Other Types of Differentiation

Cartesian equation --- An equation connecting *x* and *y*

$$y = x^3 + 4x \qquad y = x^2 + \sqrt{x} \qquad x^2 + y^2 = 9$$

Parametric equations

1. $x = 2t \qquad y = t^2 + 1$

2. $x = \sin \boldsymbol{q} + 2 \qquad y = \cos \boldsymbol{q} - 5$

3. $x = 1 + e^t \qquad y = e^{2t}$

Other Types of Differentiation

- Parametric Differentiation

Given $y = f(x)$, where

$$\begin{cases} y = u(t) \\ x = v(t), \end{cases}$$

we have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{u'(t)}{v'(t)}$$

Parametric Differentiation - Example

- Let $x = a(t - \sin t)$ and $y = a(1 - \cos t)$. Find $\dfrac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{a\sin t}{a(1-\cos t)}$$

$$= \frac{2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)}{2\sin^2\left(\frac{t}{2}\right)}$$

$$= \cot\left(\frac{t}{2}\right)$$

$$x = v(t) \qquad y = u(t)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{u'(t)}{v'(t)}$$

Pause and Think !!!

True or false ??

$$\frac{d^2y}{dx^2} = \frac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}} = \frac{u''(t)}{v''(t)}$$

Derivative – Rules of Differentiation

Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{d}{dx}(y) = \frac{d}{du}(y) \cdot \frac{du}{dx}$$

Cartesian equation --- An equation connecting x and y

$$y = x^3 + 4x$$

$$y = x^2 + \sqrt{x}$$

$$x^2 + y^2 = 9$$

$$\frac{dy}{dx} = 3x^2 + 4$$

$$\frac{dy}{dx} = 2x + \frac{1}{2\sqrt{x}}$$

Use

Implicit

Differentiation

Ordinary differentiation	Implicit differentiation
$\frac{d}{dx}(x^2) = 2x$	$\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$
$\frac{d}{dx}(x^n) = nx^{n-1}$	$\frac{d}{dx}(y^n) = ny^{n-1}\frac{dy}{dx}$
$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}(\sin y) = \cos y\frac{dy}{dx}$
$\frac{d}{dx}(e^x) = e^x$	$\frac{d}{dx}(e^y) = e^y\frac{dy}{dx}$
$\frac{d}{dx}(\ln x) = \frac{1}{x}$	$\frac{d}{dx}(\ln y) = \frac{1}{y}\frac{dy}{dx}$
$\frac{d}{dx}(x) = 1$	$\frac{d}{dx}(y) = 1\frac{dy}{dx}$

Implicit Differentiation - Example

- Find $\dfrac{dy}{dx}$ if $2y = x^2 + \sin y$.

Differentiate both sides with respect to x,

$$2\frac{dy}{dx} = 2x + \cos y \cdot \frac{dy}{dx}$$

So,

$$(2 - \cos y)\frac{dy}{dx} = 2x \Rightarrow \frac{dy}{dx} = \frac{2x}{2 - \cos y}$$

Implicit Differentiation

Pause and Think !!!

What is $\dfrac{d}{dx}x^x$, where $x > 0$?

Implicit Differentiation

What is $\dfrac{d}{dx}x^x$, where $x>0$?

Let $y = x^x$.

Then $\ln y = \ln x^x$

$= x \ln x.$

Note : $\ln a^b = b \ln a$

Differentiating both sides *w.r.t* x yields

$$\frac{1}{y}\frac{dy}{dx} = 1 + \ln x$$

So,

$$\frac{dy}{dx} = y(1+\ln x) = x^x(1+\ln x) = x^x + x^x \ln x$$

Implicit Differentiation

To differentiate $\dfrac{d}{dx} f(x)^{g(x)}$

Let $y = f(x)^{g(x)}$.

Consider $\ln y = \ln f(x)^{g(x)}$

$$= g(x) \ln f(x)$$

Implicit differentiation and product rule

Other Types of Differentiation

- Higher Order Derivatives

  Higher order derivatives are obtained when we differentiate repeatedly. Let $y = f(x)$, then the following notation is used:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}=f''(x), \quad \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)=\frac{d^3y}{dx^3}=f'''(x).$$

Other Types of Differentiation

In general, the n-th derivative is denoted by

$$\frac{d^n y}{dx^n} \quad or \quad f^{(n)}(x)$$

Higher Order Derivatives - Example

- Let $f(x)=\sqrt{x}$. Compute $f'''(x)$

$$f'(x)=\frac{1}{2}x^{-\frac{1}{2}},\ f''(x)=-\frac{1}{4}x^{-\frac{3}{2}},\ f'''(x)=\frac{3}{8}x^{-\frac{5}{2}}$$

Pause and Think !!!

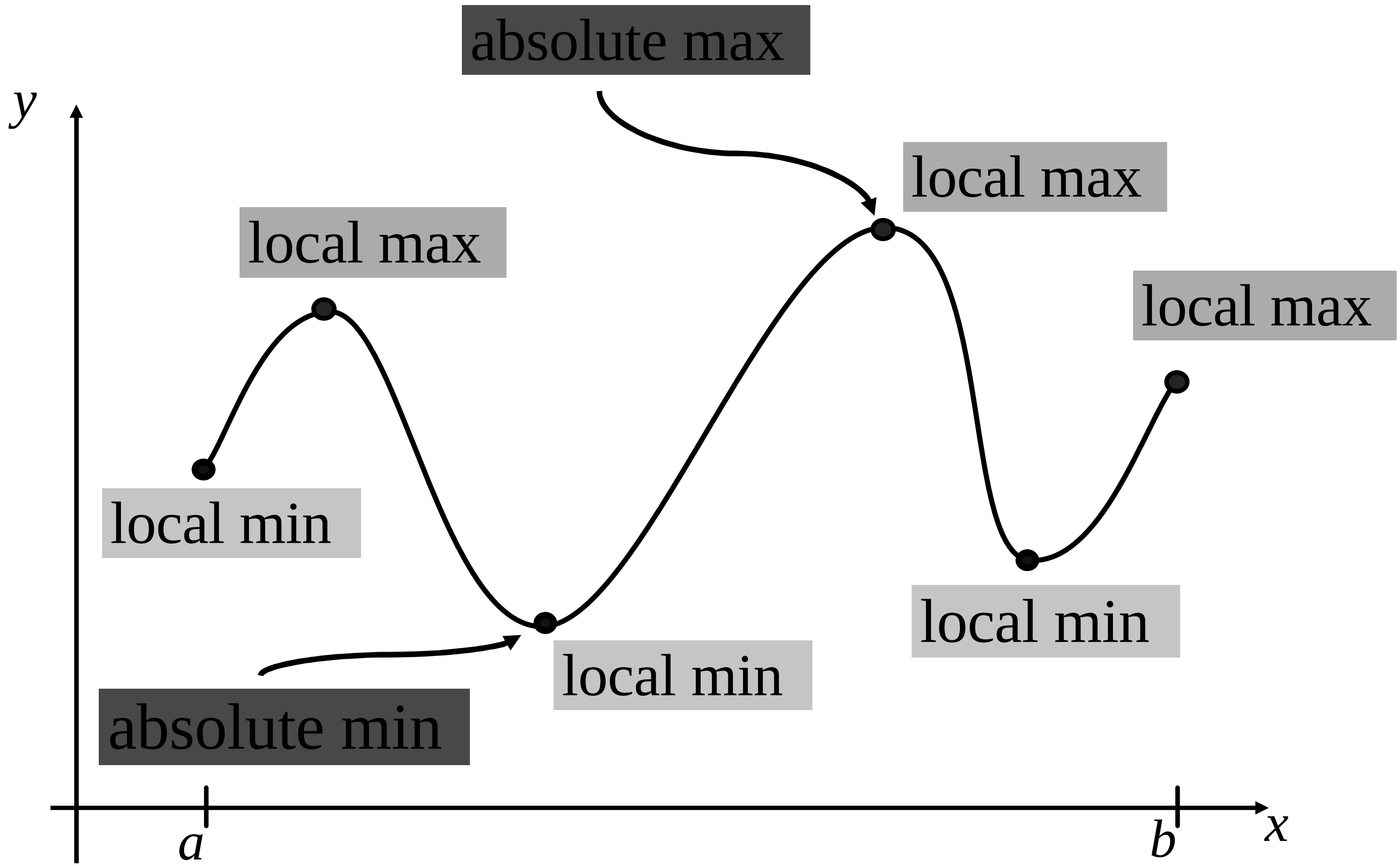
Let $f(x) = \sqrt{x}$.

What is $f^{(n)}(x)$.

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}, \; f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}, \; f'''(x) = \frac{3}{8}x^{-\frac{5}{2}}$$

Maxima and Minima

Local and absolute extremes



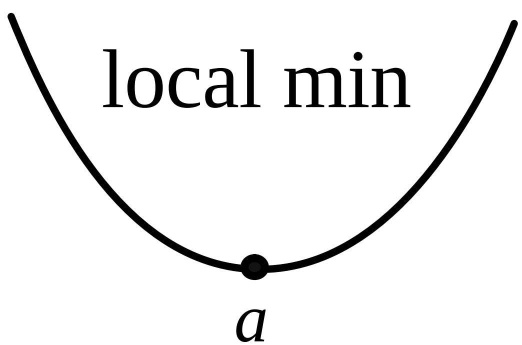
What you have done in JC/High school

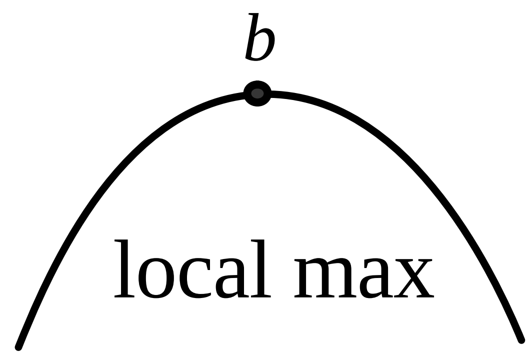
To find local max and local min of $y = f(x)$.

Step 1. Find $\dfrac{dy}{dx}$

Step 2. Set $\dfrac{dy}{dx} = 0$ and find value(s) of x

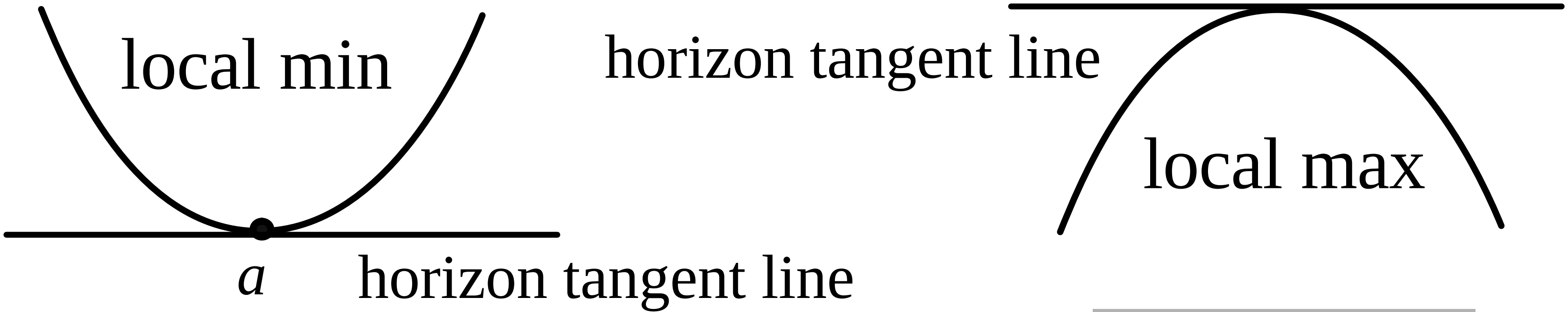
Step 3. Test for local max/local min





$\dfrac{dy}{dx} = 0$ at $x = a$

$\dfrac{dy}{dx} = 0$ at $x = b$



$\dfrac{dy}{dx} = 0 \;\text{ at } x = a$

$\dfrac{dy}{dx} = 0 \;\text{ at } x = b$

Pause and Think !!!

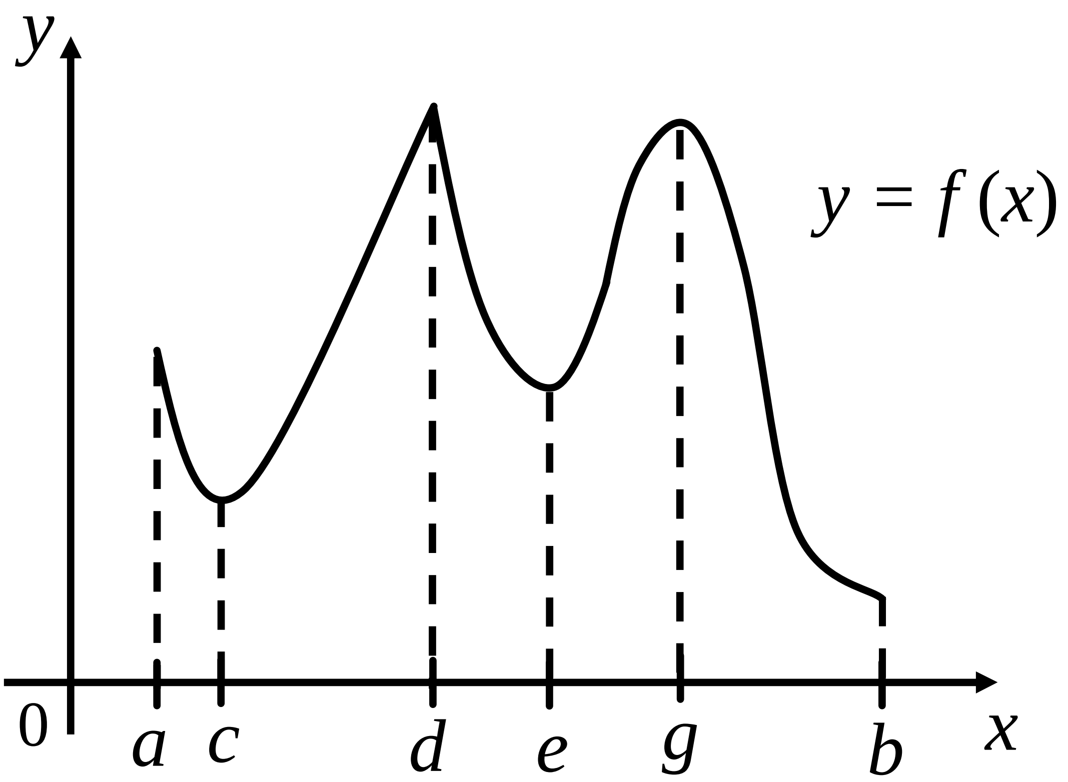
Are there any max points / min points

such that $\dfrac{dy}{dx} \neq 0$???

Pause and Think !!!

Are there any max points / min points

such that $\dfrac{dy}{dx} \neq 0$???

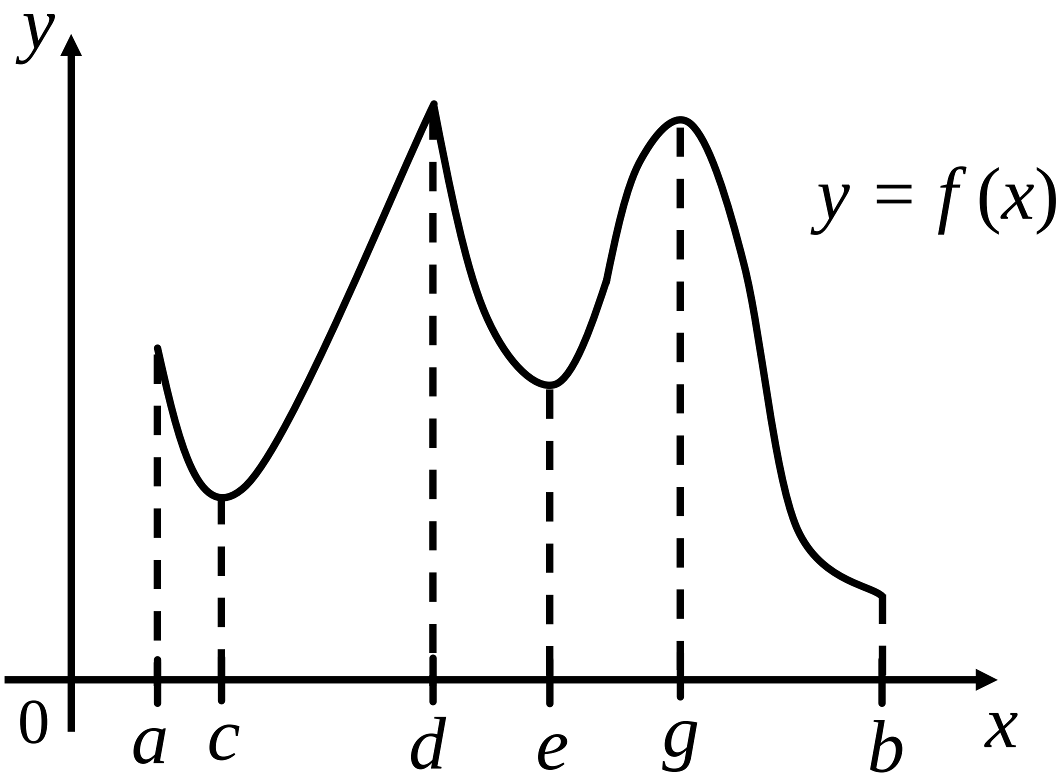

y
$y = f(x)$
0
a
c
d
e
g
b
x

Maxima and Minima

Local and absolute extremes

(i) f has a ***local (relative) maximum values*** at 'a', 'd' and 'g'.

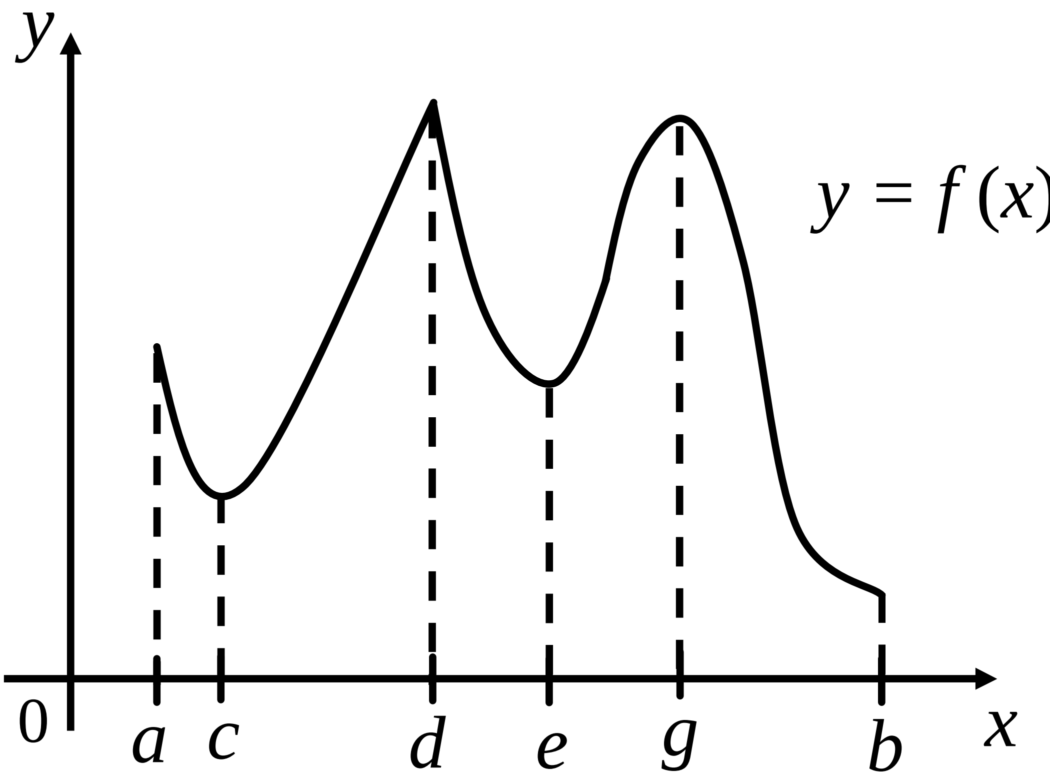
(ii) f has a ***local (relative) minimum values*** at 'c', 'e' and 'b'.



Maxima and Minima

(iii) f has the ***absolute maximum value*** at 'd'.

(iv) f has the ***absolute minimum value*** at 'b'.



Note:

(1) 'a' and 'b' are ***end points*** of the domain

(2) $f'(c) = f'(e) = f'(g) = 0$

(3) $f'(d)$ does not exist

Maxima and Minima

- Finding extreme values

Points where f can have an extreme values are

(1) Interior points where $f'(x) = 0$.

(2) Interior points where $f'(x)$ does not exist.

(3) End points of the domain of f.

Maxima and Minima

- Critical Points

An interior point of the domain of a function f where f' is zero or fails to exist is a ***critical point*** of f.

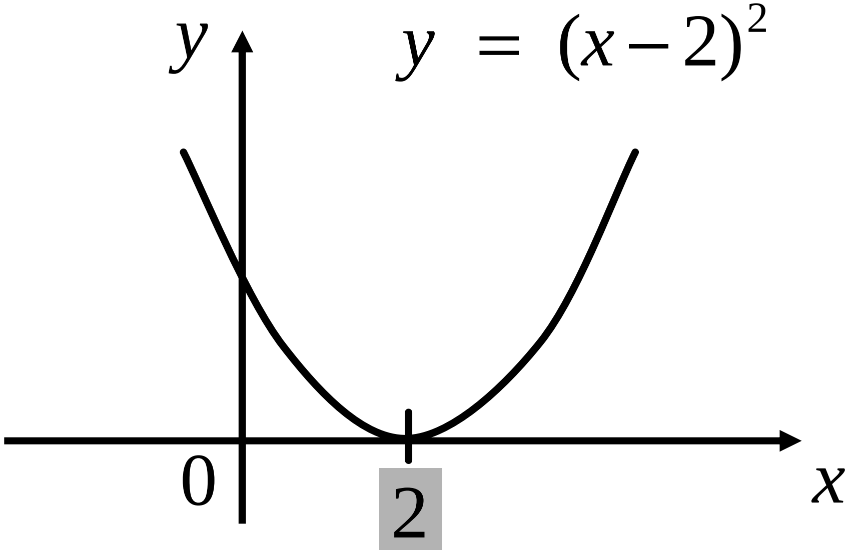
Example

$$f(x) = (x-2)^2$$

$$f'(x) = 2(x-2)$$

$$f'(x) = 0$$

$$x = 2$$



local min at $x = 2$

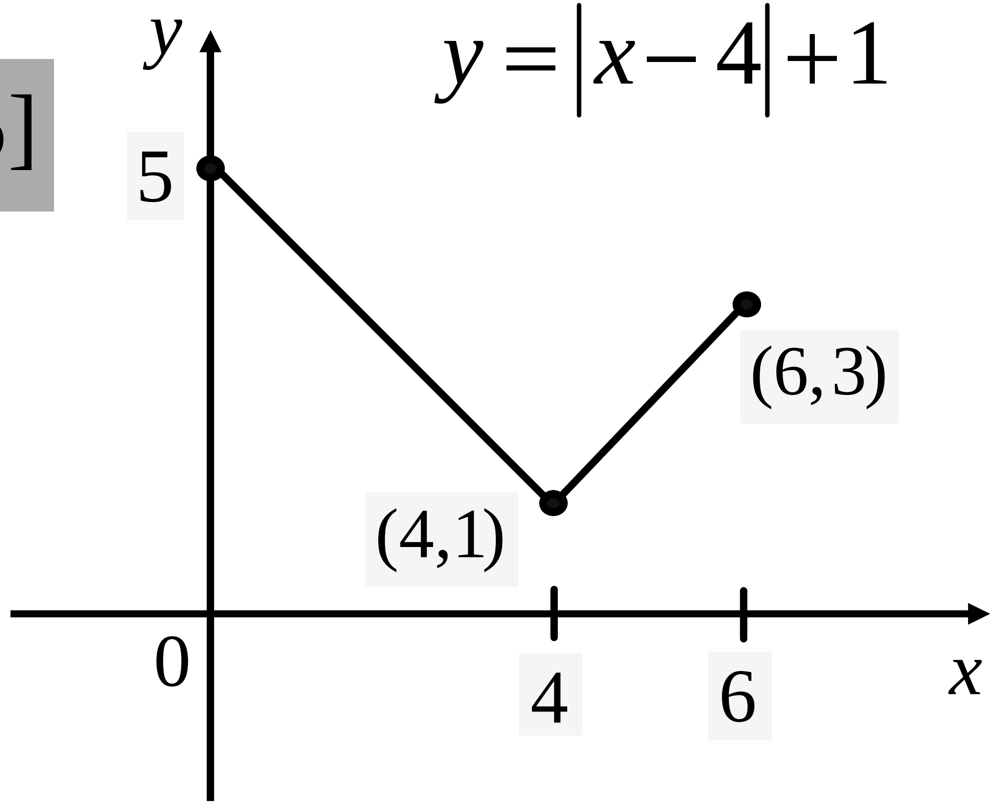
$$f'(2) = 0$$

absolute min at $x = 2$

Note : No absolute max

Example

$f(x) = |x-4| + 1 \text{ on } [0,6]$



local min at $x = 4$

$f\,'(4)$ does not exist

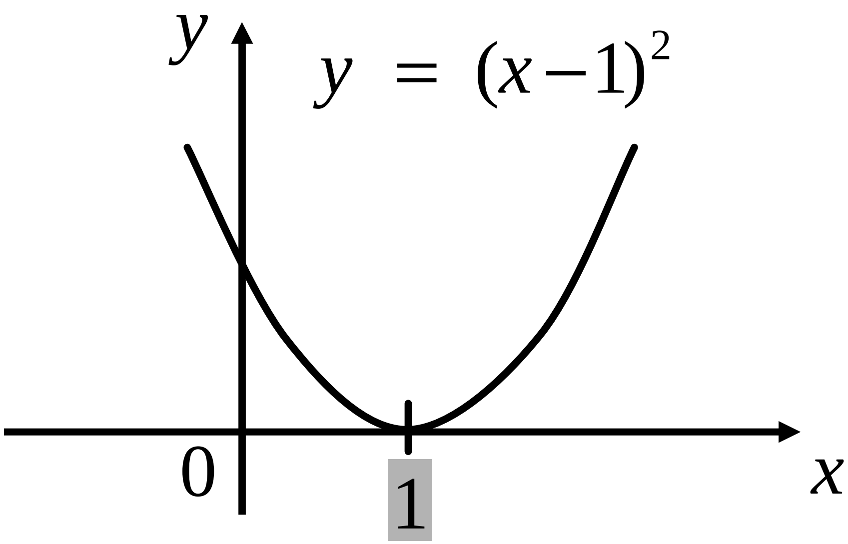
local max at $x = 0$

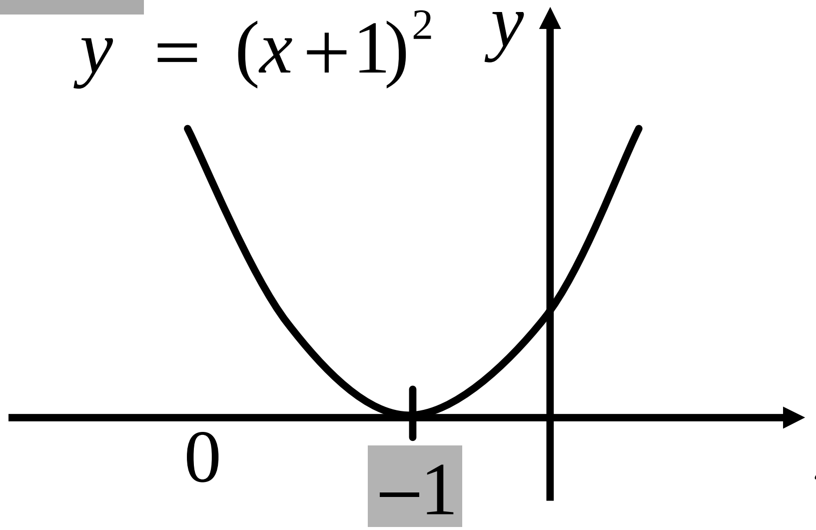
local max at $x = 6$

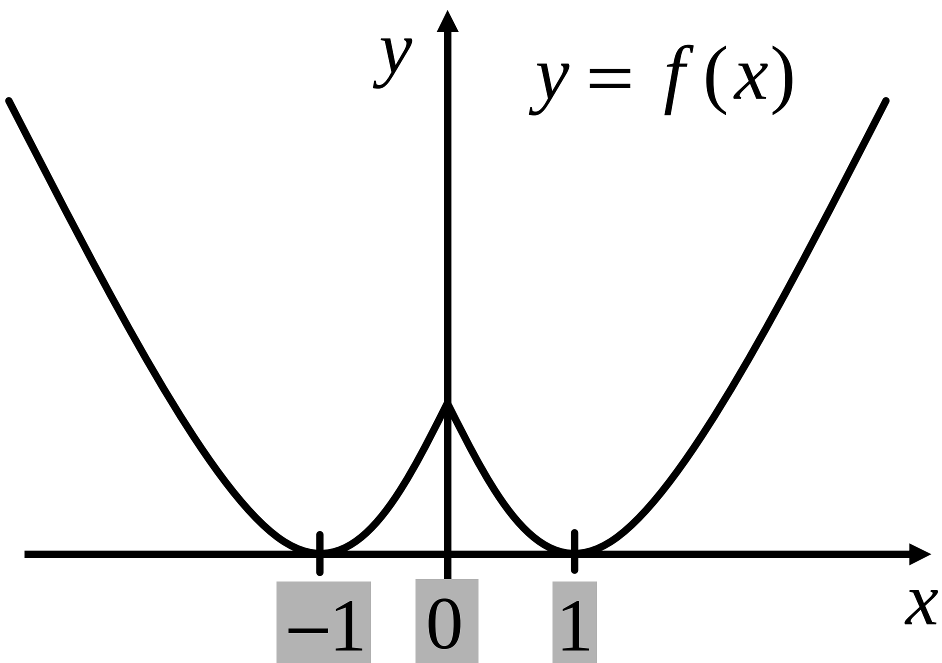
absolute min at $x = 4$

absolute max at $x = 0$

$$f(x) = \begin{cases} (x-1)^2 & \text{if } x \geq 0, \\ (x+1)^2 & \text{if } x < 0. \end{cases}$$

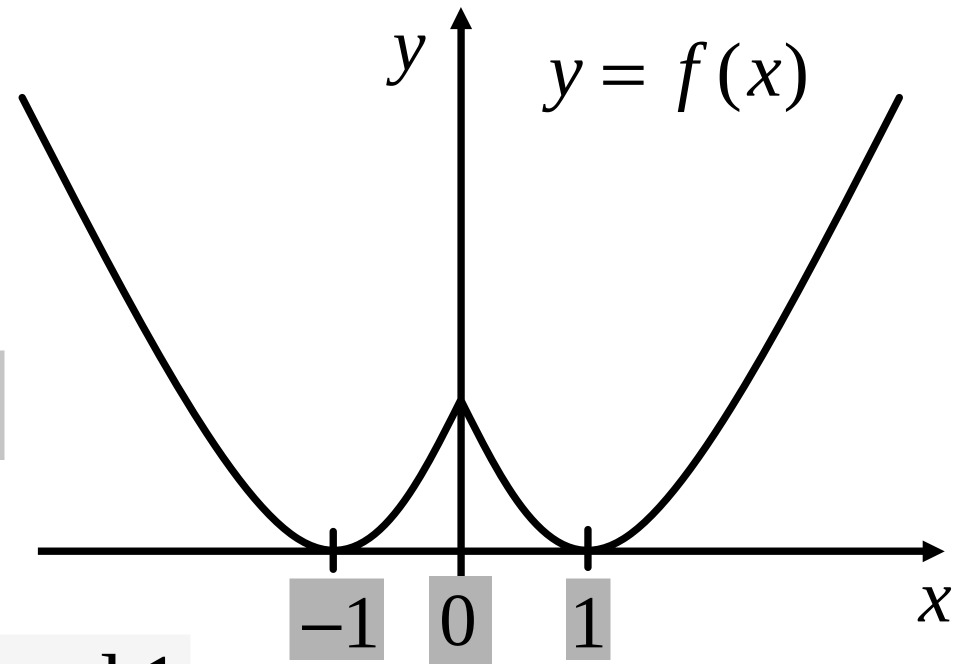






Critical Points - Example

$$f(x) = \begin{cases} (x-1)^2 & \text{if } x \geq 0, \\ (x+1)^2 & \text{if } x < 0. \end{cases}$$



Note : $f\,'(0)$ does not exist

Critical points at $x = -1, 0$ and 1

local min at $x = -1$ and 1

absolute min at $x = -1$ and 1

local max at $x = 0$

absolute max at $x = 0$

Pause and Think !!!

Question:

Must a function always have

a local maximum / local minimum

at a critical point ??