Chapter 1

A Library of Functions

1.1 Functions and Change

function – An item used to represent the dependence of one quantity upon another. $f(x)$ means f is a function of x.

domain – The input values of a function (x part of $f(x)$). Also referred to as the ***independent variable***.

range – The output values of a function (y part of $y = f(x)$). Also referred to as the *dependent variable* because it depends on x. Some variables assume discrete values, while others are continuous. Some examples of discrete variables are listed below.

1. Date
2. Cost
3. Number of ...

Note that some quantities, such as "Date" in the above list, are discretized values from a continuous variable, time. Measurement of a continuous variable results in a set of discrete values.

Domain and Range values are often written using ***Interval Notation***. This notation is used to describe the extrema of a set of numbers. The following cases are used to describe numeric sets using this notation:

$$a \leq x \leq b = [a, b] \tag{1.1}$$

$$a \leq x < b = [a, b) \tag{1.2}$$

$$a < x \leq b = (a, b] \tag{1.3}$$

$$a < x < b = (a, b) \tag{1.4}$$

If the domain is not specified, it is usually assumed to be the the set of real numbers, that is $x \in \mathbb{R}$.

Linear Function – A function is linear if the ***slope***, or rate of change of the function, is constant. ***slope*** – The rate that the dependent variable changes with respect to the independent variable.

With this, certain assumptions about the function hold:

1. The Principle of Superposition applies
2. Highest degree of function is 1

The Greek letter $\boldsymbol{\Delta}$ is used to indicate "change in" a particular variable; thus, $\boldsymbol{\Delta}x$ means "Change in x." This is commonly used to express the ***slope*** of a function, m:

$$m = \frac{Rise}{Run} = \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \tag{1.5}$$

If the magnitude of $f(x)$ increases as x increases, then $f(x)$ is classified as an ***Increasing Function***. Contrarily, if the magnitude of $f(x)$ decreases as x increases, then $f(x)$ is classified as a ***Decreasing Function***. If this is true $\forall x$ (for all values of x), then the function can also be classified as ***monotonic***.

Examples

Coming soon!™

1.2 Exponential Functions

Exponential Functions are a class of functions that can be described by:

$$P = P_0 a^t \tag{1.6}$$

where P_0 is some initial quantity (value at $t = 0$), and a is the factor by which P changes when t is increased by 1. If $a > 1$, then exponential growth occurs. If $0 < a < 1$, then exponential decay is present. These functions can also be described by their concavity. If the "opening" of the function points toward the positive Y-axis, then the function is ***concave up***. Similarly, if it points toward the negative Y-axis, then the function is ***concave down***.

Half-Life – The time required for an exponentially decaying quantity to reach 50% of the initial value, P_0.

Doubling Time – The time required for an exponentially growing quantity to reach 200% of the initial value, P_0.

Some exponential functions use the natural number, $e \approx 2.71828$ as the base of the exponential quantity. Because this ensures the base is positive, then exponential growth may also occur when:

$$P = P_0 e^{kt} \tag{1.7}$$

and exponential decay may occur when:

$$P = P_0 e^{-kt} \tag{1.8}$$

This is true if $k > 0$ and $t > 0$.

Examples

Coming soon!™

1.3 New Functions from Old

Translations of functions occur though additive operations. ***Stretches*** occur when a function is multiplied by a factor, k, when k is not 0 or 1. ***Reflections*** occur when a function is multiplied by a factor, k, when $k < 0$.

Composite Functions are functions that depend on quantities that can be described by other functions. The example given in the text is the following:

$$A = f(r) = \pi r^2$$
$$r = g(t) = 1 + t$$
$$A = \pi r^2 = \pi (1+t)^2$$
$$A = f(g(t)) = \pi (g(t))^2 = \pi (1+t)^2$$

Here, the area of a circle, which is mathematically defined by πr^2, is used as the *outside function*. $r(t)$ is the *inside function* describes the radius as a ***monotonically increasing*** quantity with time. Therefore, the area can also be described in terms of time using the definition of the ***composite function***.

A function can be classified as ***even*** or ***odd*** if it is symmetrical about the Y-axis. ***Even*** functions have the property of:

$$f(-x) = f(x) \quad \forall x \tag{1.9}$$

An example of this type of function is $\cos(x)$. On the other hand, ***odd*** functions follow:

$$f(-x) = -f(x) \quad \forall x \tag{1.10}$$

An example of this type of function is $\sin(x)$. Many functions are neither ***even*** nor ***odd***.

For a given function $f(x)$, the inverse of the function, $f^{-1}(x)$, is given by:

$$f^{-1}(y) = x \quad == \quad y = f(x) \tag{1.11}$$

Not all functions are invertible. For this to be true, $f(x)$ must be *single-valued*, or each y value uniquely corresponds to one x value. More specifically, a function has an inverse if, and only if, it intersects any horizontal line at most once. Thus, lines of constant y values cannot correspond to multiple x values for the inverse to exist.

Examples

Coming soon!™

1.4 Logarithmic Functions

Logarithmic Functions are the inverse of an exponential function, provided both functions share a common base. These functions are written out as:

$$\log_{10} x = k \quad == \quad 10^k = x \tag{1.12}$$

where k is a real number. Here, 10 is the common base of the logarithmic and exponential functions. The ***Natural Logarithm*** uses the natural number, e, as its base. It is written as:

$$\ln x = k \quad == \quad e^c = x \tag{1.13}$$

In logarithmic functions, $x > 0$ because no power of a real number results in zero, and negative values of x are infeasible for positive bases with real exponents.

The following table outlines the properties of ***logarithmic functions***:

	Base A	**Natural Logarithm**
1.	$\log(AB) = \log A + \log B$	$\ln(AB) = \ln A + \ln B$
2.	$\log\left(\frac{A}{B}\right) = \log A - \log B$	$\ln\left(\frac{A}{B}\right) = \ln A - \ln B$
3.	$\log(A^p) = p \log A$	$\ln(A^p) = p \ln A$
4.	$\log_A(A^x) = x$	$\ln e^x = x$
5.	$10^{\log x} = x$	$e^{\ln x} = x$

These types of equations are useful when solving for unknown exponents.

Examples

Coming soon!™

1.5 Trigonometric Functions

The input of the basic trigonometric functions, sin, cos, and tan are angles, which are measured in ***radians*** or ***degrees***. To convert between the two, this relationship is used:

$$1 \quad \text{radian} = \frac{\pi}{180} \quad \text{degrees} \tag{1.14}$$

The angle of 1 ***radian*** on a unit circle has an arc length of 1. Often, if no units are prescribed for an angular measurement, it is understood to be in ***radians***. The arclength equation is given by:

$$s = r\theta \tag{1.15}$$

where s is the arc length, r is the radius of the circle, and θ is the angular measurement in radians. If a point P on a circle has coordinates in an (x, y) coordinate frame, then we can use the trigonometric functions to relate it's position with the angle θ by:

$$\cos\theta = x \tag{1.16}$$

$$\sin\theta = y \tag{1.17}$$

Because the equation of a circle is given by:

$$x^2 + y^2 = r^2 \tag{1.18}$$

a substitution using the trigonometric functions can be made:

$$\cos^2\theta + \sin^2\theta = r^2 \tag{1.19}$$

For a unit circle, which as a radius $r = 1$, this reduces to the trigonometric identity:

$$\cos^2\theta + \sin^2\theta = 1 \tag{1.20}$$

As θ increases, the values of $\sin\theta$ and $\cos\theta$ oscillate between $[-1, 1]$, and repeats every 2π ***radians*** or $360°$. Thus, $\sin\theta$ and $\cos\theta$ are *periodic functions*, or functions that repeat their values after a regular interval.

The ***period*** of a *periodic function* is the length of that regular interval. The ***amplitude*** of the function is $\frac{1}{2}$ the distance between the function's maximum and minimum values. It should be noted that the sin and cos functions can be related through a ***phase shift***, or angular ***translation***:

$$\cos\theta = \sin\left(\theta + \frac{\pi}{2}\right) \tag{1.21}$$

Functions whose shape can be described using $\sin\theta$ and $\cos\theta$ are given the name ***sinusoidal functions***. To summarize their properties:

$$f(\theta) = A\sin(B\theta) \quad g(\theta) = A\cos(B\theta) \tag{1.22}$$

where $|A|$ is the ***amplitude***, $\frac{2\pi}{|B|}$ is the period. Horizontal ***translations*** occur when the argument $B\theta$ is replaced by $B\theta \pm h$. Vertical ***translations*** occur when a constant C is added to the functions:

$$f(\theta) = A\sin(B\theta) + C \quad g(\theta) = A\cos(B\theta) + C \tag{1.23}$$

The *tangent function* is used as a relationship between *sine* and *cosine* functions. It is defined as:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \tag{1.24}$$

This function has vertical *asymptotes* at points where $\cos\theta = 0$, or $\forall\theta$ defined as $\pm\frac{(2n-1)\pi}{2}$, where $n \in \mathbb{N}$, or the set of *Natural Numbers*. The tan function has a period of π ***radians***.

Trigonometric functions also have ***inverse functions***. These are used to find an angular value given the (x, y) coordinates of a point:

$$\sin x = 0.45 \tag{1.25}$$

For the inverse sin function, which is commonly written as $\arcsin\theta$, $\operatorname{a}\sin\theta$ or $\sin^{-1}\theta$, a domain of $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is used. Thus, for $y \in [-1, 1]$:

$$\arcsin y = x \quad == \quad \sin x = y, \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \tag{1.26}$$

For the inverse cos function, which is commonly written as $\arccos\theta$, $\operatorname{a}\cos\theta$ or $\cos^{-1}\theta$, the domain is also $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, but the range is $x \in [0, \pi]$. Lastly, for the inverse tan function, commonly written as $\arctan\theta$, $\operatorname{a}\tan\theta$ or $\tan^{-1}\theta$, a domain of $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is used, but has a range $y \in \mathbb{R}$.

Examples

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1.6 Powers, Polynomials, and Rational Functions

A ***power function*** is a function $f(x)$ where the ***dependent variable***, y is proportional to a power of the ***independent variable***, x:

$$f(x) = kx^p \tag{1.27}$$

where k and p are constant. Examples of these functions are the volume of a sphere:

$$V = \frac{4}{3}\pi r^3 \tag{1.28}$$

or Newton's Law of Gravitation:

$$F = kr^{-2} = \frac{k}{r^2} \tag{1.29}$$

For functions of the form x^n, where n is a positive integer, odd values of $n > 1$ pass through the origin and can assume negative values these are ***monotonically increasing functions***. Even values of $n > 1$ also pass through the origin, but $\forall x < 0$, $f(x)$ is a ***decreasing*** function, and $\forall x > 0$, $f(x)$ is ***increasing***. For large exponents n, the function value grows faster. Though ***power functions*** may equal values greater than some arbitrary ***exponential funtion***, *every* ***exponential function*** will eventually dominate *every* ***power function*** at some value of x if the base of the ***exponential function***, a, is greater than 1.

Polynomials are the sum of ***power functions*** with non-negative integer exponents:

$$y = p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0 \tag{1.30}$$

The highest exponent n in a ***polynomial*** is referred to as the***degree*** of the polynomial.

Rational Functions are ratios of ***polynomials***, $p(x)$ and $q(x)$:

$$f(x) = \frac{p(x)}{q(x)} \tag{1.31}$$

These functions may have ***horizontal or vertical asymptotes*** which occur when:

$$f(x) \to L \quad \text{as} \quad x \to \pm\infty \tag{1.32}$$

for ***horizontal asymptotes*** or

$$y \to \pm\infty \quad \text{as} \quad x \to K \tag{1.33}$$

for ***vertical asymptotes***. Here, the horizontal asymptote is $y = L$, and the ***vertical asymptote*** is $x = K$. The function behavior as $x \to \pm\infty$ is referred to as ***end behavior***.

Examples

Coming soon!™

1.7 Introduction to Continuity

This section focuses on the idea of ***continuity*** along an interval, $[a, b]$, and at a point, p. ***Continuous Functions*** have many desirable properties. For functions along an interval, a general rule of continuity is that a function is continuous along an interval $[a, b]$ if it has no breaks, jumps, or holes within that interval. Many functions are not continuous $\forall x$, such as $\frac{1}{x}$, which is undefined at $x = 0$, but is continuous for any interval $[a, b]$ that does not contain 0. ***Exponential***, ***power***, and the *sine* and *cosine* functions are continuous along any interval $[a, b]$. ***Rational functions*** are continuous on any interval that the denominator is non-zero. Functions derived via addition or multiplication of other continuous functions, and ***composite functions*** are continuous if the functions used to create them are continuous.

Intermediate Value Theorem – Assume $f(x)$ is continuous on a closed interval $[a, b]$. If k is any number between $f(a)$ and $f(b)$, then $\exists$ at least one number $c \in [a, b]$ such that $f(c) = k$.

A function is continuous if nearby values of the independent variable, x, give nearby values of the dependent variable, y. Continuity is important because it implies that small perturbations in x do not result in large changes in y. To check continuity at a point, for example, $x = 2$, check nearby values to the left and right of that point, $x = 1.99, 1.98$ and $x = 2.01, 2.02$. If $f(x)$ changes significantly, then the function has a discontinuity at $x = 2$.

Examples

Coming soon!™

1.8 Limits

The idea of the ***limit*** is fundamental to the study of Calculus. The ***limit*** makes sense of a function "approaching" a value. Limit notation is defined as:

$$\lim_{x \to c} f(x) = L \tag{1.34}$$

which means the function $f(x)$ approaches the value L as the independent variable, x approaches the value c. x is never equals c, but is infinitesimally close to it.

A function f is defined on an interval around c, except at the point $x = c$. The ***limit*** of the function $f(x)$ as x approaches c is equal to a number L, should such a limit exist, such that $f(x)$ is as close to L as we want whenever x is sufficiently close to c. The distance between $f(x)$ and L is given by:

$$\text{Distance} = |f(x) - L| \tag{1.35}$$

which we want to be sufficiently close. The Greek letter ϵ is used to refer to small numbers; thus, we want:

$$|f(x) - L| < \epsilon \tag{1.36}$$

to show limit convergence. Similarly, we want the following to be true:

$$|x - c| < \delta \tag{1.37}$$

for a chosen value of δ. The definition of the limit can then be rewritten as:

Limit – The limit $\lim_{x \to c} f(x)$ is equal to the number L, if one exists, such that $\forall \epsilon > 0$ (as small as we want), there is a $\delta > 0$ (sufficiently small) such that if $|x - c| < \delta$ and $x \neq c$, then $|f(x) - L| < \epsilon$.

Limits have the following properties, assuming all the limits on the right-hand side exist:

1. If b is constant, then $\lim_{x \to c} (bf(x)) = b (\lim_{x \to c} f(x))$
2. $\lim_{x \to c} (f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$
3. $\lim_{x \to c} (f(x)g(x)) = (\lim_{x \to c} f(x)) (\lim_{x \to c} g(x))$
4. $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$, provided $\lim_{x \to c} g(x) \neq 0$
5. For any constant k, $\lim_{x \to c} k = k$
6. $\lim_{x \to c} x = c$

Limits can be taken from both the right and the left. The general form:

$$\lim_{x \to c} f(x) = L \tag{1.38}$$

means that $f(x) \to L$ as $x \to c$ *from both directions*. Piecewise-defined functions may have different values from the right or the left if the limit is taken at a jump discontinuity. To indicate a limit from the left, the following notation is used:

$$\lim_{x \to c^-} f(x) = L^- \tag{1.39}$$

and from the right:

$$\lim_{x \to c^+} f(x) = L^+ \tag{1.40}$$

Again, the limit from the right or left is not guaranteed to converge to the same value, so L^- and L^+ are used, but L^- may equal L^+.

Limits do not exist when there is no finite number L that the function value assumes at the point the limit is taken. Sometimes, limits are taken at $\pm\infty$ to understand the ***end behavior*** of a function $f(x)$. Here ∞ does not represent a number, just sufficiently large values of x. These limits are written as:

$$\lim_{x \to \pm\infty} f(x) = L \tag{1.41}$$

if the limit, L, exists.

Using this definition of the limit, ***continuity*** of a function can be formally defined as: ***Continuity*** – A function $f(x)$ is continuous at $x = c$ if f is defined at $x = c$ and if:

$$\lim_{x \to c} f(x) = f(c) \tag{1.42}$$

If c is an endpoint of an interval $[a, b]$, then a one-sided limit is used.

Using the continuity of sums of products, we can show that any ***polynomial*** is a continuous function. The following rules can be used to determine of more complicated functions, such as $\sin\theta$, $\cos\theta$, or e^x are continuous, provided f and g are continuous on an interval $[a, b]$ and b is constant:

1. $bf(x)$ is continuous
2. $f(x) + g(x)$ is continuous
3. $f(x)g(x)$ is continuous

4. $\frac{f(x)}{g(x)}$ is continuous, provided $g(x) \neq 0$ on the interval $[a, b]$.

For ***composite functions***, if f and g are continuous and if the ***composite function*** $f(g(x))$ is defined on an interval $[a, b]$, then $f(g(x))$ is continuous on $[a, b]$.

Examples

Coming soon!™

Chapter 2

The Derivative

2.1 How do We Measure Speed?

Speed is the magnitude of ***velocity***, and is a scalar quantity. ***Velocity*** is a vector quantity, meaning it has magnitude *and* direction. Thus, velocity can be negative if an object is moving opposite the positive direction. If $s(t)$ is the position of an object at time t, then the ***average velocity*** of the object over the interval $t \in [a, b]$ is:

$$V_{avg} = \frac{\Delta s}{\Delta t} = \frac{s(b) - s(a)}{b - a} \tag{2.1}$$

This representation does not help to understand the velocity of an object at a given instant in time, only over the interval $[a, b]$. For this problem of ***instantaneous velocity***, we need to look closer at the specified time instant, t.

One way to converge to the instantaneous velocity at time t is to use the definition of the ***limit***. As the interval $[a, b]$ gets sufficiently small and encloses the time t, a two-sided limit is being taken to determine the ***instantaneous velocity***; therefore, if $s(t)$ is the position at time t, the ***instantaneous velocity*** at time $t = a$ can be defined as:

$$\lim_{h \to 0} \frac{s(a + h) - s(a)}{h} \tag{2.2}$$

This approximation is valid as $h \to 0$ because most functions, $f(x)$, appear linear for small changes between x values. This approximation yields the slope of the curve $f(x)$ at a point, $x = a$. Using this definition, we can redefine the ***average velocity*** over any time interval $t \in [a, b]$ as:

Average Velocity – the slope of the line joining the points on the graph of $s(t)$ corresponding to $t = a$ and $t = b$.

Examples

Coming soon!™

2.2 The Derivative at a Point

In this section, the ***difference quotient*** shown in Equation (2.1) is applied to functions that are not necessarily functions of time. We are still interested in intervals of length $[a, a+h]$ where h is sufficiently small. For any function f, the ***difference quotient*** is given by $\frac{\Delta f}{\Delta x}$.

The numerator of the ***difference quotient*** is the ***absolute change*** of the function, whereas the full ***difference quotient*** is the average rate of change. When the interval $[a, a+h]$ is sufficiently small, that is $h \to 0$, we arrive at the ***instantaneous rate of change***. This is referred to as the *derivative of function f at point a*, and is denoted by $f'(a)$.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{2.3}$$

If the ***limit*** in Equation (2.3) exists, then f is said to be **differentiable at a**. The ***derivative*** at point a can be interpreted as the slope of the curve $f(x)$ at point a, or the slope of the tangent line to the curve at point a if $h \ll 1$.

Examples

Coming soon!™

2.3 The Derivative Function

The previous section focused on the ***derivative*** at a fixed point, a. This section will cover how the ***derivative*** changes at different points because it is also a function of the independent variable. Using the definition of the ***derivative*** at a point, we can arrive at the following extention to functions:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{2.4}$$

Thus, for every x value that the limit in Equation (2.4) exists, we can state that the function f is *differentiable at* that x value. If there are no points that f is ***undifferentiable***, then f is *differentiable everywhere*, which is often the case.

The derivative tells us how the function $f(x)$ is changing with x. If $f' > 0$ over an interval $[a, b]$, then f is ***increasing*** over that interval. Conversely, If $f' < 0$ over an interval $[a, b]$, then f is ***decreasing*** over that interval. The magnitude of f' indicates how much the function increases ($|f'| \gg 0 \rightarrow f$ is changing rapidly; ($|f'| \ll 1 \rightarrow f$ is changing slowly).

For constant functions, $f(x) = k$, the ***derivative*** is equal to zero because the function values do not change with changes in x. For linear functions, $f(x) = mx+b$, the ***derivative*** is m because it is the slope of the tangent line to $f(x)$, which is linear. For $n - degree$ polynomials, $n > 1$, the derivative can be estimated numerically by function evaluations using Equation (2.4).

For power functions, $f(x) = x^n$, the ***Binomial Theorem*** can be used to show the ***Power Rule of Differentiation***, which is valid for $n \in \mathbb{R}$:

$$\text{If} \quad f(x) = x^n, \quad \text{then} \quad f'(x) = nx^{n-1} \tag{2.5}$$

Examples

Coming soon!™

2.4 Interpretations of the Derivative

Another commonly used notation for the ***derivative*** is ***Leibniz's Notation***:

$$f'(x) = \frac{dy}{dx} \tag{2.6}$$

Here, d suggests "*small difference in.*" An alternative form of ***Leibniz's Notation***:

$$\frac{dy}{dx} = \frac{d}{dx}(y) \tag{2.7}$$

which means "*the derivative of y with respect to x*". The terms dy and dx in this notation are often used a seperate entities in mathematics, representing infinitesimally small changes in y and x, respectively. Relating this back to previous examples, velocity can be writen as:

$$v = \frac{ds}{dt} \tag{2.8}$$

To specify the derivative at a point, c, we can write:

$$\left.\frac{dy}{dx}\right|_{x=c} \tag{2.9}$$

The units of the ***derivative*** of a function depend on the units of the dependent and independent variables, but it is always given by:

$$\frac{Y \quad \text{units}}{X \quad \text{units}} \tag{2.10}$$

Examples

Coming soon!™

2.5 The Second Derivative

Because the ***derivative*** is a function, we can also consider its ***derivative***. The ***derivative*** of the ***derivative*** results in the ***second derivative***, which is denoted by:

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) \tag{2.11}$$

The ***second derivative*** provides the same information about the ***derivative*** as the ***derivative*** provides about $f(x)$, namely:

1. If $f'' > 0$ on an interval $[a, b]$, then f' is ***increasing*** over $[a, b]$.

2. If $f'' < 0$ on an interval $[a, b]$, then f' is ***decreasing*** over $[a, b]$.

With this information, we can determine if $f(x)$ is ***concave up*** or ***concave down***.

1. If $f'' > 0$ on an interval $[a, b]$, then $f(x)$ is ***concave up*** over $[a, b]$.

2. If $f'' < 0$ on an interval $[a, b]$, then $f(x)$ is ***concave down*** over $[a, b]$.

If $f'' = 0$, then this is an ***inflection point***, where the function $f(x)$ changes the direction of curvature, or a ***saddle point***, where the ***derivative*** is zero, but concavity does not change.

Relating the ***second derivative*** back to speed and velocity, and using the definition in Equation (2.11), the acceleration of an object can be given by:

$$a(t) = v'(t) = s''(t) = \frac{d^2 s}{dt^2} = \frac{d}{dt}(v) = \frac{d}{dt}\left(\frac{ds}{dt}\right) \tag{2.12}$$

Examples

Coming soon!™

2.6 Differentiability

A function f is ***differentiable*** at x if the following ***limit*** exists:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{2.13}$$

Thus, the graph of f has a non-vertical tangent line at x. The value of the ***limit*** and the slope of the tangent line are the ***derivative*** of f at x.

Additionally, the function must be continuous at the point x and not have a ***sharp corner*** at x. ***Sharp corners*** occur when each ***one-sided limit*** approaches different values, that is $L^+ \neq L^-$. An example of this type of function is:

$$f(x) = |x| \tag{2.14}$$

which has a ***sharp corner*** at $x = 0$.

If a function $f(x)$ is ***differentiable*** at $x = a$, then $f(x)$ is ***continuous*** at $x = a$; however $f(x) = |x|$ is continuous at $x = 0$, but undifferentiable.

Examples

Coming soon!™

Chapter 3

Short-Cuts to Differentiation

This chapter will cover simple rules that are used to take the ***derivative*** for a variety of functions.

3.1 Powers and Polynomials

The ***derivative*** of a constant multiple is given by:

$$\frac{d}{dx}[cf(x)] = cf'(x) \tag{3.1}$$

If $f(x)$ is ***differentiable*** and c is constant. This is a special case of the ***Product Rule***, which will be covered shortly. Proving this rule can be done using the definition of the ***derivative***:

$$\frac{d}{dx}[cf(x)] = \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} = \lim_{h\to 0} c\frac{f(x+h)-f(x)}{h} \tag{3.2}$$

$$\lim_{h\to 0} c\frac{f(x+h)-f(x)}{h} = cf'(x) \tag{3.3}$$

The derivative of two functions, $f(x)$ and $g(x)$, added or subtracted together, provided both $f(x)$ and $g(x)$ are ***differentiable***, is given by:

$$\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x) \tag{3.4}$$

$$\frac{d}{dx}[f(x)-g(x)] = f'(x)-g'(x) \tag{3.5}$$

Again, using the definition of the ***derivative***:

$$\frac{d}{dx}\left[f(x) + g(x)\right] = \lim_{h \to 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h} \tag{3.6}$$

$$\lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h}\right] \tag{3.7}$$

where the limits of each term in Equation (3.7) approach $f'(x)$ and $g'(x)$.

For equations containing powers of x, the ***Power Rule*** can be used:

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1} \tag{3.8}$$

This is valid provided that n is constant and $n \in \mathbb{R}$. Combining the rules shown in Equations (3.1), (3.4), (3.5), and (3.8), the ***derivative*** of *any* ***polynomial*** can be taken.

Examples

Coming soon!™

3.2 The Exponential Function

Recall that ***exponential functions*** are given by:

$$f(x) = a^x \tag{3.9}$$

These are ***monotonically increasing*** functions; thus, the ***derivative*** must be strictly positive. Suppose we have a function, $f(x)$, given by:

$$f(x) = 2^x \tag{3.10}$$

Using the definition of the ***derivative***:

$$f'(x) = \lim_{h \to 0}\left(\frac{2^{x+h} - 2^x}{h}\right) = \lim_{h \to 0}\left(\frac{2^x 2^h - 2^x}{h}\right) \tag{3.11}$$

$$\lim_{h \to 0}\left(\frac{2^h - 1}{h}\right) \cdot 2^x \tag{3.12}$$

Table 3.1: Evaluations of coefficient $\frac{2^h-1}{h}$ for various values of h

h	$\frac{2^h-1}{h}$
-0.1	0.6697
-0.01	0.6908
-0.001	0.6929
0.001	0.6934
0.01	0.6956
0.1	0.7177

Using this representation, we can approach the ***limit*** of the coefficient of 2^x by taking small values of h: This suggests that the ***limit*** of $\frac{2^h-1}{h}$ as $h \to 0$ is approximately 0.693. Thus:

$$g'(x) = \frac{d}{dx}\left(0.693 \cdot 2^x\right) \tag{3.13}$$

Applying Equation (3.1) to this expression means that only the ***derivative*** of 2^x must be taken. This indicates that the ***derivative*** of an ***exponential function*** depends upon the function itself. In the general form:

$$f(x) = a^x \tag{3.14}$$

$$f'(x) = \lim_{h\to 0}\left(\frac{a^{x+h} - a^x}{h}\right) = \lim_{h\to 0}\left(\frac{a^h - 1}{h}\right) \cdot a^x \tag{3.15}$$

Looking at Table 3.2 indicates that there is a number, a, between 2 and 3 such that $\lim_{h\to 0} \frac{a^h-1}{h}$ is 1. Thus:

$$a^h - 1 \approx h, \quad h \ll 1 \tag{3.16}$$

$$a^h \approx 1 + h \tag{3.17}$$

$$a \approx (1+h)^{\frac{1}{h}} \tag{3.18}$$

Taking the ***limit*** of Equation (3.18) results in the values shown in Table 3.3 We observe that $a \approx 2.718$. This is referred to as the ***Natural Number***, e. Using the ***derivative*** relationships shown previously:

$$\frac{d}{dx}\left(e^x\right) = e^x \tag{3.19}$$

To adjust this expression for various values of a, we use a combination of the ***limit*** and ***derivative*** defintions previously presented and properties of logarithms:

$$f'(x) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = \left(\lim_{h \to 0} \frac{a^h - 1}{h} \right) a^x \tag{3.20}$$

$$a = e^{\ln a} \tag{3.21}$$

$$\lim_{h \to 0} \frac{\left(e^{\ln a}\right)^h - 1}{h} = \lim_{h \to 0} \frac{e^{(\ln a)h} - 1}{h} \tag{3.22}$$

From the definition of the ***Natural Number***, e:

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1 \tag{3.23}$$

Substituting $t = (\ln a) \cdot h$:

$$\lim_{h \to 0} \frac{e^{(\ln a)h} - 1}{h} = \lim_{t \to 0} \frac{e^t - 1}{t/\ln a} = \lim_{t \to 0} \left(\ln a \cdot \frac{e^t - 1}{t} \right) \tag{3.24}$$

$$= \ln a \left(\lim_{t \to 0} \frac{e^t - 1}{t} \right) = (\ln a) \cdot 1 = \ln a \tag{3.25}$$

Thus:

$$\frac{d}{dx} \left(a^x\right) = (\ln a) a^x \tag{3.26}$$

Examples

Coming soon!™

Table 3.2: Limit evaluations of coefficient $\frac{a^h - 1}{h}$ for various values of a

a	$\lim_{h \to 0} \frac{a^h - 1}{h}$
2	0.693
3	1.099
4	1.386
5	1.609
6	1.792
7	1.946

3.3 The Product and Quotient Rules

Recall the Greek symbol Δ refers to the change in a quantity. For a function $f(x)$, the change in the function can be written as:

$$\Delta f = f(x+h) - f(x) \tag{3.27}$$

for small values of h. Using this notation, we can arrive at the definition of the ***derivative*** as:

$$f'(x) = \lim_{h \to 0} \frac{\Delta f}{h} \tag{3.28}$$

Suppose we know the ***derivatives*** of two functions, $f(x)$ and $g(x)$, but we want to calculate the ***derivative*** of their product, $f(x)g(x)$. A proof of this is included in the text these notes are based on (see Mechanical-Engineering-Curriculum/Calculus˙I/README.md), but the ***Product Rule*** can be used to determine this ***derivative***. The proof uses the definition of the ***limit***, but the ***Product Rule*** is given as:

$$(fg)' = f'g + fg' \tag{3.29}$$

provided that $u = f(x)$ and $v = g(x)$ are ***differentiable***. Using ***Leibniz's Notation***:

$$\frac{d(uv)}{dx} = \frac{d}{dx}(uv) = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx} \tag{3.30}$$

Similarly, the ***Quotient Rule*** is the inverse of the ***Product Rule***. If a function $Q(x) = \frac{f(x)}{g(x)}$ exists, and $u = f(x)$ and $v = g(x)$ are ***differentiable*** then the ***Quotient Rule*** is given as:

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \tag{3.31}$$

Table 3.3: Evaluations of coefficient $(1+h)^{\frac{1}{h}}$ for various values of h

h	$(1+h)^{\frac{1}{h}}$
-0.001	2.7196422
-0.0001	2.7184178
-0.00001	2.7182954
0.00001	2.7182682
0.0001	2.7181459
0.001	2.7169239

Representing this in ***Leibniz's Notation***:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx} \cdot v - u\frac{dv}{dx}}{v^2} \tag{3.32}$$

Note that some expressions are likely to be easier handled by one rule over another, but equations can also be re-written such that multiplicative operations appear as division, such as:

$$\frac{1}{x} = x^{-1} \tag{3.33}$$

Here, the ***derivative*** of the right-hand side can be easily computed using the ***Quotient Rule***, but the left-hand side is in the form of the ***Power Rule***, but both expressions are equivalent.

Examples

Coming soon!™

3.4 The Chain Rule

The ***Chain Rule*** is used to take the ***derivative*** of ***composite functions***. Suppose $f(g(x))$ is a ***composite function***. Then, substituting $z = g(x)$ yields:

$$y = f(z) \tag{3.34}$$

Because z is a function of x, a small change in x, Δx will result in a small change in z, Δz. In turn, Δz results in a small change in the dependent variable, y, because $y = f(z)$. If Δx and Δz are not identically zero, then:

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta z} \cdot \frac{\Delta z}{\Delta x} \tag{3.35}$$

Because $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, as these three differentials get closer to zero the ***Chain Rule*** can be defined as:

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} \tag{3.36}$$

Furthermore, because $\frac{dy}{dz} = f'(z)$ and $\frac{dz}{dx} = g'(x)$:

$$\frac{d}{dx}\left(f\left(g\left(x\right)\right)\right) = f'\left(z\right) \cdot g'(x) \tag{3.37}$$

Back-substituting $z = g(x)$ yields:

$$\frac{d}{dx}\left(f\left(g\left(x\right)\right)\right) = f'\left(\left(g\left(x\right)\right)\right) \cdot g'(x) \tag{3.38}$$

Examples

Coming soon!™

3.5 The Trigonometric Functions

This section is concerned with the ***derivatives*** of the *sine* and *cosine* functions. Because these functions are periodic, their ***derivatives*** must also be periodic. This can be done by looking at the graph of the functions and taking the ***limit*** at multiple points for small changes from the point, h. Note that the relationships between the*sine* and *cosine* functions and the Unit Circle require x to be in ***radians***.

Taking the limit at multple points along the *sine* function's domain of $[0, 2\pi]$ results in the following relationship:

$$\frac{d}{dx}\left(\sin(x)\right) = \cos(x) \tag{3.39}$$

Following a similar process for the *cosine* function:

$$\frac{d}{dx}\left(\cos(x)\right) = -\sin(x) \tag{3.40}$$

The ***derivative*** of *cosine* is $-\sin$ because of the phase shift between the two functions. Recall that:

$$\cos(x) = \sin\left(x + \frac{\pi}{2}\right) \tag{3.41}$$

Also note that the ***Chain Rule*** must be used to differentiate the trigonometric functions because they are ***composite functions***. For the *tangent* function, which is defined as:

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \tag{3.42}$$

the ***Quotient Rule*** can be easily used. It results in the following:

$$\frac{d}{dx}\left(\tan(x)\right) = \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) \tag{3.43}$$

$$\frac{d}{dx}\left(\tan(x)\right) = \frac{(\sin(x))' \cdot (\cos(x)) - (\sin(x)) \cdot (\cos(x))'}{\cos^2(x)} \tag{3.44}$$

$$\frac{d}{dx}\left(\tan(x)\right) = \frac{\sin^2(x) + \cos^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} \tag{3.45}$$

Again, these relationships require x to be in ***radians***.

Examples

Coming soon!™

3.6 The Chain Rule and Inverse Functions

We will now apply the ***Chain Rule*** to calculate ***derivatives*** of ***logarithmic functions***, ***exponential functions***, and the ***inverse trigonometric functions***. A general rule for the ***derivative*** of any ***inverse function*** will also be shown.

Recall that, using the properties of ***logarithms***, we can write:

$$e^{\ln x} = x \tag{3.46}$$

To arrive at the ***derivative*** of $\ln x$, we will use the equation $e^{\ln x} = x$:

$$\frac{d}{dx}\left(e^{\ln x}\right) = \frac{d}{dx}\left(x\right) \tag{3.47}$$

$$e^{\ln x} \cdot \frac{d}{dx}\left(\ln x\right) = 1 \tag{3.48}$$

Thus:

$$\frac{d}{dx}\left(\ln x\right) = \frac{1}{e^{\ln x}} = \frac{1}{x} \tag{3.49}$$

For ***derivatives*** of the ***inverse trigonometric functions***, $\arcsin(x)$, $\arccos(x)$, and $\arctan(x)$, we can utilize several trigonometric identities. For the ***derivative*** of $\arctan(x)$, note that $\tan(\arctan(x)) = x$. Differentiation via the ***Chain Rule*** yields:

$$\frac{1}{\cos^2(\arctan(x))} \cdot \frac{d}{dx}\left(\arctan(x)\right) = 1 \tag{3.50}$$

$$\frac{d}{dx}\left(\arctan(x)\right) = \cos^2(\arctan(x)) \tag{3.51}$$

The trigonometric identity $1 + \tan^2(\theta) = \frac{1}{\cos^2(\theta)}$ can be used here, replacing θ with $\arctan(x)$; thus:

$$\cos^2(\arctan(x)) = \frac{1}{1 + \tan^2(\arctan(x))} = \frac{1}{1 + x^2} \tag{3.52}$$

$$\frac{d}{dx}\left(\arctan(x)\right) = \cos^2(\arctan(x)) = \frac{1}{1 + x^2} \tag{3.53}$$

Following a similar process for $\arcsin(x)$ and $\arccos(x)$:

$$\frac{d}{dx}\left(\arcsin(x)\right) = \frac{1}{\sqrt{1 - x^2}} \tag{3.54}$$

$$\frac{d}{dx}\left(\arccos(x)\right) = -\frac{1}{\sqrt{1 - x^2}} \tag{3.55}$$

For general ***inverse functions***, if a function, $f(x)$, has a ***differentiable inverse***, f^{-1}, its ***derivative***, $f\left(f^{-1}(x)\right) = x$ by the ***Chain Rule***:

$$\frac{d}{dx}\left(f\left(f^{-1}(x)\right)\right) = 1 \tag{3.56}$$

$$f'\left(f^{-1}(x)\right) \cdot \frac{d}{dx}\left(f^{-1}(x)\right) = 1 \tag{3.57}$$

Therefore:

$$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'\left(f^{-1}(x)\right)} \tag{3.58}$$

Examples

Coming soon!™

3.7 Implicit Functions

So far, the functions presented have been considered ***explicit functions*** of x, functions of the form $y = f(x)$. This section addresses ***implicit functions***, where one x value corresponds to multiple y values. An example of an ***implicit function*** is the equation of a circle:

$$x^2 + y^2 = r^2 \tag{3.59}$$

Here, y is a function of x on both the top and bottom halves of the circle, buy when the circle is considered as a whole, as in Equation (3.59), where the function does not have a tangent lineat each point, but the equation is still ***differentiable*** with respect to x:

$$\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(r^2\right) \tag{3.60}$$

Because r is a constant of the circle, ***differentiation*** results in the right-hand side of Equation (3.60) goes to zero. If y is treated as a function of x, and the ***Chain Rule*** is used:

$$2x + 2y\frac{dy}{dx} = 0 \tag{3.61}$$

Solving this for the $\frac{dy}{dx}$ yields:

$$\frac{dy}{dx} = -\frac{x}{y} \tag{3.62}$$

This relationship is valid, provided $y \neq 0$. This is expected because the line tangent to the circle at this point is vertical, so the ***slope*** is infinite.

Examples

Coming soon!™

3.8 Hyperbolic Functions

The ***Hyperbolic Functions*** are combinations of e^x and e^{-x}. *Hyperbolic Sine*, abbreviated $\sinh(x)$, and *Hyperbolic Cosine*, abbreviated $\cosh(x)$, are given by:

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \tag{3.63}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \tag{3.64}$$

Observing properties of these functions graphically, and recalling that $\frac{d}{dx}(e^x) = e^x$, the following results hold:

Table 3.4: Properties of $\sinh(x)$ and $\cosh(x)$

$\cosh(0) = 1$	$\sinh(0) = 0$
$\cosh(-x) = \cosh(x)$	$\sinh(-x) = -\sinh(x)$
$\frac{d}{dx}(\cosh(x)) = \sinh(x)$	$\frac{d}{dx}(\sinh(x)) = \cosh(x)$

Furthermore, the it can be shown that the following result is true:

$$\cosh^2(x) - \sinh^2(x) = 1 \tag{3.65}$$

The *Hyperbolic Tangent* function can also be defined, and is given by:

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \tag{3.66}$$

It's derivative is given by:

$$\frac{d}{dx}(\tanh(x)) = \frac{d}{dx}\left(\frac{\sinh(x)}{\cosh(x)}\right) = \frac{(\cosh(x))^2 - (\sinh(x))^2}{(\cosh(x))^2} \tag{3.67}$$

$$\frac{d}{dx}(\tanh(x)) = \frac{1}{\cosh^2(x)} \tag{3.68}$$

Examples

Coming soon!™

3.9 Linear Approximation and the Derivative

When zooming into the graph of a ***differentiable*** function, if the change between numbers is small enough ($\Delta x \to 0$), the graph will look more and more like a line. This idea of ***local linearity*** is commonly used to approximate the ***slope*** or ***derivative*** of a function $f(x)$ at a specific point, $x = a$

and function values near that point, $y = f(a + h)$. The ***Tangent Line Approximation*** is given by:

$$f(x) \approx f(a) + f'(a)(x - a) \tag{3.69}$$

provided that $f(x)$ is ***differentiable*** at $x = a$. The resulting error in the approximation is given by:

$$E(x) = f(x) - f(a) - f'(a)(x - a) \tag{3.70}$$

More information about this error approximation will be discussed in the Calculus II course.

Examples

Coming soon!™

3.10 Theorems about Differentiable Functions

The ***Mean Value Theorem*** states that if $f(x)$ is ***continuous*** on $[a, b]$, and is ***differentiable*** on (a, b), then a number c exists such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{3.71}$$

which is the slope of the secant line between points $x = a$, and $x = b$. This theorem does not indicate how to find c, though.

The ***Increasing Function Theorem*** states that if a function $f(x)$ is ***continuous*** on $[a, b]$, and is ***differentiable*** on (a, b), then:

1. If $f'(x) > 0 \quad \forall x \in (a, b)$, then f is ***increasing*** on $[a, b]$.
2. If $f'(x) \geq 0 \quad \forall x \in (a, b)$, then f is ***non-decreasing*** on $[a, b]$.

The ***Constant Function Theorem*** states that if a function $f(x)$ is ***continuous*** on $[a, b]$, and is ***differentiable*** on (a, b), then if $f'(x) = 0 \quad \forall x \in [a, b]$, then f is constant on $x \in [a, b]$.

The ***Racetrack Principle*** states that for two functions, $g(x)$, and $h(x)$, that are ***continuous*** on $[a, b]$, and is ***differentiable*** on (a, b), and that $g'(x) \leq h'(x) \quad \forall x \in (a, b)$:

1. If $g(a) = h(a)$, then $g(x) \leq h(x) \quad \forall x \in [a, b]$.
2. If $g(b) = h(b)$, then $g(x) \geq h(x) \quad \forall x \in [a, b]$.

Examples

Coming soon!™

Chapter 4

Using the Derivative

This chapter will cover

4.1 Using First and Second Derivatives

Recall from Chapter 2 that:

1. If $f' > 0 \ \forall x \in [a, b]$, then $f(x)$ is ***monotonically increasing*** on that interval

2. If $f' < 0 \ \forall x \in [a, b]$, then $f(x)$ is ***monotonically decreasing*** on that interval

3. If $f'' > 0 \ \forall x \in [a, b]$, then $f(x)$ is ***concave up*** on that interval

4. If $f'' < 0 \ \forall x \in [a, b]$, then $f(x)$ is ***concave down*** on that interval

We can combine these principles along with the general derivation formulas we discussed in Chapter 3.

Often, the ***local maxima*** or ***local minima*** are points of interest for a given function, if they exist. Suppose p is a point in the domain of $f(x)$; $p \in [a, b]$. Then:

1. $f(x)$ has a ***local minimum*** at p if $f(p \pm \Delta) > f(p)$.

2. $f(x)$ has a ***local maximum*** at p if $f(p \pm \Delta) < f(p)$.

These definitions of the ***maxima*** and ***minima*** are ***local*** because these relationships do not provide any information outside of the near vicinity of point p.

Local maxima or ***local minima***, as well as other important points of a function, are commonly referred to as ***critical points***. These points are identified by the ***derivative***. For any function, $f(x)$, ***critical points*** are located at points p satisfying:

1. $f'(p) = 0$

2. $f'(p)$ is undefined

The ***critical values*** of a function are found by evaluating $f(p)$ at each ***critical point***, p. Note that not every ***critical point*** is a ***maximum*** or ***minimum***.

Suppose we wanted to find all of the ***extrema*** of a function. This can be done using both $f'(x)$ and $f''(x)$. When performing the **First-Derivative Test**, assume that p is a ***critical point*** in the domain of $f(x)$; $p \in [a, b]$. Then, assuming x is increasing:

1. If $f'(x)$ changes sign from negative to positive at p, then p is a ***local minimum*** of $f(x)$.

2. If $f'(x)$ changes sign from positive to negative at p, then p is a ***local maximum*** of $f(x)$.

The **Second-Derivative Test** also provides curvature information about the function. Again, assuming that p is a ***critical point*** in the domain of $f(x)$; $p \in [a, b]$:

1. If $f'(p) = 0$ and $f''(p) > 0$, then f has a ***local minimum*** at p.

2. If $f'(p) = 0$ and $f''(p) < 0$, then f has a ***local maximum*** at p.

3. If $f'(p) = 0$ and $f''(p) = 0$, then the **Second-Derivative Test** does not result in a conclusion.

Inflection Points are defined at places that the function, f, changes ***concavity***. These occur when $f'' = 0$ or f'' is undefined.

Examples

Coming soon!™

4.2 Optimization

Some functions have maximum or minimum values across its domain, meaning the functions do not grow or decay to $\pm\infty$. These maximum or minimum values are referred to as ***extrema*** or ***optimal values***. Practical applications of finding these ***extrema*** are to minimize weight of an airplane or maximize the profit of an investment. The **Extreme Value Theorem** is used to describe when ***extrema*** exist: **Extreme Value Theorem**: If

f is ***continuous*** on the closed interval, $a \leq x \leq b$, then f has a ***global maximum*** and ***global minimum*** on that interval.

The ***critical points*** of the function are first found using the **First-Derivative Test**. Then, the function values are evaluated at each of the ***critical points*** and the endpoints of the domain, a and b.

Examples

Coming soon!™

4.3 Families of Functions

Families of Functions are defined by functions that all have similar terms. An example is the set of quadratic functions, which follow the form of:

$$f(x) = ax^2 + bx + c \tag{4.1}$$

where a, b, and c are constants. Different combinations of a, b, and c represent unique members of the ***family of functions***. Because there are an infinite amount of constant values, there are an infinite amount of quadratic functions. Different ***families of functions*** are commonly used in mathematical modeling of some system or event. Some examples of these are probabalistic theory, kinematic motion, or population density.

Examples

Coming soon!™

4.4 Optimization, Geometry, and Modeling

The practice of ***Optimization*** can be applied to a variety of problems. One common example of this in the field of geometry is minimizing or maximizing the surface area or volume of a given shape. To do this, a mathematical model of how surface area and volume change with other parameters is required. These ***optimization*** problems often have constraints, such as maximum or minimum dimensions or the shape must have have at least a certain volume.

A common technique to solve ***optimization*** problems is to apply the **First-Derivative Test** to find the ***critical points*** of a function; where $f'(x) = 0$, and then testing the possible optimal solutions for the maximum or minimum function value.

Examples

Coming soon!™

4.5 Applications to Marginality

In the world of business, decisions are often made based on ***revenue*** or ***cost*** of a project or investment. The ***derivative*** can be used to maximize ***profit***, the difference between ***revenue*** or ***cost***:

$$\text{Profit} = \text{Revenue} - \text{Cost} \tag{4.2}$$

Cost represents the total cost of producing a quantity, q, of some good. The more goods are made, the higher the ***cost***, so $C(q)$ is an increasing function.

Many ***cost functions***, $C(q)$, have ***fixed costs***, or costs that are incurred before the first good is produced. This represents things like facility cost or hardware/software or material investments to create the good.

The ***Revenue Function***, $R(q)$, represents the total value for selling a good:

$$\text{Revenue} = \text{Price} * \text{Quantity} \tag{4.3}$$

Sometimes, the price, p, of an item can depend on the quantity, q, sold. Accessing the ***profit*** of producing the next good, based on the current ***cost*** and ***revenue*** is referred to as ***Marginal Analysis***. This is performed using the definition of the ***derivative***. The ***Marginal Cost*** is given by:

$$\text{MC} = \frac{C(q+1) - C(q)}{(q+1) - q} \tag{4.4}$$

$$\text{MC} = C'(q) \approx C(q+1) - C(q)$$

Likewise, ***Marginal Revenue*** is given by:

$$\text{MR} = \frac{R(q+1) - R(q)}{(q+1) - q} \tag{4.5}$$

$$\text{MR} = R'(q) \approx R(q+1) - R(q)$$

It can be shown that the maximum ***profit*** occurs when:

$$\text{MR} = \text{MC} \tag{4.6}$$

Examples

Coming soon!™

4.6 Rates and Related Rates

Because ***derivatives*** represent rates of change, they can be used to represent the rates of different observable and quantifiable situations. This includes ideas like change in volume of a melting snowball, the distance an object travels over time due to changes in velocity and acceleration, the fluid level of a tank as it is expelling its contents, and so on.

Examples

Coming soon!™

4.7 L'Hopital's Rule, Growth, and Dominance

Suppose we want to evaluate the ***limit*** of a function exactly of a quantity as it approaches a point where the function is not defined, such as:

$$\lim_{x\to 0} \frac{e^{2x}-1}{x} \tag{4.7}$$

This yields $\frac{0}{0}$, which does not make mathematical sense. Though the ***limit*** can be approximated by taking values near the desired evaluation point, it can be calculated exactly by applying the principle of ***local linearity***.

By replacing the numerator and denominator with two functions, $f(x)$ and $g(x)$, we can approximate the limit by the ratio of the corresponding y-values. Taking $f'(x)$ and $g'(x)$ as a tangent-line approximation, we see that:

$$\frac{f(x)}{g(x)} = \frac{e^{2x}-1}{x} \approx \frac{2x}{x} = \frac{2}{1} = \frac{f'(0)}{g'(0)} \tag{4.8}$$

Thus, we arrive at:

$$\lim_{x\to 0} \frac{e^{2x}-1}{x} = 2 \tag{4.9}$$

This technique is referred to as ***L'Hopital's Rule***, which is formally defined as:

L'Hopital's Rule: if f and g are ***differentiable***, $f(a) = g(a) = 0$, and $g'(a) \neq 0$, then:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)} \tag{4.10}$$

This requires that the right-hand ***limit*** exists. ***L'Hopital's Rule*** also applies to limits that approach $\pm\infty$.

Examples

Coming soon!™

4.8 Parametric Equations

Parametric Equations are commonly used to represent motion in the xy-plane because motion is a function of time. Thus, if $x = f(t)$ and $y = g(t)$ then at time t, the object can be located at point $(f(t), g(t))$. The *parameter* of the ***parametric equations*** for this example is the value of time, t.

For straight-line motion, the object is located at an initial point, (x_0, y_0). Provided the x and y rates of change (***deriatives***) are constant, they can be represented by the quantities $a = \frac{dx}{dt}$ and $b = \frac{dy}{dt}$. Thus, at time t, the object has coordinates $(x_0 + at, y_0 + bt)$. The slope of the line the object is following is $m = \frac{b}{a}$.

The ***speed*** of this object can be quantified as well. In one unit of time, t, the object will move a units in the x-direction and b units in the y-direction; provided a and b, are constant. Applying the Pythagorean Theorem results in:

$$\text{Speed} = \frac{Distance}{Time} = \frac{\sqrt{a^2 + b^2}}{1} = \sqrt{a^2 + b^2} \tag{4.11}$$

More generally, for an object along an arbitrary curve with time-varying speed, the ***instantaneous speed*** is given by:

$$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \tag{4.12}$$

where $v_x = \frac{dx}{dt}$ is the ***instantaneous velocity*** in the x-direction and $v_y = \frac{dy}{dt}$ is the ***instantaneous velocity*** in the y-direction. Recall that ***velocity*** has both magnitude *and* direction. This introduces the idea of ***unit vectors***, which have an identity magnitude (1) and are always oriented in the x- and y-directions. For traditional xy-coordinate frames, i, and j are used to represent motion in x and y, respectively. Thus, the ***velocity vector*** is written as:

$$\overrightarrow{v} = v_x \overrightarrow{i} + v_y \overrightarrow{j} \tag{4.13}$$

For any given point (x_0, y_0), a tangent line to the curve can be given parametrically by finding the straight-line motion through (x_0, y_0) with the same x and y velocities as an object moving along the curve.

Sometimes, the curve is more interesting to observe than the object's motion through space and time. This representation is referred to as the *parameterization* of the curve. This can be helpful to graph complicated curves.

The slope and concavity of ***parametric curves*** can be obtained via the ***Chain Rule*** by thinking of the ***parametric equations*** as functions of time. The slope is given by:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \tag{4.14}$$

The second ***derivative*** can be found by a similar method. If w is a ***differentiable*** function of x, then:

$$\frac{dw}{dx} = \frac{\frac{dw}{dt}}{\frac{dx}{dt}} \tag{4.15}$$

For $w = \frac{dy}{dx}$:

$$\frac{dw}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} \tag{4.16}$$

So, by the ***Chain Rule***:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \tag{4.17}$$

Examples

Coming soon!™

Chapter 5

Key Concept: The Definite Integral

5.1 How Do We Measure Distance Traveled?

The ***distance*** formula is commonly referred to as:

$$\text{Distance} = \text{Velocity} \cdot \text{Time} \tag{5.1}$$

Recall that ***velocity*** has both magnitude *and* direction. Thus, it is important to know what direction indicates positive ***velocity***. This section will estimate ***distance*** when ***velocity*** is time-varying.

When estimating the ***distance*** traveled, it is important to know how often the ***velocity*** measurements are taken. For example, consider this table with measurements taken every 2 seconds: Higher frequency ***velocity*** measurements result in less ***distance*** estimation error, but an estimate can still be made. This results in:

$$20 \cdot 2 + 30 \cdot 2 + 38 \cdot 2 + 44 \cdot 2 + 48 \cdot 2 = 360\text{feet} \tag{5.2}$$

This serves as a lower limit because we know the car has moved at these speeds when it was measured. An upper limit considers the fact that the

Time (sec)	0	2	4	6	8	10
Velocity $\left(\frac{\text{ft}}{\text{s}}\right)$	20	30	38	44	38	50

car instantaneously accelerated to the maximum velocity from the next measurement, yielding:

$$30 \cdot 2 + 38 \cdot 2 + 44 \cdot 2 + 48 \cdot 2 + 50 \cdot 2 = 420\text{feet} \tag{5.3}$$

Thus, we can conclude that:

$$360 \leq \text{Total Distance Traveled} \leq 420\text{feet} \tag{5.4}$$

A smaller difference between the upper and lower estimates can be obtained by increasing the measurement frequency. Each time interval between measurements can be represented by a rectangle on the *Time-Velocity*-axes. As the time intervals get smaller, the rectangles become thinner. The ***limit*** of this is as the time interval approaches zero, the rectangles are infinitesimally thin, and the distance error between the lower and upper estimates approaches zero. It will be shown later that, in the ***distance*** and ***velocity*** relationship, the **area under the curve** of the *Time-Velocity*-axes is equivalent to the ***total distance*** traveled, if ***velocity*** is strictly positive.

If ***velocity*** is ever negative, then the object is traveling back towards the starting position. Thus, its ***distance*** from the starting position is decreasing, but the ***total distance*** traveled is increasing.

In the general case, let $v = f(t)$ be a non-negative ***velocity*** function, $t \geq 0$. One may wish to determine the ***distance*** traveled between times a and b. Measurements are taken at evenly spaced times, t_0, t_1, ... t_n. If $a = t_0$ and $b = t_n$, then the time interval between any two measurements is given by:

$$\Delta t = \frac{b - a}{n} \tag{5.5}$$

For each time interval, t_i, the ***distance*** traveled is given by:

$$\text{Distance} = f\left(t_i\right) \Delta t \tag{5.6}$$

Summing all of the distances between each subsequent time interval between a and b yields:

$$\text{Distance} \approx \sum_{i=0}^{n-1} f\left(t_i\right) \Delta t \tag{5.7}$$

This is a ***Left-Hand Sum*** because it includes all velocities from the left-side of the rectangular intervals. The ***Right-Hand Sum*** can be written as:

$$\textbf{Distance} \approx \sum_{i=1}^{n} f\left(t_i\right) \Delta t \tag{5.8}$$

If f is an ***increasing function***, then the ***Left-Hand Sum*** underestimates the ***total distance*** and ***Right-Hand Sum*** overestimates it. Conversely, if f is ***decreasing***, then the ***Left-Hand Sum*** overestimates the ***total distance*** and ***Right-Hand Sum*** underestimates it. For a ***monotonically increasing*** or ***monotonically decreasing*** function, the accuracy of the estimates is given by:

$$\text{Error} = |f(b) - f(a)| \cdot \Delta t \tag{5.9}$$

Examples

Coming soon!™

5.2 The Definite Integral

The ***Definite Integral*** is defined by taking the ***limit*** of the ***Left-Hand Sum*** or ***Right-Hand Sum*** as the parameter n approaches $+\infty$, provided the function $f(x)$ is ***continuous*** on $[a, b]$. This can be written as:

$$\int_a^b f(x)dx \tag{5.10}$$

The summations represented by Equations (5.7) and (5.8) are referred to as ***Riemann Sums***. The ***integrand*** is the function being integrated, $f(x)$, and the ***limits of integration*** are the endpoints of the interval, a and b.

More specifically, Equations (5.7) and (5.8) are special cases of ***Riemann Sums***. The general form of the ***Riemann Sum*** for a function, $f(x), x \in [a, b]$, is given by:

$$\sum_{i=1}^{n} f\left(c_i\right) \Delta t_i \tag{5.11}$$

where $a = t_0 < t_1 < \ldots\ t_n = b$ and, for $i = 1, 2\ \ldots,\ n$, $\Delta t_i = t_i - t_{i-1}$, and $t_{i-1} \leq c_i \leq t_i$. The ***Definite Integral*** approximates the area under the curve down to the x-axis by summing the areas of n rectangles in the ***Riemann Sum***. When the ***integrand*** is negative, the distance to the x-axis is above the curve. Because as positive sign convention is used, the resulting area above the curve found through ***integration*** is negative. This is what causes ***integrations*** such as $\int_0^{2\pi} \sin x \ dx$ to equal zero.

Examples

Coming soon!™

5.3 The Fundamental Theorem and Interpretations

The ***Fundamental Theorem of Calculus*** is written as:

If f is ***continuous*** on $[a, b]$, and $f(x) = F\prime(x)$, then:

$$\int_a^b f(x)dx = F(b) - F(a) \tag{5.12}$$

Thus, if a function f is equal to the ***rate of change*** of a quantity, then the ***definite integral*** results in the total change.

The ***integral*** can also be used to approximate the average value of a function, f, over a given interval, $[a, b]$:

$$\text{Average Value of } f = \frac{1}{b-a}\int_a^b f(x)dx \tag{5.13}$$

Lastly, the ***Fundamental Theorem of Calculus*** can be used to compute ***definite integrals*** exactly.

Examples

Coming soon!™

5.4 Theorems About Definite Integrals

So far, we have only considered the ***Definite Integral*** when $a < b$. Recall that:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x \tag{5.14}$$

Then, provided $f(x)$ is ***continuous***, for any numbers, a, b, and c:

1. $\int_b^a f(x)\ dx = -\int_a^b f(x)\ dx$

2. $\int_a^c f(x)\ dx + \int_c^b f(x)\ dx = \int_a^b f(x)\ dx$

The first result can be derived from the definition of Δx, namely:

$$\Delta x = \frac{(a-b)}{n} = -\frac{(b-a)}{n} \tag{5.15}$$

The second result is true because of the definition of Δx and that the upper limit of integration of the first is equal to the lower limit of integration for the second, c.

We can also evaluate properties of ***integrals*** for multiple functions. Suppose f and g are both continuous functions, and c is an arbitrary constant. Then:

1. $\int_a^b (f(x) \pm g(x))\ dx = \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx$

2. $\int_a^b c \cdot f(x)\ dx = c\int_a^b f(x)\ dx$

These properties hold because of the ***Principle of Superposition*** and that c is simply a scaling factor.

The area between curves can also be calculated, provided $f(x)$ lies above $g(x)$ for $a \leq x \leq b$:

$$\text{Area between } f \text{ and } g = \int_a^b (f(x) - g(x))\,dx \tag{5.16}$$

Symmetry can also be used to aid in the evaluation of ***integrals***. For ***Even***

Functions:

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx \tag{5.17}$$

and for ***Odd Functions***:

$$\int_{-a}^{a} f(x)dx = 0 \tag{5.18}$$

This is because of the definition of ***Even*** and ***Odd Functions***.

Examples

Coming soon!™