Problem Set 0

EC404, Spring 2025

[This problem set is OPTIONAL.]

This problem set is designed to help refresh your skills at solving optimization problems. This is significantly more involved than the mathematics we will use this semester. Nevertheless, it may be useful as a reminder and to "warm up" for the term. In this regard, while the solutions may not seem immediately obvious to you, with a little effort you should be able to follow them. If, even after thinking about it for awhile, you are having major difficulty following the solutions, you should probably come see me. Each problem below has a Part A and a Part B. I've provided detailed solutions to Part A. Part B is left as an exercise for you. In general, please feel welcome to discuss any of these problems with me.

<u>Question (1A)</u>: Suppose a person wants to choose α to maximize

$$u(\alpha) \quad = \quad \frac{1}{2}\ln(1000+.5\alpha) \quad + \quad \frac{1}{2}\ln(1000-.4\alpha).$$

What will she choose?

Solution: This is a simple unconstrained maximization with one variable.

The first-order condition is (reminder: the derivative of $\ln x$ is $1/x$):

$$\frac{du}{d\alpha} = 0 \qquad \Longleftrightarrow \qquad \left(\frac{1}{2}\right)\left(\frac{1}{1000 + .5\alpha}\right)(.5) + \left(\frac{1}{2}\right)\left(\frac{1}{1000 - .4\alpha}\right)(-.4) = 0.$$

And solving for α:

$$\begin{aligned}
\left(\frac{1}{2}\right)\left(\frac{1}{1000 + .5\alpha}\right)(.5) &= \left(\frac{1}{2}\right)\left(\frac{1}{1000 - .4\alpha}\right)(.4) \\
\left(\frac{1}{1000 + .5\alpha}\right)(.5) &= \left(\frac{1}{1000 - .4\alpha}\right)(.4) \\
(.5)(1000 - .4\alpha) &= (.4)(1000 + .5\alpha) \\
500 - .2\alpha &= 400 + .2\alpha \\
\alpha &= 250
\end{aligned}$$

Hence, the person will choose $\alpha = 250$.

More advanced: If you were being really careful, you'd want to confirm that the first-order condition is identifying a maximum (and not a minimum). To do so, you'd check the second-order condition to see if the objective function is concave. Here, it is easy to show that $d^2u/d\alpha^2 < 0$, and so the first-order condition indeed yields a maximum.

Question (1B): Suppose a person wants to choose α to maximize

$$u(\alpha) \quad = \quad \frac{1}{2}\frac{(1000+.5\alpha)^{1-\rho}}{1-\rho} \quad + \quad \frac{1}{2}\frac{(1000-.4\alpha)^{1-\rho}}{1-\rho}.$$

What will she choose?

Reminder: The derivative of $\frac{x^{1-\rho}}{1-\rho}$ is $x^{-\rho}$.

Note: Your answer will be an α that is a function of ρ.

Question (2A): Suppose a person wants to choose x and y to maximize

$$u(x,y) = 2x^{1/2} + 4y^{1/2}$$

subject to the constraint that $x+y=100$. What will she choose?

Solution: This is a constrained optimization problem with two variables. There are two techniques that you can use, the substitution method and the Lagrangian method. (Either technique is fine, so as long as you understand one you'll be fine.)

Substitution Method: In the substitution method, we use the constraint to substitute out one of the variables, which turns the problem into an unconstrained maximization with one variable.

Here, the constraint $x+y=100$ implies $y=100-x$. Substituting $y=100-x$ into the original objective function yields a new (unconstrained) objective function

$$\tilde{u}(x)=2x^{1/2}+4(100-x)^{1/2}.$$

The first-order condition is (reminder: the derivative of $x^{1/2}$ is $\frac{1}{2}x^{-1/2}$):

$$\frac{d\tilde{u}}{dx}=0 \qquad \Longleftrightarrow \qquad x^{-1/2}+2\,(100-x)^{-1/2}\,(-1)=0.$$

And solving for x:

$$\begin{aligned} x^{-1/2} &= 2(100-x)^{-1/2} \\ (100-x)^{1/2} &= 2x^{1/2} \\ \left[(100-x)^{1/2}\right]^2 &= \left[2x^{1/2}\right]^2 \\ 100-x &= 4x \\ x &= 20. \end{aligned}$$

Hence, the person will choose $x=20$ and $y=100-x=80$.

Lagrangian Method: We first set up the Lagrangian equation, which consists of the objective function plus a Lagrangian multiplier λ multiplied by the "constraint":

$$L = 2x^{1/2} + 4y^{1/2} + \lambda\left[100 - (x+y)\right].$$

The first-order conditions for the Lagrangian are:

$$\begin{aligned}
\frac{\partial L}{\partial x} &= 0 \quad \Longleftrightarrow \quad x^{-1/2} - \lambda = 0. \\
\frac{\partial L}{\partial y} &= 0 \quad \Longleftrightarrow \quad 2y^{-1/2} - \lambda = 0. \\
\frac{\partial L}{\partial \lambda} &= 0 \quad \Longleftrightarrow \quad 100 - (x+y) = 0.
\end{aligned}$$

The third condition just spits out the constraint (as always). The first two equations give us a relationship between x and y. Specifically, the equations yield $x^{-1/2} = \lambda$ and $2y^{-1/2} = \lambda$, which together imply $x^{-1/2} = 2y^{-1/2}$. From this, we can conclude that $y = 4x$. Substituting $y = 4x$ into the constraint yields $x + y = x + (4x) = 100$ or $x = 20$.

Hence, the person will choose $x = 20$ and $y = 4x = 80$.

Question (2B): Suppose a person wants to choose x and y to maximize

$$u(x,y) = \frac{x^{1-\rho}}{1-\rho} + 2\frac{y^{1-\rho}}{1-\rho}$$

subject to the constraint that $x + y = I$. What will she choose?

Note: Your answer will be an x and a y that are functions of ρ and I.

Question (3A): Suppose a person wants to choose x, y, and z to maximize

$$u(x,y,z) = \ln x + \delta \ln y + \delta^2 \ln z$$

subject to the constraint that $x + y + z = I$. What will she choose?

Solution: This is a constrained optimization problem with three variables. Again, you can use either the substitution method or the Lagrangian method (and as long as you understand one you'll be fine.).

Substitution Method: Again, we use the constraint to substitute out one of the variables, which here creates an unconstrained maximization with two variables.

The constraint $x+y+z=I$ implies $z=I-x-y$. Substituting $z=I-x-y$ into the original objective function yields a new (unconstrained) objective function

$$\tilde{u}(x,y)=\ln x+\delta\ln y+\delta^2\ln(I-x-y).$$

The first-order conditions for this unconstrained maximization are:

$$\begin{aligned}
\frac{\partial\tilde{u}}{\partial x} &= 0 \qquad \iff \qquad \left(\frac{1}{x}\right)+\delta^2\left(\frac{1}{I-x-y}\right)(-1)=0.\\
\frac{\partial\tilde{u}}{\partial y} &= 0 \qquad \iff \qquad \delta\left(\frac{1}{y}\right)+\delta^2\left(\frac{1}{I-x-y}\right)(-1)=0.
\end{aligned}$$

Now we have two equations in two unknowns, and we solve for x and y — there are various ways to do so, here's one: We can rewrite these equations as:

$$\begin{aligned}
\delta^2 x &= I-x-y\\
\delta y &= I-x-y.
\end{aligned}$$

These equations imply $\delta^2 x=\delta y$ or $y=\delta x$. Substituting $y=\delta x$ into the first equation yields

$$\begin{aligned}
\delta^2 x &= I-x-\delta x\\
x &= \frac{I}{1+\delta+\delta^2}.
\end{aligned}$$

Hence, the person will choose $x = \frac{I}{1+\delta+\delta^2}$, $y = \delta x = \frac{\delta I}{1+\delta+\delta^2}$, and $z = I - x - y = \frac{\delta^2 I}{1+\delta+\delta^2}$.

Lagrangian Method: Again, we first set up the Lagrangian equation:

$$L = \ln x + \delta \ln y + \delta^2 \ln z + \lambda \left[I - (x+y+z) \right].$$

The first-order conditions for the Lagrangian are:

$$\begin{aligned}
\frac{\partial L}{\partial x} &= 0 \quad \Longleftrightarrow \quad \frac{1}{x} - \lambda = 0. \\
\frac{\partial L}{\partial y} &= 0 \quad \Longleftrightarrow \quad \frac{\delta}{y} - \lambda = 0. \\
\frac{\partial L}{\partial z} &= 0 \quad \Longleftrightarrow \quad \frac{\delta^2}{z} - \lambda = 0. \\
\frac{\partial L}{\partial \lambda} &= 0 \quad \Longleftrightarrow \quad I - (x+y+z) = 0.
\end{aligned}$$

Again, the last condition just spits out the constraint. The first three equations give us relationships between x, y, and z. Specifically, the equations yield $1/x = \lambda$, $\delta/y = \lambda$, and $\delta^2/z = \lambda$, which together imply $1/x = \delta/y$ and $1/x = \delta^2/z$. From these, we can conclude that $y = \delta x$ and $z = \delta^2 x$. Substituting $y = \delta x$ and $z = \delta^2 x$ into the constraint yields $x + y + z = x + (\delta x) + (\delta^2 x) = I$ or $x = \frac{I}{1+\delta+\delta^2}$.

Hence, the person will choose $x = \frac{I}{1+\delta+\delta^2}$, $y = \delta x = \frac{\delta I}{1+\delta+\delta^2}$, and $z = \delta^2 x = \frac{\delta^2 I}{1+\delta+\delta^2}$.

Question (3B): Suppose a person wants to choose x, y, and z to maximize

$$u(x,y,z) = \frac{x^{1-\rho}}{1-\rho} + \delta \frac{y^{1-\rho}}{1-\rho} + \delta^2 \frac{z^{1-\rho}}{1-\rho}$$

subject to the constraint that $x + y + z = I$. What will she choose?