RBOT101: MATHEMATICAL FOUNDATIONS OF ROBOTICS

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Random Variables

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P2-1.

The function g simply needs to map $X = 1$ back into itself at $Y = 1$, and map $X = 0$ to $Y = -1$ since Y is meant to be Rademacher. Affine functions are the simplest function class capable of such a transformation, so we try that first. Indeed, we find that we can use

$$g(X) = 2X - 1 \tag{1}$$

although other unnecessarily complicated choices such as

$$g(X) = 8X^3 - 6X - 1 \tag{2}$$

would also work (since we are working with discrete random variables the nonlinearities of the functions do not matter, we are just matching points-to-points).

The inverse function h can be found by literally inverting the function g in the elementary function sense:

$$h(Y) = (Y + 1)/2. \tag{3}$$

For the pmf of Y, since X is Bernoulli X only takes the values 0 and 1, and therefore Y only takes the values $2(0) - 1 = -1$ and $2(1) - 1 = 1$. Thus it suffices to compute the probabilities $P(Y = 1)$ and $P(Y = -1)$:

$$P(Y = 1) = P(2X - 1 = 1) = P(X = 1) = p, \tag{4}$$

$$P(Y = -1) = P(2X - 1 = -1) = P(X = 0) = 1 - p, \tag{5}$$

and $P(Y \not\in \{1, -1\}) = 0$. This is precisely the pmf of a Rademacher random variable, so g is correct.

Carrying out a similar procedure for h shows

$$P(X = 1) = P((Y + 1)/2 = 1) = P(Y = 1) = p \tag{6}$$

$$P(X = 0) = P((Y + 1)/2 = 0) = P(Y = -1) = 1 - p, \tag{7}$$

and $P(X \not\in \{1, 0\}) = 0$. This is precisely the pmf of a Bernoulli random variable, so h is correct.

P2-2.

(a)

Define the events

$$A_i = \{\text{receiver } i \text{ succeeded in a single attempt}\} \tag{8}$$
$$B = \{\text{all } N \text{ receivers succeed in a single attempt}\} \tag{9}$$

Since the successes are independent for different receivers, we have

$$P(B) = P(A_1 \cap A_2 \cap \cdots \cap A_N) = P(A_1)P(A_2)\cdots P(A_N) = p^N \tag{10}$$

The complement of the event $S(m)$ is

$$S(m)^c = \{\text{failed transmission to at least one receiver in exactly } m \text{ attempts} \tag{11}$$

Treating "all N receivers succeed in a single attempt" as a single trial, we can use the binomial distribution to get

$$P(S(m)^c) = \binom{m}{0} \left(p^N\right)^0 \left(1 - p^N\right)^m = \left(1 - p^N\right)^m \tag{12}$$

Therefore we have the solution

$$\begin{aligned} P(m) &= P(S(m)) && (13)\\ &= 1 - P(S(m)^c) && (14)\\ &= 1 - \left(1 - p^N\right)^m && (15) \end{aligned}$$

(b)

First we consider a single receiver. Define the event and its complement

$$C_i = \{\text{receiver } i \text{ succeeds in at least one of the } m \text{ attempts}\} \tag{16}$$
$$C_i^c = \{\text{receiver } i \text{ fails in all of the } m \text{ attempts}\} \tag{17}$$

We have

$$\begin{aligned}
P(C_i) &= 1 - P(C_i^c) && (18)\\
&= 1 - \binom{m}{0} (P(A_i))^0 (1 - P(A_i))^m && (19)\\
&= 1 - \binom{m}{0} \left(p\right)^0 \left(1-p\right)^m && (20)\\
&= 1 - \left(1-p\right)^m && (21)
\end{aligned}$$

Now considering all N receivers, we have the event

$$S_d(m) = \{\text{at least one successful transmit for each of the } N \text{ receivers in exactly } m \text{ attempts }\} \tag{22}$$

Since the receivers are independent we have

$$P_d(m) = P(S_d(m)) = P(C_1 \cap C_2 \cap \cdots \cap C_N) = P(C_1)P(C_2)\cdots P(C_N) = \left[1 - \left(1-p\right)^m\right]^N \tag{23}$$

The solution is

$$\boxed{P_d(m) = \left[1 - \left(1-p\right)^m\right]^N} \tag{24}$$

(c)

For the particular problem data $p = 0.9$, $N = 5$, $m = 2$ we have

$$P(2) = 0.832 < P_D(2) = 0.951 \tag{25}$$

P2-3.

The cdf can be found directly by integrating the pdf:

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt = \int_{-\infty}^{x} \left(\begin{cases} \frac{1}{b-a} & \text{if } a \le t \le b, \\ 0 & \text{otherwise.} \end{cases} \right) dt. \tag{26}$$

Consider three disjoint and collectively exhaustive cases:

Case 1: $x < a$

$$\int_{-\infty}^{x} \left(\begin{cases} \frac{1}{b-a} & \text{if } a \le t \le b, \\ 0 & \text{otherwise.} \end{cases} \right) dt = \int_{-\infty}^{x} 0 dt = 0. \tag{27}$$

Case 2: $a \le x \le b$

$$\int_{-\infty}^{x} \left(\begin{cases} \frac{1}{b-a} & \text{if } a \le t \le b, \\ 0 & \text{otherwise.} \end{cases} \right) dt \tag{28}$$

$$= \int_{-\infty}^{a} \left(\begin{cases} \frac{1}{b-a} & \text{if } a \le t \le b, \\ 0 & \text{otherwise.} \end{cases} \right) dt + \int_{a}^{x} \left(\begin{cases} \frac{1}{b-a} & \text{if } a \le t \le b, \\ 0 & \text{otherwise.} \end{cases} \right) dt \tag{29}$$

$$= \int_{-\infty}^{a} 0 dt + \int_{a}^{x} \frac{1}{b-a} dt \tag{30}$$

$$= \frac{x-a}{b-a}. \tag{31}$$

Case 3: $x > b$

$$\int_{-\infty}^{x} \left(\begin{cases} \frac{1}{b-a} & \text{if } a \le t \le b, \\ 0 & \text{otherwise.} \end{cases} \right) dt \tag{32}$$

$$= \int_{-\infty}^{b} \left(\begin{cases} \frac{1}{b-a} & \text{if } a \le t \le b, \\ 0 & \text{otherwise.} \end{cases} \right) dt + \int_{b}^{x} \left(\begin{cases} \frac{1}{b-a} & \text{if } a \le t \le b, \\ 0 & \text{otherwise.} \end{cases} \right) dt \tag{33}$$

$$= \frac{b-a}{b-a} + \int_{b}^{x} 0 dt \tag{34}$$

$$= 1. \tag{35}$$

Altogether, the uniform cdf is the piecewise linear function

$$F_X(x) = \begin{cases} 0 & \text{if } x < a \\ \dfrac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{if } x > b \end{cases} \tag{36}$$

P2-4.

The cdf can be found directly by integrating the pdf:

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \int_{-\infty}^{x} \left(\begin{cases} \lambda \exp(-\lambda t) & \text{if } t \geq 0, \\ 0 & \text{otherwise,} \end{cases} \right) dt \tag{37}$$

Clearly, since the pdf is zero from $-\infty$ to 0, so is the cdf. Thus it suffices to consider $x > 0$:

$$\begin{aligned} \int_{-\infty}^{x} \left(\begin{cases} \lambda \exp(-\lambda t) & \text{if } t \geq 0, \\ 0 & \text{otherwise,} \end{cases} \right) dt &= \int_{0}^{x} \lambda \exp(-\lambda t)\,dt \qquad (38) \\ &= -\exp(-\lambda t)\Big|_{0}^{x} \qquad (39) \\ &= 1 - \exp(-\lambda x) \qquad (40) \end{aligned}$$

The exponential cdf is

$$F_X(x) = \begin{cases} 1 - \exp(-\lambda x) & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{41}$$

P2-5.

By definition the cdf of $Z = \max(X, Y)$ is

$$F_Z(z) = P[\max(X, Y) \leq z]. \tag{42}$$

But the event $\{\max(X, Y) \leq z\}$ is the same as the joint event $\{X \leq z, Y \leq z\}$. Hence

$$\begin{aligned}
F_Z(z) &= P[Z \leq z] && (43)\\
&= P[\max(X, Y) \leq z] && (44)\\
&= P[X \leq z, Y \leq z] && (45)\\
&= P[X \leq z]P[Y \leq z] && \text{(by independence of } X, Y\text{)}\\
&= F_X(z)F_Y(z) && (46)
\end{aligned}$$

Differentiating and using the product rule for derivatives we obtain the solution

$$\boxed{\begin{aligned}
f_Z(z) &= \frac{d}{dz}F_Z(z) && (47)\\
&= \frac{d}{dz}[F_X(z)F_Y(z)] && (48)\\
&= F_X(z)\left(\frac{d}{dz}F_Y(z)\right) + \left(\frac{d}{dz}F_X(z)\right)F_Y(z) && (49)\\
&= f_Y(z)F_X(z) + f_X(z)F_Y(z). && (50)
\end{aligned}}$$

Remark: When X, Y are not independent, we are stuck with the formula

$$F_Z(z) = P[X \leq z, Y \leq z] = F_{XY}(z, z) \tag{51}$$

where we must know the joint cdf F_{XY} to find F_Z. The pdf is left as

$$f_Z(z) = \frac{d}{dz}F_{XY}(z, z) \tag{52}$$

which can be evaluated in closed-form for certain joint distributions F_{XY}.