Fundamental Theorem of Calculus (Part II)

(II) If F is an ***antiderivative*** of f on $[a,b]$, then

$$\int_a^b f(x)\,dx = [F(x)]_a^b$$
$$= F(b) - F(a)$$

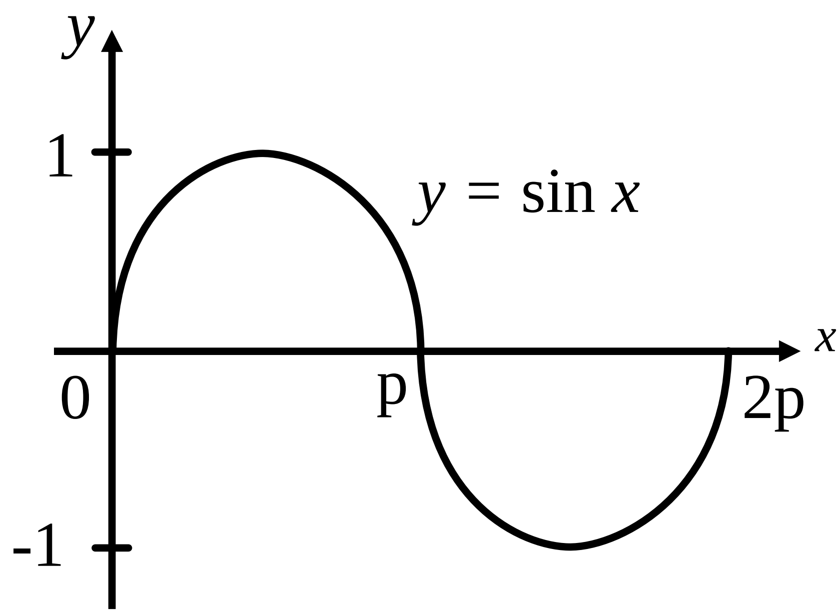
Example

Evaluate $\int_0^{2\pi} \sin x \, dx$.

$$\int_0^{2\pi} \sin x \, dx = \left[-\cos x\right]_0^{2\pi}$$

$$= -(\cos 2\pi - \cos 0)$$

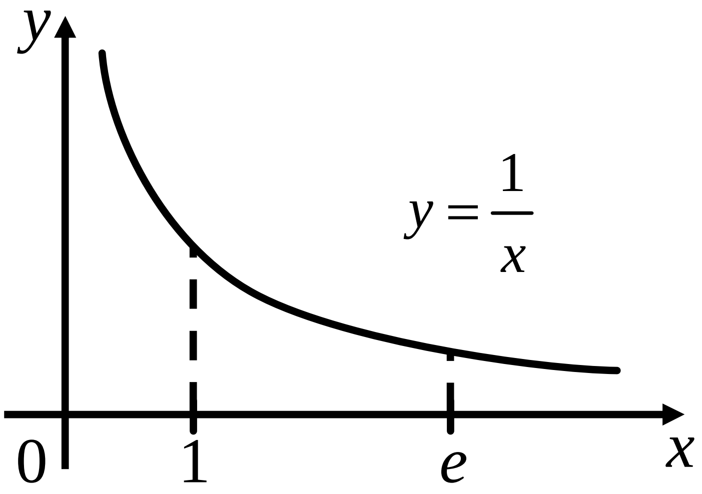
$$= 0$$



Example

Evaluate $\int_1^e \frac{1}{x}\,dx.$

$$\int_1^e \frac{1}{x}\,dx = \left[\ln x\right]_1^e$$
$$= (\ln e - \ln 1)$$
$$= 1$$



Various Integration Techniques

Integration by Substitution - Example

Evaluate $\int (x^2+2x-3)^2(x+1)\ dx$.

Let $u = x^2+2x-3$.

Then $\dfrac{du}{dx} = 2(x+1)$.

$$du = 2(x+1)dx$$

$$\tfrac{1}{2}du = (x+1)dx$$

$$\begin{aligned}\int (x^2+2x-3)^2(x+1)\ dx &= \int u^2 \frac{1}{2}\, du \\ &= \frac{1}{6}u^3 + C \\ &= \frac{1}{6}(x^2+2x-3)^3 + C\end{aligned}$$

Integration by Substitution - Example

Evaluate $\int \sin^4 x \cos x \, dx$.

Let $u = \sin x$. Then $\dfrac{du}{dx} = \cos x$.

$$du = \cos x \, dx$$

$$\begin{aligned}
\int \sin^4 x \cos x \, dx &= \int u^4 \, du \\
&= \frac{1}{5} u^5 + C \\
&= \frac{1}{5} \sin^5 x + C
\end{aligned}$$

Integration by Substitution - Example

Evaluate $\int \frac{(\ln x)^5}{x}\, dx.$

Let $u = \ln x.$ Then $\frac{du}{dx} = \frac{1}{x}.$

$$du = \frac{1}{x}\, dx$$

$$\begin{aligned} \int \frac{(\ln x)^5}{x}\, dx &= \int u^5\, du \\ &= \frac{1}{6}u^6 + C \\ &= \frac{1}{6}(\ln x)^6 + C \end{aligned}$$

Integration by Substitution - Example

Evaluate $\int e^{x+e^x}\ dx.$

Note that $e^{x+e^x} = e^x e^{e^x}.$

$e^{m+n} = e^m e^n$

Let $u = e^x.$

Then $\dfrac{du}{dx} = e^x.$

$du = e^x\ dx$

$$\begin{aligned}\int e^{x+e^x}\ dx &= \int e^x e^{e^x}\ dx \\ &= \int e^u\ du \\ &= e^u + C \\ &= e^{e^x} + C\end{aligned}$$

Integration by Substitution - Example

Evaluate $\int_0^{p/4} \tan x \sec^2 x \; dx.$

Let $u = \tan x.$ Then $\dfrac{du}{dx} = \sec^2 x.$

$$du = \sec^2 x \; dx$$

$$\begin{aligned} \int \tan x \sec^2 x \; dx &= \int u \; dx \\ &= \frac{1}{2}u^2 + C \\ &= \frac{\tan^2 x}{2} + C \end{aligned}$$

$$\int_0^{\frac{p}{4}} \tan x \sec^2 x \; dx = \left[\frac{\tan^2 x}{2} \right]_0^{\frac{p}{4}} = \frac{1}{2}$$

Integration by Substitution - Example

Evaluate $\int_0^{p/4} \tan x \sec^2 x \; dx.$

Let $u = \tan x$, then $\dfrac{du}{dx} = \sec^2 x.$

$$du = \sec^2 x \; dx$$

When $x = 0, \quad u = \tan 0 = 0$

$$x = \frac{p}{4}, \quad u = \tan\frac{p}{4} = 1$$

$$\int_0^{\frac{p}{4}} \tan x \sec^2 x \; dx = \int_0^1 u \; du$$

$$= \left[\frac{u^2}{2}\right]_0^1$$

$$= \frac{1}{2} - \frac{0}{2}$$

$$= \frac{1}{2}$$

Integration by Parts

- Recall the product rule

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

In differential form it becomes

$$d(uv) = u\,dv + v\,du$$

or, equivalently,

$$u\,dv = d(uv) - v\,du$$

Integration by Parts

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$$

Thus we have the Integration-by-parts Formula:

$$\int u\frac{dv}{dx}\ dx = uv - \int v\frac{du}{dx}\ dx.$$

$$\int u\ dv = uv - \int v\ du.$$

Integration by Parts

$$\int u \frac{dv}{dx}\, dx = uv - \int v \frac{du}{dx}\, dx.$$

$$\int u\, dv = uv - \int v\, du$$

tough

easier

Must choose u and dv correctly

The part you choose as u, you differentiate to find du

The part you choose as dv, you integrate to find v

Integration by Parts - Example

Evaluate $\int x \ln x \, dx$.

$$\int u \, dv = uv - \int v \, du.$$

Two choices :

(1)

Let $u = x$

$$\frac{du}{dx} = 1$$

$$du = dx$$

$$dv = \ln x \, dx$$

$$v = \int \ln x \, dx$$

Difficult to find v

(2)

Let $u = \ln x$

$$du = \frac{1}{x} \, dx$$

$$dv = x \, dx$$

$$v = \frac{1}{2} x^2$$

Good Choice

Integration by Parts - Example

Evaluate $\int x \ln x \, dx$.

$$\int u \, dv = uv - \int v \, du.$$

Let $u = \ln x$

$$du = \frac{1}{x} \, dx$$

$$dv = x \, dx$$

$$v = \frac{1}{2}x^2$$

$$\int x \ln x \, dx = \overbrace{\frac{1}{2}x^2}^{v} \overbrace{\ln x}^{u} - \int \overbrace{\frac{1}{2}x^2}^{v} \overbrace{\frac{1}{x} \, dx}^{du}$$

$$= \frac{1}{2}x^2 \ln x - \boxed{\frac{1}{2}\int x \, dx}$$

easy to solve

$$= \frac{1}{2}x^2 \ln x - \frac{x^2}{4} + C$$

Integration by Parts - Example

Evaluate $\int \ln x \, dx$.

$$\int u \, dv = uv - \int v \, du.$$

Let $u = \ln x$

$$du = \frac{1}{x} dx$$

$$dv = dx$$

$$v = \int 1 \, dx = x$$

$$\begin{aligned} \int \ln x \, dx &= (\ln x)x - \int x \left(\frac{1}{x} \right) dx \\ &= x \ln x - \int 1 \, dx \\ &= x \ln x - x + C \end{aligned}$$

Integration by Parts - Example

Evaluate $\int_0^1 xe^x dx$.

$$\int u\ dv = uv - \int v\ du.$$

Let $u = x$ $\qquad dv = e^x dx$

$du = dx$ $\qquad v = \int e^x\ dx = e^x$

easy to integrate

$$\int_0^1 xe^x dx = \left[xe^x\right]_0^1 - \int_0^1 e^x dx$$

$$= 1 \cdot e^1 - 0 - \left[e^x\right]_0^1$$

$$= e - (e^1 - e^0)$$

$$= 1$$

Let $u = e^x$ $\qquad dv = xdx$

$du = e^x dx$ $\qquad v = \int x\ dx = \frac{1}{2}x^2$

$$\int xe^x dx = \frac{1}{2}x^2 e^x - \int \frac{1}{2}x^2 e^x dx$$

difficult to integrate

Wrong choice of u and v

Integration by Parts - Example

Evaluate $\int x^2 e^x \, dx$.

$$\int u \, dv = uv - \int v \, du.$$

Let $u = x^2$ $\qquad dv = e^x dx$

$du = 2x \, dx$ $\qquad v = \int e^x \, dx = e^x$

$$\begin{aligned} \int x^2 e^x \, dx &= x^2 e^x - \int e^x 2x \, dx \\ &= x^2 e^x - 2\int x e^x \, dx \\ &= x^2 e^x - 2(xe^x - e^x) + C \\ &= x^2 e^x - 2xe^x + 2e^x + C \end{aligned}$$

Integration by Parts - Example

Evaluate $\int e^x \cos x \, dx$.

$$\int u \, dv = uv - \int v \, du.$$

Let $u = e^x$ $\qquad dv = \cos x \, dx$

$du = e^x \, dx \qquad v = \int \cos x \, dx = \sin x$

$$\begin{aligned}\int e^x \cos x \, dx &= e^x \sin x - \int \sin x \, e^x \, dx \\ &= e^x \sin x - \int e^x \sin x \, dx\end{aligned}$$

Need integration by parts again

To find $\int e^x \sin x \, dx.$

$$\int u \, dv = uv - \int v \, du.$$

Similarly to evaluate $\int e^x \cos x \, dx,$

let $u = e^x$ $\quad dv = \sin x \, dx$

$du = e^x \, dx$ $\quad v = \int \sin x \, dx = -\cos x$

$$\int e^x \sin x \, dx = e^x(-\cos x) - \int (-\cos x) e^x \, dx$$

$$= -e^x \cos x + \int e^x \cos x \, dx$$

Get back the integral we started with

$$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx$$

Integration by Parts - Example

$$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx$$

$$\int e^x \cos x \, dx = e^x \sin x - \boxed{\int e^x \sin x \, dx}$$

$$= e^x \sin x - \boxed{\left(-e^x \cos x + \int e^x \cos x \, dx\right)}$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

$$2\int e^x \cos x \, dx = e^x \sin x + e^x \cos x$$

$$\int e^x \cos x \, dx = \frac{1}{2}(e^x \sin x + e^x \cos x)$$

Integration by Parts - Remark

The method is suitable for other integrands such as

$x^n e^x$, $x^n \ln x$, $x^n \cos x$, $x^n \sin x$, etc.

$$\int u\,dv = uv - \int v\,du$$

tough

easier

Must choose u and dv correctly

The part you choose as u, you differentiate to find du

The part you choose as dv, you integrate to find v