Fractals and Chaos–Homework 3

Chapter 7.

Solution 8.

Consider $Q_c(1/2)$ and $-p_+$ for $c = -(5+2\sqrt{5})/4$. We have that $Q_c(1/2) = -1 - \sqrt{5}/2$ and $-p_+ = -1/2(1+\sqrt{1-4c}) = -1/2(1+\sqrt{1+2\sqrt{5}+5}) = -1/2(1+1+\sqrt{5}) = -1-\sqrt{5}/2$. Thus $-p_+ = Q_c(1/2)$ for this value of c. Let $u(c) = -1/2(1+\sqrt{-4c})$ and note that $-p_+ < u(c)$ for any negative c. We have the following:

$$\frac{du}{dc} = \frac{1}{2\sqrt{-c}}$$
$$\frac{d(Q_c(1/2))}{dc} = 1$$

For all $c < -1$, we observe that $\frac{d(Q_c(1/2))}{dc} - \frac{du}{dc} > 0$. Thus for any $\gamma < -(5+2\sqrt{5}) < -1$, we know the following inequality:

$$\int_{-(5+2\sqrt{5})}^{\gamma} \frac{d(Q_c(1/2))}{dc} - \frac{du}{dc}dc = -\int_{\gamma}^{-(5+2\sqrt{5})} \frac{d(Q_c(1/2))}{dc} - \frac{du}{dc}dc$$
$$< 0$$

Thus we know that at any value of $c < -(5+2\sqrt{5})$, we have that $Q_c(1/2) - u$ is bounded above by the value of $Q_c(1/2) - u$ at $c = -(5+2\sqrt{5})$. Previously we have shown that $-p_+ < u(c)$ for all negative c, and thus at $c = -(5+2\sqrt{5})$ since $-p_+ = Q_c(1/2)$, we have that $Q_c(1/2) < u(c)$. This gives us $Q_c(1/2) - u(c) < 0$, and we can conclude that for all $c < -(5+2\sqrt{5})$ we have $Q_c(1/2) < u(c) < -p_+$.

Solution 9.

By graphical analysis we show that all $x > 1$ or $x < 0$ diverge to $-\infty$.

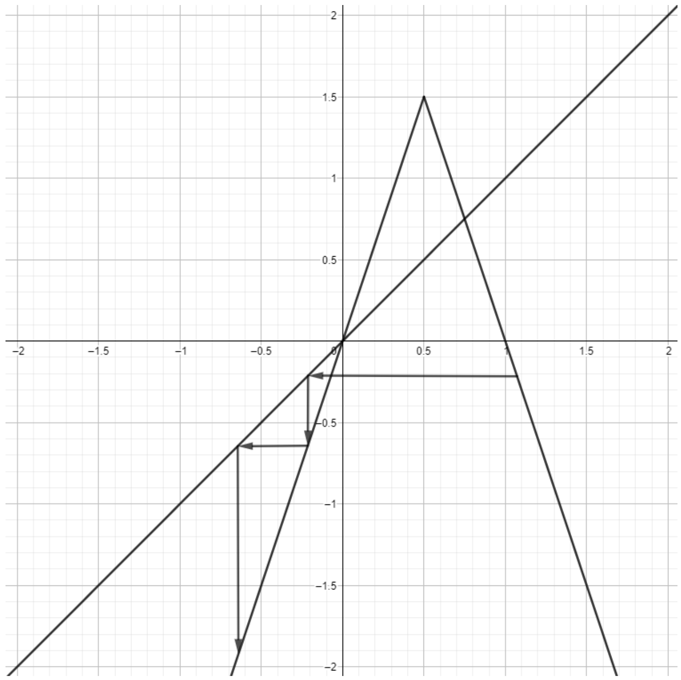

Figure 1: All $x > 1$ or $x < 0$ follow the kind of path drawn above.

Solution 11.

Consider the orbits of $3/13$ and $3/28$.

$T(3/13) = 9/13$	$T(3/28) = 9/28$
$T(9/13) = 12/13$	$T(9/28) = 27/28$
$T(12/13) = 3/13$	$T(27/28) = 3/28$

Both of these have prime period 3.

Solution 12.

If $x \in (1/3, 2/3)$, we have that $x > 1/3$, and thus $3x > 1$. On the other hand, we also have that $x < 2/3$, and thus $-3x > -2 \implies 3 - 3x > 1$. We conclude that in either case for $T(x)$, we have that $T(x) > 1$ and by our earlier result must diverge to $-\infty$.

Solution 13.

If $x \in (1/9, 2/9)$, then $x \leq 1/2$ meaning $T(x) = 3x$. This would then map $(1/9, 2/9)$ to $(1/3, 2/3)$. Likewise if $x \in (7/9, 8/9)$, then $x > 1/2$ and $T(3) = 3 - 3x$. Thus $(7/9, 8/9)$ is also mapped to $(1/3, 2/3)$. By our earlier result, both of these intervals must also diverge to $-\infty$.

Solution 16.

$F_4(x)$ is a parabola that passes through the points $(0,0)$ and $(0,1)$, with a peak at $(1/2,1)$. We note through graphical observation that this function takes both of the intervals $(0,1/2)$ and $(1/2,1)$ to $(0,1)$. Since $F_4(x)$ is continuous and begins each interval on one side of $f(x)=x$ while ending on the other, we then know that there is a x in each interval such that $F_4(x)=x$. We also note that repeated applications of F_4 will double the amount of peaks and troughs in the graph. Algebraically, we have that after n iterations, the intervals $(i/2^n,(i+1)/2^n$ will all map to $(0,1$ for integers $0\leq i\leq 2^n$. Thus $F_4^n(x)$ has at least 2^n intersections with $f(x)=x$, and $F_4(x)$ has at least 2^n points of period n.

Additional Problems

Solution 1.

As we showed in class, any element of K can be represented in base-3. Let $x\in[0,2]$. We will construct some $a,b\in K$ such that $a+b=x$. Begin with $i=1$. Let x_i be the i-th digit of x. If at our current digit, our previous digit did not require a carry to have been produced, we follow this table to emit the digits a_i, b_i, and decide if the next digit must produce a carry.

x_i	a_i	b_i	carry
0	0	0	0
1	0	0	1
2	2	0	0

If we don't need a carry we follow this table instead:

x_i	a_i	b_i	carry
0	2	0	1
1	2	2	0
2	2	2	1

At each digit we have used produced digits a_i,b_i such that they can combine with a carry to create the digit x_i with or without another carry for the previous digit. Thus through the recursive application of this method we can construct a and b for any x, and $[0,2]\subset K+K$.

In the other direction, we note that K is bounded by $[0,1]$ and thus $K+K$ is bounded by $[0,2]$. We conclude by double inclusion that $K+K=[0,2]$.

Solution 2.

To show that this modified Cantor set has positive measure, we consider the measure of all intervals removed from $[0,1]$. At the first step, an interval of length $1/6$ is removed. At each subsequent step, there are twice as many intervals removed with $1/3$ the length of the intervals from the previous step. Thus the total length is equal to the series $\Sigma_0^\infty\frac{1}{6}(\frac{2}{3})^n$. This geometric series evaluates to $1/2$, and thus the measure of the modified Cantor set has measure greater than 0.

Solution 3.

For the Sierpinsky triangle, we consider the area of the triangles that are removed at each step. This is equal to 3/4 times the area of the triangles that are removed in the previous step. Since the area removed in the first step is 1/4 of the area of the original triangle, we have that proportion of the original triangle's area that is removed is the geometric series $\Sigma_0^\infty \frac{1}{4}(\frac{3}{4})^n = 1$. Thus the entire area is removed.

We now consider the length of the sides which remain. At each step, the length of the sides which remain is equal to 3/2 times the length of the sides which remained from the previous step. Since the length of the sides which remain from the first step is some perimeter p, we have that the total sides which remain is the geometric series $\Sigma_0^\infty p(\frac{3}{2})^n = \infty$.