Tutorial 2
(Supplementary)

Q 1 (d)

$$y' + \frac{x-1}{2x} y = \frac{x}{2} e^x y^{-1}, \quad x>0$$

$$\boxed{\begin{array}{c} y' + p(x) y = q(x) y^n \\ Z = y^{1-n} \text{ get} \\ Z' + (1-n) p(x) Z = (1-n) q(x) \end{array}}$$

Bernoulli eq

Let $z = y^{1-(-1)} = y^2$.

Subst $z = y^2$ into the given ODE, get

$$z' + (2) \frac{x-1}{2x} z = (2) \frac{x}{2} e^x$$

$$\therefore z' + \frac{x-1}{x} z = x e^x$$

integrating factor $= e^{\int \frac{x-1}{x} dx}$

$$\int \frac{x-1}{x} dx = \int \left(1 - \frac{1}{x}\right) dx = x - \ln x$$

$$\therefore e^{\int \frac{x-1}{x} dx} = e^{x - \ln x} = e^x e^{-\ln x}$$

$$= e^x e^{\ln x^{-1}}$$

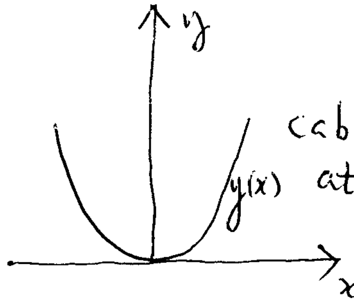
$$= e^x x^{-1} = \frac{e^x}{x}$$

$$\therefore z \frac{e^x}{x} = \int x e^x \left(\frac{e^x}{x}\right) dx = \int e^{2x} dx$$

$$= \frac{1}{2} e^{2x} + C$$

$$z = \frac{x}{2} e^x + cxe^{-x} \quad \therefore y^2 = \frac{x}{2} e^x + cxe^{-x}$$

Q2



cable held at two ends at the same height

$\therefore y(0) = 0$ initial condition

By given

$$y'(x) = \frac{\mu}{T} \int_0^x \sqrt{(y'(t))^2 + 1}\, dt$$

We want to find y.

i.e., solve the above integral equation

Need to change to solving ODE.

For convenience, let $u(x) = y'(x)$.

We have

$$u(x) = \frac{\mu}{T} \int_0^x \sqrt{(u(t))^2 + 1}\, dt \quad \cdots (1)$$

Note that $u(0) = \frac{\mu}{T} \int_0^0 \sqrt{(u(t))^2 + 1}\, dt = 0$

initial condition for u.

differentiate eq (1), get 2

$$u'(x) = \frac{\mu}{T} \sqrt{(u(x))^2 + 1}$$

i.e,

$$\frac{du}{dx} = \frac{\mu}{T}\sqrt{u^2+1}$$

$$\frac{1}{\sqrt{u^2+1}}\,du = \frac{\mu}{T}\,dx$$

$$\int \frac{1}{\sqrt{u^2+1}}\,du = \frac{\mu}{T}\int dx$$

$$\sinh^{-1} u = \frac{\mu}{T}x + c$$

By initial condition $u(0)=0$, get

$$0 = \sinh^{-1} 0 = \frac{\mu}{T}0 + c$$

$$\therefore \; c = 0$$

$$\therefore \quad \sinh^{-1} u = \frac{\mu}{T}x$$

$$\therefore \quad u = \sinh\left(\frac{\mu}{T}x\right)$$

Hence $\dfrac{dy}{dx} = \sinh\left(\dfrac{\mu}{T}x\right)$

$$\int dy = \int \sinh\left(\frac{\mu}{T}x\right)dx$$

$$y = \frac{T}{\mu}\cosh\left(\frac{\mu}{T}x\right) + c$$

By $y(0)=0$, get

$$0 = \frac{T}{\mu}\cosh\left(\frac{\mu}{T}0\right) + c$$

$$\therefore \quad 0 = \frac{T}{\mu} + c \qquad \therefore \; c = -\frac{T}{\mu}$$

$$\therefore \; y = \frac{T}{\mu}\cosh\left(\frac{\mu}{T}x\right) - \frac{T}{\mu}$$

Q 3.

Note that $\tanh x \approx 1$ when $x = 2$

Hence

$$\tanh\left(\frac{t}{T}\right) \approx 1 \quad \text{when } \frac{t}{T} = 2$$

$$c(t) = k \tanh\left(\frac{t}{T}\right) \approx k \quad \text{when } \frac{t}{T} = 2$$

i.e, need $2T$ time to reach her maximum potential

$$\frac{dp}{dt} + c(t)\,p = c(t)M$$

$$\frac{dp}{dt} = c(t)\,(M-p)$$

$$\frac{1}{M-p}\,dp = c(t)\,dt$$

$$\int \frac{1}{M-p}\,dp = \int c(t)\,dt$$

$$(-1)\int \frac{1}{M-p}\,d(M-p) = \int k \tanh\left(\frac{t}{T}\right) dt$$

$$(-1)\ln(M-p) = kT \ln\cosh\left(\frac{t}{T}\right) + c$$

$$p(0)=0 \Rightarrow c = (-1)\ln M$$

$$\therefore p = M\left[1 - \left(\operatorname{sech}\left(\frac{t}{T}\right)\right)^{kT}\right]$$