite strain 3D setting.

Viscoelastic materials are frequently modelled as a Standard Linear Solid (SLS) consisting of springs and a dashpot, as illustrated in Figure 1. An SLS implies the following constitutive

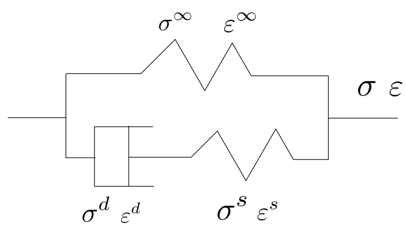


Figure 1: Schematic of the Standard Linear (viscoelstic) Solid (SLS). TODO: 1) make figure look better

relation for the stress:

$$\sigma = \hat{\sigma}\left(\varepsilon, \varepsilon^d\right) = \hat{\sigma}^\infty\left(\varepsilon\right) + \hat{\sigma}^s\left(\varepsilon - \varepsilon^d\right), \tag{1}$$

which is accompanied by the evolution equation

$$\hat{\sigma}^s\left(\varepsilon - \varepsilon^d\right) = \hat{\sigma}^d\left(\dot{\varepsilon}^d\right), \tag{2}$$

that defines the evolution of the internal (state) variable ε^d. Here, $\hat{\bullet}$ denotes a function, $\bullet^s$ and $\bullet^d$ denote spring and dashpot values, respectively, $\bullet^\infty$ denotes long time scale values, σ denotes stress, and ε denotes strain.

Throughout this contribution we use the following "sub-constitutive models" for the springs:

$$\hat{\sigma}^s\left(\varepsilon^s\right) = E^s\varepsilon^s, \tag{3}$$

$$\hat{\sigma}^\infty\left(\varepsilon\right) = E^\infty\varepsilon, \tag{4}$$

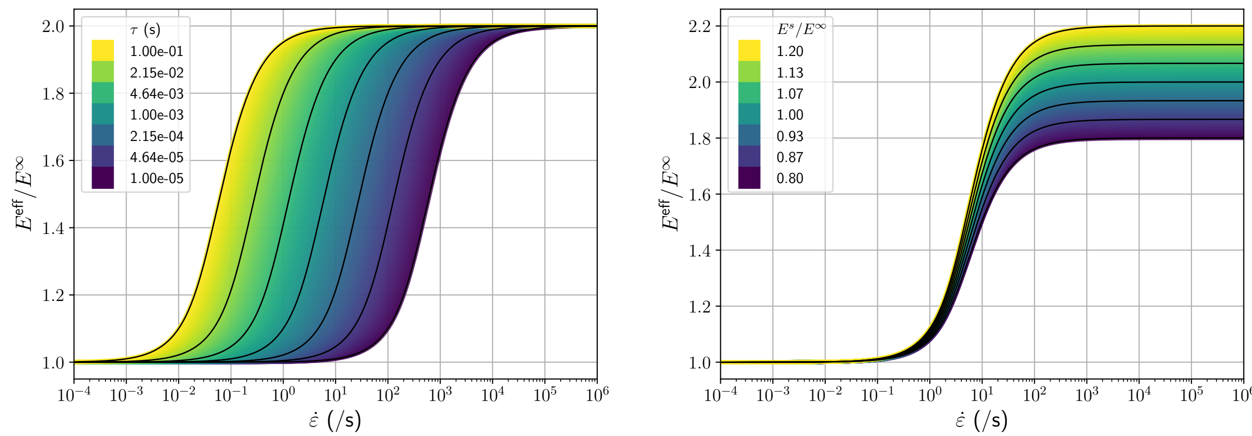
where E denotes a Young's modulus-like material parameter. If one chooses a linear sub-constitutive model for the damper, that is,

$$\hat{\sigma}^d\left(\dot{\varepsilon}^d\right) = \eta\dot{\varepsilon}^d, \tag{5}$$

where η is a constant viscosity-like material parameter, then a closed for solution to the evolution equation (2) can be obtained. This leads to the following equation for the stress:

$$\sigma = \int_{-\infty}^{t} \left[E^{\infty} + \exp\left(\frac{s-t}{\tau}\right) E^{s} \right] \dot{\varepsilon} \, \mathrm{d}s \,. \tag{6}$$

Here, t is time and $\tau = \eta / E^s$ is the "relaxation time". The effective stiffness, $E^{\text{eff}} = \sigma/\varepsilon$, observed during constant strain rate loading at a strain of $\varepsilon = 0.02\,(\text{mm/mm})$ is presented as a function of applied strain rate in Figure 2 for varying values of material parameters. The characteristic shape of the curves motivates the term "S-curves" which is used henceforth.



(a) (b)

Figure 2: Effective stiffness at a strain of $\varepsilon = 0.02\,(\text{mm/mm})$ of an SLS with constant viscosity during constant strain-rate loading for (a) different relaxation times τ with $E^s = E^{\infty}$ and (b) different ratios of E^s/E^{∞} with $\tau = 1 \times 10^{-3}\,$s.

The S-curves for linear viscoelasticity can be translated left and right by changing τ (Figure 2 (a)) and can be scaled vertically by changing E^s (Figure 2 (b)). However, the curves cannot be scaled horizontally. This proves problematic in the case of biological materials, such as bone, where a 'steeper' S-curve is observed in experimental data.

Horizontal scaling of the S-curve can be achieved by using a dashpot with varying viscosity. An example is furnished by taking inspiration from the Ostwald–de Waele model for non-Newtonian fluids; that is,

$$\begin{aligned} \hat{\sigma}^d\left(\dot{\varepsilon}^d\right) &= \hat{\eta}\left(\dot{\epsilon}^d\right)\dot{\epsilon}^d \,, \\ \hat{\eta}\left(\dot{\epsilon}^d\right) &= \eta_0 \left|\dot{\epsilon}^d\right|^{n-1} \,. \end{aligned} \tag{7}$$

Here, η_0 and n are material parameters. With such a model, S-curves can be scaled horizontally by changing n (Figure 3).

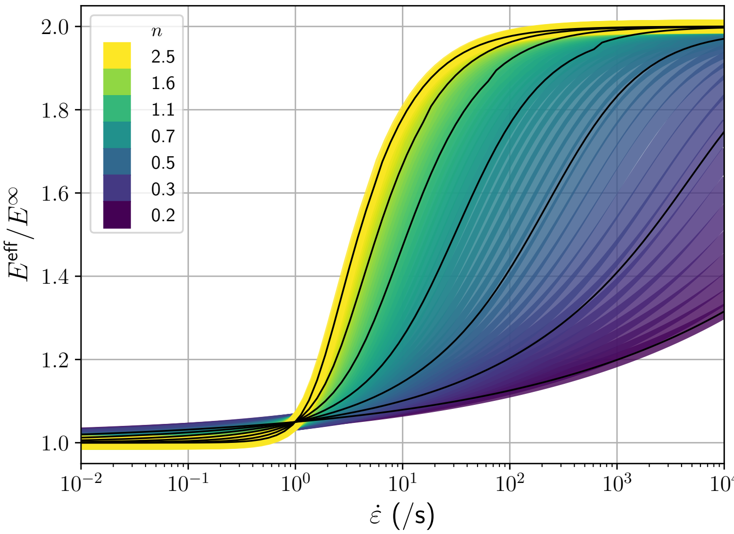


Figure 3: Material response obtained using equation (7). The "steepness" of the S-curve is altered by varying n.

However, finding a closed for solution for the evolution equation (2) with the sub-constitutive model in equation (7) is non-trivial. Furthermore, it is beneficial to have a computational framework that allows for postulating various sub-constitutive models, such as equation (7), without needing to find a closed form solution to the evolution equation for each model.

Such a framework is constructed by discretizing the time domain and using Newton-Raphson's method to solve for the value of the internal variable ε^d that satisfies equation (2) at each time step. Details of the procedure will be provided in the presentation. Additionally, various non-linear viscoelastic models fitted to bone data and an extension of the framework to a 3D finite strain setting will be presented.