A survey on different triangular norm-based fuzzy logics

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Abstract

Among various approaches to fuzzy logics, we have chosen two of them, which are built up in a similar way. Although starting from different basic logical connectives, they both use interpretations based on Frank t-norms. Different interpretations of the implication lead to different axiomatizations, but most logics studied here are complete. We compare the properties, advantages and disadvantages of the two approaches.

Key words: Fuzzy logic, many-valued logic, Frank t-norm

1 Introduction

A many-valued propositional logic with a continuum of truth values modelled by the unit interval $[0,1]$ is quite often called a *fuzzy logic.* In such a logic, the conjunction is usually interpreted by a triangular norm.

In this context, a (propositional) fuzzy logic is considered as an ordered pair $\mathcal{P} = (\mathcal{L}, \mathcal{Q})$ of a *language* (*syntax*) $\mathcal{L}$ and a *structure* (*semantics*) $\mathcal{Q}$ described as follows:

(i) The language of $\mathcal{P}$ is a pair $\mathcal{L} = (A, \mathcal{C})$, where A is an at most countable set of *atomic symbols* and $\mathcal{C}$ is a tuple of *connectives.*

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(ii) The structure of $\mathcal{P}$ is a pair $\mathcal{Q} = ([0,1], \mathcal{M})$, where $[0,1]$ is the set of *truth values*, and the tuple $\mathcal{M}$ consists of the *interpretations* (*meanings*) of the connectives in $\mathcal{C}$.

For simplicity, we fix the set A of atomic symbols throughout this paper. All fuzzy logics have the same syntax, they may differ only by the semantics. Also, the tuple of connectives always will contain at least a conjunction whose interpretation will be given by a t-norm.

The class $\mathcal{F}_{\mathcal{P}}$ of *well-formed formulas* in a fuzzy logic ($\mathcal{P}$-*formulas* for short) is defined inductively as follows:

(i) Each atomic symbol $p \in A$ is a $\mathcal{P}$-formula.

(ii) If $\Box$ is an n-ary connective and $\varphi_1, \ldots, \varphi_n$ are $\mathcal{P}$-formulas, then

$$\Box(\varphi_1, \ldots, \varphi_n)$$

is a $\mathcal{P}$-formula.

For each function $t : A \to [0,1]$, which assigns a truth value to each atomic formula, there exists always a unique *natural extension* of t to a *truth assignment* $\bar{t}_{\mathcal{P}} : \mathcal{F}_{\mathcal{P}} \to [0,1]$ which, for each atomic symbol p, for each n-ary connective $\Box$, its interpretation $\text{Meaning}_{\Box}$ and for all $\mathcal{P}$-formulas $\varphi_1, \ldots, \varphi_n$, is obtained by induction in the following canonical way:

$$\begin{aligned} \bar{t}_{\mathcal{P}}(p) &= t(p) \\ \bar{t}_{\mathcal{P}}(\Box(\varphi_1, \ldots, \varphi_n)) &= \text{Meaning}_{\Box}(\bar{t}_{\mathcal{P}}(\varphi_1), \ldots, \bar{t}_{\mathcal{P}}(\varphi_n)). \end{aligned}$$

The unit interval $[0,1]$, equipped with a triangular norm, forms a commutative, fully ordered semigroup with neutral element 1 and annihilator 0. Triangular norms were introduced in the framework of probabilistic metric spaces [34, 33, 35], based on ideas first presented in [23], and they are applied in several fields, e.g., in fuzzy sets [36], fuzzy logics [2, 14, 29] and their applications, but also in the theory of generalized measures [1, 20] and nonlinear differential and difference equations [28].

To repeat the essential definitions, a *triangular norm* (*t-norm* for short) is a commutative, associative, non-decreasing operation $T : [0,1]^2 \to [0,1]$ with neutral element 1. An immediate consequence of this definition are the following boundary conditions

$$\begin{aligned} T(x,1) &= T(1,x) = x, \\ T(x,0) &= T(0,x) = 0, \end{aligned}$$

which means that all t-norms coincide on the boundary of the unit square $[0,1]^2$.

Three important and basic t-norms are the minimum $T_{\mathbf{M}}$, the product $T_{\mathbf{P}}$ and the Łukasiewicz t-norm $T_{\mathbf{L}}$ given, respectively, by $T_{\mathbf{M}}(x,y) = \min(x,y)$, $T_{\mathbf{P}}(x,y) = xy$ and $T_{\mathbf{L}}(x,y) = \max(0, x+y-1)$.

A *triangular conorm* (*t-conorm* for short) is a commutative, associative, non-decreasing operation $S : [0,1]^2 \to [0,1]$ with $S(x,0) = x$ for all $x \in [0,1]$.

There is an obvious duality between t-norms and t-conorms. Let $N : [0,1] \to [0,1]$ be a *strong (fuzzy) negation*, i.e., an order-reversing involution. For a t-norm T, the function $S_{T,N} : [0,1]^2 \to [0,1]$ given by

$$S_{T,N}(x,y) = N(T(N(x), N(y)))$$

is a t-conorm, called the *N-dual of T*. Dually, for a t-conorm S, the function $T_{S,N} : [0,1]^2 \to [0,1]$ given by

$$T_{S,N}(x,y) = N(S(N(x), N(y)))$$

is a t-norm, called the *N-dual of S*. Moreover, we have $T_{S_{T,N},N} = T$ and $S_{T_{S,N},N} = S$.

If, in particular, we use the *standard (fuzzy) negation* $N_{\mathbf{s}} : [0,1] \to [0,1]$ defined by

$$N_{\mathbf{s}}(x) = 1 - x, \tag{1}$$

then the $N_{\mathbf{s}}$-duals of T and S are simply called *duals* thereof.

The duals of the three important t-norms are the maximum $S_{\mathbf{M}}$, the probabilistic sum $S_{\mathbf{P}}$ and the bounded sum $S_{\mathbf{L}}$ given, respectively, by $S_{\mathbf{M}}(x,y) = \max(x,y)$, $S_{\mathbf{P}}(x,y) = x + y - xy$ and $S_{\mathbf{L}}(x,y) = \min(1, x+y)$.

The family $(T_\lambda)_{\lambda\in[0,\infty]}$ of *Frank t-norms* is given by

$$T_\lambda(x,y) = \begin{cases} T_{\mathbf{M}}(x,y) & \text{if } \lambda = 0, \\ T_{\mathbf{P}}(x,y) & \text{if } \lambda = 1, \\ T_{\mathbf{L}}(x,y) & \text{if } \lambda = \infty, \\ \log_\lambda \left(1 + \dfrac{(\lambda^x - 1)(\lambda^y - 1)}{\lambda - 1}\right) & \text{otherwise.} \end{cases}$$

The duals of the Frank t-norms T_λ are the *Frank t-conorms* S_λ given by

$$S_\lambda(x,y) = \begin{cases} S_{\mathbf{M}}(x,y) & \text{if } \lambda = 0, \\ S_{\mathbf{P}}(x,y) & \text{if } \lambda = 1, \\ S_{\mathbf{L}}(x,y) & \text{if } \lambda = \infty, \\ 1 - \log_\lambda \left(1 + \dfrac{(\lambda^{1-x} - 1)(\lambda^{1-y} - 1)}{\lambda - 1}\right) & \text{otherwise.} \end{cases}$$

In [18], the Frank t-norms and t-conorms are denoted by $T_\lambda^{\mathbf{F}}$ and $S_\lambda^{\mathbf{F}}$, respectively. Since we do not work here with other families of t-norms and t-conorms, we omit the upper index $\mathbf{F}$ in this paper.

An element $x \in]0,1]$ is called a *zero divisor* of a t-norm T if there is some $y \in]0,1]$ with $T(x,y) = 0$. A t-norm T satisfies the *cancellation law* if, for all $x \in]0,1]$, $T(x,y) = T(x,z)$ implies $y = z$. A continuous t-norm T is called *Archimedean* if $T(x,x) < x$ for all $x \in]0,1[$. A continuous Archimedean t-norm is called *nilpotent* if it has at least one zero divisor $x > 0$, and *strict* otherwise. This means that a continuous Archimedean t-norm is strict if and only if it satisfies the cancellation law. The minimum $T_{\mathbf{M}} = T_0$ is the only Frank t-norm which is not Archimedean, and the Łukasiewicz t-norm $T_{\mathbf{L}} = T_\infty$ is the only nilpotent Frank t-norm; all the other Frank t-norms are strict.

The family of Frank t-norms $(T_\lambda)_{\lambda\in[0,\infty]}$ is strictly decreasing, and the family of Frank t-conorms $(S_\lambda)_{\lambda\in[0,\infty]}$ is strictly increasing with respect to the parameter λ (see [1]). Both families are continuous with respect to λ, i.e., for all $\lambda_0 \in [0, \infty]$

$$\lim_{\lambda\to\lambda_0} T_\lambda = T_{\lambda_0},$$
$$\lim_{\lambda\to\lambda_0} S_\lambda = S_{\lambda_0}.$$

Extensive overviews on Frank and other t-norms can be found in [18, 35].

2 Fuzzy logics with residual implications

A reasonable way of constructing connectives in fuzzy logics is to start with a (left) continuous t-norm T and to use the *residuum* (*R-implication*, see [4, 8, 7, 12, 31, 32, 30]) defined by

$$R_T(x, y) = \sup\, \{z \in [0, 1] \mid T(x, z) \le y\}. \tag{2}$$

as the interpretation of the implication. It is immediate that we have

$$R_T(x, y) = 1 \qquad \text{if and only if} \qquad x \le y.$$

Since in this paper we restrict our attention to Frank t-norms T_λ we shall write briefly R_λ rather than R_{T_λ}. Observe that the residuum R_λ cannot be substituted by an expression in the t-norm T_λ and other basic fuzzy logical operations.

The following approach to fuzzy logics with residual implications is described in detail in [13].

A *residuum-based propositional fuzzy logic* (*R-fuzzy logic* for short) is defined, for each $\lambda \in [0, \infty]$, as an ordered pair $\mathcal{R}_\lambda = (\mathcal{L}, \mathcal{Q}_\lambda)$ of a language $\mathcal{L}$ and a structure $\mathcal{Q}_\lambda$ described as follows:

(i) The language of $\mathcal{R}_\lambda$ is a pair $\mathcal{L} = (A, (\wedge, \to, \mathbf{0}))$, where A is an at most countable set of atomic symbols and $\wedge, \to$ and $\mathbf{0}$ are connectives which represent the conjunction, the implication and the (nulary) false statement, respectively.

(ii) The structure of $\mathcal{R}_\lambda$ is a pair $\mathcal{Q}_\lambda = ([0, 1], (T_\lambda, R_\lambda, 0))$, where $[0, 1]$ is the set of truth values, and T_λ, R_λ and 0 (the latter is the zero constant function) are the interpretations (meanings) of the conjunction, the implication and the false statement, respectively.

Recall that we assume the set A of atomic symbols to be fixed throughout the whole paper, so all R-fuzzy logics $\mathcal{R}_\lambda$ will have the same syntax.

The class of well-formed formulas $\mathcal{F}_\mathcal{R}$ in an R-fuzzy logic ($\mathcal{R}$-*formulas* for short) is constructed using the binary connectives $\wedge$ and $\rightarrow$ and the nulary connective $\mathbf{0}$:

(i) Each atomic symbol $p \in A$ is an $\mathcal{R}$-formula.

(ii) $\mathbf{0}$ is an $\mathcal{R}$-formula.

(iii) If φ and ψ are $\mathcal{R}$-formulas, then $\varphi \wedge \psi$ and $\varphi \rightarrow \psi$ are $\mathcal{R}$-formulas.

Since the class $\mathcal{F}_\mathcal{R}$ of well-formed formulas in $\mathcal{R}_\lambda$ is independent of λ, we omit this index.

In R-fuzzy logics, the unique extension of a function $t : A \to [0, 1]$ to a truth assignment $\overline{t}_{\mathcal{R}_\lambda} : \mathcal{F}_\mathcal{R} \to [0, 1]$ is given by

$$\begin{aligned}
\overline{t}_{\mathcal{R}_\lambda}(p) &= t(p),\\
\overline{t}_{\mathcal{R}_\lambda}(\mathbf{0}) &= 0,\\
\overline{t}_{\mathcal{R}_\lambda}(\varphi \wedge \psi) &= T_\lambda(\overline{t}_{\mathcal{R}_\lambda}(\varphi), \overline{t}_{\mathcal{R}_\lambda}(\psi)),\\
\overline{t}_{\mathcal{R}_\lambda}(\varphi \rightarrow \psi) &= R_\lambda(\overline{t}_{\mathcal{R}_\lambda}(\varphi), \overline{t}_{\mathcal{R}_\lambda}(\psi)).
\end{aligned}$$

The logics corresponding to the basic t-norms $T_\mathbf{M}, T_\mathbf{L}$ and $T_\mathbf{P}$, which will play a special role, are the *Gödel R-fuzzy logic* $\mathcal{R}_0 = \mathcal{R}_\mathbf{M}$, the *Łukasiewicz R-fuzzy logic* $\mathcal{R}_\infty = \mathcal{R}_\mathbf{L}$ and the *product R-fuzzy logic* $\mathcal{R}_1 = \mathcal{R}_\mathbf{P}$. In fact, only these three logics are investigated explicitly in [13].

Using the basic logical connectives $\wedge, \rightarrow$ and $\mathbf{0}$, we can define additional logical connectives in an R-fuzzy logic $\mathcal{R}_\lambda$.

The negation $\neg$ in $\mathcal{R}_\lambda$ is defined as an implication with consequence $\mathbf{0}$, i.e.,

$$\neg\varphi = \varphi \rightarrow \mathbf{0}.$$

Its interpretation is the fuzzy negation N_λ given by

$$N_\lambda(x) = R_\lambda(x, 0).$$

For $\lambda = \infty$, i.e., in the Łukasiewicz R-fuzzy logic $\mathcal{R}_\mathbf{L}$, we obtain the standard negation, i.e.,

$$N_\infty(x) = N_\mathbf{s}(x) = 1 - x.$$

Since, with the exception of the Łukasiewicz t-norm $T_\mathbf{L}$, no Frank t-norm has zero divisors, we obtain in all the other cases, i.e., for all $\lambda \in [0, \infty[$, the *Gödel (fuzzy) negation*,

$$N_\lambda(x) = N_\mathbf{G}(x) = \begin{cases} 1 & \text{if } x = 0,\\ 0 & \text{if } x > 0. \end{cases}$$

The Gödel negation is neither continuous nor involutive, so it is not a strong negation. It attains only the crisp truth values 0 and 1. This will cause problems in the interpretation of a disjunction.

A disjunction $\vee$ in an R-fuzzy logic $\mathcal{R}_\lambda$ may be defined using the de Morgan formula

$$\varphi \vee \psi = \neg(\neg\varphi \wedge \neg\psi).$$

Its interpretation is the operation $D_\lambda : [0,1]^2 \to [0,1]$ defined by

$$D_\lambda(x,y) = N_\lambda(T_\lambda(N_\lambda(x), N_\lambda(y))).$$

For the Łukasiewicz R-fuzzy logic, we obtain the Łukasiewicz t-conorm, i.e., $D_\infty = S_{\mathbf{L}}$. In all the other cases, i.e., for all $\lambda \in [0,\infty[$, we have

$$D_\lambda(x,y) = N_{\mathbf{G}}(T_\lambda(N_{\mathbf{G}}(x), N_{\mathbf{G}}(y))) = \begin{cases} 0 & \text{if } x = y = 0, \\ 1 & \text{otherwise.} \end{cases}$$

This latter operation attains only the crisp truth values 0 and 1, and it does not satisfy the boundary conditions for a t-conorm. So it is not an ideal candidate for a reasonable interpretation of the disjunction. An alternative possibility to define a disjunction in an R-fuzzy logic is presented in [6].

For $\Gamma \subseteq \mathcal{F}_\mathcal{R}$ and $K \subseteq [0,1]$, we say that Γ is *K-satisfiable* in $\mathcal{R}_\lambda$ if there exists a truth assignment $\overline{t}_{\mathcal{R}_\lambda}$ such that $\varphi \in \Gamma$ implies $\overline{t}_{\mathcal{R}_\lambda}(\varphi) \in K$. The set Γ is said to be *finitely K-satisfiable* in $\mathcal{R}_\lambda$ if each finite subset of Γ is K-satisfiable in $\mathcal{R}_\lambda$. In R-fuzzy logics $\mathcal{R}_\lambda$ with $\lambda \in [0,\infty[$, the interpretation of the implication is not continuous, so we cannot prove the compactness property analogously as for S-fuzzy logics (see Section 3, Theorem 7).

We shall use the standard definition of a tautology (called 1-tautology in [13]) in R-fuzzy logics. We say that an $\mathcal{R}$-formula φ is a *1-tautology in* $\mathcal{R}_\lambda$ if $\overline{t}_{\mathcal{R}_\lambda}(\varphi) = 1$ for all $t \in [0,1]^A$. As some theorems in the classical logic are not 1-tautologies in $\mathcal{R}_\lambda$, it is necessary to adapt the logical axioms in order to obtain a sound logic. The notion of 1-tautology in $\mathcal{R}_\lambda$ depends on the choice of λ, hence we need different axiomatizations for different R-fuzzy logics. We shall discuss them in some detail in Sections 2.1–2.4.

A set $\Gamma \subseteq \mathcal{F}_\mathcal{R}$ is said to be *closed under modus ponens* if we have $\psi \in \Gamma$ whenever $\varphi \in \Gamma$ and $\varphi \to \psi \in \Gamma$ (where $\varphi \to \psi = \neg(\varphi \wedge \neg\psi)$). The *closure* of a set $\Gamma \subseteq \mathcal{F}_\mathcal{R}$ under modus ponens is then the smallest subset of $\mathcal{F}_\mathcal{R}$ containing Γ and being closed under modus ponens.

An $\mathcal{R}$-formula φ is called an *$\mathcal{R}_\lambda$-theorem* if it belongs to the closure of the set of all logical axioms under modus ponens.

A *theory* $\mathcal{T}$ in an R-fuzzy logic $\mathcal{R}_\lambda$, $\lambda \in [0,\infty]$, is a set of $\mathcal{R}$-formulas. An $\mathcal{R}$-formula is called *$\mathcal{R}_\lambda$-provable* in $\mathcal{T}$ (in symbols $\mathcal{T} \vdash_{\mathcal{R}_\lambda} \varphi$) if it belongs to the closure of the union of $\mathcal{T}$ and the set of all axioms under modus ponens.

In all R-fuzzy logics $\mathcal{R}_\lambda$, $\lambda \in [0,\infty]$, the following deduction theorem holds:

Theorem 1 *Let $\lambda \in [0, \infty]$, $\mathcal{T}$ be a theory in the R-fuzzy logic $\mathcal{R}_\lambda$, and let φ, ψ be $\mathcal{R}$-formulas. Then we have $\mathcal{T} \cup \{\varphi\} \vdash_{\mathcal{R}_\lambda} \psi$ if and only if there is an $n \in \mathbb{N}$ such that $\mathcal{T} \vdash_{\mathcal{R}_\lambda} \varphi^n \to \psi$, where φ^n, $n \in \mathbb{N}$, is the $\mathcal{R}$-formula defined recursively as follows:*

$$\varphi^1 = \varphi,$$
$$\varphi^{n+1} = \varphi \wedge \varphi^n.$$

2.1 The Łukasiewicz R-fuzzy logic

Choosing the Łukasiewicz t-norm $T_{\mathbf{L}}$ as the interpretation of the conjunction operator in the Łukasiewicz R-fuzzy logic $\mathcal{R}_{\mathbf{L}}$, we obtain the interpretation $R_{\mathbf{L}}$ of the implication defined by

$$R_{\mathbf{L}}(x, y) = \min(1 - x + y, 1).$$

The fact that $R_{\mathbf{L}}$ is just the implication introduced in [22] justifies it to call $T_{\mathbf{L}}$ and $S_{\mathbf{L}}$ the Łukasiewicz t-norm and t-conorm, respectively, although these operations nowhere appear explicitly in the work of Łukasiewicz.

In the Łukasiewicz R-fuzzy logic $\mathcal{R}_{\mathbf{L}}$, we have the following compactness theorem (see [13]).

Theorem 2 *Let $\Gamma \subseteq \mathcal{F}_{\mathcal{R}}$, let K be a closed subset of $[0, 1]$ and $r \in [0, 1]$. The Łukasiewicz R-fuzzy logic $\mathcal{R}_{\mathbf{L}}$ has the following properties:*

(i) The set Γ is K-satisfiable in $\mathcal{R}_{\mathbf{L}}$ if and only if it is finitely K-satisfiable in $\mathcal{R}_{\mathbf{L}}$.

(ii) If Γ is $\{r\}$-satisfiable in $\mathcal{R}_{\mathbf{L}}$, then there exists a maximal number $r^* \in [0, 1]$ such that Γ is $\{r^*\}$-satisfiable in $\mathcal{R}_{\mathbf{L}}$.

The Łukasiewicz R-fuzzy logic $\mathcal{R}_{\mathbf{L}}$ is axiomatizable; its set of axioms (see [13]) is given as follows:

[A1] $(\alpha \to \beta) \to [(\beta \to \gamma) \to (\alpha \to \gamma)]$,

[A2] $(\alpha \wedge \beta) \to \alpha$,

[A3] $(\alpha \wedge \beta) \to (\beta \wedge \alpha)$,

[A4] $[\alpha \wedge (\alpha \to \beta)] \to [\beta \wedge (\beta \to \alpha)]$,

[A5a] $[\alpha \to (\beta \to \gamma)] \to [(\alpha \wedge \beta) \to \gamma]$,

[A5b] $[(\alpha \wedge \beta) \to \gamma] \to [\alpha \to (\beta \to \gamma)]$,

[A6] $[(\alpha \to \beta) \to \gamma] \to [((\beta \to \alpha) \to \gamma) \to \gamma]$,

[A7] $\mathbf{0} \to \alpha$,

[L4] $[(\alpha \to \beta) \to \beta] \to [(\beta \to \alpha) \to \alpha]$.

Observe that the classical deduction theorem (Theorem 4) does not hold in the Łukasiewicz R-fuzzy logic, only the weaker Theorem 1. The Łukasiewicz R-fuzzy logic $\mathcal{R}_{\mathbf{L}}$ is sound and complete, i.e., the set of $\mathcal{R}_{\mathbf{L}}$-theorems and the set of 1-tautologies in $\mathcal{R}_{\mathbf{L}}$ coincide.

There is an alternative formulation of the Łukasiewicz fuzzy logic, based only on the implication $\to$ and the false statement $\mathbf{0}$ as basic connectives. The conjunction $\wedge$ is then considered as a derived connective,

$$\varphi \wedge \psi = \neg(\varphi \to \neg\psi).$$

This conjunction is interpreted by the Łukasiewicz t-norm $T_{\mathbf{L}}$, so the interpretation remains the same. In this approach, there is an axiomatization with the following four axioms:

[L1] $\quad \alpha \to (\beta \to \alpha),$

[L2] $\quad (\alpha \to \beta) \to [(\beta \to \gamma) \to (\alpha \to \gamma)],$

[L3] $\quad (\neg\alpha \to \neg\beta) \to (\beta \to \alpha),$

[L4] $\quad [(\alpha \to \beta) \to \beta] \to [(\beta \to \alpha) \to \alpha].$

Notice that [L1] and [L3] are just the axioms [C1] and [C3] of the classical logic (see also Section 3), respectively, and that [L2] (which is equal to [A1]) is weaker than [C2]. The closure of all axioms of the forms [L1]–[L4] under modus ponens gives exactly all $\mathcal{R}_{\mathbf{L}}$-theorems which do not contain the conjunction $\wedge$.

The corresponding algebraic model of the Łukasiewicz R-fuzzy logic $\mathcal{R}_{\mathbf{L}}$ is an MV-algebra [17].

2.2 The Gödel R-fuzzy logic

Choosing the minimum t-norm $T_{\mathbf{M}}$ as the interpretation of the conjunction operator in the Gödel R-fuzzy logic $\mathcal{R}_{\mathbf{G}}$, we obtain the interpretation $R_{\mathbf{M}}$ of the implication defined by

$$R_{\mathbf{M}}(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise.} \end{cases}$$

The R-implication $R_{\mathbf{M}}$ (called the *Gödel fuzzy implication*) is not continuous in the points (x,x) with $x \in [0,1[$. It gives rise to the Gödel negation $N_{\mathbf{G}}$.

In the Gödel R-fuzzy logic $\mathcal{R}_{\mathbf{G}}$, we have at least the following compactness theorem (see [13]):

Theorem 3 *A set $\Gamma \subseteq \mathcal{F}_{\mathcal{R}}$ is $\{1\}$-satisfiable in $\mathcal{R}_{\mathbf{G}}$ if and only if it is finitely $\{1\}$-satisfiable in $\mathcal{R}_{\mathbf{G}}$.*

The Gödel R-fuzzy logic $\mathcal{R}_{\mathbf{G}}$ is axiomatizable (see [5, 10, 13]); its axioms are [A1]–[A7] together with

[G] $\alpha \to (\alpha \wedge \alpha)$.

The axioms [A2] and [G] imply that the conjunction must be interpreted by an idempotent operation. Since the minimum $T_{\mathbf{M}}$ is the only idempotent t-norm, it is the only t-norm for the interpretation of a logic with these axioms.

The Gödel R-fuzzy logic $\mathcal{R}_{\mathbf{G}}$ is the only R-fuzzy logic in which the classical deduction theorem holds, a special case of Theorem 1 (which is also valid for the Gödel R-fuzzy logic, because the conjunction is interpreted by the minimum which is idempotent):

Theorem 4 *Let $\mathcal{T}$ be a theory in the R-fuzzy logic $\mathcal{R}_{\mathbf{G}}$, and let φ, ψ be $\mathcal{R}$-formulas. Then we have*

$$\mathcal{T} \cup \{\varphi\} \vdash_{\mathcal{R}_{\mathbf{G}}} \psi \qquad \textit{if and only if} \qquad \mathcal{T} \vdash_{\mathcal{R}_{\mathbf{G}}} \varphi \to \psi.$$

The Gödel R-fuzzy logic $\mathcal{R}_{\mathbf{G}}$ is sound and complete. The corresponding algebraic model of the Gödel R-fuzzy logic $\mathcal{R}_{\mathbf{G}}$ is a Heyting algebra satisfying one additional condition (see [12, 13] for details).

2.3 The product R-fuzzy logic

Choosing the product t-norm $T_{\mathbf{P}}$ as the interpretation of the conjunction operator in the product R-fuzzy logic $\mathcal{R}_{\mathbf{P}}$, we obtain the interpretation $R_{\mathbf{P}}$ of the implication defined by

$$R_{\mathbf{P}}(x, y) = \begin{cases} 1 & \text{if } x \leq y, \\ \dfrac{y}{x} & \text{otherwise.} \end{cases}$$

The R-implication $R_{\mathbf{P}}$ (also called the *Goguen fuzzy implication*) is not continuous in the point $(0, 0)$. It gives rise to the Gödel negation $N_{\mathbf{G}}$, the same as for the Gödel R-fuzzy logic.

It seems to be an open problem whether a compactness theorem analogous to Theorem 3 holds for the product R-fuzzy logic.

The product R-fuzzy logic $\mathcal{R}_{\mathbf{P}}$ is axiomatizable [14]; its axioms are [A1]–[A7] together with

[P1] $\neg\neg\gamma \to [[(\alpha \wedge \gamma) \to (\beta \wedge \gamma)] \to (\alpha \to \beta)]$,

[P2] $\neg(\alpha \wedge \alpha) \to \neg\alpha$.

The axiom [P1] means that the conjunction satisfies the cancellation law. So only t-norms satisfying the cancellation law (e.g., strict t-norms) are acceptable candidates for the interpretation of a logic with the axioms [P1] and [P2].

The product R-fuzzy logic does not satisfy the classical deduction theorem (Theorem 4), only the weaker Theorem 1. The product R-fuzzy logic $\mathcal{R}_{\mathbf{P}}$ is sound and complete. The corresponding algebraic model of the product R-fuzzy logic $\mathcal{R}_{\mathbf{P}}$ is called a product algebra (see [13, 14]).

2.4 Other R-fuzzy logics

What was said about the product R-fuzzy logic $\mathcal{R}_{\mathbf{P}} = \mathcal{R}_1$, remains essentially valid also for all R-fuzzy logics $\mathcal{R}_\lambda$ with $\lambda \in \,]0, \infty[$. Due to the representation theorem for strict t-norms (see, e.g., [21, 26]), there is an automorphism (i.e., an order-preserving bijection, also called a multiplicative generator) $h_\lambda : [0,1] \to [0,1]$ such that, for all $x, y \in [0,1]$,

$$h_\lambda(T_\lambda(x,y)) = T_{\mathbf{P}}(h_\lambda(x), h_\lambda(y)). \tag{3}$$

The automorphism h_λ represents a change of the scale of the unit interval which transforms T_λ into the product t-norm $T_{\mathbf{P}}$. It transforms the corresponding R-implication R_λ into the Goguen fuzzy implication $R_{\mathbf{P}}$. The Gödel negation $N_{\mathbf{G}}$, however, is preserved under the automorphism h_λ. The whole structure is (up to the change of scale represented by h_λ) exactly the same as in the product R-fuzzy logic $\mathcal{R}_{\mathbf{P}}$.

Again the problem remains open whether a compactness theorem analogous to Theorem 3 holds for the R-fuzzy logics $\mathcal{R}_\lambda$ with $\lambda \in \,]0, \infty[$.

All R-fuzzy logics $\mathcal{R}_\lambda$ with $\lambda \in \,]0, \infty[$ are axiomatizable by the same axioms as the product R-fuzzy logic $\mathcal{R}_{\mathbf{P}}$, i.e., by [A1]–[A7] together with [P1] and [P2]. They do not satisfy the classical deduction theorem (Theorem 8), only Theorem 1, but they are sound and complete.

Let us summarize some properties which are common for all R-fuzzy logics. The general form of the completeness theorem is as follows:

Theorem 5 *For each $\lambda \in [0, \infty]$, the R-fuzzy logic $\mathcal{R}_\lambda$ is sound and complete, i.e., the set of $\mathcal{R}_\lambda$-theorems and the set of 1-tautologies in $\mathcal{R}_\lambda$ coincide.*

The *validation set* $V_{\mathcal{R}_\lambda}(\varphi)$ of a given $\mathcal{R}$-formula φ in $\mathcal{R}_\lambda$ is defined as

$$V_{\mathcal{R}_\lambda}(\varphi) = \{\bar{t}_{\mathcal{R}_\lambda}(\varphi) \mid t \in [0,1]^A\}.$$

Notice that this notion depends on the choice of λ. In contrast to the situation of S-fuzzy logics (see Section 3), the validation set $V_{\mathcal{R}_\lambda}(\varphi)$ of an $\mathcal{R}$-formula φ in $\mathcal{R}_\lambda$ with $\lambda \in [0, +\infty[$ is not neccesarily a subinterval of $[0,1]$, so an analogy of Proposition 10 holds only for the Łukasiewicz R-fuzzy logic.

An $\mathcal{R}$-formula φ is called an $\mathcal{R}_\lambda$-*contradiction* if $\neg\varphi$ is an $\mathcal{R}_\lambda$-theorem, and φ is called an $\mathcal{R}_\lambda$-*contingency* if it is neither an $\mathcal{R}_\lambda$-theorem nor an $\mathcal{R}_\lambda$-contradiction. These notions depend on λ because of different axiomatizations. We have the following characterization by the validation sets:

Theorem 6 *Let $\lambda \in [0, \infty]$, $\mathcal{R}_\lambda$ be an R-fuzzy logic, φ an $\mathcal{R}$-formula, and $V_{\mathcal{R}_\lambda}(\varphi)$ its validation set. Then we have:*

(i) φ is an $\mathcal{R}_\lambda$-theorem if and only if $V_{\mathcal{R}_\lambda}(\varphi) = \{1\}$;

(ii) φ is an $\mathcal{R}_\lambda$-contradiction if and only if $V_{\mathcal{R}_\lambda}(\varphi) = \{0\}$;

(iii) φ is an $\mathcal{R}_\lambda$-contingency if and only if $\{0\} \neq V_{\mathcal{R}_\lambda}(\varphi) \neq \{1\}$.

If, for an $\mathcal{R}$-formula φ and a truth assignment $\bar{t}_{\mathcal{R}_\lambda}$, we have $\bar{t}_{\mathcal{R}_\lambda}(\varphi) \in \,]0,1[$, then φ is an $\mathcal{R}_\lambda$-contingency. Observe that this condition is not necessary in R-fuzzy logics since there are $\mathcal{R}_\lambda$-contingencies φ with $\bar{t}_{\mathcal{R}_\lambda}(\varphi) \notin \,]0,1[$ for any truth assignment $\bar{t}_{\mathcal{R}_\lambda}$. For example, if we take $\lambda \in [0,\infty[$ and an arbitrary atomic symbol p, then for the $\mathcal{R}$-formula $\varphi = \neg p$ we obtain $V_{\mathcal{R}_\lambda}(\varphi) = \{0,1\}$.

3 S-fuzzy logics

Another way to construct propositional fuzzy logics was presented in [2], where the full details and proofs of most of the theorems can be found. Further results are proved in [15].

To start with, let T_λ be the Frank t-norm with index $\lambda \in [0,\infty]$, $N_{\mathbf{s}}$ the standard negation given by (1), and S_λ the Frank t-conorm dual to T_λ.

An *S-fuzzy logic* [2] is defined, for each $\lambda \in [0,\infty]$, as an ordered pair $\mathcal{S}_\lambda = (\mathcal{L}, \mathcal{Q}_\lambda)$ of a language $\mathcal{L}$ and a structure $\mathcal{Q}_\lambda$ described as follows:

(i) The language of $\mathcal{S}_\lambda$ is a pair $\mathcal{L} = (A, (\neg, \wedge))$, where A is an at most countable set of *atomic symbols* and $\neg$ and $\wedge$ are *connectives* which, as usual, are called *negation* and *conjunction*, respectively.

(ii) The structure of $\mathcal{S}_\lambda$ is a pair $\mathcal{Q}_\lambda = ([0,1], (N_{\mathbf{s}}, T_\lambda))$, where $[0,1]$ is the set of *truth values*, and $N_{\mathbf{s}}$ and T_λ are the *interpretations* of the negation $\neg$ and the conjunction $\wedge$, respectively.

Again we fix the set A of atomic symbols, so all S-fuzzy logics $\mathcal{S}_\lambda$ have the same syntax, and they may differ only by their semantics. Therefore, there isno need to index the language $\mathcal{L}$ by the parameter λ.

For $\lambda = 0$, we obtain the *min-max S-fuzzy logic* $\mathcal{S}_0 = \mathcal{S}_{\mathbf{M}}$. For $\lambda = \infty$, we obtain the *Łukasiewicz S-fuzzy logic* $\mathcal{S}_\infty = \mathcal{S}_{\mathbf{L}}$. In these cases, we use the indices $\mathbf{M}$ and $\mathbf{L}$ also for the corresponding structures, etc.

The class $\mathcal{F}_{\mathcal{S}}$ of *well-formed formulas* in an S-fuzzy logic (*$\mathcal{S}$-formulas* for short) is defined inductively as follows:

(i) Each atomic symbol $p \in A$ is an $\mathcal{S}$-formula.

(ii) If φ is an $\mathcal{S}$-formula, then $\neg\varphi$ is an $\mathcal{S}$-formula.

(iii) If φ and ψ are $\mathcal{S}$-formulas, then $\varphi \wedge \psi$ is an $\mathcal{S}$-formula.

Since the class $\mathcal{F}_{\mathcal{S}}$ of well-formed formulas in $\mathcal{S}_\lambda$ is independent of λ, we can omit this index.

In S-fuzzy logics, the unique extension of a function $t : A \to [0,1]$ to a truth assignment $\bar{t}_{\mathcal{S}_\lambda} : \mathcal{F}_{\mathcal{S}} \to [0,1]$ is given by

$$
\begin{aligned}
\bar{t}_{\mathcal{S}_\lambda}(p) &= t(p) \\
\bar{t}_{\mathcal{S}_\lambda}(\neg\varphi) &= N_{\mathbf{s}}(\bar{t}_{\mathcal{S}_\lambda}(\varphi)), \\
\bar{t}_{\mathcal{S}_\lambda}(\varphi \wedge \psi) &= T_\lambda(\bar{t}_{\mathcal{S}_\lambda}(\varphi), \bar{t}_{\mathcal{S}_\lambda}(\psi)).
\end{aligned}
$$

Starting with the basic logical connectives $\neg$ and $\wedge$, we can define additional logical connectives in an S-fuzzy logic $\mathcal{S}_\lambda$. The *disjunction* $\vee$ in $\mathcal{S}_\lambda$ is defined using the de Morgan formula

$$\varphi \vee \psi = \neg(\neg\varphi \wedge \neg\psi).$$

For the interpretation of the disjunction, we obtain

$$\bar{t}_{\mathcal{S}_\lambda}(\varphi \vee \psi) = N_{\mathbf{s}}(T_\lambda(N_{\mathbf{s}}(\bar{t}_{\mathcal{S}_\lambda}(\varphi)), N_{\mathbf{s}}(\bar{t}_{\mathcal{S}_\lambda}(\psi)))) = S_\lambda(\bar{t}_{\mathcal{S}_\lambda}(\varphi), \bar{t}_{\mathcal{S}_\lambda}(\psi)),$$

so the disjunction $\vee$ is interpreted by the t-conorm S_λ which is dual to T_λ.

The *implication* $\to$ in $\mathcal{S}_\lambda$ is defined as

$$\varphi \to \psi = \neg(\varphi \wedge \neg\psi).$$

This is one of numerous formulas which are equivalent to the (unique) implication in the classical logic. In fuzzy logics, these formulas are not necessarily equivalent, hence the choice of implication becomes important. For the interpretation of the implication, we obtain

$$\bar{t}_{\mathcal{S}_\lambda}(\varphi \to \psi) = N_{\mathbf{s}}(T_\lambda(\bar{t}_{\mathcal{S}_\lambda}(\varphi), N_{\mathbf{s}}(\bar{t}_{\mathcal{S}_\lambda}(\psi)))) = S_\lambda(N_{\mathbf{s}}(\bar{t}_{\mathcal{S}_\lambda}(\varphi)), \bar{t}_{\mathcal{S}_\lambda}(\psi)).$$

Thus the logical implication $\to$ is interpreted by the binary operation $I_\lambda : [0,1]^2 \to [0,1]$ defined by

$$I_\lambda(x,y) = S_\lambda(N_{\mathbf{s}}(x), y),$$

which is often called the *S-implication* induced by the t-norm T_λ. This notion is the main reason why we call the corresponding logic an S-fuzzy logic. Notice that, for all $\lambda \in [0,\infty[$,

$$I_\lambda(x,y) = 1 \qquad \text{if and only if} \qquad (x = 0 \text{ or } y = 1). \tag{4}$$

Only for the Łukasiewicz S-implication $I_{\mathbf{L}} = I_\infty$ we have

$$I_{\mathbf{L}}(x,y) = 1 \qquad \text{if and only if} \qquad x \leq y. \tag{5}$$

Observe that the S-implication $I_{\mathbf{L}}$ coincides with the R-implication $R_{\mathbf{L}}$. So the interpretation of logical connectives in the Łukasiewicz S-fuzzy logic $\mathcal{S}_{\mathbf{L}}$ and the Łukasiewicz R-fuzzy logic $\mathcal{R}_{\mathbf{L}}$ is identical (although not the same connectives are considered as the basic ones).

One difference between the two Łukasiewicz fuzzy logics $\mathcal{R}_{\mathbf{L}}$ and $\mathcal{S}_{\mathbf{L}}$ is that the nulary connective $\mathbf{0}$ was not considered a basic $\mathcal{S}$-formula. Nevertheless, it can be introduced as a derived logical connective putting, e.g., $\mathbf{0} = \neg\varphi \wedge \varphi$ for a fixed $\mathcal{S}$-formula φ, so these formulas are semantically equivalent in the Łukasiewicz S-fuzzy logic $\mathcal{S}_{\mathbf{L}}$.

Satisfiability (as well as finite satisfiability) in S-fuzzy logics is defined analogously to R-fuzzy logics, i.e., for $\Gamma \subseteq \mathcal{F}_{\mathcal{S}}$ and $K \subseteq [0,1]$ we say that Γ is *K-satisfiable* in $\mathcal{S}_\lambda$ if there exists a truth assignment $\bar{t}_{\mathcal{S}_\lambda}$ such that $\varphi \in \Gamma$ implies $\bar{t}_{\mathcal{S}_\lambda}(\varphi) \in K$. The set Γ is said to be *finitely K-satisfiable* in $\mathcal{S}_\lambda$ if each finite subset of Γ is K-satisfiable in $\mathcal{S}_\lambda$.

An important feature of S-fuzzy logics $\mathcal{S}_\lambda$ is that they have the *compactness property* (see [2, Theorem 3.3, Proposition 3.6]):

Theorem 7 *Let $\lambda \in [0, \infty]$ and let $\mathcal{S}_\lambda$ be an S-fuzzy logic. Then for each $\Gamma \subseteq \mathcal{F}_{\mathcal{S}}$, for each closed subset K of $[0,1]$ and for each $r \in [0,1]$ we have:*

(i) The set Γ is K-satisfiable in $\mathcal{S}_\lambda$ if and only if it is finitely K-satisfiable in $\mathcal{S}_\lambda$.

(ii) If Γ is $\{r\}$-satisfiable in $\mathcal{S}_\lambda$, then there exists a maximal number $r^* \in [0,1]$ such that Γ is $\{r^*\}$-satisfiable in $\mathcal{S}_\lambda$.

In analogy to the classical two-valued logic, an $\mathcal{S}$-formula φ is said to be a *logical axiom* if, for some $\alpha, \beta, \gamma \in \mathcal{F}_{\mathcal{S}}$, φ has one of the following forms:

[C1] $\quad \alpha \to (\beta \to \alpha)$,

[C2] $\quad [\alpha \to (\beta \to \gamma)] \to [(\alpha \to \beta) \to (\alpha \to \gamma)]$,

[C3] $\quad (\neg\alpha \to \neg\beta) \to (\beta \to \alpha)$.

Since the implication is considered a derived logical connective in $\mathcal{S}_\lambda$, its use in the axioms should be avoided. To be precise, we should rewrite the axioms [C1]–[C3] using only the basic connectives $\neg$ and $\wedge$. However, this would lead to less intuitive expressions. The form of axioms makes no difference in the notions depending on the axiomatic system, so we use the standard form of axioms known from the classical logic, which was also used in [2].

An $\mathcal{S}$-formula φ is called an $\mathcal{S}$-*theorem* if it belongs to the closure of the set of all logical axioms under modus ponens. This notion is the same in all S-fuzzy logics $\mathcal{S}_\lambda$, so it is not indexed by λ. The notions of a theory $\mathcal{T}$ in an S-fuzzy logic $\mathcal{S}_\lambda$ and of a formula φ which is $\mathcal{S}$-provable in $\mathcal{T}$ (in symbols $\mathcal{T} \vdash_{\mathcal{S}} \varphi$) are defined analogously to R-fuzzy logics. However, $\mathcal{S}$-provability of a formula φ is independent of the choice of the particular S-fuzzy logic $\mathcal{S}_\lambda$. For all S-fuzzy logics, we have the classical deduction theorem:

Theorem 8 *Let $\lambda \in [0, \infty]$, $\mathcal{T}$ be a theory in the S-fuzzy logic $\mathcal{S}_\lambda$, and let φ, ψ be $\mathcal{S}$-formulas. Then we have*

$$\mathcal{T} \cup \{\varphi\} \vdash_{\mathcal{S}} \psi \qquad \textit{if and only if} \qquad \mathcal{T} \vdash_{\mathcal{S}} \varphi \to \psi.$$

In any of the S-fuzzy logics $\mathcal{S}_\lambda$, a truth assignment $\bar{t}_{\mathcal{S}_\lambda}$ evaluates some $\mathcal{S}$-theorems, even the axioms, by values less than 1. Therefore, in order to achieve soundness and completeness of an S-fuzzy logic, the notion of tautology has to be adapted accordingly. We say that an $\mathcal{S}$-formula φ is a *tautology in* $\mathcal{S}_\lambda$ if $\bar{t}_{\mathcal{S}_\lambda}(\varphi) > 0$ for all $t \in [0,1]^A$. Notice that this notion depends on the choice of λ.

Theorem 9 *For each $\lambda \in [0, \infty[$, the S-fuzzy logic $\mathcal{S}_\lambda$ is sound and complete, i.e., the set of $\mathcal{S}$-theorems and the set of tautologies in $\mathcal{S}_\lambda$ coincide.*

An analogue of Theorem 9 for the Łukasiewicz S-fuzzy logic $\mathcal{S}_{\mathbf{L}}$ does not hold because, as a consequence of the existence of zero divisors of $T_{\mathbf{L}}$, it is not sound. For instance, the formula $(p \wedge p) \vee (\neg p \wedge \neg p)$, where p is an atomic symbol, is an $\mathcal{S}$-theorem which is not a tautology in $\mathcal{S}_{\mathbf{L}}$. Indeed, if we choose $t(p) = 0.5$, we obtain $\bar{t}_{\mathcal{S}_{\mathbf{L}}}((p \wedge p) \vee (\neg p \wedge \neg p)) = 0$.

The non-soundness of the Łukasiewicz S-fuzzy logic $\mathcal{S}_{\mathbf{L}}$ seems to contradict the soundness of the Łukasiewicz R-fuzzy logic $\mathcal{R}_{\mathbf{L}}$ (in particular since, in some sense, these two logics are even "semantically equivalent" [19]). The reason is that the notions of a theorem are different (as a consequence of different axiomatizations) in these two logics, and that the tautologies in $\mathcal{S}_{\mathbf{L}}$ are not necessarily 1-tautologies in $\mathcal{R}_{\mathbf{L}}$.

In S-fuzzy logics, even more can be said in terms of the validation set $V_{\mathcal{S}_\lambda}(\varphi)$ of an $\mathcal{S}$-formula φ in $\mathcal{S}_\lambda$

$$V_{\mathcal{S}_\lambda}(\varphi) = \{\bar{t}_{\mathcal{S}_\lambda}(\varphi) \mid t \in [0,1]^A\}.$$

Proposition 10 *For each $\lambda \in [0, \infty]$, each S-fuzzy logic $\mathcal{S}_\lambda$ and for each $\mathcal{S}$-formula φ, the validation set $V_{\mathcal{S}_\lambda}(\varphi)$ is a closed subinterval of $[0,1]$ such that either $0 \in V_{\mathcal{S}_\lambda}(\varphi)$ or $1 \in V_{\mathcal{S}_\lambda}(\varphi)$.*

An $\mathcal{S}$-formula φ is called an $\mathcal{S}$-*contradiction* if $\neg\varphi$ is an $\mathcal{S}$-theorem, and φ is called an $\mathcal{S}$-*contingency* if it is neither an $\mathcal{S}$-theorem nor an $\mathcal{S}$-contradiction.

For S-fuzzy logics $\mathcal{S}_\lambda$ which are different from the Łukasiewicz S-fuzzy logic $\mathcal{S}_{\mathbf{L}}$ we can give a more specific characterization of the validation sets:

Theorem 11 *Let $\lambda \in [0, \infty[$, $\mathcal{S}_\lambda$ be an S-fuzzy logic, φ an $\mathcal{S}$-formula, and $V_{\mathcal{S}_\lambda}(\varphi)$ its validation set. Then we have:*

(i) φ is an $\mathcal{S}$-theorem if and only if, for some $a \in]0,1[$, $V_{\mathcal{S}_\lambda}(\varphi) = [a, 1]$;

(ii) φ is an $\mathcal{S}$-contradiction if and only if, for some $b \in]0,1[$, $V_{\mathcal{S}_\lambda}(\varphi) = [0, b]$;

(iii) φ is an $\mathcal{S}$-contingency if and only if $V_{\mathcal{S}_\lambda}(\varphi) = [0, 1]$.

For the min-max S-fuzzy logic $\mathcal{S}_{\mathbf{M}}$, we have an even stronger result (see [2, Corollary 5.3]):

Theorem 12 *Let φ be an $\mathcal{S}$-formula in the min-max S-fuzzy logic $\mathcal{S}_{\mathbf{M}}$, and $V_{\mathcal{S}_{\mathbf{M}}}(\varphi)$ its validation set. Then we have:*

(i) φ is an $\mathcal{S}$-theorem if and only if $V_{\mathcal{S}_{\mathbf{M}}}(\varphi) = [0.5, 1]$;

(ii) φ is an $\mathcal{S}$-contradiction if and only if $V_{\mathcal{S}_{\mathbf{M}}}(\varphi) = [0, 0.5]$;

(iii) φ is an $\mathcal{S}$-contingency if and only if $V_{\mathcal{S}_{\mathbf{M}}}(\varphi) = [0, 1]$.

No analogue of Theorem 11 holds for the Łukasiewicz S-fuzzy logic $\mathcal{S}_{\mathbf{L}}$. Also, for $\lambda \in]0, \infty[$ there is no chance for a strengthening of Theorem 11 since, in contrast to Theorem 12(i), we have the following result [15]:

Theorem 13 *Let $\lambda \in]0, \infty[$, $\mathcal{S}_\lambda$ be an S-fuzzy logic and let D be the set of all numbers $a \in [0,1]$ such that there exists an $\mathcal{S}$-theorem φ_a with $V_{\mathcal{S}_\lambda}(\varphi_a) = [a, 1]$. Then D is a dense subset of $[0,1]$.*

4 A comparison

During the introduction and discussion of R- and S-fuzzy logics, we already mentioned some of their similarities and differences. We shall summarize this knowledge and add a comparison from other viewpoints.

Both approaches can be formalized in a way similar to the classical logic. They use different sets of logical connectives. The missing basic connectives cannot always be substituted by derived connectives. Different interpretations of the implication cause the main difference in semantics. Both approaches work with logics which are truth functional; the truth assignment is calculated for a compound formula uniquely from the evaluation of its subformulas. The two approaches use the same single deduction rule — modus ponens — but they are based on different axiomatizations and, therefore, they have to work with a different notion of tautology in order to achieve soundness and completeness.

In an R-fuzzy logic, the basic connectives are the conjunction $\wedge$, the implication $\to$ and the false statement $\mathbf{0}$, the negation $\neg$ is a derived connective. Except for the Łukasiewicz R-fuzzy logic $\mathcal{R}_{\mathbf{L}}$, a connective dual to the conjunction (by the de Morgan formula) does not give a reasonable disjunction. We also have some kind of non-symmetry, because we have a conjunction (interpreted by a t-norm) without a corresponding disjunction interpreted by the dual t-conorm.

There is one observation restricting the latter disadvantage: The formula $\varphi \wedge (\varphi \to \psi)$ (in any R-fuzzy logic) has many properties of an (idempotent) conjunction of φ and ψ, and it is interpreted by the minimum t-norm (due to the properties of R-implications). In the Gödel R-fuzzy logic $\mathcal{R}_{\mathbf{G}}$ (and only in it), this formula is semantically equivalent to $\varphi \wedge \psi$. Further, the formula

$$[(\varphi \to \psi) \to \psi] \wedge [[(\varphi \to \psi) \to \psi] \to [(\psi \to \varphi) \to \varphi]]$$

has properties of an (idempotent) disjunction of φ and ψ, and it is interpreted by the maximum t-conorm. So we have a disjunction (interpreted by a t-conorm) in any R-fuzzy logic, but it is dual to the (basic) conjunction only in the Gödel R-fuzzy logic $\mathcal{R}_{\mathbf{G}}$. In the Łukasiewicz R-fuzzy logic $\mathcal{R}_{\mathbf{L}}$, we have a disjunction dual to the basic conjunction. So the problems with a disjunction arise only in R-fuzzy logics $\mathcal{R}_\lambda$ with $\lambda \in]0, \infty[$ since all these logics are (up to an automorphism of $[0, 1]$) equivalent to the product R-fuzzy logic $\mathcal{R}_{\mathbf{P}}$.

The use of an R-implication for the interpretation of the implication causes some problems; except for the Łukasiewicz R-fuzzy logic, the corresponding R-implication is not continuous. In this case (i.e., in $\mathcal{R}_\lambda$ with $\lambda \in [0, \infty[$) the negation is interpreted by the Gödel negation $N_{\mathbf{G}}$ which is not strong and attains crisp values only. This seems to decrease the applicability of such a logic.

On the other hand, the use of an R-implication allows very nice and deep logical results to be proven. New (weaker) sets of axioms of the Gödel, Łukasiewicz and product R-fuzzy logics led to quite new axiomatizations.

The main disadvantage of R-fuzzy logics $\mathcal{R}_\lambda$ with $\lambda \in]0, \infty[$ — the absence of a disjunction dual to the conjunction — led recently to a new concept, an R-fuzzy logic with an involutive negation (see [6]). In this approach, the negation $\neg$ becomes an additional basic connective interpreted by a strong negation N. This negation can be used in the de Morgan formula defining a dual disjunction which is interpreted by the N-dual t-conorm. Then $\neg\varphi$ does not necessarily coincide with $\varphi \to \mathbf{0}$. Also this logic is axiomatizable, sound and complete. On the other hand, the system of axioms is more complicated, and also one new deduction rule has to be added.

In an S-fuzzy logic, the basic connectives are the conjunction $\wedge$ and the negation $\neg$, and we can derive an implication $\to$ and a disjunction $\vee$, as well as the other usual logical connectives, in analogy to the classical logic. However, the nulary operation $\mathbf{0}$, i.e., the false statement as a constant, can be obtained only in the Łukasiewicz S-fuzzy logic $\mathcal{S}_{\mathbf{L}}$ (e.g., as $\neg\varphi \wedge \varphi$ for an arbitrary formula φ). In all other S-fuzzy logics, i.e., in $\mathcal{S}_\lambda$ for $\lambda \in [0, \infty[$, there is no formula which is evaluated to zero by any truth assignment. This disadvantage can be easily eliminated by adding $\mathbf{0}$ as a basic nulary connective. No serious problems arise, only the validation sets may become singletons $\{0\}$ or $\{1\}$ and the formulation of Theorems 11 and 12 has to be generalized to include this case.

In Section 2.4, we argued that all R-fuzzy logics $\mathcal{R}_\lambda$ with $\lambda \in]0, \infty[$ are equivalent to the product R-fuzzy logic $\mathcal{R}_{\mathbf{P}} = \mathcal{R}_1$ up to an automorphism $h_\lambda : [0,1] \to [0,1]$. This argument does not work in S-fuzzy logics. The automorphism h_λ satisfying (3) is the same. However, it need not preserve the standard negation $N_{\mathbf{s}}$, because the equality $h_\lambda(N_{\mathbf{s}}(x)) = N_{\mathbf{s}}(h_\lambda(x))$ does not hold in general. In this case, there is still a t-conorm S satisfying

$$h_\lambda(S_\lambda(x,y)) = S_{\mathbf{P}}(h_\lambda(x), h_\lambda(y)),$$

which is not necessarily the ($N_{\mathbf{s}}$-)dual of T_λ but the N-dual of T_λ, where the strong negation N is given by

$$N(x) = h_\lambda^{-1}(N_{\mathbf{s}}(h_\lambda(x))).$$

As a consequence, the S-fuzzy logics $\mathcal{S}_\lambda$ with $\lambda \in]0, \infty[$ have basically different semantics.

Proposition 10 works because all connectives in an S-fuzzy logic have continuous interpretations. On the other hand, the choice of an S-implication as the interpretation of the implication causes serious problems from the logical point of view.

The most important disadvantage of S-fuzzy logics seems to be that their syntax is essentially the syntax of the classical logic and it does not bring anything new. Using the standard system of axioms of the classical logic, we obtain as $\mathcal{S}$-theorems exactly the theorems of the classical logic. Only the semantics is different. Except for the Łukasiewicz S-fuzzy logic $\mathcal{S}_{\mathbf{L}}$, there seems to be no chance to find an axiomatization allowing a completeness theorem for *1-tautologies* in $\mathcal{S}_\lambda$ with $\lambda \in [0, \infty[$ (i.e., for formulas φ such that $\bar{t}_{\mathcal{S}_\lambda}(\varphi) = 1$ for all truth assignments $\bar{t}_{\mathcal{S}_\lambda}$) to be proven. The problem is in equation (4); the S-implication does not give the value 1 for non-crisp arguments. In fact, without adding the nulary connective $\mathbf{0}$, there are even no 1-tautologies in $\mathcal{S}_\lambda$ with $\lambda \in [0, \infty[$.

Concluding remarks

We have discussed two main approaches to propositional fuzzy logics based on Frank t-norms, the R-fuzzy logics (where the basic connectives are conjunction, implication and the false statement) and the S-fuzzy logics (where negation and conjunction are basic connectives). We have seen that the main difference is the interpretation of the implication (by the residuum in R-fuzzy logics and by an S-implication in S-fuzzy logics).

In both approaches we have studied the important issues of compactness, deduction, axiomatization, soundness and completeness.

We tried to compare R-fuzzy logics and S-fuzzy logics by pointing out the advantages and disadvantages of the two concepts.

We only mention that infintary S-fuzzy logics were introduced in [2] (see also [19]). The introduction of infinitary R-fuzzy logics seems to be an open field of research since we did not find any study of this subject in the literature.

Finally, it should be noted that there are many other approaches to $[0,1]$-valued logics starting from different points of view, some of which are described in detail in [16, 27, 29]. For a rather extensive overview, see [11].

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