MA1506

Mathematics II

Systems of First Order ODEs

7.1 Romeo and Juliet

R(t), Romeo's feelings

J(t), Juliet's feelings

Initial feelings

$$\frac{dR}{dt} = aJ, \qquad R(0) = \alpha$$
$$\frac{dJ}{dt} = -bR, \qquad J(0) = \beta$$

a, b >0

System of simultaneous first order ODE

Linear, i.e. easy to solve

7.1 Romeo and Juliet

$$\begin{aligned} \frac{dR}{dt} &= aJ, \qquad R(0) = \alpha \\ \frac{dJ}{dt} &= -bR, \qquad J(0) = \beta \end{aligned}$$

could be complex

Try $R = Ae^{\lambda t}$, $J = Be^{\lambda t}$,

But final solution must be real

$$\begin{aligned} A\lambda e^{\lambda t} &= aBe^{\lambda t} \\ B\lambda e^{\lambda t} &= -bAe^{\lambda t} \end{aligned} \Rightarrow \begin{aligned} A\lambda &= aB \\ B\lambda &= -bA. \end{aligned}$$

$$\boxed{\lambda^2 = -ab < 0}$$

7.1 Romeo and Juliet $R = Ae^{\lambda t}, J = Be^{\lambda t},$

$$\boxed{\lambda^2 = -ab < 0}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\Rightarrow \begin{aligned} R(t) &= C\cos(\sqrt{ab}t) + D\sin(\sqrt{ab}t), \\ J(t) &= E\cos(\sqrt{ab}t) + F\sin(\sqrt{ab}t). \end{aligned}$$

$$\begin{aligned} R(0) &= C & \dot{R}(0) &= \sqrt{ab}D \\ J(0) &= E & \dot{J}(0) &= \sqrt{ab}F \end{aligned}$$

7.1 Romeo and Juliet

$$R = Ae^{\lambda t}, J = Be^{\lambda t},$$

$$\begin{aligned}\frac{dR}{dt} &= aJ, \qquad R(0) = \alpha \\ \frac{dJ}{dt} &= -bR, \qquad J(0) = \beta\end{aligned}$$

$$R(0) = C = \alpha \qquad D = \frac{\dot{R}(0)}{\sqrt{ab}} = \beta\sqrt{\frac{a}{b}}$$

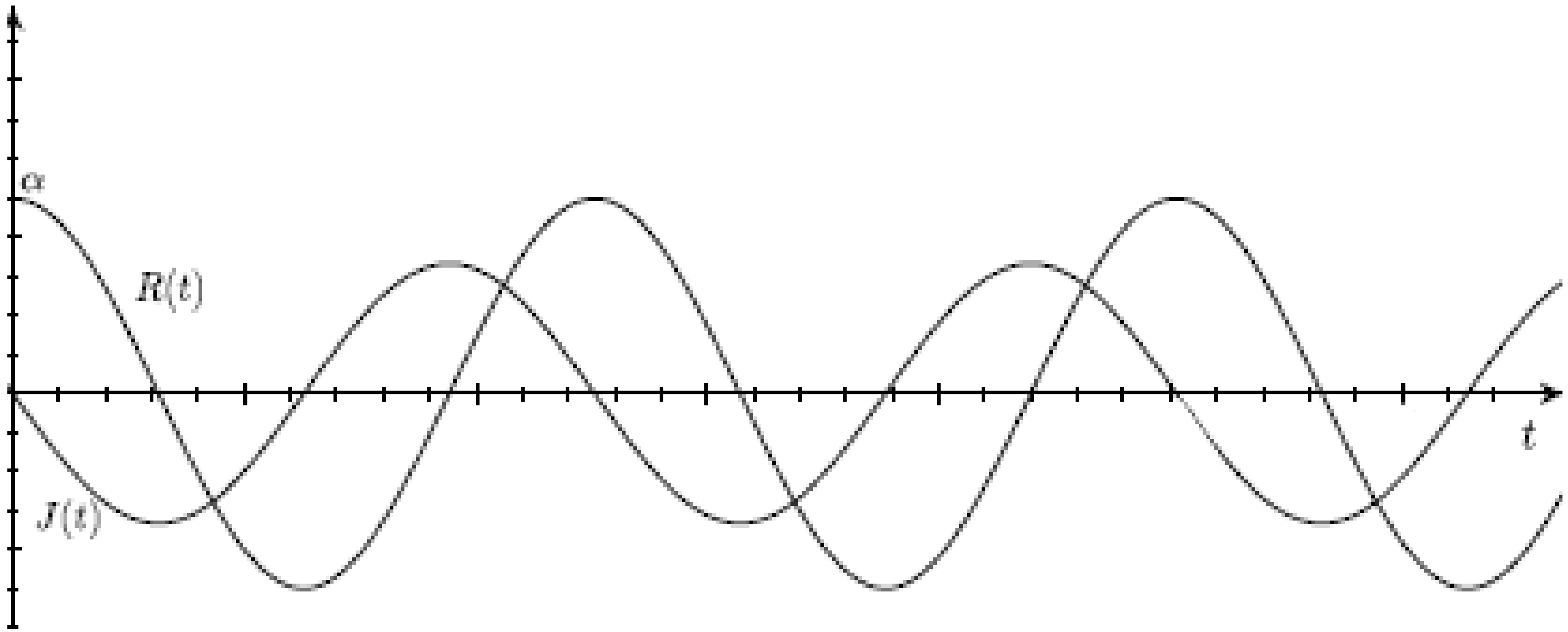
$$J(0) = E = \beta \qquad F = \frac{\dot{J}(0)}{\sqrt{ab}} = -\alpha\sqrt{\frac{b}{a}}$$

$$\begin{aligned}R(t) &= \alpha\cos(\sqrt{ab}t) + \beta\sqrt{\frac{a}{b}}\sin(\sqrt{ab}t), \\ J(t) &= \beta\cos(\sqrt{ab}t) - \alpha\sqrt{\frac{b}{a}}\sin(\sqrt{ab}t).\end{aligned}$$

7.1 Romeo and Juliet

$$R(t) = \alpha \cos(\sqrt{ab}t),$$
$$J(t) = -\alpha\sqrt{\frac{b}{a}}\sin(\sqrt{ab}t).$$

$$\boxed{\alpha > 0, \beta = 0}$$



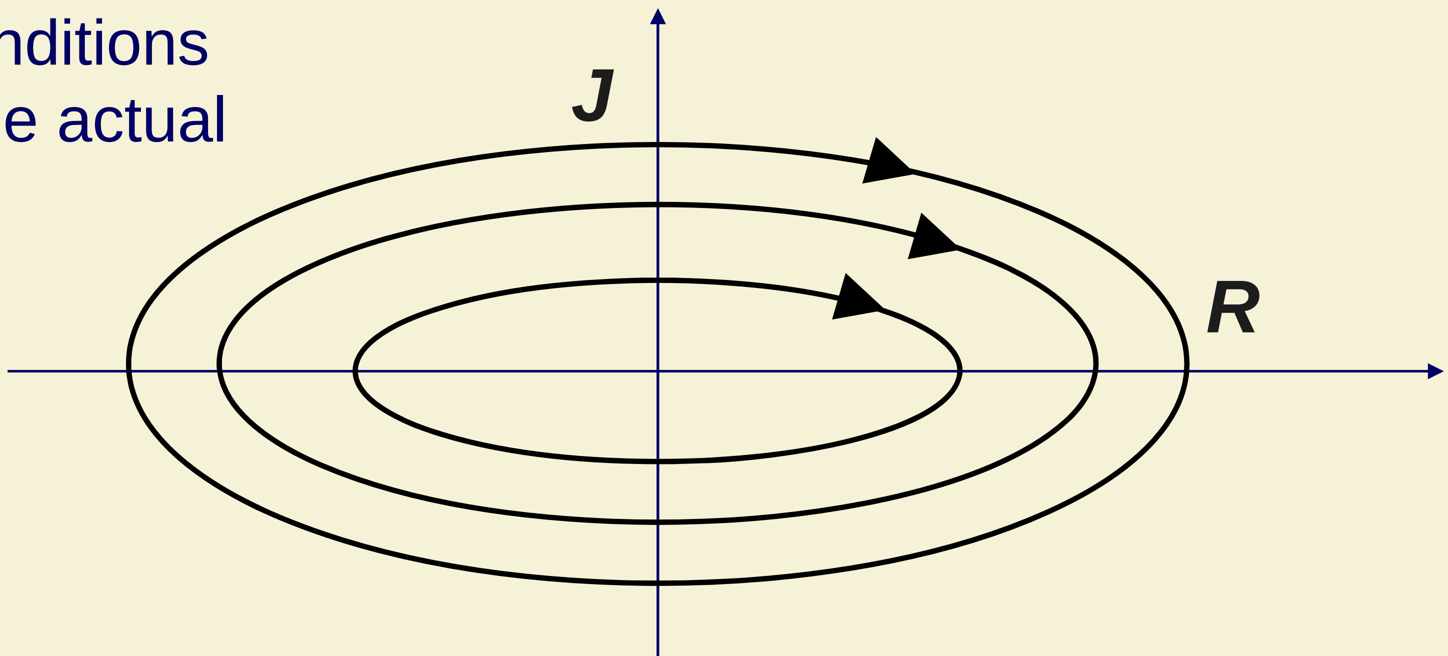
7.1 Romeo and Juliet $R = Ae^{\lambda t}, J = Be^{\lambda t},$

$$\left.\begin{aligned} R(t) &= \alpha \cos(\sqrt{ab}t), \\ J(t) &= -\alpha\sqrt{\frac{b}{a}}\sin(\sqrt{ab}t). \\ \alpha > 0, \beta = 0 & \end{aligned}\right\}$$

Eliminate ***t***

$$\frac{R^2}{R_{max}^2} + \frac{J^2}{J_{max}^2} = 1$$

Initial conditions determine actual ellipse

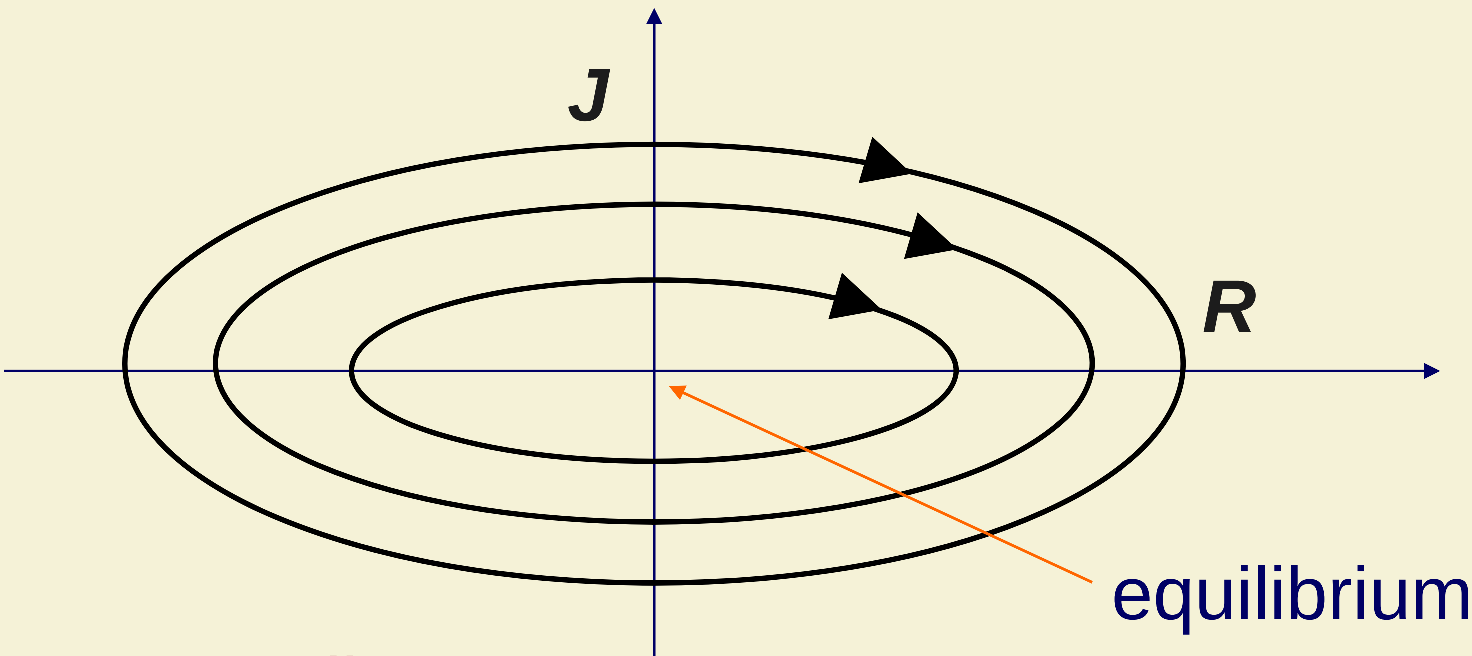


phase plane diagram

7.1 Romeo and Juliet $R = Ae^{\lambda t}, J = Be^{\lambda t},$

Can Romeo and Juliet have a steady relationship?

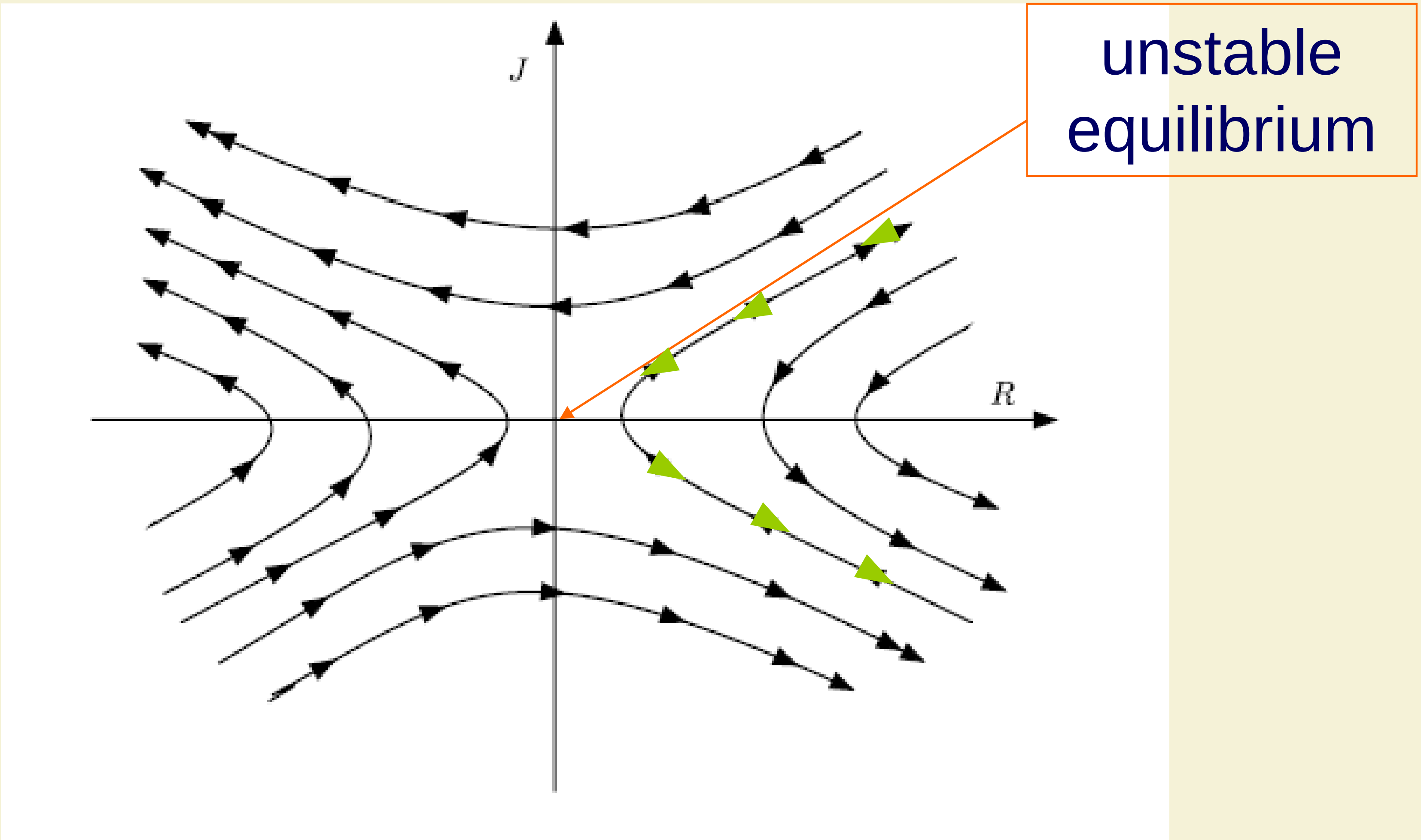
Stable Equilibrium ***R = J = 0***



phase plane diagram

Phase Plane (qualitative vs quantitative)

Easy to detect equilibrium



7.2 Solving Linear System of ODEs

$\frac{dx}{dt} = ax + by$, $\frac{dy}{dt} = cx + dy$ *a,b,c,d* constants

$$\boxed{\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}}$$

Try $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix} = e^{rt}\vec{u}_0$ $\vec{u}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ constant

$$re^{rt}\vec{u}_0 = Be^{rt}\vec{u}_0 \Rightarrow \boxed{B\vec{u}_0 = r\vec{u}_0.}$$

eigenvalue/eigenvector

Linear Systems of ODEs

$$(B - rI)\vec{u}_0 = \vec{0} \Rightarrow \det(B - rI) = 0$$

$$\Rightarrow \begin{vmatrix} a - r & b \\ c & d - r \end{vmatrix} = (a - r)(d - r) - bc = 0$$

$$\Rightarrow r^2 - (a + d)r + ad - bc = 0$$

$$\Rightarrow r = \frac{1}{2}\left(a + d \pm \sqrt{(a + d)^2 - 4(ad - bc)}\right)$$

$$\Rightarrow \boxed{r = \frac{1}{2}\left(Tr(B) \pm \sqrt{(Tr(B))^2 - 4\det(B)}\right)}$$

Linear Systems of ODEs

$$\boxed{r = \frac{1}{2}\left(Tr(B) \pm \sqrt{(Tr(B))^2 - 4\,\mathrm{det}(B)}\right)}$$

unless $(Tr(B))^2 = 4\,\mathrm{det}\,B$

Two solutions: r_1 , r_2 possibly complex

$$\vec{u}(t) = c_1 e^{r_1 t}\vec{u}_1 + c_2 e^{r_2 t}\vec{u}_2$$

Example 1

$$\left.\begin{aligned} \frac{dx}{dt} &= -4x+3y \\ \frac{dy}{dt} &= -2x+y \end{aligned}\right\} \quad B = \begin{bmatrix} -4 & 3 \\ -2 & 1 \end{bmatrix}$$

$$Tr(B) = -3, \det B = 2$$

$$r = \frac{1}{2}\left(-3 \pm \sqrt{9-8}\right) = -1, -2$$

$$\begin{bmatrix} -4 & 3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = -\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \Rightarrow -3x_0 + 3y_0 = 0.$$

$$\Rightarrow \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Example 1

$$\left.\begin{aligned}\frac{dx}{dt} &= -4x+3y\\ \frac{dy}{dt} &= -2x+y\end{aligned}\right\} \quad B=\begin{bmatrix}-4 & 3\\ -2 & 1\end{bmatrix}$$

$$Tr(B)=-3, \det B = 2$$

$$r=\frac{1}{2}\left(-3\pm\sqrt{9-8}\right)=-1,-2$$

$$\begin{bmatrix}-4 & 3\\ -2 & 1\end{bmatrix}\begin{bmatrix}x_0\\ y_0\end{bmatrix}=-2\begin{bmatrix}x_0\\ y_0\end{bmatrix}\Rightarrow -2x_0+3y_0=0.$$

$$\Rightarrow \quad \begin{bmatrix}x_0\\ y_0\end{bmatrix}=\begin{bmatrix}3\\ 2\end{bmatrix}$$

Example 1

$$\begin{aligned} \frac{dx}{dt} &= -4x + 3y \\ \frac{dy}{dt} &= -2x + y \end{aligned} \quad \Bigg\} \quad B = \begin{bmatrix} -4 & 3 \\ -2 & 1 \end{bmatrix}$$

$$Tr(B) = -3, \det B = 2$$

Gen sol $$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

arbitrary constants

Example 2

$$\left.\begin{aligned}\frac{dx}{dt} &= 4x - 5y \\ \frac{dy}{dt} &= 2x - 2y\end{aligned}\right\} \qquad B = \begin{bmatrix} 4 & -5 \\ 2 & -2 \end{bmatrix}$$

$$Tr(B) = 2, \det B = 2$$

$$r = \frac{1}{2}\left(2 \pm \sqrt{4-8}\right) = 1 \pm i$$

eigenvectors $\begin{bmatrix} 5 \\ 3-i \end{bmatrix}, \begin{bmatrix} 5 \\ 3+i \end{bmatrix}$

$$\Rightarrow \quad \vec{u}(t) = c_1 e^{(1+i)t}\begin{bmatrix} 5 \\ 3-i \end{bmatrix} + c_2 e^{(1-i)t}\begin{bmatrix} 5 \\ 3+i \end{bmatrix}.$$

Complex Numbers Note:

1) $z = x + iy \ , \ \bar{z} = x - iy$

2) $z + \bar{z} = 2x$ real

3) $\overline{ab} = \bar{a}\bar{b}$

4) $c_1 a + \bar{c_1}\bar{a}$ real

Example 2

$$\vec{u}(t) = c_1 e^{(1+i)t} \begin{bmatrix} 5 \\ 3-i \end{bmatrix} + c_2 e^{(1-i)t} \begin{bmatrix} 5 \\ 3+i \end{bmatrix}.$$

Only want real part: $c_1 = \alpha + i\beta, c_2 = \alpha - i\beta$

$$\begin{aligned} x &= 5(\alpha + i\beta)e^t(\cos t + i \sin t) \\ &\quad +5(\alpha - i\beta)e^t(\cos t - i \sin t) \end{aligned}$$

$$\Rightarrow \quad x = 10(\alpha \cos t - \beta \sin t)e^t$$

$$\begin{aligned} y &= (\alpha + i\beta)e^t(\cos t + i \sin t)(3-i) \\ &\quad +(\alpha - i\beta)e^t(\cos t - i \sin t)(3+i) \end{aligned}$$

Example 2

$$x = 10(\alpha \cos t - \beta \sin t)e^t$$

$$\begin{aligned} y &= (\alpha + i\beta)e^t(\cos t + i \sin t)(3 - i) \\ &\quad + (\alpha - i\beta)e^t(\cos t - i \sin t)(3 + i) \end{aligned}$$

$$\begin{aligned} y &= (3\alpha + \beta + 3i\beta - i\alpha)e^t(\cos t + i \sin t) \\ &\quad + (3\alpha + \beta - 3i\beta + i\alpha)e^t(\cos t - i \sin t) \end{aligned}$$

$$y = 6(\alpha \cos t - \beta \sin t)e^t + 2\left(\beta \cos t + \alpha \sin t\right)e^t$$

$$y = 2(3\alpha + \beta)e^t \cos t + 2(\alpha - 3\beta)e^t \sin t$$

Qualitative Aspects (Example 2)

$$x = 10(\alpha \cos t - \beta \sin t)e^t$$

$$y = 2(3\alpha + \beta)e^t \cos t + 2(\alpha - 3\beta)e^t \sin t$$

$$e^t, \underbrace{e^{-t}, \sin t, \cos t}$$

Bounded as $t \to \infty$ Stable behaviour

slight perturbation is ok

$e^t \to \infty$ as $t \to \infty$ unstable behaviour

Example 1

$$\left.\begin{aligned}\frac{dx}{dt} &= -4x+3y\\ \frac{dy}{dt} &= -2x+y\end{aligned}\right\}\qquad B=\begin{bmatrix}-4 & 3\\ -2 & 1\end{bmatrix}$$

$$Tr(B)=-3,\det B=2$$

Gen sol $\begin{bmatrix}x\\ y\end{bmatrix}=c_1e^{-t}\begin{bmatrix}1\\ 1\end{bmatrix}+c_2e^{-2t}\begin{bmatrix}3\\ 2\end{bmatrix}$

stable solution

$e^{t},\quad e^{2t}$

$e^{-t},\quad e^{2t}$ unstable

$e^{t},\quad e^{-2t}$

Qualitative Aspects

$$r = \frac{1}{2}\left(Tr(B) \pm \sqrt{(Tr(B))^2 - 4\det(B)}\right)$$

For real roots $(TrB)^2 > 4\det B$

For two negative roots 1) $TrB < 0$

2) $TrB + \sqrt{(TrB)^2 - 4\det B} < 0$

$$\sqrt{(TrB)^2 - 4\det B} < -TrB$$

$$(TrB)^2 - 4\det B < (TrB)^2$$

$$\det B > 0$$

Qualitative Aspects

Conclusion: For two real eigenvalues

$$\left.\begin{aligned} Tr B &< 0 \\ \det B &> 0 \end{aligned}\right\}$$ Stable system:

Example 2

$$\left.\begin{aligned}\frac{dx}{dt} &= 4x - 5y \\ \frac{dy}{dt} &= 2x - 2y\end{aligned}\right\} \qquad B = \begin{bmatrix} 4 & -5 \\ 2 & -2 \end{bmatrix}$$

$$Tr(B) = 2, \det B = 2$$

$$x = 10(\alpha \cos t - \beta \sin t)e^t$$

$$y = 2(3\alpha + \beta)e^t \cos t + 2(\alpha - 3\beta)e^t \sin t$$

Complex eigenvalues: $1 \pm i$

$$e^{(1\pm i)t} = e^t(\cos t \pm i \sin t)$$

For complex eigenvalues

$$r = \frac{1}{2}\left(Tr(B) \pm \sqrt{(Tr(B))^2 - 4\det(B)}\right)$$
$$= \Phi \pm i\Psi$$

$$e^{(\Phi \pm i\Psi)t} = e^{\Phi t}(\cos(\Psi t) \pm i\sin(\Psi t))$$

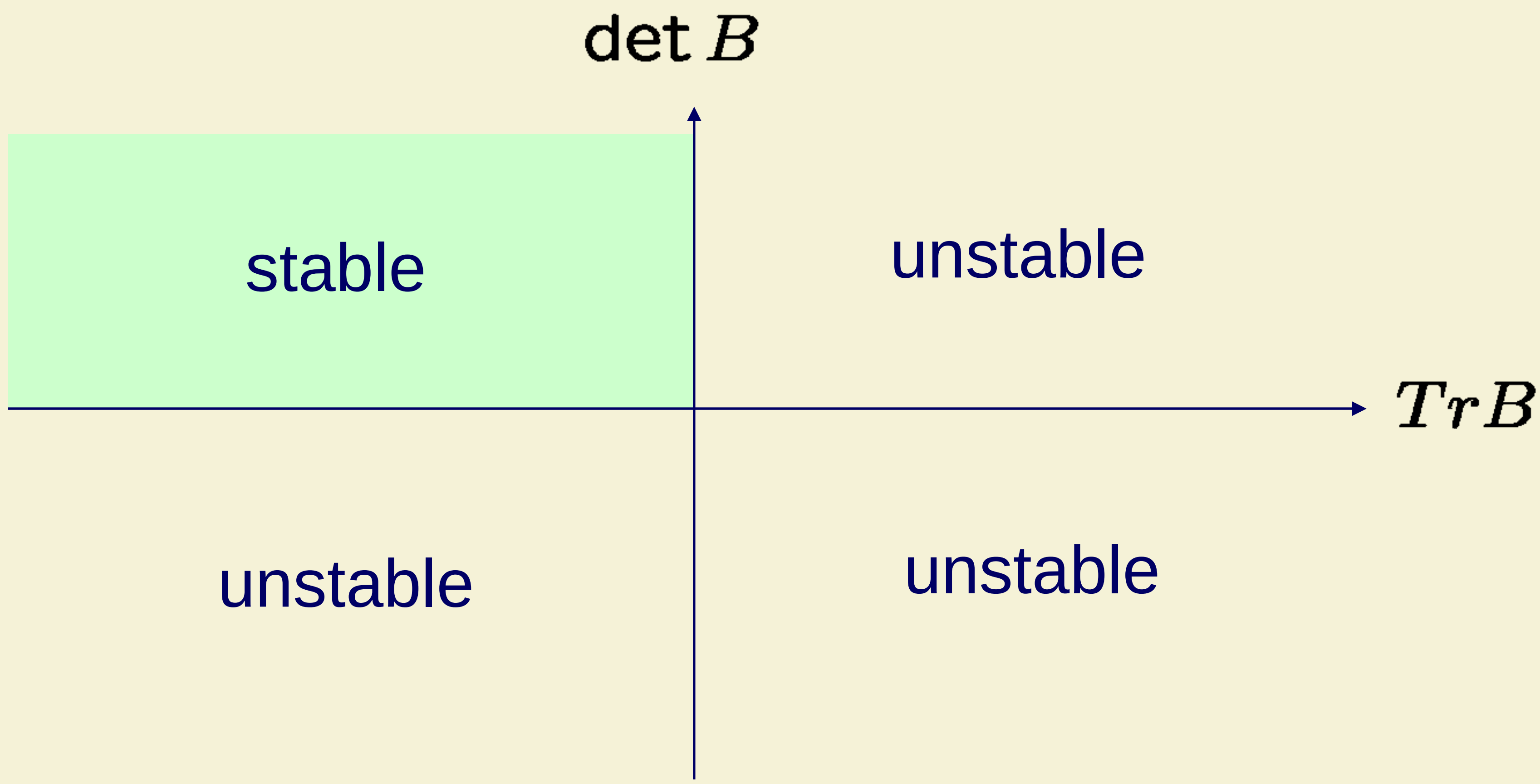
want $\Phi \leq 0$ $\Rightarrow$ $\boxed{TrB \leq 0}$

$$(TrB)^2 < 4\det B$$

Greater than 0

In all cases

$$r = \frac{1}{2}\left(Tr(B) \pm \sqrt{(Tr(B))^2 - 4\det(B)}\right)$$



Why System of 1st Order ODE?

$$\boxed{x'' - 3x' + 2x = 0}$$

$$\Rightarrow \quad \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1) = 0$$

$$x = Ae^{2t} + Be^{t}$$

$$y = x' \Rightarrow y' = x''$$

$$\boxed{\begin{aligned} x' &= y \\ y' &= 3y - 2x \end{aligned}} \quad \Rightarrow \quad \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Why always exponential function?

General form of homogenous d.e.

$$\boxed{\frac{d}{dt}\vec{u} = B\vec{u}}$$

Treat differentiation as a transformation,

What is an eigenvector?

$$D\vec{f} = \lambda\vec{f} \quad \Rightarrow \quad f = e^{\lambda t}$$

7.3 Phase Plane: Real Eigenvalues

$$\boxed{\frac{d\vec{u}}{dt} = B\vec{u}}$$

Aim: Obtain the phase plane from ***B***

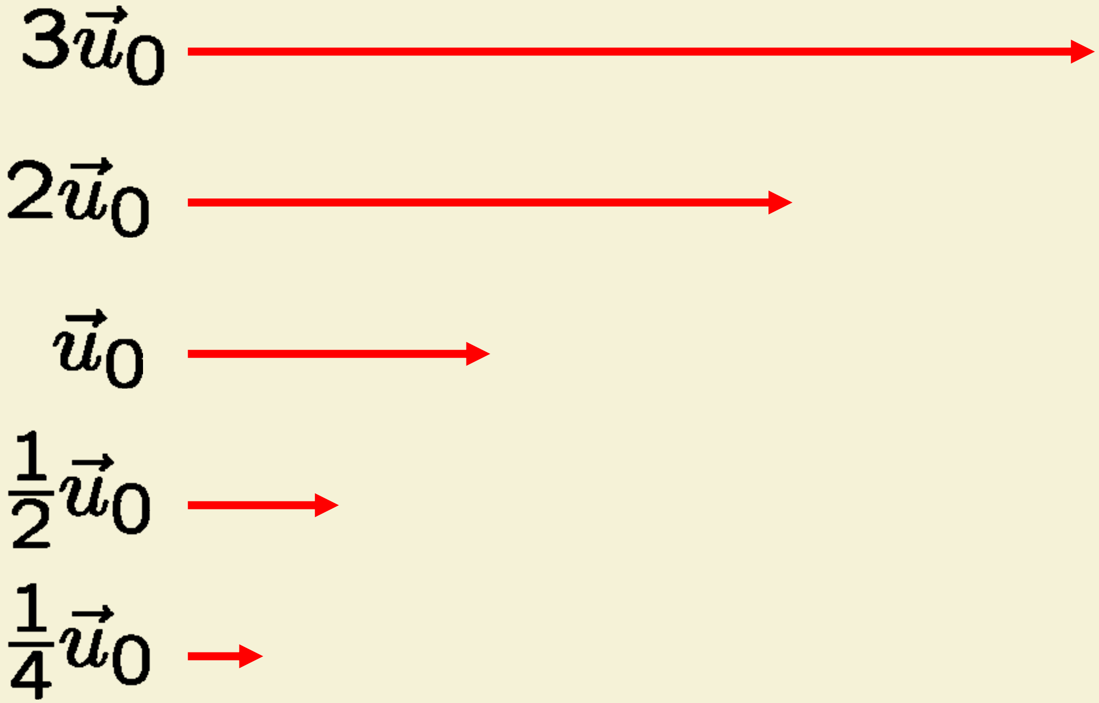
eigenvalue

Solution: $\vec{u} = \underbrace{e^{rt}}_{\text{scalar}}\vec{u}_0$

eigenvector

7.3 Phase Plane: Real Eigenvalues

Multiplying a scalar = stretching / shrinking



7.3 Phase Plane: Real Eigenvalues

$$e^{rt}\vec{u}_0$$

Scalar changes with time

Stretches if ***r > 0***

Shrinks if ***r < 0***

Eigenvectors are straight lines on phase planes

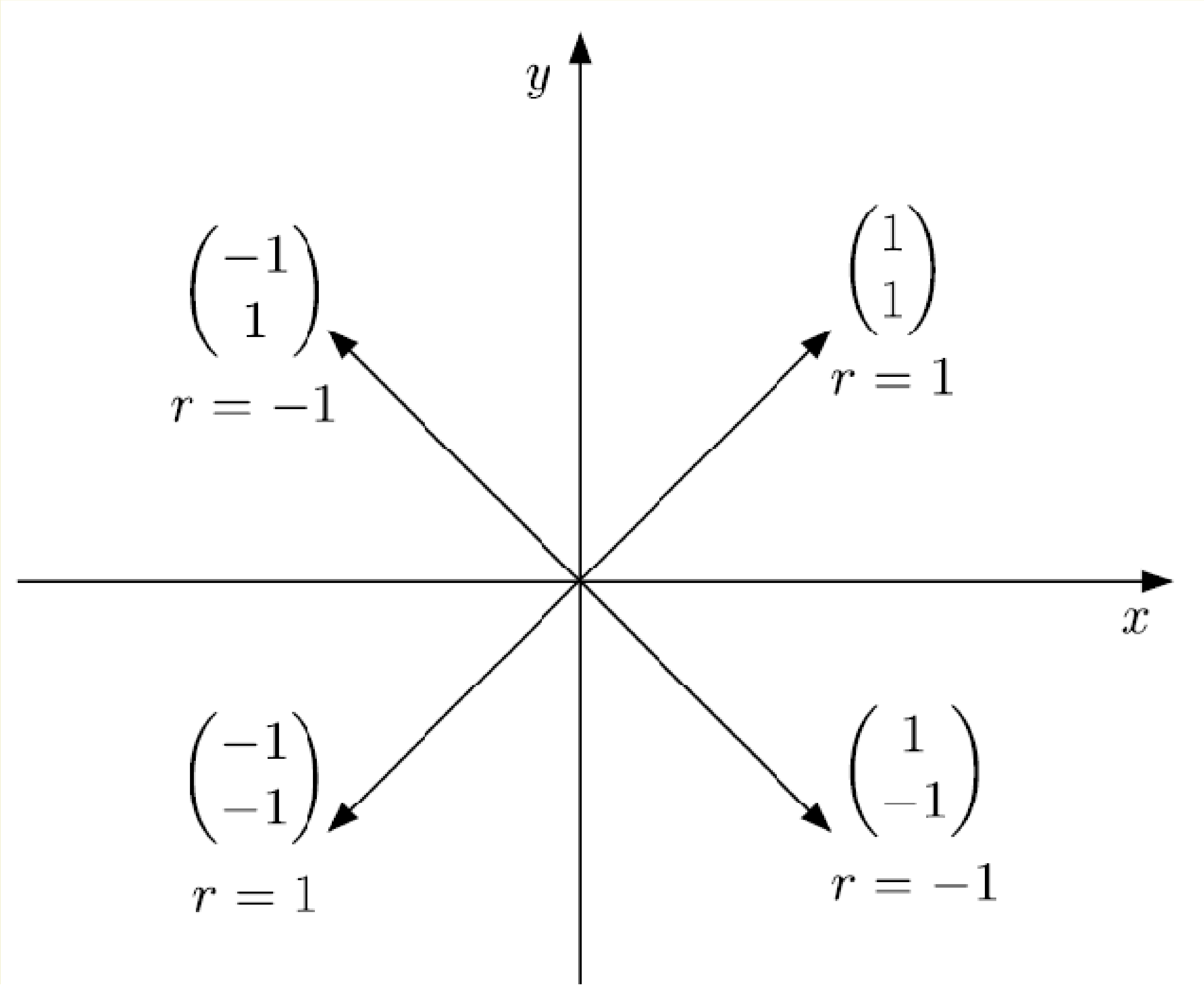
7.3 Example

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = - \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

7.3 Eigenvectors (Not phase plane)



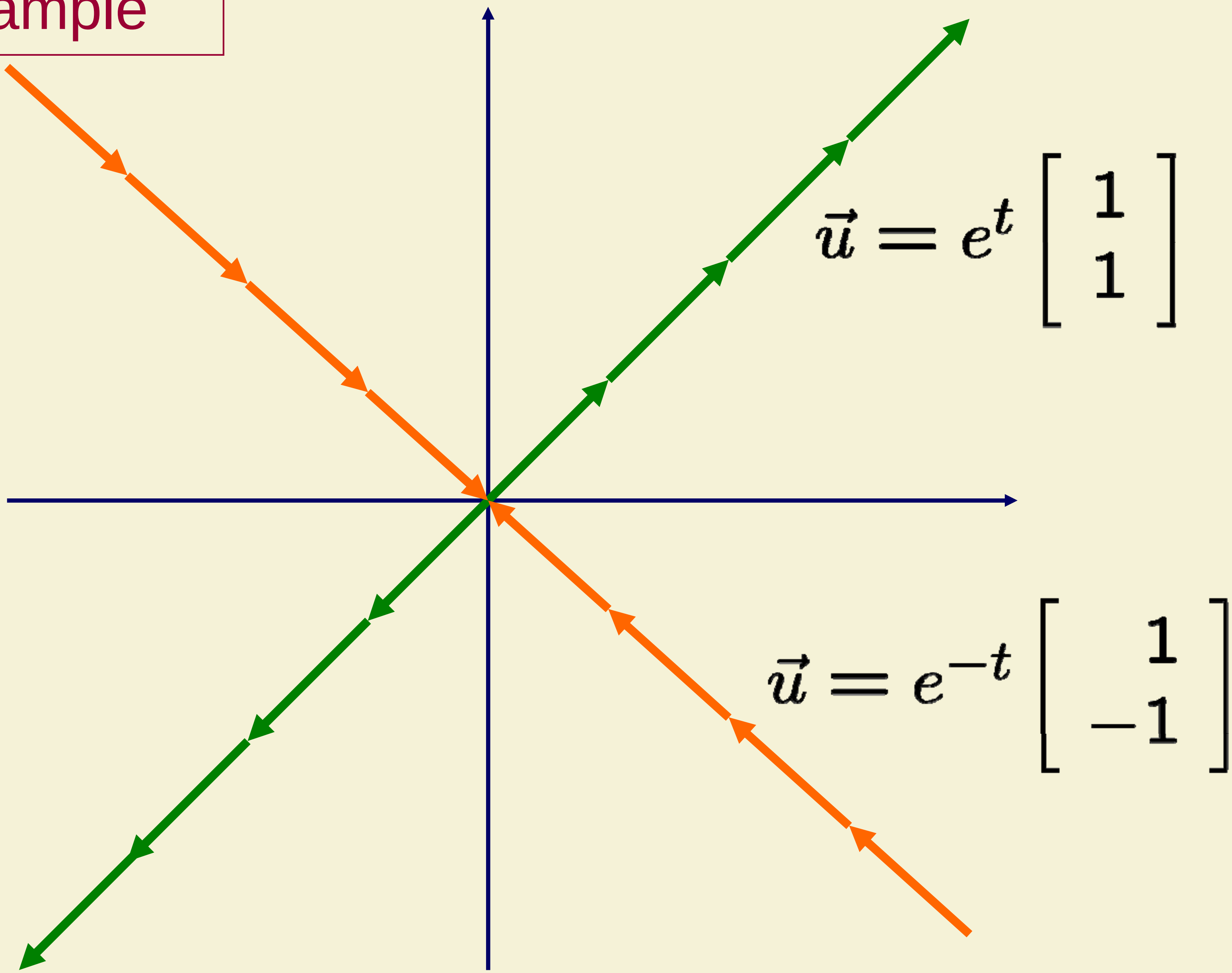
7.3 Example

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

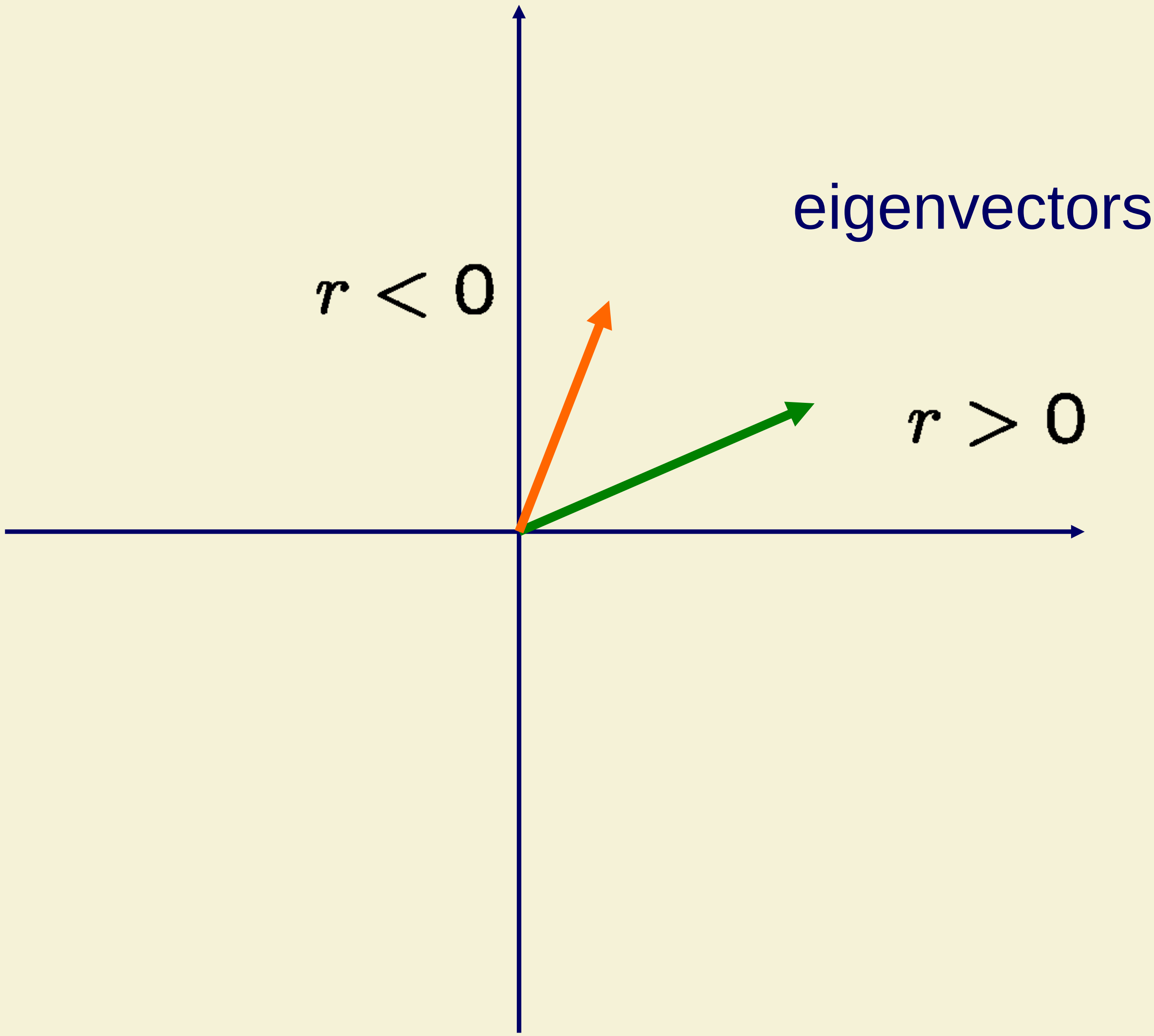
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \vec{u} = e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -\begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \vec{u} = e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

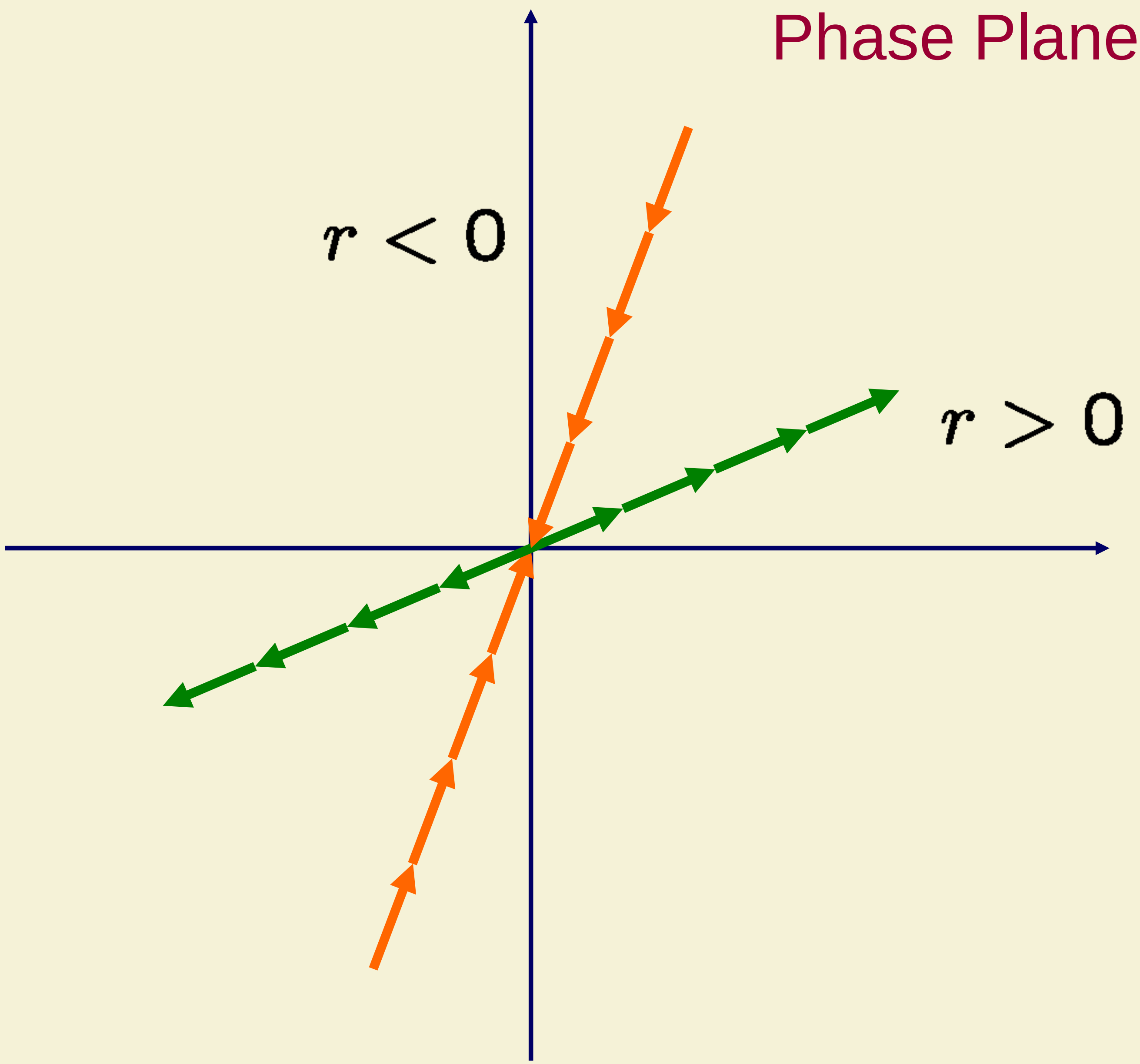
7.3 Example


$\vec{u} = e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$\vec{u} = e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Real Eigenvalues, Opposite signs

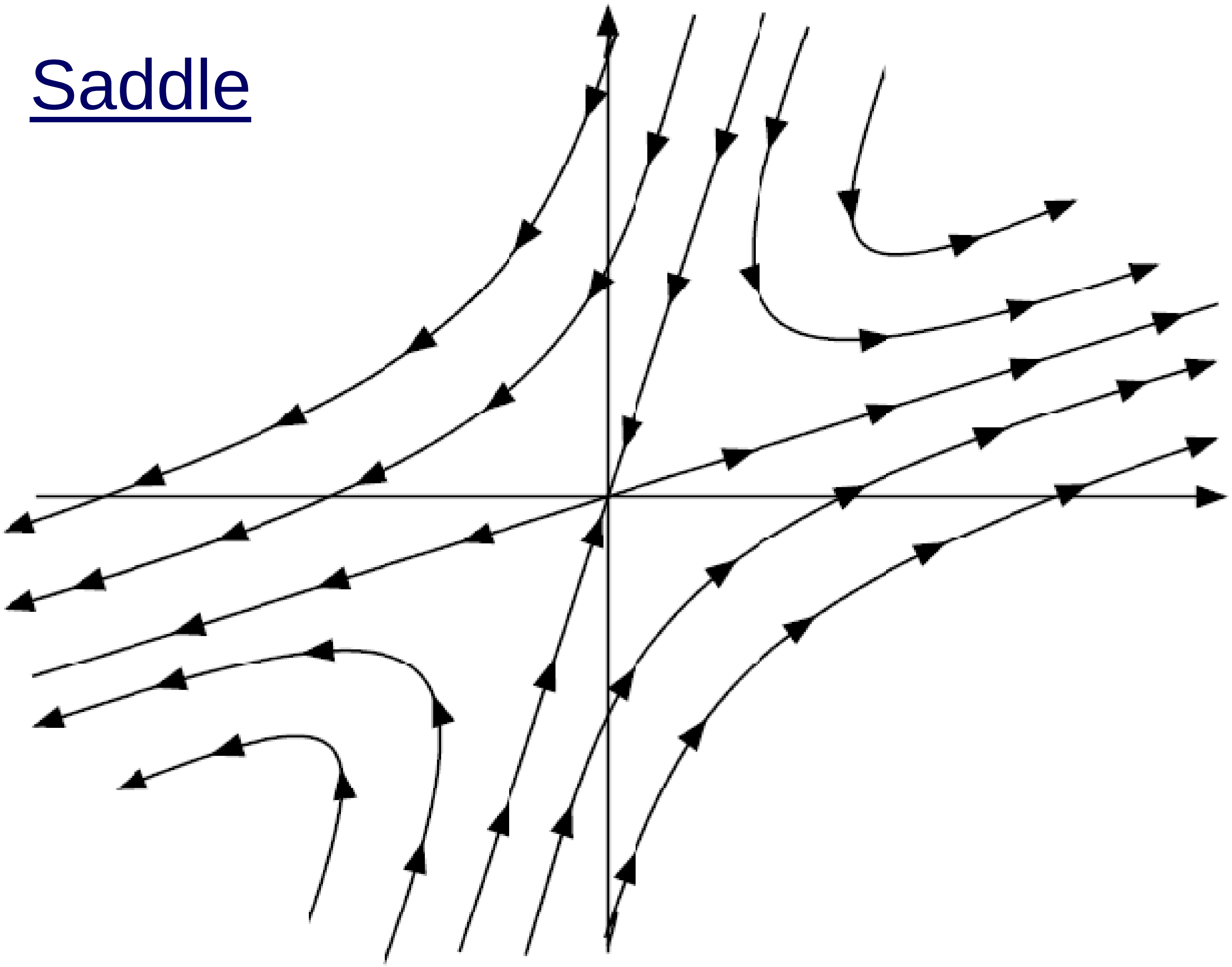


Real Eigenvalues, Opposite signs

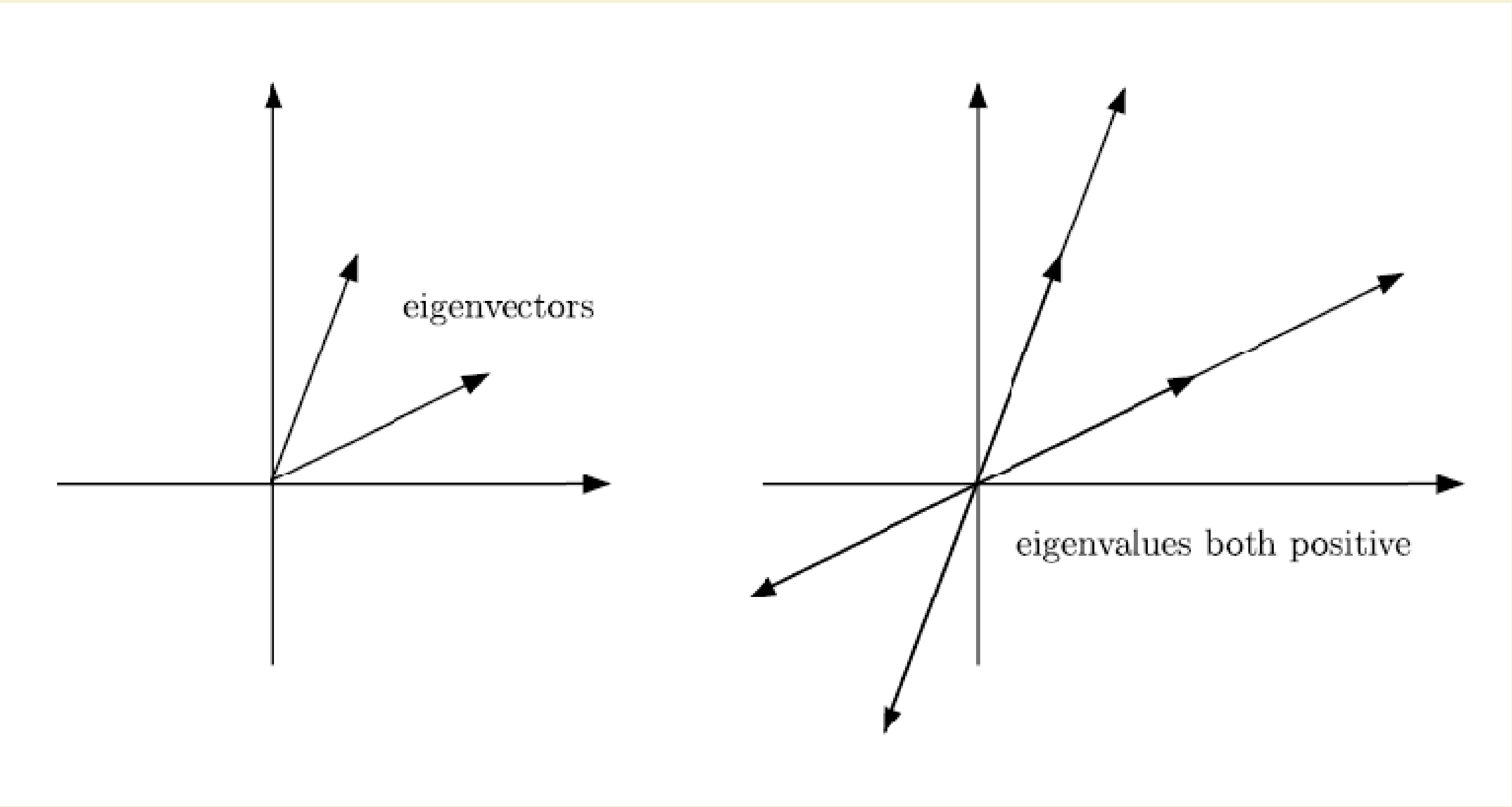


Real Eigenvalues, Opposite signs

Saddle



Real Eigenvalues, Same signs



Real Eigenvalues, Same signs

$$\frac{dx}{dt} = 2x \qquad \frac{dy}{dt} = y \quad \Rightarrow \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Eigenvalues: 2, 1

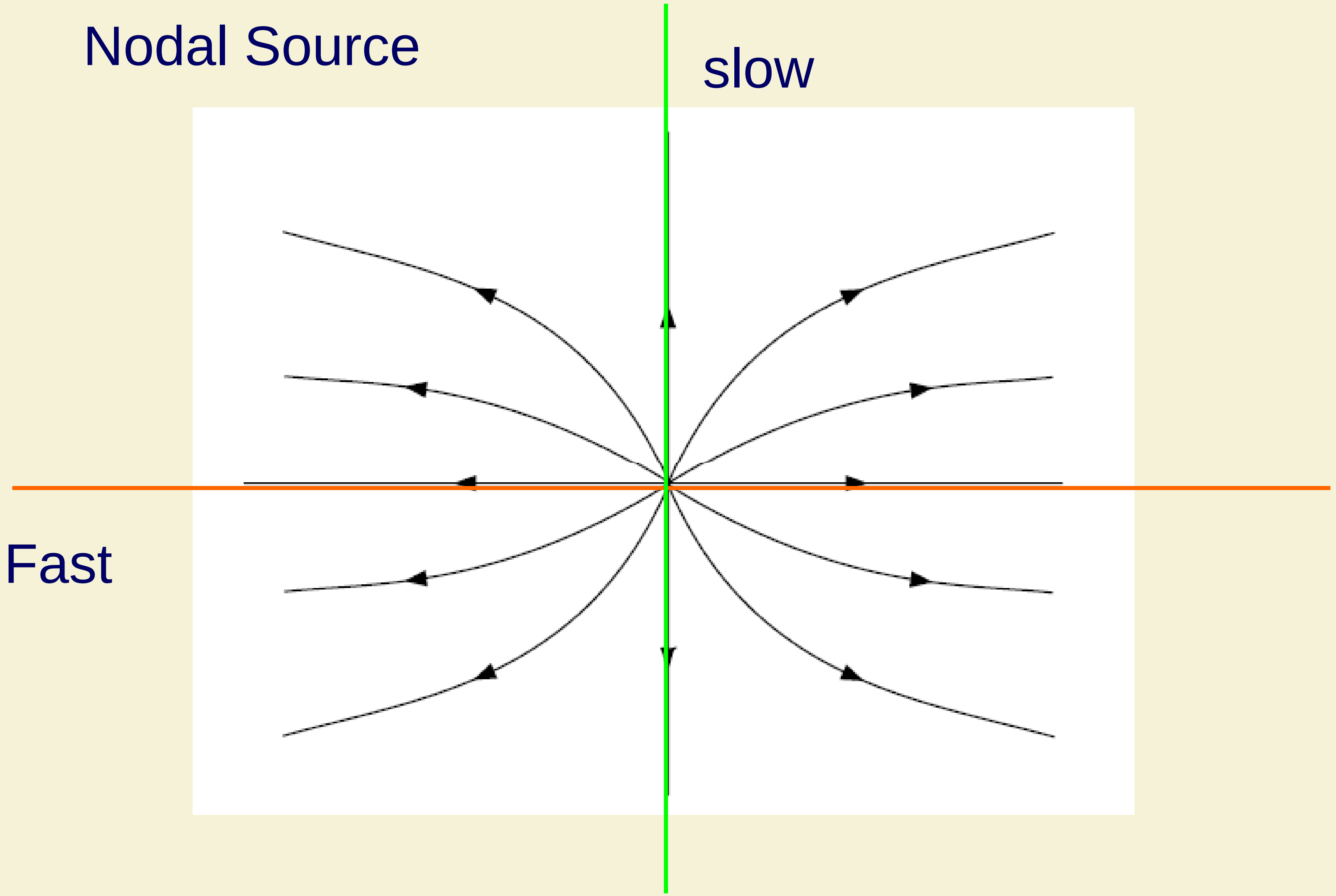
fast $\rightarrow$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Solution: $x = x_0 e^{2t}, \quad y = y_0 e^{t}$

$\Rightarrow \quad x = ky^2$ Parabolas!

Real Eigenvalues, both positive

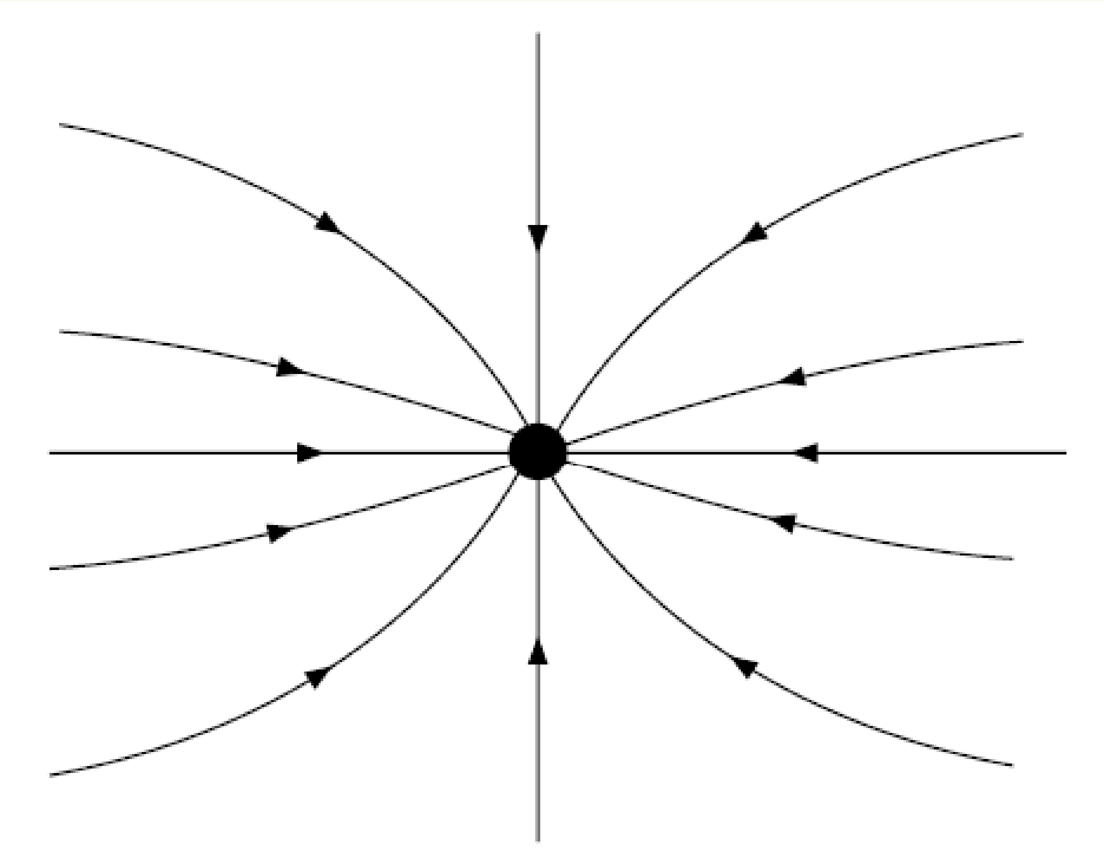
Nodal Source



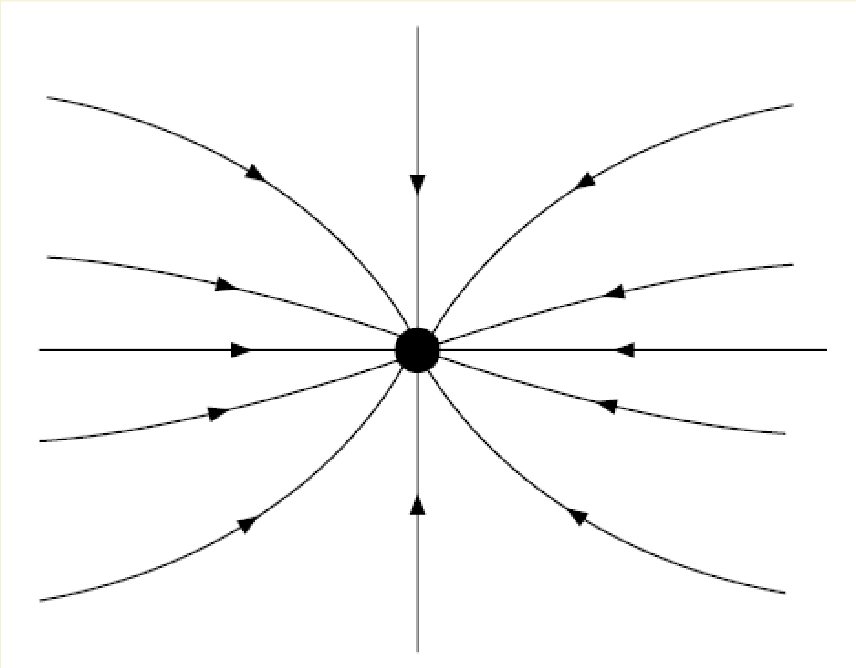
Real Eigenvalues, both negative

Nodal Sink

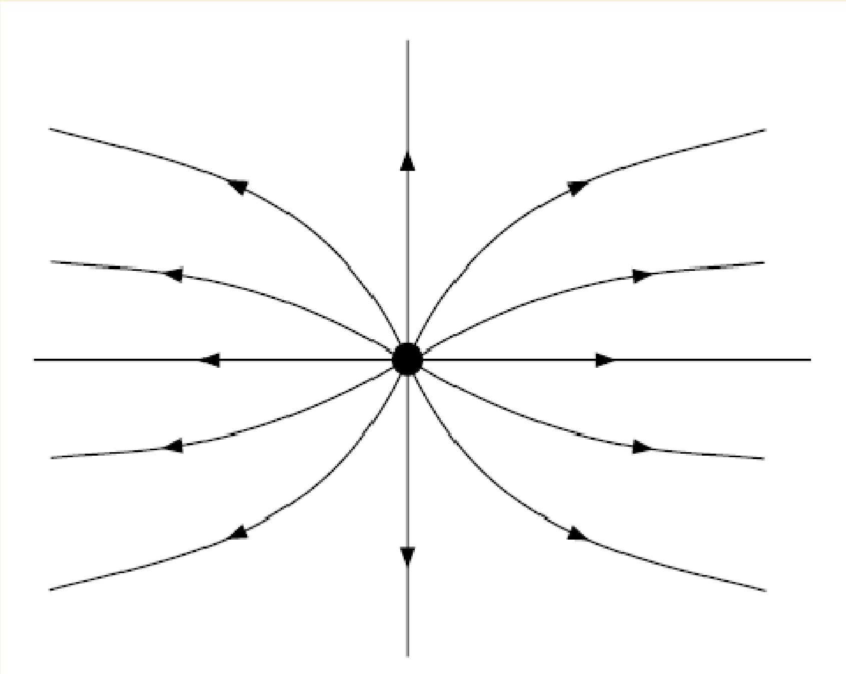
$$\frac{dx}{dt} = -2x$$
$$\frac{dy}{dt} = -y$$



Equilibriums



Stable Equilibrium



Unstable Equilibrium

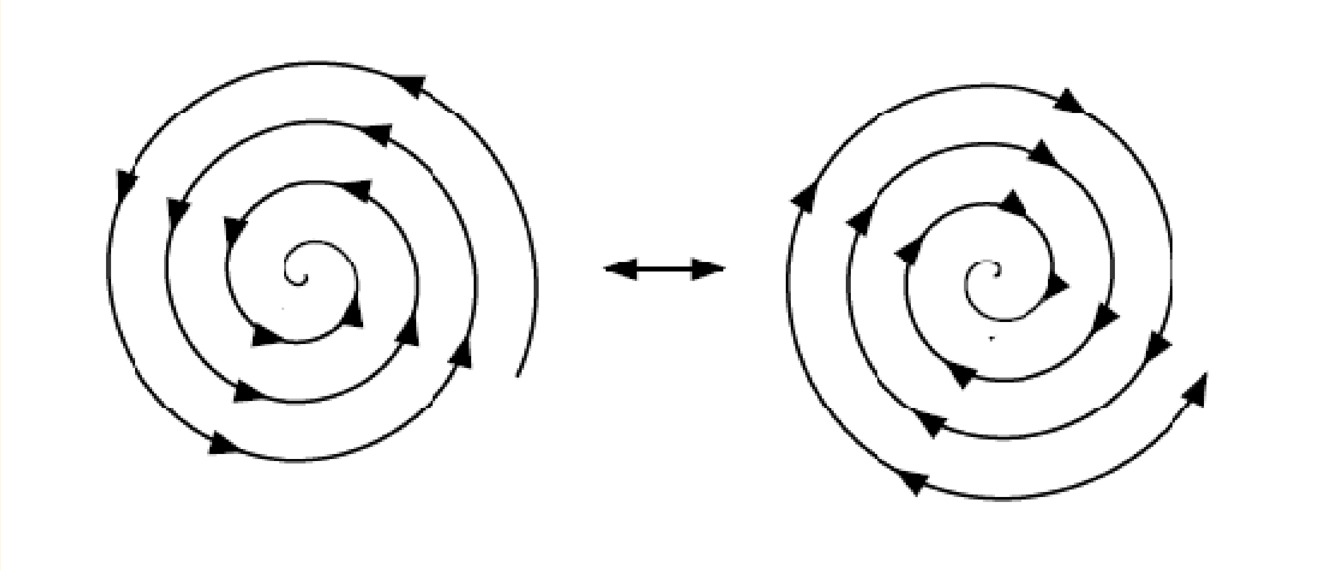
Summary

Real Eigenvalues

- Opp signs : Saddles
- Both > 0 : Nodal source
- Both < 0 : Nodal sink

7.4 Phase Plane: Complete Classification

6 Types

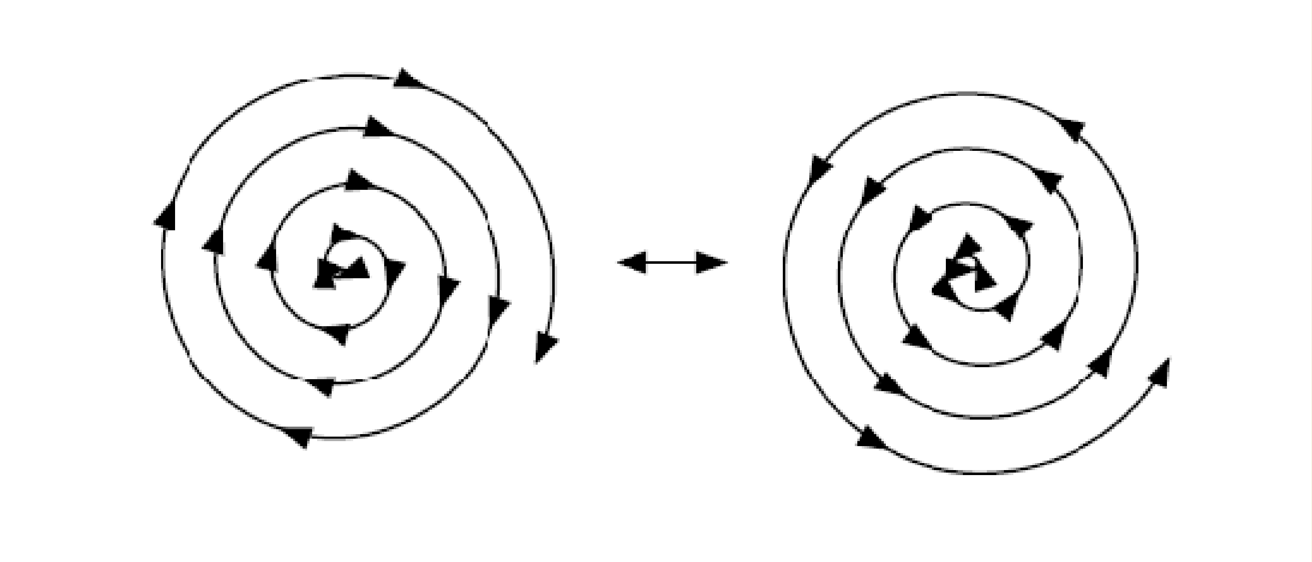


1) Spiral Sink: (Clockwise or anticlockwise)

Trajectories spiralling towards equilibirum

7.4 Phase Plane: Complete Classification

6 Types

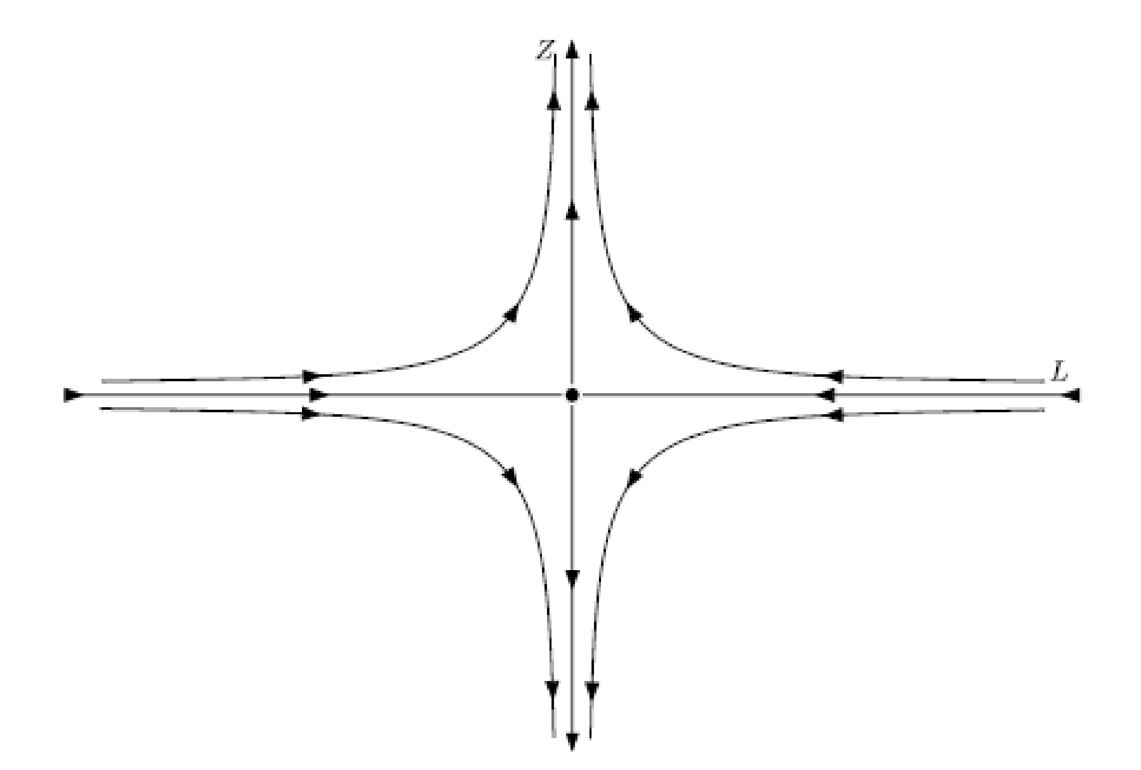


2) Spiral Source: (Clockwise or anticlockwise)

Trajectories spiralling away from equilibirum

7.4 Phase Plane: Complete Classification

6 Types



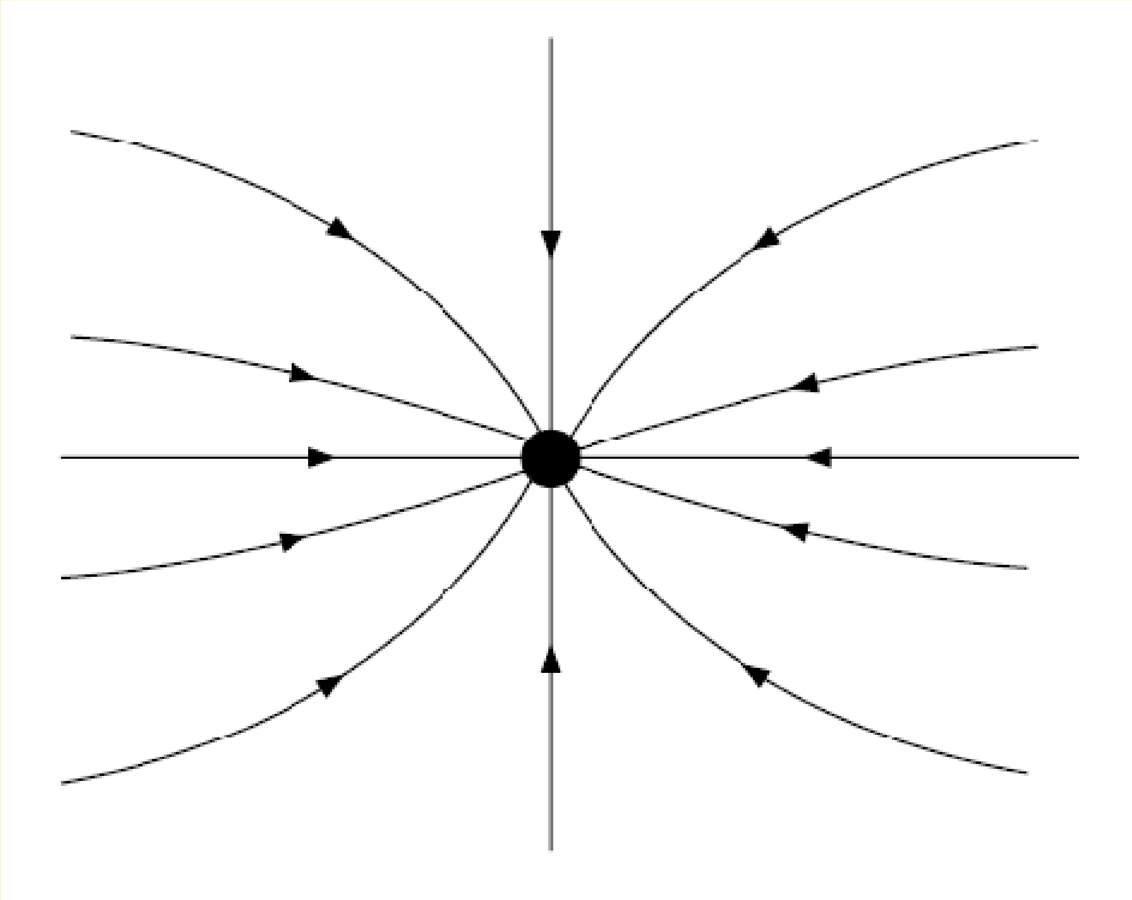
3) Saddle:

Some Trajectories towards equilibirum

Some Trajectories away from equilibirum

7.4 Phase Plane: Complete Classification

6 Types

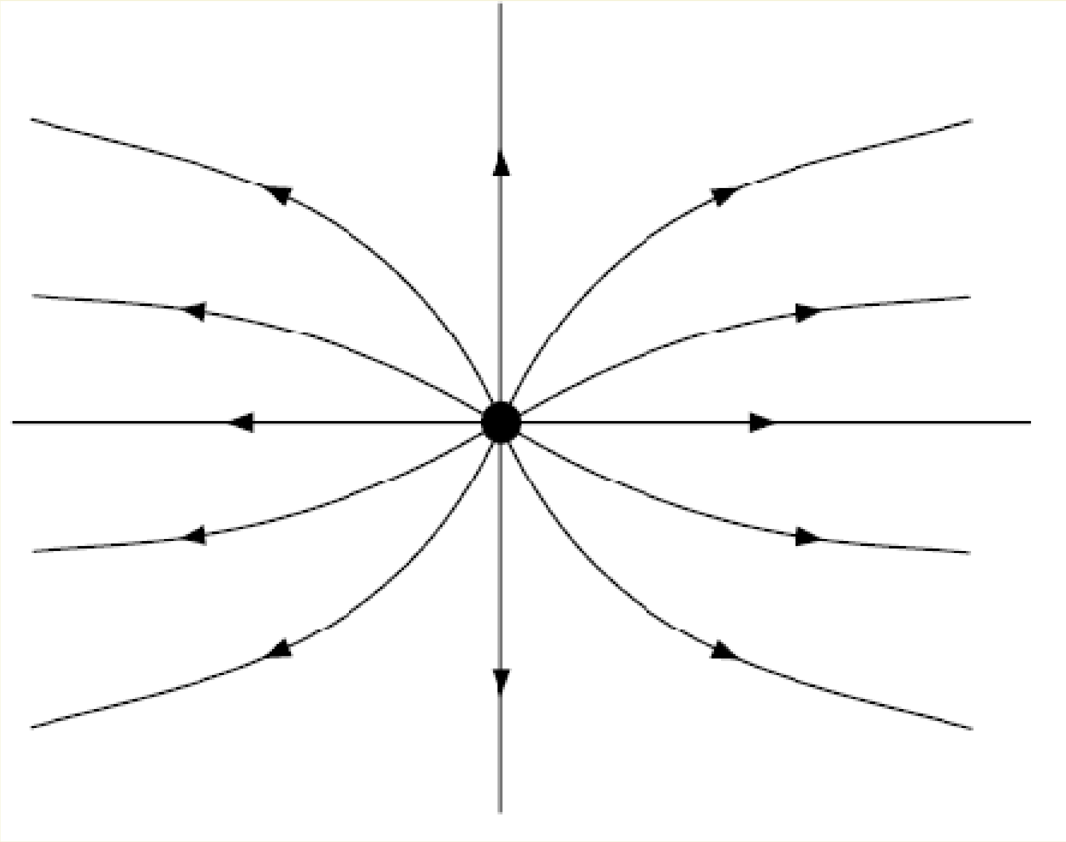


4) Nodal Sink:

Trajectories towards equilibirum

7.4 Phase Plane: Complete Classification

6 Types

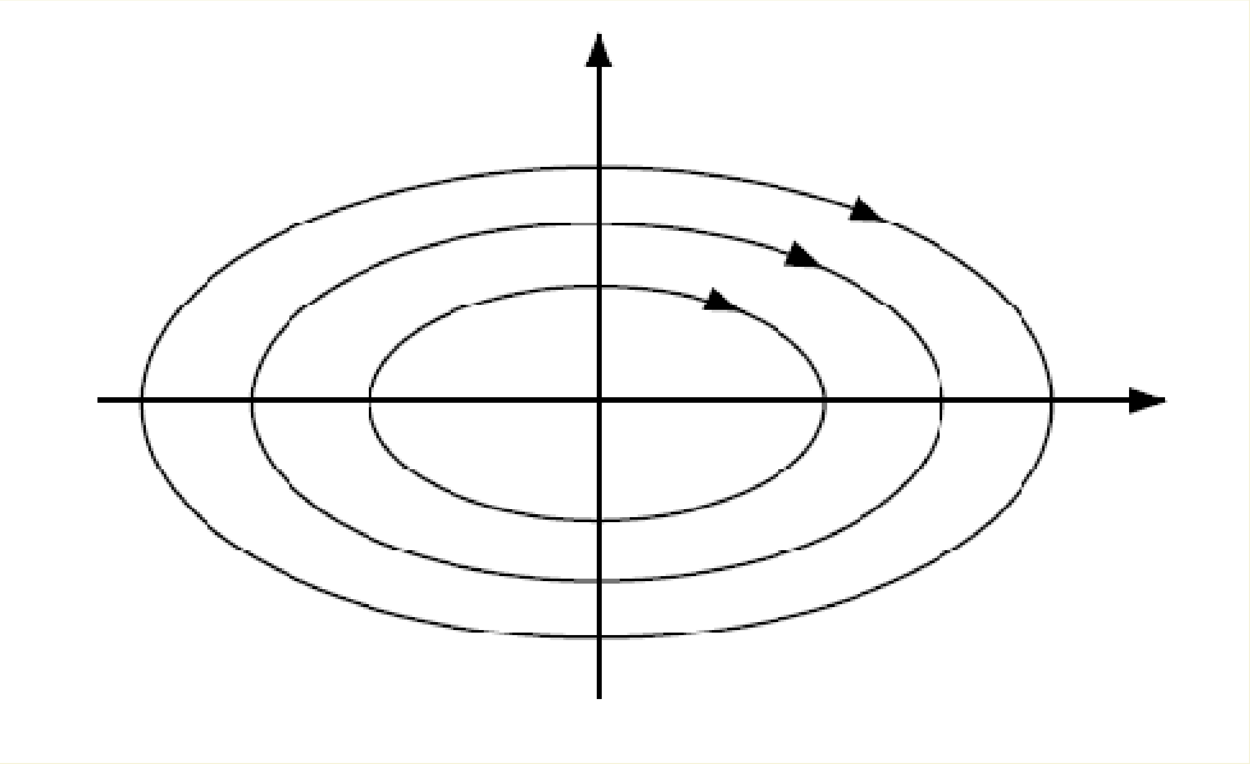


5) Nodal Source:

Trajectories away from equilibirum

7.4 Phase Plane: Complete Classification

6 Types



6) Centre:

Trajectories orbiting around equilibirum

Eigenvalues

$$r = \frac{1}{2}\left(Tr(B) \pm \sqrt{(Tr(B))^2 - 4\det(B)}\right)$$

Real roots $(TrB)^2 > 4\det B$

Nodal Source	Both > 0	$TrB > 0$	$\det B > 0$
Nodal Sink	Both < 0	$TrB < 0$	$\det B > 0$
Saddle	Opp Signs		$\det B < 0$

Complex Eigenvalues

$$r = \frac{1}{2}\left(Tr(B) \pm \sqrt{(Tr(B))^2 - 4\det(B)}\right)$$

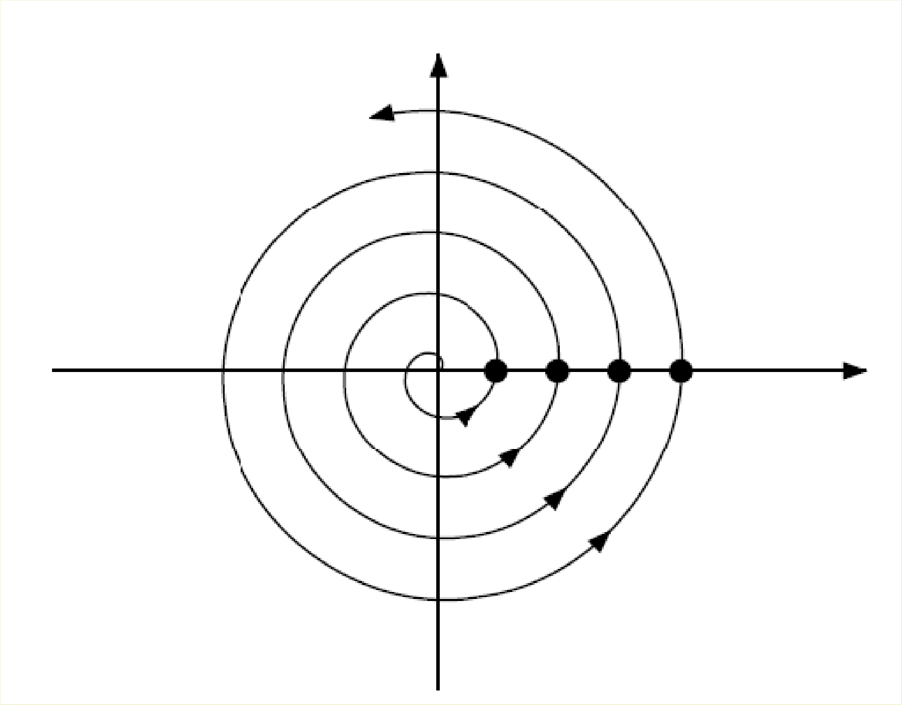
$$= \Phi \pm i\Psi$$

$$e^{(\Phi \pm i\Psi)t} = \underbrace{e^{\Phi t}}_{\text{Stretch / Shrink}} \underbrace{(\cos \Psi t + i \sin \Psi t)}_{\text{Rotating portion}}$$

Complex Eigenvalues

$TrB > 0$

$$e^{(\Phi \pm i\Psi)t} = e^{\Phi t}(\cos \Psi t + i \sin \Psi t)$$



Spiral Source	$TrB > 0$
Spiral Sink	$TrB < 0$
Centre	$TrB = 0$

Summary

$$\begin{aligned} \frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy \end{aligned} \quad \Rightarrow \quad \frac{d\vec{u}}{dt} = B\vec{u}$$

Method 1: Find Eigenvalues

EIGENVALUES

- both real
  - BOTH $> 0 \rightarrow$ Nodal Source
  - BOTH $< 0 \rightarrow$ Nodal Sink
  - Opposite Signs $\rightarrow$ Saddle
- both complex
  - Real Part $> 0 \rightarrow$ Spiral Source
  - Real Part $< 0 \rightarrow$ Spiral Sink
  - Real Part $= 0 \rightarrow$ Centre

Summary

equilibrums: x= y=0

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$

$$\Rightarrow \quad \frac{d\vec{u}}{dt} = B\vec{u}$$

Method 2: $$r = \frac{1}{2}\left(Tr(B) \pm \sqrt{(Tr(B))^2 - 4\det(B)}\right)$$

$(TrB)^2 - 4\det B$

$+$

- $TrB > 0$, $\det B > 0 \rightarrow$ Nodal Source
- $TrB < 0$, $\det B > 0 \rightarrow$ Nodal Sink
- $\det B < 0 \rightarrow$ Saddle

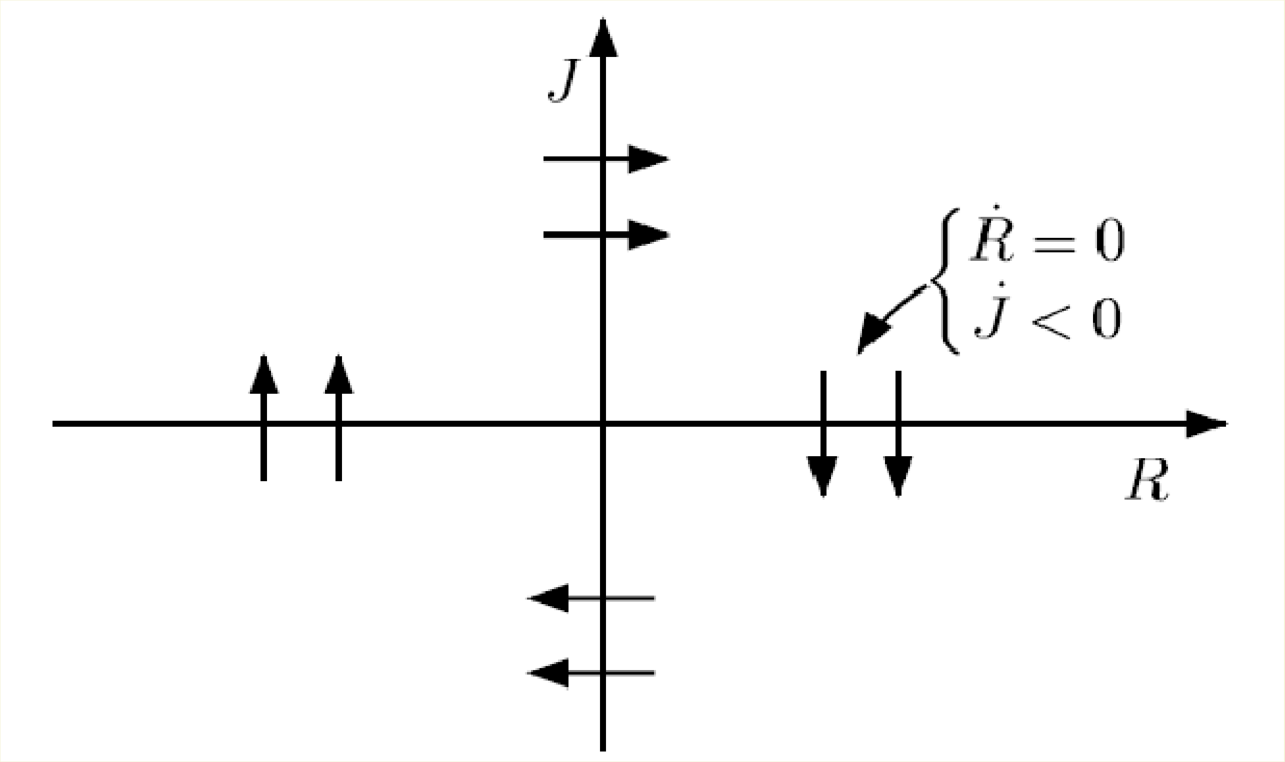
$-$

- $TrB > 0 \rightarrow$ Spiral Source
- $TrB < 0 \rightarrow$ Spiral Sink
- $TrB = 0 \rightarrow$ Centre

Example: Romeo + Juliet

$$B = \begin{bmatrix} 0 & a \\ -b & 0 \end{bmatrix} \Rightarrow TrB = 0, \det B = ab > 0$$

Complex Eigenvalues $\Rightarrow$ Centre



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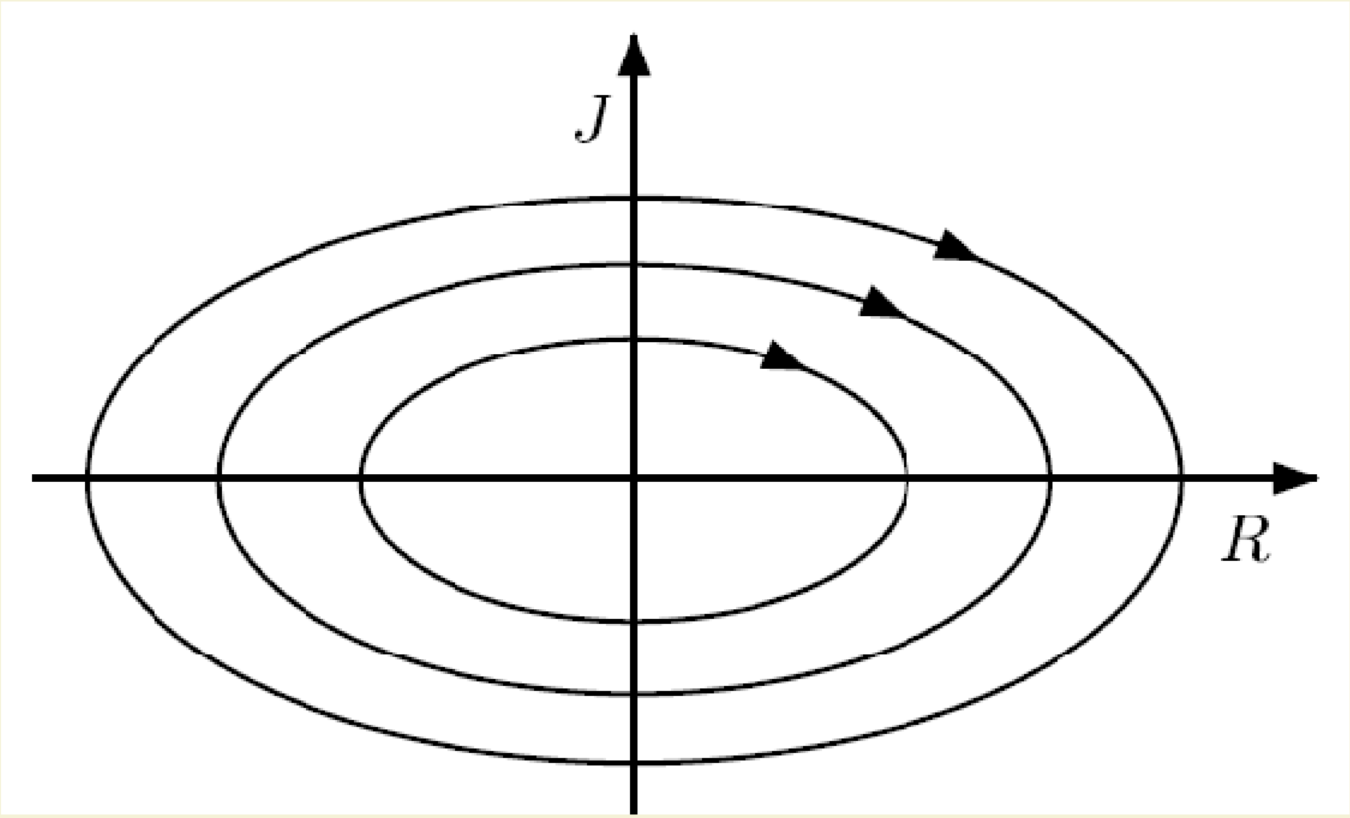
Complex Eigenvalues ➡ Centre



Example:

$$B = \begin{bmatrix} -4 & 3 \\ -2 & 1 \end{bmatrix} \Rightarrow \quad TrB = -3, \det B = 2$$

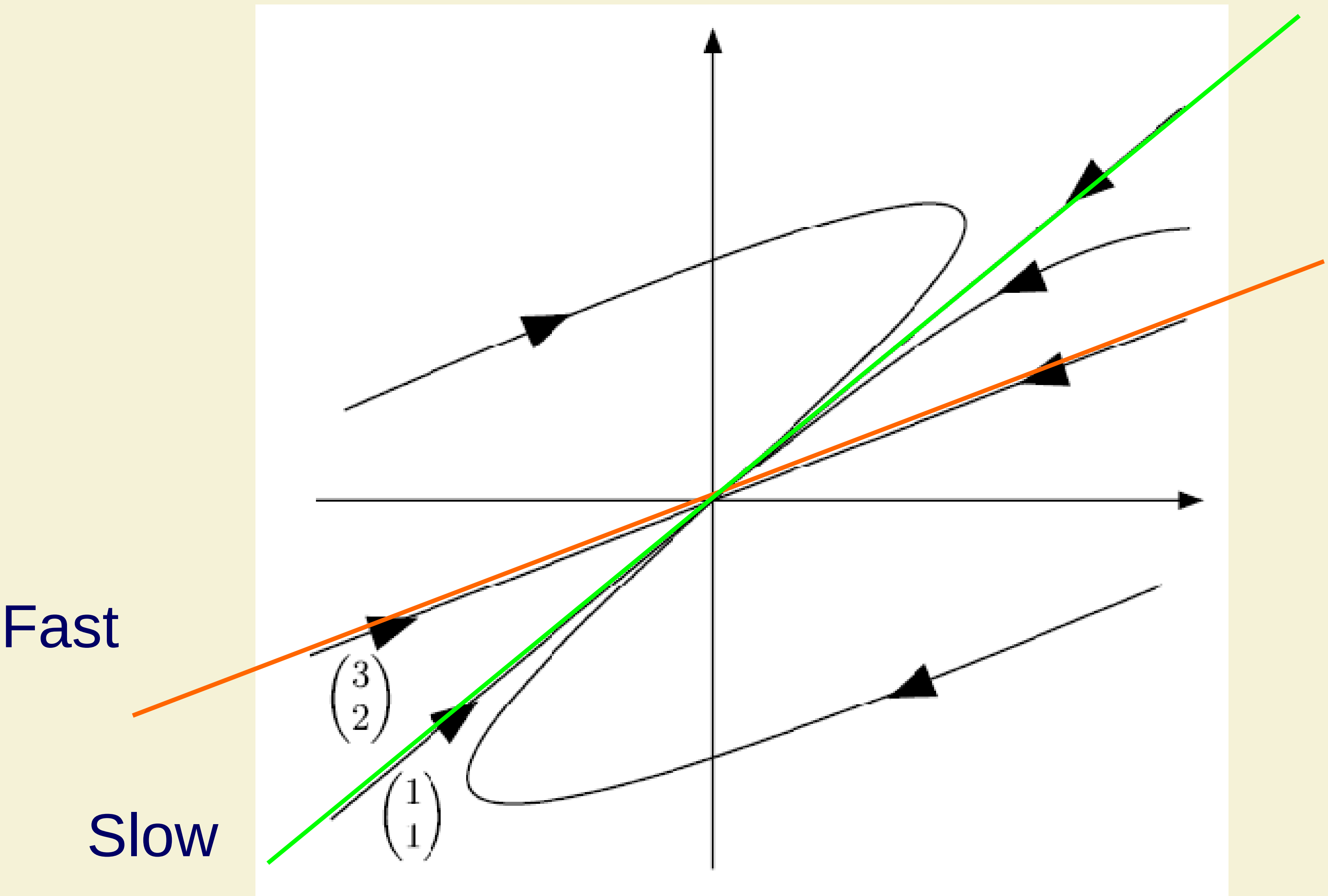
Real Eigenvalues ➡ Nodal Sink

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_+ e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_- e^{-2t} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

fast

Example:

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_+ e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_- e^{-2t} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$



Web Application

- http://www.aw-bc.com/ide/idefiles/media/JavaTools/lnclmtrx.html

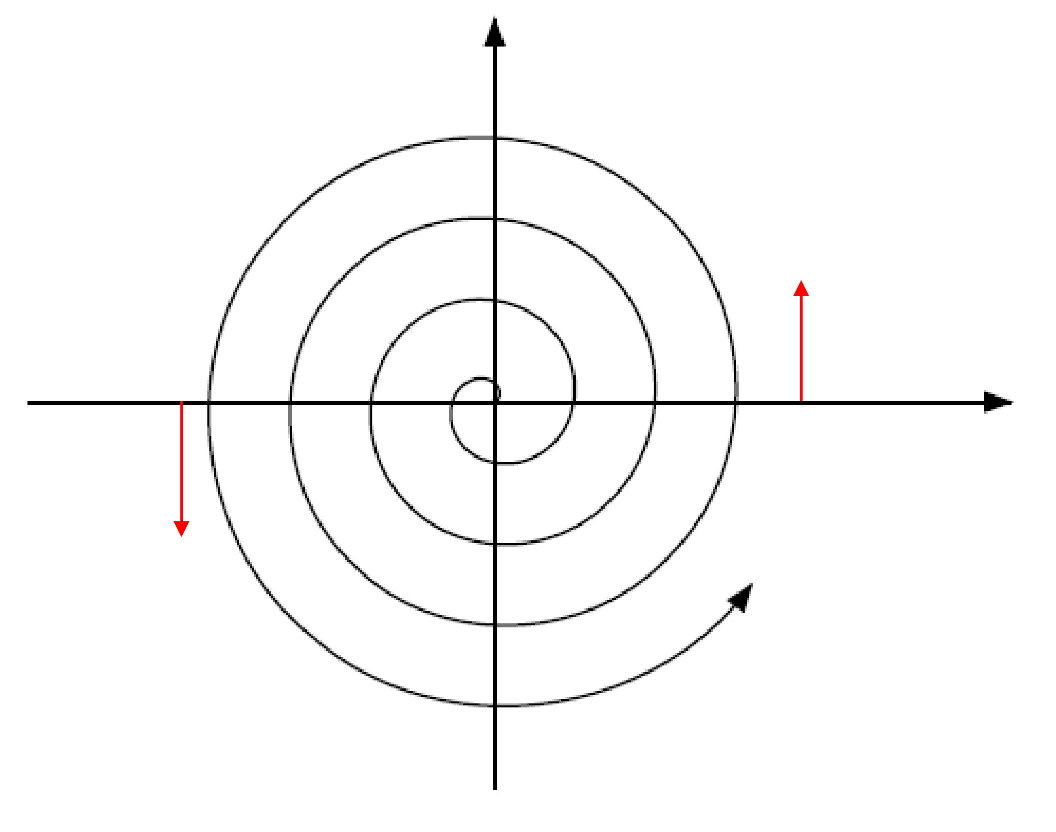
Example:

$$B = \begin{bmatrix} 4 & -5 \\ 2 & -2 \end{bmatrix} \Rightarrow TrB = 2, \det B = 2$$

Complex Eigenvalues ➡ Spiral Source

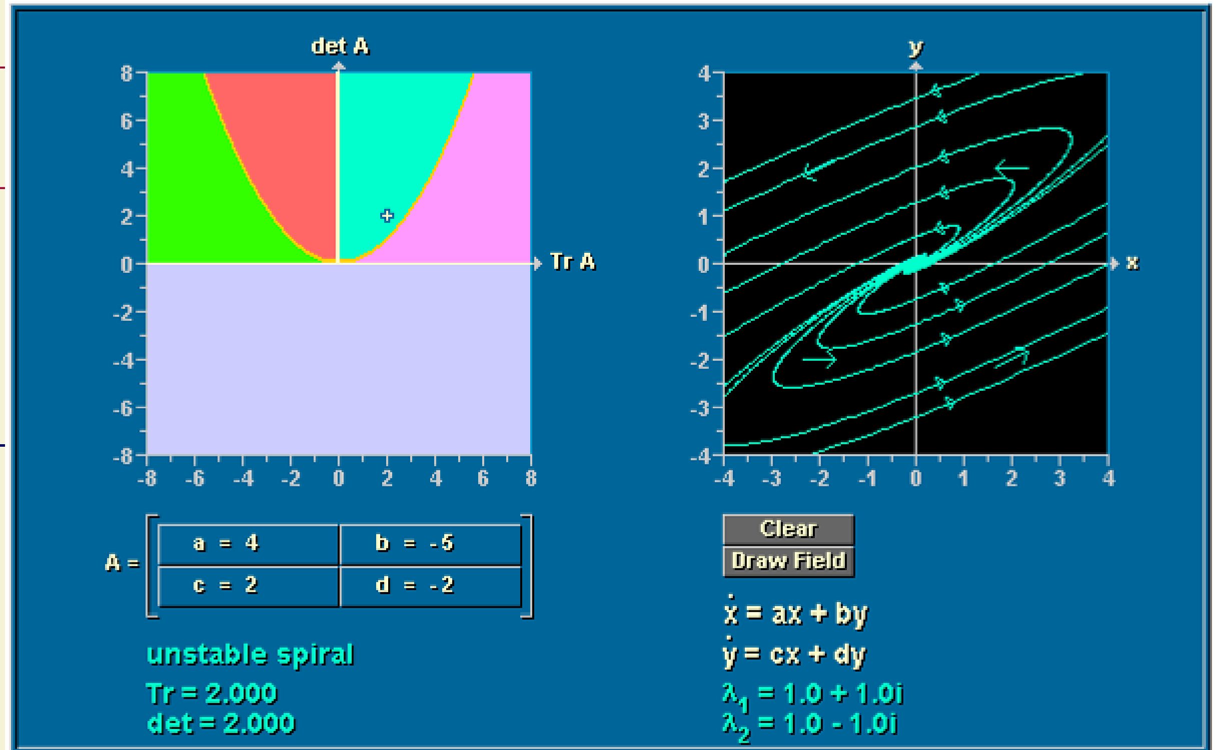
$$\frac{dy}{dt} = 2x - 2y$$

$$\left.\frac{dy}{dt}\right|_{y=0} = 2x$$



More precisely

$$B = \begin{bmatrix} 4 & -5 \\ 2 & -2 \end{bmatrix}$$

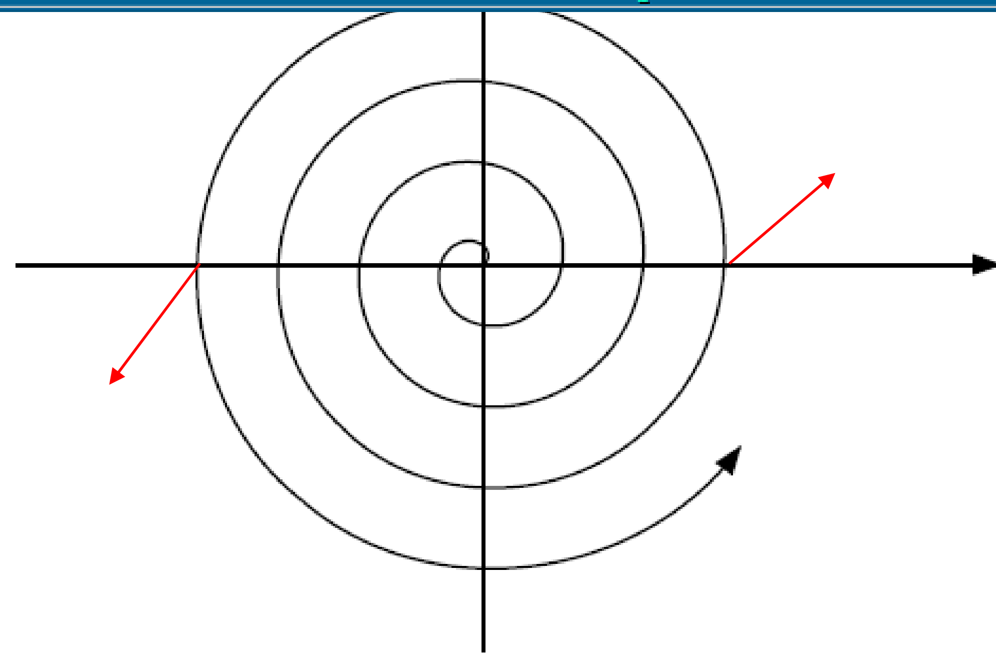


$$\frac{dx}{dt} = 4x - 5y$$

$$\left.\frac{dx}{dt}\right|_{y=0} = 4x$$

$$\frac{dy}{dt} = 2x - 2y$$

$$\left.\frac{dy}{dt}\right|_{y=0} = 2x$$



Warfare $\frac{dG}{dt} = -G - 0.75M \quad \frac{dM}{dt} = -G$

$$\begin{bmatrix} -1 & -0.75 \\ -1 & 0 \end{bmatrix} \Rightarrow TrB = -1, \det B = -0.75$$

$\Rightarrow$ Saddle

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix}$$

Eigenvectors

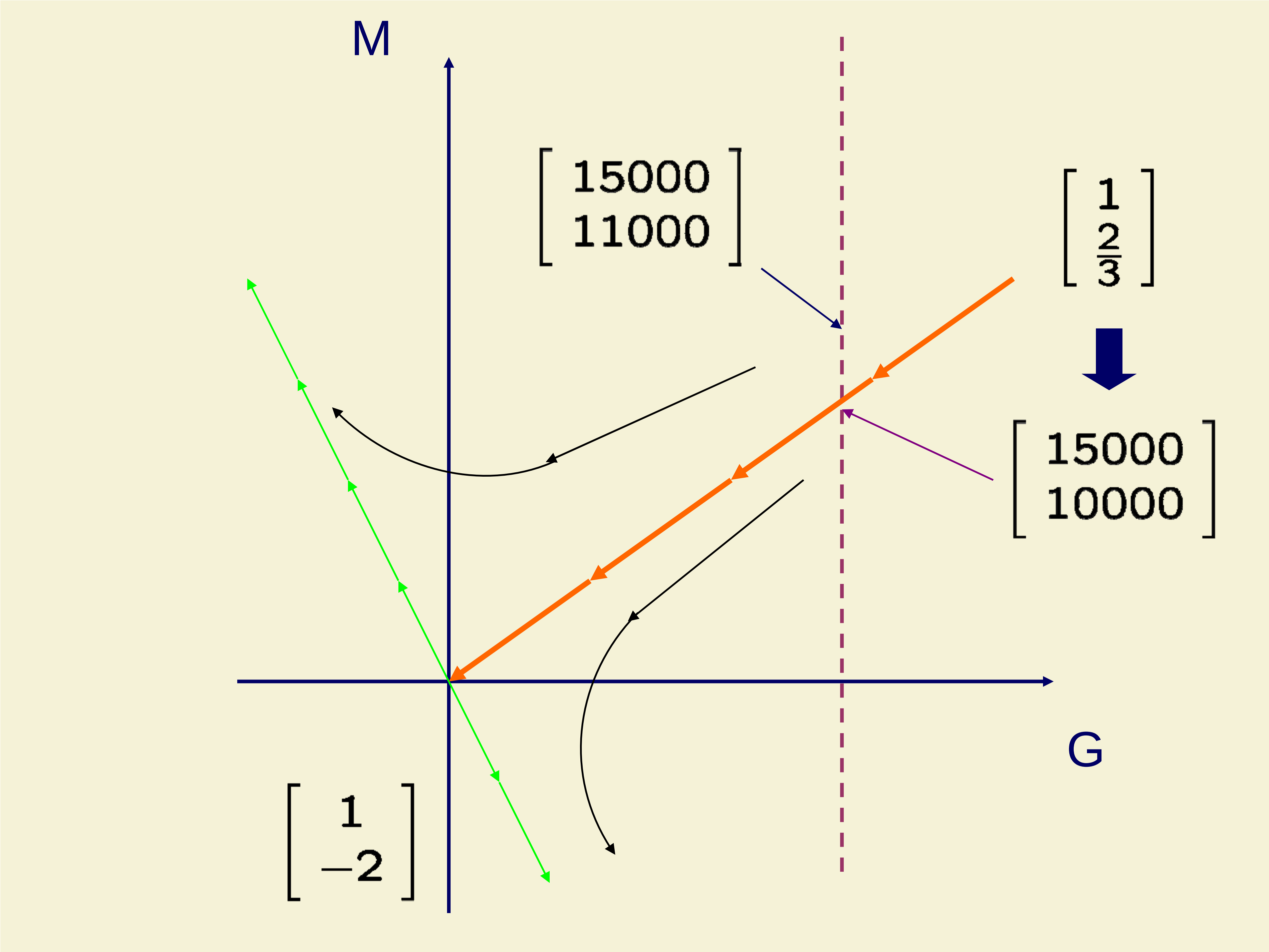
How to check?

1/2

$$\begin{bmatrix} -1 & -0.75 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

-3/2

$$\begin{bmatrix} -1 & -0.75 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix} = \lambda_2 \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix}$$



M
G
$\begin{bmatrix} 15000 \\ 11000 \end{bmatrix}$
$\begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix}$
$\begin{bmatrix} 15000 \\ 10000 \end{bmatrix}$
$\begin{bmatrix} 1 \\ -2 \end{bmatrix}$