Choice under Uncertainty (Lecture 1a)

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Our Outline:

#1: Carefully Define Your Environment

My view: it is impossible to operate without precise definitions of things.

- E.g., if we don't define what it means for someone to "get sick", we can have annoying conversations about appropriate health policy.

This rings especially true when we try to deal with "intuitive" things like "risk".

> Your task: understand the definitions in this course. They are **precise**.

Guiding Questions

1. What are the arguments of the utility function?
2. Can a risk-loving person choose \$10 for sure over a gamble between \$0 and \$14? Why or why not?
3. What assumptions do economists typically make about how risk aversion depends on wealth?

(*) Using the definition of risk aversion from the class, how might we think about vacccine hesitancy?

A Historical Perspective

The study of risk is perhaps the oldest element of "decision science" or behavioral economics.

Renaissance, scientists and mathematicians such as Girolamo Cardano mused about probability and concocted puzzles around games of chance.

- Franciscan monk Luca Pacioli proposed "the problem of points", which asks how one should divide the stakes in an incomplete game.
- "Fast forward" to 1654: French mathematicians Blaise Pascal and Pierre de Fermat developed a way to determine the likelihood of each possible result of a simple game (balla, which had fascinated Pacioli).
- "Fun" fact: the problem of points led directly to the first mathematical proofs by induction and the development of Pascal's infamous triangle.

Today's lecture will cover insights from the mid 18th century to modern approaches.

Definition: A *lottery* (or *gamble* or *risky prospect*) is a set of possible outcomes and a probability of each outcome occurring.

Example A: If you go to a roulette table and place $10 on **black**, you have just purchased a lottery. With probability $18/38$ you will receive $20, and with probability $20/38$ you will receive $0.

Example B: When you buy a new car, you are buying a lottery. For instance, it might be that with probability $1/2$ it will be a car that you love (high value), with probability $2/5$ it will be a car that only adequately serves your needs (low value), and with probability $1/10$ it will be a "lemon" (worthless).

> Side note: why do we call them lemons? Lemons are fine. Were other citrus fruits considered?

Example C: More abstractly, you might get x_1 with probability p_1, x_2 with probability p_2, and x_3 with probability p_3 (where $p_1 + p_2 + p_3 = 1$).

Ways to write lotteries:

- As a probability tree.
- As a vector of outcome-probability pairs:

Example A: Payoff =(\$20,18/38 ;\$0,20/38).

Example B: Car Value =(high,1/2; low,2/5; worthless,1/10).

- As a type of equation (see board in class):

Example B:

Car Value	Probability
High	1/2
Low	2/5
Worthless	1/10

In the realm of choice under uncertainty, we examine people's choices of lotteries.

Next step: Develop models of behavior.

- Suppose you face a choice between two lotteries. How do you decide?

(Not a rhetorical question... how do you? Reflect for a moment.)

One model of behavior: People choose the option with the largest "expected value."

Definition: The *expected value* of a lottery $\mathbf{x} = (x_1, p_1; \ldots; x_n, p_n)$ is

$$EV(\mathbf{x}) \equiv \sum_{i=1}^{n} p_i \, x_i.$$

E.g. the expected value of a 50/50 gamble in which you might win $10 and otherwise win nothing is $5.

St. Petersburg Paradox (Bernoulli, 1713)

Consider the following bet: I'm going to flip a coin, and I'm going to keep on flipping it until I flip a HEADS. Then you'll be paid as a function of how many times we flip. Specifically:

- If I immediately flip a HEADS, you get $2.
- If I flip one TAILS and then a HEADS, you get $4.
- If I flip two TAILS and then a HEADS, you get $8.
- If I flip three TAILS and then a HEADS, you get $16.
- And so forth....

Simple Question: How much are you willing to pay for this bet?

How much would an expected-value maximizer pay?

- The EV of this bet is ∞, but people are unwilling to pay much for it. Hence, EV is not a good description of people's choices.

St. Petersburg Paradox

> "Now it is highly probable that any increase in wealth, no matter how insignificant, will always result in an increase in utility which is inversely proportionate to the quantity of goods already possessed." Bernouilli, 1738 (translated in 1954)

- People have **diminishing marginal utility** of money.
- Just because the gamble is worth infinite money does not mean it is infinitely valuable to a would-be gambler.

Immediate implication: People should maximize *utility*, not money.

> What the f*^# does that mean? Aren't we just moving the goalposts?

Implication 2: Maximizing utility implies people turn down many risks.

What's Missing from EV?

Simple answer: risk aversion.

Definition: A person is *risk-averse* if for any (risky) lottery $\mathbf{x}$ she prefers to have $EV(\mathbf{x})$ for sure instead of lottery $\mathbf{x}$.

Definition: A person is *risk-neutral* if for any (risky) lottery $\mathbf{x}$ she is indifferent between $EV(\mathbf{x})$ for sure vs. lottery $\mathbf{x}$.

Definition: A person is *risk-loving* if for any (risky) lottery $\mathbf{x}$ she prefers to have lottery $\mathbf{x}$ instead of $EV(\mathbf{x})$ for sure.

$\Longrightarrow$ Evidence suggests people are risk-averse.

$\Longrightarrow$ Also your intuition suggests people are risk-averse.

...except it's not true all the time and we'll discuss this later.

A second model of behavior, building on economic models and Bernoulli's intuition: People choose the option with the largest "expected utility".

Definition: The *expected utility* of a lottery $\mathbf{x} = (x_1, p_1; \ldots; x_n, p_n)$ is

$$EU(\mathbf{x}) \equiv \sum_{i=1}^{n} p_i \, u(x_i).$$

for some function $u : \mathbb{R} \to \mathbb{R}$.

Reminder: $u(x)$ is a mathematical representation of *observable choices*.

Could an expected-utility maximizer (EU maximizer) choose

$(100, \frac{1}{2}; 0, \frac{1}{2})$ over $(100, \frac{1}{2}; 100, \frac{1}{2})$?

Could an expected-utility maximizer (EU maximizer) choose

$(100, \frac{1}{2}; 0, \frac{1}{2})$ over $(100, \frac{1}{2}; 50, \frac{1}{2})$?

Could an expected-utility maximizer (EU maximizer) choose

$(100, \frac{1}{2}; 0, \frac{1}{2})$ over $(200, \frac{1}{2}; 0, \frac{1}{2})$?

Point: If we put no restrictions on the utility function $u(x)$, then expected-utility theory can explain **any** individual choice

> ...and we're left with a pretty unsatisfying theory of behavior.

Three Comments:

(1) We say that expected-utility theory can "explain" a choice if there exists a utility function $\boldsymbol{u(x)}$ such that an EU maximizer with utility function $\boldsymbol{u(x)}$ would make that choice.

(2) Even when we put no restrictions on the utility function $\boldsymbol{u(x)}$, although an individual choice cannot violate EU, combinations of choices can violate EU.

E.g.:

$$(100, \tfrac{1}{2}; 0, \tfrac{1}{2}) \succ (200, \tfrac{1}{2}; 0, \tfrac{1}{2})$$

AND

$$(100, \tfrac{1}{3}; 500, \tfrac{2}{3}) \prec (200, \tfrac{1}{3}; 500, \tfrac{2}{3})$$

(3) When we put **restrictions** on the utility function $\boldsymbol{u(x)}$, then even an individual choice could violate EU.

A natural restriction: We virtually always assume that **more is better** --- that is, u is increasing in x.

Definition: Lottery $\mathbf{x}$ *dominates* lottery $\mathbf{y}$ (or equivalently, lottery $\mathbf{y}$ *is dominated by* lottery $\mathbf{x}$) if for any amount z the probability of getting at least z in lottery $\mathbf{x}$ is (weakly) larger than the probability of getting at least z in lottery $\mathbf{y}$.

Examples:

$(200, \frac{1}{2}; 0, \frac{1}{2})$ vs. $(100, \frac{1}{2}; 0, \frac{1}{2})$

$(200, \frac{1}{2}; 0, \frac{1}{2})$ vs $(150, \frac{1}{2}; 50, \frac{1}{2})$

$(200, \frac{1}{3}; 150, \frac{1}{3}, 75, \frac{1}{3})$ vs. $(150, \frac{1}{2}; 75, \frac{1}{2})$

A Modestly Uninteresting Punchline

Result: Under expected utility with more is better, a person will never choose a dominated lottery.

To incorporate risk-aversion (or risk-seeking), we must assume more about the utility function.

Result: Under expected-utility theory, a person is risk-averse if and only if her utility function is concave: $u''(x) < 0$

Result: Under expected-utility theory, a person is risk-neutral if and only if her utility function is linear: $u''(x) = 0$

Result: Under expected-utility theory, a person is risk-loving if and only if her utility function is convex: $u''(x) > 0$

(1) The utility function is unique up to a positive affine transformation.

Definition: The function $\tilde{u}(x)$ is a *positive affine transformation* of the function $u(x)$ if $\tilde{u}(x) = a \cdot u(x) + b$ for some $a > 0$ and any b.

Definition: $\tilde{u}(x)$ and $u(x)$ *reflect the same preferences* if for any pair of lotteries $\mathbf{x}$ and $\mathbf{y}$

$$EU(\mathbf{x}) > EU(\mathbf{y}) \text{ if and only if } E\tilde{U}(\mathbf{x}) > E\tilde{U}(\mathbf{y}).$$

Theorem: $\tilde{u}(x)$ and $u(x)$ reflect the same preferences *if and only if* $\tilde{u}(x)$ is a positive affine transformation of $u(x)$.

(2) Can $u(x)$ be negative? Can $EU(\mathbf{x})$ be negative?

- Answer: Sure. Consider an affine transformation w/ $b = -10000$.
- Point: The level of utility doesn't matter, only the comparisons.

(3) Integration: $\boldsymbol{EU}$ operates on *final wealth*.

So thus far we have been doing things a bit sloppily.

Proper way to use expected-utility theory:

Let $\boldsymbol{w}$ be a person's *wealth*.

Let $\mathbf{x} = (x_1, p_1; \ldots; x_n, p_n)$ be a lottery (or gamble or risky prospect).

- Note: $\mathbf{x}$ yields *income* $\boldsymbol{x_i}$ with probability $\boldsymbol{p_i}$.

EU theory says evaluate prospect $\mathbf{x}$ according to:

$$EU(\mathbf{x}; w) \equiv \sum_{i=1}^{n} p_i \, u(w + x_i).$$

(4) Closely related: EU operates on *reduced lotteries*.

- Example: If you face the possibility of both lottery $\mathbf{x} = (100, 1/2; -90, 1/2)$ and $\mathbf{y} = (95, 1/2; -85, 1/2)$, you must first combine these into a reduced lottery before applying EU.

If they are independent, the reduced lottery is

$$(\ 195, 1/4\ ;\ 15, 1/4\ ;\ 5, 1/4\ ; -175, 1/4\ ).$$

If they have perfect positive correlation, the reduced lottery is

$$(\ 195, 1/2\ ;\ \ -175, 1/2\ ).$$

If they have perfect negative correlation, the reduced lottery is

$$(\ 15, 1/2\ ;\ 5, 1/2\ ).$$

(5) There is a natural mathematical way to describe the degree of risk aversion. (At least, it is natural in a way.)

Goal: We'd like to make statements of the form: "If person A is more risk-averse than person B, then A is less likely to accept a particular gamble."

Question: How to formally define "more risk averse" ?

Definition: The *coefficient of risk aversion* at wealth w is

$$r(w) = \frac{-u''(w)}{u'(w)}.$$

Suppose w_A is person A's wealth and w_B is person B's wealth. If $r(w_A) > r(w_B)$ then person A is less likely than person B to accept a risky gamble.

Note: A person's risk aversion depends on both her utility function and her wealth.

$$r(w) = \frac{-u''(w)}{u'(w)}.$$

How is one's willingness to accept a gamble likely to depend on my wealth?

- Economists often *assume* risk aversion decreases with wealth:

$$\uparrow w \Longrightarrow \downarrow r(w).$$

Definition The *constant relative risk aversion (CRRA)* utility function is

$$u(z) = \frac{(z)^{1-\rho}}{1-\rho} \qquad \text{for } \rho \in [0, \infty).$$

Some Features:

1. For $\rho = 0$, equivalent to $u(x) = x$, or risk neutrality.
2. For $\rho = 1$, equivalent to $u(x) = \ln x$.
3. Note that $r(w) = \rho/w$, hence $\uparrow \rho \implies \uparrow r(w)$ and $\uparrow w \implies \downarrow r(w)$.

Features of CRRA Utility

The functional form implies that a person feels the same about the following gambles:

(\$11, 1/2; -\$10, 1/2) from wealth \$1000.

(\$110, 1/2; -\$100, 1/2) from wealth \$10000

(\$1100, 1/2; -\$1000, 1/2) from wealth \$100000.

Empirical studies suggest ρ in the 1-5 range. (You'll explore what this means a bit later.)

Suppose you have wealth \$1000. (Sorry, poor students)

But there is a 10% chance that you will suffer a loss of \$250.

- Hence, you face the lottery $(\,1000, .9\,;\,750, .1\,)$.

Insurance agent:

> I will fully insure you if you pay me \$p

Question 1: What lottery do you face?

Applying Models to Real-World Settings

Implication: If you buy insurance, you face the lottery: ($1000-p, 1).

Question 2: What is your willingness to pay for full insurance?

$$(\ 1000 - p^*, 1 \ ) \sim (\ 1000, .9 \ ; \ 750, .1 \ )$$

or

$$u(1000 - p^*) = (.9) \ u(1000) + (.1) \ u(750).$$

This was perhaps unrealistic. Suppose instead:

- To insure against the entire \$250 loss, it costs \$p.
- To insure against proportion α of the \$250 loss, it costs αp.

Question 3: What is the optimal α?

Because α is continuous, use calculus:

- Step 1: Derive final lottery as a function of α.
- Step 2: Calculate expected utility as a function of α.
- Step 3: Use calculus to solve for the optimal α.

As a function of the $\boldsymbol{\alpha}$ you choose, final lottery is:

$$\mathbf{x}(\alpha) = (\ 1000 - \alpha p\ ,\ .9\ ; 750 + (250 - p)\alpha\ ,\ .1\ ).$$

As a function of the $\boldsymbol{\alpha}$ you choose, expected utility is:

$$EU(\mathbf{x}(\alpha)) = (.9)\ u(1000 - \alpha p)\ +\ (.1)\ u(750 + (250 - p)\alpha).$$

Definition: Insurance is *actuarially fair* if the premium equals the expected payout.

Definition: Insurance is *actuarially unfair* if the premium is larger than the expected payout; insurance is *actuarially overfair* if the premium is smaller than the expected payout.

Let's assume insurance is actuarially fair.

- Back to our example: actuarially fair insurance means $\boldsymbol{p} = \mathbf{(.10)(250)} =$ \$25.

Substituting $p = 25$ into $EU(\mathbf{x}(\alpha))$:

$$EU(\mathbf{x}(\alpha)) = (.9)u(1000 - 25\alpha) + (.1)u(750 + 225\alpha).$$

Now we need to do some calculus!

. . . or do we?

Let's first check the edge cases: $\alpha = 1$ and $\alpha = 0$.

$$EU(\mathbf{x}(1)) = (.9)u(1000 - 25) + (.1)u(750 + 225).$$

$$= u(975)$$

And likewise:

$$EU(\mathbf{x}(0)) = (.9)u(1000) + (.1)u(750).$$

Question 4: What does the above imply?

Definition: A person is *risk-averse* if for any (risky) lottery $\mathbf{x}$ she prefers to have $EV(\mathbf{x})$ for sure instead of lottery $\mathbf{x}$.

Implication: If insurance is actuarially fair, and if you are risk averse, then you will choose to *fully* insure.

Moreover a risk-averse consumer will reach the same conclusion if insuring against $25,000 loss

...or $250,000 loss

...or $25 loss.

(A slightly different approach) Recall from... some math class you took:

Definition: Function f is *concave* if for any x, y and any $\alpha \in [0, 1]$

$$f((1-\alpha)x + \alpha y) \geq (1-\alpha)f(x) + \alpha f(y)$$

Suppose that u from the previous slide was concave. Reminder:

$$EU(\mathbf{x}(\alpha)) = (.9)u(1000 - 25\alpha) + (.1)u(750 + 225\alpha).$$

Applying concavity yields

$$u(.9(1000 - 25\alpha) + .1(750 + 225\alpha) \geq (.9)u(1000 - \alpha 25) + (.1)u(750 + 225\alpha)$$

$$\Rightarrow u(975) \geq (.9)u(1000 - \alpha 25) + (.1)u(750 + 225\alpha)$$

Suppose you have wealth of \$1000 --- you are, after all, a student--- and that you plan to invest it for one year and then spend it. There exist two assets in the world in which you might invest:

- A *risk-free asset* that yields a certain return of \$0.
- A *risky asset* that yields a return of 50% with probability $1/2$ and a return of -40% with probability $1/2$.

Let α be the amount you invest in the risky asset, so $1000 - \alpha$ is the amount you invest in the risk-free asset. (Assume $0 \leq \alpha \leq 1000$.)

- If do well, final wealth $= (1000 - \alpha)(1.00) + \alpha 1.50) = 1000 + \alpha(.50)$.
- If do poorly, final wealth $= (1000 - \alpha)(1.00) + \alpha(0.60) = 1000 - \alpha(.40)$.

As a function of the $\boldsymbol{\alpha}$ you choose, you face lottery:

$$\mathbf{x}(\alpha) = (\ 1000 + \alpha(.50)\ ,\ \frac{1}{2}\ ;\ 1000 - \alpha(.40)\ ,\ \frac{1}{2}\ ).$$

As a function of the $\boldsymbol{\alpha}$ you choose, your expected utility is:

$$EU(\mathbf{x}(\alpha)) = \frac{1}{2}\, u(1000 + \alpha(.50))\ +\ \frac{1}{2}\, u(1000 - \alpha(.40)).$$

Many people view expected utility as a good *normative* theory. Here's one reason why:

Axioms (properties about a person's choices over lotteries):

1. Preferences are complete and transitive.
2. Preferences are continuous.
3. *Independence* axiom: If $\mathbf{x} \succsim \mathbf{y}$, then for any lottery $\mathbf{z}$:

$$(1-\alpha)\mathbf{x} + \alpha\mathbf{z} \succsim (1-\alpha)\mathbf{y} + \alpha\mathbf{z} \quad \textbf{for any} \quad 0 < \alpha < 1$$

.

Theorem: A person's behavior can be described by expected utility theory if and only if her choices over lotteries are complete, transitive, continuous, and satisfy the independence axiom.

Expected Utility -- Conclusion

EU might be a desirable normative theory. You can decide.

However, EU seems to fail as a positive (descriptive) theory. That's where we'll go next....