**MA1506 Tutorial 2 Solutions**

Question 1.

(1a)

$$y'+(1+\frac{1}{x})y=\frac{1}{x}e^{-x}$$

Integrating factor is $\exp\int(1+\frac{1}{x})=\exp(x+\ln x)=xe^{x}$ (in general, $y'+P(x)y\Rightarrow$ multiply by $\exp\int P\Rightarrow y'e^{\int P}+Pye^{\int P}=\frac{d}{dx}(ye^{\int P}))$

So $\frac{d}{dx}(yxe^{x})=1(=xe^{x}\times\frac{1}{x}e^{-x})$

$\Rightarrow yxe^{x}=x+c\Rightarrow y=e^{-x}+cx^{-1}e^{-x}$

(1b)

$$\exp\int-(1+\frac{3}{x})=\exp(-x-3\ln|x|)=\frac{1}{x^3}e^{-x}$$

$$\Rightarrow\frac{d}{dx}(y\frac{1}{x^3}e^{-x})=(x+2)\frac{1}{x^3}e^{-x}$$

$$\frac{y}{x^3}e^{-x}=\int\frac{e^{-x}}{x^2}+2\int\frac{e^{-x}}{x^3}+c$$

$$=\int\frac{e^{-x}}{x^2}+\frac{-e^{-x}}{x^2}-\int\frac{e^{-x}}{x^2}+c$$

$$=\frac{-e^{-x}}{x^2}+c$$

$y=-x+cx^3e^{x}$

since $y(1)=e-1=-1+ce\Rightarrow c=1$

$$y=-x+x^3e^{x}$$

(1c)

This kind is called a Bernoulli equation -- set

$z=y^2 \qquad z'=2yy' \qquad y'=z'/2y$

$$\frac{z'}{2y}+y+\frac{x}{y}=0\Rightarrow\frac{1}{2}z'+z+x=0\Rightarrow z'+2z=-2x\Rightarrow\frac{d}{dx}(e^{2x}z)=-2xe^{+2x}$$

$$\Rightarrow ze^{2x}=(-x+\frac{1}{2})e^{2x}+c\Rightarrow y^2=\frac{1}{2}-x+ce^{-2x}$$

(1d)

Since $2yy' = (y^2)'$ we define $Y = y^2$, $Y' + (1 - \frac{1}{x})Y = xe^x$, $\exp\int(1 - \frac{1}{x}) = \frac{1}{x}e^x$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{x}e^x Y\right) = e^{2x} \Rightarrow \frac{1}{x}e^x Y = \frac{1}{2}e^{2x} + c$$

$$\Rightarrow y^2 = \frac{1}{2}xe^x + cxe^{-x}$$

Question 2.

Define v = dy/dx. Then our equation is

$$v = \frac{\mu}{\mathrm{T}} \int_0^x \sqrt{(v)^2 + 1}\ dt\,.$$

Notice that v(0) = 0. Now by the fundamental theorem of calculus

$$v' = \frac{\mu}{T}\sqrt{v^2 + 1}$$

This is a separable differential equation with initial condition v(0) = 0. Separating the variables and integrating [remember that $\cosh^2 - \sinh^2 = 1$] we find that v = sinh(μx/T). Integrating, we get

$$y = \frac{T}{\mu}\cosh(\frac{\mu}{T}x) + C\,,$$

where C must be – T/μ since y(0) = 0 and cosh(0) = 1. Thus we have

$y = \frac{T}{\mu}\cosh(\frac{\mu}{T}x) - \frac{T}{\mu}$. The shape of the graph of this function is indeed “U-shaped”. [Draw it!]

Question 3.

The constant C has units of 1/time. Solving the equation [either as linear first order or as a separable equation] we get

$$P = M - Me^{-Ct}$$

Clearly P will approach M more rapidly if C is large; that is, C measures how rapidly the student is able to learn. Thus the equation indeed expresses the idea that the student’s performance improves more slowly as she approaches **her** maximum possible performance.

As the years go by and the student becomes more familiar with the methods of mastering mathematics, her rate of learning new things might be expected to improve; but surely there is an upper bound to how much she can improve. The tanh function is a simple way of representing this since it always increases but is bounded above. [Remind the students of the shape of tanh(x) if necessary.] Then K represents her maximum possible speed of learning [since tanh is asymptotic to 1], and T measures the amount of time required for her to realise her maximum potential. [Note that K has units of 1/time, while of course T has units of time, so KT is dimensionless.] The equation can now be written as

$$\frac{dP}{dt} + K\tanh(t/T)(P - M) = 0\,.$$

It's convenient now to define Q = P – M, so the equation is

$$\frac{dQ}{dt} + K\tanh(t/T)Q = 0.$$

This is a first-order equation with integrating factor $\cosh^{KT}(t/T)$, so we have

$$Q\cosh^{KT}(t/T) = \mathrm{A},$$

where A is a constant. Since we are assuming that P(0) = 0, we have Q(0) = - M, so we have A = - M, thus finally

$$P = M\left[1 - \operatorname{sech}^{KT}(t/T)\right].$$

Students are encouraged to graph examples of such questions using [for example] the software available free at http://www.graphmatica.com/

Question 4.

The constant K measures the rapidity with which the rumour will spread. It depends on how interesting the rumour is, how much the students like to gossip, how gullible they are, etc. The right hand side of the equation is designed to be small both near R = 1 and near R = 1300, when indeed the rumour can be expected to spread slowly either because not enough or too many people have heard it.

We have

$$\frac{dR}{dt} - 1300KR = -KR^2\,.$$

This is a Bernoulli equation as discussed in the notes. We solve it by defining Z = 1/R, which transforms the equation into a linear one:

$$\frac{dZ}{dt} + 1300KZ = K \, ,$$

with solution

$$\frac{1}{R} = \frac{1}{1300} + C\exp(-1300Kt) \, .$$

The problem says that this highly interesting rumour was started by one student, so R(0) = 1. Thus C = 1299/1300. Hence

$$\frac{1}{R} = \frac{1}{1300} + \frac{1299}{1300}\exp(-1300Kt) \, .$$

Of course as t tends to infinity, R tends to 1300.

Question 5.

Suppose we write the equation governing the Uranium as

$$\frac{dU}{dt} = -k_U U$$

where U represents the number of Uranium atoms, etc, as usual. The half-life of Uranium can be used to compute the decay rate constant, as in the lecture notes, and similarly for Thorium. From the lecture notes we have

$$\frac{T}{U} = \frac{k_U}{k_T - k_U}[1 - \exp((k_U - k_T)t)]$$

and so if T/U is 0.1 we know everything in this equation except t. Solving for t, you should find that the answer is approximately 39.5 thousand years.