Causal Inference Theory with Information Algebras (1/2): Introducing the Witsenhausen Intrinsic Model

Causality in Practice
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Outline of the presentation

Witsenhausen intrinsic model [15']

Classification of information structures [10']

Witsenhausen intrinsic model [15']

Witsenhausen intrinsic model [15']

Agents, actions, Nature, configuration space, information σ-algebras

Agents, action spaces and Nature space

- Let A be a (finite or infinite) set,
  whose elements are called agents (or decision-makers)
- With each agent $a \in A$ is associated a measurable space

$$\Big(\underbrace{\mathbb{U}_a}_{\substack{\text{set of}\\ \text{actions}\\ \text{for agent } a}} , \underbrace{\mathcal{U}_a}_{\substack{\sigma\text{-algebra}\\ \subset 2^{\mathbb{U}_a}}} \Big)$$

- With Nature is associated a measurable space

$$(\Omega, \mathcal{F})$$

(at this stage of the presentation, we do not need to equip $(\Omega, \mathcal{F})$ with a probability distribution, as we only focus on information)

The configuration space is a product space

Configuration space

The configuration space is the product space

$$\mathbb{H} = \prod_{a \in A} \mathbb{U}_a \times \Omega$$

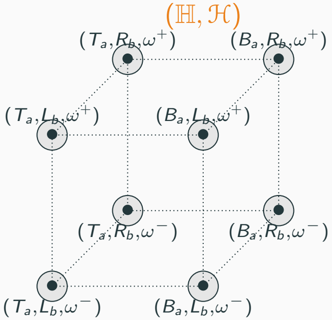
equipped with the product σ-algebra, called configuration σ-algebra

$$\mathcal{H} = \bigotimes_{a \in A} \mathcal{U}_a \otimes \mathcal{F}$$

so that $(\mathbb{H}, \mathcal{H})$ is a measurable space

Example of configuration space

$\mathbb{U}_a = \{T_a, B_a\}$, $\mathbb{U}_b = \{R_b, L_b\}$, $\Omega = \{\omega^+, \omega^-\}$
$\mathcal{U}_a = 2^{\mathbb{U}_a}$, $\mathcal{U}_b = 2^{\mathbb{U}_b}$, $\mathcal{F} = 2^{\Omega}$



- product configuration space

$$\mathbb{H} = \prod_{a \in A} \mathbb{U}_a \times \Omega$$

- product configuration σ-algebra

$$\mathcal{H} = \bigotimes_{a \in A} \mathcal{U}_a \otimes \mathcal{F}$$

represented by
the partition of its atoms

Information σ-algebras

Information σ-algebras express dependencies

Information σ-algebra of an agent

The information σ-algebra of agent $a \in A$ is a σ-field

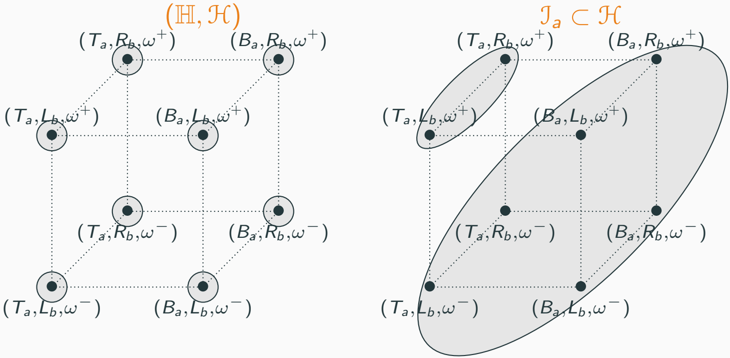
$$\mathcal{I}_a \subset \mathcal{H} = \bigotimes_{a \in A} \mathcal{U}_a \otimes \mathcal{F}$$

which is a sub σ-algebra of the product configuration σ-algebra

- The sub σ-algebra $\mathcal{I}_a$ of the configuration σ-algebra $\mathcal{H}$ represents the information available to agent a when the agent chooses an action
- Therefore, the information of agent a may depend
    - on the states of Nature
    - and on other agents' actions

In the finite case, information σ-algebras are represented by the partition of its atoms

The information σ-algebra of agent $a \in A$
is a sub σ-algebra $\mathcal{I}_a \subset \mathcal{H} = \bigotimes_{a \in A} \mathcal{U}_a \otimes \mathcal{F}$
which can, in the finite case, be represented by the partition of its atoms



Elements of an atom cannot be distinguished by the agent a

Definition of the W-model (2 basic objects, possibly 1 axiom)

W-model

A W-model $\big(A, (\Omega, \mathcal{F}), (\mathbb{U}_a, \mathcal{U}_a)_{a \in A}, (\mathcal{I}_a)_{a \in A}\big)$
consists of 2 basic objects

(W-BO1a) the sample space $(\Omega, \mathcal{F})$

(W-BO1b) the collection $(\mathbb{U}_a, \mathcal{U}_a)_{a \in A}$ of agents' action spaces

(W-BO2) the collection $(\mathcal{I}_a)_{a \in A}$ of agents' information sub σ-algebras of $\mathcal{H} = \bigotimes_{a \in A} \mathcal{U}_a \otimes \mathcal{F}$

and (possibly) 1 axiom imposed on them

(W-Axiom1) for all agent $a \in A$, absence of self-information holds

$$\mathcal{I}_a \subset \{\emptyset, \mathbb{U}_a\} \otimes \bigotimes_{b \in A \setminus \{a\}} \mathcal{U}_b \otimes \mathcal{F}$$

To avoid paradoxes, we can consider W-models that display absence of self-information

Absence of self-information

A W-model displays absence of self-information when

$$\mathcal{I}_a \subset \underbrace{\{\emptyset, \mathbb{U}_a\}}_{\text{not one's own action}} \otimes \underbrace{\bigotimes_{b \in A \setminus \{a\}} \mathcal{U}_b}_{\text{other agents' actions}} \otimes \mathcal{F}\ , \quad \forall a \in A$$

- Absence of self-information means that the information of agent a can only depend on the states of Nature
  and on all the other agents' actions, but not on his own action
- Absence of self-information makes sense
  as we have distinguished an individual from an agent
  (else, it would lead to paradoxes)

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ON INFORMATION STRUCTURES, FEEDBACK AND CAUSALITY*

H. S. WITSENHAUSEN†

Abstract. A finite number of decisions, indexed by $\alpha \in A$, are to be taken. Each decision amour to selecting a point in a measurable space $(U_\alpha, \mathscr{F}_\alpha)$. Each decision is based on some information f back from the system and characterized by a subfield $\mathscr{I}_\alpha$ of the product space $(\prod_\alpha U_\alpha, \prod_\alpha \mathscr{F}_\alpha)$. T decision function for each α can be any function γ_α measurable from $\mathscr{I}_\alpha$ to $\mathscr{F}_\alpha$.

A property of the $\{\mathscr{I}_\alpha\}_{\alpha \in A}$ is defined which assures that the setup has a causal interpretatio This property implies that for any combination of choices of the γ_α, the closed loop equations have unique solution.

The converse implication is false, when card $A > 2$.

1. Introduction. In control-oriented works on dynamic games (in particula stochastic control problems) one usually finds a "dynamic equation" describi the evolution of a "state" in response to decision (control) variables of the playe and to random variables. One also finds "output equations" which define outp variables for a player as functions of the state, decision and random variabl Then the information structure is defined by allowing each decision variable to any desired (measurable) function of the output variables generated for that play

Witsenhausen intrinsic model [15']

Examples

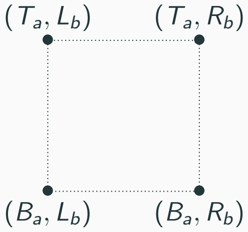
Alice and Bob

"Alice and Bob" configuration space

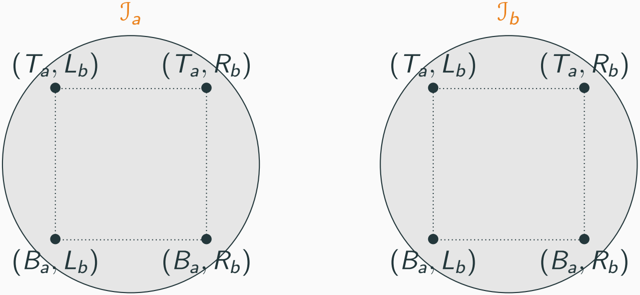
Example

- no Nature
- two agents a (Alice) and b (Bob)
- two possible actions each $\mathbb{U}_a = \{T_a, B_a\}$, $\mathbb{U}_b = \{R_b, L_b\}$
- product configuration space (4 elements)

$$\mathbb{H} = \{T_a, B_a\} \times \{R_b, L_b\}$$



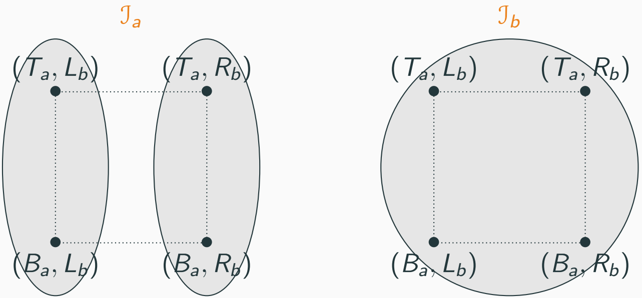
"Alice and Bob" information partitions



- $\mathcal{I}_a = \{\emptyset, \{T_a, B_a\}\} \otimes \{\emptyset, \{R_b, L_b\}\}$
  Alice knows nothing
- $\mathcal{I}_b = \{\emptyset, \{T_a, B_a\}\} \otimes \{\emptyset, \{R_b, L_b\}\}$
  Bob knows nothing

Alice knows Bob's action

"Alice and Bob" information partitions



- $\mathcal{I}_b = \{\emptyset, \{T_a, B_a\}\} \otimes \{\emptyset, \{R_b, L_b\}\}$
  Bob knows nothing
- $\mathcal{I}_a = \{\emptyset, \{T_a, B_a\}\} \otimes \{\emptyset, \{R_b\}, \{L_b\}, \{R_b, L_b\}\}$
  Alice knows what Bob does
  (as she can distinguish between Bob's actions $\{R_b\}$ and $\{L_b\}$)

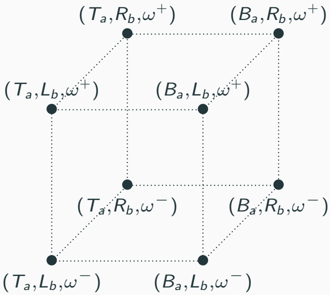
Alice, Bob and a coin tossing

"Alice, Bob and a coin tossing" configuration space

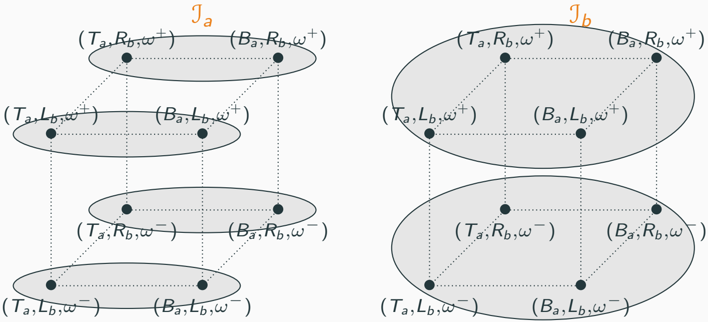
Example

- two states of Nature $\Omega = \{\omega^+, \omega^-\}$ (heads/tails)
- two agents a and b
- two possible actions each: $\mathbb{U}_a = \{T_a, B_a\}$, $\mathbb{U}_b = \{R_b, L_b\}$
- product configuration space (8 elements)

$$\mathbb{H} = \{T_a, B_a\} \times \{R_b, L_b\} \times \{\omega^+, \omega^-\}$$



"Alice, Bob and a coin tossing" information partitions

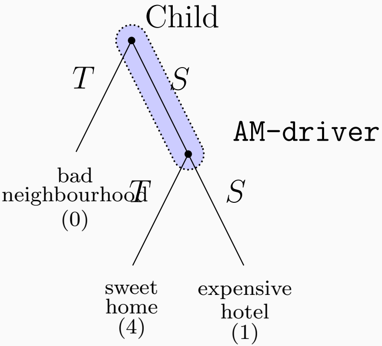


$$\mathcal{I}_b = \overbrace{\{\emptyset, \{T_a, B_a\}\}}^{\text{Bob does not know what Alice does}} \otimes \{\emptyset, \mathbb{U}_b\} \otimes \overbrace{\{\emptyset, \{\omega^+\}, \{\omega^-\}, \{\omega^+, \omega^-\}\}}^{\text{Bob knows Nature's move}}$$

$$\mathcal{I}_a = \{\emptyset, \mathbb{U}_a\} \otimes \underbrace{\{\emptyset, \{R_b\}, \{L_b\}, \{R_b, L_b\}\}}_{\text{Alice knows what Bob does}} \otimes \underbrace{\{\emptyset, \{\omega^+\}, \{\omega^-\}, \{\omega^+, \omega^-\}\}}_{\text{Alice knows Nature's move}}$$

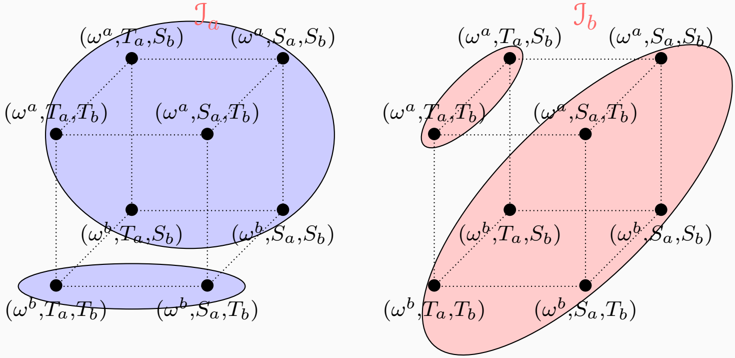
Absent-minded driver

Absent-minded driver



- S=Stay, T=Turn
- "paradox" that raised a problem in game theory
- the player looses public time, as plays "SS" "ST" cross the information set twice
- cannot be modelled *per se* in tree models (violates "no-AM" axiom)

A W-model for the absent-minded driver



$$\mathcal{I}_a = \{\emptyset, \overbrace{\underbrace{\mathbb{U}_a \times \mathbb{U}_b \times \{\omega_a\}}_{\substack{\text{agent } a \text{ is whether} \\ \text{the first one to act}}} \cup \underbrace{\{S_b\} \times \mathbb{U}_a \times \{\omega_b\}}_{\substack{\text{or he acts second after} \\ \text{agent } b \text{ has chosen } S}}}^{\text{agent } a \text{ makes a move}}, \overbrace{\underbrace{\{T_b\} \times \mathbb{U}_a \times \{\omega_b\}}_{\substack{\text{agent } b \text{ chose } T \\ \text{and finished the game}}}}^{\text{agent } a \text{ doesn't make a move}}, \mathbb{H}\}$$

$$\mathcal{I}_b = \{\emptyset, \mathbb{U}_a \times \mathbb{U}_b \times \{\omega_b\} \cup \{S_a\} \times \mathbb{U}_b \times \{\omega_a\}, \{T_a\} \times \mathbb{U}_b \times \{\omega_a\}, \mathbb{H}\}$$

What land have we covered?
What comes next?

- The stage is in place; so are the actors
  - agents
  - Nature
  - information
- How can actors play?
  - strategies
  - solvability

Witsenhausen intrinsic model [15']

Strategies and solvability property

Information is the fuel of W-strategies

W-strategy of an agent

A (pure) W-strategy of agent a is a mapping

$$\lambda_a : (\mathbb{H}, \mathcal{H}) \to (\mathbb{U}_a, \mathcal{U}_a)$$

which is measurable w.r.t. the information σ-algebra $\mathcal{I}_a$, that is,

$$\underbrace{\lambda_a^{-1}(\mathcal{U}_a)}_{\substack{\sigma\text{-algebra} \\ \text{generated by} \\ \text{W-strategy } \lambda_a}} \subset \underbrace{\mathcal{I}_a}_{\substack{\text{information} \\ \sigma\text{-algebra} \\ \text{of agent } a}}$$

This condition expresses the property that
a W-strategy $\lambda_a : (\mathbb{H}, \mathcal{H}) \to (\mathbb{U}_a, \mathcal{U}_a)$ for agent a
can only depend on the information $\mathcal{I}_a$ available to the agent

For instance, $\lambda_a^{-1}(\mathcal{U}_a) \subset \underbrace{\{\emptyset, \mathbb{H}\}}_{\text{no information}} \iff \lambda_a$ is constant on $\mathbb{H}$

Examples of W-strategies

Consider a W-model with two agents a and b,
and suppose that the σ-algebras $\mathcal{U}_a$, $\mathcal{U}_b$ and $\mathcal{F}$ contain the singletons

- Absence of self-information

$$\mathcal{I}_a \subset \{\emptyset, \mathbb{U}_a\} \otimes \mathcal{U}_b \otimes \mathcal{F} \; , \;\; \mathcal{I}_b \subset \mathcal{U}_a \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F}$$

Then, W-strategies λ_a and λ_b have the form

$$\lambda_a(\cancel{u_a}, u_b, \omega) = \widetilde{\lambda_a}(u_b, \omega) \; , \;\; \lambda_b(u_a, \cancel{u_b}, \omega) = \widetilde{\lambda_b}(u_a, \omega)$$

- Sequential W-model

$$\mathcal{I}_a = \{\emptyset, \mathbb{U}_a\} \otimes \mathcal{U}_b \otimes \mathcal{F} \; , \;\; \mathcal{I}_b = \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F}$$

Then, W-strategies λ_a and λ_b have the form

$$\lambda_a(\cancel{u_a}, u_b, \omega) = \widetilde{\lambda_a}(u_b, \omega) \; , \;\; \lambda_b(\cancel{u_a}, \cancel{u_b}, \omega) = \widetilde{\lambda_b}(\omega)$$

Set of W-strategies

Set of W-strategies of an agent

We denote the set of (pure) W-strategies of agent a by

$$\Lambda_a = \left\{ \lambda_a : (\mathbb{H}, \mathcal{H}) \to (\mathbb{U}_a, \mathcal{U}_a) \,\middle|\, \lambda_a^{-1}(\mathcal{U}_a) \subset \mathcal{I}_a \right\}$$

and the set of W-strategies of all agents is

$$\Lambda = \Lambda_A = \prod_{a \in A} \Lambda_a$$

Structural causal and Witsenhausen intrinsic models

Structural causal model	*Witsenhausen intrinsic model*
exogeneous variables	Nature $\omega \in \Omega$ (meas. space $(\Omega, \mathcal{F})$)
exogeneous distribution	
index of endogeneous variables	agent $a \in A$
domain of endogeneous variables	action set $\mathbb{U}_a$ (meas. space $(\mathbb{U}_a, \mathcal{U}_a)$)
	configuration space
	$\mathbb{H} = \prod_{a \in A} \mathbb{U}_a \times \Omega$, $\mathcal{H} = \bigotimes_{a \in A} \mathcal{U}_a \otimes \mathcal{F}$
	information σ-algebras $\{\mathcal{I}_a\}_{a \in A} \subset \mathcal{H}$
functional relation	W-strategy $\lambda_a : (\mathbb{H}, \mathcal{H}) \to (\mathbb{U}_a, \mathcal{U}_a)$
	$\lambda_a^{-1}(\mathcal{U}_a) \subset \mathcal{I}_a$, $\forall a \in A$
causal mechanism	W-strategy profile $\{\lambda_a\}_{a \in A}$

Solvability

Solvability

- In the Witsenhausen's intrinsic model,
  agents make actions in an order which is not fixed in advance
- Briefly speaking, solvability
  is the property that, for each state of Nature,
  the agents' actions are uniquely determined by their W-strategies

Solvability problem

The solvability problem consists in finding

- for any collection $\lambda = \{\lambda_a\}_{a \in A} \in \Lambda_A$ of W-strategies
- for any state of Nature $\omega \in \Omega$

actions $u \in \mathbb{U}_A$ satisfying
the implicit ("closed loop") equation

$$u = \lambda(u, \omega)$$

or, equivalently, the family of "closed loop" equations

$$u_a = \lambda_a(\{u_b\}_{b \in A}, \omega) \ , \ \ \forall a \in A$$

Solvability property

A W-model displays the solvability property when

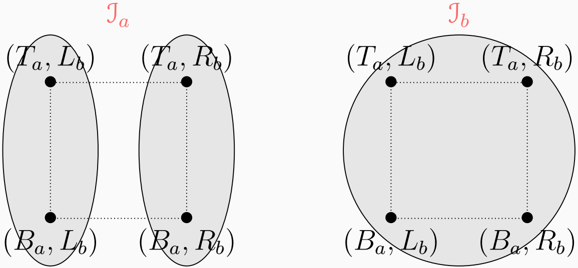
$$\forall \lambda = (\lambda_a)_{a \in A} \in \Lambda_A \;, \;\; \forall \omega \in \Omega \;, \;\; \exists ! u \in \mathbb{U}_A \;, \;\; u = \lambda(u, \omega)$$

or, equivalently,

$$\forall \lambda = (\lambda_a)_{a \in A} \in \Lambda_A \;, \;\; \forall \omega \in \Omega \;, \;\; \exists ! u \in \mathbb{U}_A$$

$$u_a = \lambda_a(\{u_b\}_{b \in A}, \omega) \;, \;\; \forall a \in A$$

Solvability is a property of the information structure



Sequential W-model

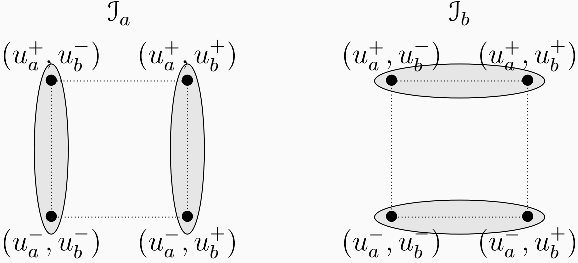
$$\mathcal{I}_a = \{\emptyset, \mathbb{U}_a\} \otimes \mathcal{U}_b \otimes \mathcal{F} \ , \ \ \mathcal{I}_b = \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F}$$

The closed-loop equations

$$u_a = \lambda_a(\cancel{u_a}, u_b, \omega) = \widetilde{\lambda}_a(u_b, \omega) \ , \ \ u_b = \lambda_b(\cancel{u_a}, \cancel{u_b}, \omega) = \widetilde{\lambda}_b(\omega)$$

always displays a unique solution (u_a, u_b),
whatever $\omega \in \Omega$ and W-strategies λ_a and λ_b

Solvability is a property of the information structure



W-model with deadlock

$$\mathcal{I}_a = \{\emptyset, \mathbb{U}_a\} \otimes \mathcal{U}_b \ , \ \ \mathcal{I}_b = \mathcal{U}_a \otimes \{\emptyset, \mathbb{U}_b\}$$

The closed-loop equations

$$u_a = \lambda_a(\cancel{u_a}, u_b) = \tilde{\lambda}_a(u_b) \ , \ \ u_b = \lambda_b(u_a, \cancel{u_b}) = \tilde{\lambda}_b(u_a)$$

may display zero solutions, one solution or multiple solutions, depending on the W-strategies λ_a and λ_b

Solvability makes it possible to define a solution map from states of Nature towards configurations

Suppose that the solvability property holds true

Solution map

We define the solution map

$$S_\lambda : \Omega \to \mathbb{H}$$

that maps states of Nature towards configurations, by

$$(u,\omega) = S_\lambda(\omega) \iff u = \lambda(u,\omega)\ , \ \forall (u,\omega) \in \mathbb{U}_A \times \Omega$$

We include the state of Nature ω in the image of $S_\lambda(\omega)$, so that we map the set Ω towards the configuration space $\mathbb{H}$, making it possible to interpret $S_\lambda(\omega)$ as a configuration driven by the W-strategy λ (in classical control theory, a state trajectory is produced by a policy)

In the sequential case, the solution map is given by iterated composition

- In the sequential case

$$\mathcal{I}_a = \{\emptyset, \mathbb{U}_a\} \otimes \mathcal{U}_b \otimes \mathcal{F} \ , \ \ \mathcal{I}_b = \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F}$$

- W-strategies λ_a and λ_b have the form

$$\lambda_a(\cancel{u_a}, u_b, \omega) = \widetilde{\lambda}_a(u_b, \omega) \ , \ \ \lambda_b(\cancel{u_a}, \cancel{u_b}, \omega) = \widetilde{\lambda}_b(\omega)$$

- so that the solution map is

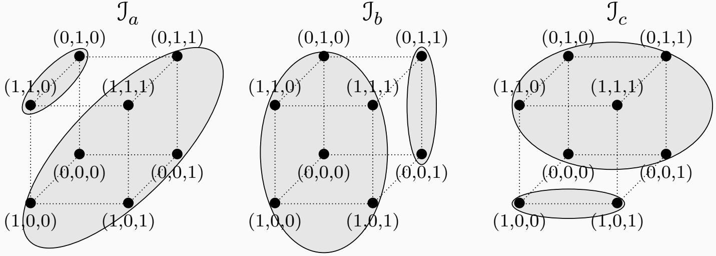
$$S_\lambda(\omega) = \Big(\widetilde{\lambda}_a\Big(\widetilde{\lambda}_b(\omega), \omega\Big), \widetilde{\lambda}_b(\omega), \omega\Big)$$

- because the system of equations $u = \lambda(\omega, u)$ here writes

$$u_a = \lambda_a(\cancel{u_a}, u_b, \omega) = \widetilde{\lambda}_a(u_b, \omega) \ , \ \ u_b = \lambda_b(\cancel{u_a}, \cancel{u_b}, \omega) = \widetilde{\lambda}_b(\omega)$$

Solvable noncausal example Witsenhausen [1971]

- No Nature, $A = \{a, b, c\}$, $\mathbb{U}_a = \mathbb{U}_b = \mathbb{U}_c = \{0, 1\}$
- Set of configurations $\mathbb{H} = \{0, 1\}^3$, and information fields
  $\mathcal{I}_a = \sigma(u_b(1 - u_c))$, $\mathcal{I}_b = \sigma(u_c(1 - u_a))$, $\mathcal{I}_c = \sigma(u_a(1 - u_b))$
- The "game" can be played but… cannot be started (no first agent)



What comes next?

- Causality (as an ingredient for solvability)
- Classification of information structures

Classification of information structures

Classification of information structures

Causality [5']

Causal configuration orderings: "Alice and Bob"

- no Nature, two agents a (Alice) and b (Bob)
- two possible actions each $\mathbb{U}_a = \{u_a^+, u_a^-\}$, $\mathbb{U}_b = \{u_b^+, u_b^-\}$
- configuration space $\mathbb{H} = \{u_a^+, u_a^-\} \times \{u_b^+, u_b^-\}$ (4 elements)
- set of total orderings (2 elements: a plays first or b plays first)
  $\Sigma^2 = \left\{ (ab) = \begin{pmatrix} \sigma:\{1,2\}\to\{a,b\} \\ \sigma(1)=a \\ \sigma(2)=b \end{pmatrix}, (ba) = \begin{pmatrix} \sigma:\{1,2\}\to\{a,b\} \\ \sigma(1)=b \\ \sigma(2)=a \end{pmatrix} \right\}$

Consider the following information structure:

- $\mathcal{I}_b = \{\emptyset, \{u_a^+, u_a^-\}\} \otimes \{\emptyset, \{u_b^+, u_b^-\}\}$
  Bob knows nothing
- $\mathcal{I}_a = \{\emptyset, \{u_a^+, u_a^-\}\} \otimes \{\emptyset, \{u_b^+\}, \{u_b^-\}, \{u_b^+, u_b^-\}\}$
  Alice knows what Bob does

We say that the constant configuration-ordering

- $\varphi(h) = (ab)$, for all $h \in \mathbb{H}$ (a plays first) is noncausal
- $\varphi(h) = (ba)$, for all $h \in \mathbb{H}$ (b plays first) is causal

Partial orderings

We denote $[\![1,k]\!] = \{1,\dots,k\}$ for $k \in \mathbb{N}^*$

Partial orderings

The sets of (partial) orderings of order k are the

$$\Sigma^k = \left\{\kappa : [\![1,k]\!] \to A \,\middle|\, \kappa \text{ is an injection} \right\}, \quad \forall k \in \mathbb{N}^*$$

The set of finite orderings is

$$\Sigma = \bigcup_{k\in\mathbb{N}^*} \Sigma^k$$

Range, cardinality, last element, first elements

For any partial ordering $\kappa \in \Sigma$, we define

the range $\|\kappa\|$ of the ordering κ as the subset of agents

$$\|\kappa\| = \big\{\kappa(1), \ldots, \kappa(k)\big\} \subset A \,, \;\; \forall \kappa \in \Sigma^k$$

the cardinality $|\kappa|$ of the ordering κ as the integer

$$|\kappa| = k \in [\![1, |A|]\!] \,, \;\; \forall \kappa \in \Sigma^k$$

the last element $\kappa_\star$ of the ordering κ as the agent

$$\kappa_\star = \kappa\big(k\big) \in A \,, \;\; \forall \kappa \in \Sigma^k$$

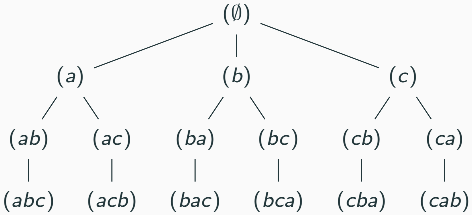
the first elements κ_- of the ordering κ to the first $k-1$ elements

$$\kappa_- = \kappa_{|\{1,\ldots,k-1\}} \in \Sigma^{k-1} \,, \;\; \forall \kappa \in \Sigma^k$$

The tree of partial orderings

There is a natural order on the set $\Sigma = \bigcup_{k \in \mathbb{N}^*} \Sigma^k$ of partial orderings

$$(\emptyset) \succeq (a) \succeq (ab) \succeq (abc)$$



Configuration-orderings

When there is a finite or countable number $|A|$ of agents, the set of total orderings is

$$\Sigma^{|A|} = \left\{ \kappa : [\![1, |A|]\!] \to A \,\middle|\, \kappa \text{ is a bijection} \right\}$$

Configuration-ordering

A configuration-ordering is a mapping

$$\varphi : \underbrace{\mathbb{H}}_{\text{configurations}} \to \underbrace{\Sigma^{|A|}}_{\text{total orderings}}$$

Configurations compatible with a partial ordering

- For any $k \in \mathbb{N}^*$, there is a natural restriction mapping ψ_k

$$\psi_k : \Sigma^{|A|} \to \Sigma^k \ , \ \ \rho \mapsto \rho_{|[\![1,k]\!]}$$

  which is the restriction of any (total) ordering of A to $[\![1,k]\!]$

- The configurations $h \in \mathbb{H}$
  that are compatible with a partial ordering $\kappa \in \Sigma$ belong to

$$\mathbb{H}^{\varphi}_{\kappa} = \left\{ h \in \mathbb{H} \,\middle|\, \underbrace{\psi_{|\kappa|}\big(\varphi(h)\big)}_{\substack{\text{partial ordering of} \\ \text{the first } |\kappa| \text{ agents}}} = \kappa \right\}$$

Causality

Causal W-model

A W-model is causal if there exists (at least one) configuration-ordering $\varphi : \mathbb{H} \to \Sigma^{|A|}$ with the property that, for any $\kappa = (\kappa_-, \kappa_\star) \in \Sigma$

$$\overbrace{\underbrace{\mathbb{H}^{\varphi}_{\kappa}}_{\substack{\text{agents}\\ \text{ordered by } \kappa}} \cap G}^{\substack{\text{information}\\ \text{of the last agent } \kappa_\star}} \in \overbrace{\mathcal{F} \otimes \mathcal{U}_{\|\kappa_-\|}}^{\substack{\text{depends at most on actions}\\ \text{of agents having lower order rank}}} , \quad \forall G \in \mathcal{I}_{\kappa_\star}$$

We also say that $\varphi : \mathbb{H} \to \Sigma^{|A|}$ is a causal configuration-ordering

Information comes first,
(possible) causal ordering comes second

If a W-model has no nonempty static team, it cannot be causal

A causal but nonsequential system

- We consider a set of agents $A = \{a, b\}$ with

$$\mathbb{U}_a = \{u_a^1, u_a^2\} \ , \ \ \mathbb{U}_b = \{u_b^1, u_b^2\} \ , \ \ \Omega = \{\omega^1, \omega^2\}$$

- The agents' information fields are given by

$$\mathcal{I}_a = \sigma(\{u_a^1, u_a^2\} \times \{u_b^1, u_b^2\} \times \{\omega^2\}, \{u_a^1, u_a^2\} \times \{u_b^1\} \times \{\omega^1\})$$
$$\mathcal{I}_b = \sigma(\{u_a^1, u_a^2\} \times \{u_b^1, u_b^2\} \times \{\omega^1\}, \{u_a^1\} \times \{u_b^1, u_b^2\} \times \{\omega^2\})$$

- When the state of Nature is ω^2, agent a only sees ω^2, whereas agent b sees ω^2 and the action of a: thus a acts first, then b
- The reverse holds true when the state of Nature is ω^1
- A non constant configuration-ordering mapping $\varphi : \mathbb{H} \to \{(a, b), (b, a)\}$ is defined by (for any couple (u_a, u_b))

$$\varphi\Big((u_a, u_b, \omega^2)\Big) = (a, b) \ \text{ and } \ \varphi\Big((u_a, u_b, \omega^1)\Big) = (b, a)$$

- The system is causal but not sequential

Causality implies solvability

Proposition Witsenhausen [1971]

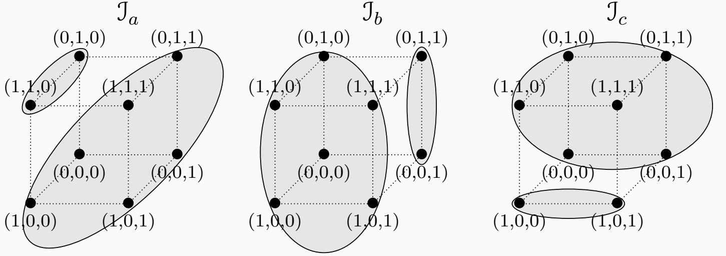
Causality implies (recursive) solvability
with a measurable solution map

$$S_\lambda = \widetilde{S}_\lambda^{(|A|)} \circ \cdots \circ \widetilde{S}_\lambda^{(1)} \circ S_\lambda^{(0)}$$

Kuhn's extensive form of a game encapsulates causality in the tree

Solvable noncausal example Witsenhausen [1971]

- No Nature, $A = \{a, b, c\}$, $\mathbb{U}_a = \mathbb{U}_b = \mathbb{U}_c = \{0, 1\}$
- Set of configurations $\mathbb{H} = \{0, 1\}^3$, and information fields
  $\mathcal{I}_a = \sigma(u_b(1 - u_c))$, $\mathcal{I}_b = \sigma(u_c(1 - u_a))$, $\mathcal{I}_c = \sigma(u_a(1 - u_b))$
- The "game" can be played but... cannot be started (no first agent)



Classification of information structures

Binary relations between agents [5’]

Handling subgroups of agents by means of cylindric extensions

Cylindric extension of a subgroup of agents

For any subset $B \subset A$ of agents, we define

$$\mathcal{H}_B = \bigotimes_{b \in B} \mathcal{U}_b \otimes \bigotimes_{a \notin B} \{\emptyset, \mathbb{U}_a\} \otimes \mathcal{F}$$

$$\mathcal{U}_B = \bigotimes_{b \in B} \mathcal{U}_b \otimes \bigotimes_{a \notin B} \{\emptyset, \mathbb{U}_a\} \subset \bigotimes_{a \in A} \mathcal{U}_a$$

$$\mathcal{H}_B = \mathcal{U}_B \otimes \mathcal{F} = \bigotimes_{b \in B} \mathcal{U}_b \otimes \bigotimes_{a \notin B} \{\emptyset, \mathbb{U}_a\} \otimes \mathcal{F} \subset \mathcal{H}$$

$$\big(\text{when } B \neq \emptyset\big) \qquad h_B = \{h_b\}_{b \in B} \in \prod_{b \in B} \mathbb{U}_b \ , \ \ \forall h \in \mathbb{H}$$

$$\big(\text{when } B \neq \emptyset\big) \qquad \lambda_B = \{\lambda_b\}_{b \in B} \in \prod_{b \in B} \Lambda_b \ , \ \ \forall \lambda \in \Lambda$$

Typology of W-models

- Static team
- Station
- Sequential W-model
- Partially nested W-model
- Quasiclassical W-model
- Classical W-model
- Hierarchical W-model
- Parallel coordinated W-model

Precedence relation $\mathfrak{P}$

What are the agents whose actions might affect the information of a focal agent?

- The precedence binary relation identifies the agents whose actions affect the observations of a given agent
- For a given agent $a \in A$, we consider the set $\mathcal{P}_a \subset 2^A$ of subsets $C \subset A$ of agents such that

$$\mathfrak{I}_a \subset \mathfrak{U}_C \otimes \mathfrak{F} = \bigotimes_{c \in C} \mathfrak{U}_c \otimes \bigotimes_{b \not\in C} \{\emptyset, \mathbb{U}_b\} \otimes \mathfrak{F}$$

- Any subset $C \in \mathcal{P}_a$ contains agents whose actions affect the information $\mathfrak{I}_a$ available to the focal agent a
- As the set $\mathcal{P}_a$ is stable under intersection, the following definition makes sense

The precedence relation $\mathfrak{P}$

Precedence relation $\mathfrak{P}$

1. For any agent $a \in A$, we define the subset $\mathfrak{P}a \subset A$ of agents

$$\mathfrak{P}a = \bigcap_{\substack{C \subset A \\ \mathcal{I}_a \subset \mathcal{U}_C \otimes \mathcal{F}}} C$$

   that is, as the smallest subset $C \subset A$ such that $\mathcal{I}_a \subset \mathcal{U}_C \otimes \mathcal{F}$

2. We define a precedence binary relation $\mathfrak{P}$ on A by

$$b \, \mathfrak{P} \, a \iff b \in \mathfrak{P}a$$

   and we say that b is a predecessor of a (or a precedent of a)

In other words, the action of any predecessor of an agent affects the information of this agent: any agent is influenced by its predecessors (when they exist, because $\mathfrak{P}a$ might be empty)

$$\mathcal{I}_a \subset \mathcal{U}_{\mathfrak{P}a} \otimes \mathcal{F}$$

Characterization of the predecessors of a focal agent

- For any agent $a \in A$, the subset $\mathfrak{P}a$ of agents is the smallest subset $C \subset A$ such that

$$\mathcal{I}_a \subset \mathcal{U}_C \otimes \mathcal{F}$$

- In other words, $\mathfrak{P}a$ is characterized by

$$\mathcal{I}_a \subset \mathcal{U}_{\mathfrak{P}a} \otimes \mathcal{F} \text{ and } \left(\mathcal{I}_a \subset \mathcal{U}_C \otimes \mathcal{F} \Rightarrow \mathfrak{P}a \subset C\right)$$

Potential for signaling

- Whenever $\mathfrak{P}a \neq \emptyset$, there is a potential for signaling, that is, for information transmission
- Indeed, any agent b in $\mathfrak{P}a$ influences the information $\mathcal{I}_a$ upon which agent a bases its actions
- Therefore, whenever agent b is a predecessor of agent a, the former can, by means of its actions, send a signal to the latter
- In case $\mathfrak{P}a = \emptyset$, the actions of agent a depend, at most, on the state of Nature, and there is no room for signaling

Iterated predecessors

- Let $C \subset A$ be a subset of agents
- We introduce the following subsets of agents

$$\mathfrak{P}C = \bigcup_{b \in C} \mathfrak{P}b \,, \;\; \mathfrak{P}^0 C = C \;\text{ and }\; \mathfrak{P}^{n+1} C = \mathfrak{P}\mathfrak{P}^n C \,, \;\; \forall n \in \mathbb{N}$$

  that correspond to the iterated predecessors of the agents in C
- When C is a singleton $\{a\}$, we denote $\mathfrak{P}^n a$ for $\mathfrak{P}^n\{a\}$

Successor relation $\mathfrak{P}^{-1}$

Successor relation $\mathfrak{P}^{-1}$

The converse of the precedence relation $\mathfrak{P}$
is the successor relation $\mathfrak{P}^{-1}$ characterized by

$$b \, \mathfrak{P}^{-1} \, a \iff a \, \mathfrak{P} \, b$$

Quite naturally, b is a successor of a iff a is a predecessor of b

Subsystem relation $\mathfrak{S}$

A subsystem is a subset of agents closed w.r.t. information

We define the information $\mathcal{I}_C \subset \mathcal{H}$ of the subset $C \subset A$ of agents by

$$\mathcal{I}_C = \bigvee_{b \in C} \mathcal{I}_b$$

that is, the smallest σ-fields that contains all the σ-fields $\mathcal{I}_b$, for $b \in C$

Subsystem

A nonempty subset C of agents in A is a subsystem if the information field $\mathcal{I}_C$ at most depends on the actions of the agents in C, that is,

$$\mathcal{I}_C \subset \mathcal{U}_C \otimes \mathcal{F}$$

Thus, the information received by agents in C depends upon states of Nature and actions of members of C only

Generated subsystem

- The subsystem $\overline{C}$ generated by a nonempty subset C of agents in A is the intersection of all subsystems that contain C, that is, the smallest subsystem that contain C
- A subset $C \subset A$ is a subsystem iff it coincides with the generated subsystem, that is,

$$C \text{ is a subsystem} \iff C = \overline{C}$$

The subsystem relation $\mathfrak{S}$

Subsystem relation $\mathfrak{S}$

We define the subsystem relation $\mathfrak{S}$ on A by

$$b \,\mathfrak{S}\, a \iff \overline{\{b\}} \subset \overline{\{a\}} \ , \ \ \forall (a, b) \in A^2$$

Therefore, $b \,\mathfrak{S}\, a$ means that

- agent b belongs to the subsystem generated by agent a
- or, equivalently, that the subsystem generated by agent a contains the one generated by agent b

The subsystem relation $\mathfrak{S}$ is a preorder

Proposition (Witsenhausen [1975])

The subsystem relation $\mathfrak{S}$ is a preorder,
namely it is reflexive and transitive

Proposition

1. *A subset $C \subset A$ is a subsystem iff $\mathfrak{P}C \subset C$, that is, iff the predecessors of agents in C belong to C:*

$$C \text{ is a subsystem} \iff \overline{C} = C \iff \mathfrak{P}C \subset C$$

2. *For any agent $a \in A$, the subsystem generated by agent a is the union of $\{a\}$ and of all its iterated predecessors, that is,*

$$\overline{\{a\}} = \bigcup_{n \in \mathbb{N}} \mathfrak{P}^n a$$

Information-memory relation $\mathfrak{M}$

The information-memory relation $\mathfrak{M}$

Information-memory relation $\mathfrak{M}$

1. With any agent $a \in A$, we associate
   the subset $\mathfrak{M}a$ of agents who pass on their information to a,
   that is,
   $$\mathfrak{M}a = \left\{ b \in A \,\middle|\, \mathfrak{I}_b \subset \mathfrak{I}_a \right\}$$
2. We define an information memory binary relation $\mathfrak{M}$ on A by
   $$b \, \mathfrak{M} \, a \iff b \in \mathfrak{M}a \iff \mathfrak{I}_b \subset \mathfrak{I}_a \,, \quad \forall (a, b) \in A^2$$

- When $b \, \mathfrak{M} \, a$, we say that
  agent b information is remembered by or passed on to agent a,
  or that agent b is an informer of agent a, or that
  the information of agent b is embedded in the information of agent a
- When agent b belongs to $\mathfrak{M}a$,
  the information available to b is also available to agent a

The information memory relation $\mathfrak{M}$ is a preorder

Proposition

The information memory relation $\mathfrak{M}$ is a preorder,
namely $\mathfrak{M}$ is reflexive and transitive

Action-memory relation $\mathfrak{D}$

The action-memory relation $\mathfrak{D}$

We recall that the action subfield $\mathcal{D}_b$ is

$$\mathcal{D}_b = \mathcal{U}_b \otimes \bigotimes_{c \neq b} \{\emptyset, \mathbb{U}_c\} \otimes \{\emptyset, \Omega\}$$

Action-memory relation
[Carpentier, Chancelier, Cohen, and De Lara, 2015]

1. With any agent $a \in A$, we associate

$$\mathfrak{D}a = \left\{ b \in A \,\middle|\, \mathcal{D}_b \subset \mathcal{I}_a \right\}$$

   the subset of agents b whose action is passed on to a

2. We define a action-memory binary relation $\mathfrak{D}$ on A by

$$b \,\mathfrak{D}\, a \iff b \in \mathfrak{D}a \iff \mathcal{D}_b \subset \mathcal{I}_a \,, \quad \forall (a, b) \in A^2$$

$\mathfrak{D} \subset \mathfrak{P}$

From

$$\mathcal{D}_{\mathfrak{D}a} = \mathcal{U}_{\mathfrak{D}a} \otimes \{\emptyset, \Omega\} \subset \mathcal{I}_a \subset \mathcal{U}_{\mathfrak{P}a} \otimes \mathcal{F}$$

we conclude that

$$\mathfrak{D}a \subset \mathfrak{P}a \,, \quad \forall a \in A$$

or, equivalently, that

$$\mathfrak{D} \subset \mathfrak{P}$$

- When $b \mathfrak{D} a$, we say that the action of agent b is remembered by or passed on to agent a, or that the action of agent b is embedded in the information of agent a
- If $b \mathfrak{D} a$, the action made by agent b is passed on to agent a and, by the fact that $\mathfrak{D} \subset \mathfrak{P}$, b is a predecessor of a
- However, the agent b can be a predecessor of a, but its influence may happen without passing on its action to a

What land have we covered?
What comes next?

With these four relations

- precedence relation $\mathfrak{P}$
- subsystem relation $\mathfrak{S}$
- information-memory relation $\mathfrak{M}$
- action-memory relation $\mathfrak{D}$

we can provide a typology of systems (W-models)

Classification of information structures

Typology of systems

Static team

Static team

Static team

A static team is a subset C of A such that $\mathfrak{P}C = \emptyset$, that is, agents in C have no predecessors

- A static team necessarily is a subset of the largest static team defined by

$$A_0 = \{a \in A \mid \mathcal{I}_a \subset \bigotimes_{b \in A} \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F} = \{a \in A \mid \mathfrak{P}a = \emptyset\}$$

- When the whole set A of agents is a static team, any agent $a \in A$ has no predecessor: $\mathfrak{P}a = \emptyset$, $\forall a \in A$
- A system is static if the set A of agents is a static team

Static team made of two agents

Two agents a, b form a static team iff

$$\mathcal{I}_a \subset \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F} \;, \;\; \mathcal{I}_b \subset \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F}$$

There is no interdependence between the actions of the agents, just a dependence upon states of Nature

Station and sequential system

Station

A station is a subset of agents such that the set of information fields of these agents is totally ordered under inclusion (i.e., nested)

Station

A subset C of agents in A is a station

- iff the information-memory relation $\mathfrak{M}$ induces a total order on C (i.e., it consists of a chain of length $m = \mathrm{card}(C)$)
- iff there exists an ordering $(a_1, \ldots, a_m)$ of C such that

$$\mathcal{I}_{a_1} \subset \cdots \subset \mathcal{I}_{a_k} \subset \mathcal{I}_{a_{k+1}} \subset \cdots \subset \mathcal{I}_{a_m}$$

or, equivalently, that

$$a_{k-1} \in \mathfrak{M}a_k \ , \ \forall k = 2, \ldots, m$$

In other words, in a station,
the antecessor $k-1$ is necessarily an informer of k

A station with two agents

$$\mathcal{I}_a = \{\emptyset, \Omega, \{\omega^1\}, \{\omega^2\}\} \times \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \mathbb{U}_b\}$$

$$\mathcal{I}_b = \{\emptyset, \Omega, \{\omega^1\}, \{\omega^2\}\} \times \{\emptyset, \mathbb{U}_a, \{u_a^1\}, \{u_a^2\}\} \otimes \{\emptyset, \mathbb{U}_b\}$$

$\mathcal{I}_a \subset \mathcal{I}_b$ may be interpreted in different ways

- one may say that
  agent a communicates its own information to agent b.
- If agent a is an individual at time $t = 0$,
  while agent b is the same individual at time $t = 1$,
  one may say that the information is not forgotten with time
  (memory of past knowledge)

Sequential system

Sequential system

A system is sequential if there exists an ordering $(a_1, \ldots, a_{|A|})$ of A such that each agent a_k is influenced at most by the previous (former or antecessor) agents $a_1, \ldots, a_{k-1}$, that is,

$$\mathfrak{P}a_1 = \emptyset \text{ and } \mathfrak{P}a_k \subset \{a_1, \ldots, a_{k-1}\}\ , \ \forall k = 2, \ldots, |A|$$

In other words, in a sequential system, predecessors are necessarily antecessors

Example of sequential system with two agents

The set of agents $A = \{a, b\}$ with information fields given by

$$\mathcal{I}_a = \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F} \;, \;\; \mathcal{I}_b = \mathcal{U}_a \otimes \{\emptyset, \mathbb{U}_b\} \otimes \{\emptyset, \Omega\}$$

forms a sequential system where

- agent a precedes agent b, because
  $\mathfrak{P}a = \emptyset$ and $\mathfrak{P}b = \{a\}$
- but $\mathcal{I}_a$ and $\mathcal{I}_b$ are not comparable:
  agent a observes only the state of Nature,
  whereas agent b observes only agent a's action

Example of sequential system with two agents

$$\mathcal{I}_a = \{\emptyset, \Omega, \{\omega^1\}, \{\omega^2\}\} \times \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \mathbb{U}_b\}$$

$$\mathcal{I}_b = \{\emptyset, \Omega, \{\omega^1\}, \{\omega^2\}\} \times \{\emptyset, \mathbb{U}_a, \{u_a^1\}, \{u_a^2\}\} \otimes \{\emptyset, \mathbb{U}_b\}$$

The system is sequential as

1. agent a observes the state of Nature
   and makes its action accordingly
2. agent b observes both agent a's action and the state of Nature
   and makes its action accordingly

Partially nested systems

Partially nested system

Partially nested system

A partially nested system is one for which
the precedence relation is included in the information-memory relation,
that is,

$$\mathfrak{P} \subset \mathfrak{M}$$

- In a partially nested system, if agent a is a predecessor of agent b — hence, a can influence b — then agent b knows what agent a knows
- In a partially nested system, any agent has access to the information of those agents who are its predecessors (and thus influence its own information)
- In other words, in a partially nested system, predecessors are necessarily informers

Quasiclassical system

Quasiclassical system

A system is quasiclassical

- iff it is sequential and partially nested
- iff there exists an ordering $(a_1, \dots, a_{|A|})$ of A such that $\mathfrak{P}a_1 = \emptyset$ and

$$\mathfrak{P}a_k \subset \{a_1, \dots, a_{k-1}\} \text{ and } \mathfrak{P}a_k \subset \mathfrak{M}a_k \ , \quad \forall k = 2, \dots, |A|$$

In other words, in a quasiclassical system,
predecessors are necessarily antecessors and
predecessors are necessarily informers

Classical system

A system is classical

- iff there exists an ordering $(a_1, \ldots, a_{|A|})$ of A for which it is both sequential and such that $\mathcal{I}_{a_k} \subset \mathcal{I}_{a_{k+1}}$ for $k = 1, \ldots, n-1$ (station property)
- iff there exists an ordering $(a_1, \ldots, a_{|A|})$ of A such that $\mathfrak{P}a_1 = \emptyset$ and for $k = 2, \ldots, |A|$,

$$\mathfrak{P}a_k \subset \{a_1, \ldots, a_{k-1}\} \subset \{a_1, \ldots, a_{k-1}, a_k\} \subset \mathfrak{M}a_k$$

In other words, in a classical system,
predecessors are necessarily antecessors and
antecessors are necessarily informers

- A classical system is necessarily partially nested because $\mathfrak{P}a_k \subset \mathfrak{M}a_k$ for $k = 1, \ldots, n$
- Hence, a classical system is quasiclassical

A classical system with two agents

- The set of agents $A = \{a, b\}$ with information fields given by

$$\mathcal{I}_a = \{\otimes\mathcal{F}\emptyset, \mathbb{U}_a\} \otimes \mathcal{U}_b \ , \ \ \mathcal{I}_b = \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F}$$

  forms a classical system
- Indeed, first, the system is sequential as b precedes a because $\mathfrak{P}b = \emptyset$ and $b \in \mathfrak{P}a$:
  - agent b observes the state of Nature and makes its action accordingly
  - agent a observes both agent b's decision and the state of Nature and makes its action based on that information
- Second, one has that $\mathcal{I}_b \subset \mathcal{I}_a$ ($b \in \mathfrak{M}a$): agent b communicates its own information to agent a

Subsystem inheritence

Theorem (Witsenhausen [1975])

Any of the properties static team, sequentiality, quasiclassicality, classicality, causality of a system is shared by all its subsystems

Hierarchical and parallel systems

Hierarchical system (Ho-Chu)

A system is hierarchical when the set A of agents
can be partitioned in (nonempty) disjoint sets $A_0, \ldots, A_K$ as follows

$$A_0 = \{a \in A \mid \mathfrak{P}a = \emptyset\}$$
$$A_1 = \{a \in A \mid a \notin A_0 \text{ and } \mathfrak{P}a \subset A_0\}$$
$$A_{k+1} = \{a \in A \mid a \notin \bigcup_{i=1}^{k} A_j \text{ and } \mathfrak{P}a \subset \bigcup_{i=1}^{k} A_j\}$$

for $k = 2, \ldots, K$

Agents in A_0 form the largest static team ($\mathfrak{P}A_0 = \emptyset$)

Parallel coordinated systems

Parallel coordinated system

A system is parallel coordinated
when the set A of agents can be partitioned
in (nonempty) disjoint sets A_0, A_1, . . . , A_K as follows

- A_0 is the largest static team ($\mathfrak{P}A_0 = \emptyset$)
- every subset $A_1 \cup A_0$, . . . , $A_K \cup A_0$ is a subsystem

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