Spectral Deferred Correction Report

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1 Method

We want to solve a first order ODE with initial conditions of the form

$$\mathbf{u}'(t) = \mathbf{f}(\mathbf{u}, t), \quad t \in [a, b], \quad t_0 = a, \quad \mathbf{u}_0 = \mathbf{u}(a). \tag{1}$$

We begin by writing the ODE in Picard integral form by integrating both sides with respect to time from a to t

$$\mathbf{u}(t) = \mathbf{u}(a) + \int_a^t \mathbf{f}(\mathbf{u}(s), s)\,\mathrm{ds}. \tag{2}$$

We are trying to find an approximate function for $\mathbf{u}(t)$ which we denote by $\hat{\mathbf{u}}(t)$. Approximate values for $\hat{\mathbf{u}}(t)$ at various nodes in $t \in [a, b]$ are initially calculated via Euler's method. Two important metrics in this method are the residuals $\mathbf{r}(t)$ and errors $\mathbf{e}(t)$

$$\mathbf{r}(t) = \mathbf{u}(a) - \hat{\mathbf{u}}(t) + \int_a^t \mathbf{f}(\hat{\mathbf{u}}(s), s)\,\mathrm{ds} \tag{3}$$

$$\mathbf{e}(t) = \mathbf{u}(t) - \hat{\mathbf{u}}(t). \tag{4}$$

After substituting equation (4) into (2) we obtain the following

$$\hat{\mathbf{u}}(t) + \mathbf{e}(t) = \mathbf{u}(a) + \int_a^t \mathbf{f}(\hat{\mathbf{u}}(s) + \mathbf{e}(s), s)\,\mathrm{ds}. \tag{5}$$

Subtracting $\hat{\mathbf{u}}(t)$ from both sides while adding and subtracting $\int_a^t \mathbf{f}(\hat{\mathbf{u}}(s), s)\,\mathrm{ds}$ we arrive at

$$\mathbf{e}(t) = \mathbf{u}(a) - \hat{\mathbf{u}}(t) + \int_a^t \mathbf{f}(\hat{\mathbf{u}}(s), s)\,\mathrm{ds} - \int_a^t \mathbf{f}(\hat{\mathbf{u}}(s), s)\,\mathrm{ds} + \int_a^t \mathbf{f}(\hat{\mathbf{u}}(s) + \mathbf{e}(s), s)\,\mathrm{ds}. \tag{6}$$

Then we substitute equation (3) into equation (6)

$$\mathbf{e}(t) = \mathbf{r}(t) + \int_a^t \mathbf{f}(\hat{\mathbf{u}}(s) + \mathbf{e}(s), s) - \mathbf{f}(\hat{\mathbf{u}}(s), s)\,\mathrm{ds} \tag{7}$$

$$\mathbf{e}'(t) = \mathbf{r}'(t) + \mathbf{G}(\hat{\mathbf{u}}(t), \mathbf{e}(t), t) \tag{8}$$

where $\mathbf{G}(\hat{\mathbf{u}}(s), \mathbf{e}(s), s) = \mathbf{f}(\hat{\mathbf{u}}(s) + \mathbf{e}(s), s) - \mathbf{f}(\hat{\mathbf{u}}(s), s)$. We can then calculate approximate values of the error function at various nodes t_i where $t_{i+1} = t_i + h_i$ using Euler's method

$$\hat{\mathbf{u}}(t_{i+1}) = \hat{\mathbf{u}}(t_i) + h_i \mathbf{f}(\hat{\mathbf{u}}(t_i), t_i), \quad \hat{\mathbf{u}}(t_0) = \mathbf{u}_0 \tag{9}$$

$$\hat{\mathbf{e}}(t_{i+1}) = \hat{\mathbf{e}}(t_i) + h_i \mathbf{G}(\hat{\mathbf{u}}(t_i), \hat{\mathbf{e}}(t_i), t_i) + \mathbf{r}(t_{i+1}) - \mathbf{r}(t_i), \quad \hat{\mathbf{e}}(t_0) = 0. \tag{10}$$