NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2007-2008

MA1505 MATHEMATICS I

November 2007 Time allowed: 2 hours

Question 1 (a) [5 marks]

Find the slope of the tangent to the curve $y^2 = x^3 + 2x^2 - 20$ at the point $(3, 5)$.

Answer 1(a)	$\frac{39}{10}$

$$y^2 = x^3 + 2x^2 - 20$$

$$2yy' = 3x^2 + 4x$$

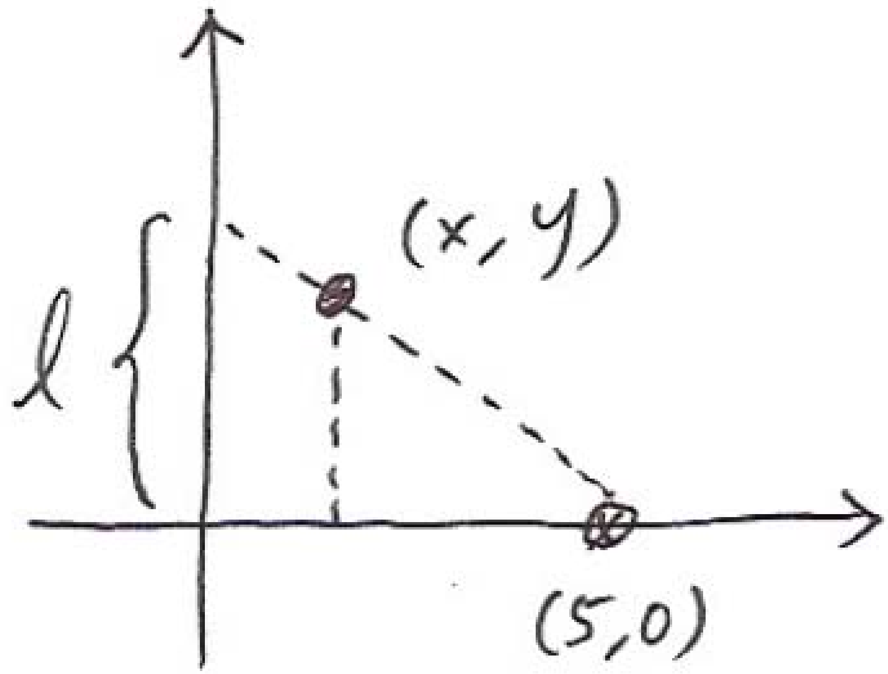
$$x = 3,\ y = 5 \implies 10y' = 27 + 12 = 39$$

$$\therefore\ y' = \frac{39}{10}$$

Question 1 (b) [5 marks]

A lamp is located at the point $(5, 0)$ in the xy-plane. An ant is crawling in the first quadrant of the plane and the lamp casts its shadow onto the y-axis. How fast is the ant's shadow moving along the y-axis when the ant is at position $(1, 2)$ and moving so that its x-coordinate is increasing at a rate of $\frac{1}{2}$ units/sec and its y-coordinate is decreasing at a rate of $\frac{1}{5}$ units/sec?

Answer 1(b)	$\frac{1}{16}$


(x, y)
l
(5,0)

$$\frac{l}{y} = \frac{5}{5-x}$$

$$l = \frac{5y}{5-x}$$

$$\frac{dl}{dt} = \frac{5\frac{dy}{dt}(5-x) + 5y\frac{dx}{dt}}{(5-x)^2}$$

$x=1,\ y=2,\ \frac{dx}{dt} = \frac{1}{2},\ \frac{dy}{dt} = -\frac{1}{5}$

$$\Rightarrow \frac{dl}{dt} = \frac{5\left(-\frac{1}{5}\right)(5-1) + 5(2)\left(\frac{1}{2}\right)}{(5-1)^2} = \underline{\underline{\frac{1}{16}}}$$

Question 2 (a) [5 marks]

Find the **exact** value of the integral

$$\int_0^{\sqrt{101}} 2x^3 e^{x^2}\, dx.$$

Express your answer in terms of e.

Answer 2(a)	$100e^{101} + 1$

Let $u = x^2$

$$\Rightarrow \int_0^{\sqrt{101}} 2x^3 e^{x^2} dx = \int_0^{101} u e^u \, du$$

$$= \int_0^{101} u \, d(e^u)$$

$$= \left[u e^u \right]_0^{101} - \int_0^{101} e^u \, du$$

$$= 101 e^{101} - \left[e^u \right]_0^{101}$$

$$= \underline{\underline{100 e^{101} + 1}}$$

Question 2 (b) [5 marks]

Find a degree three polynomial to approximate the function

$$f(x) = \ln(1 + \sin x)$$

near $x = 0$.

Answer 2(b)	$x - \frac{1}{2}x^2 + \frac{1}{6}x^3$

$$f(x) = \ln(1+\sin x) \Rightarrow f(0) = 0$$

$$f'(x) = \frac{\cos x}{1+\sin x} \Rightarrow f'(0) = 1$$

$$f''(x) = \frac{-\sin x(1+\sin x) - \cos^2 x}{(1+\sin x)^2}$$

$$= \frac{-\sin x - 1}{(1+\sin x)^2} = \frac{-1}{1+\sin x} \Rightarrow f''(0) = -1$$

$$f'''(x) = \frac{\cos x}{(1+\sin x)^2} \Longrightarrow f'''(0) = 1$$

$$\therefore f(x) \approx 0 + x - \frac{1}{2}x^2 + \frac{1}{6}x^3$$

$$= \underline{\underline{x - \frac{1}{2}x^2 + \frac{1}{6}x^3}}$$

Question 3 (a) [5 marks]

Let $f(x) = |\sin x|$ for all $x \in (-\pi, \pi)$, and $f(x + 2\pi) = f(x)$ for all x. Let

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be the Fourier Series which represents $f(x)$. Let m denote a fixed positive integer. Find the **exact** value of a_{2m}. Express your answer in terms of m in the simplest form.

Answer 3(a)	$-\dfrac{4}{(4m^2-1)\pi}$

Note that f is even.

$$
\begin{aligned}
a_{2m} &= \frac{2}{\pi}\int_0^{\pi} \sin x \cos 2mx \, dx \\
&= \frac{1}{\pi}\int_0^{\pi} \{\sin(2m+1)x - \sin(2m-1)x\} dx \\
&= \frac{1}{\pi}\left[-\frac{1}{2m+1}\cos(2m+1)x + \frac{1}{2m-1}\cos(2m-1)x\right]_0^{\pi} \\
&= \frac{1}{\pi}\left\{\frac{(-1)^{2m+2}}{2m+1} + \frac{1}{2m+1} + \frac{(-1)^{2m-1}}{2m-1} - \frac{1}{2m-1}\right\} \\
&= \frac{1}{\pi}\left\{\frac{2}{2m+1} - \frac{2}{2m-1}\right\} \\
&= -\frac{4}{(4m^2-1)\pi}
\end{aligned}
$$

Question 3 (b) [5 marks]

Find the shortest distance from the point $(-1, 1, 2)$ to the plane

$$2x + 3y - z - 10 = 0.$$

Answer 3(b)	$\dfrac{11}{\sqrt{14}}$

$$d = \frac{|2(-1) + 3(1) - (2) - 10|}{\sqrt{4+9+1}} = \underline{\underline{\frac{11}{\sqrt{14}}}}$$

Question 4 (a) [5 marks]

Let L_1 be a straight line which passes through the point $(-1, 0, 1)$ and suppose that L_1 is perpendicular to the plane $2x - y + 7z = 12$. Let L_2 be the line $\mathbf{r}(t) = (2+t)\,\mathbf{i} + (-4+2t)\,\mathbf{j} + (18-3t)\,\mathbf{k}$. Find the coordinates of the point of intersection of L_1 and L_2.

Answer 4(a)	$(3, -2, 15)$

$$L_1: (x, y, z) = (-1, 0, 1) + s(2, -1, 7)$$

$$= (-1+2s, -s, 1+7s)$$

$$\begin{cases} 2+t = -1+2s & \text{------ ①} \\ -4+2t = -s & \text{------ ②} \\ 18-3t = 1+7s & \text{------ ③} \end{cases}$$

$① + 2② \Rightarrow -6 + 5t = -1$

$\Rightarrow t = 1$

$\therefore$ point of intersection $= \underline{\underline{(3, -2, 15)}}$

Question 4 (b) [5 marks]

Let $f(x, y) = \ln(\tan x + \tan y)$, with $0 < x, y < \frac{\pi}{2}$. Find the value of

$$(\sin 2x)\frac{\partial f}{\partial x} + (\sin 2y)\frac{\partial f}{\partial y}.$$

Your answer should be a number.

Answer 4(b)	2

$$\frac{\partial f}{\partial x} = \frac{\sec^2 x}{\tan x + \tan y}$$

$$\frac{\partial f}{\partial y} = \frac{\sec^2 y}{\tan x + \tan y}$$

$$\sin 2x \frac{\partial f}{\partial x} + \sin 2y \frac{\partial f}{\partial y} = \frac{2\tan x}{\tan x + \tan y} + \frac{2\tan y}{\tan x + \tan y}$$

$$= \underline{\underline{2}}$$

Question 5 (a) [5 marks]

Let n be a positive integer. Find the directional derivative of

$$f(x, y) = x^2 - xy + y^n$$

at the point $(2, 1)$ in the direction of the vector joining the point $(2, 1)$ to the point $(6, 4)$. Express your answer in terms of n.

Answer 5(a)	$\frac{3n+6}{5}$

$$\vec{u} = \frac{(6,4)-(2,1)}{\|(6,4)-(2,1)\|} = \frac{(4,3)}{5} = \left(\frac{4}{5}, \frac{3}{5}\right)$$

$$\nabla f = (f_x, f_y) = (2x-y, -x+ny^{n-1})$$

$$\nabla f(2,1) = (3, n-2)$$

$$D_{\vec{u}} f(2,1) = \nabla f(2,1) \cdot \vec{u}$$

$$= \frac{12}{5} + \frac{3(\pi - 2)}{5}$$

$$= \frac{3\pi + 6}{5}$$

Question 5 (b) [5 marks]

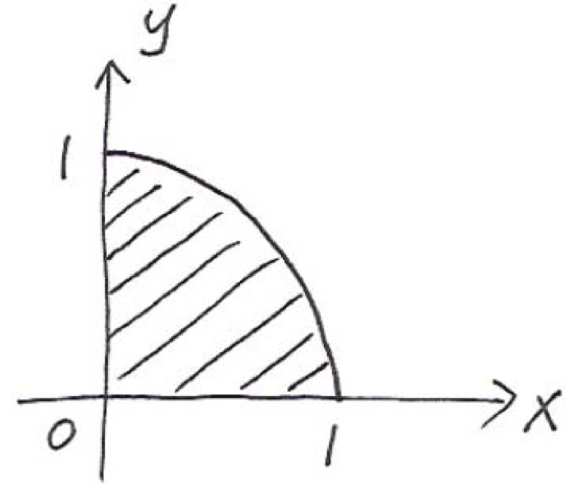
Evaluate

$$\iint_D x\,dA,$$

where D is the finite plane region in the first quadrant bounded by the two coordinate axes and the curve $y = 1 - x^2$.

Answer 5(b)	$\frac{1}{4}$

$$\iint_D x\,dx\,dy$$



$$= \int_0^1 \int_0^{1-x^2} x\,dy\,dx$$

$$= \int_0^1 \left[x\,y\right]_{y=0}^{y=1-x^2} dx$$

$$= \int_0^1 x(1-x^2)\,dx$$

$$= \int_0^1 (x - x^3)\,dx = \left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_0^1 = \underline{\underline{\frac{1}{4}}}$$

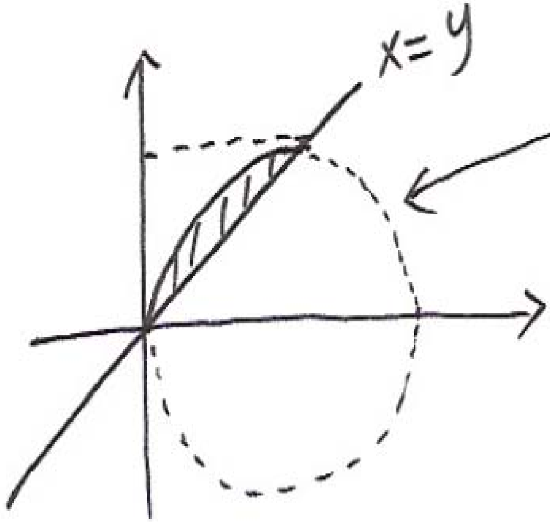
Question 6 (a) [5 marks]

Find the **exact** value of the integral

$$\int_0^1 \int_{1-\sqrt{1-y^2}}^{y} y e^{\left(x^2 - \frac{2}{3}x^3\right)} \, dx \, dy.$$

Express your answer in terms of e.

Answer 6(a)	$\frac{1}{2}(e^{1/3} - 1)$



$$x = 1-\sqrt{1-y^2} \iff \sqrt{1-y^2} = 1-x$$

$$\iff 1-y^2 = (1-x)^2$$

$$= 1-2x+x^2$$

$$\iff y^2 = 2x - x^2$$

$$\text{Given integral} = \int_0^1 \int_x^{\sqrt{2x-x^2}} y e^{(x^2-\frac{2}{3}x^3)}\,dy\,dx$$

$$= \int_0^1 \left[\frac{1}{2} y^2 e^{(x^2-\frac{2}{3}x^3)}\right]_{y=x}^{y=\sqrt{2x-x^2}} dx$$

$$= \frac{1}{2}\int_0^1 (2x-2x^2)\, e^{(x^2-\frac{2}{3}x^3)}\,dx$$

$$= \frac{1}{2} e^{(x^2-\frac{2}{3}x^3)}\Big|_0^1$$

$$= \underline{\underline{\frac{1}{2}\left(e^{1/3}-1\right)}}$$

Question 6 (b) [5 marks]

Let a be a positive constant. Evaluate the line integral

$$\int_C (x^2 + y^2 + z^2)\, ds,$$

where C is the circular helix given by $x = a\cos t$, $y = a\sin t$, $z = t$, $0 \le t \le a$.

Answer 6(b)	$\frac{4}{3}a^3\sqrt{1+a^2}$

$$C: \vec{r}(t) = (a\cos t,\ a\sin t,\ t)$$

$$\vec{r}\,'(t) = (-a\sin t,\ a\cos t,\ 1)$$

$$\|\vec{r}\,'(t)\| = \sqrt{1+a^2}$$

$$\int_C (x^2+y^2+z^2)\,ds = \int_0^a (a^2+t^2)\sqrt{1+a^2}\,dt$$

$$= \sqrt{1+a^2}\left[a^2 t + \frac{1}{3}t^3\right]_0^a$$

$$= \underline{\underline{\frac{4}{3}a^3\sqrt{1+a^2}}}$$

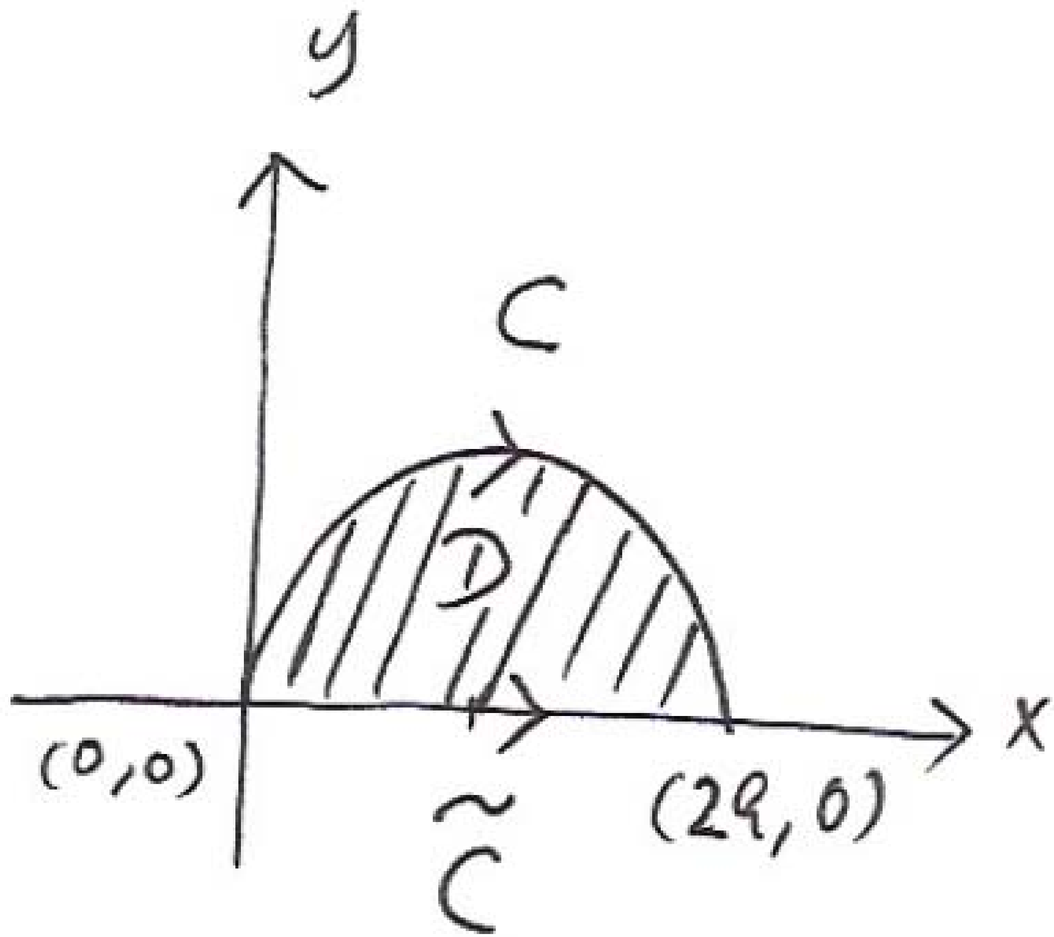
Question 7 (a) [5 marks]

Let a be a positive constant. Evaluate the line integral

$$\int_C (2xe^{\sin y} + 3x^2y^2 + ay)\,dx + (x^2e^{\sin y}\cos y + 2x^3y + 2ax + 1)\,dy,$$

where C is the semicircle, centered at $(a, 0)$ with radius a, in the first quadrant joining $(0, 0)$ to $(2a, 0)$.

Answer 7(a)	$4a^2 - \frac{1}{2}\pi a^3$



y
C
D
x
(0,0)
(2a,0)
$\tilde{C}$

Let $\tilde{C} : \vec{r}(t) = (t, 0), \quad 0 \le t \le 2a.$

Then $\partial D = \tilde{C} - C$

Apply Green's Theorem to D:

Let $P = 2x e^{\sin y} + 3x^2 y^2 + ay$

$Q = x^2 e^{\sin y} \cos y + 2x^3 y + 2ax + 1$

$$\oint_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D (2xe^{\sin y}\cos y + 6x^2y + 2a)$$
$$- (2xe^{\sin y}\cos y + 6x^2y + a)\,dA$$
$$= \iint_D a\,dA = a(\text{area } D) = \frac{1}{2}\pi a^3$$

$$\therefore \int_{\tilde{C}} - \int_{C} Pdx + Qdy = \frac{1}{2}\pi a^3$$

$$\therefore \int_{C} Pdx + Qdy = \left\{ \int_{\tilde{C}} Pdx + Qdy \right\} - \frac{1}{2}\pi a^3 = \left\{ \int_0^{2a} 2t\, dt \right\} - \frac{1}{2}\pi a^3$$

$$= \underline{\underline{4a^2 - \frac{1}{2}\pi a^3}}$$

Question 7 (b) [5 marks]

Evaluate the surface integral

$$\iint_S \mathbf{F} \bullet d\mathbf{S},$$

where $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and S is the portion of the paraboloid $z = 1 - x^2 - y^2$ lying on and above the xy plane. The orientation of S is given by the outer normal vector.

Answer 7(b)	$\dfrac{3\pi}{2}$

$S: \vec{r}(u,v) = (u, v, 1-u^2-v^2)$

$\vec{r}_u = (1, 0, -2u), \quad \vec{r}_v = (0, 1, -2v)$

$\vec{r}_u \times \vec{r}_v = 2u\vec{i} + 2v\vec{j} + \vec{k}.$

at $(0,0,1)$, $\vec{r}_u \times \vec{r}_v = \vec{k}$ points outwards.

$$\iint_S \vec{F}\cdot dS = \iint_{u^2+v^2\le 1} \{2u^2+2v^2+(1-u^2-v^2)\}\,du\,dv$$

$$= \int_0^{2\pi}\int_0^1 (1+r^2)\,r\,dr\,d\theta$$

$$= 2\pi\left[\frac{1}{2}r^2+\frac{1}{4}r^4\right]_0^1$$

$$= \underline{\underline{\frac{3\pi}{2}}}$$

Question 8 (a) [5 marks]

By using Stokes' Theorem, or otherwise, find the **exact** value of the surface integral

$$\int\int_S (\nabla \times \mathbf{F}) \bullet d\mathbf{S},$$

where S is the hemisphere $x^2 + y^2 + z^2 = 16$ lying on and above the xy plane, and $\mathbf{F} = \left(x^2 + y - 4e^z\right)\mathbf{i} + \left(3xy\cos^2 z\right)\mathbf{j} + \left(2e^{xy}\sin z + x^2yz^3\right)\mathbf{k}$. The orientation of S is given by the outer normal vector. Express your answer in terms of π.

Answer 8(a)	-16π

Let C : $\vec{r}(t) = 4\cos t\,\vec{i} + 4\sin t\,\vec{j} + 0\vec{k}$, $0 \leq t \leq 2\pi$.

Note that the orientation of C is anti-clockwise and this matches with the outer normal orientation of S.

By Stokes' Theorem

$$\iint_S (\nabla \times \vec{F}) \cdot dS = \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_0^{2\pi} \{ (16\cos^2 t + 4\sin t - 4)(-4\sin t) + 48\sin t \cos t\,(4\cos t) \} dt$$

$$= \int_0^{2\pi} (128\cos^2 t \sin t - 16\sin^2 t + 16\sin t)\, dt$$

$$= \left[-\frac{128}{3}\cos^3 t \right]_0^{2\pi} - 8\int_0^{2\pi} (1 - \cos 2t)\, dt$$

$$= \underline{\underline{-16\pi}}$$

Question 8 (b) [5 marks]

Find a solution of the form $u(x, y) = F(ax + y)$, where a is a constant and F is a differentiable single variable function, to the partial differential equation

$$u_x - 2u_y = 0,$$

that satisfies the condition $u(x, 0) = \cos x$.

Answer 8(b)	$u(x,y) = \cos \frac{2x+y}{2}$

$$u_x = aF'(ax+y)$$

$$u_y = F'(ax+y)$$

$$u_x - 2u_y = 0 \Rightarrow aF'(ax+y) - 2F'(ax+y) = 0$$

$$\Rightarrow a = 2$$

$$\therefore \quad u(x,y) = F(2x+y)$$

$$u(x,0)=\cos x \Rightarrow F(2x)=\cos x$$

$$\Rightarrow F(x)=\cos\frac{x}{2}$$

$$\therefore u(x,y)=F(2x+y)=\cos\frac{2x+y}{2}$$

$$\therefore \underline{\underline{u(x,y)=\cos\frac{2x+y}{2}}}$$