NATIONAL UNIVERSITY OF SINGAPORE

EXAMINATION

ST2334 PROBABILITY AND STATISTICS

Semester 2: AY 2003-2004

April 2004 - Time Allowed: 2 Hours

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INSTRUCTIONS TO THE CANDIDATES

1. This examination paper contains **FOUR (4) QUESTIONS** and comprises 3 pages.

2. Answer **ALL** questions. The total mark for this paper is 60.

3. Students are allowed to use 4 A-4 pages with notes, written both sides.

4. Non-programmable calculators may be used.

5. The table of the Cumulative Normal Distribution is in the last page.

1) Let X, Y be random variables with joint density

$$f(x,y) = e^{-(x+y)}, x > 0, \ y > 0.$$

a) Compute the probability $P[X < Y]$. [4 points]
b) Compute the probability $P[X < Y | X < 2Y]$. [3 points]
c) Compute the covariance of X, Y. [2 points]

2) a) Let X, Y be independent Poisson(λ) random variables. Show that the random variable $W = X + Y$ is a Poisson(2λ) random variable using moment generating functions (in order to get credit). [5 points]
b) Suppose that the moment generating function of the random variable U is given by $\phi(t) = e^{3(e^t-1)}$. Compute, if it is possible, the probability $P[U = 0]$. Explain in order to get full credit. [4 points]

3) Let $X_1, \ldots, X_n$ be i.i.d. random variables with density function

$$f(x|\theta) = (\theta + 1)x^\theta, 0 \le x \le 1, \theta > -1.$$

a) Determine a one-dimensional sufficient statistic. Explain. [4 points]
b) Find the moments estimate $\tilde{\theta}_n$ of θ. [6 points]
c) Find the maximum likelihood estimate $\hat{\theta}_n$ of θ. [6 points]
d) Calculate the Fisher's information $I_{X_1}(\theta)$ contained in X_1. [4 points]
e) For n large, give the form of a 95% Confidence Interval for θ. [4 points]

4) Let $X_1, \ldots, X_n$ be *i.i.d.* random variables with density

$$f(x|\theta) = \theta e^{-\theta x}, \ x > 0.$$

a) Derive explicitly the rejection region for the most powerful (MP) α-level likelihood ratio test of the hypotheses

$$H_0 : \theta = .5, \text{ against } H_1 : \theta = .25 \qquad [5 \text{ points}].$$

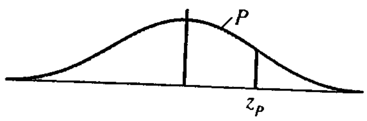
b) Determine as explicitly as possible the equation that will determine the critical value c of the α-level rejection region. You may use that $Y = \sum_{i=1}^n X_i$ follows a Gamma distribution with parameters θ and n, and density $g_Y(y) = \frac{\theta^n y^{n-1} e^{-\theta y}}{\Gamma(n)}$, $y > 0$. Partial credit will be given for writing the equation using an integral, without expressing it as a sum of simple terms. [6 points].

c) Determine the power of the test. [4 points].
d) Explain why the test obtained in a) is (or is not) uniformly most powerful (UMP) at level α for testing the hypotheses

$$H_0 : \theta = .5, \text{ against } H_1^* : \theta \le .25 \qquad [3 \text{ points}].$$

END OF THE PAPER

Cumulative Normal Distribution—Values of P Corresponding to z_p for the Normal Curve



z is the standard normal variable. The value of P for $-z_p$ equals 1 minus the value of P for $+z_p$; for example, the P for -1.62 equals $1 - .9474 = .0526$.

z_p	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998