RBOT101: MATHEMATICAL FOUNDATIONS OF ROBOTICS

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Probability Theory

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P1-1.

Translate given information into math:

$$P[A|B] = P[A^c|B^c] = 95\%, \qquad P[B] = 0.5\% \tag{1}$$

Use the law of total probability to find $P[A^c]$, the probability of the test saying a patient does not have cancer:

$$\begin{aligned} P[A^c] &= P[A^c|B]P[B] + P[A^c|B^c]P[B^c] & (2)\\ &= (100 - 95\%)(0.5\%) + (95\%)(100\% - 0.5\%) & (3)\\ &= 94.55\% & (4) \end{aligned}$$

Alternately, in the example in the slides we found $P[A]$ so we could have simply used the rule of complementary events

$$P[A^c] = 1 - P[A] = 100\% - 5.45\% = 94.55\%$$

Now use Bayes' theorem:

$$P[B|A^c] = \frac{P[A^c|B]P[B]}{P[A^c]} = \frac{(5\%)(0.5\%)}{94.55\%} \approx 0.0264\% \tag{5}$$

This is a very small probability, so the patient does not have too much to worry about!

P1-2.

Define the events

$$A = \text{examinee knew the correct answer,} \tag{6}$$
$$B = \text{examinee did not know the correct answer and therefore guessed,} \tag{7}$$
$$C = \text{examinee answered correctly.} \tag{8}$$

We are given the following probability information:

$$P(A) = p, \tag{9}$$
$$P(B) = 1 - p, \tag{10}$$
$$P(C|A) = 1, \tag{11}$$
$$P(C|B) = 1/m. \tag{12}$$

Now we have

$$\begin{aligned} P(A|C) &= \frac{P(C|A)P(A)}{P(C)} && \text{(Bayes' theorem)} \\ &= \frac{P(C|A)P(A)}{P(C|A)P(A) + P(C|B)P(B)} && \text{(Law of total probability)} \\ &= \frac{(1)(p)}{(1)(p) + (1/m)(1-p)} && (13) \\ &= \frac{p}{p + \frac{1-p}{m}} && (14) \end{aligned}$$

The probability that an examinee knew the answer to a question, given that he or she has correctly answered it, is

$$P(A|C) = \frac{1}{1 + \frac{1-p}{pm}}. \tag{15}$$

For the specific choice $m = 4$ and $p = 50\%$ we have

$$P(A|C) = \frac{1}{1 + \frac{50\%}{(50\%)(4)}} = 80\%. \tag{16}$$

P1-3.

Define the events

$$
\begin{aligned}
A &= \text{chip was manufactured by machine A,} && (17)\\
B &= \text{chip was manufactured by machine A,} && (18)\\
C &= \text{chip was manufactured by machine A,} && (19)\\
D &= \text{chip was defective.} && (20)
\end{aligned}
$$

Use the law of total probability to find $P(D)$ as

$$
\begin{aligned}
P(D) &= P(D|A)P(A) + P(D|B)P(B) + P(D|C)P(C) && (21)\\
&= (0.05)(0.25) + (0.04)(0.35) + (0.02)(0.40) && (22)\\
&= 0.0345 && (23)
\end{aligned}
$$

Using Bayes' theorem with the given information to find the conditional probabilities:

$$P(A|D) = \frac{P(D|A)P(A)}{P(D)} = \frac{(0.05)(0.25)}{0.0345} = 0.363 \tag{24}$$

$$P(B|D) = \frac{P(D|A)P(A)}{P(D)} = \frac{(0.04)(0.35)}{0.0345} = 0.406 \tag{25}$$

$$P(C|D) = \frac{P(D|A)P(A)}{P(D)} = \frac{(0.02)(0.40)}{0.0345} = 0.232 \tag{26}$$

The solution is

$$
\begin{aligned}
P(A|D) &= 0.363 && (27)\\
P(B|D) &= 0.406 && (28)\\
P(C|D) &= 0.232 && (29)
\end{aligned}
$$

P1-4.

Define the events

$$B = \text{failure of breathing apparatus} \tag{30}$$
$$M = \text{failure of monitor system} \tag{31}$$
$$D = \text{death.} \tag{32}$$

We are given the probability information

$$P_B = 0.05, \quad P_M = 0.10, \quad B \text{ and } M \text{ independent} \tag{33}$$

The probability of dying without the monitor system is just

$$P(D) = P(B) = P_B = 0.05 \tag{34}$$

The probability of dying with the monitor system is

$$\begin{aligned}
P(D) &= P(B \cup M) && (35)\\
&= P(B)P(M) && \text{(product rule since } B, M \text{ independent)}\\
&= P_B P_M && (36)\\
&= (0.05)(0.10) && (37)\\
&= 0.005 && (38)
\end{aligned}$$

Clearly $P(B \cup M) = 0.005 < 0.05 = P(B)$ so the monitor system provides a substantial reduction in mortality; Professor X was incorrect in his assessment that the monitor system is useless even with $P_M > P_B$.