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Statistical Analysis of a Spatial Birth-and-Death Process Model with a View to Modelling Linear Dune Fields

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ABSTRACT. Statistical inference for parametric models of spatial birth-and-death processes is discussed in detail. In particular, a flexible and statistically tractable parametric class of such processes, defined on the real line, is presented and analysed by likelihood and partial likelihood methods. The suggested methods are illustrated by applying them to two sets of data given in the form of aerial photographs from the Kalahari Desert.

Key words: graphical methods, likelihood methods, linear dune fields, modelling and model control, nearest-neighbour birth-and-death processes on the real line, partial likelihood

1. Introduction

Spatial birth-and-death processes have been used in spatial statistics as a tool for simulating spatial point processes, which can be thought of as the equilibrium distribution towards which a spatial birth-and-death process converges (Ripley, 1977, 1981; Diggle, 1983; Stoyan *et al.*, 1987; Baddeley & Møller, 1989). Recently, simpler and more effective Metropolis–Hastings algorithms have been introduced, see Geyer & Møller (1993). In the present paper it is demonstrated that spatial birth-and-death processes are useful for modelling certain dynamical spatial processes. A flexible and statistically tractable parametric class of such processes, defined on the real line, is presented. The discussion is mainly focused on modelling a particular phenomenon, linear dune fields, but the model presented is obviously applicable in many other contexts, and most of the methods proposed can be generalized straightforwardly to other birth-and-death process models. For general results on spatial birth-and-death processes, see Preston (1977), for processes on the real line in particular, see Holley & Stroock (1978). A simple spatial birth process was used to analyse geological data in Fiksel & Stoyan (1983) and Fiksel (1984).

Linear dune fields are described in section 2, where we also present two sets of data given in the form of aerial photographs from the Kalahari Desert. Some results on spatial birth-and-death processes on the real line that are needed later are given in section 3, where our parametric sub-class is also defined and some of its probabilistic properties are discussed. Approximate proper and partial likelihood functions are derived in section 4. The partial likelihood functions are also relevant to a semi-parametric model more general than the model of section 3. It is proved that the maximum likelihood estimates exist with probability one, and conditions for the existence of the maximum partial likelihood estimates are formulated and discussed. In section 5 the aerial photographic data on linear dune fields are analysed by means of the model presented in section 3. Finally, in section 6 methods, mostly graphical, for checking the consistency of our model with the data are proposed and applied to the dune data.

This paper is a revised version of Møller & Sørensen (1990), where extensions to higher dimensions are considered and asymptotic properties of the estimators and of test statistics are discussed in the spirit of Barndorff-Nielsen & Sørensen (1994).

2. Data on linear dune fields

The data to be modelled and analysed in the present paper consist of two aerial photographs of different parts of a field of vegetated linear dunes in the Kalahari Desert. A linear dune field is a large collection of parallel nearly linear sand dunes. The height of the dunes on the aerial photographs is 2–15 metres, and their spacing is typically 200–500 metres. They can be several kilometres long.

The dunes are parallel to the direction of the prevailing wind. It is believed that dunes start in the lee of relatively densely vegetated sand patches, behind which sand can accumulate. According to this theory the dune will, provided the sand supply is sufficient, grow in the downwind direction as a result of a positive feed-back mechanism between vegetation and sand deposition (Tsoar & Møller, 1986). Viewed in the downwind direction the dune field has certain dynamics: as new dunes start, some of the old dunes that started further upwind will tend to end due to the limited sand supply. Most often they simply end, but on some occasions two dunes merge and continue downwind as one dune. The latter event is called a y-junction. It is assumed that in this way an equilibrium pattern develops in the wind direction. Presumably the new dunes do not start at completely random locations: there must be a higher tendency for dune development at places distant from other dunes where the sand supply is relatively large. The dynamics briefly outlined here will be modelled by means of spatial birth-and-death processes in the next section.

The two aerial photographs analysed in this paper were chosen because they contained only one or two y-junctions, so that this complication could be safely ignored. This was done only for the sake of simplicity of presentation. As discussed briefly in the next section, y-junctions can be built easily into our model. The birth-and-death process model assumes that the dunes are parallel line segments. The actual dune fields are, of course, not exactly like that, so as a preliminary step in the analysis of the data, parallel straight line segments were fitted to the dunes on each of the two aerial photographs by the usual linear regression. The two photographs were originally called image 14a and image 15. We use these labels for future references. The fitted line segments for image 14a are given in Fig. 1. The field of view in the figure is about 11×10.5 kilometres. The corresponding figure for image 15 looks similar and is omitted here. Information on, for instance dune spacing, needed in the statistical analysis, was calculated from the position of the line segments.

The preliminary fitting of line segments to the dunes is not necessary for the partial likelihood approach presented in section 4, because the partial likelihood function does not depend on what happens between transitions. Also the likelihood approach could be applied without the preliminary fitting of lines if a more complicated model were used, in which points were allowed to move along deterministic trajectories between their birth and death.

3. Nearest-neighbour birth-and-death models on the real line

In this section we construct a stochastic model for the linear dune fields, cf. section 2. The model is a nearest-neighbour birth-and-death process on the real line. Such processes are briefly reviewed in subsection 3.1 and the model is presented in subsection 3.2.

3.1. Birth-and-death processes on the real line

We shall consider finite birth-and-death processes on the real line, i.e. time homogeneous processes $\{x(t): t \geqslant 0\}$ where a state is a finite set of points contained in a bounded interval and where a transition is either the addition of a new point or the deletion of an existing

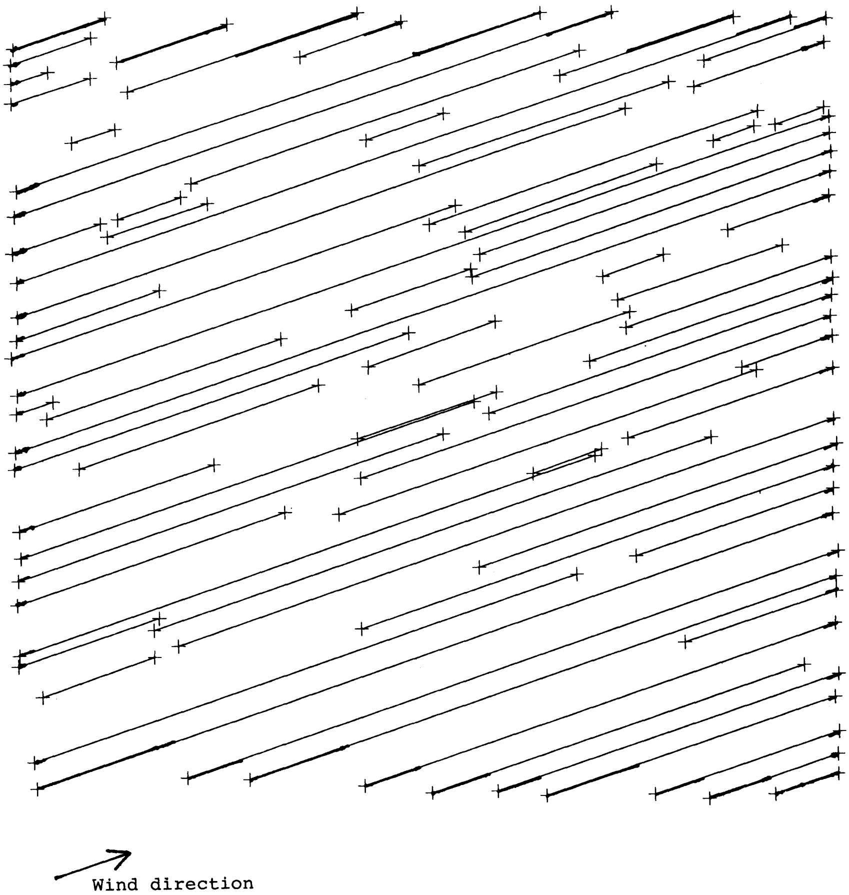

Fig. 1. Parallel line segments fitted to the dunes on image 14a by linear regression. The thick line segments constitute the boundary, cf. subsection 4.1.

point. More formally, let S be a bounded open interval on the real line and let, for each $n = 0, 1, 2, \ldots$, Ω_n be the set of all point configurations $x = (x_1, \ldots, x_n) \in S^n$ with $x_1 < \ldots < x_n$. Especially, Ω_0 consists of the empty point configuration denoted by 0. The state space of $x(t)$ is $\Omega = \bigcup_{n=0}^{\infty} \Omega_n$ and a transition from a state in Ω_n is either to Ω_{n+1} (a birth) or to Ω_{n-1} (a death). Let $\mathscr{B}$ be the Borel σ-field on S, $\mathscr{B}_n$ the corresponding σ-field on Ω_n, and $\mathscr{F}$ the σ-field on Ω generated by $\{\mathscr{B}_n : n \geqslant 0\}$.

We shall henceforth assume that the process $\{x(t): t \geqslant 0\}$ is described, in a way indicated below, by two measurable and non-negative functions $b(x, \xi)$ and $d(x, \xi)$, $(x, \xi) \in \Omega \times S$, where $b(x, \cdot)$ is Lebesgue integrable for all $x \in \Omega$ and where $d(0, \cdot) \equiv 0$. If $x = (x_1, \ldots, x_n) \in \Omega$, write $x \backslash x_i$ for $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. Further, let $x_0 < x_{n+1}$ denote the endpoints of the interval S. When $\xi \in S$ with $x_i < \xi < x_{i+1}$, we write $x \cup \xi$ for $(x_1, \ldots, x_i, \xi, x_{i+1}, \ldots, x_n)$. For $x \in \Omega$ and $A \in \mathscr{B}$ we define

$$B(x, A) = \int_A b(x, \xi)\, d\xi,$$

$$\beta(x) = B(x, S),$$

$$\delta(x) = \begin{cases} \sum_{i=1}^{n} d(x \backslash x_i, x_i) & \text{if } x = (x_1, \ldots, x_n) \text{ and } n > 0, \\ 0 & \text{if } x = 0, \end{cases}$$

$$\alpha(x) = \beta(x) + \delta(x).$$

Finally, let $t_1 < t_2 < \ldots$ denote the transition times for the process and set $t_0 = 0$.

Now we can define the spatial birth-and-death process in the following way: Given that $x(t_{j-1}) = x$ with $x = (x_1, \ldots, x_n) \in \Omega_n$ and given the history of the process before time t_{j-1}, we require that

$$t_j - t_{j-1} \quad \text{is exponentially distributed with mean} \quad 1/\alpha(x), \tag{3.1}$$

$$x(t_j) \in \Omega_{n+1} \quad \text{with probability} \quad \beta(x)/\alpha(x) \quad \text{and}$$

$$x(t_j) \in \Omega_{n-1} \quad \text{with probability} \quad \delta(x)/\alpha(x), \tag{3.2}$$

$$\text{given that} \quad x(t_j) \in \Omega_{n+1}, \quad \text{i.e. that} \quad x(t_j) = x(t_{j-1}) \cup \xi,$$

$$\text{then} \quad \xi \in A \quad \text{with probability} \quad B(x, A)/\beta(x), \tag{3.3}$$

$$\text{given that} \quad x(t_j) \in \Omega_{n-1}, \quad \text{i.e. that} \quad x(t_j) = x(t_{j-1}) \backslash \xi,$$

$$\text{then} \quad \xi = x_i \quad \text{with probability} \quad d(x \backslash x_i, x_i)/\delta(x). \tag{3.4}$$

Thus, given an initial configuration $x(0)$, simulation of the process $x(t)$ within a finite time interval is straightforward using (3.1)–(3.4).

Preston (1977) gives simple conditions for the unique existence of the above process and for its convergence to a limit distribution. Møller (1989) gives conditions ensuring a geometrically fast convergence. The limit distribution is the unique equilibrium distribution of the process. The density f: $\Omega \to [0, \infty]$ of this equilibrium distribution with respect to the measure μ on $\mathscr{F}$ given by

$$\mu(F) = 1_F(0) + \sum_{n=1}^{\infty} \int_{x_0 < x_1 < \ldots < x_n < x_{n+1}} \cdots \int 1_F((x_1, \ldots, x_n)) \, dx_1 \ldots dx_n$$

can be determined as follows when the process is time-reversible. Assume that $d(\cdot, \cdot) > 0$ and define

$$\gamma(x, \xi) = b(x, \xi)/d(x, \xi).$$

Furthermore, assume that a density f with respect to μ is defined inductively by

$$f(x \cup \xi) = \gamma(x, \xi) f(x) \tag{3.5}$$

and that

$$\int \beta(x) f(x) \mu(dx) < \infty. \tag{3.6}$$

The solution to (3.5) is well-defined if and only if a balance condition holds, viz.

$$\gamma(x \cup \xi, \eta)\gamma(x, \xi) = \gamma(x \cup \eta, \xi)\gamma(x, \eta). \tag{3.7}$$

Under the conditions imposed, the spatial birth-and-death process is time-reversible and f is the equilibrium density, cf. Preston (1977, th. 8.1).

In this paper we shall concentrate on nearest-neighbour birth-and-death processes, i.e. processes where $b(x, \xi)$ and $d(x, \xi)$ depend only on ξ and its nearest left and right neighbours in x. Hence, the equilibrium density (3.5) is seen to be Markov with respect to the sequential neighbour relation, cf. Baddeley and Møller (1989). We write $b(x_i, \xi, x_{i+1})$ for $b(x, \xi)$ and $d(x_i, \xi, x_{i+1})$ for $d(x, \xi)$ if $x_i < \xi < x_{i+1}$. Moreover, we define $H(x_i, x_{i+1}) = B(x, (x_i, x_{i+1}))$, which depends only on x_i and x_{i+1}. With this definition (3.3) naturally splits into the following two steps:

$$\begin{aligned}&\text{given that} \quad x(t_j) \in \Omega_{n+1}, \quad \text{i.e. that} \quad x(t_j) = \\ &x(t_{j-1}) \cup \xi, \quad \text{then} \quad x_i < \xi < x_{i+1} \quad \text{with probability} \\ &H(x_i, x_{i+1})/\beta(x), \quad i = 0, \ldots, n,\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}&\text{given that} \quad x(t_j) = x(t_{j-1}) \cup \xi \in \Omega_{n+1} \quad \text{with} \\ &x_i < \xi < x_{i+1}, \quad \text{then } \xi \text{ has a density on} \quad (x_i, x_{i+1}) \\ &\text{given by} \quad h(\xi \mid x_i, x_{i+1}) = b(x_i, \xi, x_{i+1})/H(x_i, x_{i+1}),\end{aligned} \tag{3.9}$$

where, in the first step,

$$\beta(x) = \sum_{i=0}^{n} H(x_i, x_{i+1}). \tag{3.10}$$

3.2. Nearest-neighbour birth-and-death models for linear dune fields

For ease of presentation we consider a nearest-neighbour birth-and-death model for the linear dune fields only in the case of no y-junctions in detail. At the end of this section we briefly discuss the case where y-junctions are present.

Linear dunes are parallel to the direction of the prevailing wind, and they grow in this direction. It is therefore natural to define a "time-axis" for a birth-and-death process parallel to the direction of the dunes and with time increasing in the prevailing wind direction. The state of the process $x(t)$ at "time" t is the intersection points between the dunes and a bounded interval S of a line perpendicular to the wind direction that goes through the point t on the "time-axis". A birth and a death for the process $\{x(t): t \geq 0\}$ correspond respectively to the events that a new dune begins and that a dune ends. It is a very reasonable assumption that the birth or death of a dune is an event that is only affected by the presence of the two neighbouring dunes, so a nearest-neighbour birth-and-death process provides a good approximation to the dynamics of the dune field. Moreover, since there is only a limited supply of sand, the interaction between neighbouring dunes is expected to be a repulsion.

Let us now describe the model we propose. First consider the case where a birth occurs. In (3.8) and (3.9) we choose

$$H(x_i, x_{i+1})/\beta(x) = (x_{i+1} - x_i)^\gamma \Big/ \sum_{j=0}^{n} (x_{j+1} - x_j)^\gamma, \tag{3.11}$$

$$h(\xi \mid x_i, x_{i+1}) = \frac{\xi_i^{\alpha-1}(1-\xi_i)^{\beta-1}}{B(\alpha, \beta)(x_{i+1} - x_i)}, \tag{3.12}$$

where $\gamma \in \mathbb{R}$, α, $\beta > 0$, $B(\alpha, \beta)$ is the Beta function and

$$\xi_i = (\xi - x_i)/(x_{i+1} - x_i)$$

is the relative distance of the new dune at ξ to its nearest left neighbour dune. The choice of (3.11) and (3.12) is because these are simple, yet flexible, expressions with an obvious physical interpretation. In particular, the expected repulsion is obtained if $\gamma > 0$ and $\alpha, \beta > 1$. If $\alpha = \beta = \gamma = 1$ there is no interaction. The assumption that ξ_i is Beta-distributed could be replaced by the more general parametric models on the unit interval proposed by Barndorff-Nielsen & Jørgensen (1990). That generalization would not change anything essential in the rest of the paper. Since we require that $b(x, \xi)$ depends only on ξ and its neighbouring dunes in x, (3.8)–(3.12) give

$$\beta(x) = k \sum_{j=0}^{n} (x_{j+1} - x_j)^{\gamma} \tag{3.13}$$

for some positive parameter k, i.e.

$$b(x_i, \xi, x_{i+1}) = \frac{k}{B(\alpha, \beta)} (x_{i+1} - x_i)^{\gamma - 1} \xi_i^{\alpha - 1} (1 - \xi_i)^{\beta - 1}. \tag{3.14}$$

Secondly, we choose

$$d(x_i, \xi, x_{i+1}) = c(x_{i+1} - x_i)^{\varphi} (\xi - x_i)^{\varepsilon} (x_{i+1} - \xi)^{\delta} \tag{3.15}$$

with $c \geqslant 0$. The following conditions imply repulsion:

$$k, c, \gamma > 0; \quad \alpha, \beta > 1; \quad \epsilon, \delta < 0; \quad \varphi + \varepsilon + \delta \leqslant 0. \tag{3.16}$$

The latter bound is obtained by rewriting (3.15) as

$$d(x_i, \xi, x_{i+1}) = c(x_{i+1} - x_i)^{\varphi + \varepsilon + \delta} \xi_i^{\varepsilon} (1 - \xi_i)^{\delta}.$$

In the Appendix weak sufficient conditions are discussed ensuring the existence and convergence of the process defined by (3.14) and (3.15). It is intuitively clear that the process explodes with a positive probability if $\gamma < 0$, because then dunes tend to be born between neighbouring dunes which are close, so that there is a positive feed-back. For $\gamma \geqslant 0$ the process exists uniquely no matter how the density $h(\cdot \mid \cdot)$ and the function $d(\cdot, \cdot)$ are specified, since the proof of the unique existence is only based on (3.13). Furthermore, it is proved that the process converges as $t \to \infty$ to a limit distribution which is the unique equilibrium distribution, if $\varepsilon, \delta, \varphi + \varepsilon + \delta \leqslant 0$ and either (i) $\gamma > 0$ or (ii) $\gamma = 0$ and $\varphi + \varepsilon + \delta < 0$ or (iii) $\gamma = \varphi = \varepsilon = \delta = 0$ and $c > k$. The rate of convergence is in fact geometrically fast, see the Appendix.

Finally, we observe that the balance condition (3.7) holds if and only if

$$\alpha - \varepsilon = \beta - \delta = \alpha + \beta + \varphi - \gamma. \tag{3.17}$$

We shall denote the common value of these three quantities by κ. When (3.17) holds, (3.5) defines the density

$$f(x) = ab^n \prod_{i=0}^{n} (x_{i+1} - x_i)^{\kappa - 1}, \quad x = (x_1, \ldots, x_n), \tag{3.18}$$

where a is a normalizing constant and $b = k/(cB(\alpha, \beta))$. Condition (3.6) is satisfied for $\gamma \geqslant 0$, cf. (A.2) in the Appendix. Thus, if the conditions given above for convergence towards equilibrium hold, (3.18) is the limit distribution of the process. Notice that under (3.18) and conditional on n, the distribution at one particular time of the relative distances $(x_{i+1} - x_i)/(x_{n+1} - x_0)$, $i = 0, \ldots, n$, between neighbouring dunes is simply a Dirichlet distribution with parameters $(\kappa, \ldots, \kappa)$ (n times). Therefore, (3.18) is a density if and only if $\kappa > 0$.

A model comprising y-junctions can be built as a straightforward generalization of the model above. At a y-junction two dunes merge, so after this event the number of dunes has gone down by one. Thus a y-junction is a transition from Ω_n to Ω_{n-1} (for some n) in the general birth-and-death process setup discussed in subsection 3.1, and this type of transition can be modelled in a simple way. Models of this more general type are also covered by Preston's (1977) theory. As long as we retain our original birth rate (3.13), the birth-and-death process will exist and be unique no matter how we specify the rate of dune merger and the probability distribution of the location of the resulting joint dune. This follows from the discussion above of the unique existence of the process without y-junctions. The convergence towards a limit distribution, however, depends on these specifications.

4. Likelihood inference

In this section we derive an approximate proper likelihood function and a partial likelihood function for the nearest-neighbour birth-and-death process model described in subsection 3.2. The partial likelihood function is also relevant to a semi-parametric model more general than the one defined in the previous section and is thus in this sense model robust. It is proved that the maximum likelihood estimates exist with probability one, and conditions for the existence of the maximum partial likelihood estimates are discussed. For consideration of asymptotic properties of estimators and test statistics, see Møller & Sørensen (1990).

4.1. Likelihood functions

For a given aerial photograph of a linear dune field and for $x(t) = (x_1, \ldots, x_n)$, let $x_j < \ldots < x_{j+p}$ ($j \geqslant 0, j+p \leqslant n+1$) be the positions of the dunes which can be seen on the aerial photograph at time t and define $y(t) = (x_{j+1}, \ldots, x_{j+p-1})$, $z(t) = (x_1, \ldots, x_{j-1}, x_{j+p+1}, \ldots, x_n)$, the "boundary" $\delta(t) = (x_j, x_{j+p})$, and the "interior" $\text{int}(t) =]x_j, x_{j+p}[$. Here $\text{int}(t) = \emptyset$ exactly when $p = 0$ or $\delta(t) = \emptyset$ (the latter case occurs when nothing is observed). Let t_0 and t_m be the first time and the last time the interior is non-empty. Thus, at time $t \in [t_0, t_m]$, $y(t)$ is the position of the dunes in the interior which together with their neighbouring dunes can be observed on the aerial photograph, see Fig. 1.

In order to write down the approximate likelihood function we need the following notation. Let $t_1 < \ldots < t_{m-1}$ be the transition times for $\{(y(t), \delta(t))\colon t_0 < t < t_m\}$. Furthermore, let

$$\mathscr{C} = \{1, \ldots, m\},$$

$$\mathscr{B} = \{i \in \mathscr{C}\colon \text{a birth happens for } y(t) \text{ at time } t = t_i\},$$

$$\mathscr{D} = \{i \in \mathscr{C}\colon \text{a death happens for } y(t) \text{ at time } t = t_i\},$$

and for $i \in \mathscr{C}$ let

$$\Delta_i = t_i - t_{i-1},$$

$$(y_{i1}, \ldots, y_{in_i}) = y(t_{i-1}) \quad \text{and} \quad (y_{i0}, y_{i(n_i+1)}) = \partial(t_{i-1})$$

$$\text{where } y_{i0} < y_{i1} < \ldots < y_{i(n_i+1)},$$

$$r_{ij} = y_{i(j+1)} - y_{ij} \quad \text{for } j = 0, \ldots, n_i,$$

$$s_{ij} = y_{i(j+1)} - y_{i(j-1)} = r_{i(j-1)} + r_{ij} \quad \text{for } j = 1, \ldots, n_i.$$

For $i \in \mathscr{B}$ we define

$\xi^{(i)} =$ position of the dune starting at time t_i,

j_i such that $y_{ij_i} < \xi^{(i)} < y_{i(j_i+1)}$,

$\xi_i = (\xi^{(i)} - y_{ij_i})/(y_{i(j_i+1)} - y_{ij_i})$,

and for $i \in \mathscr{D}$ we let

j_i be such that y_{ij_i} is the position of the dune which dies at time t_i.

Suppose the entire dune field has been observed for $t \in [t_0, t_m]$, i.e. that $\delta(t) = (x_0, x_{n+1})$ for all $t \in [t_0, t_m]$. Then the likelihood function is given by

$$L = \prod_{i \in \mathscr{C}} \exp\left(-\Delta_i \sum_{j=0}^{n_i} H(y_{ij}, y_{i(j+1)}) - \Delta_i \sum_{j=1}^{n_i} d(y_{i(j-1)}, y_{ij}, y_{i(j+1)})\right)$$
$$\times \prod_{i \in \mathscr{B}} b(y_{ij_i}, \xi^{(i)}, y_{i(j_i+1)}) \tag{4.1}$$
$$\times \prod_{i \in \mathscr{D}} d(y_{i(j_i-1)}, y_{ij_i}, y_{i(j_i+1)}).$$

However, on the aerial photographs, we consider here, the boundary $\delta(t)$ does not correspond to the endpoints of S, but changes randomly with t. When this is the case, several of the factors in the likelihood function depend on quantities that have not been observed. We define our approximate likelihood function as what is left when all factors depending on some unobserved quantity have been deleted. With the notation described above, the approximate likelihood function when $\delta(t)$ is not constantly equal to the end-points is given by (4.1).

In the case where $d(x_{j-1}, x_j, x_{j+1}) = D_1(x_{j-1}, x_j) + D_2(x_j, x_{j+1})$, it is not difficult to see that (4.1) is the conditional likelihood for the process $\{y(t) : t \in [t_0, t_m]\}$ given $y(t_0)$ and $\{\delta(t) : t \in [t_0, t_m]\}$, and that this conditional process is independent of the "exterior" process $\{z(t) : t \in [t_0, t_m]\}$. However, our specification (3.15) of the death intensity d does not satisfy this additivity condition, so for our model there is some dependence between y and z if we condition on δ, and (4.1) is not exactly a conditional likelihood function.

There is, however, another interpretation of (4.1). Again we think of $\{\delta(t) : t \in [t_0, t_m]\}$ as fixed, but when describing the dynamics of the process y, we do not condition on future values of δ. Specifically, we think of $\{y(t) : t \in [t_0, t_m]\}$ as a realization of the following process. Define a counting process $\{N(t) : t \in [t_0, t_m]\}$ by specifying its intensity

$$\lambda(t) = \lambda_1(t) + \lambda_2(t), \tag{4.2}$$

$$\lambda_1(t) = \sum_{j=0}^{n(t)} H(y_j(t), y_{j+1}(t)), \tag{4.3}$$

$$\lambda_2(t) = \sum_{j=1}^{n(t)} d(y_{j-1}(t), y_j(t), y_{j+1}(t)), \tag{4.4}$$

where $y(t-) = (y_1(t), \ldots, y_{n(t)}(t))$ and $\delta(t-) = (y_0(t), y_{n(t)+1}(t))$. At every jump time of N the state of the process y is modified. Conditionally on $\{(y(s), \delta(s)) : s \in [t_0, t_m)\}$ and on the fact that a modification happens, we specify the distribution determining what modification occurs as follows: either a new dune starts or an existing dune ends. The first event has conditional probability $\lambda_1(t)/\lambda(t)$, the second $\lambda_2(t)/\lambda(t)$. If a new dune begins, the probability density function of its position is $b(y_j(t), \xi, y_{j+1}(t))/\lambda_1(t)$ for $y_j(t) < \xi < y_{j+1}(t)$,

$j = 0, \ldots, n(t)$. If a dune ends, the probability that the ending dune is the one at $y_j(t)$ is $d(y_{j-1}(t), y_j(t), y_{j+1}(t))/\lambda_2(t)$, $j = 1, \ldots, n(t)$. The state of y is also changed if one of its points becomes a boundary point. In a somewhat loose way we can think of this model for y as obtained by conditioning only on the past and present value of δ. With this model for y the likelihood function is exactly equal to (4.1). There is a positive probability that dunes are born at positions and have life-times which are inconsistent with future values of δ. This is, however, not a practical problem because the observed dune field will always be consistent with the future values of δ.

By combining (4.1) with (3.14) and (3.15) we get the approximate likelihood function L for our model, i.e.

$$L = k^{\#\mathscr{B}} c^{\#\mathscr{D}} \exp\left(-k \sum_{i \in \mathscr{C}} \Delta_i \sum_{j=0}^{n_i} r_{ij}^{\gamma} - c \sum_{i \in \mathscr{C}} \Delta_i \sum_{j=1}^{n_i} s_{ij}^{\varphi}\, r_{i(j-1)}^{\varepsilon}\, r_{ij}^{\delta}\right)$$

$$\times \prod_{i \in \mathscr{B}} \left[r_{ij_i}^{\gamma} \frac{\xi_i^{\alpha-1}(1-\xi_i)^{\beta-1}}{B(\alpha, \beta)} \right] \prod_{i \in \mathscr{D}} [s_{ij_i}^{\varphi}\, r_{i(j_i-1)}^{\varepsilon}\, r_{ij_i}^{\delta}] \tag{4.5}$$

(ignoring a constant of proportionality). This factorizes as

$$L = L_1(k, \gamma) L_2(c, \varphi, \varepsilon, \delta) L_3(\gamma) L_4(\alpha, \beta) L_5(\varphi, \varepsilon, \delta), \tag{4.6}$$

where

$$L_1 = \left(k \sum_{i \in \mathscr{C}} \Delta_i \sum_{j=0}^{n_i} r_{ij}^{\gamma}\right)^{\#\mathscr{B}} \exp\left(-k \sum_{i \in \mathscr{C}} \Delta_i \sum_{j=0}^{n_i} r_{ij}^{\gamma}\right), \tag{4.7}$$

$$L_2 = \left(c \sum_{i \in \mathscr{C}} \Delta_i \sum_{j=1}^{n_i} s_{ij}^{\varphi}\, r_{i(j-1)}^{\varepsilon}\, r_{ij}^{\delta}\right)^{\#\mathscr{D}} \exp\left(-c \sum_{i \in \mathscr{C}} \Delta_i \sum_{j=1}^{n_i} s_{ij}^{\varphi}\, r_{i(j-1)}^{\varepsilon}\, r_{ij}^{\delta}\right), \tag{4.8}$$

$$L_3 = \prod_{i \in \mathscr{B}} \left(r_{ij_i}^{\gamma} \Big/ \sum_{h \in \mathscr{C}} \Delta_h \sum_{j=0}^{n_h} r_{hj}^{\gamma} \right), \tag{4.9}$$

$$L_4 = \prod_{i \in \mathscr{B}} \frac{\xi_i^{\alpha-1}(1-\xi_i)^{\beta-1}}{B(\alpha, \beta)}, \tag{4.10}$$

$$L_5 = \prod_{i \in \mathscr{D}} \left(s_{ij_i}^{\varphi}\, r_{i(j_i-1)}^{\varepsilon}\, r_{ij_i}^{\delta} \Big/ \sum_{h \in \mathscr{C}} \Delta_h \sum_{j=1}^{n_h} s_{hj}^{\varphi}\, r_{h(j-1)}^{\varepsilon}\, r_{hj}^{\delta} \right). \tag{4.11}$$

The profile likelihood function $\tilde{L}$ for the parameters α, β, γ, φ, ε and δ is shown in subsection 4.2 to be

$$\tilde{L} = L_3(\gamma) L_4(\alpha, \beta) L_5(\varphi, \varepsilon, \delta), \tag{4.12}$$

so the maximum likelihood estimates for γ, (α, β) and $(\varphi, \varepsilon, \delta)$ can be found from L_3, L_4 and L_5, respectively. For simplicity we write profile likelihood function and maximum likelihood estimate rather than profile approximate likelihood function and maximum approximate likelihood estimate.

We shall now introduce a partial likelihood function in the sense of Cox (1975) which, as argued below, turns out to be of a form similar to $\tilde{L}$ in (4.12), namely

$$L_p = L_{p3}(\gamma) L_{p4}(\alpha, \beta) L_{p5}(\varphi, \varepsilon, \delta). \tag{4.13}$$

Here $L_{p4} = L_4$ whereas L_{p3} and L_{p5} can be regarded as alternatives to L_3 and L_5 for estimating γ and $(\varphi, \varepsilon, \delta)$, respectively. Note that (4.12) and (4.13) still hold if L_4 is replaced by the likelihood function based on another choice of parametric model than the family of Beta-distributions given by (4.10).

The partial likelihood function is obtained by arguing that the waiting times $\mathscr{T} = \{\Delta_i: i \in \mathscr{C}\}$ and the observations whether each particular transition is a birth or a death do not contain much information about the parameters γ, φ, ε and δ, and obviously no information on α and β. The distribution of this part of the observations depends on the parameters γ, φ, ε and δ in quite an intricate way and is mainly determined by k and c. In the general setup we obtain the partial likelihood function by first multiplying over all births, the conditional density of the position of the new dune given the past, i.e. given the configuration immediately prior to the birth, and secondly the conditional probability of the observed death given the past, over all deaths. Hence we arrive at the following expression for the partial likelihood function:

$$L_p = \prod_{i \in \mathscr{B}} \frac{H(y_{ij_i}, y_{i(j_i+1)})}{\sum_{j=0}^{n_i} H(y_{ij}, y_{i(j+1)})} \prod_{i \in \mathscr{B}} h(\xi_i \mid y_{ij_i}, y_{i(j_i+1)}) \prod_{i \in \mathscr{D}} \frac{d(y_{i(j_i-1)}, y_{ij_i}, y_{i(j_i+1)})}{\sum_{j=0}^{n_i} d(y_{i(j-1)}, y_{ij}, y_{i(j+1)})}.$$

With our particular specifications (3.14) and (3.15), L_p is given by (4.13) with

$$L_{p3} = \prod_{i \in \mathscr{B}} \left(r_{ij_i}^{\gamma} \Big/ \sum_{j=0}^{n_i} r_{ij}^{\gamma} \right), \tag{4.14}$$

$$L_{p5} = \prod_{i \in \mathscr{D}} \left(s_{ij_i}^{\varphi}\, r_{i(j_i-1)}^{\varepsilon}\, r_{ij_i}^{\delta} \Big/ \sum_{j=1}^{n_i} s_{ij}^{\varphi}\, r_{i(j-1)}^{\varepsilon}\, r_{ij}^{\delta} \right). \tag{4.15}$$

An appealing property of the partial likelihood function (4.13) is that the same expression would be obtained in the much larger class of semi-parametric models obtained by allowing k and c to be functions depending on the time and on the entire past of the process. Thus also non-Markovian birth-and-death processes are included. The only restriction on k and c is that the process should not explode. For instance, it suffices to require that k is bounded. Dependence on time would, in the linear dune field context considered in this paper, correspond to a spatial inhomogeneity in the wind direction.

4.2. Estimation

We recall that the nearest-neighbour birth-and-death model given by (3.14) and (3.15) exists if and only if $\gamma \geqslant 0$, so the range of variation of the parameters in the full model is given by

$$H_0: k, c > 0, \quad \gamma \geqslant 0, \quad \alpha, \beta > 0, \quad \varphi, \delta, \varepsilon \in \mathbb{R}.$$

In this subsection we shall discuss maximum likelihood estimation under H_0 and under the following simplifications,

$$H_1: \alpha = \beta,$$

$$H_2: \varepsilon = \delta,$$

$$H_3: \varphi = -\varepsilon - \delta,$$

$$H_4: \alpha - \varepsilon = \beta - \delta = \alpha + \beta + \varphi - \gamma = \kappa > 0.$$

The hypotheses H_1 and H_2 specify that the interaction is symmetric between a dune and its nearest left and right neighbouring dunes, cf. (3.12) and (3.15). The hypothesis H_3 states that the tendency of a dune to die depends only on the relative distances to its neighbouring dunes, cf. (3.15). The hypothesis H_4 is the balance condition (3.17). Under this condition the spatial birth-and-death process is time-reversible, i.e. the pattern of line segments looks the same whether you walk through the dune field in the wind direction or against the wind.

Rejection of this hypothesis would substantiate the theory that linear dune fields develop in the wind direction. Note that under H_3 the balance condition H_4 simplifies to $\alpha - \varepsilon = \beta - \delta = \gamma = \kappa$.

For fixed α, β, γ, φ, ε and δ the likelihood function L attains, under any of the hypotheses H_0–H_4, its maximal value at (k, c) given by

$$k(\gamma) = \#\mathscr{B} \Big/ \sum_{i \in \mathscr{C}} \Delta_i \sum_{j=0}^{n_i} r_{ij}^{\gamma},$$

$$c(\varphi, \varepsilon, \delta) = \#\mathscr{D} \Big/ \sum_{i \in \mathscr{C}} \Delta_i \sum_{j=0}^{n_i} s_{ij}^{\varphi}\, r_{i(j-1)}^{\varepsilon}\, r_{ij}^{\delta}$$

cf. (4.6)–(4.8). Thus, given estimates $\hat{\gamma}$, $\hat{\varphi}$, $\hat{\varepsilon}$, $\hat{\delta}$ under any of the hypotheses we estimate k and c by

$$\hat{k} = k(\hat{\gamma}), \quad \hat{c} = c(\hat{\varphi}, \hat{\varepsilon}, \hat{\delta}). \tag{4.16}$$

Notice that the profile likelihood

$$L(\gamma, \alpha, \beta, \varphi, \varepsilon, \delta, k(\gamma), c(\varphi, \varepsilon, \delta))$$

is proportional to

$$L_3(\gamma) L_4(\alpha, \beta) L_5(\varphi, \varepsilon, \delta),$$

because $L_1(\gamma, k(\gamma))$ and $L_2(\varphi, \varepsilon, \delta, c(\varphi, \varepsilon, \delta))$ are parameter independent. In particular, this proves (4.12). The maximum likelihood estimates of the parameters γ, φ, ε and δ under the hypotheses H_0–H_3 and the likelihood ratio test statistics for H_2, H_3 and other hypotheses involving only these parameters do not depend on our particular choice of L_4.

Now, let us discuss the existence of the maximum likelihood estimates and the partial likelihood estimates. Since L_4 is simply the likelihood function for the Beta-distributed sample $\{\xi_i\colon i \in \mathscr{B}\}$, cf. (4.10), it suffices under any of the hypotheses H_0–H_3 to consider the likelihood functions L_3 and L_5 and the alternative partial likelihood functions L_{p3} and L_{p5}. These functions are all of the form

$$m(\theta_1, \ldots, \theta_p) = \prod_{i \in \mathscr{F}} \left(1 + \sum_{j \in \mathscr{A}_i} a_{ij} \prod_{k=1}^{p} v_{ijk}^{\theta_k}\right)^{-1}, \tag{4.17}$$

where $\mathscr{F}$ and the $\mathscr{A}_i$ are finite sets, $p < \infty$, $a_{ij} > 0$, $\theta_k \in \mathbb{R}$ and $v_{ijk} > 0$ for all i, j, k, compare with (4.9), (4.11), (4.14) and (4.15). For instance, (4.9) is of the form (4.17) with $\mathscr{F} = \mathscr{B}$, $\mathscr{A}_i = \{(h, j)\colon h \in \mathscr{C}, j = 1, \ldots, n_h\} \backslash \{(i, j_i)\}$, $p = 1$, $a_{i(h,j)} = \Delta_h / \Delta_i$ and $v_{i(h,j)k} = r_{hj}/r_{ij_i}$. Likelihood functions of the form (4.17) were studied by Jacobsen (1989). From his results follows that the function $m\colon \mathbb{R}^p \to \mathbb{R}_+$ is strictly log-concave and attains its maximal value at a unique point if and only if there does not exist $(\theta_1, \ldots, \theta_p) \neq (0, \ldots, 0)$ such that

$$\prod_{k=1}^{p} v_{ijk}^{\theta_k} \geqslant 1 \quad \text{for all} \quad j \in \mathscr{A}_i \quad \text{and} \quad i \in \mathscr{F}. \tag{4.18}$$

First consider the case $m(\gamma) = L_3(\gamma)$. Here (4.18) is satisfied if and only if $r_{hj}^{\gamma} \geqslant r_{ij_i}^{\gamma}$ for all $(h, j) \in \mathscr{A}_i$ and all $i \in \mathscr{B}$, in particular for all (h, j_h), $h \in \mathscr{B} \backslash \{i\}$. This is only possible in case r_{ij_i} is the same for all $i \in \mathscr{B}$, which happens with probability 0 provided $n_i > 1$ for some $i \in \mathscr{C}$. If this is the case, $L_3(\cdot)$ has a unique maximum $\hat{\gamma}$. Note, however, that $\hat{\gamma}$ might be negative which corresponds to an exploding process. Analogously it is seen that a sufficient condition that the maximum likelihood estimates of the other parameters exist almost surely under any of the hypotheses H_0–H_3 is that $\#\mathscr{D} > 4$. Note that all distances are distinct with probability one. In practice two or more distances might coincide, and in extreme cases the maximum likelihood estimates might be non-existent, for this reason. Since the profile likelihood functions (4.9)–(4.11) are log-concave, the estimates can easily be found by the Newton–Raphson algorithm, and then inserted in (4.16).

Under each of the hypotheses H_0–H_3 the maximum partial likelihood estimates based on (4.14) and (4.15) exist with a probability less than one. For instance, it is seen from (4.18) that the maximum partial likelihood estimate of γ does not exist precisely when either $r_{ij_i} \leqslant r_{ij}$ for all $i \in \mathscr{B}$ and $j = 0, \dots, n_i$ or $r_{ij_i} \geqslant r_{ij}$ for all $i \in \mathscr{B}$ and $j = 0, \dots, n_i$. That is, the only situations in which the maximum partial likelihood estimate of γ does not exist are when all births take place in the largest interdune space available and when all births take place in the smallest dune interval available. The probability of this event tends to zero as $\# \mathscr{B}$ tends to infinity. Analogously, it follows that the maximum partial likelihood estimates of the parameters φ, ε and δ exist on an event which is complicated, but easily described in terms of (4.18). Under H_2 and H_3 it is seen that the maximum partial likelihood estimate does not exist precisely when either all deaths happen to the dune for which $r_{i(j_i-1)} r_{ij_i} s_{\bar{ij_i}}^{-2}$ is largest, or all deaths happen to the dune with the minimum value of $r_{i(j_i-1)} r_{ij_i} s_{\bar{ij_i}}^{-2}$. The probability of this event tends to zero as $\# \mathscr{D}$ tends to infinity.

Under the hypothesis H_4, combined with one or more of the hypotheses H_0–H_3, the profile likelihood function $\tilde{L}$ in (4.12) can, for fixed α and β, be shown to have the form (4.17). By (4.18) it attains its unique maximum with probability one if $\# \mathscr{B} > 1$ and $\# \mathscr{D} > 1$. Therefore, in order to find the maximum likelihood estimates under H_4, one can simply investigate

$$\max_{\gamma \geqslant 0,\, \kappa > 0} L_3(\gamma) L_4(\alpha, \beta) L_5(\gamma + \kappa - \alpha - \beta, \alpha - \kappa, \beta - \kappa)$$

as a function of (α, β). For each value of (α, β) the maximum over γ and κ can be determined by the Newton–Raphson algorithm.

It seems likely that the asymptotic normality of the maximum likelihood estimators and the partial maximum likelihood estimators can be proven using the fact that the score functions and partial score functions under the various models are martingales under the model for the process y given by (4.2)–(4.4) and the description below these formulae. This and other asymptotic problems are discussed, but not proven, in Møller & Sørensen (1990).

5. Analysis of aerial photographic data

In this section we analyse the aerial photographic data statistically, using the nearest-neighbour birth-and-death process model discussed in the previous sections. It should be noted

Table 1. *Maximum likelihood estimates of the model parameters for the data in image 14a and image 15 under the hypotheses H_0, $H_1 + H_2$ and $H_1 + H_2 + H_3$. The confidence intervals are based on the observed information*

	α	β	γ	δ	ε	φ	k	c
Image 14a								
H_0	2.87	2.80	3.13 ± 0.64	-1.52 ± 1.07	-1.63 ± 1.23	2.29 ± 2.30	$8.93 \cdot 10^{-8}$	$5.48 \cdot 10^{-3}$
$H_1 + H_2$	2.83		3.13 ± 0.64	-1.54 ± 1.05		2.19 ± 2.18	$8.93 \cdot 10^{-8}$	$6.13 \cdot 10^{-3}$
$H_1 + H_2 + H_3$	2.83		3.13 ± 0.64	-1.37 ± 1.05			$8.93 \cdot 10^{-8}$	$3.19 \cdot 10^{-4}$
Image 15								
H_0	2.32	2.28	2.16 ± 0.58	-1.92 ± 0.85	-2.08 ± 0.91	3.26 ± 1.68	$1.91 \cdot 10^{-7}$	$2.19 \cdot 10^{-3}$
$H_1 + H_2$	2.30		2.16 ± 0.58	-1.96 ± 0.81		3.19 ± 1.65	$1.91 \cdot 10^{-7}$	$2.24 \cdot 10^{-3}$
$H_1 + H_2 + H_3$	2.30		2.16 ± 0.58	-1.74 ± 0.86			$1.91 \cdot 10^{-7}$	$1.13 \cdot 10^{-4}$

that proofs are not given in this paper of asymptotic results, justifying our use of confidence intervals based on the observed information and of chi-square approximations to distributions of test statistics. Asymptotic results are discussed in Møller & Sørensen (1990).

Table 1 gives the maximum likelihood estimates of the model parameters for the data in image 14a and image 15 under the hypotheses H_0, $H_1 + H_2$ and $H_1 + H_2 + H_3$. Confidence intervals are given for the most important parameters. These intervals are based on the observed information evaluated at the maximum likelihood estimates. The interval for γ is derived from the profile information, which is known to give the correct results, see Richards (1961) and Patefield (1977).

The estimates in Table 1 satisfy condition (3.16), implying that the interaction between neighbouring dunes is a repulsion as expected. Also the condition which implies geometrically fast convergence into equilibrium, cf. subsection 3.2, is satisfied. A glance at Table 1 is enough to expect the hypotheses of symmetry H_1 and H_2 to be acceptable, and indeed the likelihood ratio test statistic Q for H_1 under H_0 is $-2\log Q = 0.05$ for image 14a and $-2\log Q = 0.01$ for image 15. The corresponding observed levels of significance, using the $\chi^2(1)$-distribution, are 82% and 92%. When testing H_2 under H_0 we find $-2\log Q = 0.09$ for image 14a and $-2\log Q = 0.30$ for image 15, corresponding to 76% and 59% levels of significance, respectively. The fact that all four levels of significance are quite large is slightly worrying. It might be an indication that the asymptotic chi-square distributions are somewhat inaccurate. This question could be investigated by simulation.

It is not obvious from Table 1 whether H_3, which states that the tendency of a dune to die depends only on the relative distances to its neighbouring dunes, is acceptable or not, and the likelihood ratio tests do not give a clear answer to the question. The test statistics for testing H_3 under H_2 are, for the two images, $-2\log Q = 3.60$ and $-2\log Q = 3.65$, respectively. Both these values correspond to an observed significance level of about 6%, so we do not have much faith in H_3.

In Table 2 the partial likelihood estimates are given together with confidence intervals based on the observed partial likelihood information. These estimates are rather different from the maximum likelihood estimates, but they also satisfy the conditions for repulsion between neighbouring dunes and for geometrically fast convergence into equilibrium. It is interesting that the estimates of the sum $\delta + \varepsilon + \varphi$ vary only by 0.1 at most between the two tables. The confidence intervals in Table 2 are consistently larger than those in Table 1, indicating a loss of information in using the partial likelihood function. We shall discuss the partial likelihood estimates further in section 6.

Table 2. *Maximum partial likelihood estimates under the hypotheses H_0, H_2 and $H_2 + H_3$ of γ, δ, ε and φ and the corresponding estimates of k and c obtained by substitution in (4.16). The estimates are based on the data in image 14a and image 15. The confidence intervals are based on the observed partial likelihood information*

	γ	δ	ε	φ	k	c
Image 14a						
H_0	3.89 ± 0.97	-2.30 ± 1.55	-2.39 ± 1.55	3.89 ± 3.09	$5.22 \cdot 10^{-9}$	$1.28 \cdot 10^{-3}$
H_2	3.89 ± 0.97	-2.34 ± 1.51		3.88 ± 3.10	$5.22 \cdot 10^{-9}$	$1.26 \cdot 10^{-3}$
$H_2 + H_3$	3.89 ± 0.97	-2.23 ± 1.53			$5.22 \cdot 10^{-9}$	$7.82 \cdot 10^{-5}$
Image 15						
H_0	2.32 ± 0.68	-1.81 ± 1.45	-1.94 ± 1.49	2.93 ± 2.73	$9.24 \cdot 10^{-8}$	$3.83 \cdot 10^{-3}$
H_2	2.32 ± 0.68	-1.85 ± 1.43		2.88 ± 2.73	$9.24 \cdot 10^{-8}$	$4.01 \cdot 10^{-3}$
$H_2 + H_3$	2.32 ± 0.68	-1.50 ± 1.29			$9.24 \cdot 10^{-8}$	$1.67 \cdot 10^{-4}$

Under the hypothesis $H_1 + H_2$, the structure of a linear dune field is described by six parameters k, c, α, γ, δ and $\varphi + 2\delta$ which we can use for comparing the two images 14a and 15. The maximum likelihood estimates of the last parameter are -0.89 (image 14a) and -0.73 (image 15), so in this respect the two parts of the dune field do not differ. The variation of $\hat{\delta}$ is obviously not significant. On the other hand, the estimates of γ vary considerably from one image to the other.

Let us finally consider the balance condition H_4. Under H_1 and H_2 the hypothesis H_4 states that $\alpha + \varphi + \varepsilon - \gamma = 0$. From Table 1 we see that the maximum likelihood estimates of this sum are 0.35 and 1.37 for image 14a and image 15, respectively. The likelihood ratio statistics for testing H_4 under $H_1 + H_2$ are $-2 \log Q = 0.15$ and $-2 \log Q = 3.41$ for the two images, respectively. If we evaluate these statistics in the $\chi^2(1)$-distribution, we find that the observed levels of significance are 70% and 7% for image 14a and image 15, respectively, which sheds doubt on the hypothesis of reversibility in the part of the dune field shown on image 15.

6. Model control

In this section we propose various plots for checking how the data fit our model. The graphical techniques developed are tried out on our dune field data, image 14a and image 15. We work here under the model for the observed process y given by (4.2)–(4.4) and the description below these formulae, i.e. the boundary process δ is considered fixed, but we do not condition on future values of δ when describing the dynamics of y.

The exponential distribution of the waiting times between events can be checked by means of a P-P plot of the observed waiting times for the y process. The latter waiting times are not distributed exponentially, however. For each $i \in \mathscr{B} \cup \mathscr{D}$ define the number $k(i) < i$ by $k(i) \in \mathscr{B} \cup \mathscr{D}$ and $k(i) < j < i \Rightarrow j \notin \mathscr{B} \cup \mathscr{D}$. Thus the time between the birth or death at time t_i and the last non-boundary event before t_i is $t_i - t_{k(i)}$. The integrated hazard of the distribution of this waiting time evaluated at $t_i - t_{k(i)}$ is

$$\Lambda_i = \sum_{j=k(i)+1}^{i} \Delta_j \left\{ \sum_{m=0}^{n_j} H(y_{jm}, y_{j(m+1)}) + \sum_{m=1}^{n_j} d(y_{j(m-1)}, y_{jm}, y_{j(m+1)}) \right\}, \tag{6.1}$$

conditionally on the state at time $t_{k(i)}$. Here H and d are given by (3.11), (3.13) and (3.15). Hence $\exp(-\Lambda_i)$ is, under the same condition, uniformly distributed in the interval [0, 1]. Since this distribution does not depend on the condition, it follows that unconditionally the random variables $\exp(-\Lambda_i)$, $i \in \mathscr{B} \cup \mathscr{D}$, are uniformly distributed in [0, 1]. It can also be seen that these random variables are independent. In Fig. 2 P-P plots based on the observed values of $\exp(-\Lambda_i)$ using the maximum likelihood estimates of the parameters under the full model are given for the two images. There are no systematic departures from the identity line except for the behaviour near zero.

The probability that event number i is a birth given that $i \in \mathscr{B} \cup \mathscr{D}$ and conditionally on the state at time t_{i-1} is

$$p_i = \frac{\sum_{j=0}^{n_i} H(y_{ij}, y_{i(j+1)})}{\sum_{j=0}^{n_i} H(y_{ij}, y_{i(j+1)}) + \sum_{j=1}^{n_i} d(y_{i(j-1)}, y_{ij}, y_{i(j+1)})} \tag{6.2}$$

Consider the normalized birth indicators

$$Z_i = \frac{1_{\{i \in \mathscr{B}\}} - p_i}{\{p_i(1-p_i)\}^{1/2}} \quad i \in \mathscr{B} \cup \mathscr{D}. \tag{6.3}$$

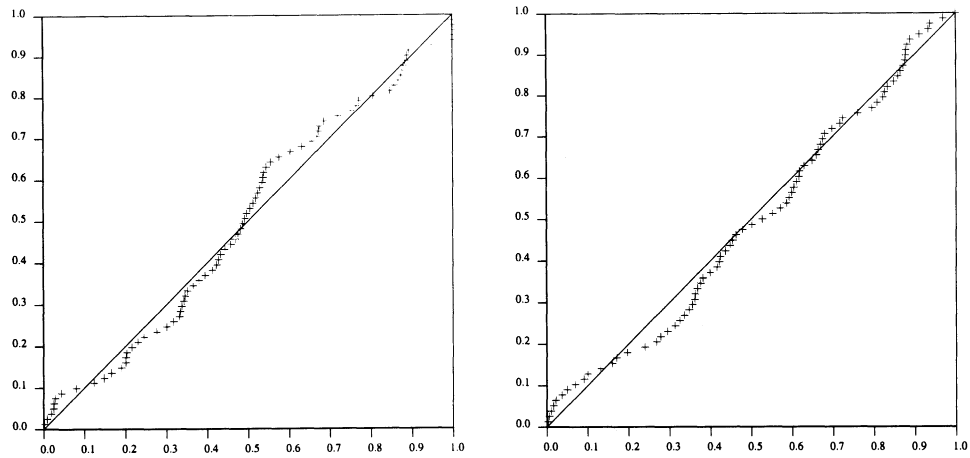

Fig. 2. Image 14a (left) and image 15 (right). Graphical check of the exponential distribution of the waiting times by means of a *P-P* plot of the observed values of $\exp(-\Lambda_i)$.

A martingale central limit theorem indicates (Møller & Sørensen, 1990) that if our model is correct, then

$$u = (\#\mathscr{B} + \#\mathscr{D})^{-1/2} \sum_{i \in \mathscr{B} \cup \mathscr{D}} Z_i \tag{6.4}$$

is asymptotically standard normally distributed as $\#\mathscr{B} + \#\mathscr{D}$ tends to infinity. Thus u can be used as a goodness-of-fit test statistic. For image 14a we find under H_0 that $u = 0.49$, while $u = -0.07$ for image 15.

Next we shall investigate whether (3.11) is a reasonable model for the probability that a new dune starts in the ith interdune interval given that a birth does take place. The distribution function of the length r_{ij_i} (see section 4) of the interval in which the birth labelled $i \in \mathscr{B}$ takes place is

$$F_i(x) = \sum_{j=0}^{n_i} r_{ij}^{\gamma} 1_{\{r_{ij} \leqslant x\}} \left[\sum_{m=0}^{n_i} r_{im}^{\gamma} \right]^{-1} \tag{6.5}$$

conditionally on the state at time t_{i-1}. Appealing to the law of large numbers for martingales (Møller & Sørensen, 1990), we expect that $H(x) \doteq F(x)$ as the number of births tends to infinity, where

$$H(x) = (\#\mathscr{B})^{-1} \sum_{i \in \mathscr{B}} 1_{\{r_{ij_i} \leqslant x\}} \tag{6.6}$$

and

$$F(x) = (\#\mathscr{B})^{-1} \sum_{i \in \mathscr{B}} F_i(x). \tag{6.7}$$

We can therefore make a generalized *P-P* plot by plotting the points $(H(r_{ij_i}), F(r_{ij_i}))$, $i \in \mathscr{B}$. This is done in Fig. 3 using the maximum likelihood estimates under the full model.

A similar generalized *P-P* plot can be made for the deaths. By arguments analogous to those given for (6.6) and (6.7) we expect that, for large values of $\#\mathscr{D}$, $G(x) \doteq D(x)$. Here

$$G(x) = (\#\mathscr{D})^{-1} \sum_{i \in \mathscr{D}} 1_{\{s_{ij_i} \leqslant x\}} \tag{6.8}$$

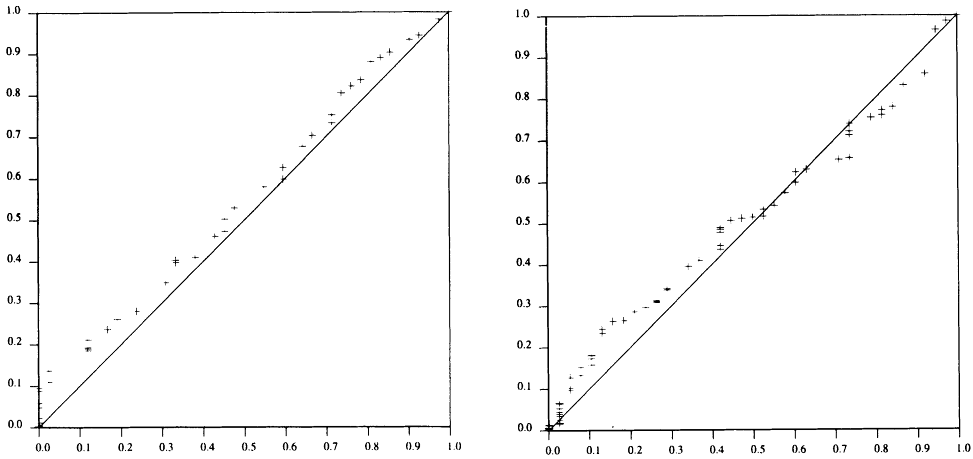

Fig. 3. Image 14a (left) and image 15 (right). Graphical check of (3.11) by means of a generalized *P-P* plot of $(H(r_{ij_i}), F(r_{ij_i}))$, $i \in \mathscr{B}$, given by (6.6) and (6.7). The maximum likelihood estimates of the parameters are used.

and

$$D(x) = (\#\mathscr{D})^{-1} \sum_{i \in \mathscr{D}} D_i(x), \tag{6.9}$$

where

$$D_i(x) = \sum_{j=1}^{n_i} s_{ij}^{\varphi}\, r_{i(j-1)}^{\varepsilon}\, r_{ij}^{\delta}\, 1_{\{s_{ij} \leq x\}} \left[\sum_{m=0}^{n_i} s_{im}^{\varphi}\, r_{i(m-1)}^{\varepsilon}\, r_{im}^{\delta} \right]^{-1} \tag{6.10}$$

is the distribution function of s_{ij_i} given the state at time t_{i-1}. Another possibility is to base the generalized *P-P*-plot on r_{ij_i}/s_{ij_i}, $i \in \mathscr{D}$, instead of s_{ij_i}, $i \in \mathscr{D}$. In Fig. 4 the points are plotted using the maximum likelihood estimates under the full model.

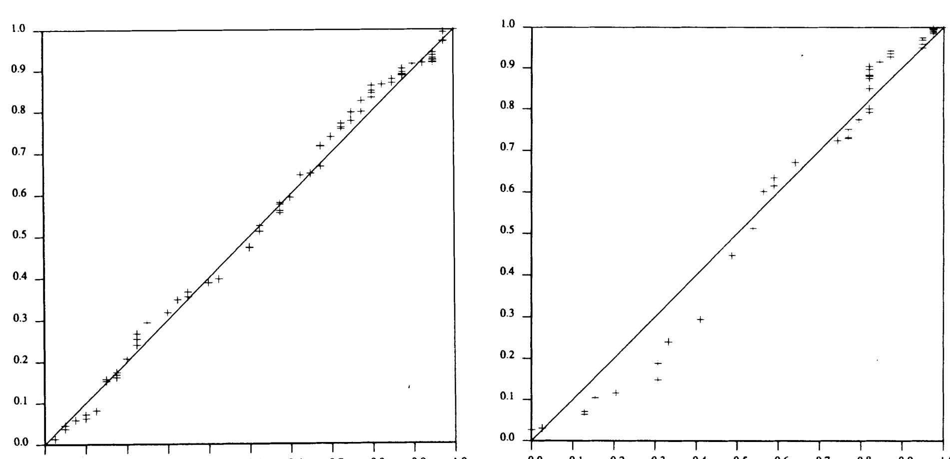

Fig. 4. Image 14a (left) and image 15 (right). Graphical check of (3.4) with d defined by (3.15) by means of a generalized *P-P* plot of $(G(s_{ij_i}), D(s_{ij_i}))$, $i \in \mathscr{D}$, given by (6.8) and (6.9). The maximum likelihood estimates of the parameters are used.

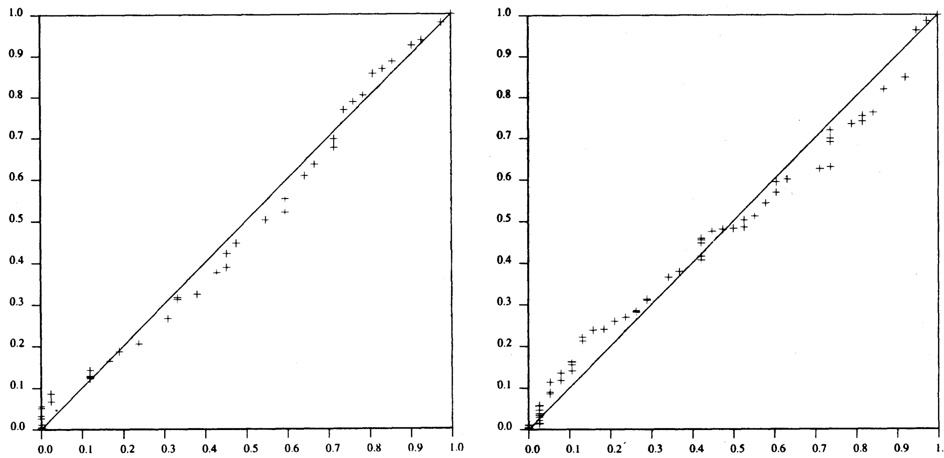

Fig. 5. Same as Fig. 3, but the maximum partial likelihood estimates of the parameters are used.

The fit in Fig. 3 is not quite satisfactory. The curves tend to be systematically above the identity line, because of the behaviour near zero. A somewhat better fit is obtained by using the maximum partial likelihood estimates as shown in Fig. 5. This is to be expected since the partial likelihood function does not depend on the waiting times, but is based only on the aspects of the model that are checked by means of the generalized *P-P* plots in Figs. 3–5. The fit in Fig. 4 also can be improved by using maximum partial likelihood estimates.

Finally, the assumption that the relative position of a new-born dune is Beta-distributed was checked by a traditional histogram, see Møller & Sørensen (1990). The fit was found to be satisfactory—most so for image 14a.

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Appendix. Existence and convergence of the model given by (3.14) and (3.15)

The spatial birth-and-death process given by (3.14) and (3.15) exists uniquely if

$$\beta_n > 0 \quad \text{for all} \quad n \geqslant 0 \quad \text{and} \quad \sum_{n=0}^{\infty} 1/\beta_n = \infty \tag{A.1}$$

where

$$\beta_n = \sup_{x \in \Omega_n} \beta(x),$$

see Preston (1977). By (3.13),

$$\beta_n = k \sup_{x \in \Omega_n} \sum_{i=0}^{n} (x_{i+1} - x_i)^{\gamma}.$$

From this we get

$$\beta_n = \begin{cases} k|S|^{\gamma} \quad (n+1)^{1-\gamma} & \text{if } 0 \leqslant \gamma \leqslant 1 \\ k|S|^{\gamma} & \text{if } \gamma > 1 \end{cases} \tag{A.2}$$

so (A.1) is seen to hold for $\gamma \geqslant 0$. Here $|S|$ denotes the length of the interval S.

The spatial birth-and-death process converges in distribution as $t \to \infty$ and the limit of the process is the unique equilibrium distribution provided the following conditions hold (Preston, 1977):

$$\delta_n > 0 \quad \text{for all} \quad n \geqslant 1 \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\beta_0 \ldots \beta_{n-1}}{\delta_1 \ldots \delta_n} < \infty, \tag{A.3}$$

$$\sum_{n=1}^{\infty} \frac{\delta_1 \ldots \delta_n}{\beta_1 \ldots \beta_n} = \infty, \tag{A.4}$$

where

$$\delta_n = \inf_{x \in \Omega_n} \delta(x).$$

We have

$$\delta_n = c \inf_{x \in \Omega_n} \sum_{i=1}^{n} (x_i - x_{i-1})^{\varepsilon} (x_{i+1} - x_i)^{\delta} (x_{i+1} - x_{i-1})^{\varphi},$$

cf. (3.15). For ε, δ and $\sigma = \varphi + \varepsilon + \delta$ all $\leqslant 0$, a lower limit of δ_n is obtained by

$$\delta_n \geqslant c \inf_{x \in \Omega_n} \sum_{i=1}^{n} (x_{i+1} - x_{i-1})^{\sigma}.$$

Using first Jensen's inequality and next that $\sigma \leqslant 0$ we get

$$\frac{1}{n} \sum_{i=1}^{n} (x_{i+1} - x_{i-1})^{\sigma} \geqslant \left[\frac{1}{n} \sum_{i=1}^{n} (x_{i+1} - x_{i-1})\right]^{\sigma} \geqslant n^{-\sigma}(2|S|)^{\sigma},$$

and so

$$\delta_n \geqslant c n^{1-\sigma}(2|S|)^{\sigma}. \tag{A.5}$$

Hence, if $\gamma \geqslant 0$ and $\varepsilon, \delta, \varphi + \varepsilon + \delta \leqslant 0$, (A.2) and (A.5) give

$$\sum_{n=1}^{\infty} \frac{\beta_0 \ldots \beta_{n-1}}{\delta_1 \ldots \delta_n} \leqslant \sum_{n=1}^{\infty} m^n (n!)^{\sigma - \min(1, \gamma)}$$

and

$$\sum_{n=1}^{\infty} \frac{\delta_1 \ldots \delta_n}{\beta_1 \ldots \beta_n} \geqslant \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{m^{-n}(n!)^{\gamma-\sigma}}{(n+1)^{1-\gamma}} & \text{for } 0 \leqslant \gamma \leqslant 1 \\ \displaystyle\sum_{n=1}^{\infty} m^{-n}(n!)^{1-\sigma} & \text{for } \gamma > 1 \end{cases}$$

where

$$m = k|S|^{\gamma-\sigma}/(c2^{\sigma}).$$

Thus (A.3) and (A.4) follow if ε, δ, $\varphi + \varepsilon + \delta \leqslant 0$ and either (i) $\gamma > 0$ or (ii) $\gamma = 0$ and $\varphi + \varepsilon + \delta < 0$ or (iii) $\gamma = \varphi = \varepsilon = \delta = 0$ and $c > k$. In fact under these conditions the convergence is geometrically fast. This follows easily by combining (A.2), (A.5) and (A.6) with cor. 3.2 in Møller (1989).