MA 1505 Mathematics I

Tutorial 1 Solutions

1. Note that $(g \circ f)(x) = \sqrt{|3 - \frac{6}{x}|}$ and $(f \circ g)(x) = \frac{6}{\sqrt{|3-x|}}$.

2. (a) $y = \dfrac{ax+b}{cx+d}\,,\quad y' = \dfrac{a(cx+d) - c(ax+b)}{(cx+d)^2} = \dfrac{ad-bc}{(cx+d)^2}$ (use quotient rule)

   (b) $y = \sin^n x \cos mx\,,\quad y' = n\sin^{n-1} x \cos x \cos mx - m \sin^n x \sin mx$
   (use product rule and chain rule)

   (c) $y = e^{x^2+x^3} \quad y' = e^{x^2+x^3}(2x+3x^2)$ (use chain rule)

   (d) $y = x^3 - 4(x^2 + e^2 + \ln 2)\,,\quad y' = 3x^2 - 8x$ (note that e^2 and $\ln 2$ are constants)

   Similarly, we find the derivatives in (e) - (h).

   (e) $-2\sin\theta(\cos\theta - 1)^{-2}$ (use quotient and chain rule)

   (f) $\sqrt{t}\sec^2(2\sqrt{t}) + \tan(2\sqrt{t})$ (use product and chain rule)

   (g) $\frac{2\sqrt{\theta+1}+1}{2\sqrt{\theta+1}}\cos(\theta + \sqrt{\theta+1})$ (use chain rule)

   (h) $4\tan x \sec x - \csc^2 x$ (use quotient rule)

3. Let $V_c(t)$ be the volume of coffee in the cone at time t and $V_p(t)$ be the volume of coffee in the pot at time t.

   Note that the rate of volume change in the cone $\dfrac{dV_c}{dt}$ is equal the rate of volume change in the pot $\dfrac{dV_p}{dt}$.

   Let $h_c(t)$ be the level of coffee in the cone at time t and $h_p(t)$ be the level of coffee in the pot at time t.

   (a) $V_p = \text{base area} \times h_p = 9\pi h_p$.

$$\begin{aligned} \frac{dV_p}{dt} &= 9\pi\frac{dh_p}{dt} \\ \Rightarrow \quad 10 &= 9\pi\frac{dh_p}{dt} \\ \Rightarrow \quad \frac{dh_p}{dt} &= \frac{10}{9\pi} \end{aligned}$$

(b) $V_c = \dfrac{1}{3}\text{base area} \times h_c = \dfrac{1}{3}\pi r^2 h_c = \dfrac{1}{3}\pi(\dfrac{h_c}{2})^2 h_c = \dfrac{\pi h_c^3}{12}$.

Note that the base radius r of the cone is half that of the height h_c.

$$\begin{aligned} \frac{dV_c}{dt} &= \frac{\pi h_c^2}{4}\frac{dh_c}{dt} \\ \Rightarrow \quad 10 &= \frac{\pi 5^2}{4}\frac{dh_c}{dt} \\ \Rightarrow \quad \frac{dh_c}{dt} &= \frac{8}{5\pi} \end{aligned}$$

4. (a) $x^{2/3} + y^{2/3} = a^{2/3}$. Differentiating the equality we get

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0.$$

Since $0 < x < a$ and $0 < y$, we have

$$\begin{aligned} \frac{dy}{dx} &= -\frac{y^{1/3}}{x^{1/3}} = -\frac{\sqrt{a^{2/3} - x^{2/3}}}{x^{1/3}} = -\sqrt{(\frac{a}{x})^{2/3} - 1}; \\ \frac{d^2y}{dx^2} &= -\frac{1}{2}\frac{1}{\sqrt{(\frac{a}{x})^{2/3} - 1}}(-\frac{2}{3})a^{2/3}x^{-5/3} = \frac{a^{2/3}}{3x^{5/3}\sqrt{(\frac{a}{x})^{2/3} - 1}} = \frac{a^{2/3}}{3x^{4/3}\sqrt{a^{2/3} - x^{2/3}}}. \end{aligned}$$

(b) $y = (\sin x)^{\sin x}$, $0 < x < \frac{\pi}{2}$, so $\sin x > 0$.

$$\begin{aligned} \ln y &= \sin x \ln \sin x, \quad \frac{y'}{y} = \cos x \ln \sin x + \cos x, \quad y' = y(1 + \ln \sin x)\cos x, \\ y'' &= y'(1 + \ln \sin x)\cos x + y\left[(1 + \ln \sin x)(-\sin x) + \frac{\cos^2 x}{\sin x}\right] \\ &= y(1 + \ln \sin x)^2 \cos^2 x + y\left[\frac{\cos^2 x}{\sin x} - (1 + \ln \sin x)\sin x\right]. \end{aligned}$$

Hence

$$\begin{aligned} y' &= (\sin x)^{\sin x}(1 + \ln \sin x)\cos x, \\ y'' &= (\sin x)^{\sin x}\left[(1 + \ln \sin x)^2 \cos^2 x + \frac{\cos^2 x}{\sin x} - (1 + \ln \sin x)\sin x\right]. \end{aligned}$$

(c) $x = a\cos t$, $y = a\sin t$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\cos t}{-a\sin t} = -\cot t, \\ \frac{d^2y}{dx^2} &= \frac{d}{dx}(\frac{dy}{dx}) = \frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(-\cot t)}{\frac{dx}{dt}} = \frac{\frac{1}{\sin^2 t}}{-a\sin t} = -\frac{1}{a\sin^3 t}. \end{aligned}$$

5. **(a)** $y = \dfrac{x+1}{x^2+1}$, $x \in [-3, 3]$.

$y' = \dfrac{2-(x+1)^2}{(x^2+1)^2}$ and $y' = 0$ if $x = -1 \pm \sqrt{2}$.

So critical points are $x = -1 \pm \sqrt{2}$ and endpoints are $x = \pm 3$.

$$y' \begin{cases} < 0 & \text{if } -3 \le x < -1-\sqrt{2}, \\ = 0 & \text{if } x = -1-\sqrt{2}, \\ > 0 & \text{if } -1-\sqrt{2} < x < -1+\sqrt{2}, \\ = 0 & \text{if } x = -1+\sqrt{2}, \\ < 0 & \text{if } -1+\sqrt{2} < x \le 3. \end{cases}$$

Hence y is decreasing in $[-3, -1-\sqrt{2})$, increasing in $(-1-\sqrt{2}, -1+\sqrt{2})$, and decreasing in $(-1+\sqrt{2}, 3]$.

So local min is:

$$y(-1-\sqrt{2}) = -\frac{1}{2(\sqrt{2}+1)}, \quad y(3) = \frac{2}{5}$$

and local max is:

$$y(-1+\sqrt{2}) = \frac{1}{2(\sqrt{2}-1)}, \quad y(-3) = -\frac{1}{5}.$$

Since

$$-\frac{1}{2(\sqrt{2}+1)} < -\frac{1}{5} < \frac{2}{5} < \frac{1}{2(\sqrt{2}-1)},$$

so absolute min. is $\displaystyle\min_{x\in[-3,3]} y = -\frac{1}{2(\sqrt{2}+1)}$ at $x = -1-\sqrt{2}$

and absolute max. is $\displaystyle\max_{x\in[-3,3]} y = \frac{1}{2(\sqrt{2}-1)}$ at $x = -1+\sqrt{2}$.

(b) $y = (x-1)\sqrt[3]{x^2}$, $x \in (-\infty, \infty)$.

$$y' = x^{2/3} + \frac{2}{3}(x-1)x^{-1/3} = \frac{5x-2}{3x^{1/3}}$$

and $y' = 0$ if $x = \frac{2}{5}$.

Note that y' does not exist at $x = 0$.

So the critical points are $x = 0$ and $x = \frac{2}{5}$.

$$y' \begin{cases} > 0 & \text{if } x < 0, \\ \text{does not exist} & \text{if } x = 0, \\ < 0 & \text{if } 0 < x < \frac{2}{5}, \\ = 0 & \text{if } x = \frac{2}{5}, \\ > 0 & \text{if } x > \frac{2}{5}. \end{cases}$$

Hence y is increasing in $(-\infty, 0)$, decreasing in $(0, \frac{2}{5})$, and increasing in $(\frac{2}{5}, \infty)$.

So local max. is $y(0) = 0$

and local min. is $y(\frac{2}{5}) = -\frac{3}{5}(\frac{2}{5})^{2/3}$.

Since $\lim_{x\to-\infty} y = -\infty$, $\lim_{x\to\infty} y = \infty$, so there is no absolute extremes.

6. Let x be the distance between B and C. Suppose the energy that it takes to fly over land is 1 unit per km, then it will take 1.4 unit per km to fly over water.

   The total energy is given by the function

   $$f(x) = 1.4\sqrt{5^2 + x^2} + (13 - x).$$

   Then

   $$f'(x) = \frac{1.4x - \sqrt{5^2 + x^2}}{\sqrt{5^2 + x^2}}.$$

   Solving $f'(x) = 0$, we have $x = 5.103$ and the First Derivative Test shows that this point is an absolute minimum.

7. Let x m be the distance from the shadow to the foot of the lamp post. Using similar triangles in the diagram on the next page, we have

   $$\frac{s}{9} = \frac{15}{x}.$$

   Therefore,

   $$sx = 135$$

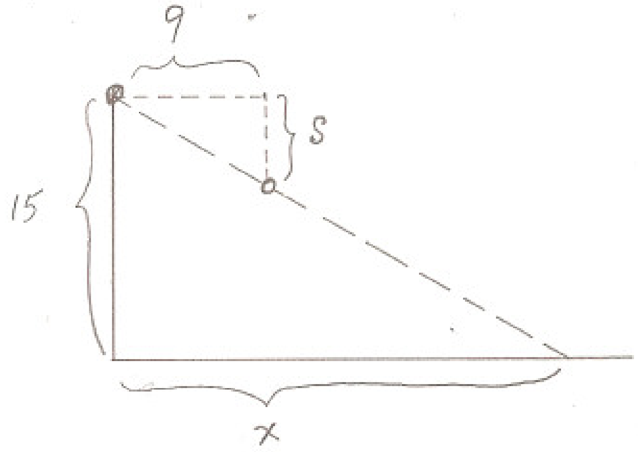
   and hence

   $$x = \frac{135}{4.9t^2}.$$

   Differentiate this with respect to t and then substitute $t = 0.5$ to solve for $\frac{dx}{dt}$, we find that

   $$\frac{dx}{dt} = -440.8,$$

   and so the speed is 440.8 m/sec. (The - sign indicates that the shadow is moving towards the foot of the lamp post.)


9
5
15
x