3. ESD, PSD and Bandwidth

3.1 Energy Spectral Density (ESD) - - - *a.k.a. Energy Spectrum*

Consider : $x(t) \leftarrow \begin{pmatrix}\text{Fourier}\\ \text{Transform}\end{pmatrix} \rightarrow X(f)$

ESD of $x(t)$: $\boldsymbol{E_x(f) \triangleq |X(f)|^2}$ (3.1)

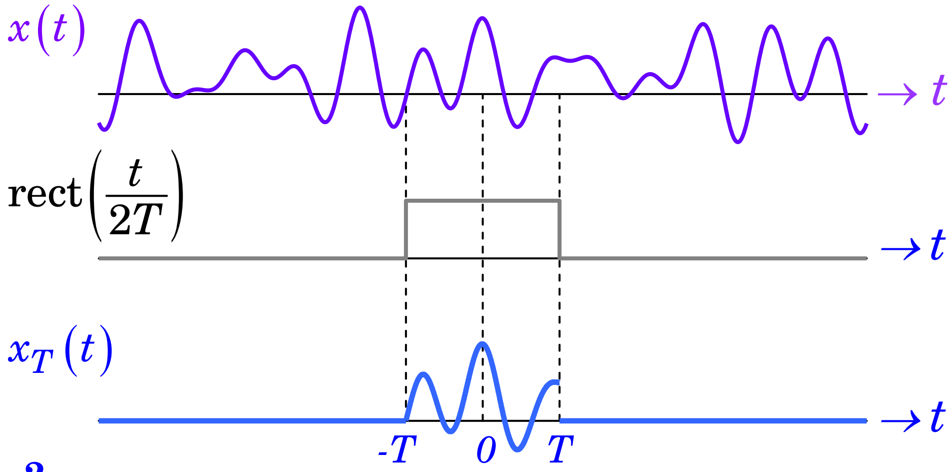
Total energy of $x(t)$: $E = \underbrace{\int_{-\infty}^{\infty} |x(t)|^2\,dt = \int_{-\infty}^{\infty} |\boldsymbol{X(f)}|^{\boldsymbol{2}}\,df}_{\textbf{Rayleigh Energy Theorem}} = \int_{-\infty}^{\infty} \boldsymbol{E_x(f)}\,df$ (3.2)

Properties of $E_x(f)$: $\begin{cases} E_x(f) \geq 0 \quad \forall f \\ \text{Energy signal: } 0 < \int_{-\infty}^{\infty} E_x(f)\,df < \infty \\ \text{Power signal: } \int_{-\infty}^{\infty} E_x(f)\,df = \infty \end{cases}$ (3.3)

3.2 Power Spectral Density (PSD) - - - *a.k.a. Power Spectrum*

Consider:

$$\begin{cases} x_T(t) = x(t)\,\mathrm{rect}\left(\dfrac{t}{2T}\right) \\ x_T(t) \Leftarrow \left(\text{Fourier Transform}\right) \Rightarrow X_T(f) \\ \lim\limits_{T\to\infty} x_T(t) = x(t) \end{cases}$$



PSD of $x(t)$: $$\boldsymbol{P_x(f) \triangleq \lim_{T\to\infty} \frac{1}{2T}\left|X_T(f)\right|^2} \tag{3.4}$$

Average power of $x(t)$: $$\begin{cases} P = \underbrace{\lim\limits_{T\to\infty}\dfrac{1}{2T}\displaystyle\int_{-\infty}^{\infty}\left|x_T(t)\right|^2 dt = \lim\limits_{T\to\infty}\dfrac{1}{2T}\displaystyle\int_{-\infty}^{\infty}\left|X_T(f)\right|^2 df}_{\textbf{Parseval Power Theorem}} \\ = \displaystyle\int_{-\infty}^{\infty} P_x(f)\,df \end{cases} \tag{3.5}$$

Properties of $P_x(f)$: $$\begin{cases} P_x(f) \ge 0 \quad \forall f \\ \text{Energy signal}: \displaystyle\int_{-\infty}^{\infty} P_x(f)\,df = 0 \quad \left[\text{implies } P_x(f) = 0 \quad \forall f\right] \\ \text{Power signal}: \ 0 < \displaystyle\int_{-\infty}^{\infty} P_x(f)\,df < \infty \end{cases} \tag{3.6}$$

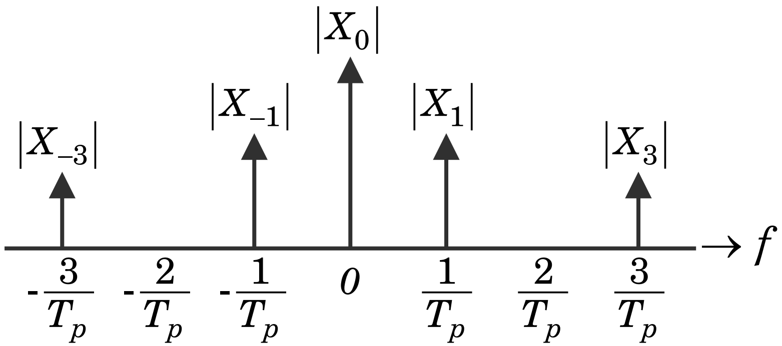
3.3 ESD and PSD of a Periodic Signal $x_p(t)$ of Period T_p

♠ Magnitude Spectrum of $x_p(t)$:

$$|X(f)| = \sum_{k=-\infty}^{\infty} |X_k| \delta\left(f - \frac{k}{T_p}\right) \qquad \text{where} \qquad X_k = \frac{1}{T_p} \int_{\gamma}^{\gamma+T_p} x_p(t) \exp\left(-j2\pi \frac{k}{T_p} t\right) dt$$

Magnitude Spectrum $|\boldsymbol{X}(\boldsymbol{f})|$

(Continuous-frequency)

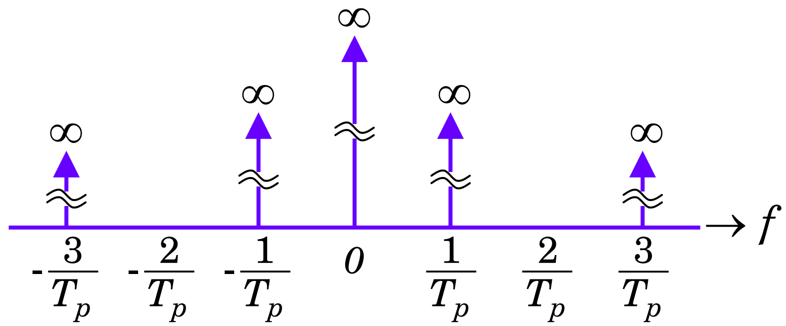


♠ ESD of $x_p(t)$:

$$E_x(f) = \underbrace{\sum_{k=-\infty}^{\infty} |X_k|^2 \delta^2\left(f-\frac{k}{T_p}\right) = \sum_{k=-\infty}^{\infty}\left(\infty\cdot|X_k|^2\right)\delta\left(f-\frac{k}{T_p}\right)}_{\textit{because } \delta^2(\bullet)=\infty\cdot\delta(\bullet)} \cdots \left[\begin{array}{l}\textit{with the understanding}\\ \textit{that } (\infty\cdot 0=0)\end{array}\right] \tag{3.7}$$

$\boldsymbol{E_x(f)}$

(Continuous-frequency)



♠ Total Energy of $x_p(t)$:

$$E = \underbrace{\int_{-\infty}^{\infty} |x_p(t)|^2\, dt = \int_{-\infty}^{\infty} E_x(f)\, df}_{\textbf{Rayleigh Energy Theorem}} = \infty \tag{3.8}$$

Proof **(optional)***:*

$$E=\int_{-\infty}^{\infty}\left|x_p(t)\right|^2dt=\int_{-\infty}^{\infty}\mathfrak{I}^{-1}\left\{\sum_{k=-\infty}^{\infty}X_k\delta\left(f-\frac{k}{T_p}\right)\right\}\left[\mathfrak{I}^{-1}\left\{\sum_{l=-\infty}^{\infty}X_l^*\delta\left(f-\frac{l}{T_p}\right)\right\}\right]^*dt$$

$$=\int_{-\infty}^{\infty}\begin{bmatrix}\displaystyle\int_{-\infty}^{\infty}\sum_{k=-\infty}^{\infty}X_k\delta\left(f-\frac{k}{T_p}\right)\cdot e^{j2\pi ft}df\\ \displaystyle\times\int_{-\infty}^{\infty}\sum_{l=-\infty}^{\infty}X_l^*\delta\left(\tilde{f}-\frac{l}{T_p}\right)\cdot e^{j2\pi\tilde{f}t}d\tilde{f}\end{bmatrix}dt$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}X_kX_l^*\delta\left(f-\frac{k}{T_p}\right)\delta\left(\tilde{f}-\frac{l}{T_p}\right)\cdot\underbrace{\int_{-\infty}^{\infty}e^{j2\pi\left(f-\tilde{f}\right)t}dt}_{\delta\left(f-\tilde{f}\right)}\right\}d\tilde{f}df$$

$$=\int_{-\infty}^{\infty}\left\{\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}X_kX_l^*\delta\left(f-\frac{k}{T_p}\right)\delta\left(f-\frac{l}{T_p}\right)\right\}df$$

$$=\int_{-\infty}^{\infty}\left\{\sum_{k=-\infty}^{\infty}\left|X_k\right|^2\int_{-\infty}^{\infty}\delta^2\left(f-\frac{k}{T_p}\right)\right\}df$$

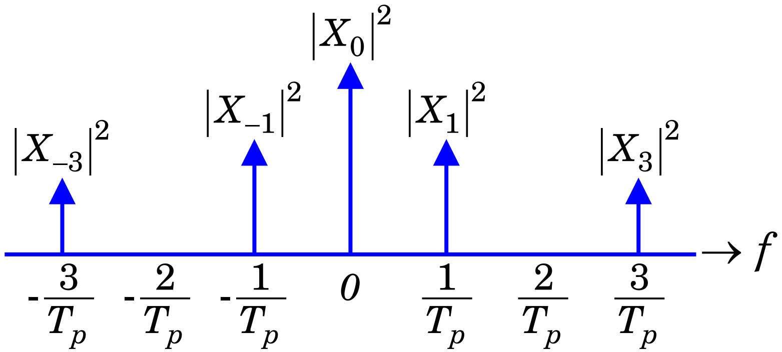
$$=\int_{-\infty}^{\infty}\underbrace{\left\{\sum_{k=-\infty}^{\infty}\left(\infty\cdot\left|X_k\right|^2\right)\delta\left(f-\frac{k}{T_p}\right)\right\}}_{ESD:\ E_x(f)}df=\infty\quad\cdots\quad \textit{since}\quad\delta^2(\bullet)=\infty\cdot\delta(\bullet)$$

♠ PSD of $x_p(t)$:

$$P_x(f) = |X_k|^2 \, \delta\left(f - \frac{k}{T_p}\right) \tag{3.9}$$

$P_x(f)$

(Continuous-frequency)



♠ Average power of $x_p(t)$:

$$P = \underbrace{\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} P_x(f) \, df}_{\textbf{Parseval Power Theorem}} = \sum_{k=-\infty}^{\infty} |X_k|^2 \tag{3.10}$$

***Proof* (optional):**

$$P=\lim_{T\to\infty}\frac{1}{2T}\int_{-\infty}^{\infty}\left|x_T(t)\right|^2 dt=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left|x(t)\right|^2 dt$$

$$=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\mathfrak{I}^{-1}\left\{\sum_{k=-\infty}^{\infty}X_k\delta\left(f-k/T_p\right)\right\}\left[\mathfrak{I}^{-1}\left\{\sum_{l=-\infty}^{\infty}X_l\delta\left(f-l/T_p\right)\right\}\right]^* dt$$

$$=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left[\int_{-\infty}^{\infty}\sum_{k=-\infty}^{\infty}X_k\delta\left(f-k/T_p\right)\cdot e^{j2\pi ft}df\right]\left[\int_{-\infty}^{\infty}\sum_{l=-\infty}^{\infty}X_l^*\delta\left(\tilde{f}-l/T_p\right)\cdot e^{-j2\pi\tilde{f}t}d\tilde{f}\right]dt$$

$$=\int_{-\infty}^{\infty}\left\{\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}X_kX_l^*\delta\left(f-k/T_p\right)\left[\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left\{\int_{-\infty}^{\infty}\delta\left(\tilde{f}-l/T_p\right)e^{j2\pi\left(f-\tilde{f}\right)t}d\tilde{f}\right\}dt\right]\right\}df$$

$$=\int_{-\infty}^{\infty}\left\{\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}X_kX_l^*\delta\left(f-k/T_p\right)\left[\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}e^{j2\pi\left(f-l/T_p\right)t}dt\right]\right\}df$$

$$=\int_{-\infty}^{\infty}\left\{\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}X_kX_l^*\delta\left(f-k/T_p\right)\underbrace{\left[\lim_{T\to\infty}\operatorname{sinc}\left(2T\left(f-l/T_p\right)\right)\right]}_{\left(1\text{ if }f=l/T_p\right)\text{ and }\left(0\text{ if }f\neq l/T_p\right)}\right\}df$$

$$=\int_{-\infty}^{\infty}\underbrace{\left\{\sum_{k=-\infty}^{\infty}\left|X_k\right|^2\delta\left(f-k/T_p\right)\right\}}_{PSD:\ P_x(f)}df=\sum_{k=-\infty}^{\infty}\left|X_k\right|^2\int_{-\infty}^{\infty}\delta\left(f-k/T_p\right)df=\sum_{k=-\infty}^{\infty}\left|X_k\right|^2$$

Example 3-1:

$$\underbrace{\left(x(t) = 2\exp(-4t)u(t) \right)}_{\textit{ENERGY SIGNAL}} \rightleftarrows \left(X(f) = \frac{2}{4 + j2\pi f} \right)$$

ESD : $E_x(f) = \left| X(f) \right|^2 = \dfrac{4}{16 + 4\pi^2 f^2}$

Total Energy :
$$\begin{cases} E = \int_{-\infty}^{\infty} E_x(f)\,df = \underbrace{\int_{-\infty}^{\infty} \dfrac{4}{16 + 4\pi^2 f^2}\,df = \int_0^{\infty} 4\exp(-8t)\,dt}_{\substack{\textbf{Rayleigh Energy Theorem:}\ \textit{In this case,} \\ \textit{it is easier to integrate in the time-domain}}} \\ \quad = \left. \dfrac{4\exp(-8t)}{-8} \right|_0^{\infty} = 0.5 \end{cases}$$

$$\left. \begin{array}{ll} \text{PSD} & : P_x(f) = 0 \\ \text{Average Power} & : P = \int_{-\infty}^{\infty} P_x(f)\,df = 0 \end{array} \right\} \textit{Holds for all Energy Signals}$$

Example 3-2:

$$\underbrace{\begin{pmatrix} x(t) = 2 + 6\cos(16\pi t) \\ \quad = 2 + 3e^{j16\pi t} + 3e^{-j16\pi t} \end{pmatrix}}_{\textit{POWER SIGNAL}} \rightleftarrows \begin{pmatrix} X(f) = 2\delta(f) + 3\delta(f-8) + 3\delta(f+8) \\ X(k) = \begin{cases} 2; & k = 0 \\ 3; & |k| = 1 \\ 0; & |k| > 1 \end{cases} \end{pmatrix}$$

ESD $\quad : E_x(f) = \infty \cdot 4\delta(f) + \infty \cdot 9\delta(f-8) + \infty \cdot 9\delta(f+8)$

Total Energy $\quad : E = \int_{-\infty}^{\infty} E_x(f)\,df = \infty \quad \cdots\cdots\cdots\cdots$ *Holds for all Power Signals*

PSD $\quad : P_x(f) = 4\delta(f) + 9\delta(f-8) + 9\delta(f+8)$

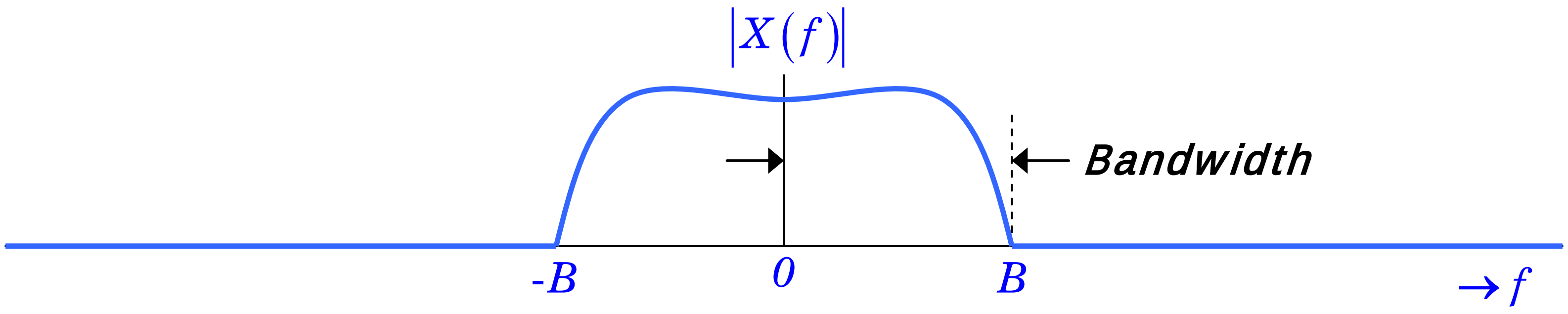
Average Power $\quad : P = \int_{-\infty}^{\infty} P_x(f)\,df = 22 \quad \cdots \quad \left(= \sum_k |X_k|^2 \right)$

3.4 Bandwidth

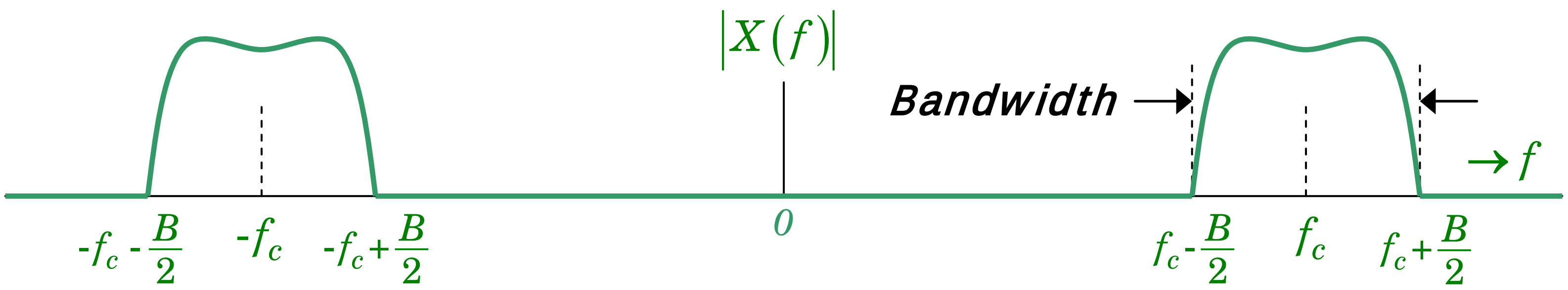
The bandwidth of a $x(t)$ is a measure of the width of a range of frequencies occupied by $|X(f)|$.

■ Band-limited signals

LOWPASS SIGNAL $x(t)$: $|X(f)| = 0; \quad |f| > B$



BANDPASS SIGNAL $x(t)$: $|X(f)| = 0; \quad \left[|f| < (f_c - B/2)\right]$ **or** $\left[|f| > (f_c + B/2)\right]$



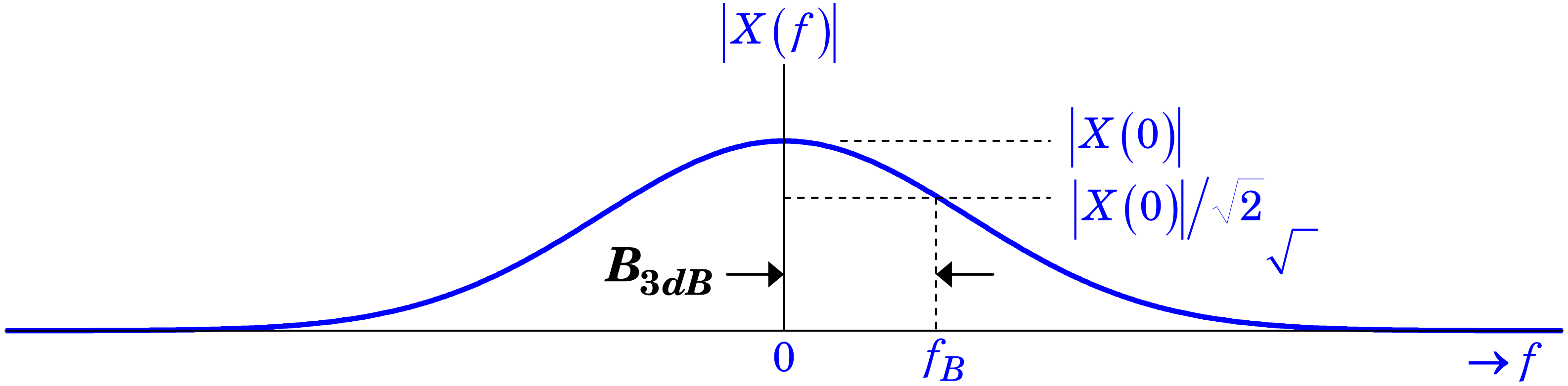
■ Signals with unrestricted band

In general, practical signals are seldom strictly bandlimited, but have infinite frequency extent. For such signals, from the signal processing and system design standpoints, it is often useful to define a bandwidth measure to include only the important part of the signal spectrum.

- **$3dB$ BANDWIDTH $\left(B_{3dB}\right)$ of Energy Signals**

LOWPASS SIGNAL $x(t)$:

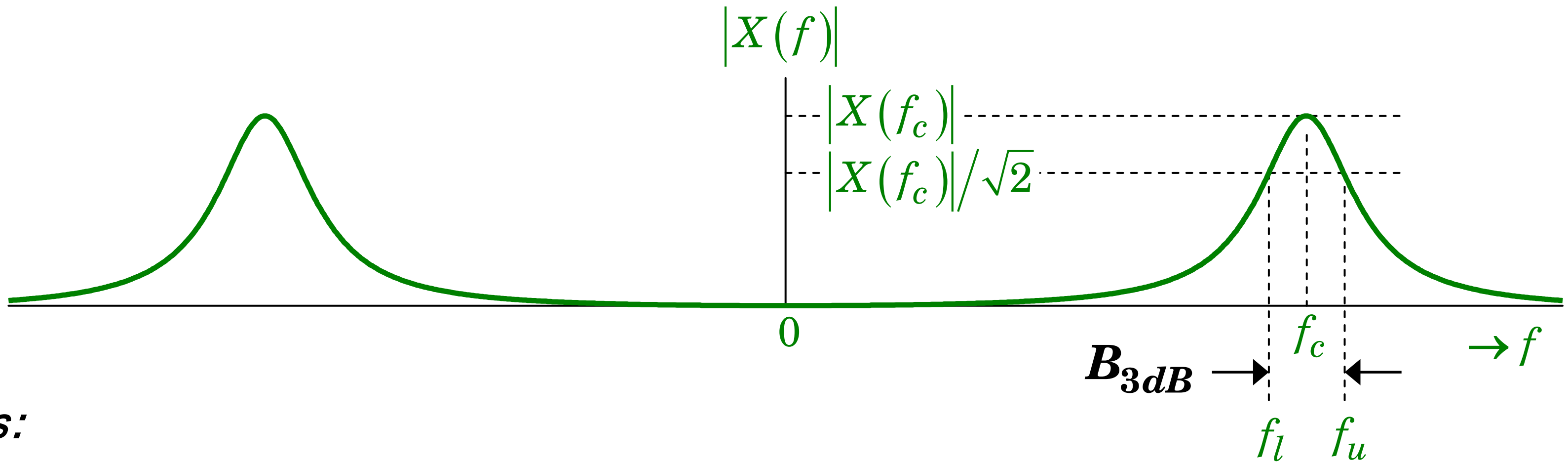
- The 3dB bandwidth of $x(t)$ is defined as the frequency, f_B, at which $|X(f)| = |X(0)|/\sqrt{2}$ **first** occurs when f is increased from 0:



- f_B is called the $3dB$ frequency because $10\log_{10}\left(\dfrac{|X(f_B)|^2}{|X(0)|^2}\right) = 10\log_{10}\left(\dfrac{1}{2}\right) \simeq -3.01dB$.

BANDPASS SIGNAL $x(t)$:

- f_c: Center frequency of the bandpass signal.
- The 3dB bandwidth of $x(t)$ is defined as $f_u - f_l$, where:
  - f_u is the frequency at which $|X(f)| = \dfrac{|X(f_c)|}{\sqrt{2}}$ **first** occurs when f is increased from f_c
  - f_l is the frequency at which $|X(f)| = \dfrac{|X(f_c)|}{\sqrt{2}}$ **first** occurs when f is decreased from f_c



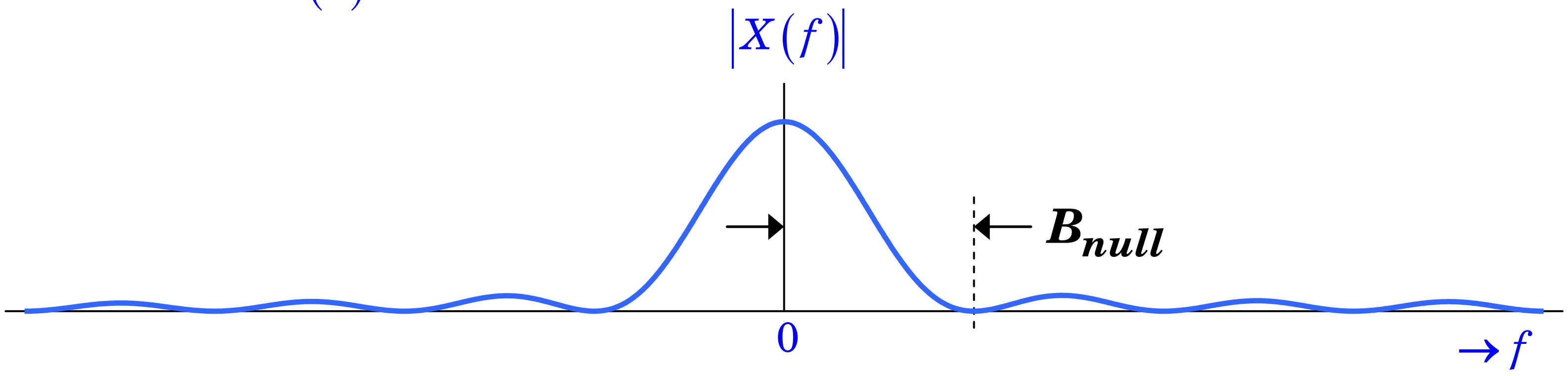
Remarks:

Bandwidth is defined in the positive frequency range.

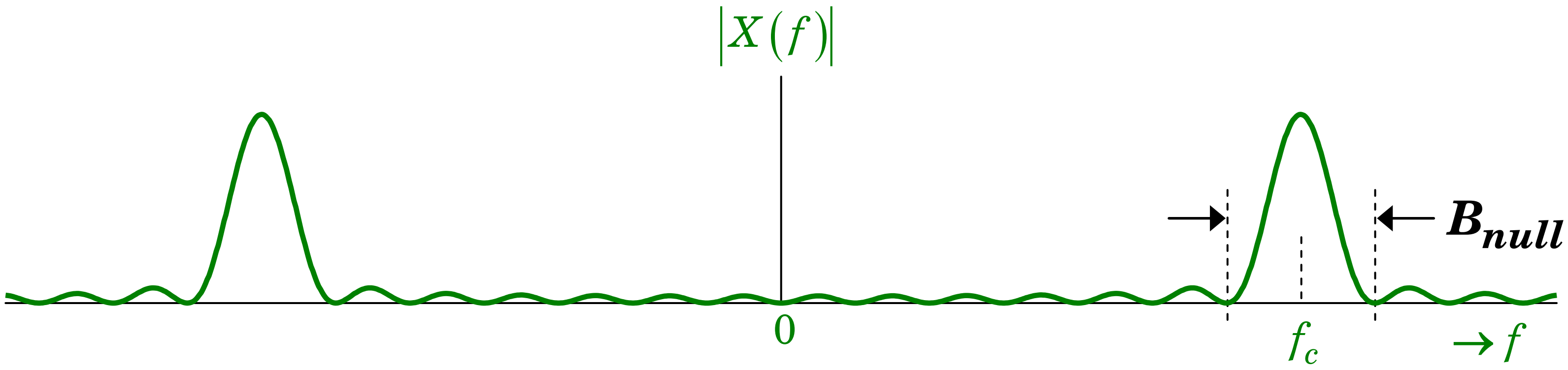
The $3dB$ bandwidth definition is a widely accepted criterion for measuring a signal bandwidth, but it may become ambiguous and nonunique with signals having multiple-peak magnitude spectrum.

- 1^{st}**-null BANDWIDTH** $\left(B_{null}\right)$ **of Energy Signals** - *Another bandwidth definition*

LOWPASS SIGNAL $\boldsymbol{x(t)}$:



BANDPASS SIGNAL $\boldsymbol{x(t)}$:



Example 3-3:

$$\left.\begin{array}{l}\textbf{\textit{Lowpass}}\\\textbf{\textit{Gaussian}}\\\textbf{\textit{Pulse}}\end{array}\right\}:\ \left[x(t)=\exp\left(-\frac{t^2}{2\sigma^2}\right)\right]\rightleftarrows\left[X(f)=\left(2\pi\sigma^2\right)^{0.5}\exp\left(-2\sigma^2\pi^2f^2\right)\right];\ \sigma>0$$

σ *is a measure of the time-spread of* $x(t)$*.*

3-dB bandwidth (f_B) ***of*** $x(t)$***:***

$$\left.\begin{aligned}\left|\frac{X(f_B)}{X(0)}\right|^2&=\frac{2\pi\sigma^2\exp\left(-4\sigma^2\pi^2f_B^2\right)}{2\pi\sigma^2}\\&=\exp\left(-4\sigma^2\pi^2f_B^2\right)=\frac{1}{2}\end{aligned}\right\}\ \Rightarrow\ f_B=\left(\frac{\ln(2)}{4\sigma^2\pi^2}\right)^{0.5}$$

Time-bandwidth product σf_B ***of*** $x(t)$***:***

$$\sigma f_B=\left(\frac{\ln(2)}{4\pi^2}\right)^{0.5}=\textbf{\textit{constant}}$$

Comment: *Time-spread is inversely proportional to bandwidth* $\left[\sigma\propto\frac{1}{f_B}\right]$*. This is an illustration of the time-frequency domain scaling exchange.*