Causal Inference with Information Fields

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Inferring the potential consequences of an unobserved event is a fundamental scientific question. To this end, Pearl's celebrated do-calculus provides a set of inference rules to derive an interventional probability from an observational one. In this framework, the primitive causal relations are encoded as functional dependencies in a Structural Causal Model (SCM), which maps into a Directed Acyclic Graph (DAG) in the absence of cycles. In this paper, by contrast, we capture causality without reference to graphs or functional dependencies, but with information fields. The three rules of do-calculus reduce to a unique sufficient condition for conditional independence: the topological separation, which presents some theoretical and practical advantages over the d-separation. With this unique rule, we can deal with systems that cannot be represented with DAGs, for instance systems with cycles and/or 'spurious' edges.

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1 INTRODUCTION

As the world shifts toward more and more data-driven decision-making, causal inference is taking more space in applied sciences, statistics and machine learning. This is because it allows for better, more robust decision-making, and provides a way to interpret the data that goes beyond correlation [1]. For instance, causal inference provides a language to describe and solve Simpson's paradox, which embodies the "correlation is not causation" principle as can be found in any "Statistics 101" basic course. The main concern in causal inference is to compute post-intervention probability distributions from observational data. For this purpose, graphical models are practical because they allow representing assumptions easily and benefit from an extensive scientific literature.

In his seminal work, Pearl builds on graphical models [2] to introduce the so-called do-calculus. Several extensions to this do-calculus have been proposed recently [3–6]. As asserted by Pearl, language is an important element in this research program [7]. Causal graphical models move the focus from joint probability distributions to functional dependencies thanks to the Structural Causal Model (SCM) framework. By leveraging the concept of information sets, we bring a new, complementary view to the causal reasoning toolbox.

So, we introduce a general, unifying framework for causal inference that may be used for both recursive and nonrecursive systems [8] (i.e. with and without cycles). But the cost for this conceptual generalization is a bit of abstraction: in what we propose, the structure is implicit, and there are no arrows. In particular, while DAGs modeling does not rely directly on random variables but on joint probability distributions (see [9], footnote 3 or [10] Appendix A),

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our approach requires going back to the classical primitives of probabilistic models: sample sets, σ-fields, measurable maps and random variables.

The present paper, however, has been written so that the main messages can be understood with the usual graphical concepts used in the field of causal inference. In particular, the notion of Topological Separation is explained for the specific case of DAGs, Theorem 10 and Examples 8, 9 and 13 should be readable without the concept of information field.

Roughly speaking the information fields are useful in this paper to introduce the notions of (1) well-posedness (Remark 3), (2) context specific independence (3) intervention variable and (4) conditional precedence. Moreover, they constitute a key technical concept for the proofs (see the companion paper [11]).

Related work and contributions. We extend the causal modeling toolbox thanks to two notions: information fields and topological separation. The concept of information field extends the expressiveness of the Structural Causal Model, and allows for instance to naturally encode context specific independence [5]. The topological separation is visually practical because it just requires to check that two sets are disjoints (see the last three examples). By contrast, the d-separation requires to check that *all* the paths that connect two variables are blocked. Moreover, as its name suggests, the topological separation has a theoretical interpretation. Our main result is Theorem 10, which is a generalization of do-calculus that can be applied in particular to nonrecursive systems [12], and which subsumes several recent results. We pinpoint the novelty of our approach with our last example, a system with cycles where our framework identifies a probabilistic independence that the framework developed in [13] (for cycles) does not.

For the sake of readability we state and illustrate some of our key results without proofs, which are provided in the companion paper [11] alongside other results. Section 2 provides a few reminders on probability theory, and then explains how one can move from the standard Structural Causal Model (SCM) to our proposal of Information Dependency Model (IDM) with the help of information fields. Section 3 introduces the notion of conditional precedence, which in particular allows us to encode intervention variables in the IDM. Section 4 contains the definition of Topological Separation. Section 5 contains our main result, which states that Topological Separation implies conditional independence. We then explain why this theorem subsumes several recent results and Pearl's do-calculus. We also provide an example for which a recently published paper [13] on causality and cycles does not identify a conditional independence, but our framework does.

2 INFORMATION FIELDS

We start with a few reminders from probability theory. A *σ-field* (henceforth sometimes referred to as *field*) over a set $\mathbb{D}$ is a subset $\mathcal{D} \subset 2^{\mathbb{D}}$, containing $\mathbb{D}$, and which is stable under complementation and countable union. The trivial σ-field over the set $\mathbb{D}$ is $\{\emptyset, \mathbb{D}\}$. The complete σ-field over the set $\mathbb{D}$ is $2^{\mathbb{D}}$. When $\mathcal{D}' \subset \mathcal{D}$ are two σ-fields over the set $\mathbb{D}$, we say that $\mathcal{D}'$ is a *subfield* of $\mathcal{D}$. If two sets $\mathbb{D}_1, \mathbb{D}_2$ are equipped with σ-fields $\mathcal{D}_1, \mathcal{D}_2$, we denote by $\mathcal{D}_1 \otimes \mathcal{D}_2$ the *product σ-field* on $\mathbb{D}_1 \times \mathbb{D}_2$ generated by the rectangles $\{D_1 \times D_2 | D_i \in \mathcal{D}_i\}$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathbb{U}, \mathcal{U})$ a set and a σ-field over this set. Probability theory defines a *random variable* as a measurable mapping from $(\Omega, \mathcal{F})$ to $(\mathbb{U}, \mathcal{U})$.

Let $\mathbb{A}$ be a finite set and $\Omega = \times_{a \in \mathbb{A}} \Omega_a$ a sample space and $\mathcal{F}_a$ be a σ-field over Ω_a. Let Pr_a a probability over $(\Omega_a, \mathcal{F}_a)$ and $\mathrm{Pr} = \bigotimes_{a \in \mathbb{A}} \mathrm{Pr}_a$. Let $(\mathbb{U}_a)_{a \in \mathbb{A}}$ be a collection of sets, with $\mathcal{U}_a$ being a σ-field over $\mathbb{U}_a$ for all a. We are interested in some random variables $(U_a)_{a \in \mathbb{A}}$ such that U_a is valued in $\mathbb{U}_a$.

It is standard to model causal hypotheses using **Structural Causal Models (SCMs)**[10]. An SCM consists of a list of assignments $(\lambda_a)_{a\in A}$ alongside a parental mapping $P : \mathbb{A} \to 2^{\mathbb{A}}$ such that

$$U_a(\omega) = \lambda_a(U_{P(a)}(\omega), \omega_a) \quad \forall \omega \in \Omega \quad \forall a \in \mathbb{A} \tag{1}$$

where ω_a is the projection of ω on Ω_a.

To get the **graphical representation** of an SCM –in $(\mathbb{A}, \mathbb{A}\times\mathbb{A})$– we draw an arrow $a \to b$ whenever $a \in P(b)$. Usually, the graphical representation is assumed to be a DAG, which means that the parental mapping induces a partial order. We will not need this hypothesis here. More often than not, the reasoning is made on the graphical representation which is uniquely defined by the parental mapping, so that the assignements functions do not even need to be specified. For a given applied problem, the SCM is derived from expert knowledge, assumptions and data analysis methods. The SCM is a central tool in causal analysis but its graphical representation does not naturally account for situations such as Context Specific Independence [5], where some edges are spurious.

We may wish for more flexibility, so we call *configuration space* the product space [1]

$$\mathbb{H} = \prod_{a\in\mathbb{A}} \mathbb{U}_a \times \Omega \ . \tag{2}$$

The *configuration field* $\mathcal{H} = \bigotimes_{a\in\mathbb{A}} \mathcal{U}_a \otimes \mathcal{F}$ is a σ-field over $\mathbb{H}$ (with $\mathcal{F} = \bigotimes_{a\in\mathbb{A}} \mathcal{F}_a$). We then extend the definition of SCM thanks to the following observation: we can express the SCM by saying that λ_a is a map from $\mathbb{H}$ to $\mathbb{U}_a$ ($\lambda_a : (\mathbb{H}, \mathcal{H}) \to (\mathbb{U}_a, \mathcal{U}_a)$) while imposing that λ_a "only depends on $U_{P(a)}$ and ω_a". It is standard (see [15, Chap. 1 p. 18]) in probability theory that such property is – under mild assumptions– equivalent to a measurability constraint on the random variable U_a. Hence (1) can be restated as

$$\lambda_a^{-1}(\mathcal{U}_a) \subset \bigotimes_{b\in P(a)} \mathcal{U}_b \otimes \bigotimes_{b\notin P(a)} \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F}_a \otimes \bigotimes_{b\neq a} \{\emptyset, \Omega_b\}, \tag{3}$$

or, with a slight abuse of notations that we will sometimes use throughout this presentation[2]

$$\lambda_a^{-1}(\mathcal{U}_a) \subset \bigotimes_{b\in P(a)} \mathcal{U}_b \otimes \mathcal{F}_a \ . \tag{4}$$

Informally, an information field is anything one may want to see on the right-hand side of Equation (4). For instance, consider the case where $\mathbb{A} = \{a, b, c\}$. If $\lambda_a^{-1}(\mathcal{U}_a) \subset \{\emptyset, \Omega_a\} \otimes \{\emptyset, \Omega_b\} \otimes \{\emptyset, \Omega_c\} \otimes \mathcal{F}_a \otimes \{\emptyset, \Omega_b\} \otimes \{\emptyset, \Omega_c\}$, that we abusively write $\lambda_a^{-1}(\mathcal{U}_a) \subset \mathcal{F}_a$, this means that $\lambda_a(u_a, u_b, u_c, \omega_a, \omega_b, \omega_c) = \lambda_a(\cancel{u_a}, \cancel{u_b}, \cancel{u_c}, \omega_a, \cancel{\omega_b}, \cancel{\omega_c})$ only depends on ω_a, that is, only depends on its own "source of uncertainty" (the field $\mathcal{F}_a$). If (abusively) $\lambda_b^{-1}(\mathcal{U}_b) \subset \mathcal{F}_a \otimes \mathcal{U}_c$, this means that $\lambda_b(u_a, u_b, u_c, \omega_a, \omega_b, \omega_c) = \lambda_b(\cancel{u_a}, \cancel{u_b}, u_c, \omega_a, \cancel{\omega_b}, \cancel{\omega_c})$ only depends on (u_c, ω_a), that is, only depends on the uncertainty ω_a (the field $\mathcal{F}_a$) and on the variable u_c (the field $\mathcal{U}_c$). More complex examples will be given later.

We thus extend the definition for SCMs. We propose the name *Information Dependency Model*

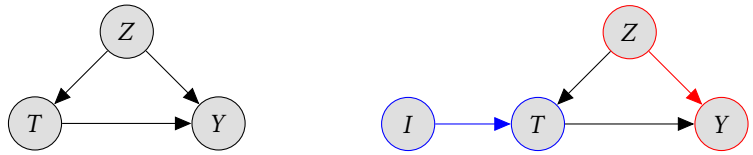
Definition 1 (Information Dependency Model). *An Information Dependency Model is a collection $(\mathcal{I}_a)_{a\in\mathbb{A}}$ of subfields of $\mathcal{H}$ such that, for $a \in \mathbb{A}$, $\mathcal{I}_a \subset \bigotimes_{b\in\mathbb{A}} \mathcal{U}_b \otimes \mathcal{F}_a$. The subfield $\mathcal{I}_a$ is called the* ***information field*** *of a.*

The SCM is now defined by the **measurability property**

$$\lambda_a^{-1}(\mathcal{U}_a) \subset \mathcal{I}_a \quad \forall a \in \mathbb{A} \ . \tag{5}$$

[1] also called *hybrid space* [14], hence the $\mathbb{H}$ notation
[2] we omit the trivial fields in the product

Property (5) expresses in a very general way that the random variable U_a may only depend upon the information $\mathcal{I}_a$ available to the random variable. For a given applied problem, like for the SCM, the IDM can be derived from expert knowledge, assumptions and data analysis methods. In particular, any SCM can be mapped into an IDM.



(a) DAG.

(b) Modeling intervention with the intervention variable I.

Fig. 1. Common cause.

Example 2 (Common cause). *First, to better understand how DAGs – and more generally SCMs can be modeled with information fields, we provide a detailed instance for a set of random variables that can be represented by the DAG in Figure 1.* ***Such an effort is not required in practice****, because the measurability properties are fully specified by the DAG for such a simple instance. Let* $\mathbb{A} = \{Z, T, Y\}$*. To simplify the exposition, we suppose that the values of each of the three random variables represented on the DAG belong to* $\{0, 1\}$*. Then,* $\mathbb{U}_Z = \mathbb{U}_T = \mathbb{U}_Y = \{0, 1\}$*, each equipped with the complete field* $\mathcal{U}_Z = \mathcal{U}_T = \mathcal{U}_Y = \big\{\emptyset, \{0\}, \{1\}, \{0, 1\}\big\}$*. We take* $\Omega = \{0, 1\}^3$ *as Nature set, equipped with the complete field* $\mathcal{F} = 2^\Omega$ *made of all subsets of* Ω*. We write* $\Omega = \Omega_Z \times \Omega_T \times \Omega_Y$*, where* $\Omega_Z = \Omega_T = \Omega_Y = \{0, 1\}$*, and* $\mathcal{F} = \mathcal{F}_Z \otimes \mathcal{F}_T \otimes \mathcal{F}_Y$*, where* $\mathcal{F}_Z = \mathcal{F}_T = \mathcal{F}_Y = \big\{\emptyset, \{0\}, \{1\}, \{0, 1\}\big\}$*. To represent, for instance, the arrows pointing to* Y *in the DAG in Figure 1 (as well as implicit assumptions about information on Nature), we require that the information field* $\mathcal{I}_Y$ *satisfies* $\mathcal{I}_Y \subset \{\emptyset, \Omega_Z\} \otimes \{\emptyset, \Omega_T\} \otimes \mathcal{F}_Y \otimes \mathcal{U}_Z \otimes \mathcal{U}_T \otimes \{\emptyset, \mathbb{U}_Y\}$*. This relation expresses that the information of* Y *depends at most on its own "source of uncertainty" (the field* $\mathcal{F}_Y$*) and on the decisions of both* Z *and* T *(the field* $\mathcal{U}_Z \otimes \mathcal{U}_T$*). Again, the effort of describing explicitly the information field is not required in the case of DAGs, because the mapping from DAGs to IDMs is trivial. On the other hand the IDM allows to express more sophisticated hypotheses.*

Remark 3 (Solvability). *When the system is recursive –i.e. when it admits a fixed causal ordering and can be represented by a DAG – there is no question of well-posedness. One can simulate a sample of random variables by first generating the variables that do not have ancestors, and then following the graph along their descendants. Such procedure is not possible for the more general case of nonrecursive systems, and we need an additional property to ensure that the system is well defined: we present the* ***solvability*** *property in the companion paper [11]. We need in particular to exclude cases such as self-information (that is* $a \in P(a)$*), and more generally case where the system of equations* (1) *could have several solutions (consider for instance* $x := y$ *and* $y := x$*) or no solution at all .*

3 CONDITIONAL PRECEDENCE

In this section, we exploit the flexibility of the information field to extend the definition of precedence. For any $B \subset \mathbb{A}$, let $\mathcal{H}_B = \mathcal{F} \otimes \bigotimes_{b \in B} \mathcal{U}_b \subset \mathcal{H}$. In our extended definition of SCM (the Information Dependency Model), we do not specify a precedence relation: the primitives are the information fields, and the notion of precedence is deduced from those fields. For instance, the traditional *precedence relation* – or parental relation– on $\mathbb{A}$ writes

$$\mathcal{P}a = \bigcap_{B \in \mathbb{A}; \mathcal{I}_a \subset \mathcal{H}_B} B. \tag{6}$$

One can check that $P(a) = \mathcal{P}a$ when $P(a)$ is minimal: the relation $U_a(\omega) = \lambda_a(U_{P(a)}(\omega), \omega_a)$ implies that $\mathcal{I}_a \subset \mathcal{H}_{P(a)}$, moreover the minimality means that it is the smallest subset of $\mathbb{A}$ satisfying such constraint. So on a DAG, $b \in \mathcal{P}a$ means that there is an arrow from b to a.

Here is how the information field allows to extend the definition of precedence.

Definition 4 (Conditional Precedence). *For any subset $H \subset \mathbb{H}$ of configurations, and any subset $W \subset \mathbb{A}$, we set*

$$\mathcal{P}_{W,H}a = \bigcap_{B \in \mathbb{A}; \mathcal{I}_a \cap H \subset \mathcal{H}_{B \cup W} \cap H} B \,, \tag{7}$$

and call it the precendence conditioned on (W, H).

Tikka et al. [5] manage to summarize the three rules of do-calculus thanks to the notions of context specific independence and labeled DAGs. Our definition allows us to reproduce their approach.

Example 5 (Context Specific Independence). *In order to model spurious edges, Tikka et al. [5] rely on so-called* labeled DAGs *that can be turned into a context specific DAG by removing the arcs that are deactivated (spurious) in the context of interest. In the formalism that we propose, such context is represented by a subset of $\mathbb{H}$. Indeed, if we denote by $H \in \mathbb{H}$ the context for which an arc (a, b) is deactivated (in the language of [5]), we encode this by the following two properties*

$$a \notin \mathcal{P}_{\emptyset,H}b \tag{8a}$$

$$a \in \mathcal{P}_{\emptyset,\mathbb{H}\setminus H}b \,. \tag{8b}$$

Such a property can be encoded in the information set of b $\mathcal{I}_b$.

For the reader familiar with [5], it is then easy to guess how we are going to model intervention variables.

Example 6 (Intervention variables). *To introduce the possibility to intervene on a variable, we use a simple procedure. Suppose we are interested in an intervention profile $\hat{\lambda}_Z$ for a subset $Z \subset \mathbb{A}$. For this purpose, we consider a new family of fields $\hat{\mathcal{I}}_z \subset \mathcal{H}$, for $z \in Z$ and we suppose that $\hat{\lambda}_Z$ is $\hat{\mathcal{I}}_z$-measurable, for any $z \in Z$. Then, we enrich the model as follows: (i) we introduce a new* intervention variable *I, equipped with $\Omega_I = \{0, 1\}$ and $\mathbb{U}_I = \{0, 1\}$, and who only has access to its private information in Ω_I; (ii) we straightforwardly adapt the information fields for $\mathbb{A} \setminus (Z \cup I)$ and the probability $\mathbb{P}$; (iii) we replace the information field $\mathcal{I}_z$ by $\big(\{0\} \otimes \mathcal{I}_z\big) \cup \big(\{1\} \otimes \hat{\mathcal{I}}_z\big)$, for $z \in Z$.*

More formally, we introduce the new model $\big(\tilde{\mathbb{A}}, (\tilde{\Omega}, \tilde{\mathcal{F}}), \big\{\tilde{\mathbb{U}}_a, \tilde{\mathcal{U}}_a\big\}_{a \in \tilde{\mathbb{A}}}, \big\{\tilde{\mathcal{I}}_a\big\}_{a \in \tilde{\mathbb{A}}}\big)$, where $\tilde{\mathbb{A}} = \mathbb{A} \cup \{I\}$, $\tilde{\Omega} = \Omega \times \{0, 1\}$, $\tilde{\mathbb{U}}_I = \{0, 1\}$, $\tilde{\mathbb{U}}_a = \mathbb{U}_a$ for any $a \in \mathbb{A}$, and

$$\tilde{\mathcal{I}}_a = \mathcal{I}_a \otimes \{\emptyset, \mathbb{U}_I\} \,, \quad \forall a \in \mathbb{A} \setminus Z \,, \tag{9a}$$

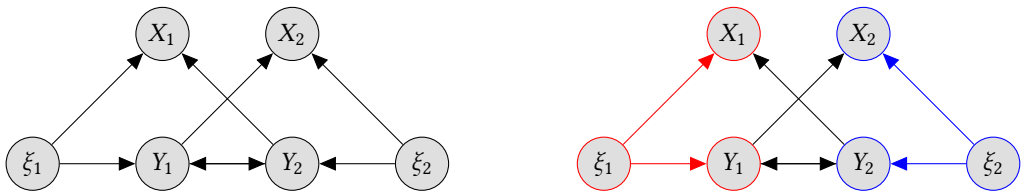
$$\tilde{\mathcal{I}}_z = \hat{\mathcal{I}}_z \otimes \mathcal{U}_I \,, \quad \forall z \in Z \,, \tag{9b}$$

$$\mathcal{I}_I = \bigotimes_{a \in \mathbb{A}} \{\emptyset, \Omega_a\} \otimes \big\{\emptyset, \{0\}, \{1\}, \{0, 1\}\big\} \otimes \bigotimes_{a \in \mathbb{A}} \{\emptyset, \mathbb{U}_a\} \,. \tag{9c}$$

We also extend the probability $\mathbb{P}$ as a product probability $\tilde{\mathbb{P}} = \mathbb{P} \otimes \mu$ on $\tilde{\Omega}$, where μ is a full support probability on $\{0, 1\}$.

4 TOPOLOGICAL SEPARATION

Next we introduce the topological separation, which can be seen generalization of the d-separation.



(a) Original graph.

(b) topological separation is checked on the initial graph $W_{X_i} = Y_i$.

Fig. 2. Topological separation is easy to check: nonrecursive system.

For any subsets $B \subset \mathbb{A}$ and $B_j \subset \mathbb{A}$, $j = 1, \ldots, n$, we write $B_1 \sqcup \cdots \sqcup B_n = B$ when we have both $B_j \cap B_k = \emptyset$ for all $j \neq k$ and $B_1 \cup \cdots \cup B_n = B$. In addition, we denote by $\overline{B}$ the smallest subset of $\mathbb{A}$ that contains B and its own predecessors under $\mathcal{P}_{W,H}$. As explained in the companion paper [11] (see also [16]), $\overline{B}$ is the **topological closure** of B under a topology induced by $\mathcal{P}_{W,H}$.

Definition 7 (Topological Separation). *Let $H \subset \mathbb{H}$ and $B, C, W \subset \mathbb{A}$. We say that B and C are (conditionally)* topologically separated *(w.r.t. (W, H)), denoted by $B \underset{t}{\perp\!\!\!\perp} C \mid (W, H)$, if there exists $W_B, W_C \subset W$ such that*

$$W_B \sqcup W_C = W \text{ and } \overline{B \cup W_B} \cap \overline{C \cup W_C} = \emptyset \ . \tag{10}$$

Observe that the condition is on the *existence* of a partition of the set of variables over which we want to condition.

On a DAG, when $H = \mathbb{H}$, we have topological separation of B and C with respect to W when there is a partition (W_B, W_C) of W such that the sets of ancestors of $B \cup W_B$ and $C \cup W_C$ – using $\mathcal{P}_{W,H}(a) = P(a) \setminus W$ – are disjoint.

Because it can be proved that d-separation and topological separation are equivalent on a DAG, we think this definition is very handy even for DAGs. Indeed, (1) the partition (W_B, W_C) can be derived mechanically (see companion paper [11]), (2) once the partition is given, it is usually much quicker to check that the ancestors sets are disjoints than checking that all the path between B and C are blocked by W.

Example 8 (Topological separation is easy to check: recursive system). *The DAG in Figure 4 (left) illustrate why this notion is practical. If one want to check that Y_1 and Y_2 are d-separated by W, one need to check that* ***every path*** *that goes from Y_1 to Y_2 are blocked by W. By contrast, the topological separation can be checked visually on Figure 4 (right) by setting $W_{Y_1} = W$, $W_{Y_2} = \emptyset$ and checking that the red and blue sets are closed and do not intersect.*

Example 9 (Topological separation is easy to check: nonrecursive system). *We display in Figure 2 a nonrecursive system for which we check (for X_1 and X_2 with respect to Y_1 and Y_2) our proposal of topological separation. This is – in our humble opinion – simpler to check than Forré et al. 's σ-separation [13] because there are less intermediate steps.*

5 INDEPENDENCE AND DO-CALCULUS WITH INFORMATION FIELDS

In this section, we suppose the random variables valued in finite sets for the sake of simplicity. We can now state our version of Pearl's three rules of do-calculus. The statement looks like a simple sufficient condition for conditional independence thanks to the fact that we encode the intervention variables in the information fields.

Theorem 10 (Do-calculus).

$$Y \underset{t}{\perp\!\!\!\perp} Z \mid (W, H) \implies \Pr(U_Y | U_W, U_{\overline{Z}}, H) = \Pr(U_Y | U_W, H) \ . \tag{11}$$

We stress the conciseness of Theorem 10 — permitted by the notions introduced in this paper — as we now show that it implies the three rules of Pearl, as well as two recent results.

Proposition 11. *Rule 1 from [5] can be deduced from Theorem 10. In particular, Theorem 10 subsumes Pearl's do-calculus from [17].*

Remark 12. *In the same manner we could derive the 3 rules of Theorem 1 (do-calculus for stochastic interventions) in [6].*

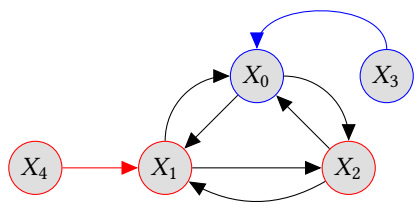


Fig. 3. X_3 and X_4 are independent conditioned on (X_0, X_1, X_2) but not independent if we only condition on (X_0, X_1). Visual proof of topological separation:
$W_{X_4} = \{X_1, X_2\}$ and $W_{X_3} = \{X_0\}$

Example 13. *The last example is inspired by the work of Witsenhausen [14] on causality. This example provides arguments to explain why it is well posed (solvable). It is depicted in Figure 3 and corresponds to the following nonrecursive binary SCM (N_i are independent, binary noise variables, $\oplus$ is the XOR operator):*

$$X_0 = (X_1.(\neg X_2)) \oplus (N_0 \oplus X_3) \quad \textit{and} \quad X_1 = (X_2.(\neg X_0)) \oplus (N_1 \oplus X_4)$$
$$X_2 = (X_0.(\neg X_1)) \oplus N_2, \quad X_3 = N_3 \quad \textit{and} \quad X_4 = N_4.$$

*The random variables X_3 and X_4 are topologically separated by (X_0, X_1, X_2) –note that X_2 is needed – , **hence** X_3 **and** X_4 **are independent conditioned on** (X_0, X_1, X_2) but not independent if we only condition on (X_0, X_1).*

Observe that the intuition that we could equivalently replace X_0, X_1 and X_2 by a unique variable W is misleading: with such a change, we would get a collider $X_4 \to W \leftarrow X_3$ over which we are conditioning, which would make X_4 and X_3 non blocked with respect to W.

Let us try to apply the elegant recent result of Forré et al. (Theorem 5.2 from [13]) on conditional independence in the presence of cycles. We first observe that the Directed Mixed Graph (DMG) induced by the Input/output Structural Causal Model (ioSCM) associated to our example (see Definition 2.3 and 5.1 ibid) looks like the graph of Figure 3. Second, we observe that X_0, X_1 and X_2 belong to the same strongly connected component S (see [13]), in the sense that they are all ancestors and descendants of each other. Third, let us consider the walk $X_4 \to X_1 \leftarrow X_0 \leftarrow X_3$. Forré et al. provide a condition for a walk to be open in definition 4.2. By apply this definition, we see that $X_4 \to X_1 \leftarrow X_0 \leftarrow X_3$ is $\{X_0, X_1, X_2\}$-σ-open because:

- *$X_4 \to X_1 \leftarrow X_0$ satisfies the collider definition (4.2, (a) in [13]) because $X_1 \in \{X_0, X_1, X_2\}$*
- *$X_1 \leftarrow X_0 \leftarrow X_3$ satisfies the left chain condition because $X_0 \in \{X_0, X_1, X_2\} \cap S$, where S is the strongly connected component of X_1.*

hence this walk is $\{X_0, X_1, X_2\}$-σ-open (see Definition 4.2 ibid). Hence it seems that Theorem 5.2 from [13] could not be used to state that X_3 and X_4 are independent conditioned on (X_0, X_1, X_2).

We illustrate our theory with a numerical exact computation, taking the N_i as binomial variables of parameter 0.1. *We solve the cycle by enumerating the 8 possible combinations of values for X_0, X_1 and X_2 and selecting the only admissible one. The results are shown in Table* **??** *and Table* **??**.

This example illustrates the novelty of the IDM approach.

6 DISCUSSION

In this paper, we simplify and generalize the do-calculus by leveraging the concept of information field. The do-calculus is reduced to one rule. We underline that the results come from the information structure, not the probability. Also, because our approach is not based on graphical models, our work provides a new proof of Pearl's original result. For most cases, one only needs to understand the notion of inverse image to work with information fields on top of SCMs and DAGs. In exchange, information fields provide a compact, unifying and versatile language that brings new intuitions on the causal structure of the problem.

For instance, we illustrated why the Topological Separation is practical: once the partition of the conditioning variables known, checking that an intersection is empty is easier than checking a blocking condition on a collection of paths.

Also, we were able to recover a few recent results – how to handle spurious edges [5], how to handle stochastic interventions [6], how to handle cycles [13]– , and we think that the Information Dependency Model is a powerful technical tool for investigating potential new extensions of already existing results in the field of causal inference. Moreover, many of those papers require the introduction of ad hoc frameworks. The Information Dependency Model is a good candidate to bring uniformity and consistency. It can be a temporary detour to introduce new notions, for instance the definition of Conditional Precedence 4 would have been harder to express with the SCM as primitive.

In addition, we presented and solved an example that cannot be handled easily with the current state of the literature.

Last, we mention that the notion of well-posedness we use was introduced in [16] half a century ago for another field of applied mathematics. It is interesting to observe that this notion could serve a new purpose in the field of causal inference.

Further work includes drawing connections with other research programs, such as Proposition 11 or questions related to identification [18–20], using the framework developed in this paper. Also, it would be interesting to study the connections of this work with [12, 13].

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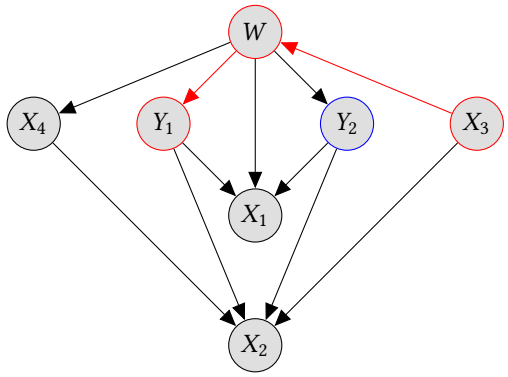

Fig. 4. Topological separation is easy to check.

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