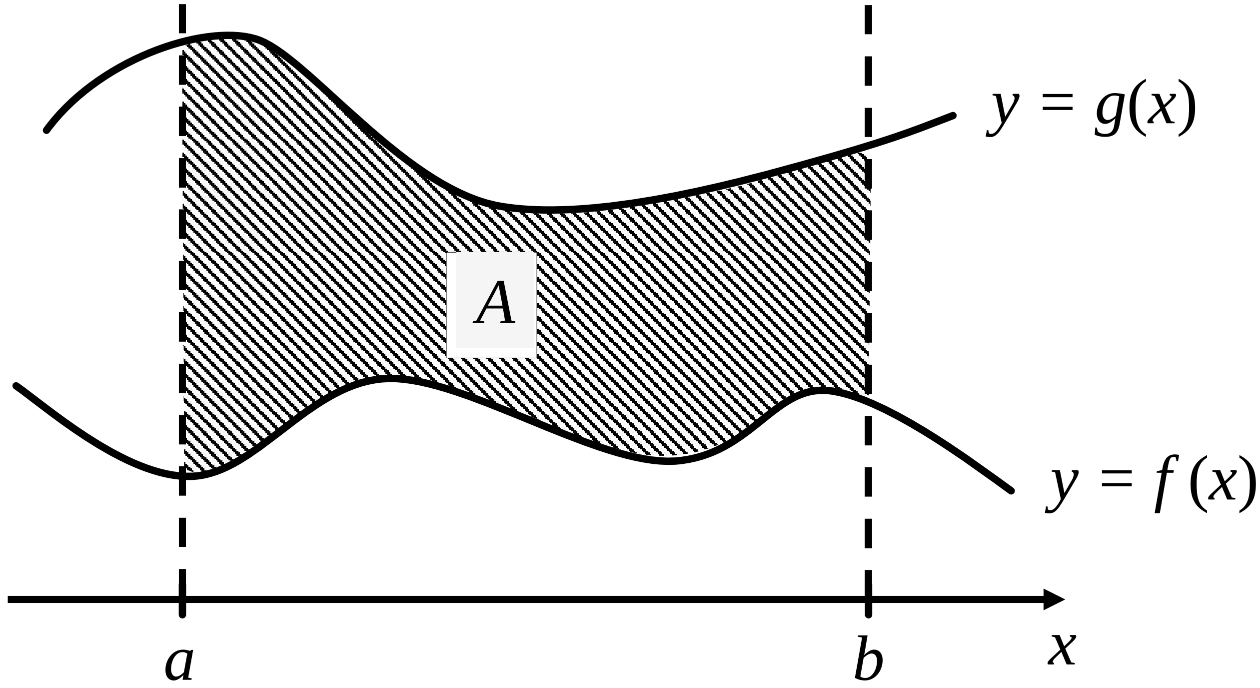
Application of Integration

Area between two curves

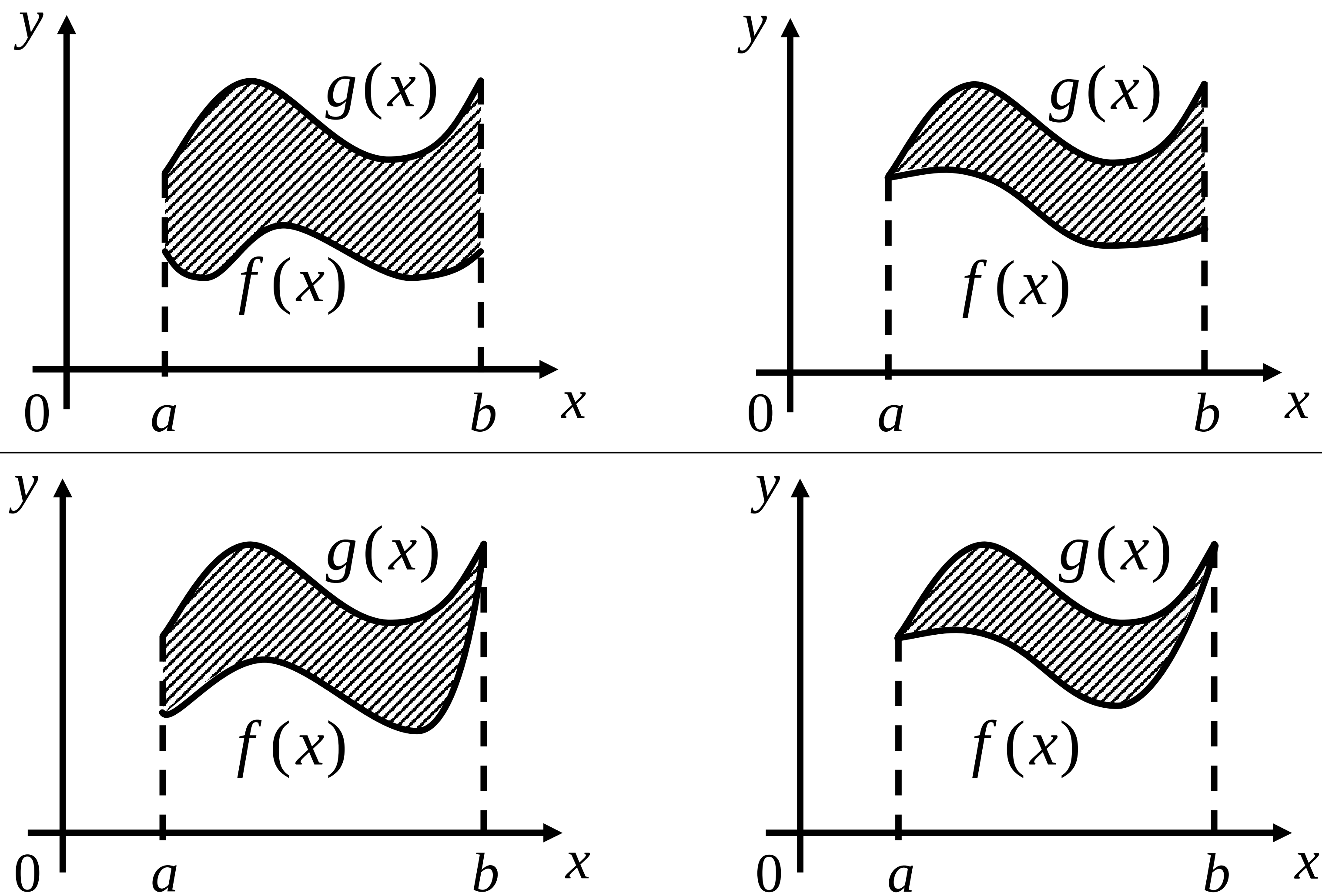


$$A = \int_a^b (g(x) - f(x))\,dx$$

Top curve − bottom curve

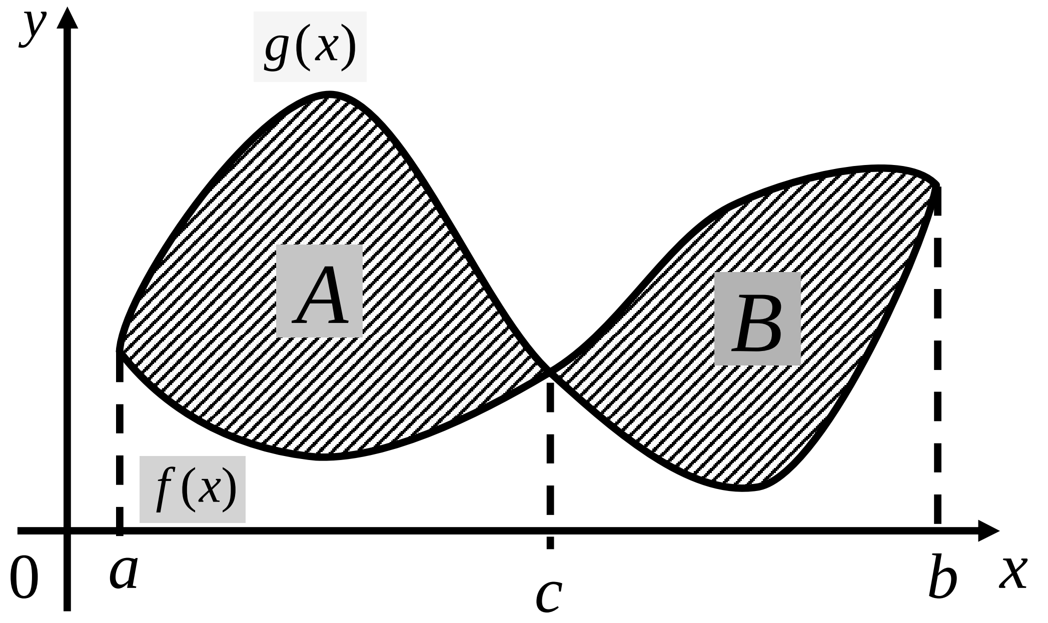
Area between two curves

Consider $g(x) \geq f(x)$



Area between the two curves $= \int_a^b g(x) - f(x)\, dx$

Area between two curves

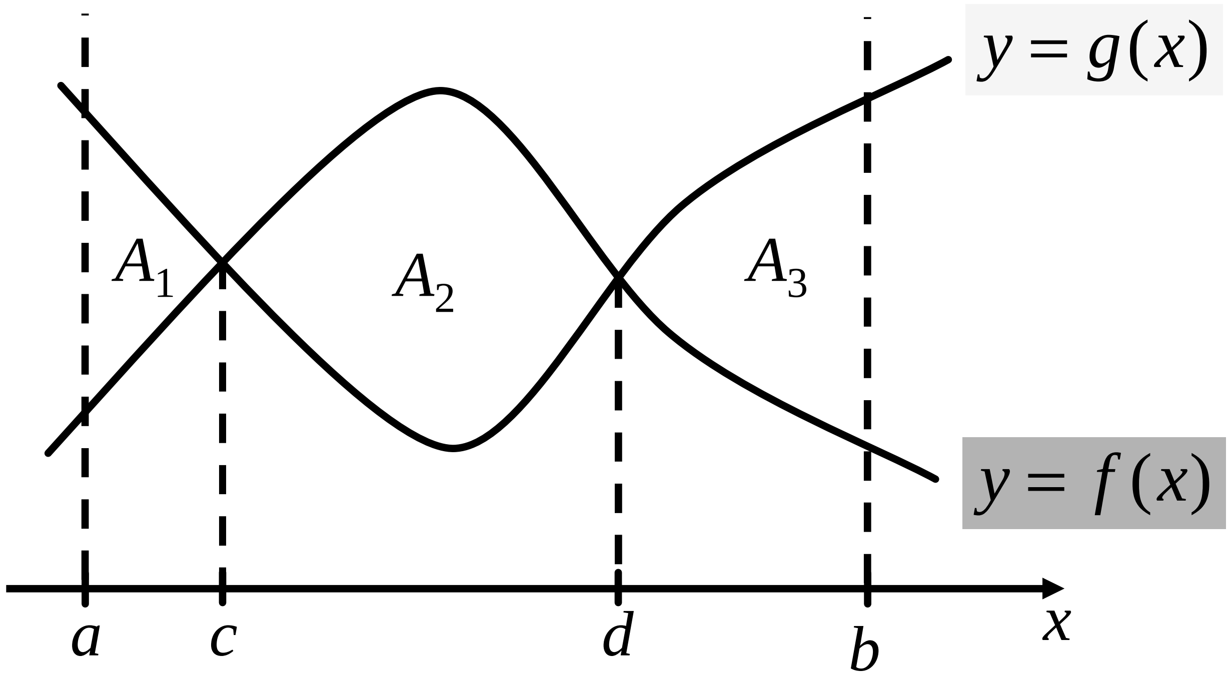


(find)

$$A = \int_a^c g(x) - f(x)\,dx$$

$$B = \int_c^b f(x) - g(x)\,dx$$

Area between two curves



$$A_1 + A_2 + A_3 = \int_a^c (g(x) - f(x))\, dx + \int_c^d (f(x) - g(x))\, dx$$
$$+ \int_d^b (g(x) - f(x))\, dx$$

Find the area enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

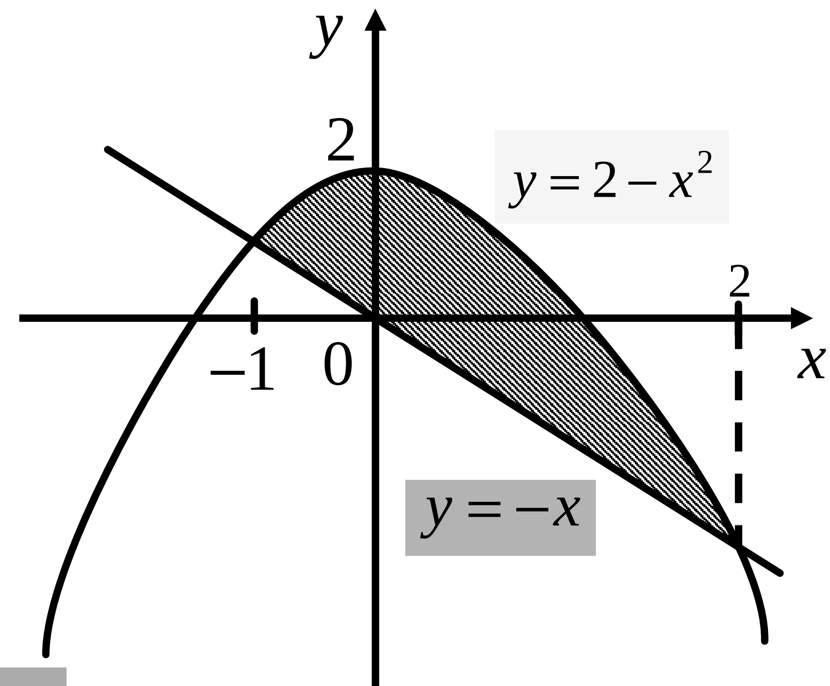
Consider

$$-x = 2 - x^2$$

$$x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0$$

$$x = 2 \text{ or } x = -1$$



$$\text{Area} = \int_{-1}^{2} 2 - x^2 - (-x)\ dx$$

$$= \int_{-1}^{2} 2 - x^2 + x\ dx$$

$$= \left[2x - \frac{x^3}{3} + \frac{x^2}{2} \right]_{-1}^{2} = 4\frac{1}{2} \text{ units}^2$$

Find the area of the region in the first quadrant bounded by the curves $y=\sqrt{x}$ and $y=x-2$.

Consider $\quad x-2=\sqrt{x}$

$$x-\sqrt{x}-2=0$$

$$\left(\sqrt{x}\right)^2-\sqrt{x}-2=0$$

$$(\sqrt{x}-2)(\sqrt{x}+1)=0$$

$$\sqrt{x}=2 \quad \text{or} \quad \sqrt{x}=-1 \ \text{(Not possible)}$$

$$x=4$$

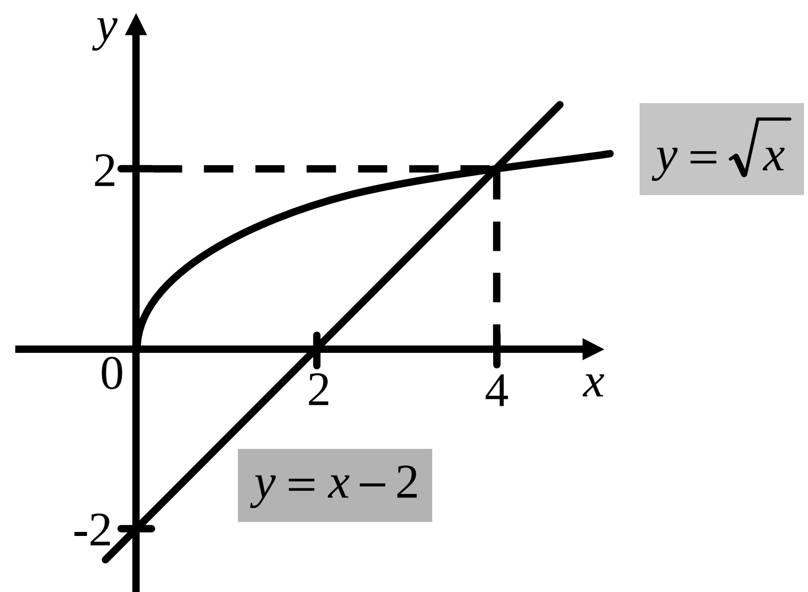
$y=\sqrt{x} \quad$ Domain : $x\geq 0$

$\sqrt{x}=$ take positive root

$$\text{Area}=\int_0^4 \sqrt{x}-(x-2)\,dx$$

$$=\left[\frac{2}{3}x^{3/2}-\frac{1}{2}x^2+2x\right]_0^4$$

$$=\frac{16}{3}\ \text{units}^2$$



Pause and Think !!!

1. Find the area of the region in the first quadrant bounded by the curves $y = \sqrt{x}$ and $y = x - 2$.

2. Find the area bounded by the curves $y = \sqrt{x}$, $y = x - 2$ and the y-axis.

What is the difference between the two questions ???

Find the area bounded by the curves $y=\sqrt{x}$, $y=x-2$ and the y-axis.

Consider $\quad x-2=\sqrt{x}$

$$x-\sqrt{x}-2=0$$

$$\left(\sqrt{x}\right)^2-\sqrt{x}-2=0$$

$$(\sqrt{x}-2)(\sqrt{x}+1)=0$$

$$\sqrt{x}=2 \quad \text{or} \quad \sqrt{x}=-1 \ \text{(Not possible)}$$

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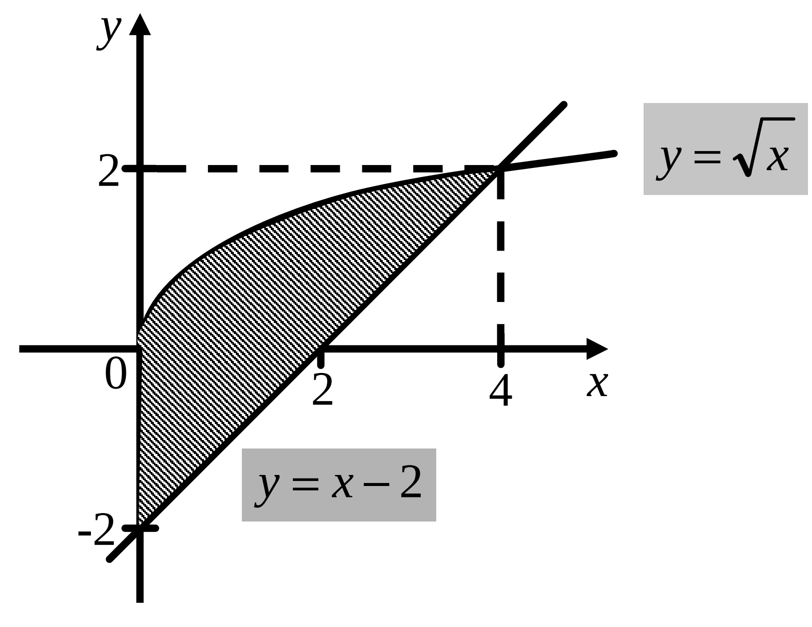
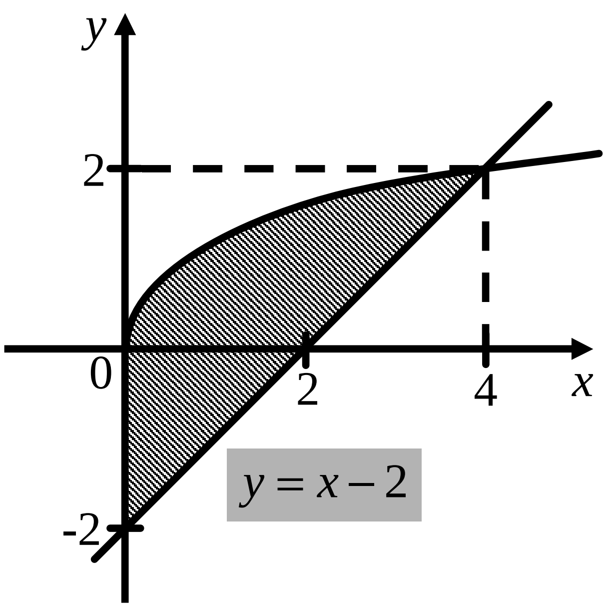
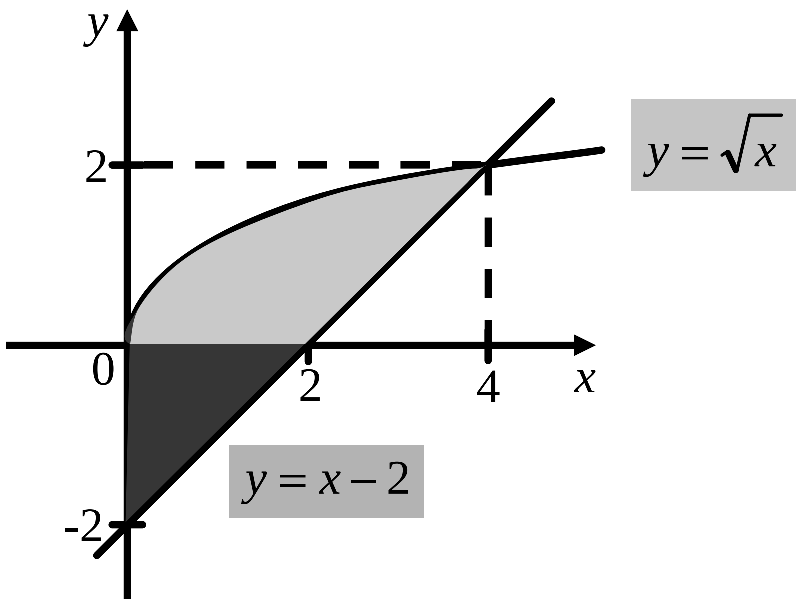


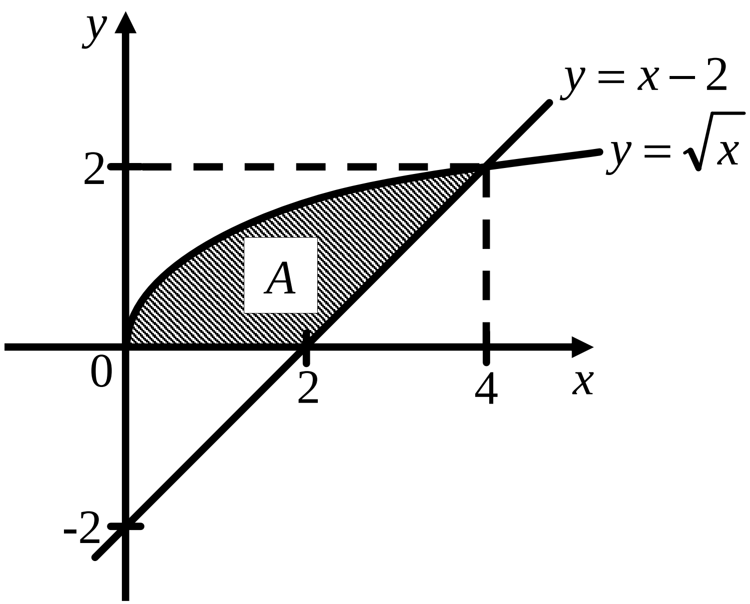
Pause and Think !!!

1. Find the area of the region in the first quadrant bounded by the curves $y = \sqrt{x}$ and $y = x - 2$.



Find the area of the region in the first quadrant bounded by the curves $y=\sqrt{x}$ and $y=x-2$.





$$\text{Area} = \int_0^4 \sqrt{x}-(x-2)\;dx$$

$$=\left[\frac{2}{3}x^{3/2}-\frac{1}{2}x^2+2x\right]_0^4$$

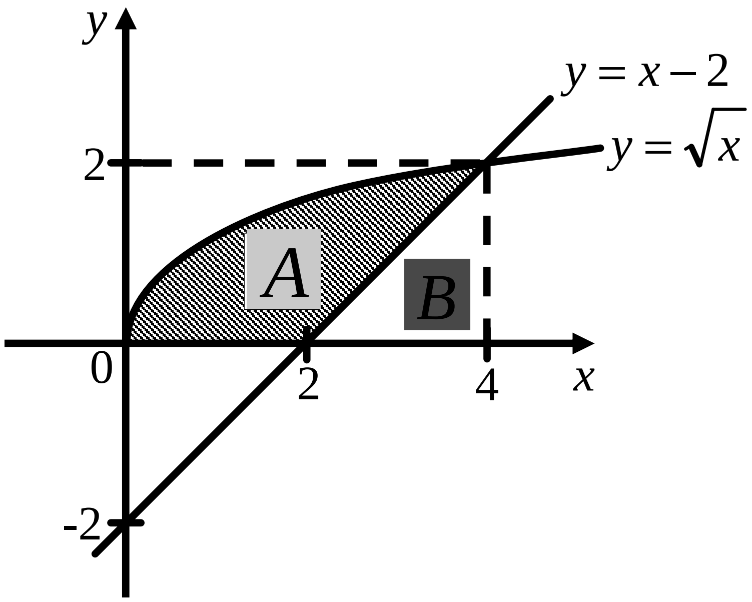
$$=\frac{16}{3}\ \text{units}^2$$

$$\text{Area } A=\frac{16}{3}-\text{Area of red triangle}$$

$$=\frac{16}{3}-\frac{1}{2}\times 2\times 2$$

$$=\frac{10}{3}$$

Find the area of the region in the first quadrant bounded by the curves $y=\sqrt{x}$ and $y=x-2$.



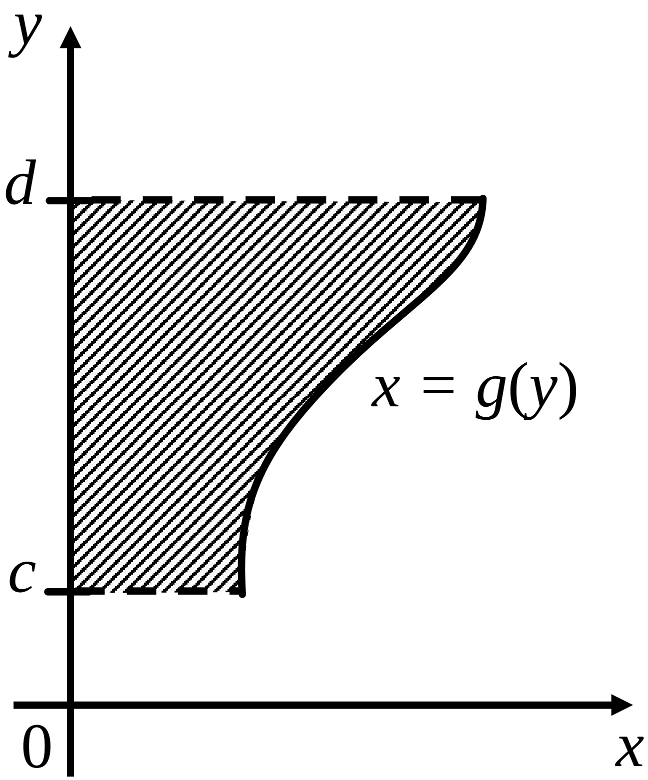
Method 2

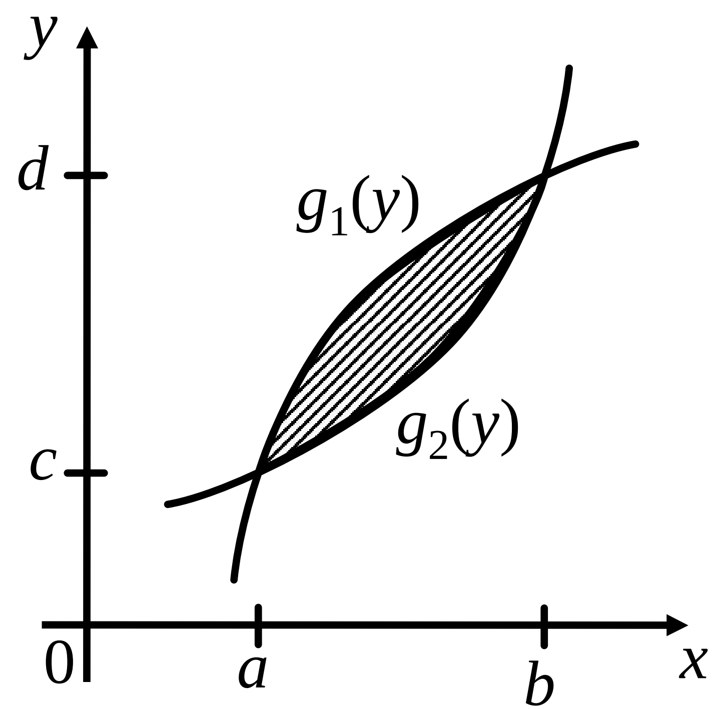
$$\text{Area } A=\int_0^4 \sqrt{x}\,dx-\text{Area of triangle } B$$

$$=\left[\frac{2}{3}x^{3/2}\right]_0^4-\frac{1}{2}\times 2\times 2$$

$$=\frac{10}{3}\text{ units}^2$$

Area between two curves





$$\text{Area} = \int_c^d g(y)\,dy$$

$$\text{Area} = \int_c^d g_2(y) - g_1(y)\,dy$$

Find $y = c$ and $y = d$

Find the area of the region in the first quadrant bounded by the curves $y = \sqrt{x}$ and $y = x - 2$.

Consider $x - 2 = \sqrt{x}$

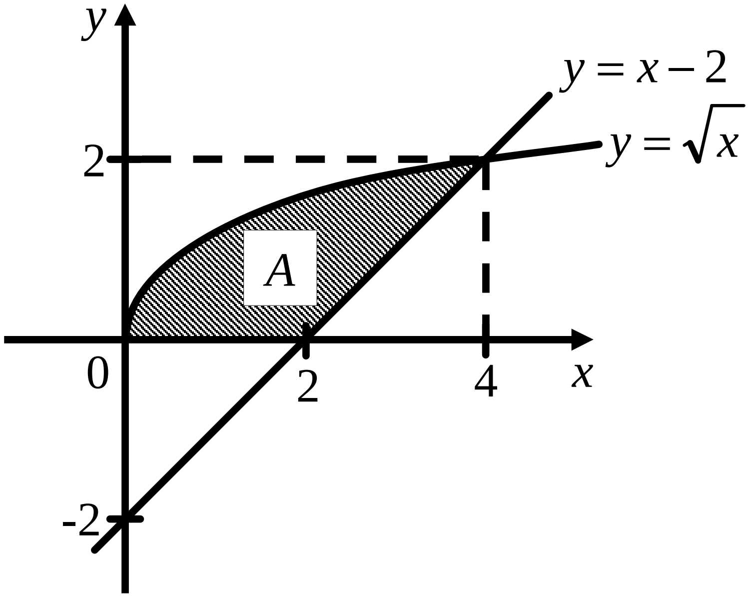
$$x - \sqrt{x} - 2 = 0$$

$$\left(\sqrt{x}\right)^2 - \sqrt{x} - 2 = 0$$

$$(\sqrt{x} - 2)(\sqrt{x} + 1) = 0$$

$\sqrt{x} = 2$ or $\sqrt{x} = -1$ (Not possible)

$$x = 4$$



When $x = 4$, $y = \sqrt{4} = 2$

$y = \sqrt{x} \quad \rightarrow \quad x = y^2$

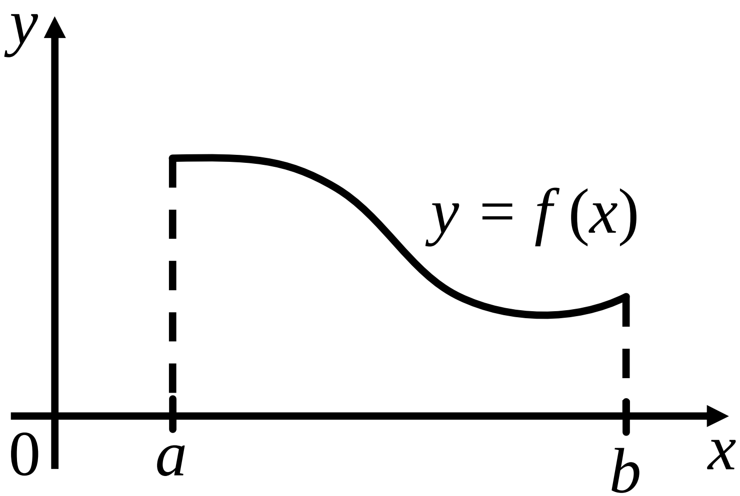
$y = x - 2 \quad \rightarrow \quad x = y + 2$

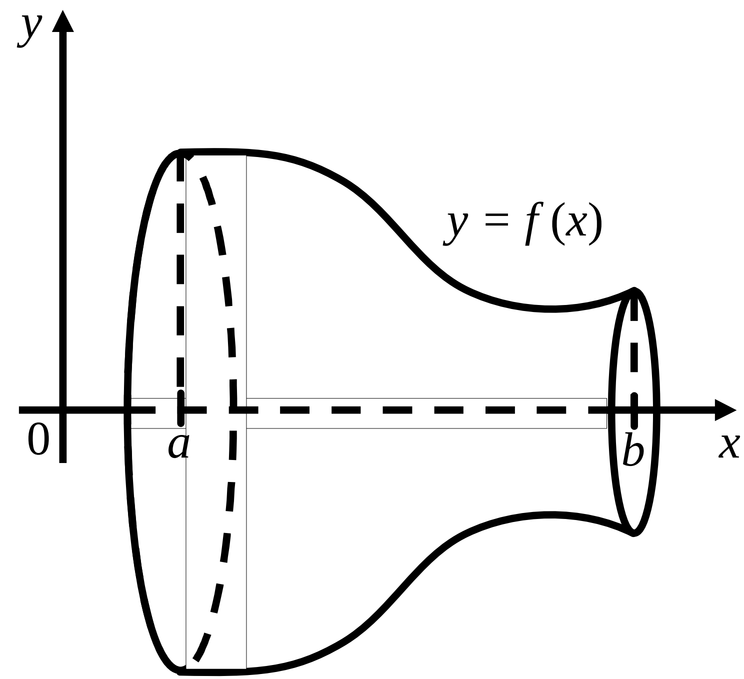
$$A = \int_0^2 ((y+2) - y^2)\ dy$$

$$= \frac{10}{3} \text{ units}^2$$

Volume of Solids of Revolution

(I) About *x*-axis





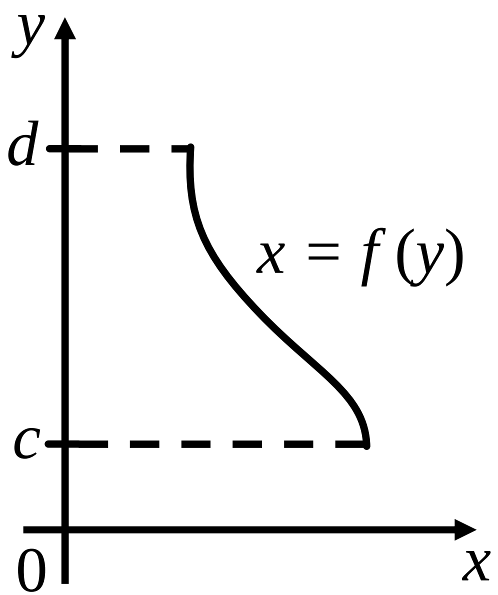
$$V = \int_a^b p\, y^2 \, dx$$

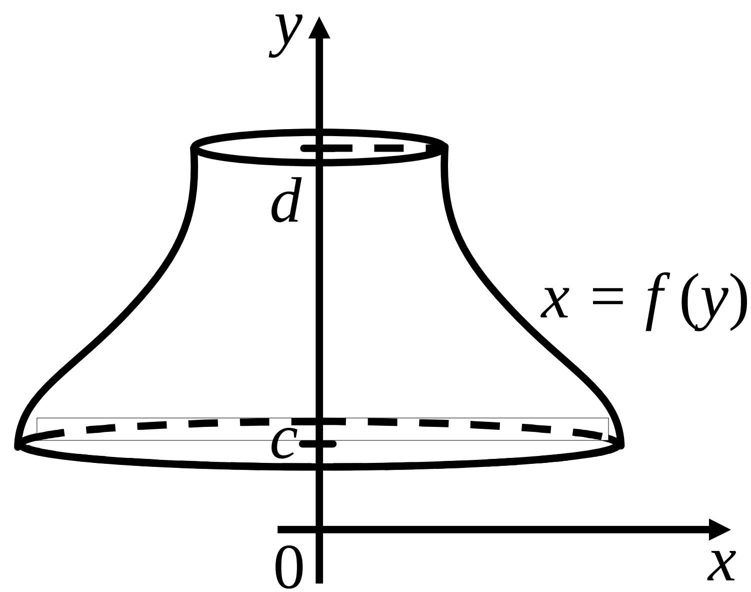
or

$$V = \int_a^b p [\, f(x)]^2 \, dx$$

Volume of Solids of Revolution

(II) About y-axis





$$V = \int_c^d p\, x^2 \; dy$$

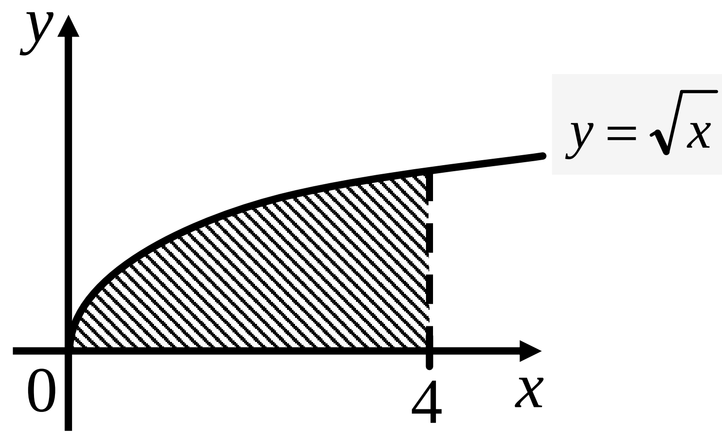
or

$$V = \int_c^d p\,[g(y)]^2 \; dy$$

Example

The region between $y = \sqrt{x}$, $0 \le x \le 4,$ and the x-axis is revolved about the x-axis. Find the volume generated.

$$
\begin{aligned}
V &= p\int_a^b y^2 \, dx \\
&= p\int_0^4 \left(\sqrt{x}\right)^2 \, dx \\
&= p\int_0^4 x \, dx \\
&= 8p \text{ units}^3
\end{aligned}
$$

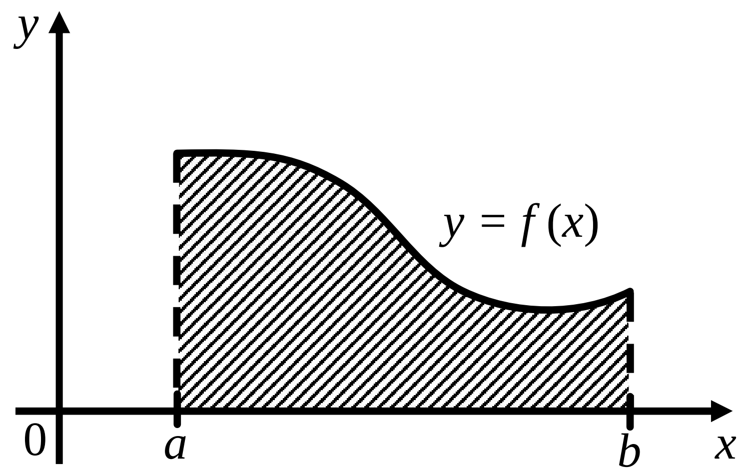


Volume of Solids of Revolution

Volume of solid generated by revolving about the x-axis from $x = a$ to $x = b$ is:

$$V = \int_a^b p\,[\,f\,(x)]^2\,dx$$

$$= \int_a^b p y^2\,dx$$



Can only use this formula if you revolve about $x-$axis

Note : revolving about x-axis

is the same as

revolving about the line $y = 0$

$$V = \int_a^b p[\,f(x)]^2\,dx$$
$$= \int_a^b p y^2\,dx$$

Note : revolving about x-axis

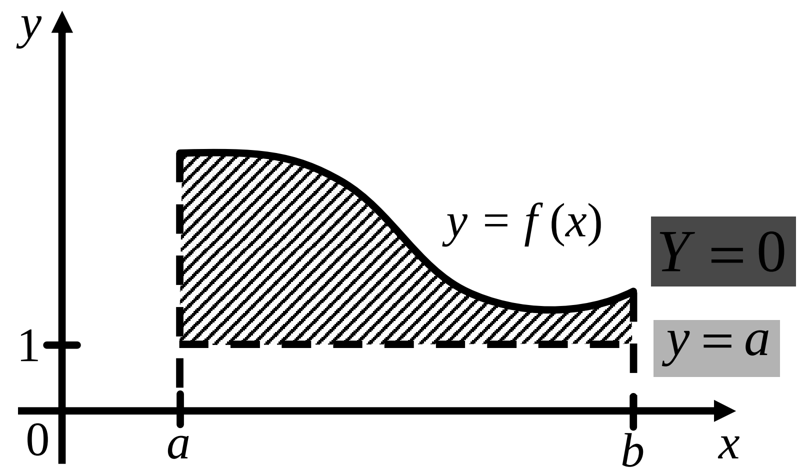
is the same as

revolving about the line $y = 0$

In some questions, we may be revolving about $y = a$ instead of the x-axis

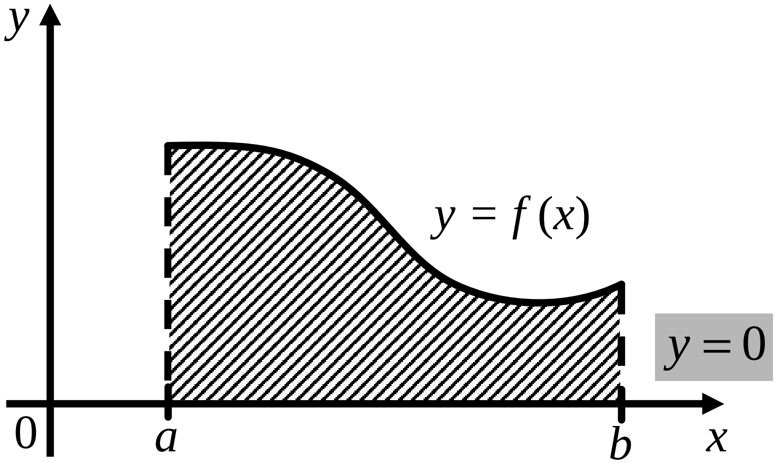
Question:

How to modify the formula to find volume ???



Answer : Shift the $x-$axis by letting $Y = y - a$

The line $y = a$ becomes $Y = 0$ the new $X -$axis



Note : revolving about x-axis

is the same as

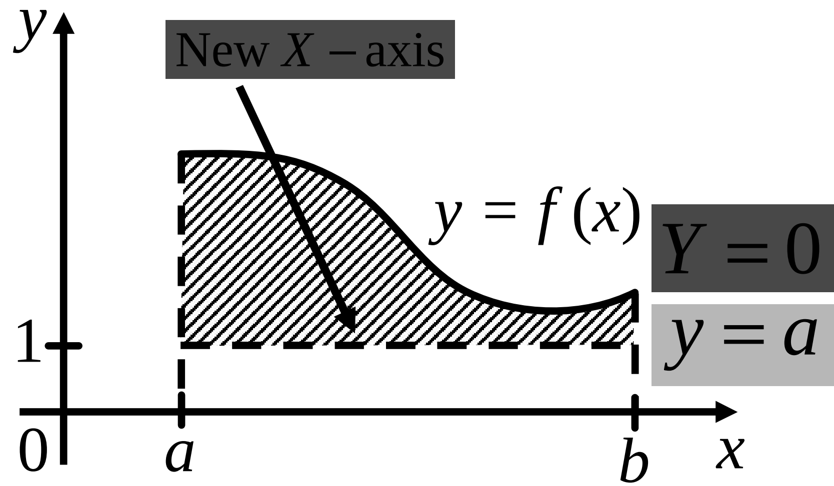
revolving about the line $y = 0$

$$V = \int_a^b p[f(x)]^2\,dx$$

$$= \int_a^b p y^2\,dx$$

Can only use this formula

if you revolve about $x-$axis



Question:

How to modify the formula

to find volume ???

Answer : Shift the $x-$axis by letting

$$Y = y - a$$

$$V = \int_a^b p Y^2\,dx$$

When $y = a$, $Y = 0$

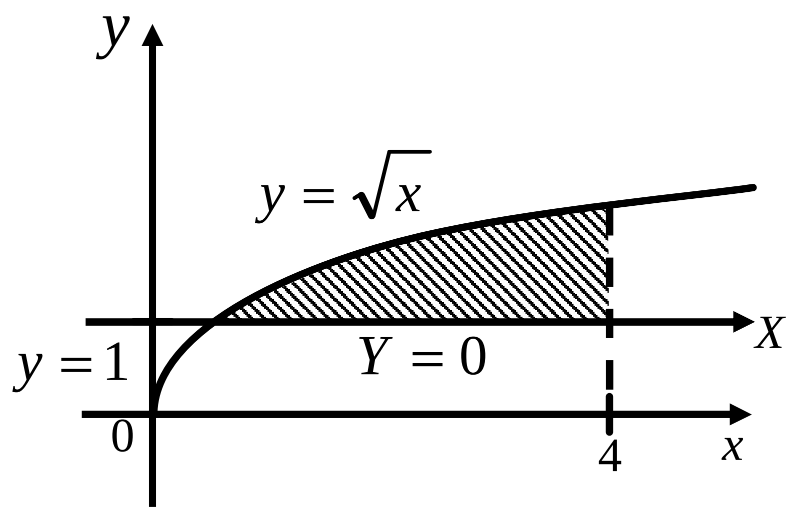
Find the volume of the solid generated by revolving the region bounded by $y=\sqrt{x}$ and the lines $y=1$ and $x=4$ about the line $y=1$.

Let $Y=y-1.$

$y=\sqrt{x}$

$$Y=y-1$$
$$=\sqrt{x}-1$$

$$V=p\int_1^4 Y^2\ dx$$
$$=p\int_1^4\left(\sqrt{x}-1\right)^2\ dx$$
$$=p\int_1^4 (x-2\sqrt{x}+1)\ dx$$
$$=p\left[\frac{x^2}{2}-\frac{4}{3}x^{3/2}+x\right]_1^4$$
$$=\frac{7p}{6}\ \text{units}^3$$



Example

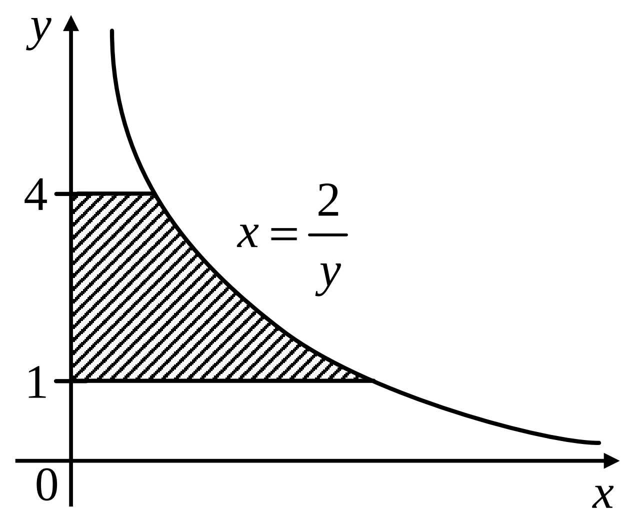
Find the volume of the solid generated by revolving the region bounded by $x = \frac{2}{y}$, $y = 1$ and $y = 4$ about the y-axis.

$$V = p\int_1^4 x^2\,dy$$

$$= p\int_1^4 \left(\frac{2}{y}\right)^2 dy$$

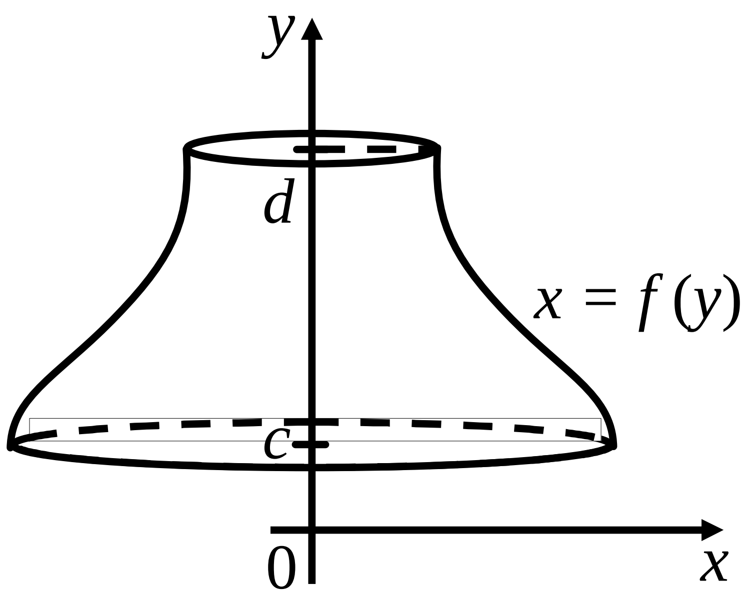
$$= 4p\int_1^4 y^{-2}\,dy$$

$$= 4p\left[\frac{y^{-1}}{-1}\right]_1^4 = 3p \text{ units}^3$$



Revolve about y-axis

Same as the line $x = 0$



$$V = \int_c^d p\,[g(y)]^2\,dy$$

$$V = \int_c^d p\,x^2\,dy$$

Revolve about the line $x = b$

Let $X = x - b$

$$V = \int_c^d p\,X^2\,dy$$

Past Exam Question

- Find the value of $\displaystyle\lim_{x\to 3^+}\frac{x^2\int_3^x\sqrt{t^3+9}\,dt}{|3-x|}$.

$$\lim_{x\to 3^+}\frac{x^2\int_3^x\sqrt{t^3+9}\,dt}{|3-x|}=\lim_{x\to 3^+}\frac{x^2\int_3^x\sqrt{t^3+9}\,dt}{x-3}$$

$$=\lim_{x\to 3^+}\frac{2x\int_3^x\sqrt{t^3+9}\,dt+x^2\left(\sqrt{x^3+9}\right)}{1}$$

$$=54$$

End