CS4243
Computer Vision
&
Pattern Recognition

Maths Fundamentals

Taylor Series

The Taylor series of a function f(x) at x=a is given by

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^3(a)}{3!}(x-a)^3 + \cdots$$

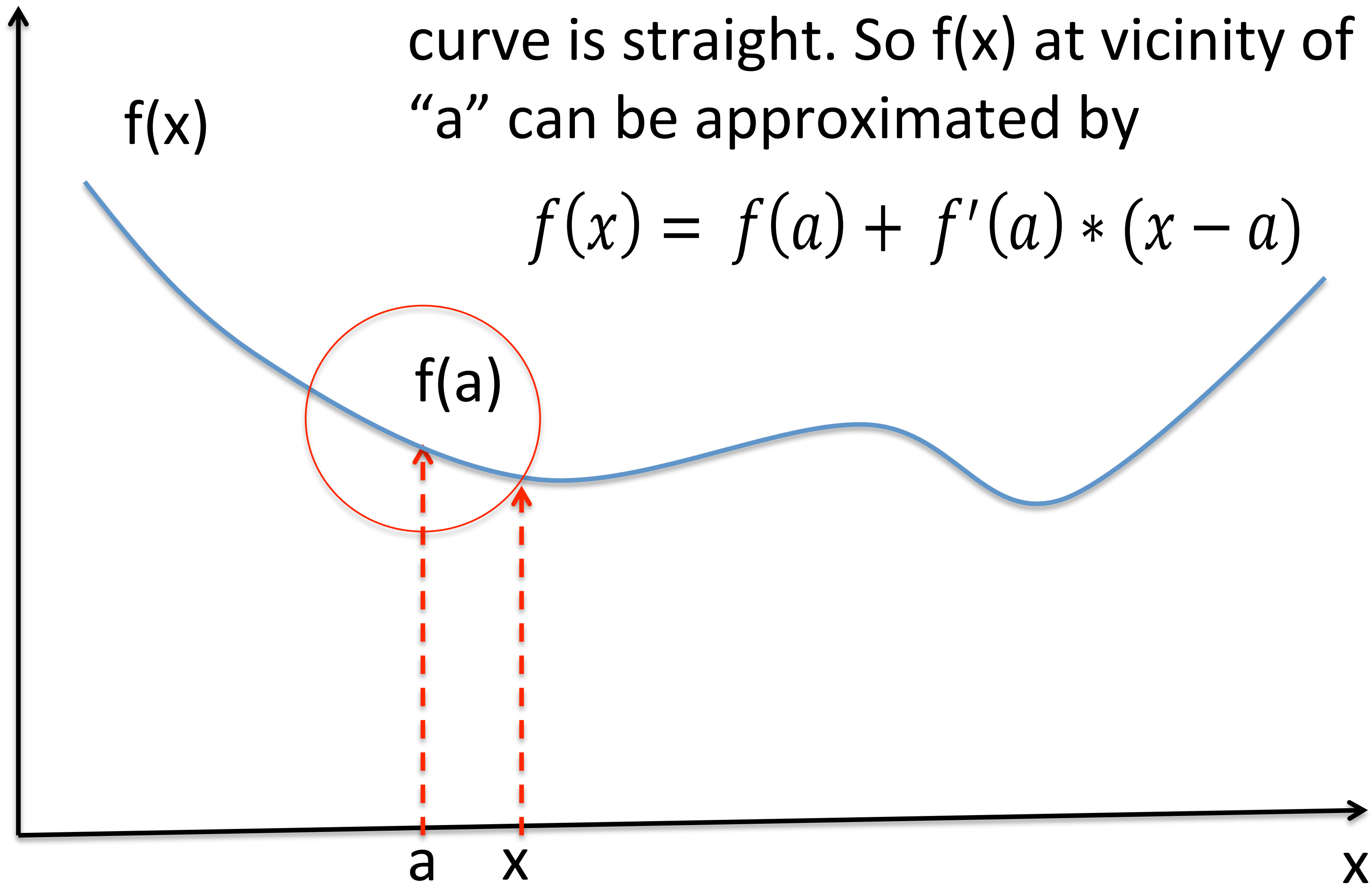
written compactly

$$f(x) = \sum_{n=0}^{\propto} \frac{f^n(a)}{n!}(x-a)^n$$

What do we mean by “ignoring higher order terms” ?

In the vicinity of “a”, can assume curve is straight. So f(x) at vicinity of “a” can be approximated by

$$f(x) = f(a) + f'(a) * (x - a)$$



Linear Algebra

A quantity which is characterized by magnitude and direction is called a vector.

We represent a vector in *N*-dimensional space by an *N*x1 tuple:

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_N \end{bmatrix}$$

Transpose of a Vector

The transpose of a vector changes it from Nx1 to 1xN, or from 1xN to Nx1.

Example:

$$a^T = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_N \end{bmatrix}$$

Magnitude of a Vector

$$\|a\| = \sqrt{a_1^2 + a_2^2 + a_3^2 + \cdots + a_N^2}$$

Multiplication of Vectors

Scalar Multiplication

$$b = \alpha\, a$$

$$= \begin{bmatrix} \alpha a_1 \\ \alpha a_2 \\ \alpha a_3 \\ \vdots \\ \alpha a_N \end{bmatrix}$$

where α is a scalar

Multiplication of Vectors

Vector Multiplication (dot-product)

$$
\begin{aligned}
c &= a \cdot b \\
&= a^T b \\
&= \sum_{i=1}^{N} a_i b_i
\end{aligned}
$$

note that c is a scalar and it is also given by

$$c = \|a\| \, \|b\| \cos\theta$$

Multiplication of Vectors

Vector Multiplication (cross-product)

Let $a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$, and $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

the vector cross product is given by

$$c = a \wedge b$$

$$= \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= (a_2 b_3 - a_3 b_2)\, i \quad - \quad (a_1 b_3 - a_3 b_1)\, j \quad + \quad (a_1 b_2 - a_2 b_1)\, k$$

the magnitude of vector ***c*** is given by

$$\|c\| = \|a\| \, \|b\| \sin\theta$$

The direction of *c* is perpendicular to both *a* and *b*.

Properties of Vector Products

If the unit vectors i¸ j, k are the three orthogonal axis of a right-handed coordinate system, then

$$i \cdot i = j \cdot j = k \cdot k = 1$$

$$i \cdot j = j \cdot k = k \cdot i = 0$$

$$i \wedge i = j \wedge j = k \wedge k = 0$$

$$i \wedge j = -j \wedge i = k$$

$$j \wedge k = -k \wedge j = i$$

$$k \wedge i = -i \wedge k = j$$

We also have the following results

$$a \cdot b \wedge c = b \cdot c \wedge a = c \cdot a \wedge b$$

$$a \wedge (b \wedge c) = (a \cdot c)b - (a \cdot b)c$$

Linearly Dependent and Linearly Independent

A set of vectors $\{x_1, x_2, \cdots, x_m\}$ is linearly dependent if exist a set of scalars $\{\alpha_1, \alpha_2, \cdots, \alpha_m\}$, not all zero, such that

$$\sum_{i=1}^{m} \alpha_i x_i = 0$$

If the only way to satisfy the equation is to have

$\alpha_i = 0 \quad \forall i$, then we say that the set of vectors

$\{x_1, x_2, \cdots, x_m\}$ are linearly independent.

Basis Set

The set of vectors that can be used to represent all Nx1 vectors is called a basis vector set. The basis vector set is said to "span" the Nx1 vector space.

For example, if $\{v_i\}_{1 \le i \le N}$ is a basis set, then any Nx1 vector x can be written as

$$x = \sum_{i=1}^{N} c_i v_i$$

where c_i is a scalar.

Vector Space

A real vector space is a set of vectors together with rules for vector addition and multiplication by real numbers. The addition and multiplication must produce vectors that are within the space. In other words, if we add any vectors in the space, their sum is in the space. If we scale any vector, it still remains in the space.

Schwarz Inequality

$$\left|a^T b\right| \leq \|a\| \|b\|$$

Matrix Algebra

A matrix of dimensions M by N is a rectangular block of numbers (real or complex) with M rows and N columns.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{pmatrix}$$

Transpose

$$A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{M1} \\ a_{12} & a_{22} & \cdots & a_{M2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1N} & a_{2N} & \cdots & a_{MN} \end{pmatrix}$$

Diagonal Matrix

A diagonal matrix is a matrix with all off-diagonal entries equal to zero.

Identity Matrix

An identity Matrix, written as *I*, is a diagonal matrix will all its diagonal entries equal to one.

Symmetric Matrix

A symmetric matrix is a square matrix whose transpose is equal to itself, i.e. $A = A^T$

Skew Symmetric Matrix

A skew symmetric matrix is a square matrix whose transpose is equal to negative of itself, i.e.

$$A = -A^T$$

Determinants

For a 2x2 matrix *A*, its determinant is given by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= a_{11}a_{22} - a_{12}a_{21}$$

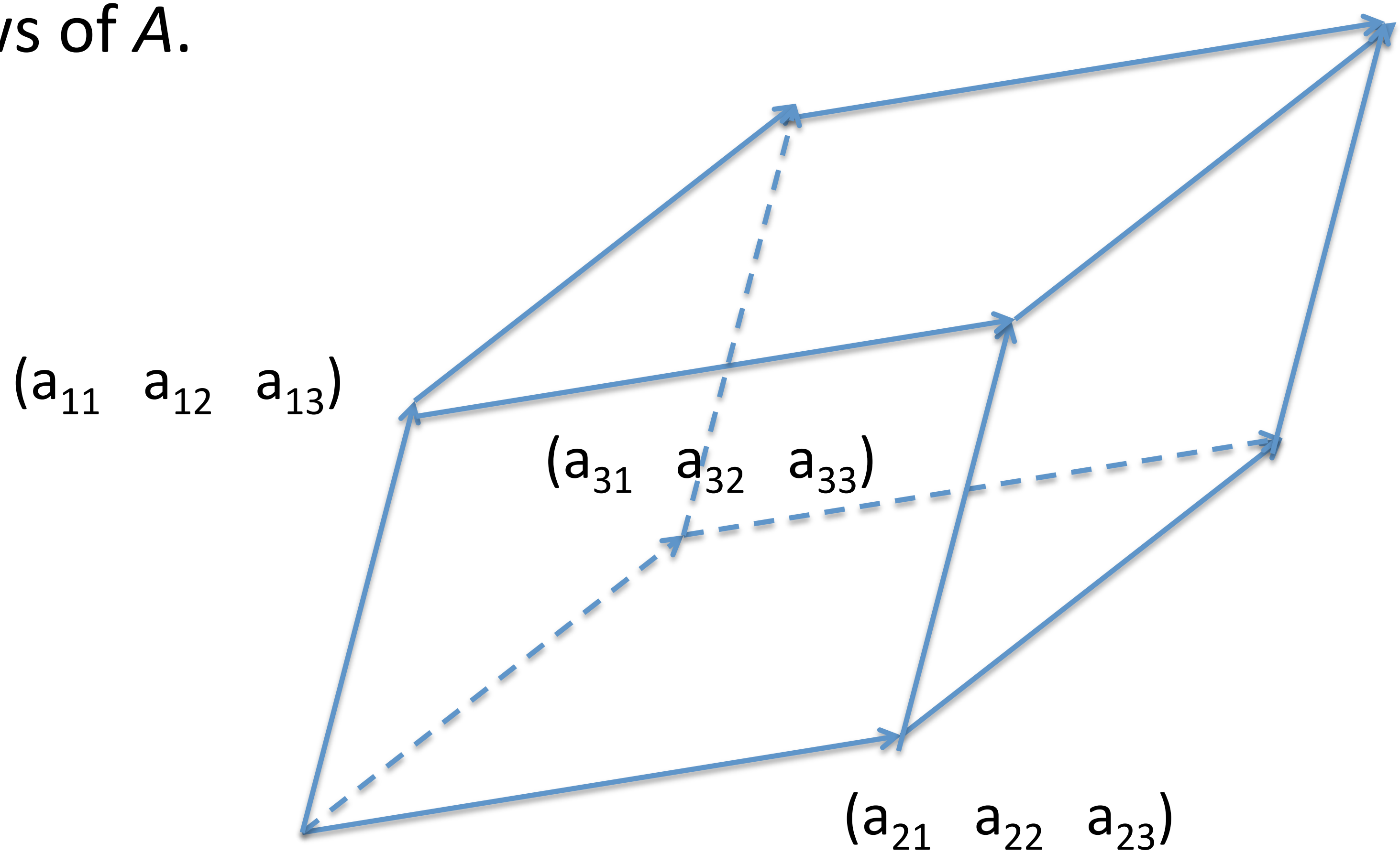
Determinants

For a 3x3 matrix *A*, its determinant is given by

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21} - a_{31}a_{22}a_{13} - a_{12}a_{21}a_{33} - a_{11}a_{32}a_{23}$$

The determinant of a matrix *A* equals the volume of a parallelepipe. The edges of the parallelepipe come from the rows of *A*.



Properties of Determinants

- $$|A| = \frac{1}{|A^{-1}|}$$

- $$|A| = |A^T|$$

- $$|AB| = |A||B|$$

- If any two rows of *A* are interchanged, the sign of its determinant is changed.

Rank of a Matrix

The rank of a matrix indicates the number of linearly independent rows (or columns) in the matrix.

Let $r(A)$ represents the rank of a matrix *A*. Then

$$r(AB) \le r(A)$$

$$r(AB) \le r(B)$$

which also means that

$$r(AB) \le \min(r(A), r(B))$$

Some General Properties of Matrices

$$IA = AI = A$$

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$(AB)^T = B^T A^T$$

$$(A^{-1})^T = (A^T)^{-1}$$

$$(AB)C = A(BC)$$

$$A(B+C) = AB + AC$$

$$(B+C)D = BD + CD$$

$AB \neq BA$ in general

Null Space

The nullspace of a matrix *A* consists of all vectors *x* such that $Ax = 0$.

If *A* is rank-deficient (i.e. not full rank), then there is a non-zero solution for *x*.

Solution of a Linear System

Given the matrix *A* and vector *b*,
the problem $Ax = b$ has a solution if and only if the vector *b* can be expressed as a linear combination of the columns of *A*.

If *A* is a square matrix and invertible, then the solution is

$$x = A^{-1}b$$

Overdetermined system:

If A is a rectangular matrix of size M by N, where $M>N$,

and if $A^T A$ is invertible (i.e. square and full

rank), then the linear least squares solution to x is given

by

$$x = (A^T A)^{-1} A^T b$$

Eigenvalues and Eigenvectors

If x is an eigenvector of A and λ is the eigenvalue, then

$$Ax = \lambda x$$

where A is a matrix, x a vector, and λ a scalar

Properties of Eigenvalues and Eigenvectors

The sum of all the eigenvalues of the matrix *A* equals the sum of the diagonal entries of *A* (note: sum of diagonal entries is also known as trace), i.e.

$$\sum_{i=1}^{n} \lambda_i = trace(A)$$

The product of all the eigenvalues of the matrix *A* equals the determinant of *A*.

Eigenvectors corresponding to different eigenvalues are linearly independent.

Eigenvectors of a real symmetric matrix are orthogonal.

Eigenvalues of a real symmetric matrix are also real.

Diagonalization of Matrix

If the n by n matrix A has n linearly independent eigenvectors, and if we form a matrix S using these eigenvectors as the columns of S, then $S^{-1}AS$ is a diagonal matrix i.e.

$$S^{-1}AS = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

Singular Value Decomposition

Any *m* by *n* matrix *A* can be decomposed into

$$A = U \Sigma V^T$$

The columns of *U* are eigenvectors of AA^T

U is an *m* by *m* matrix

The columns of *V* are eigenvectors of $A^T A$

V is an *n* by *n* matrix

The entries on the diagonal of $\sum$ are known as the singular values. They are the square roots of the eigenvalues of both AA^T and $A^T A$

The number of non-zero singular values equals the rank of matrix *A*.

Reference:

"Linear Algebra and its Applications" by Gilber Strang

Python Commands

- ```
  import numpy as np
  import numpy.linalg as la
  ```
- `A = np.matrix(np.random.rand(3,3))`
- `invA = la.inv(A)`
- `K = np.cross(I, J)`
- `detA = la.det(A)`
  ```

Python Commands

- eigvalues, eigvectors = la.eig(A)

  note: eigenvalues given by Python are not sorted
  can use np.argsort(eigvalues) to sort

- U,S,VT = la.svd(A)

- rankA = la.matrix_rank(A)