Space Debris

Abstract

In this essay, we will be delving into the topic of space debris and its capture. We will start by developing equations for the motion of orbiting space debris, originating from Newton's equations. Then we will progress onto modelling the trajectory of orbiting objects to develop a program that finds the closest distance of a spacecraft to a piece of debris. Our model for orbiting objects will allow us to further create a program that will approximate the amount of fuel and thrust required for a spacecraft to reach the debris in orbit from given positions. We will continue to apply our programs to find optimal parameters that minimise fuel usage while getting the spacecraft within a metre of the space debris. Towards the end of this essay, we will look at the change of atmospheric density at different altitudes and evaluate a function for density by fitting it against real data. Finally, we will develop a predictive function for the time of debris re-entry from expressions for energy dissipation and altitude change. Our function will account for the effect of atmospheric density and other conditions such as mass, area and altitude of the debris.

Throughout this essay, we will use Python to generate our plots and programs which we will refer to and utilise for conclusions about our models. We will strengthen our understanding of space debris capture and explore the mathematics behind the process.

1 Introduction

The capture of space debris is an often overlooked aspect of maintaining our modern civilisation. While many applaud our efforts to explore the universe and launch rocket-powered machines into the cosmos, few pause to consider the lingering remnants of these endeavours.

In more than 60 years of space activities, some more than 56450 objects have been tracked in orbit, of which about 28160 remain in space. A small fraction of these remaining objects (about 4000) are intact, operational satellites [1]. By the nature of gravity, these objects are due to re-enter our atmosphere and in the process could cause damage to both our planet and space endeavours.

The main threat of space debris is that posed to our satellites, as the debris changes its orbit it can collide with our functional satellites and cause damage to them, potentially destroying them. An increasingly more relevant issue caused by space debris is its contribution to global warming, as debris that has re-entered our atmosphere will burn up on its approach to Earth causing increased emissions. In order to combat these issues, humanity has developed space debris capture programs starting from 1979 by NASA [2], but how difficult is it to capture debris? To give an idea of how difficult space debris capture is, the first mission dedicated to testing space debris capture methods was only launched in 2018 [3]. We will analyse how we can capture debris along with the complexities that affect such processes.

2 The fall of space debris

Space debris comes from old satellites or rockets and remain in our orbit until they gradually fall back to Earth. This has become much more prevalent due to the major increase in satellites and space missions over the more recent decades with its growth being exponential. As of December 31, 2022, there are 6,718 operational satellites in the Earth's orbit [4] [5].

In order to approach the question of how can we clean up space debris, we need to find a way of modelling orbiting objects. To do this we will use the set of equations laid out by Sir Isaac Newton:

$$\begin{aligned} m\ddot{x} &= -\frac{GM_E m}{x^2+y^2}\frac{x}{\sqrt{x^2+y^2}} + F_x(t), \\ m\ddot{y} &= -\frac{GM_E m}{x^2+y^2}\frac{y}{\sqrt{x^2+y^2}} + F_y(t). \end{aligned} \tag{1}$$

Where $G = 6.673 \times 10^{-11}$ is Newton's gravitational constant, M_E is the mass of the Earth and m being the mass of the spacecraft. F_x and F_y are the forces exerted on the object along the x and y axis respectively. These forces are generated from the thrusters (in the case of the spacecraft which we will later address), or the friction forces from the very thin atmosphere or solar wind.

An easy approach to solving such equations is using polar coordinates in which we will assume orbits around the Earth are circular, to which we can use radii and angles to represent an object position.

$$x = r\sin(\theta), \quad y = r\cos(\theta). \tag{2}$$

We want to express the second order differential equations in terms of $\ddot{r}$ and $\ddot{\theta}$ thus we need to express $\ddot{x}$ and $\ddot{y}$ in terms of r and θ: **Q1**

$$\begin{aligned} \ddot{x} &= \ddot{r}\sin(\theta) + 2\dot{r}\cos(\theta)\dot{\theta} - r\sin(\theta)\dot{\theta}^2 + rcos(\theta)\ddot{\theta}, \\ \ddot{y} &= \ddot{r}\cos(\theta) - 2\dot{r}\sin(\theta)\dot{\theta} - r\cos(\theta)\dot{\theta}^2 - r\sin(\theta)\ddot{\theta}. \end{aligned}$$

We can then form a relation between $\ddot{x}$, $\ddot{y}$ and $\ddot{r}$, $\ddot{\theta}$:

$$m\ddot{x}\sin(\theta) + m\ddot{y}\cos(\theta) = m(\ddot{r} - r\dot{\theta}^2). \tag{3}$$

Then we substitute (3) and (2) into (1) to get:

$$\begin{aligned} m(\ddot{r} - r\dot{\theta}^2) &= -\frac{GM_E m}{r^2}\sin^2(\theta) - \frac{GM_E m}{r^2}\cos^2(\theta) + F_x(t)\sin(\theta) + F_y(t)\cos(\theta) \\ &= -\frac{GM_E m}{r^2} + F_r \\ &\Rightarrow \quad \ddot{r} = -\frac{GM_E}{r^2} + \frac{F_r}{m} + r\ddot{\theta}^2. \end{aligned}$$

Similarly:

$$\begin{aligned} m\ddot{x}\cos(\theta) - m\ddot{y}\sin(\theta) &= \\ m(2\dot{r}\dot{\theta} + r\ddot{\theta}) &= -\frac{GM_E m}{r^2}\sin(\theta)\cos(\theta) + \frac{GM_E m}{r^2}\sin(\theta)\cos(\theta) + F_x\cos(\theta) - F_y\sin(\theta) \\ &\Rightarrow \quad \ddot{\theta} = -2\frac{\dot{\theta}\dot{r}}{r} + \frac{F_\theta}{mr}. \end{aligned}$$

This leaves us with the second order differential equations:

$$\ddot{r} = -\frac{GM_E}{r^2} + \frac{F_r}{m} + r\ddot{\theta}^2, \tag{4}$$

$$\ddot{\theta} = -2\frac{\dot{\theta}\dot{r}}{r} + \frac{F_\theta}{mr}. \tag{5}$$

Where $F_r = F_x \sin(\theta) + F_y \cos(\theta)$ and $F_\theta = F_x \cos(\theta) - F_y \sin(\theta)$.

Now that we have a pair of equations that allow us to find the relative motion and position of a spacecraft, we require an expression for the speed of a piece of debris.
Utilising the equations above we can set $F_r = F_\theta = 0$, giving the simplest solution as

$$r = r_0, \quad \theta = \omega_0 t + \theta_0.$$

Which corresponds to a circular orbit of period

$$T_0 = \frac{2\pi}{\omega_0}, \quad \omega_0 = \sqrt{\frac{GM_E}{r_0^3}}, \tag{6}$$

Where $r_0 = R_E + h$ (R_E and h being the Earth's radius and the altitude above the Earth's surface respectively) and ω_0 are both constants.

Now we can derive the expression for the speed of a piece of debris using our simple solution (where we assume a circular orbit as previously established).
We can derive the speed, v, from its expression of distance and time: Q2

$$v = \frac{distance}{time} = \frac{2\pi r}{T_0}$$

Inserting (6) into our expression we get

$$\begin{aligned} v = r_o\omega_0 &= r_0\sqrt{\frac{GM_E}{r_0^3}} \\ &= \sqrt{\frac{GM_E}{r_0}} \end{aligned} \tag{7}$$

Using (6), we can calculate the time it takes for a piece of debris to travel around the Earth. Let us consider two cases which we will calculate the period time of utilising (6):

- for an orbit of 330km above the Earth's surface we get a period of 90.99 minutes/ 5459.39 seconds (2 decimal places) or 90 minutes and 59 seconds.
- For and orbit of 435km we get a period of 93.14 minutes / 5588.23 (2 decimal places) seconds or 93 minutes and 8 seconds.

Here we use the previously stated constant G, $R_E = 6.370 \times 10^6$ (radius of the Earth) and $M_E = 5.97 \times 10^24$ from [6] to limited significant figures.
The specific cases relate to the lower and upper orbit of the international space station.

When comparing our calculated periods to the average period of the international space station [7] we find them suitable giving they are around 90 and 93 minutes.

3 Space debris capture

In order to model the capture of debris we will assume our spacecraft must get within one meter of the debris. Our previous set of second order differentials are useful in interpreting the motion of a space craft or debris in orbit, however, we will require a new set of differential equations in terms of relative coordinates for our model. We will utilise these equations in our future programs to model the trajectory of the spacecraft over periods of time.

Let us consider our reference trajectory (the position of the debris) as:

$$r = r_0, \quad \theta = \omega_0 t. \tag{8}$$

In which $r_0 = R_E + h$ as previously used in (6).
Now we will express the position of the active spacecraft as the relative coordinates

$$r = r_0 + z(t), \quad \theta = \omega_0 t + \phi(t). \tag{9}$$

Inserting (9) into (4) and (5) we get

$$\ddot{z} = -\frac{GM_E}{(r_0+z)^2} + (r_0+z)(\omega_0+\dot{\phi})^2 + \frac{F_r}{m}, \tag{10}$$

$$\ddot{\phi} = -2\frac{(\omega_0+\dot{\phi})\dot{z}}{(r_0+z)} + \frac{F_0}{m(r_0+z)}. \tag{11}$$

Converting (10) and (11) into a system of first order differentials we are left with: **Q3**

$$\begin{aligned}
\dot{z} &= V_z \\
\dot{V}_z &= -\frac{GM_E}{(r_0+z)^2} + (r_0+z)(\omega_0+\dot{\phi})^2 + \frac{F_r}{m} \\
\dot{\phi} &= V_\phi \\
\dot{V}_\phi &= -2\frac{(\omega_0+\dot{\phi})\dot{z}}{(r_0+z)} + \frac{F_0}{m(r_0+z)}.
\end{aligned} \tag{12}$$

We can calculate the distance between the spacecraft and reference trajectory considering the x and y components for the reference trajectory and space craft **Q4**

$$d = \sqrt{((r_0+z(t))\cos(\omega_0 t+\phi(t)) - r_0\cos(\omega_0 t))^2 + ((r_0+z(t))\sin(\omega_0 t+\phi(t)) - r_0\sin(\omega_0 t))^2}$$

$$\Rightarrow \quad d = \sqrt{z^2 + (r_0\phi)^2 + 2r_0 z(1-\cos(\phi))} \qquad \text{(using trigonometric identities)} \tag{13}$$

Where $z(t)$ is the change in orbital radius with respect to time.
We will test our formula against cases for which d can be easily computed:

- $z = 0, \quad \phi = 0$. Here it is obvious that $d = 0$ and holds true for our formula.
- $z = a, \quad \phi = 0$. Our formula yields a result of $d = a$ which is to be expected as the only path difference is that of the radius by z = a.
- $z = 0, \phi = a$. Our formula yields $d = r_0 a$. If we assumed small a the first order Taylor expansion would give us

  $$d = \sqrt{0 + (r_0 a)^2 + 2r_0 \times 0 \times (\frac{a}{2})^2} = r_0 a$$

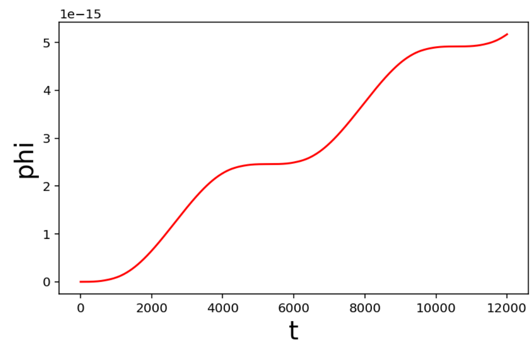
  which is the expected results as if the only change in polar coordinates is by angle ϕ then the actual distance moved is the arc length $r_0\phi$.

- $z = 0, \quad \phi = \pi/2$. We get $d = \frac{r_0 \pi}{2}$ once again holding true for the arc length distance covered.
- $z = a, \quad \phi = \pi$ where a is an arbitrary distance. Here we get $d = \sqrt{a^2 + (r_0\pi)^2 + 4r_0 a}$.

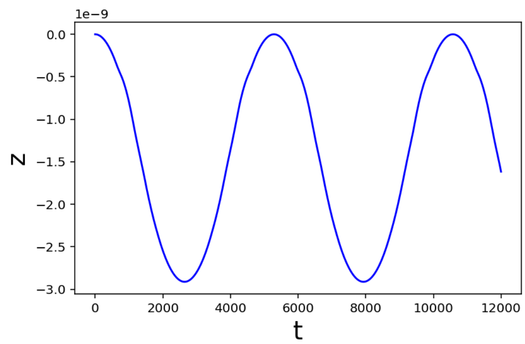
We have now established an expression for the distance between the debris and a system of first order differential equations for our spacecraft. We can create a python program and name it gravity.py which will solve our system of equations (12) and generate plots that show us the motion of a spacecraft over a period of time. We test this program against 3 cases, each having different initial parameters of $z, \quad \dot{z}, \quad \phi, \quad \dot{\phi}$, to show how it effects the motion of the spacecraft.

To get an understanding of the effect different positions in orbit have on the motion of objects we will look at these 3 cases with their plots.

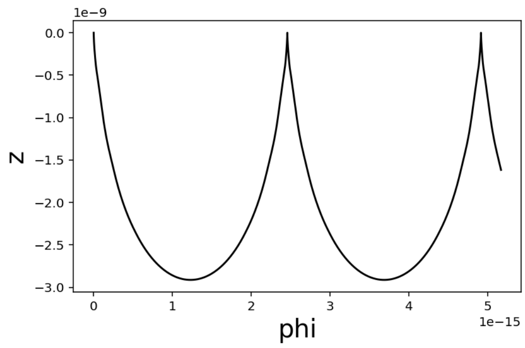
3.1 Case A: Initial Parameters $z(0) = \dot{z}(0) = \phi(0) = \dot{\phi}(0) = 0$



(a) Change in position on a fixed orbit over a 200-minute period.

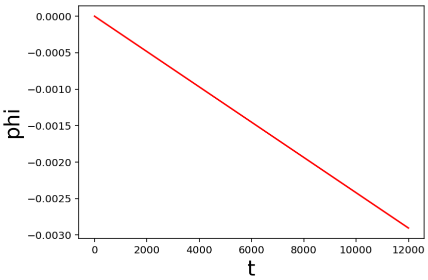


(b) Change in radial orbit over a 200-minute period.

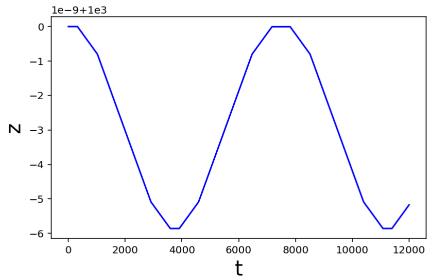


(c) Change in radial position with respect to its orbital position over a 200-minute period.

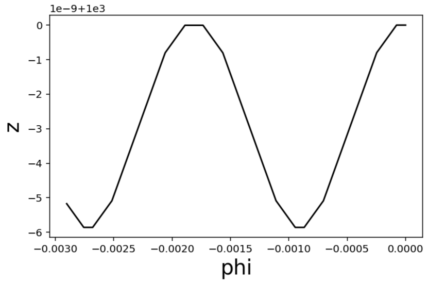
3.2 Case B: Initial Parameters $z(0) = 1000$, $\phi(0) = \dot{z}(0) = 0$, $\dot{\phi} = \sqrt{\frac{GM_E}{(r_0+z_0)^3}}$



(a) Change in position on a fixed orbit over a 200-minute period.

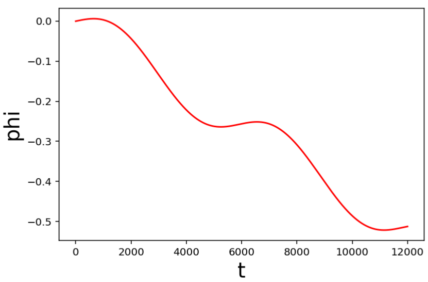


(b) Change in radial orbit over a 200-minute period.

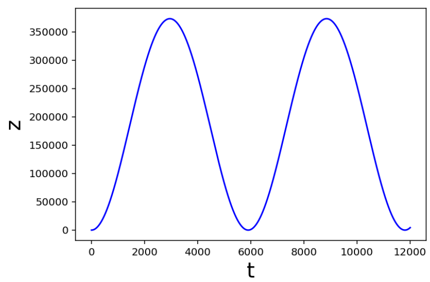


(c) Change in radial position with respect to its orbital position over a 200-minute period.

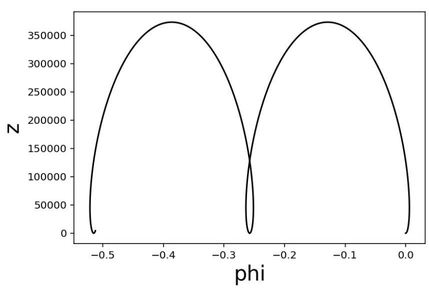
3.3 Case C: Initial Parameters $z(0) = \phi(0) = \dot{z}(0) = 0$, $\dot{\phi} = \frac{100}{(r_0+z_0)}$



(a) Change in position on a fixed orbit over a 200-minute period.



(b) Change in radial orbit over a 200-minute period.



(c) Change in radial position with respect to its orbital position over a 200-minute period.

In gravity.py we also developed functions using (13) to find the closest distance to a piece of debris. We will create another program called spacecraft.py which considers the continuous use of the previously mentioned thrusters that push the spacecraft into a different orbit and parallel to its orbit. We will test this program with 3 cases which will give us the trajectory plot of the spacecraft for each as well as the closest distance and its time it which it reaches said distance after it starts its thrusters, t_{min}. The spacecraft will have an initial position such that it is on a circular orbit 1km lower than the piece of debris and about 2km behind.

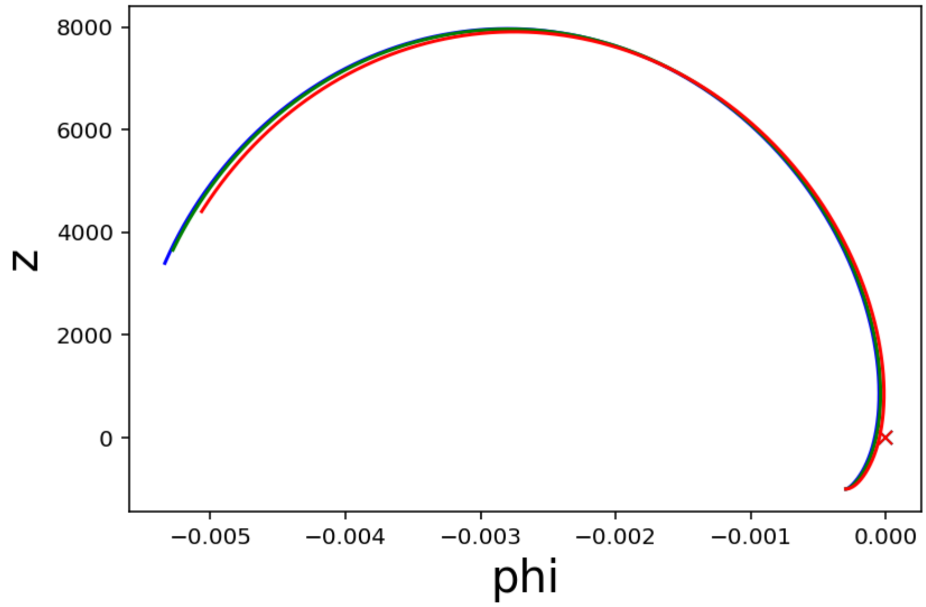


Figure 4: Trajectory plot for spacecraft.py

Here we have 3 paths of motion for a spacecraft on spacecraft.py, the conditions and distances are as follows (to appropriate significant figures) with t_{thrust} being the period of thrust activation:

- Blue: F_r=50 F_θ=100 t_{thrust}=100
  distance=499.61, t_{min}=524.80.
- Green: F_r=25 F_θ=50 t_{thrust}=200
  distance=432.74, t_{min}=567.90.
- Red: F_r=10 F_θ=20 t_{thrust}=500
  distance=279.71, t_{min}=693.40.

4 Manoeuvring the spacecraft

We have determined how close a spacecraft can get to a piece of debris with a set amount of thrust with our program spacecraft.py but in a real life scenario there is a lot more manoeuvring that comes into play.

Q5

Let us consider how close a spacecraft can get to a piece of debris by just giving it a steady push for a finite time. If we take $F_r = 50$ and $F_\theta = 100$ and adjust t_{thrust} (period of thrust activation) from our program we find the minimum distance is 220.63 meters (2 d.p) with $t_{thrust} = 350$ seconds.

Coming back to our main question - how difficult is it to capture debris? Space debris capture is not as simple as aiming straight at it. In the 1960s, during the Gemini missions, NASA conducted rendezvous and docking procedures that highlighted the challenges of approaching objects in space. It seemed as if they were getting further away from their target when they simply tried to propel directly towards it [8].

The reason why you cant simply aim straight at the debris to reach it is due to the difference in orbits and relative velocities. The tangential velocities are dependent on their position in orbit as such they will not stay in line long enough to simply propel directly towards the debris when in different levels of orbit.

Let us try and reach the piece of debris by adjusting our F_r and t_{thrust} (thruster that pushes the spacecraft radially, where a positive value represents a push into a higher orbit and negative to a lower). We calculate the fuel consumed by

$$Fuel = (|F_r| + |F_\theta|)t_{thrust}$$

in units kgm/s. We will fix the pushing force parallel to the orbit, F_θ, at a value of 100N. Keeping the maximum time of thrust to $t_{max} = 4000s$, we want to reach the piece of debris within a meter. Using our program, there are many parameters that get us within one meter of the debris, for instance $F_r = 32N$, $F_\theta = 100N$ and $t_{thrust} = 266s$ however, it is not very fuel efficient with the required fuel equalling $35112kgm/s$. A more optimal solution we can find is $F_r = 1.89N$, $F_\theta = 100N$ and $t_{thrust} = 73.3s$, this gets the spacecraft well within a meter at $D_{min} = 0.279$ (3 s.f) whilst only requiring $7468.537kgm/s$ of fuel.

Here we can see how the spacecraft gets extremely close to the piece of debris (red cross).
The green trajectory relating to the case with parameters, $F_r = 32N$ and $t_{thrust} = 266s$. Whilst blue refers to our more efficient approach, $F_r = 1.89N$ and $t_{thrust} = 73.3s$. The difference in radial path is significant but unnecessary as can be seen between the cases with significant difference in F_r.

Q6

Given the results of the program under specific parameters we can determine the most effective strategy to space debris capture if we do not assume that the spacecraft activates its thrusters when it is 2km behind the debris, but rather when it is the best time to do so.

Based upon the most fuel efficient result where $F_r = 1.89N$ and $F_\theta = 100N$, it would be best to activate the radial affecting thrusters when the orbit alignment is the most optimal and then focus fuel usage on increasing the velocity parallel to the orbit to reach the spacecraft. Given these specific parameters where the radial push is less significant, we can estimate that the fuel

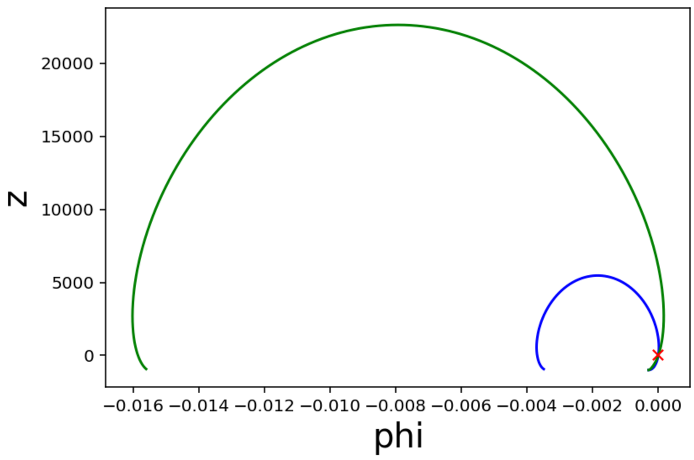


Figure 5: Trajectory of spacecraft using our two examples.

needed for this procedure to be around $7470 kgm/s$ (to 3 significant figures). One might think that if the spacecraft was timed to enter the orbit of the debris well enough then only a radial push would be needed and no push parallel to the orbit, however our program shows that the force F_θ is also dependent on radial position and thus if the spacecraft where to catch up to the debris optimally it would need some radial adjustment.

If we were to assume that the ejection speed of the thrusters gas is $1m/s$ we can calculate that the mass of fuel required is around 7.47kg (3 significant figures) for a piece of debris on an orbit 1km of that of the spacecraft. In this scenario the spacecraft will have to try and get within 2km behind the debris on the same orbit.
If the spacecraft has 500kg of fuel it could theoretically capture approximately 66 pieces of debris in the range of 300 to 500km above the Earth's surface., however this would be limited to the condition that the debris would be 1km different in orbit and 2km behind each other.

5 Atmospheric influence on debris

The Earth's atmosphere extends far into space and although it is very thin a few kilometres above the Earth's surface it can affect debris capture [9]. As debris travel at a high speed they are progressively slowed down by the Earth's atmosphere until they reach a denser section where they eventually burn out. We will try and estimate what the friction force is and how the altitude of debris might decrease due to it. For simplicity, we will assume that debris are on a circular orbit at all times and that the orbit will decrease very slowly due to the weak friction force.

We will consider the energy of the debris and the rate at which it dissipates. We previously computed the speed of a piece of debris (7). We will define the radius of the orbit as $r_h = R_E + h$, as well as its gravitational and kinetic energies as

$$P_h = -\frac{GmM_E}{r_h}, \quad K_h = \frac{GmM_E}{2r_h} \tag{14}$$

respectively.
We shall call its total energy

$$E_h = \quad P_h + K_h = -\frac{GmM_E}{2r_h} \tag{15}$$

At 300km the speed of atmospheric particles is smaller than the orbiting speed of a satellite, refer to appendix 2. In order to estimate the friction force on a piece of debris, we can assume that the molecules in the atmosphere are at rest and that their relative speed with the debris is the orbital speed of the debris.

As the debris travels through the atmosphere it will collide with $v_h A \rho_{At}$ kg of molecules where A is the cross-section of the piece of debris and ρ_{At} is the atmospheric density in kg/m^3. The molecules collide with the debris they carry of energy and momentum, and so by newtons third law a force is exerted on the debris which we can calculate as

$$F_{fr} = 2A\rho_{At}v_h^2. \tag{16}$$

The energy loss from the collisions is equal to the force times the distance travelled and thus the power dissipation, $\frac{dE_h}{dt}$, is given as the product of the force times the speed of the piece of the debris

$$\frac{dE_h}{dt} = -2A\rho_{At}v_h^3. \tag{17}$$

The average density of the high atmosphere can be found in [9] and can be fitted well with the function

$$\rho_{At}(h) = ae^{-hl} + B(\frac{h}{h_0})^{-\sigma} \tag{18}$$

where h is expressed in km, ρ in kg/m^3 and

$$a = 1.946kg/m^3, \quad l = 0.15/km, \quad B = 8.82 \times 10^7 kg/m^3, \sigma = 7.57, \quad h_0 = 1km. \tag{19}$$

We can write a program called fit_high_atmosphere.py which fits our density function against real data (18).
For our fitted function we keep B fixed and obtain the parameter values

$$a = 0.5347kg/m^3, \quad l = 0.1243/km, \quad \sigma = 7.104 \tag{20}$$

(all to 4 significant figures).

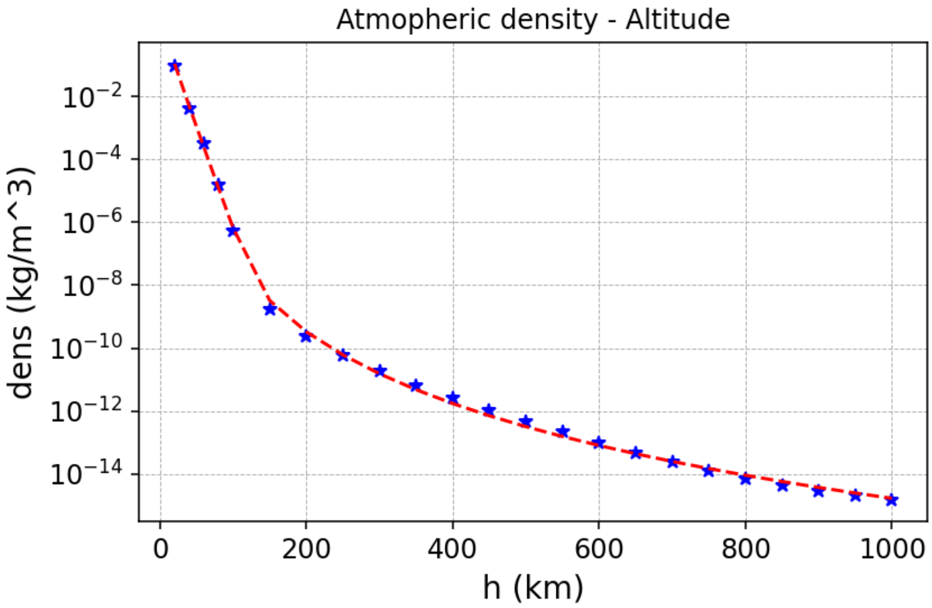


Figure 6: Logarithmic plot of given data and the fitted curve of our function.

Here we can see that our function is fitting for all altitudes and thus is reliable for use in our later derivations.

We can then convert (19) to measure h in meters and we get

$$a = 1.946kg/m^3, \quad l = 1.5\times 10^{-4}/m, \quad B = 4.63\times 10^{30}kg/m^3, \quad \sigma = 7.57, \quad h_0 = 1000m. \tag{21}$$

We can now start to develop expressions that will help us model the time of re-entry with atmospheric influence using expressions we developed in previous sections.

Using (15) and the expression for v (7), we can expression v in terms of E_h. First lets look at the derivation for K_h, K_h (or kinetic energy) is equal to $\frac{mv^2}{2}$, we can rewrite this using (7) to $K_h = \frac{GmM_E}{2r_h}$ which is equal to $-E_h$. **Q7**
Thus we can write

$$\begin{aligned} E_h = -K_h &= -\frac{mv^2}{2} \\ \Rightarrow v = \sqrt{-\frac{2E_h}{m}}. \end{aligned} \tag{22}$$

We will now use (15) to compute $\frac{dE}{dt}$ as a function of $\frac{dr_h}{dt}$

$$\frac{dE}{dt} = \frac{GmM_E}{2r_h^2} \times \frac{dr_h}{dt} \qquad \text{(by chain rule).}$$

We can also express $\frac{dE}{dt}$ as a function of G, M_E and r_h, as well as, R_E, A and $\rho_{At}(h)$ by using (17) and our Expression for v.

$$\begin{aligned}
\frac{dE}{dt} &= -2A\rho_{At}v_h^3 \\
&= -F_{fr} \times v_h \qquad \text{(using 16)} \\
&= -2A\rho_{At}(h)(\frac{GM_E}{R_E+h})^{\frac{3}{2}}
\end{aligned} \tag{23}$$

(given $h = r_h - R_E$). Now lets express r_h as a function of the energy E_h by using (15),

$$r_h = -\frac{GmM_E}{2E_h}.$$

Utilising this we can compute $\frac{dr_h}{dt} = \frac{dh}{dt}$ (as $h = r_h - R_E$ where R_E is the constant which is the Earth's radius) as a function of $A, \rho_{At}(h), G, M_E, m$ and h. From our expression of $\frac{dE}{dt}$ in terms of $\frac{dr_h}{dt}$, we can rearrange and use our newly established identities and expressions to get

$$\begin{aligned}
\frac{dr_h}{dt} = \frac{dh}{dt} &= \frac{\frac{dE}{dt} \times 2r_h^2}{GmM_E} \\
&= -\frac{4A\rho v_h^3(-\frac{GmM_E}{2E_h})^2}{GmM_E} = -\frac{A\rho v_h^3 GmM_E}{E_h^2} = \frac{2A\rho GM_E\sqrt{\frac{GM_E}{r_h}}}{E_h} \\
&= -\frac{4A\rho\sqrt{GM_E(R_E+h)}}{m}
\end{aligned} \tag{24}$$

This is a very useful function as we now have a first order differential equation which we can separate and integrate to solve for r_h in terms of t. We will apply this integral to solve for the times of re-entry and thus want to integrate from $h = h_0$, the initial altitude, to $h = 0$ the surface of the Earth.

$$\int 1dt = \int_{h_0}^{0} -\frac{m}{4A\rho\sqrt{GM_E(R_E+h)}}dh$$

Here we will use the approximations $r_h \approx R_E$ and $\rho_{At}(h) \approx Bh^{-\sigma}$ in order to be able to calculate this integral. This approximation consists in understanding the density of the atmosphere below 150km and so will lead to overestimating the time of re-entry by a very small amount.

$$\begin{aligned}
t &\approx \int_0^{h_0} \frac{m}{4ABh^{-\sigma}\sqrt{GM_ER_E}}dh \quad \text{(given that when } t=0, \quad h=0\text{, for time of re-entry)} \\
&\approx \frac{h_0^{\sigma+1}}{4B\sqrt{GM_ER_E}(\sigma+1)} \times \frac{m}{A}
\end{aligned} \tag{25}$$

This allows us to predict the re-entry times of pieces of debris in the atmosphere as a function of its latitude and the ration m/A.
According to a NASA podcast, Mark Matney states "so, at an object at the ISS orbit, which is about 400 kilometers at 250 miles, it would take several months for a piece of debris, or possibly a few years, to decay out of that orbit. Where NASA typically flies its scientific satellites at about 700 kilometers or 450 miles, it would take several decades to decay out. And objects only a little bit higher, to 1,000 kilometers or about 600 miles, will take centuries".[11] We will take some example cases of debris and plot a figure which we will compare to this statement to test our prediction function.

- Case 1: Consider an aluminium bolt $\rho = 2700kg/m^3$. We will consider the bolt to be a cube and calculate $\frac{m}{A} = 27$.

- Case 2: An aluminium rod of with $\frac{m}{A} = 54$.
- Case 3: A square aluminium plate with $\frac{m}{A} = 1.35$.
- Case 4: Gemini spacecraft with $\frac{m}{A} = 544.66$ (to 2 significant figures).

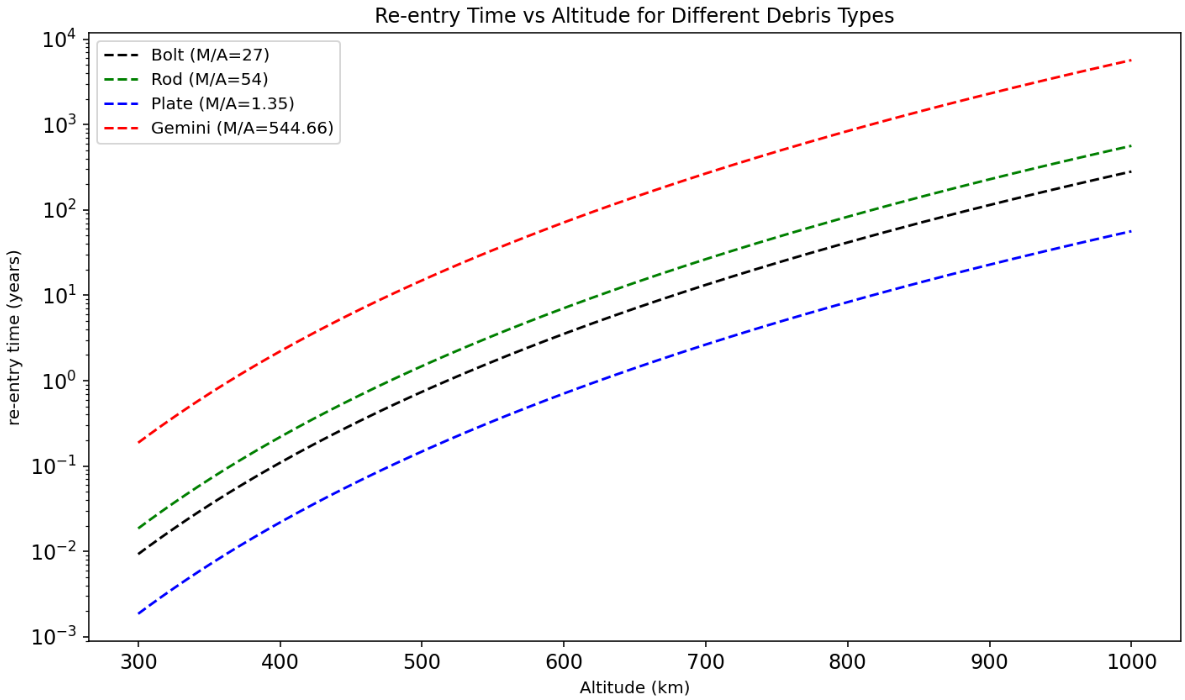


Figure 7: Logarithmic plot of re-entry time against altitude

Our graph shows us that our prediction does not completely agree with NASA's observations. Our prediction appears to only agree with NASA's observations above an altitude of around 700km.
Our prediction presents that the larger $\frac{m}{A}$ ratio debris take longer to re-enter the atmosphere.

Objects with a greater mass will retain more energy and momentum after a collision due to conservation laws, so the higher the mass of an object the longer we would expect it to remain in orbit. However, we also need to consider the frequency of collisions, the more frequent the collisions are the more energy and momentum is lost, so we would expect objects with less surface area to remain in orbit longer. As such it is clear that our plot is a good representation for the affect of the $\frac{m}{A}$ ratio on re-entry time.
Whilst our prediction is not good for low orbits it does clearly present the interaction of the atmosphere with different pieces of debris.

Our underestimation at lower orbits can be attributed to the approximation for density in our integration step, which was a significantly larger constant than we would expect for the density. We know from our graph (6) that the density has logarithmic behaviour and we can calculate the density at around 700km to find it fits this approximate value, however it fails for lower altitudes. Naturally, our plot would deviate from the podcast's statement due to this.

6 Conclusion

In conclusion, we started this essay by introducing Newton's equations which we then used to develop equations of motion for objects in orbit in terms of polar coordinates. We then created expressions for the period of an orbit and the velocity of an orbiting object.

We then started consider the capture of debris using a spacecraft and its complexities. We developed a system of first order differential equations which we could use to model a spacecraft's motion and developed a function to find the distance between the spacecraft and a piece of debris. Applying our system of equations and our distance function we developed a program called gravity.py which found the shortest distance between the spacecraft and debris given certain initial conditions. This then allowed us to create another function called spacecraft.py which considered the use of two thrusters (one for a radial push and the other for a push parallel to the orbit) on the spacecraft to see how close we could get the spacecraft to the debris under specific amounts of force over a specific amount of time. In this instance we considered an initial position 1km below in orbit and 2km behind from the debris.

We then took our consideration of thrusters a step further by contemplating manoeuvrability of the spacecraft. Once again by using spacecraft.py, we found a set amount of force from each thruster with a specific time of activation that could get us within a meter of the debris given the initial position being the same. Our program allowed us to compare parameters to find the most fuel efficient approach which we used to challenge questions such as how much fuel is required to on the spacecraft? Given a specific amount of fuel how much debris could we collect in a given range of orbital distance? We discovered that high amounts of fuel are not necessary for debris capture and a spacecraft should utilise its own orbit to get within a certain radius of the debris, then manoeuvrer towards it using its two thrusters.

Then in the final stages of this essay we delved into the effect of the atmosphere on debris re-entry which is a vital concept that we utilise to know which debris pose more of an issue than others. We considered the effect of particle collisions that affect the energy of the debris and created and expression for the dissipation of that energy as well as the force exerted. We moved on to fit a function that relates the atmospheric density to the altitude above the Earth's surface using real data in a program called fit_high_atmosphere.py. Our program yielded a good fit for our function. To lead us into our final model we developed expressions for energy dissipation, $\frac{dE}{dt}$ in terms or the rate of change of the radial position, $\frac{dr_h}{dt}$ of our debris, and incorporated our previous expression for speed which we then found in terms of energy. This allowed us to sub in expressions for $\frac{dE}{dt}$ and $\frac{dr_h}{dt}$ to leave us with a first order differential equation for $\frac{dh}{dt}$.
Gradually, we solved this differential to be able to predict the re-entry time of debris given their altitude and ratio of mass to area, $\frac{m}{A}$. Our prediction was found to give a reliable representation of the effect the ratio, $\frac{m}{A}$, has on debris re-entry time but also yielded flaws below 700km of altitude due to the lack of a more appropriate approximation used for our density function. Our prediction could be refined to account for many factors such as: change in particle temperature at different orbits, effects of drag, solar activity and elliptical orbits [12].

Our approach has its limitations, a massive one being the fact that we have assumed a spherical orbit throughout this essay despite it being elliptical in reality but, overall, we have become more proficient in our understanding of space debris and how we can capture it. Space debris will only continue to increase over time if we do not increase our efforts to reduce it, thankfully organisations like NASA and the ESA are continuing to tackle this problem.

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7 Appendix 2

At 300km, the atmosphere is mostly made out of Nitrogen, Helium and oxygen [10]. and the density of gas is approximately $\rho = 10^{-11} kg/m^3$ [9]. Moreover, the temperature at high altitude is approximately 1500K at 300km and drops to 300K around 100km. At such altitude, the average speed of a gas molecules in one direction can then be estimated using

$$\frac{k_B T}{2} = \frac{mv^2}{2},$$

where $k_B = 1.38 \times 10^{-23} J/K$ and T is the temperature in Kelvin. As for Nitrogen $m_N = 2.32 \times 10{-}26kg$ we have $v_N \approx \sqrt{k_B t m_N} \approx 950 m/s$. On the other hand, using (6) the orbital speed at 300km is $v_h \approx 7700 m/s$, or in other words, much larger than the average thermal speed of gas molecule.