Chapter 3
Integration

Overview

- Integral
  - Indefinite Integral
  - Definite Integral
- Fundamental Theorem of Calculus
- Various Integration Techniques
  - Integration by Substitution
  - Integration by Parts

Overview

- <u>Application of Integration</u>
  - Area between two curves
  - Volume of Solids of Revolution

Integrals

Indefinite Integral

Let $f(x) = 3x^2$.

Then $\int 3x^2 dx = x^3 + C$

We call

$x^3 + C$ or $\int 3x^2 dx$

the indefinite integral of $3x^2$

The ***indefinite integral*** of f $w.r.t$ x

$= \int f(x)\, dx$

We call

$$x^3 + C \quad \text{or} \quad \int 3x^2 dx$$

the indefinite integral of $3x^2$

If we fix a value to C, we get an antiderivative of $f(x)$.

$$\left.\begin{array}{l} x^3 + 1 \\ x^3 + 2 \\ x^3 + 3 \\ x^3 + \text{ a fix number} \end{array}\right\} \text{antiderivatives of } f(x) = 3x^2.$$

If we fix a value to C, we get an antiderivative of $f(x)$.

$$\left.\begin{array}{l} x^3+1 \\ x^3+2 \\ x^3+3 \\ x^3+\text{ a fix number} \end{array}\right\}$$

antiderivatives of $f(x)=3x^2$.

Note :

$$\frac{d}{dx}(x^3+1)=3x^2 \qquad \frac{d}{dx}(x^3+2)=3x^2 \qquad \frac{d}{dx}(x^3+3)=3x^2$$

$$\frac{d}{dx}(x^3+\text{a fix number})=3x^2$$

If we differentiate antiderivative of $f(x)$, the answer is $f(x)$.

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$$F(x) \xrightarrow[\text{Differentiation}]{} F'(x) = f(x)$$

Reverse Procedure

A function F is called an ***antiderivative*** of a function f on an interval I if

$$F'(x) = f(x) \quad \text{for all } x \in I$$

Indefinite Integral

The indefinite integral of f $w.r.t$ x

$$= \int f(x)$$

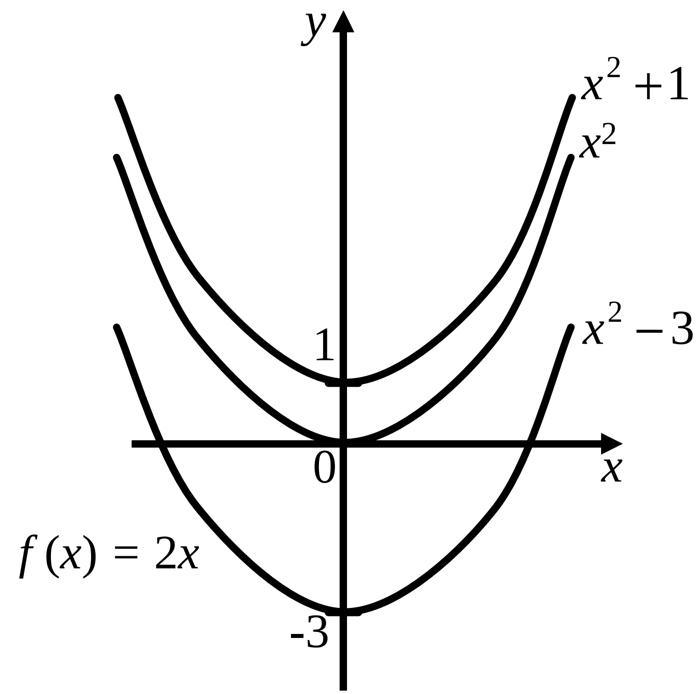
$=$ the set of all *antiderivatives* of f

Indefinite Integral - Remark

The geometrical interpretation of the process on ***integration*** is to find all curves

$$y = F(x) + C$$

which have their slopes $f(x)$ at x.



Indefinite Integral

If F is an ***antiderivative*** of f on I, then $F+C$ is also an ***antiderivative*** of f on I and every ***antiderivative*** of f on I is of this form.

$$\int f(x)\,dx = F(x) + C$$

Integral sign **Integrand** **Constant of integration**

If $F'(x) = G'(x)$ for all $x \in (a,b)$, then there exists C such that

$$G(x) = F(x) + C \quad \text{for all } x \in (a,b)$$

Example:

$$F(x) = x^3 + 2009 \qquad G(x) = x^3 + 1$$

$$F'(x) = 3x^2 \qquad G'(x) = 3x^2$$

Therefore, $F'(x) = G'(x)$.

Note :

$$\begin{aligned} F(x) &= x^3 + 2009 \\ &= x^3 + 1 + 2008 \\ &= G(x) + 2008 \end{aligned}$$

If $F'(x) = G'(x)$ for all $x \in (a,b)$, then there exists C such that

$$G(x) = F(x) + C \text{ for all } x \in (a,b)$$

Pause and Think !!!

How to prove trigonometric identities using the above result?

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$\cot^2 x + 1 = \csc^2 x$$

If $F'(x) = G'(x)$ for all $x \in (a,b)$, then there exists C such that

$$G(x) = F(x) + C \quad \text{for all } x \in (a,b)$$

To prove: $\sin^2 x + \cos^2 x = 1$

Let $F(x) = \sin^2 x$ $\qquad G(x) = -\cos^2 x$

Then $F'(x) = 2\sin x \cos x$ $\qquad G'(x) = -2\cos x(-\sin x) = 2\sin x \cos x$

Thus $F'(x) = G'(x)$

Hence $F(x) = G(x) + C$

Therefore $\sin^2 x = -\cos^2 x + C$

Question: How to find C ??

To prove: $\sin^2 x + \cos^2 x = 1$

Let $F(x) = \sin^2 x$ $\quad G(x) = -\cos^2 x$

Then $F'(x) = 2\sin x \cos x$ $\quad G'(x) = -2\cos x(-\sin x)$

$$= 2\sin x \cos x$$

Thus $F'(x) = G'(x)$

Hence $F(x) = G(x) + C$

Therefore $\sin^2 x = -\cos^2 x + C$

Question: How to find C ??

Put $x = 0$, $\quad \sin^2 0 = -\cos^2 0 + C$

$$0 = -1 + C$$

$C = 1$ $\quad$ Therefore $\sin^2 x = -\cos^2 x + 1$

Integral Formulae

$$1.\ \int x^n\, dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1,\ n \text{ rational}$$

$$\int 1\, dx = \int dx = x + C \qquad (\text{Special case}, n = 0)$$

$$2.\ \int \sin kx\, dx = -\frac{\cos kx}{k} + C$$

$$3.\ \int \cos kx\, dx = \frac{\sin kx}{k} + C$$

$$4.\ \int \sec^2 x\, dx = \tan x + C$$

$$\boxed{5.\ \int \csc^2 x\, dx = -\cot x + C}$$

Integral Formulae

6. $\int \sec x \tan x \, dx = \sec x + C$

7. $\int \csc x \cot x \, dx = -\csc x + C$

8. $\int \frac{1}{x} \, dx = \ln x + C$

9. $\int a^x \, dx = \frac{a^x}{\ln a} + C, \quad a \neq 1$

10. $\int e^x \, dx = e^x + C$

Rules for Indefinite Integration

1. $\int kf(x)\, dx = k\int f(x)\, dx$

   where k is a constant (independent of x)

2. $\int -f(x)\, dx = -\int f(x)\, dx$

   (Rule 1 with $k = -1$)

3. $\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$

Indefinite Integral - Example

Find the curve in the xy - plane which passes through the point $(9,4)$ and whose slope at each point (x, y) is $3\sqrt{x}$.

The curve is given by $y = y(x)$, satisfying

(i) $\dfrac{dy}{dx} = 3\sqrt{x}$ and (ii) $y(9) = 4$

When $x = 9,\ y = 4$

The curve is given by $y = y(x)$, satisfying

When $x = 9,\ y = 4$

(i) $\dfrac{dy}{dx} = 3\sqrt{x}$ and (ii) $y(9) = 4$

Solving (i), we get

$$y = \int 3\sqrt{x}\ dx = 3\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C = 2x^{\frac{3}{2}} + C$$

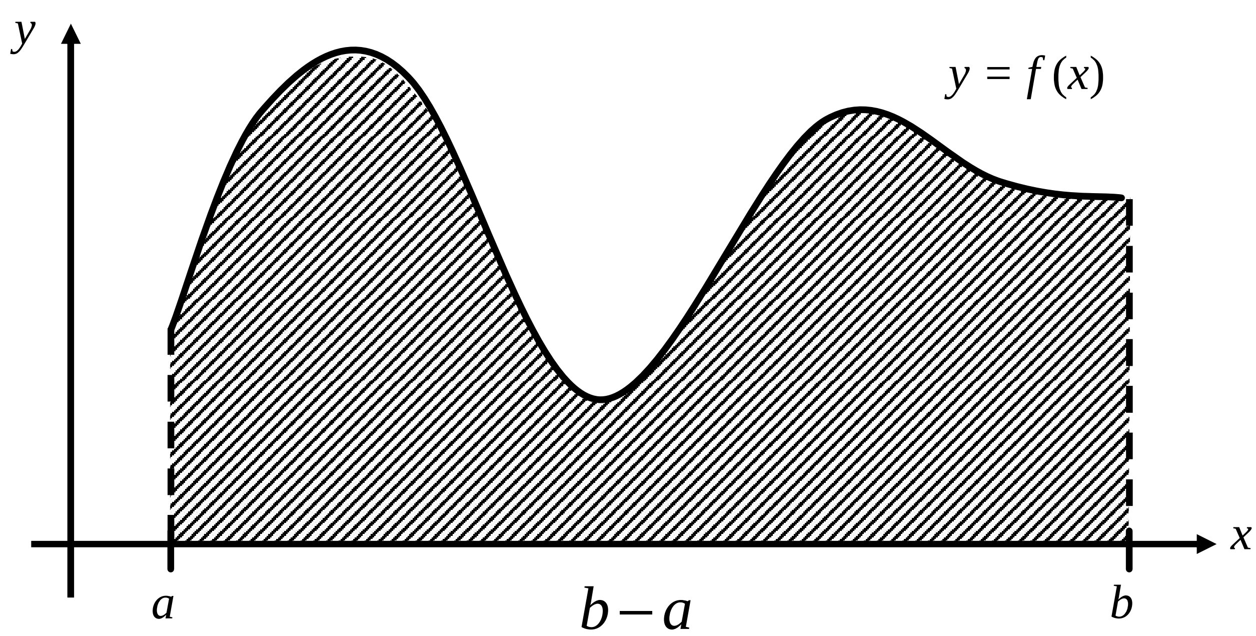
By (ii),

$$4 = 2(9)^{\frac{3}{2}} + C = 2(27) + C$$

$$C = 4 - 54 = -50$$

Hence $y = 2x^{\frac{3}{2}} - 50.$

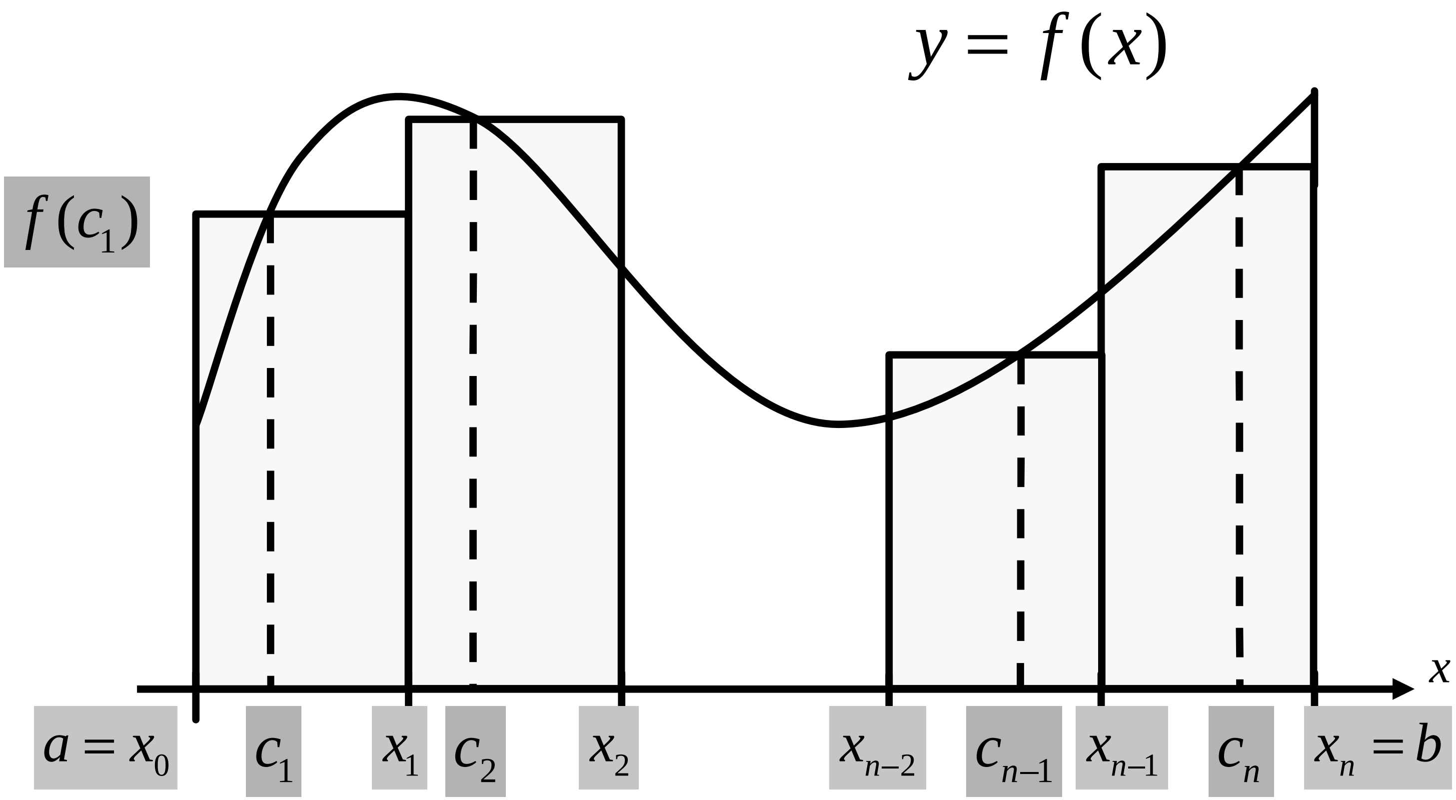
Integrals



Area under curve

$$A = \int_a^b f(x)\,dx$$

Riemann Integrals

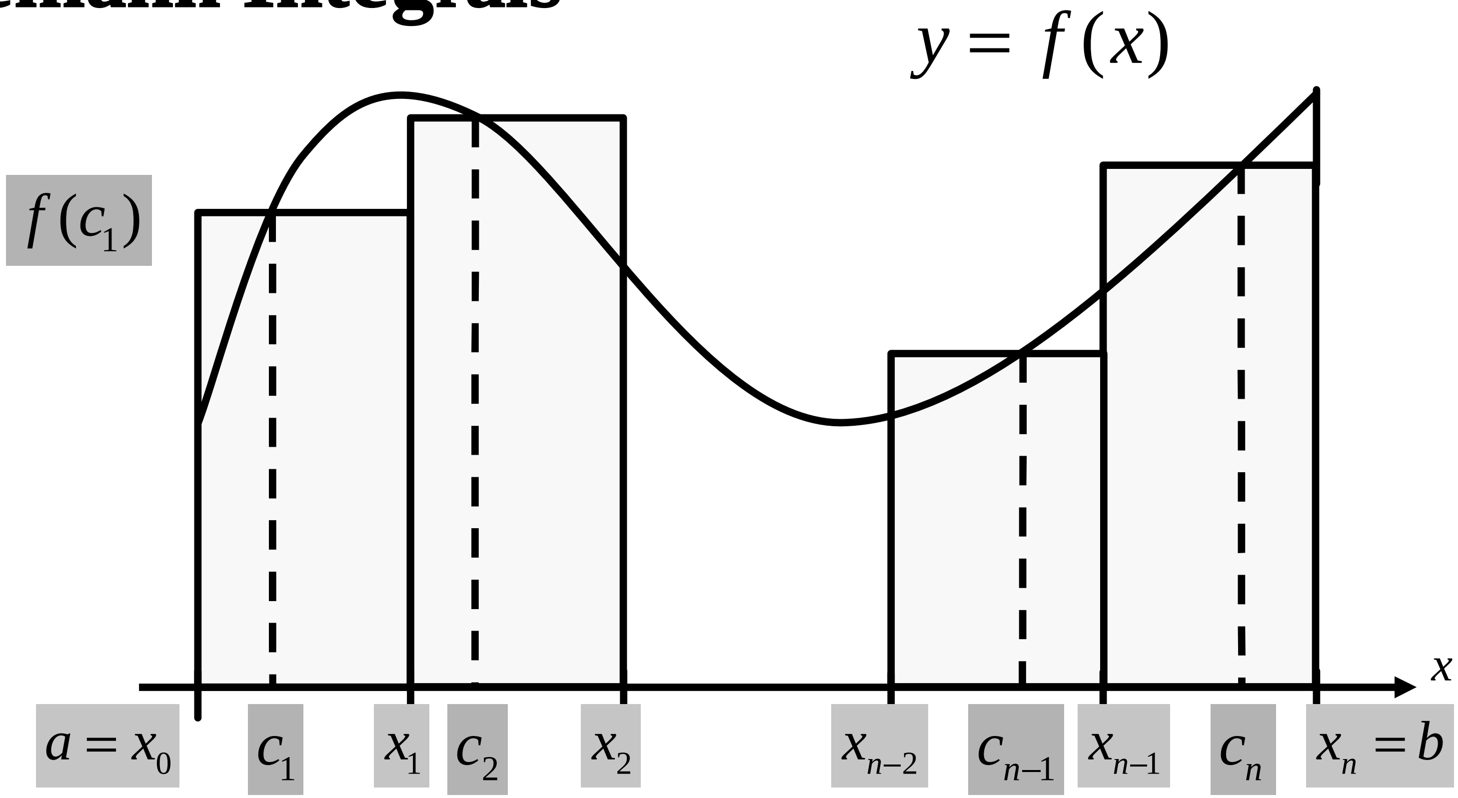


Divide $[a,b]$ into n equal intervals

Length of each interval $= \Delta x = \dfrac{b-a}{n}$

Area of rectangles $= f(c_1)\Delta x + f(c_2)\Delta x + \cdots + f(c_n)\Delta x$

Riemann Integrals

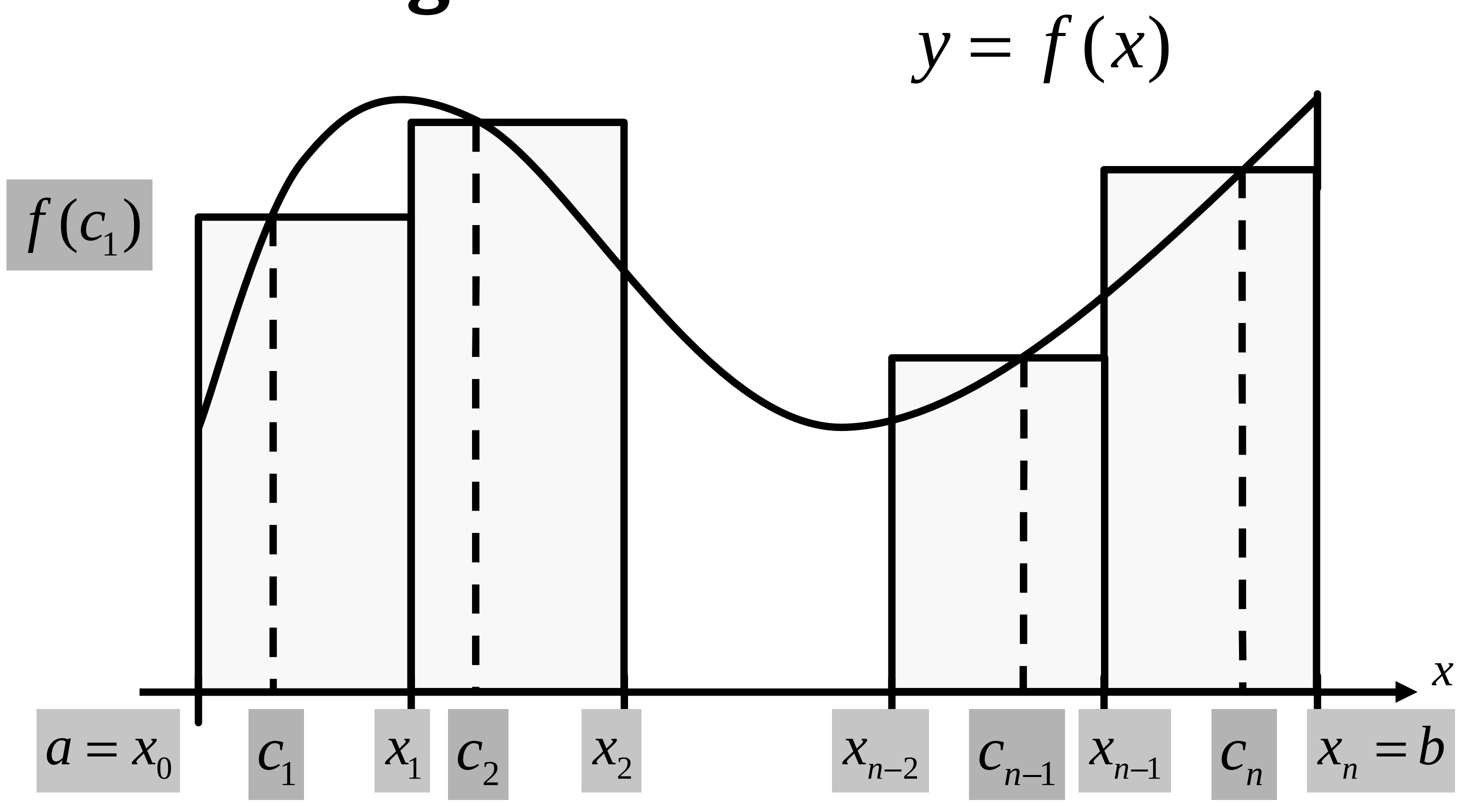


The ***area*** under the curve of $y = f(x)$ from a to b

$$\approx \sum_{k=1}^{n} f(c_k)\,\Delta x$$

Riemann sum of f on $[a,b]$

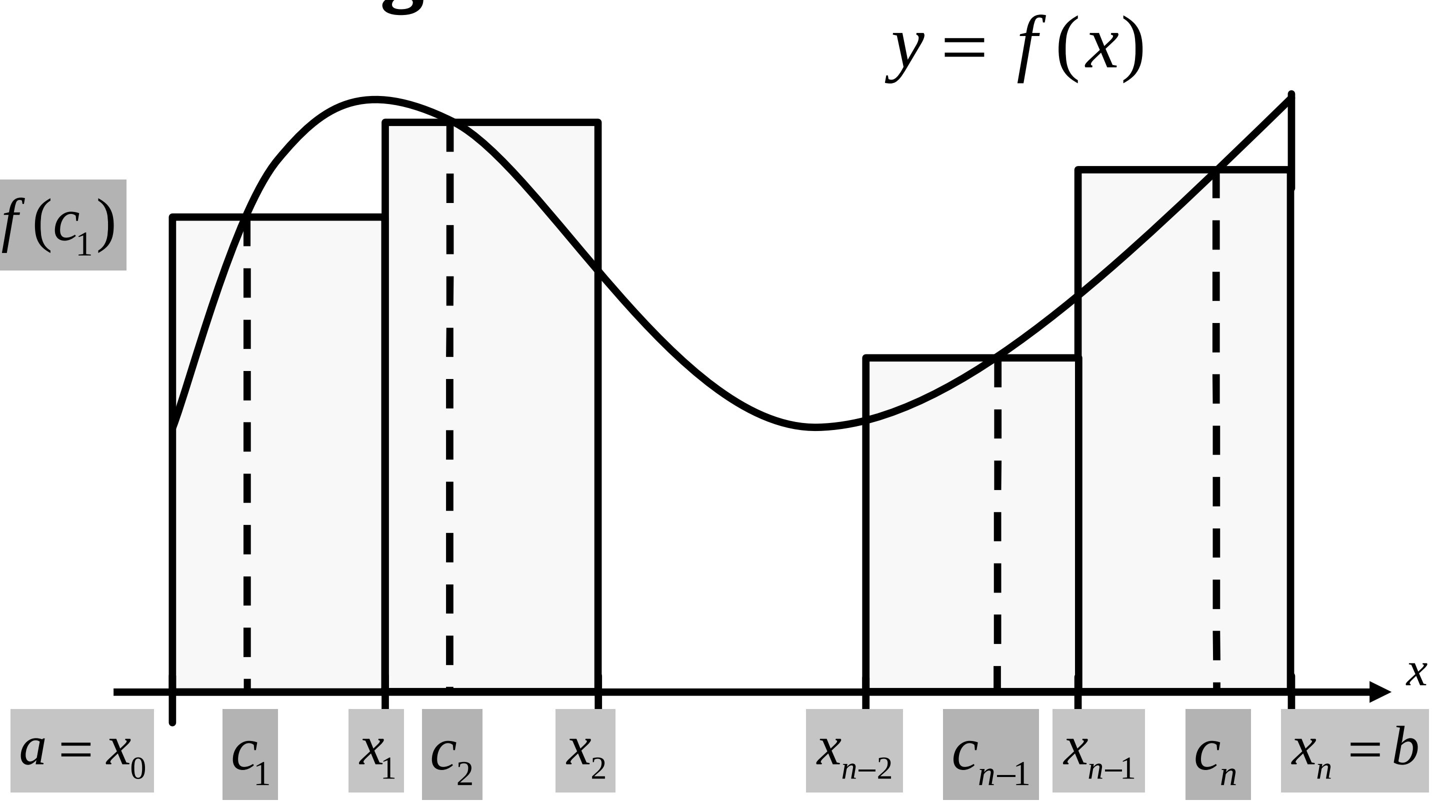
Riemann Integrals



When $n \to \infty$, we have

Area of rectangles $\to$ Area under the curve $f(x)$ from $x = a$ to $x = b$.

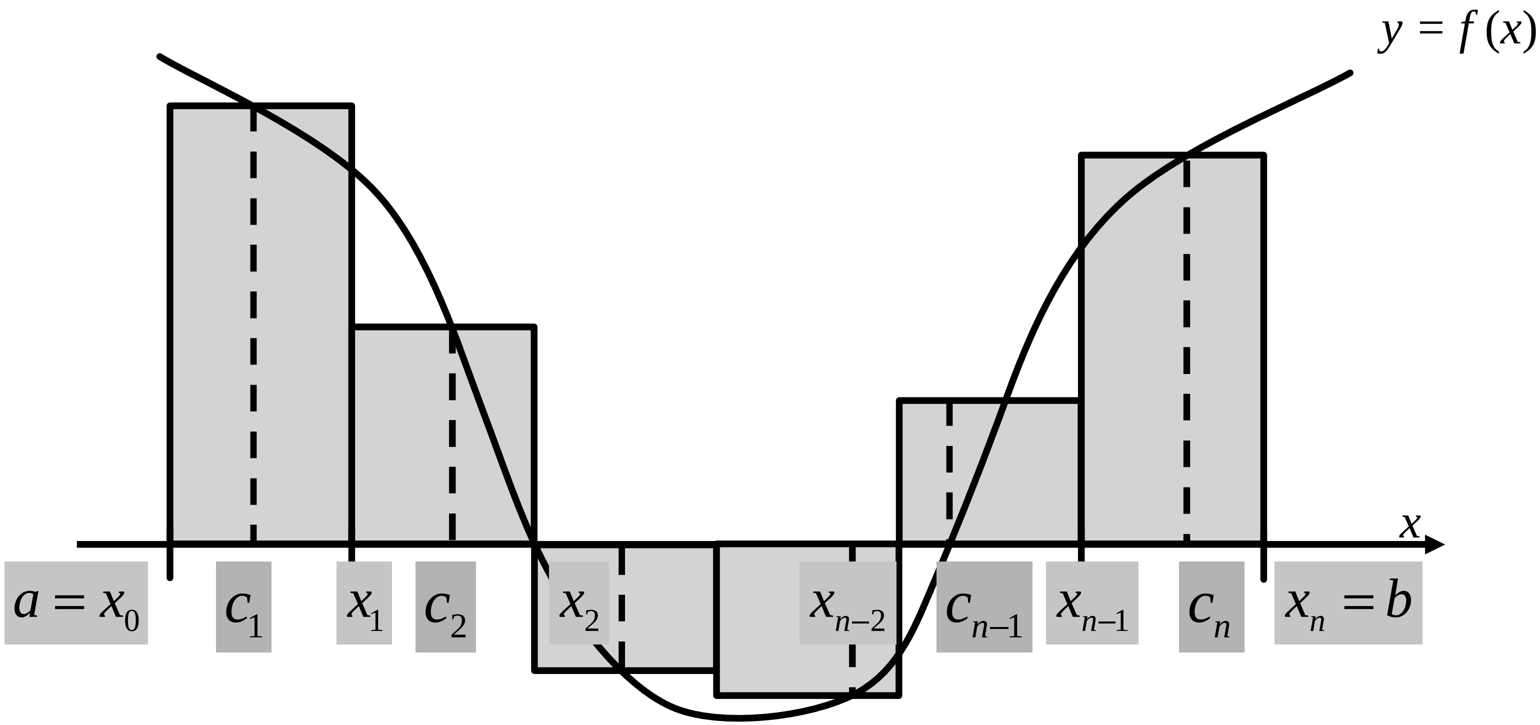
Riemann Integrals



Let $n \to \infty$

The exact area *A* is given by $\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\,\Delta x$

Riemann Integrals



We write

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\,\Delta x$$

and call it the ***Riemann integral*** (or ***definite integral***) of f over $[a,b]$.

Riemann Integrals - Terminology

$$\int_a^b f(x)\,dx$$

$[a,b]:$ the interval of integration

$a:$ lower limit of integration

$b:$ upper limit of integration

$x:$ variable of integration

$f(x):$ the integrand

Note :

x is a dummy variable, i.e.,

$$\int_a^b f(x)\;dx = \int_a^b f(u)\;du = \int_a^b f(t)\;dt, \text{ etc.}$$

Rules of algebra for Definite Integrals

$$1. \int_a^a f(x)\,dx = 0$$

$$2. \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$3. \int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx, \quad \text{where } k \text{ is a constant}$$

$$\text{In particular, } \int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$$

Take $k = -1$

Rules of algebra for Definite Integrals

$$4.\ \int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$$5.\ \text{If } f(x) \geq g(x) \text{ on } [a,b], \text{ then } \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

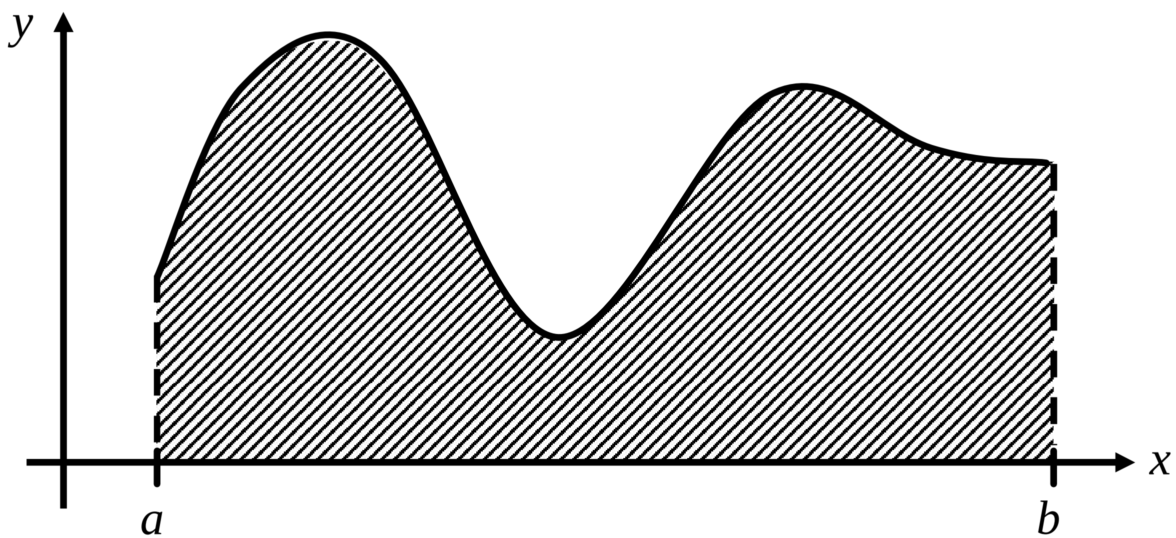
$$6.\ \text{If } f(x) \geq 0 \text{ on } [a,b], \text{ then } \int_a^b f(x)\,dx \geq 0$$

Integration preserve inequality sign
(See Rules 5 and 6)

Integration preserve inequality sign

(See Rules 5 and 6)

6. If $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)\, dx \geq 0$

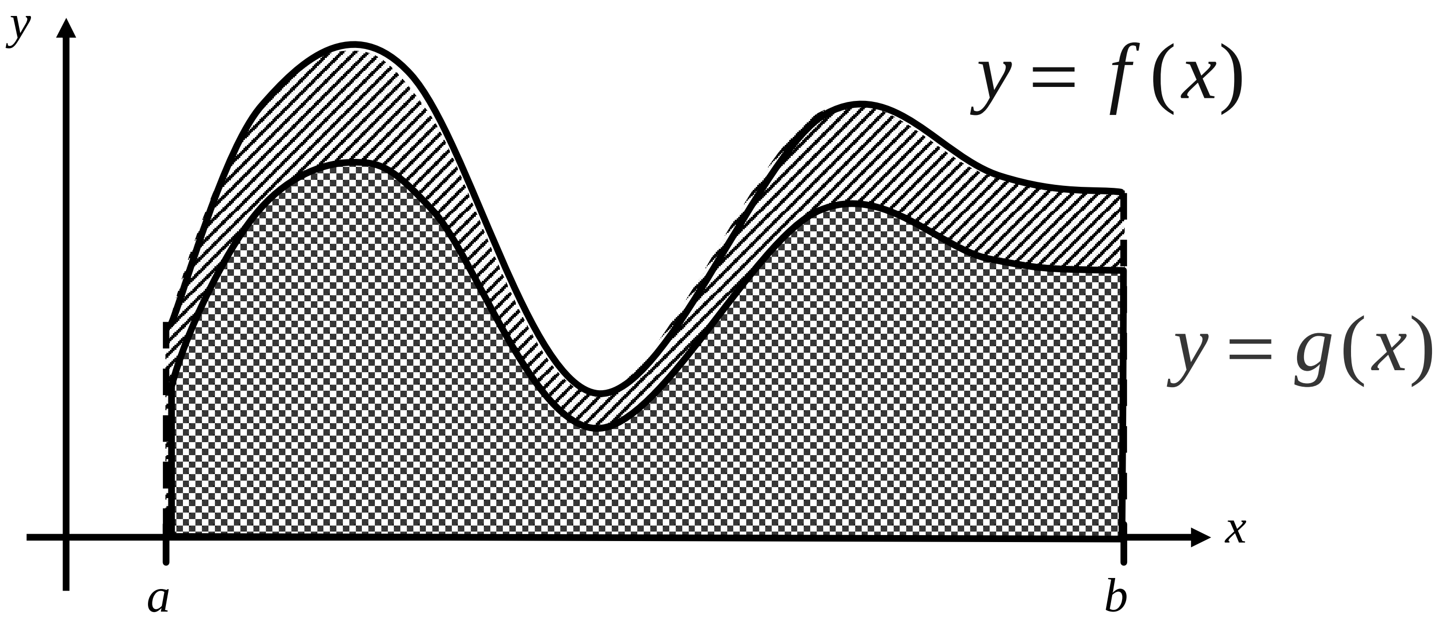


Area under the curve of $f(x)$ $A = \int_a^b f(x)\, dx$

Integration preserve inequality sign

(See Rules 5 and 6)

5. If $f(x) \geq g(x)$ on $[a, b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$



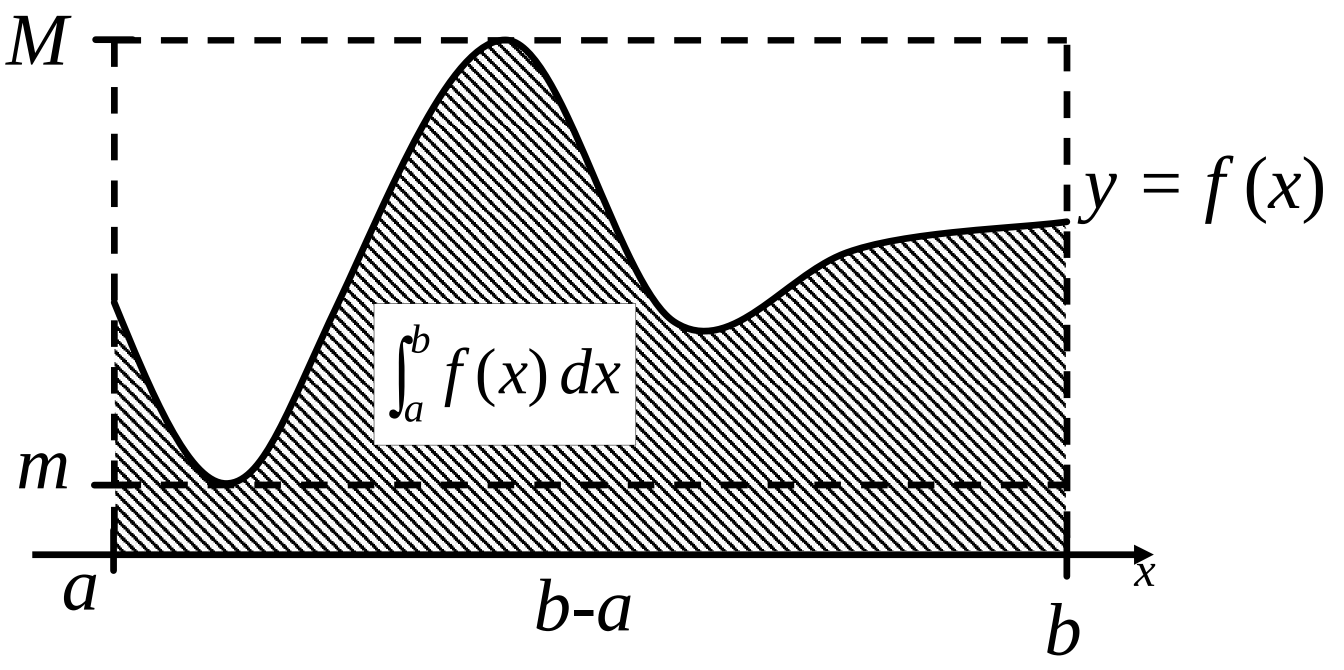
Area under the curve of $f(x)$

$$A = \int_a^b f(x)\,dx$$

Rules of algebra for Definite Integrals

7. If M and m are maximum and minimum values of f on $[a,b]$ respectively,

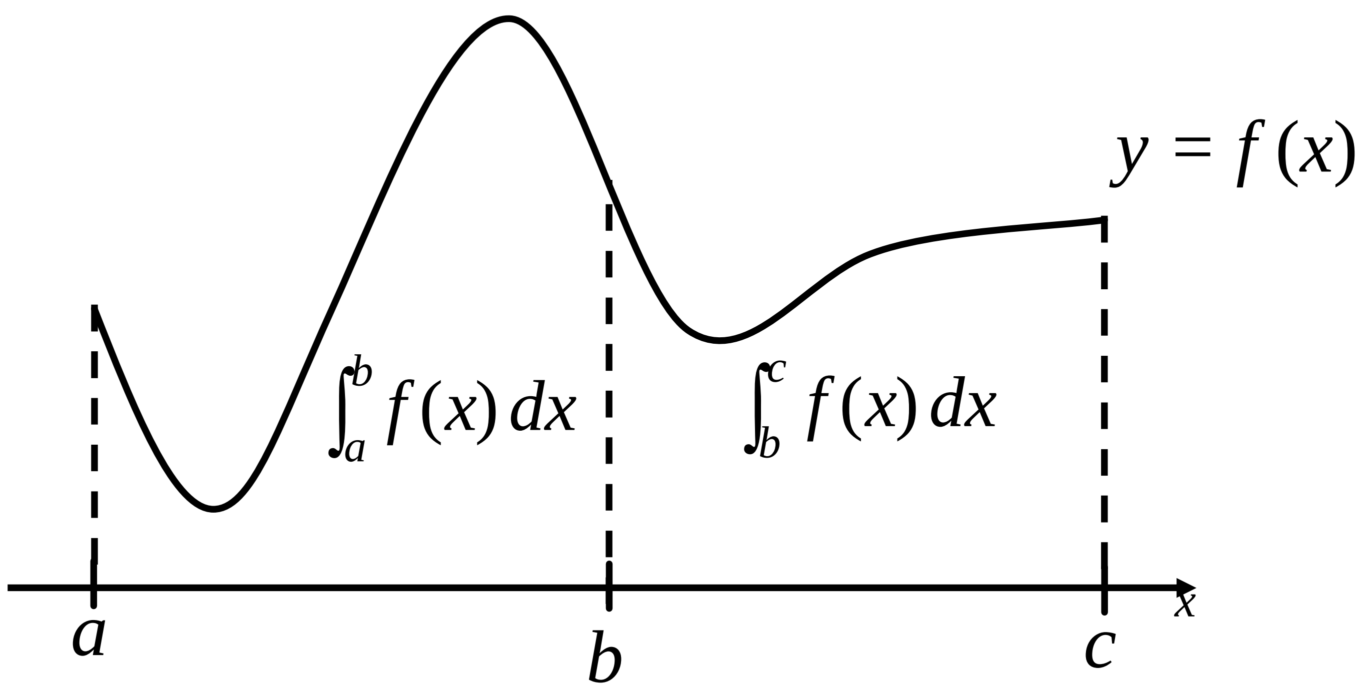
$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$



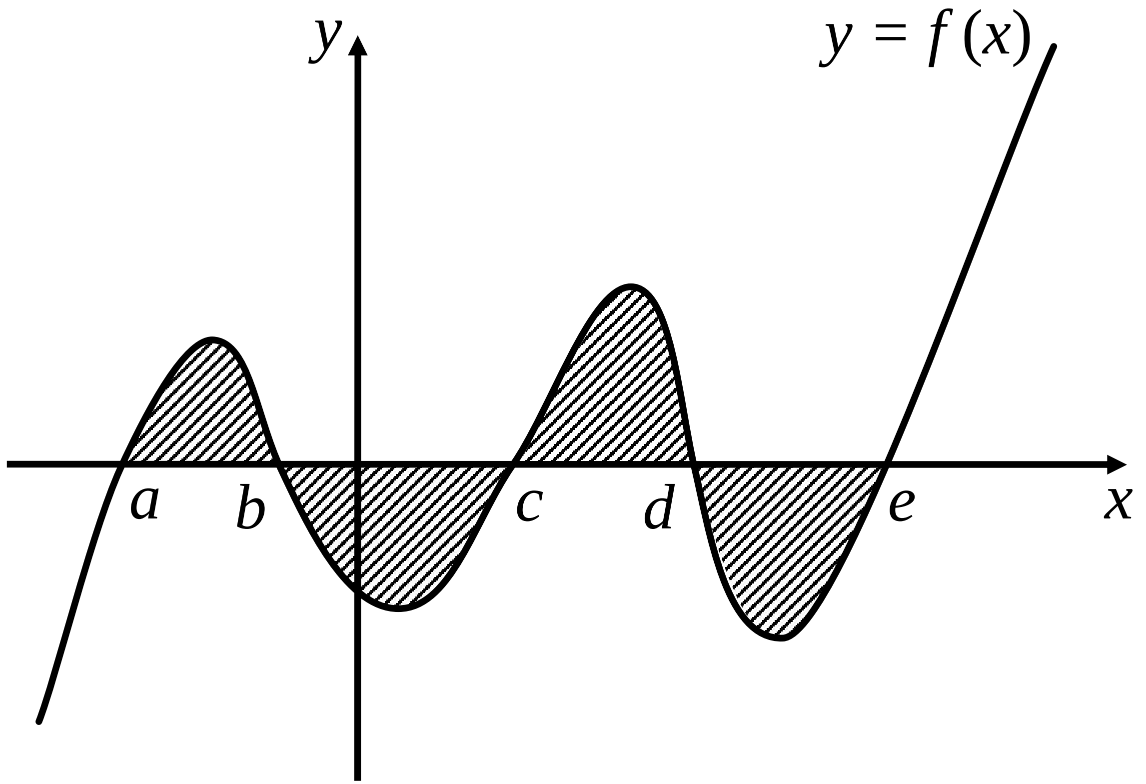
Rules of algebra for Definite Integrals

8. If f is continuous on the interval joining a, b and c, then

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



Note



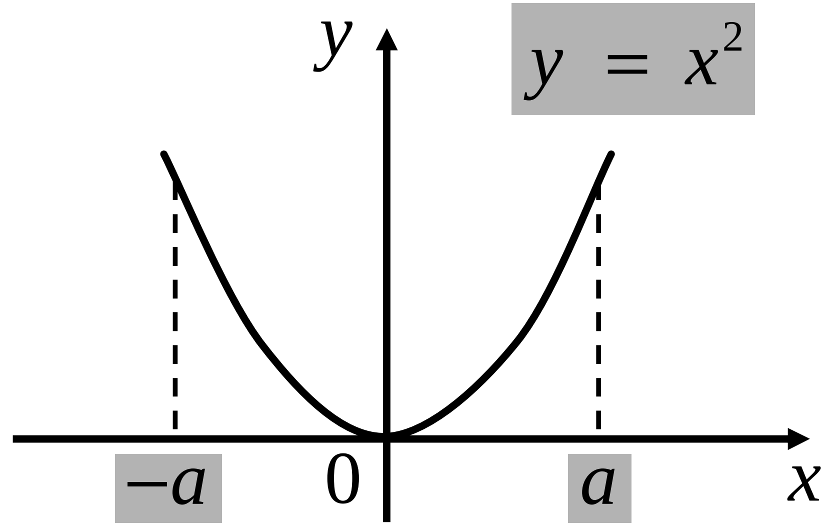
$$\int_a^b f(x)\,dx = +\text{ve}, \int_b^c f(x)\,dx = -\text{ve},$$

$$\int_c^d f(x)\,dx = +\text{ve}, \int_d^e f(x)\,dx = -\text{ve}.$$

Rules of algebra for Definite Integrals

Even function : $f(-x) = f(x)$

Example:



$f(x) = x^2$

$f(-x) = (-x)^2$

$= x^2$

$= f(x)$

Therefore, $f(x) = x^2$ is an even function

Note that : $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

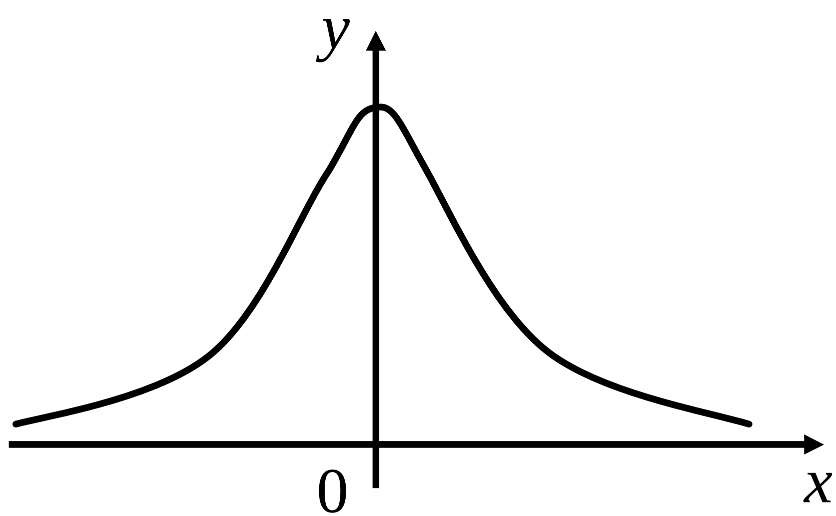
Rules of algebra for Definite Integrals

Even function : $f(-x) = f(x)$

To check a given function $f(x)$ is even function.

Need to check that $f(-x) = f(x)$.

The graph of an even function is symmetrical about $y-$axis



$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

Rules of algebra for Definite Integrals

Odd function : $f(-x) = -f(x)$

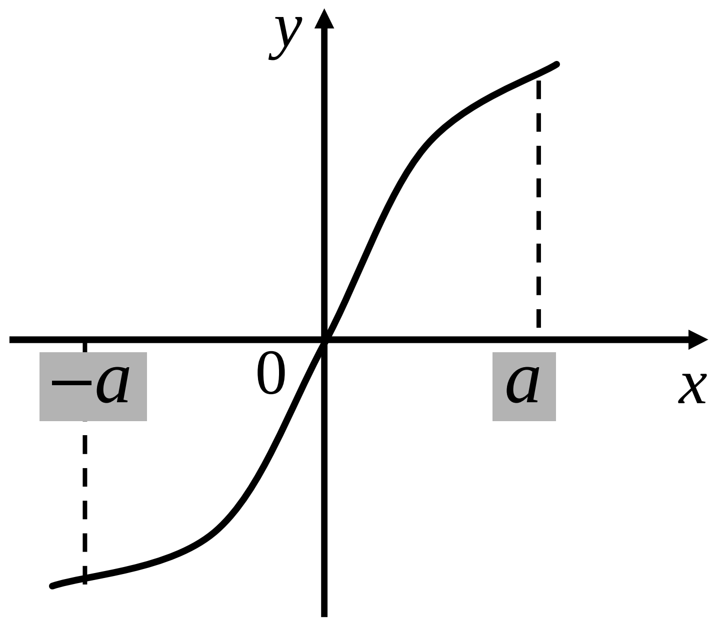
Example:

$$f(x) = x^3$$

$$f(-x) = (-x)^3$$

$$= -x^3$$

$$= -f(x)$$



Therefore, $f(x) = x^3$ is an odd function

Note that : $$\int_{-a}^{a} f(x)\,dx = 0$$

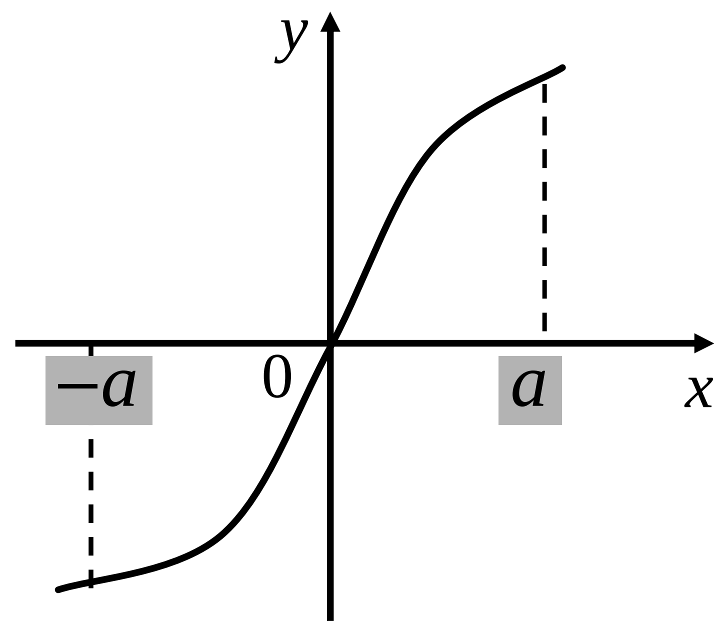
Rules of algebra for Definite Integrals

Odd function : $f(-x) = -f(x)$

To check a given function $f(x)$ is odd function.

Need to check that $f(-x) = -f(x)$.

The graph of an odd function is symmetrical about origin



$$\int_{-a}^{a} f(x)\, dx = 0$$

Question :

What is the difference between finding the value of an integral and finding area bounded?

To find value of integral:

$$\begin{aligned}\int_0^p \cos x\,dx &= [\sin x]_0^p \\ &= \sin p - \sin 0 \\ &= 0 - 0 \\ &= 0\end{aligned}$$

Question :

What is the difference between finding the value of an integral and finding area bounded?

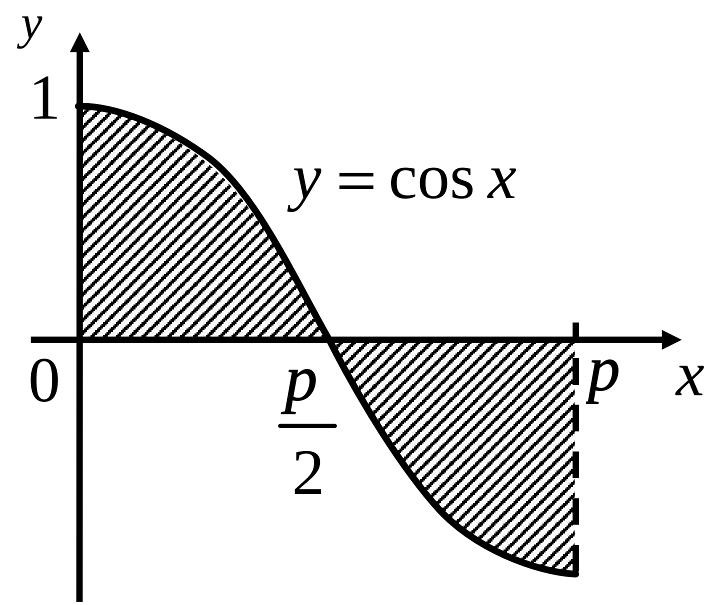
To find shaded area:

$$\int_0^{\frac{p}{2}} \cos x \, dx = [\sin x]_0^{\frac{p}{2}}$$

$$= \sin\frac{p}{2} - \sin 0 = 1$$

$$\int_{\frac{p}{2}}^{p} \cos x \, dx = [\sin x]_{\frac{p}{2}}^{p}$$

$$= \sin p - \sin\frac{p}{2} = -1$$



$$\text{Shaded Area} = 1 + |-1| = 2$$

Fundamental Theorem of Calculus

Pause and Think !!!

How to find $\dfrac{d}{dx}\int_{-p}^{x} \cos t \, dt$?

Pause and Think !!!

How to find $\dfrac{d}{dx}\int_{-p}^{x} \cos t \, dt$?

$$\int_{-p}^{x} \cos t \, dt = \left[\sin t\right]_{-p}^{x}$$

$$= \sin x - \sin(-p)$$

$$\frac{d}{dx}\int_{-p}^{x} \cos t \, dt = \frac{d}{dx}[\sin x - \sin(-p)]$$

$$= \cos x - 0 \text{, since } \sin(-p) \text{ is a constant}$$

$$= \cos x$$

Pause and Think !!!

How to find $\dfrac{d}{dx}\int_{-p}^{x} \cos t \, dt$?

$$\int_{-p}^{x} \cos t \, dt = [\sin t]_{-p}^{x}$$
$$= \sin x - \sin(-p)$$

$$\frac{d}{dx}\int_{-p}^{x} \cos t \, dt = \frac{d}{dx}[\sin x - \sin(-p)]$$
$$= \cos x - 0, \text{ since } \sin(-p) \text{ is a constant}$$
$$= \cos x$$

(1) $\dfrac{d}{dx}$ and $\int_{a}^{x} \, dt$ "cancel" each other

(2) replace t by x

Fundamental Theorem of Calculus (Part I)

replace t by x

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

(1) $\dfrac{d}{dx}$ and $\displaystyle\int_a^x \; dt$ "cancel" each other

(2) replace t by x

(3) lower limit must be a constant

Fundamental Theorem of Calculus (Part I)

Let f be a ***continuous*** function on $[a,b]$.

(I) Let $G(x)=\int_a^x f(t)\,dt,$ then

$$\frac{d}{dx}G(x)=f(x)$$

i.e., $$\frac{d}{dx}\int_a^x f(t)\,dt=f(x)$$

Example

$$\frac{d}{dx}\int_0^x \frac{1}{1+t^2}\,dt = \frac{1}{1+x^2}$$

(1) $\dfrac{d}{dx}$ and $\displaystyle\int_a^x \; dt$ "cancel" each other

(2) replace t by x

Chain Rule

$$\frac{d}{dx}(e^{x}) = e^{x}$$

$$\frac{d}{dx}(e^{3x}) = e^{3x}(3)$$

$$\frac{d}{dx}(e^{x^2}) = e^{x^2}(2x)$$

$$\frac{d}{dx}(e^{\sin x}) = e^{\sin x}(\cos x)$$

Pause and Think !!!

How to find $\dfrac{d}{dx}\displaystyle\int_{-p}^{x^2} \cos t \, dt$?

Pause and Think !!!

How to find $\dfrac{d}{dx}\int_{-p}^{x}\cos t\,dt$?

How to find $\dfrac{d}{dx}\int_{-p}^{x^2}\cos t\,dt$?

$$\int_{-p}^{x}\cos t\,dt = [\sin t]_{-p}^{x}$$
$$= \sin x - \sin(-p)$$

$$\int_{-p}^{x^2}\cos t\,dt = [\sin t]_{-p}^{x^2}$$
$$= \sin x^2 - \sin(-p)$$

$$\frac{d}{dx}\int_{-p}^{x}\cos t\,dt = \frac{d}{dx}[\sin x - \sin(-p)]$$
$$= \cos x\ ,\quad \sin(-p)\ \text{constant}$$
$$= \cos x$$

$$\frac{d}{dx}\int_{-p}^{x^2}\cos t\,dt = \frac{d}{dx}[\sin x^2 - \sin(-p)]$$
$$= \cos x^2(2x)$$

(1) $\dfrac{d}{dx}$ and $\int_{a}^{x} dt$ "cancel" each other

(2) replace t by x

(1) $\dfrac{d}{dx}$ and $\int_{a}^{x} dt$ "cancel" each other

(2) replace t by x^2

(3) $\frac{d}{dx}(x^2) = 2x$

$$\frac{d}{dx}\int_{-p}^{x} \cos t \; dt = \cos x$$

$$\frac{d}{dx}\int_{-p}^{3x} \cos t \; dt = \cos 3x(3)$$

$$\frac{d}{dx}\int_{-p}^{x^2} \cos t \; dt = \cos x^2(2x)$$

$$\frac{d}{dx}\int_{-p}^{x} \cos t \; dt = \cos x$$

$$\frac{d}{dx}\int_{-p}^{3x} \cos t \; dt = \cos 3x(3)$$

$$\frac{d}{dx}\int_{-p}^{x^2} \cos t \; dt = \cos x^2(2x)$$

Pause and Think !!!

$$\frac{d}{dx}\int_{3x}^{x^2} \cos t \; dt = \quad ???$$

Pause and Think !!!

$$\frac{d}{dx}\int_{3x}^{x^2} \cos t \; dt = \quad ???$$

$$\int_a^b f(x)\; dx + \int_b^c f(x)\; dx = \int_a^c f(x)\; dx$$

$$\int_{3x}^{x^2} \cos t \; dt = \int_{3x}^{-p} \cos t \; dt + \int_{-p}^{x^2} \cos t \; dt$$

$$= -\int_{-p}^{3x} \cos t \; dt + \int_{-p}^{x^2} \cos t \; dt$$

$$\int_a^b - f(x)\; dx = -\int_a^b f(x)\; dx$$

$$\frac{d}{dx}\int_{3x}^{x^2} \cos t \; dt = -\frac{d}{dx}\int_{-p}^{3x} \cos t \; dt + \frac{d}{dx}\int_{-p}^{x^2} \cos t \; dt$$

$$= -\cos 3x(3) + \cos x^2(2x)$$

$$= 2x\cos x^2 - 3\cos 3x$$

Example

$$\frac{d}{dx}\int_0^5 \sqrt{t^3+1}\,dt = \frac{d}{dx}(\text{constant})$$
$$= 0$$

$$\frac{d}{dx}\left(\int_0^x \sin\sqrt{t}\,dt\right) = \sin\sqrt{x}$$

$$\frac{d}{dx}\int_1^{x^4} \frac{t}{\sqrt{t^3+2}}\,dt = \frac{x^4}{\sqrt{(x^4)^3+2}}(4x^3)$$
$$= \frac{4x^7}{\sqrt{x^{12}+2}}$$

Pause and Think !!!

Find

(a) $$\lim_{x \to 0} \frac{\int_0^x \sin t \, dt}{x^2}$$

(b) $$\lim_{x \to 0} \frac{\int_0^x t \sin t \, dt}{x^2}$$

(c) $$\lim_{x \to 0} \frac{\int_0^x x \sin t \, dt}{x^2}$$

Pause and Think !!!

Find

$$(a) \qquad \frac{d}{dx}\int_a^x t\, f(t)\, dt$$

$$(b) \qquad \frac{d}{dx}\int_a^x x\, f(t)\, dt$$

Question: What is the difference between (a) and (b) ???