Projective geometry

Projective geometry is used for its simplicity in formalism.

A point in projective n-space, P^n, is given by a (n+1) -vector of coordinates x=(x0, x_1 ,….., x_n)^T. At least one of these coordinates should differ from zero. These coordinates are called homogeneous coordinates. Two points represented by (n+1) vectors x and y are equal if and only if there exists a nonzero scalar k such that x = ky. This will be indicated by x ~ y.

Often the points with coordinate $x_n = 0$ are said to be at infinity.

The projective plane

The projective plane is the projective space $\mathcal{P}^2$. A point of $\mathcal{P}^2$ is represented by a 3-vector $\mathtt{m} = [x\, y\, w]^\top$. A line $\mathtt{l}$ is also represented by a 3-vector. A point $\mathtt{m}$ is located on a line $\mathtt{l}$ if and only if

$$\mathtt{l}^\top \mathtt{m} = 0 \ .$$

This equation can however also be interpreted as expressing that the line $\mathtt{l}$ passes through the point $\mathtt{m}$. This symmetry in the equation shows that there is no formal difference between points and lines in the projective plane. This is known as the principle of *duality*. A line $\mathtt{l}$ passing through two points $\mathtt{m}_1$ and $\mathtt{m}_2$ is given by their vector product $\mathtt{m}_1 \times \mathtt{m}_2$. This can also be written as

$$\mathtt{l} \sim [\mathtt{m}_1]_\times \mathtt{m}_2 \text{ with } [\mathtt{m}_1]_\times = \begin{bmatrix} 0 & w_1 & -y_1 \\ -w_1 & 0 & x_1 \\ y_1 & -x_1 & 0 \end{bmatrix} \ .$$

The dual formulation gives the intersection of two lines, i.e. the intersection of two lines **l** and **l'** can be obtained by **x**= **l** × **l'**.

Projective 3-space

Projective 3D space is the projective space $\mathcal{P}^3$. A point of $\mathcal{P}^3$ is represented by a 4-vector $\mathtt{M} = [X\, Y\, Z\, W]^\top$. In $\mathcal{P}^3$ the dual entity of a point is a plane, which is also represented by a 4-vector. A point $\mathtt{M}$ is located on a plane Π if and only if

$$\Pi^\top \mathtt{M} = 0 \ .$$

The stratification of 3D geometry

(The rest of this document are optional topics--for your intellectual satisfaction only)

Usually the world is perceived as a Euclidean 3D space. In some cases (e.g. starting from images) it is not possible or desirable to use the full Euclidean structure of 3D space. It can be interesting to only deal with the more restricted and thus simpler structure of projective geometry. An intermediate layer is formed by the affine geometry. These structures can be thought of as different geometric strata which can be overlaid on the world. The simplest being projective, then affine, next metric and finally Euclidean structure.

This concept of stratification is closely related to the groups of transformations acting on geometric entities and leaving invariant some properties of configurations of these elements. Attached to the projective stratum is the group of projective transformations, attached to the affine stratum is the group of affine transformations, attached to the metric stratum is the group of similarities and attached to the Euclidean stratum is the group of Euclidean transformations (Rotation, reflection, translation). It is important to notice that these groups are subgroups of each other, e.g. the metric group is a subgroup of the affine group and both are subgroups of the projective group.

An important aspect related to these groups are their invariants. An *invariant* is a property of a configuration of geometric entities that is not altered by any transformation belonging to a specific group, e.g. length and angle remain invariant under Euclidean transformations. Invariants therefore correspond to the measurements that one can do considering a specific stratum of geometry. For example, it makes sense to talk about such length and angle in the Euclidean space.

Affine Transformation

An *affine transformation* is usually presented as follows:

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \end{bmatrix} \text{ with } \det(a_{ij}) \neq 0$$

Using homogeneous coordinates, this can be rewritten as follows $\mathtt{M}' \sim \mathbf{T}_A \mathtt{M}$ with

$$\mathbf{T}_A \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

where **M'** and **M** are the homogeneous representation of $\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix}$ and $\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$ respectively.

You should be able to show the above.

Euclidean Transformation

Euclidean transformations have 6 degrees of freedom, 3 for orientation and 3 for translation. A Euclidean transformation has the following form

$$\mathbf{T}_E \sim \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_X \\ r_{21} & r_{22} & r_{23} & t_Y \\ r_{31} & r_{32} & r_{33} & t_Z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{B39}$$

with r_{ij} representing the coefficients of an orthonormal matrix. If $\mathbf{R}$ is a rotation matrix (i.e. $\mathbf{R}\mathbf{R}^\top = \mathbf{I}$ and $\det \mathbf{R} = 1$) then, this transformation represents a rigid motion in space. Note that an Euclidean transformation is a subgroup of Affine transformation.

Projective Transformation

A projective transformation of 3D space can be represented by a 4×4 invertible matrix

$$\mathbf{T}_P \sim \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{bmatrix} \tag{B21}$$

This transformation matrix is only defined up to a nonzero scale factor. Note that the Affine and Euclidean transformation are both subgroups of Projective transformation.

Table 1: *Number of degrees of freedom, transformations and invariants corresponding to the different geometric strata*

ambiguity	DOF	transformation	invariants
projective	15	$\mathbf{T}_P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{bmatrix}$	cross-ratio
affine	12	$\mathbf{T}_A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$	relative distances along direction parallelism *plane at infinity*

metric	7	$\mathbf{T}_M = \begin{bmatrix} \sigma r_{11} & \sigma r_{12} & \sigma r_{13} & t_x \\ \sigma r_{21} & \sigma r_{22} & \sigma r_{23} & t_y \\ \sigma r_{31} & \sigma r_{32} & \sigma r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$	relative distances angles *absolute conic*
Euclidean	6	$\mathbf{T}_E = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$	absolute distances

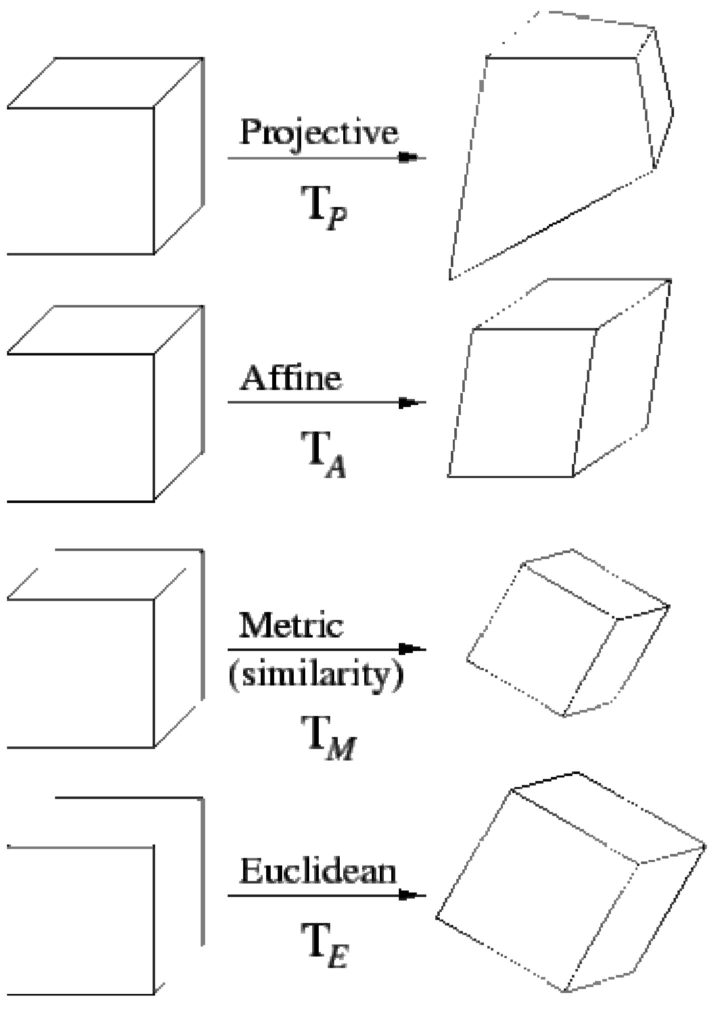

Figure 1: *Shapes which are equivalent to a cube for the different geometric ambiguities*