Chapter 6
Three Dimensional Space

Overview

- The Cartesian Coordinate System
- Vectors
  - Definitions
  - Angle between 2 vectors
  - Scalar or Dot Product
  - Properties of Scalar Product

Overview

- <u>Unit Vectors</u>
  - Projection
- <u>Vector Product</u>
  - Properties of Vector Product
- <u>Lines in 3-D Space</u>
  - Vector Equation of a Line
  - Parametric Equation of a Line

Overview

- Planes
  - Equation for Plane
  - Distance from Point to Plane

- Vector Functions of One Variable
  - Limits and Continuity
  - Derivatives of Vector Functions
  - Definite Integral of a Vector Function

Overview

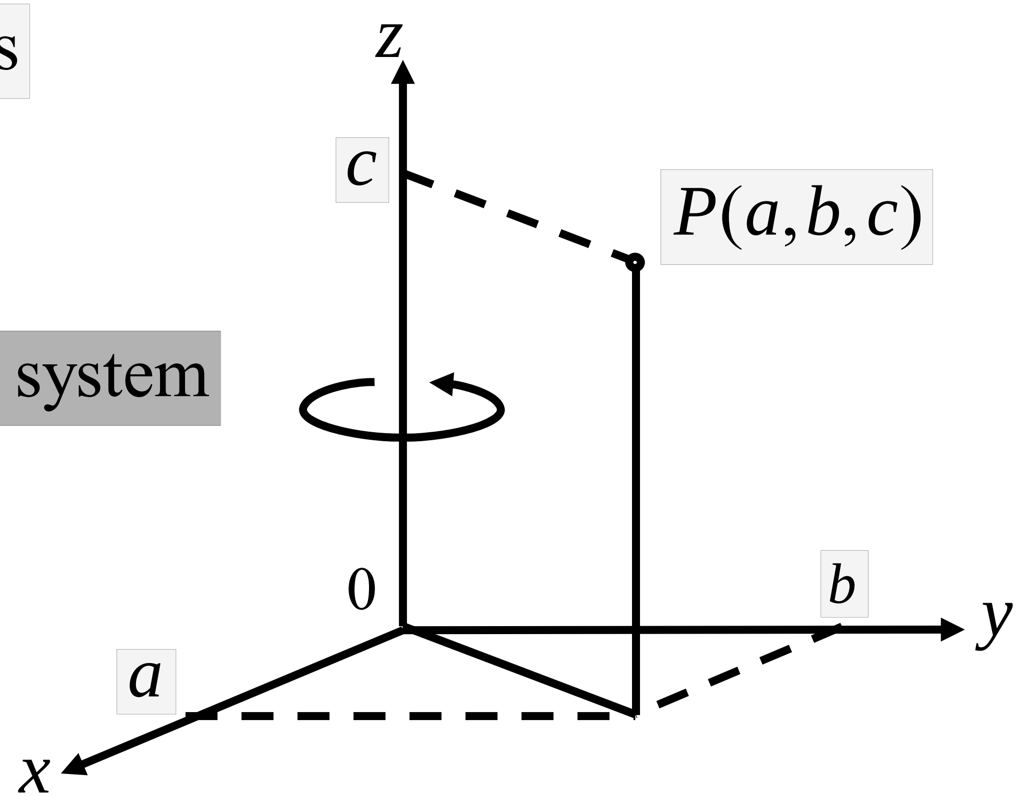
- Space Curves
  - Smooth Curves
  - Tangent Vector and Line to a Curve
  - Arc Length of a Space Curve

The Cartesian Coordinate System

The Cartesian Coordinate System

Rectangular Coordinates

Right-handed coordinates system

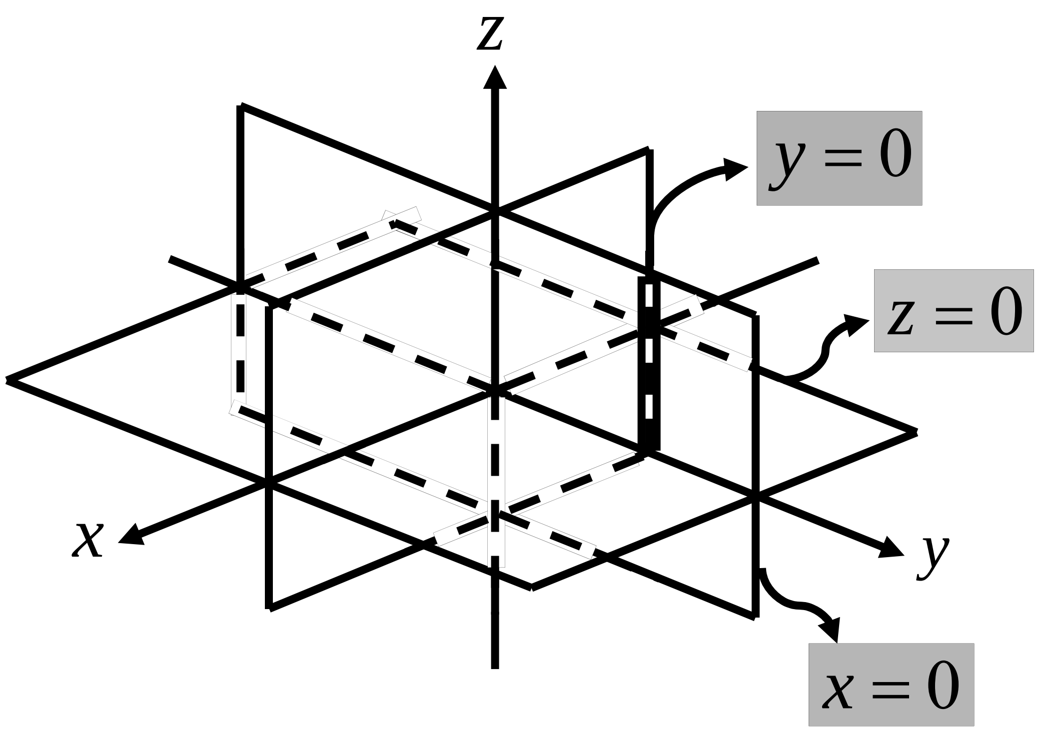


If we rotate the $x-$axis counterclockwise toward the $y-$axis, then a right-handed screw will move in the positive z direction.

The Cartesian Coordinate System

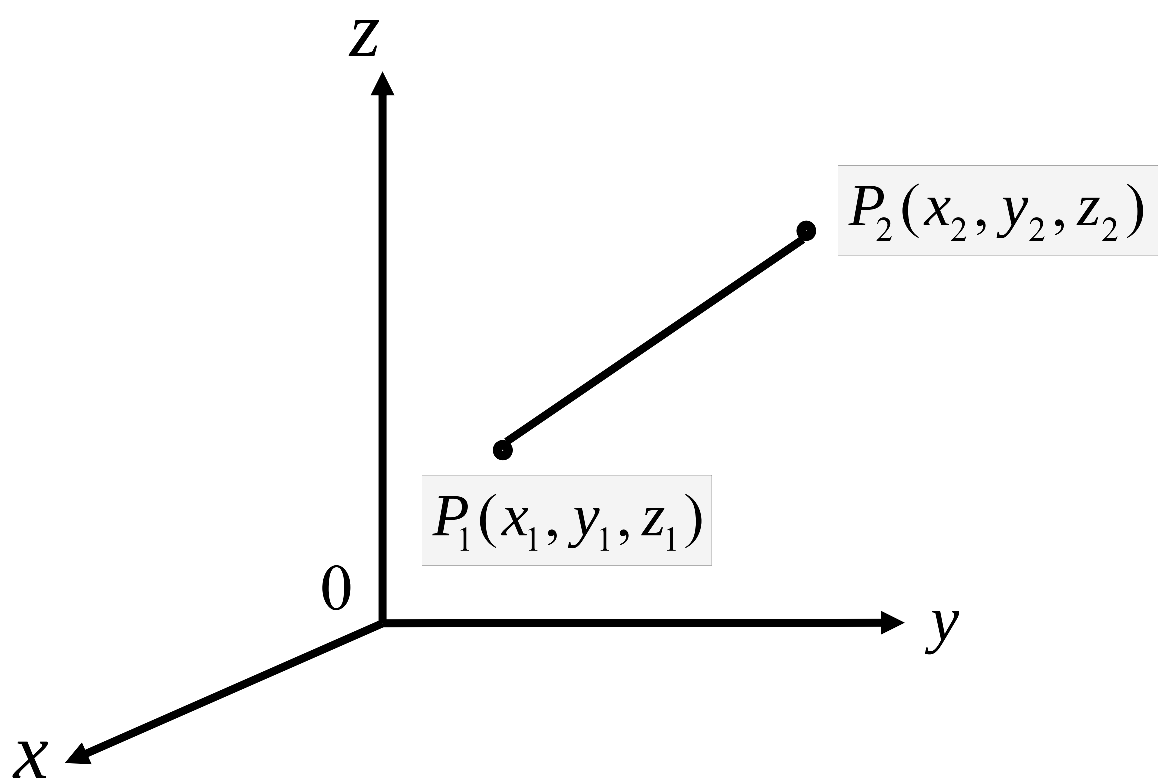
Planes : $x = 0, y = 0, z = 0$

Eight Octants



The Cartesian Coordinate System

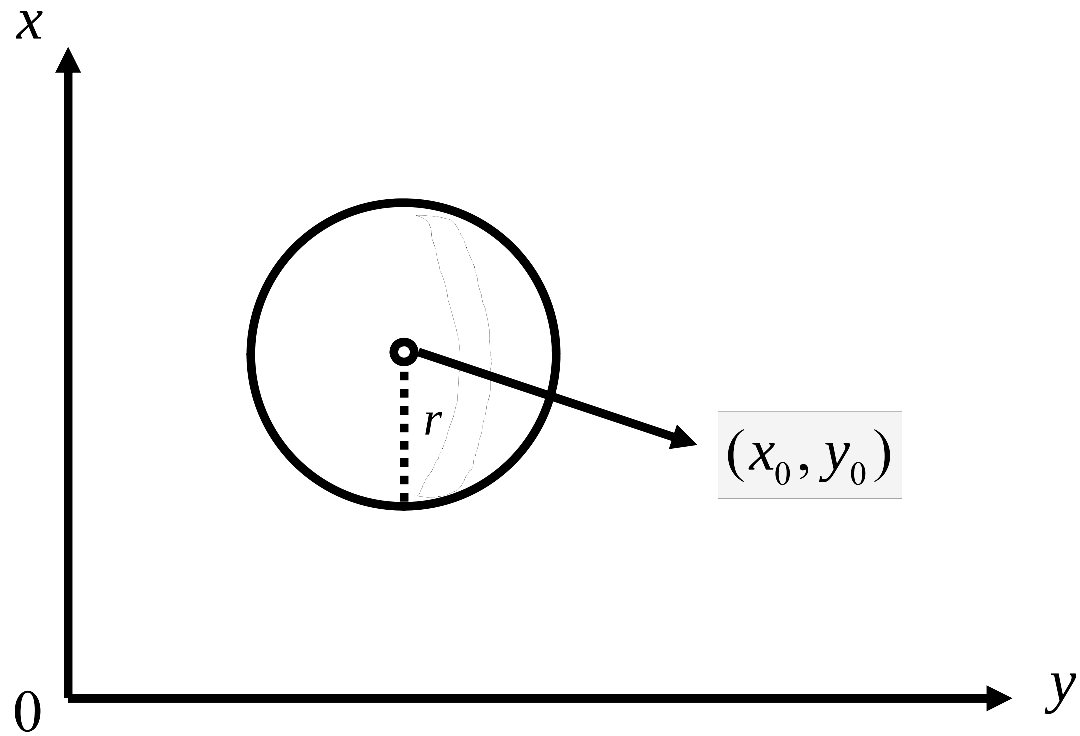
Distane between two points



$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The Cartesian Coordinate System

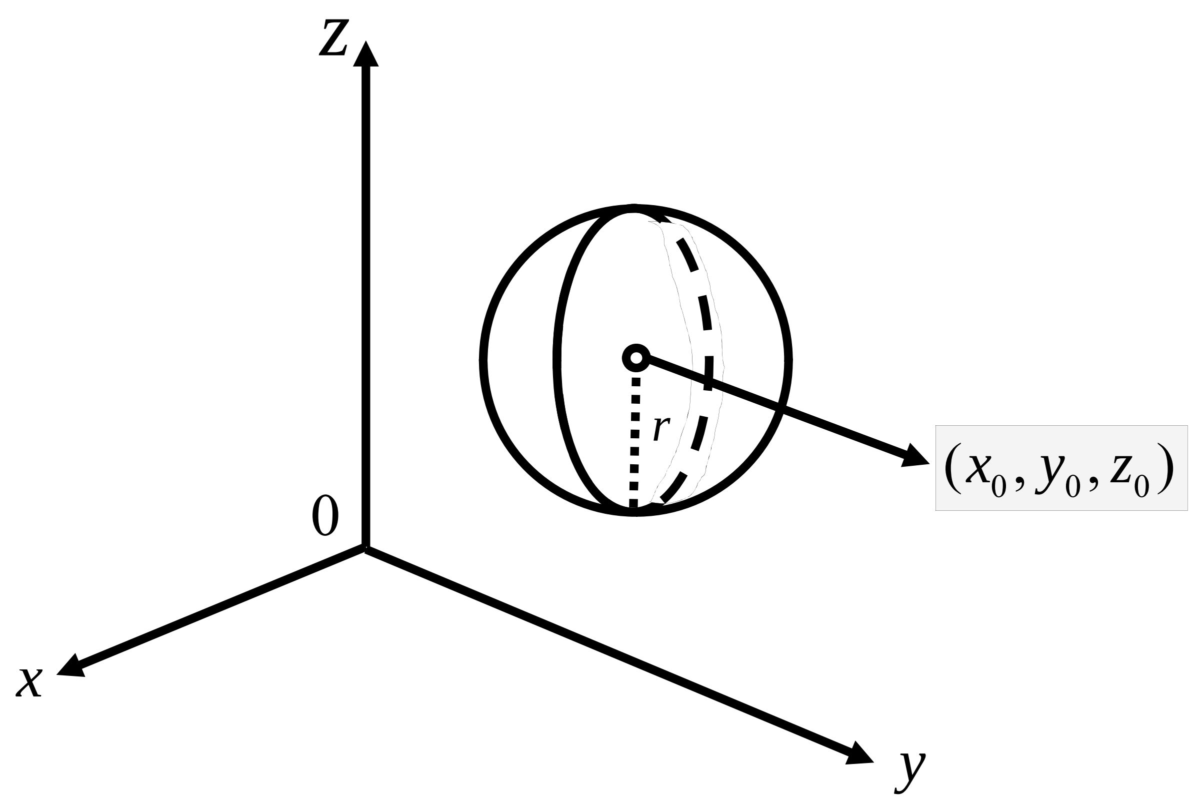
Equation of circle of radius r and center (x_0, y_0)



Equation of circle: $(x - x_0)^2 + (y - y_0)^2 = r^2$

The Cartesian Coordinate System

Equation of the sphere of radius r and center (x_0, y_0, z_0)



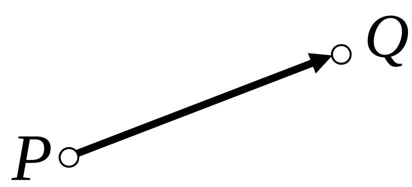
$$\text{Equation of sphere}: (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$

Vectors

Vectors – Definitions

A directed line segment PQ

(initial point)

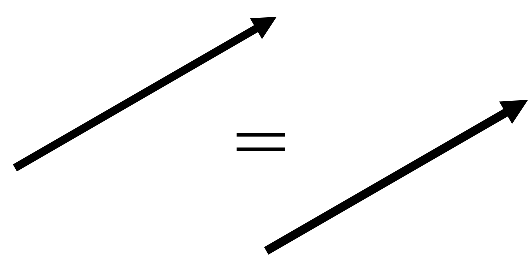


(terminal point)

Direction: direction of the arrow

Magnitude: length of the line segment

Two vectors are equal if they have the same direction and length.



Vectors – Definitions

The ***position*** vector of $P\ (x_0, y_0, z_0)$:

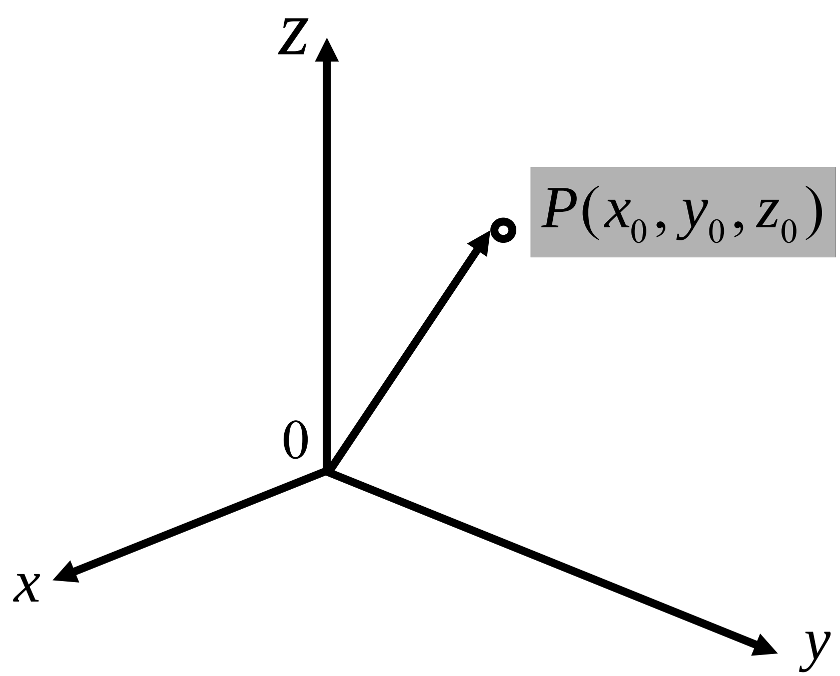
$$\overrightarrow{OP} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

The ***length*** of $\overrightarrow{OP}$ is denoted by :

$$\|\overrightarrow{OP}\| = \sqrt{x_0^2 + y_0^2 + z_0^2}$$

(***magnitude***)

The ***zero*** vector is $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

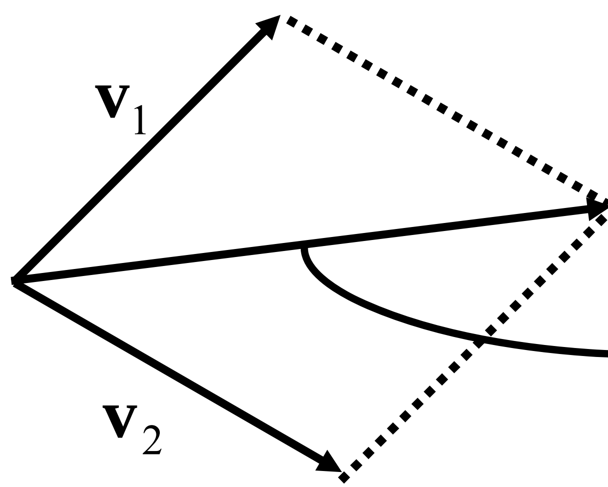


Vectors – Definitions

Addition

$$\mathbf{v}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

$$\mathbf{v}_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$



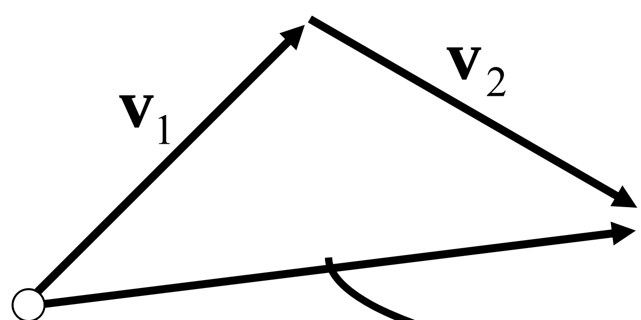
$$\mathbf{v}_1 + \mathbf{v}_2 = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$$

Vectors – Definitions

Addition

$$\mathbf{v}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

$$\mathbf{v}_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

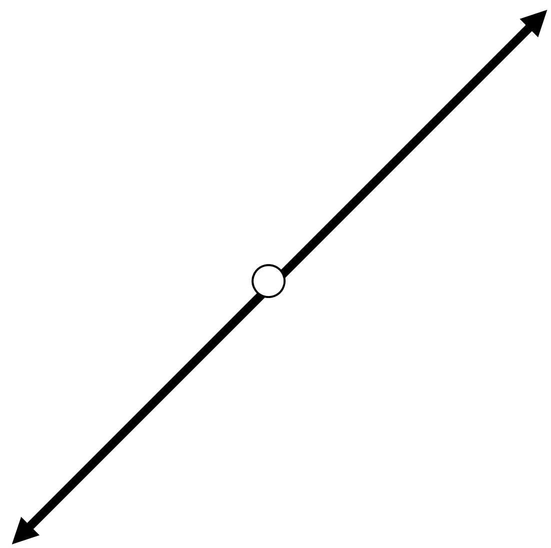


$$\mathbf{v}_1 + \mathbf{v}_2 = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$$

Vectors – Definitions

Negative

$$-\mathbf{v}_1 = \begin{pmatrix} -x_1 \\ -y_1 \\ -z_1 \end{pmatrix}$$



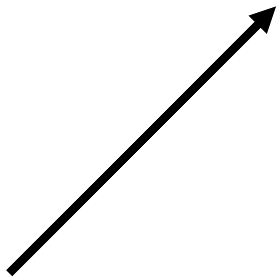
$$\mathbf{v}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

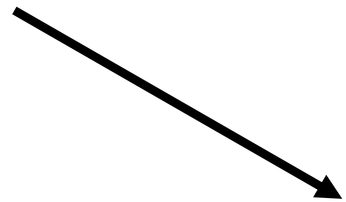
Same magnitude but opposite direction

Vectors – Definitions

Note : $\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{v}_1 + (-\mathbf{v}_2)$
$= -\mathbf{v}_2 + \mathbf{v}_1$

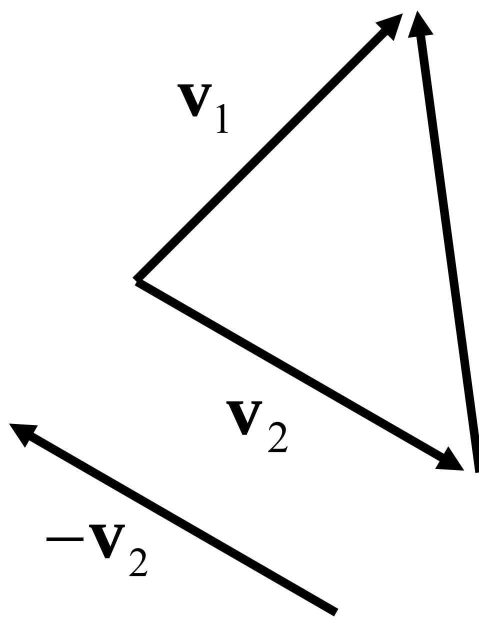
Difference





$$\mathbf{v}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

$$\mathbf{v}_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$



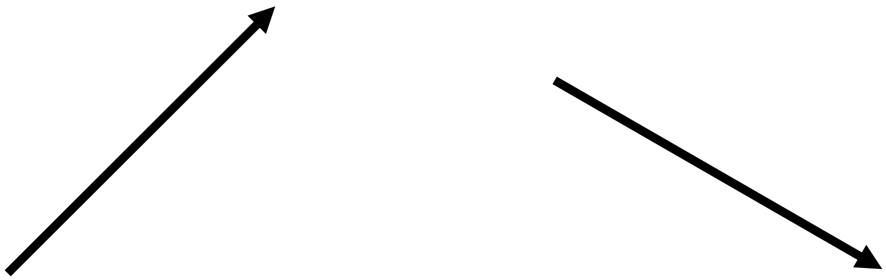
$$\mathbf{v}_1 - \mathbf{v}_2 = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \end{pmatrix}$$

Vectors – Definitions

Note : $\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{v}_1 + (-\mathbf{v}_2)$

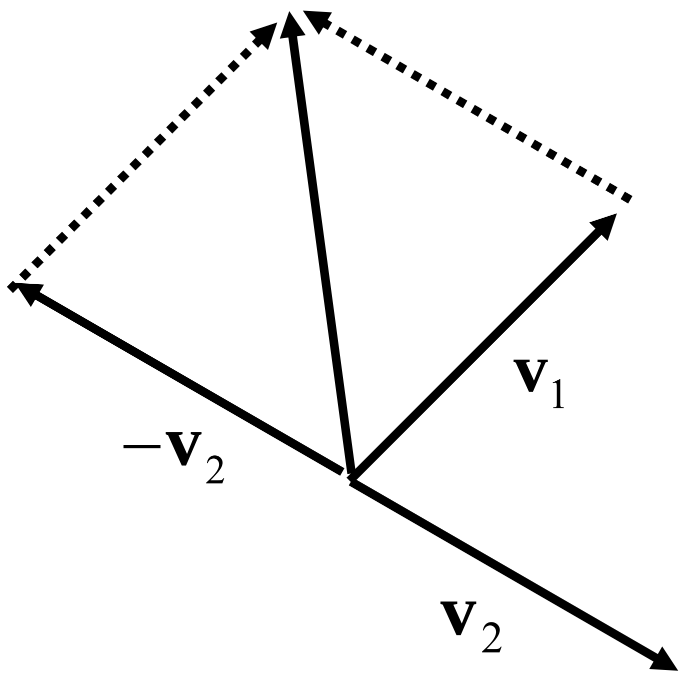
$= -\mathbf{v}_2 + \mathbf{v}_1$

Difference



$$\mathbf{v}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

$$\mathbf{v}_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$



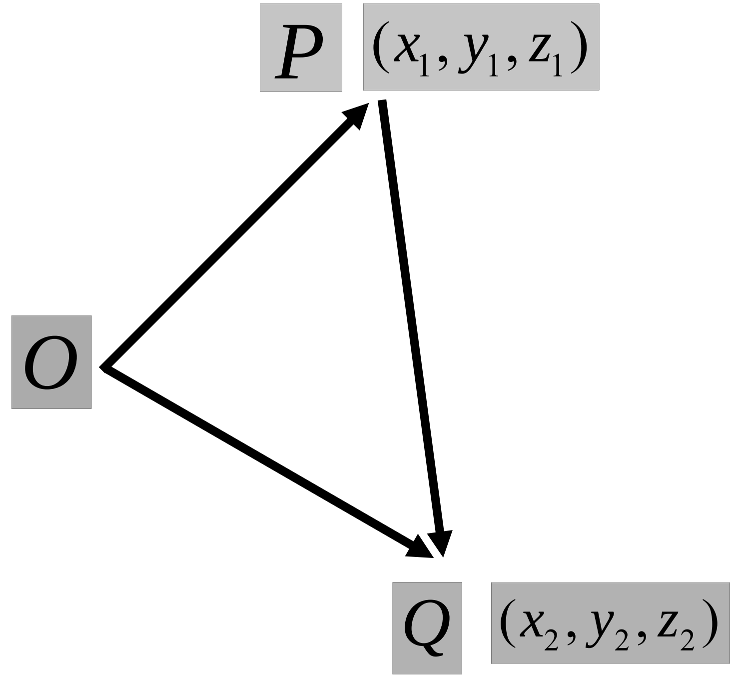
$$\mathbf{v}_1 - \mathbf{v}_2 = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \end{pmatrix}$$

Vectors – Definitions

Difference

$$\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$



$$\overrightarrow{PQ} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \end{pmatrix}$$

Wrong !!

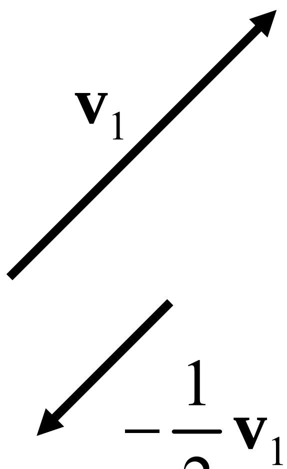
$$\overrightarrow{PQ} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix}$$

Correct

Vectors – Definitions

Scalar Multiplication

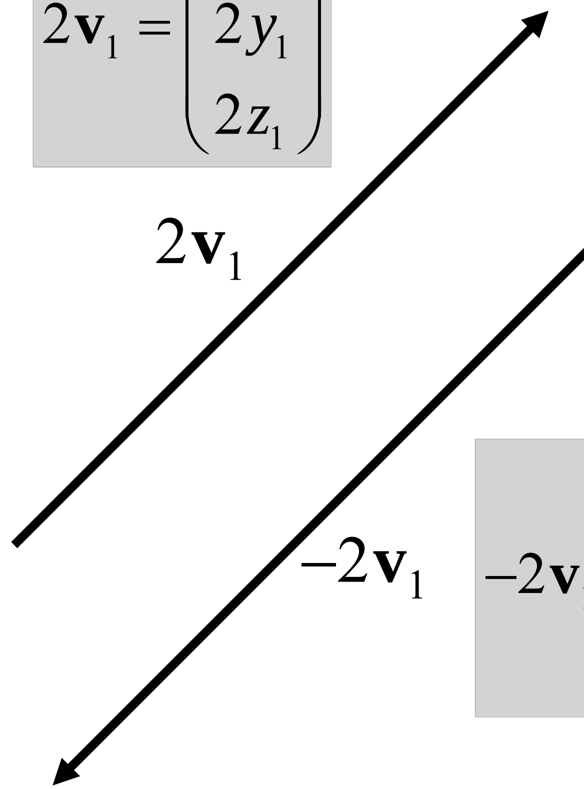
$$\mathbf{v}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$



$$-\frac{1}{2}\mathbf{v}_1 = \begin{pmatrix} -\frac{1}{2}x_1 \\ -\frac{1}{2}y_1 \\ -\frac{1}{2}z_1 \end{pmatrix}$$

$$2\mathbf{v}_1 = \begin{pmatrix} 2x_1 \\ 2y_1 \\ 2z_1 \end{pmatrix}$$

$$c\mathbf{v}_1 = \begin{pmatrix} cx_1 \\ cy_1 \\ cz_1 \end{pmatrix}$$



$$-2\mathbf{v}_1 = \begin{pmatrix} -2x_1 \\ -2y_1 \\ -2z_1 \end{pmatrix}$$

If $c > 0$, then $c\mathbf{v}_1$ and $\mathbf{v}_1$ point in the same direction.

If $c < 0$, then $c\mathbf{v}_1$ and $\mathbf{v}_1$ point in the opposite direction.

Angle between 2 vectors

$$\mathbf{v}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

$$\|\mathbf{v}_1\|^2 = x_1^{\;2} + y_1^{\;2} + z_1^{\;2}$$

$$\mathbf{v}_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

$$\|\mathbf{v}_2\|^2 = x_2^{\;2} + y_2^{\;2} + z_2^{\;2}$$

$$\mathbf{v}_1 - \mathbf{v}_2 = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \end{pmatrix}$$

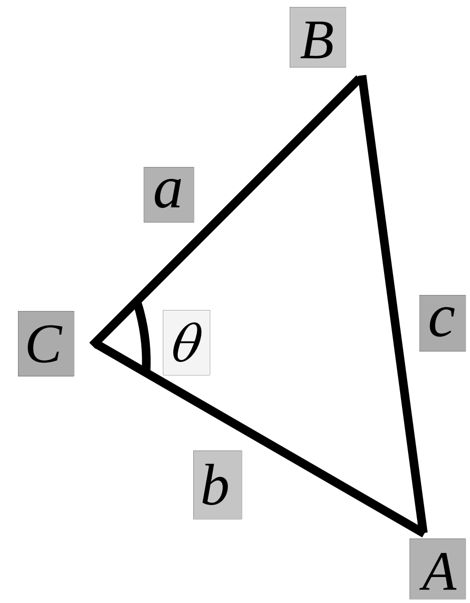
$$\| v_1 - v_2 \|^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$$

$$\| v_1 - v_2 \|^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$$

$$= x_1^{\;2} - 2x_1x_2 + x_2^{\;2} + y_1^{\;2} - 2y_1y_2 + y_2^{\;2} + z_1^{\;2} - 2z_1z_2 + z_2^{\;2}$$

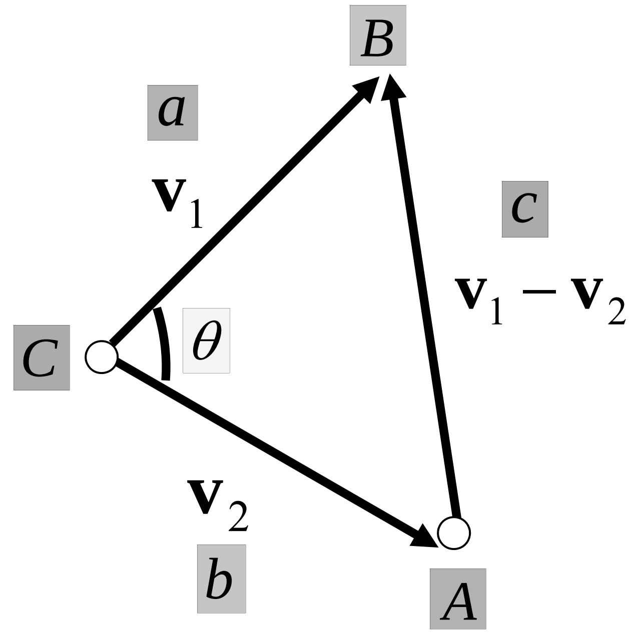
Cosine Rule

$$\cos\theta = \frac{a^2 + b^2 - c^2}{2ab}$$



Angle between 2 vectors

$$\cos\theta = \frac{a^2 + b^2 - c^2}{2ab}$$



$$\cos\theta = \frac{\|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2 - (\|\mathbf{v}_1 - \mathbf{v}_2\|)^2}{2\|\mathbf{v}_1\| \; \|\mathbf{v}_2\|}$$

$$\|\mathbf{v}_1\|^2 = x_1{}^2 + y_1{}^2 + z_1{}^2$$

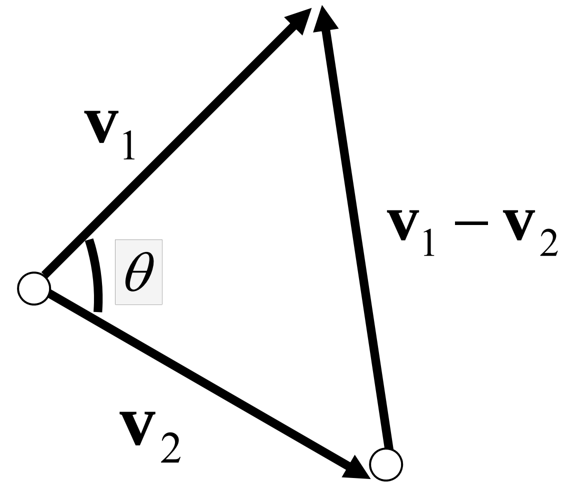
$$\|\mathbf{v}_2\|^2 = x_2{}^2 + y_2{}^2 + z_2{}^2$$

$$\|v_1 - v_2\|^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$$

$$= x_1{}^2 - 2x_1x_2 + x_2{}^2 + y_1{}^2 - 2y_1y_2 + y_2{}^2 + z_1{}^2 - 2z_1z_2 + z_2{}^2$$

$$\cos\theta = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\|\mathbf{v}_1\| \; \|\mathbf{v}_2\|}$$

Scalar or Dot Product



$$\cos\theta = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\|\mathbf{v}_1\| \ \|\mathbf{v}_2\|}$$

$$\mathbf{v}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

Define $\quad \mathbf{v}_1 \cdot \mathbf{v}_2 = x_1x_2 + y_1y_2 + z_1z_2$

Then

$$\cos\theta = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{\|\mathbf{v}_1\| \ \|\mathbf{v}_2\|} \quad (0 \le \theta \le 180°)$$

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \|\mathbf{v}_1\| \ \|\mathbf{v}_2\| \cos\theta \quad (0 \le \theta \le 180°)$$

$\cos 90° = 0$

$\mathbf{v}_1$ perpendicular to $\mathbf{v}_2$ if and only if $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$

Scalar or Dot Product - Example

Let $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$.

$$\| \mathbf{v}_1 \| = \sqrt{2^2 + 4^2 + 5^2} = \sqrt{45}$$

$$\| \mathbf{v}_2 \| = \sqrt{(-1)^2 + 2^2 + 3^2} = \sqrt{14}$$

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = x_1 x_2 + y_1 y_2 + z_1 z_2$$

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = (2)(-1) + (4)(2) + (5)(3) = 21$$

$$\cos\theta = \frac{21}{\sqrt{45}\sqrt{14}} = \frac{\sqrt{7}}{\sqrt{10}}$$

$$\cos\theta = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{\| \mathbf{v}_1 \| \;\; \| \mathbf{v}_2 \|} \quad (0 \le \theta \le 180°)$$

Scalar or Dot Product - Example

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = x_1 x_2 + y_1 y_2 + z_1 z_2$$

The vectors $\mathbf{w}_1 = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}$ and $\mathbf{w}_2 = \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}$ are perpendicular since their dot product

$$\mathbf{w}_1 \cdot \mathbf{w}_2 = (2)(4) + (-5)(2) + (1)(2) = 0.$$

Properties of Scalar Product

If $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are vectors in xyz-space and c is a real number, then

(a) $\mathbf{v}_1 \cdot \mathbf{v}_1 = \|\mathbf{v}_1\|^2 \geq 0$

(b) $\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_2 \cdot \mathbf{v}_1$

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = x_1x_2 + y_1y_2 + z_1z_2$$

(c) $(\mathbf{v}_1 + \mathbf{v}_2) \cdot \mathbf{v}_3 = \mathbf{v}_1 \cdot \mathbf{v}_3 + \mathbf{v}_2 \cdot \mathbf{v}_3$

(d) $(c\mathbf{v}_1) \cdot \mathbf{v}_2 = \mathbf{v}_1 \cdot (c\mathbf{v}_2) = c(\mathbf{v}_1 \cdot \mathbf{v}_2)$.

Properties of Scalar Product

(a) $\mathbf{v}_1 \cdot \mathbf{v}_1 = \ \| \mathbf{v}_1 \|^2 \quad \geq 0$

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = x_1 x_2 + y_1 y_2 + z_1 z_2$$

$$\mathbf{v}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

$$\| \mathbf{v}_1 \|^2 = x_1^{\ 2} + y_1^{\ 2} + z_1^{\ 2}$$

$$\mathbf{v}_1 \cdot \mathbf{v}_1 = x_1 x_1 + y_1 y_1 + z_1 z_1$$

$$= x_1^{\ 2} + y_1^{\ 2} + z_1^{\ 2}$$

$$= \ \| \mathbf{v}_1 \|^2$$

Unit Vectors

Unit Vectors

Vectors of length 1

The standard unit vectors are

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = x_1 x_2 + y_1 y_2 + z_1 z_2$$

Note that :

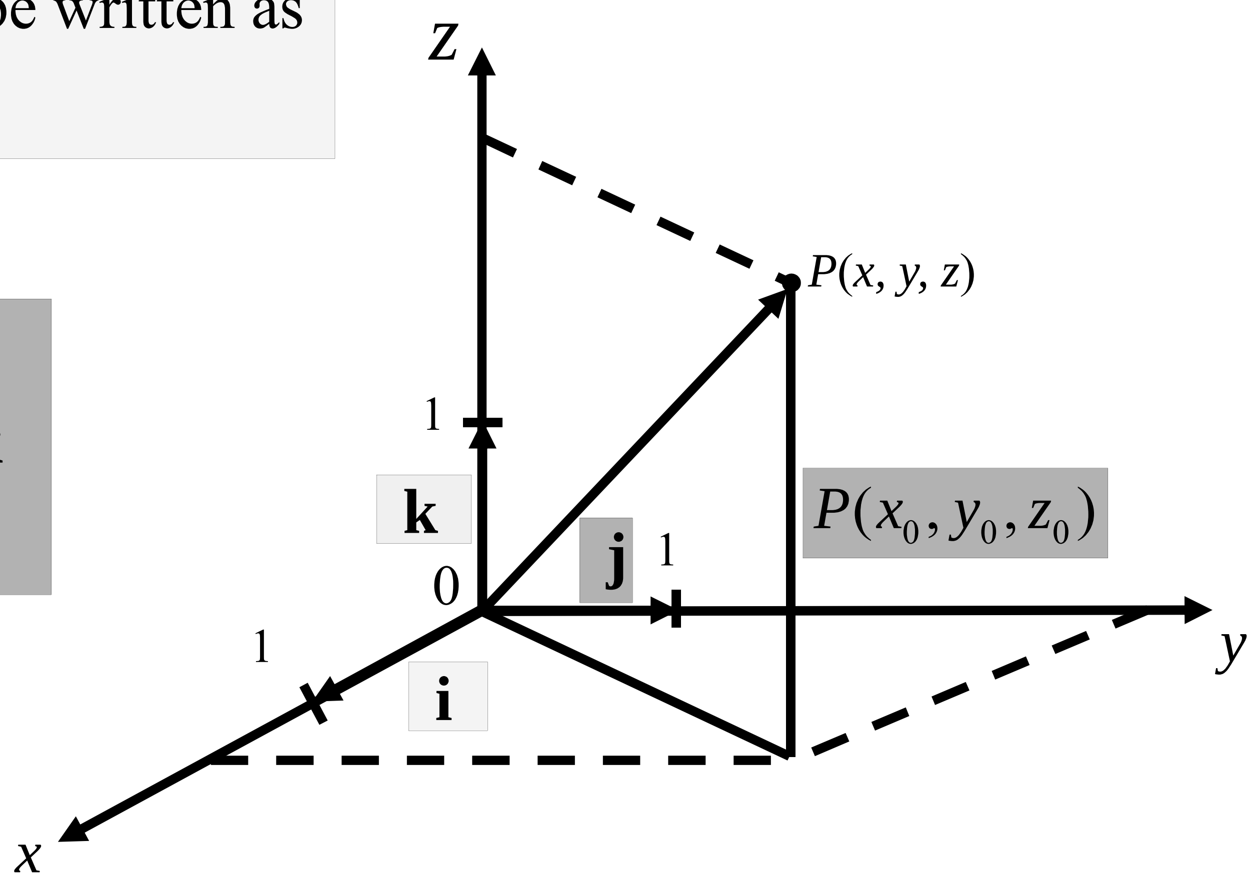
$$\mathbf{i}.\mathbf{j} = 0 \qquad \mathbf{j}.\mathbf{k} = 0 \qquad \mathbf{k}.\mathbf{i} = 0$$

Unit Vectors

Note that :

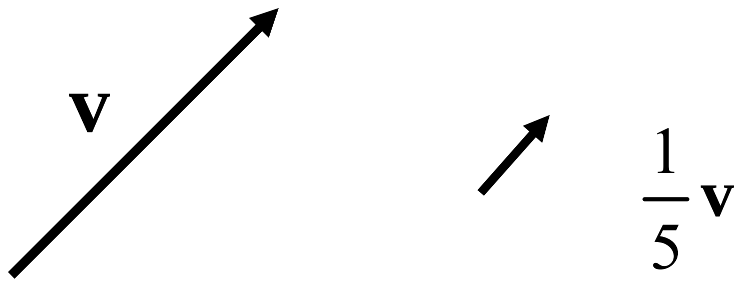
every vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ can be written as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$



Unit Vectors

Vectors of length 1



Suppose $||\mathbf{v}|| = 5$, then $\frac{1}{5}\mathbf{v}$ will have length 1.

To find unit vector: $\dfrac{1}{||\,\mathbf{v}\,||}\mathbf{v}$

Example

Suppose $||\mathbf{v}|| = 5$.

Find a vector with length 7 and in the direction $\mathbf{v}$.

$\frac{1}{5}\mathbf{v}$ is of length 1

Answer: $7\left(\frac{1}{5}\mathbf{v}\right) = \frac{7}{5}\mathbf{v}$

Unit Vectors

To find unit vector: $\dfrac{1}{\|\mathbf{v}\|}\mathbf{v}$

Let $\mathbf{w} = \begin{pmatrix} 4 \\ -5 \\ 22 \end{pmatrix} = 4\mathbf{i} - 5\mathbf{j} + 22\mathbf{k}.$

The unit vector with the same direction as **w** is

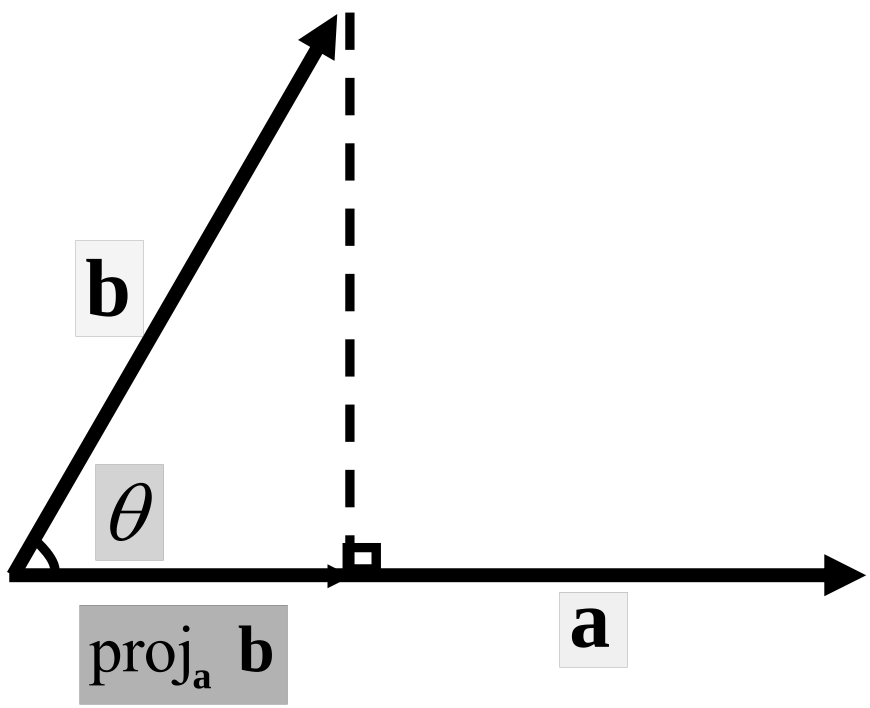
$$\frac{1}{\|\mathbf{w}\|}\mathbf{w} = \frac{1}{\sqrt{4^2 + 5^2 + 22^2}}(4\mathbf{i} - 5\mathbf{j} + 22\mathbf{k})$$

$$= \frac{4}{\sqrt{525}}\mathbf{i} - \frac{5}{\sqrt{525}}\mathbf{j} + \frac{22}{\sqrt{525}}\mathbf{k}$$

Projection

Let **a** and **b** be vectors.

The ***projection*** of **b** onto **a** ($\text{proj}_{\mathbf{a}}\mathbf{b}$) is shown below:

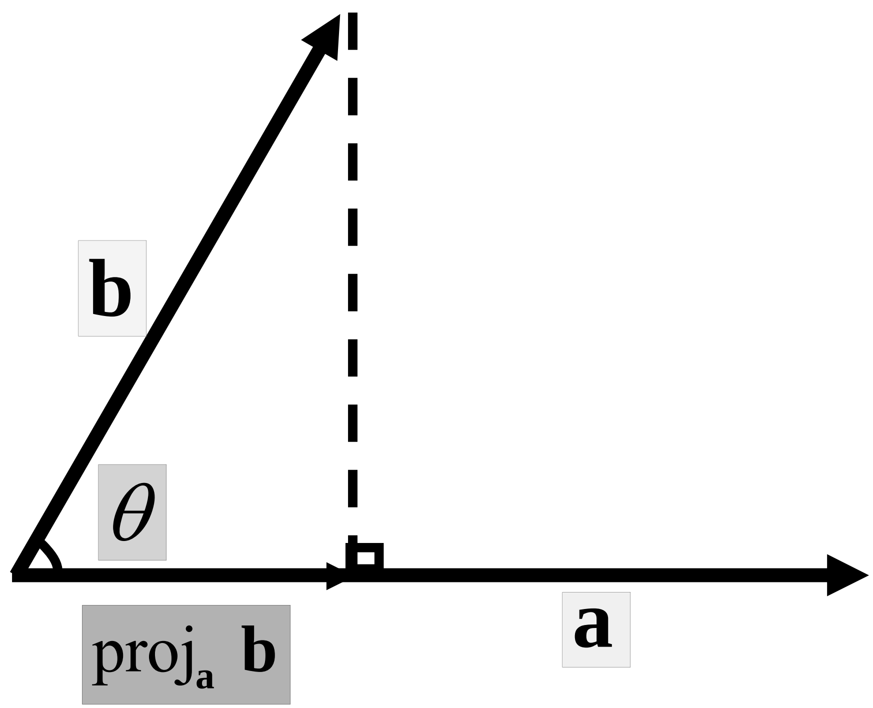


Question : How to find $\text{proj}_{\mathbf{a}}\mathbf{b}$???

Projection

Let **a** and **b** be vectors.

The ***projection*** of **b** onto **a** ($\mathrm{proj}_{\mathbf{a}}\mathbf{b}$) is shown below:



Note that

$\mathrm{proj}_{\mathbf{a}}\mathbf{b}$ is the vector in red

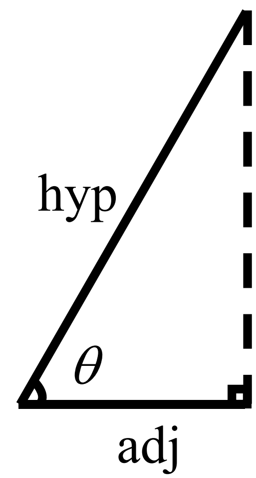
$\|\mathrm{proj}_{\mathbf{a}}\mathbf{b}\|$ is the length of projection, the length of the vector in red

Question : How to find $\text{proj}_{\mathbf{a}}\mathbf{b}$???

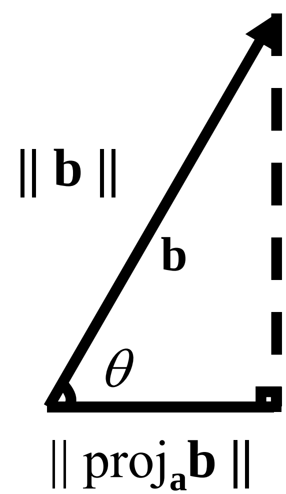
Note that

$$\frac{\|\text{proj}_{\mathbf{a}}\mathbf{b}\|}{\|\mathbf{b}\|} = \cos\theta$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}}$$



Thus, $\|\text{proj}_{\mathbf{a}}\mathbf{b}\| = \|\mathbf{b}\|\cos\theta$ ------ (1)

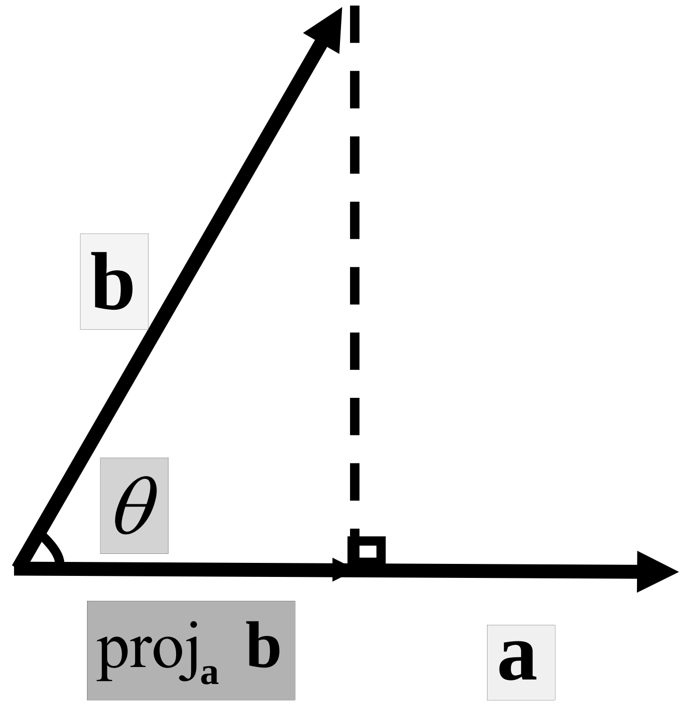


Recall that $\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\ \|\mathbf{b}\|\cos\theta$

$$\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|} = \|\mathbf{b}\|\cos\theta \quad \text{------ (2)}$$

From (1) and (2), we have length of projection $\|\text{proj}_{\mathbf{a}}\mathbf{b}\| = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}$

Question : How to find $\text{proj}_{\mathbf{a}}\mathbf{b}$???



From (1) and (2), $\|\text{proj}_{\mathbf{a}}\mathbf{b}\| = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}$

Unit vector in the direction $\mathbf{a}$: $\dfrac{1}{\|\mathbf{a}\|}\mathbf{a}$

Thus, $\text{proj}_{\mathbf{a}}\mathbf{b} = \|\text{proj}_{\mathbf{a}}\mathbf{b}\|$ (unit vector along $\mathbf{a}$)

$$= \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}\,\frac{1}{\|\mathbf{a}\|}\mathbf{a}$$

$$= \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a}$$

$$\boxed{\text{proj}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a}}$$

$\text{proj}_{\mathbf{a}}\mathbf{b}$ is the vector in red

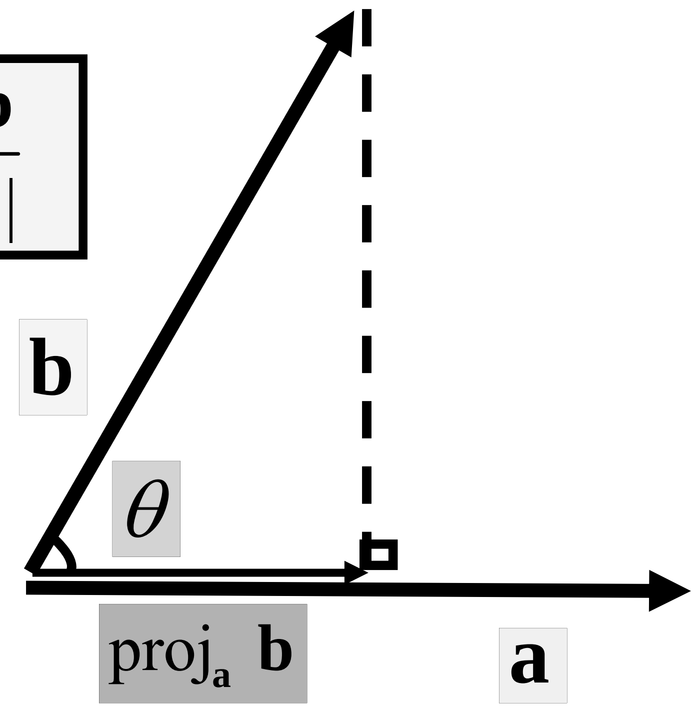
$\|\text{proj}_{\mathbf{a}}\mathbf{b}\|$ is the length of projection, the length of the vector in red

The ***projection*** of $\mathbf{b}$ onto $\mathbf{a}$ $(\text{proj}_{\mathbf{a}}\mathbf{b})$ is :

The length of projection $\|\text{proj}_{\mathbf{a}}\mathbf{b}\| = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}$

Unit vector : $\dfrac{1}{\|\mathbf{a}\|}\mathbf{a}$

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a}$$



The ***projection*** of $\mathbf{a}$ onto $\mathbf{b}$ $(\text{proj}_{\mathbf{b}}\mathbf{a})$ is :

The length of projection $\|\text{proj}_{\mathbf{b}}\mathbf{a}\| = \dfrac{\mathbf{b}\cdot\mathbf{a}}{\|\mathbf{b}\|}$

Unit vector : $\dfrac{1}{\|\mathbf{b}\|}\mathbf{b}$

$$\text{proj}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{b}\cdot\mathbf{a}}{\|\mathbf{b}\|^2}\mathbf{b}$$

Projection - Example

Find the projection of $\mathbf{a} = 2\mathbf{i} + 5\mathbf{j}$ onto the vector $\mathbf{b} = \mathbf{i} + \mathbf{j}$.

The length of projection $\|\text{proj}_{\mathbf{b}}\mathbf{a}\| = \dfrac{\mathbf{b}\cdot\mathbf{a}}{\|\mathbf{b}\|}$

Unit vector : $\dfrac{1}{\|\mathbf{b}\|}\mathbf{b}$

$\text{proj}_{\mathbf{b}}\mathbf{a} = \dfrac{\mathbf{b}\cdot\mathbf{a}}{\|\mathbf{b}\|^2}\mathbf{b}$

The length of projection of $\mathbf{a}$ onto $\mathbf{b}$ is $\dfrac{(\mathbf{a}\cdot\mathbf{b})}{\|\mathbf{b}\|} = \dfrac{(2\mathbf{i}+5\mathbf{j})\cdot(\mathbf{i}+\mathbf{j})}{\sqrt{1^2+1^2}} = \dfrac{7}{\sqrt{2}}$.

A unit vector along $\mathbf{b}$ is $\dfrac{\mathbf{i}+\mathbf{j}}{\|\mathbf{i}+\mathbf{j}\|} = \dfrac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}$

Hence the projection of $\mathbf{a}$ onto $\mathbf{b}$ is $\dfrac{7}{\sqrt{2}}\dfrac{\mathbf{i}+\mathbf{j}}{\sqrt{2}} = \dfrac{7}{2}\mathbf{i} + \dfrac{7}{2}\mathbf{j}$.

PAUSE AND THINK !!!

Let **a** and **b** be two given vectors.

How to express vector **b** as the sum of vectors parallel to **a** and perpendicular to **a** ???

PAUSE AND THINK !!!

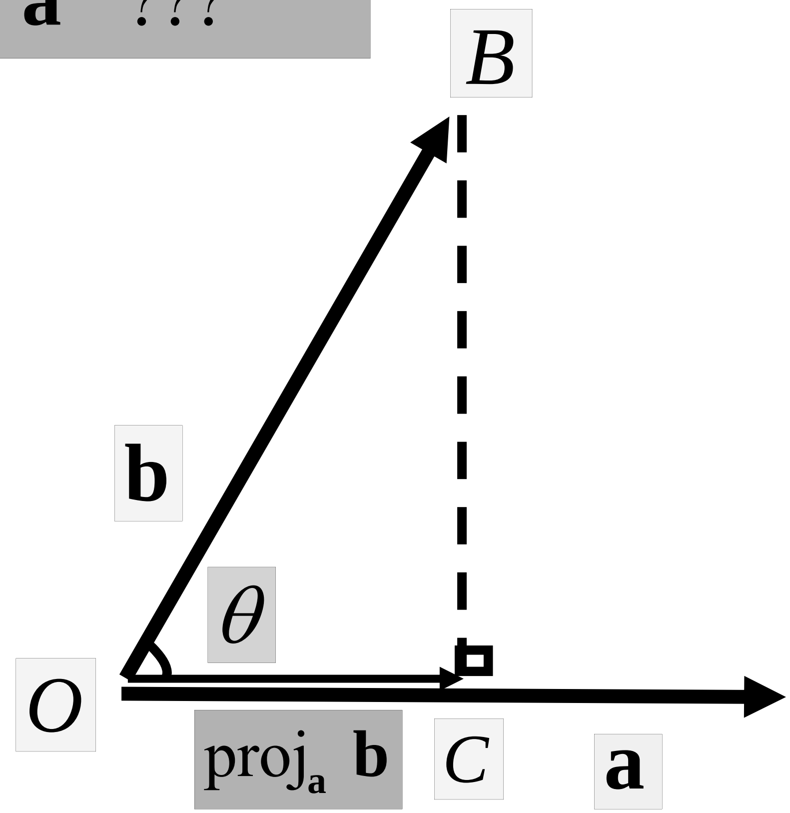
How to express vector **b** as the sum of vectors parallel to **a** and perpendicular to **a** ???

$$\boxed{\mathrm{proj}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a}}$$

$$\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB}$$

$$\begin{aligned}\overrightarrow{CB} &= \overrightarrow{OB} - \overrightarrow{OC}\\ &= \mathbf{b} - \mathrm{proj}_{\mathbf{a}}\mathbf{b}\end{aligned}$$

Note: $\overrightarrow{CB}$ is perpendicular to **a**



Answer : $\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB} = (\mathrm{proj}_{\mathbf{a}}\mathbf{b}) + (\mathbf{b} - \mathrm{proj}_{\mathbf{a}}\mathbf{b})$

parallel to **a**

perpendicular to **a**

Vector Product

Vector Product

Let $\mathbf{v}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$.

Then their ***vector product*** or ***cross product*** is the vector

$$\mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

$$= (y_1 z_2 - y_2 z_1)\mathbf{i} - (x_1 z_2 - x_2 z_1)\mathbf{j} + (x_1 y_2 - x_2 y_1)\mathbf{k}$$

Recall that $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} = y_1 z_2 - y_2 z_1$$

$$\begin{matrix} + & - & + \\ \mathbf{i} & \mathbf{j} & \mathbf{k} \end{matrix}$$

Vector Product - Example

Let $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 3 \\ -1 \\ -3 \end{pmatrix}$.

$$\mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 2 \\ 3 & -1 & -3 \end{vmatrix}$$

$$= -\mathbf{i} + 12\mathbf{j} - 5\mathbf{k}$$

Recall that $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$\begin{matrix} + & - & + \\ \mathbf{i} & \mathbf{j} & \mathbf{k} \end{matrix}$$

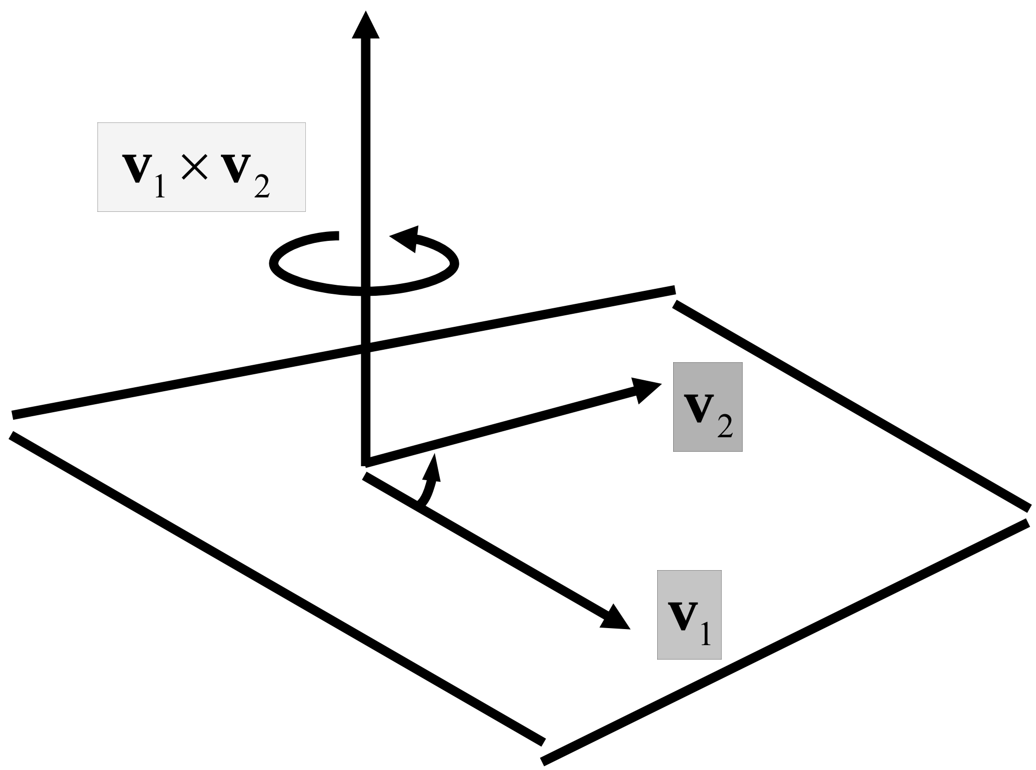
Vector Product

Let $\mathbf{v}_1 = (x_1, y_1, z_1)$ and $\mathbf{v}_2 = (x_2, y_2, z_2)$.

Then $\mathbf{v}_1 \times \mathbf{v}_2 = (y_1 z_2 - y_2 z_1, -x_1 z_2 + x_2 z_1, x_1 y_2 - x_2 y_1)$.

It can be checked that

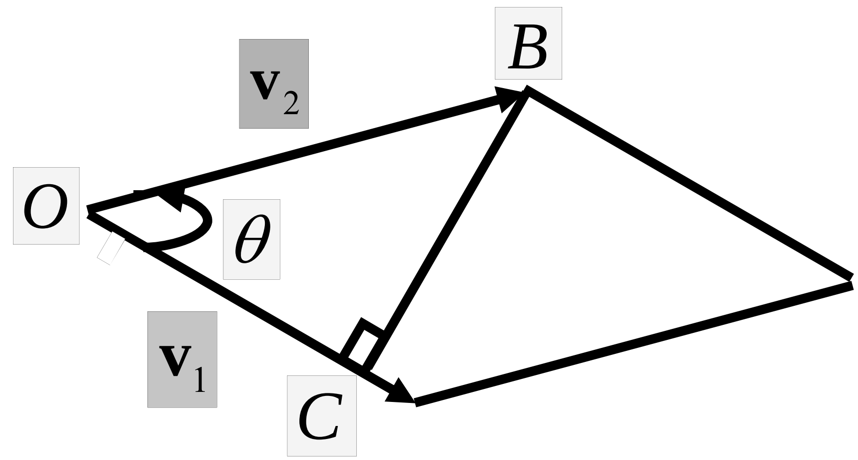
$$(\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_1 = 0 = (\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_2.$$



Right hand

Vector Product

$$\|\mathbf{v}_1 \times \mathbf{v}_2\| = \|\mathbf{v}_1\| \ \|\mathbf{v}_2\| \sin\theta$$



$$\frac{BC}{OB} = \sin\theta$$

$$BC = OB\sin\theta$$

$$= \|\mathbf{v}_2\| \sin\theta$$

$$\begin{aligned}\text{Area of parallelogram} &= \text{Base} \times \text{height}\\ &= \|\mathbf{v}_1\| \ BC\\ &= \|\mathbf{v}_1\| \|\mathbf{v}_2\| \sin\theta\\ &= \|\mathbf{v}_1 \times \mathbf{v}_2\|\end{aligned}$$

$$\mathbf{v}_1 // \mathbf{v}_2 \Leftrightarrow \mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{0}$$

$$\sin 0° = 0$$

$$\sin 180° = 0$$

Properties of Vector Product

Let $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ be vectors in *xyz*-space and let c be a real number.

(a) $\mathbf{v}_1 \times \mathbf{v}_2 = -\mathbf{v}_2 \times \mathbf{v}_1$.

(b) $\mathbf{v}_1 \times (\mathbf{v}_2 + \mathbf{v}_3) = \mathbf{v}_1 \times \mathbf{v}_2 + \mathbf{v}_1 \times \mathbf{v}_3$.

(c) $(\mathbf{v}_1 + \mathbf{v}_2) \times \mathbf{v}_3 = \mathbf{v}_1 \times \mathbf{v}_3 + \mathbf{v}_2 \times \mathbf{v}_3$.

(d) $c(\mathbf{v}_1 \times \mathbf{v}_2) = (c\mathbf{v}_1) \times \mathbf{v}_2 = \mathbf{v}_1 \times (c\mathbf{v}_2)$.

(e) $\mathbf{v}_1 \times \mathbf{v}_1 = \mathbf{0}$.

(f) $\mathbf{0} \times \mathbf{v}_1 = \mathbf{v}_1 \times \mathbf{0} = \mathbf{0}$.

Properties of Vector Product

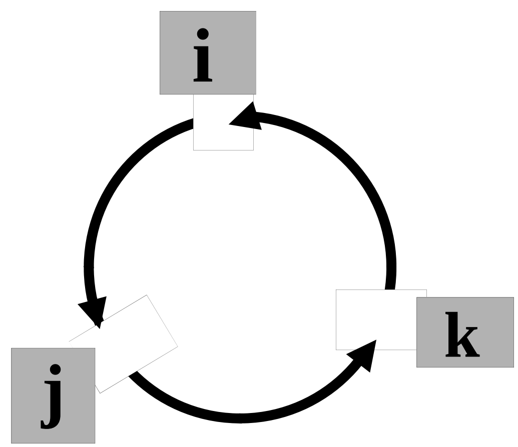
(e) $\mathbf{v}_1 \times \mathbf{v}_1 = \mathbf{0}.$

$$\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = \mathbf{0}.$$

$$\mathbf{i}\times\mathbf{j} = \mathbf{k}$$

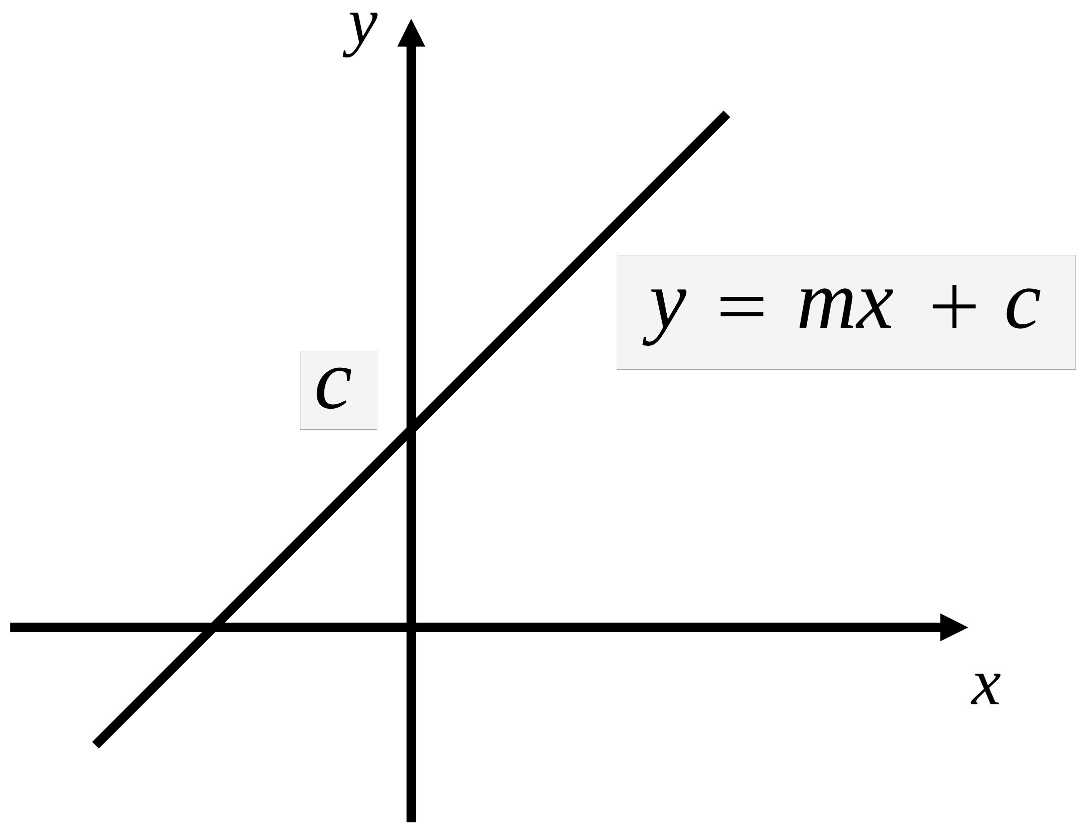
$$\mathbf{j}\times\mathbf{k} = \mathbf{i}$$

$$\mathbf{k}\times\mathbf{i} = \mathbf{j}$$



Lines in 3-D Space

Linear Equation in 2 Variables



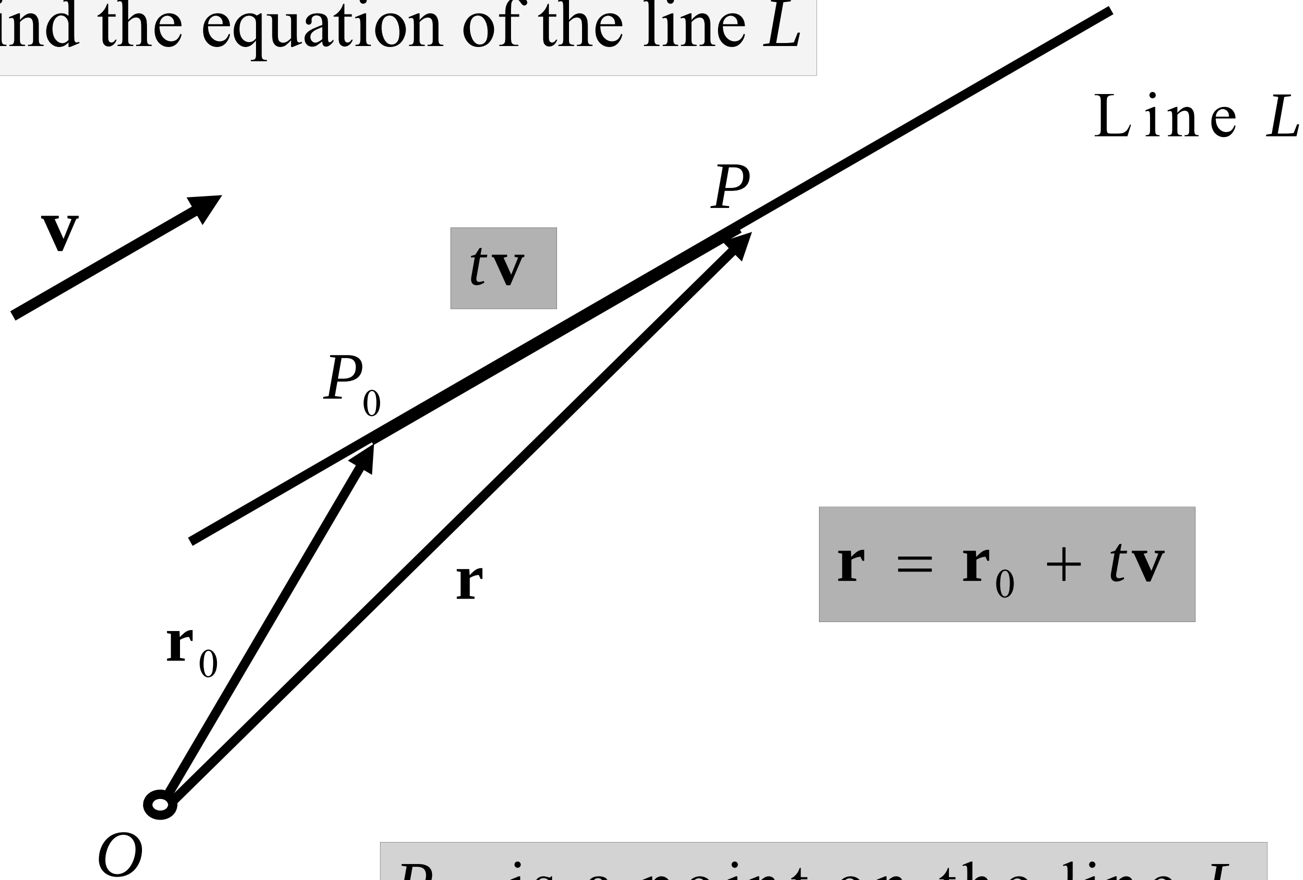
To determine a line, we need

gradient m

y – intercept c

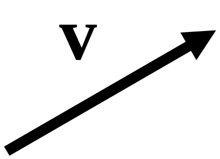
Vector Equation of a Line

Find the equation of the line L



$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

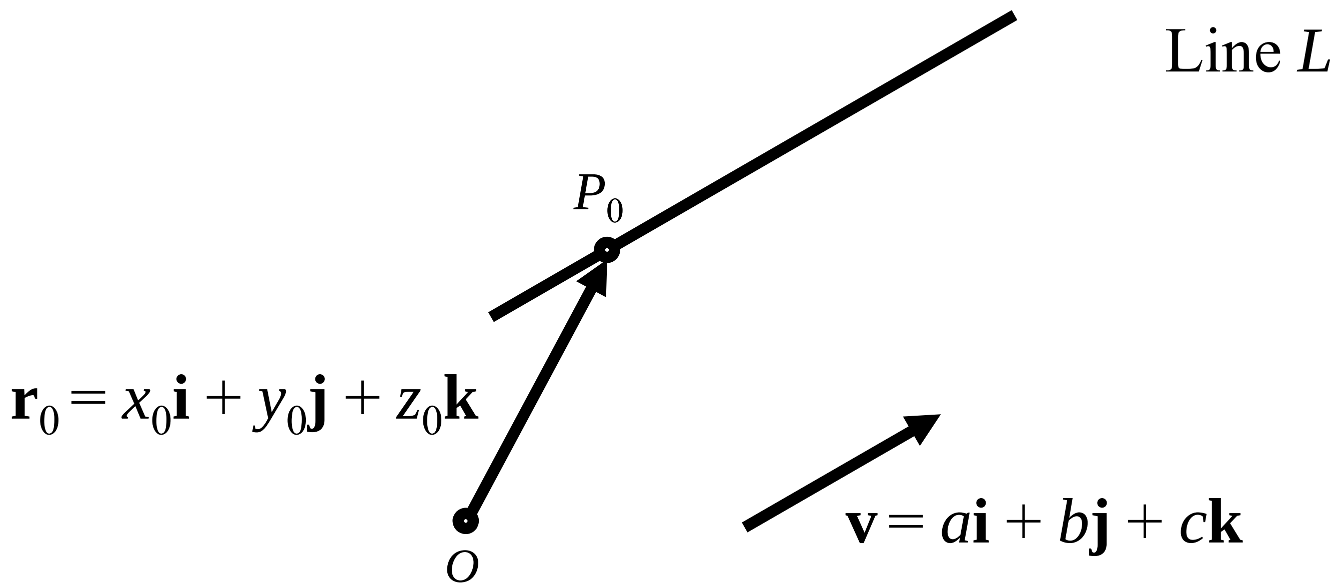
P_0 is a point on the line L



$\mathbf{v}$ is called direction vector of the line

Different values of t gives different points on the line L.

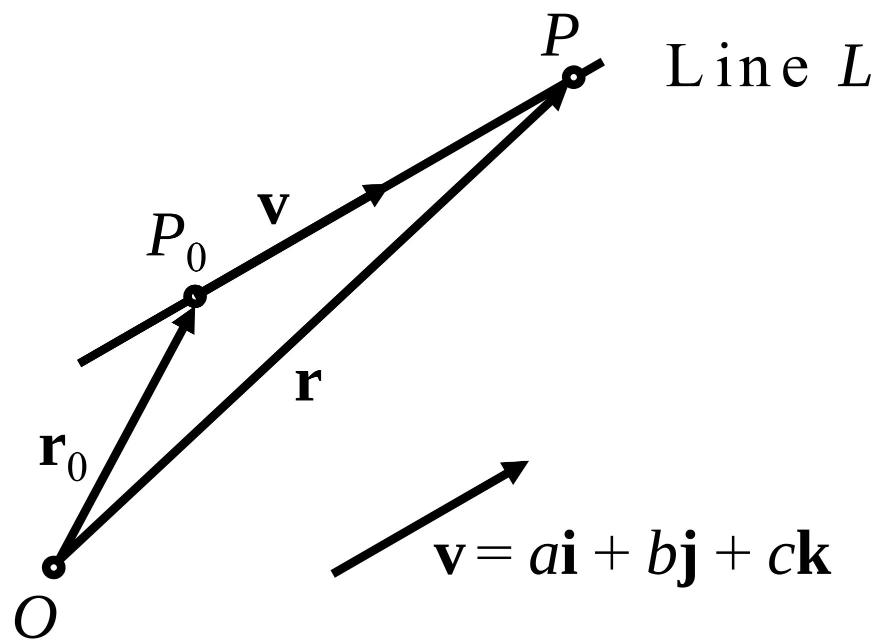
Find the equation of the line L



The point P_0 bring you to the line L.

Vector Equation of a Line

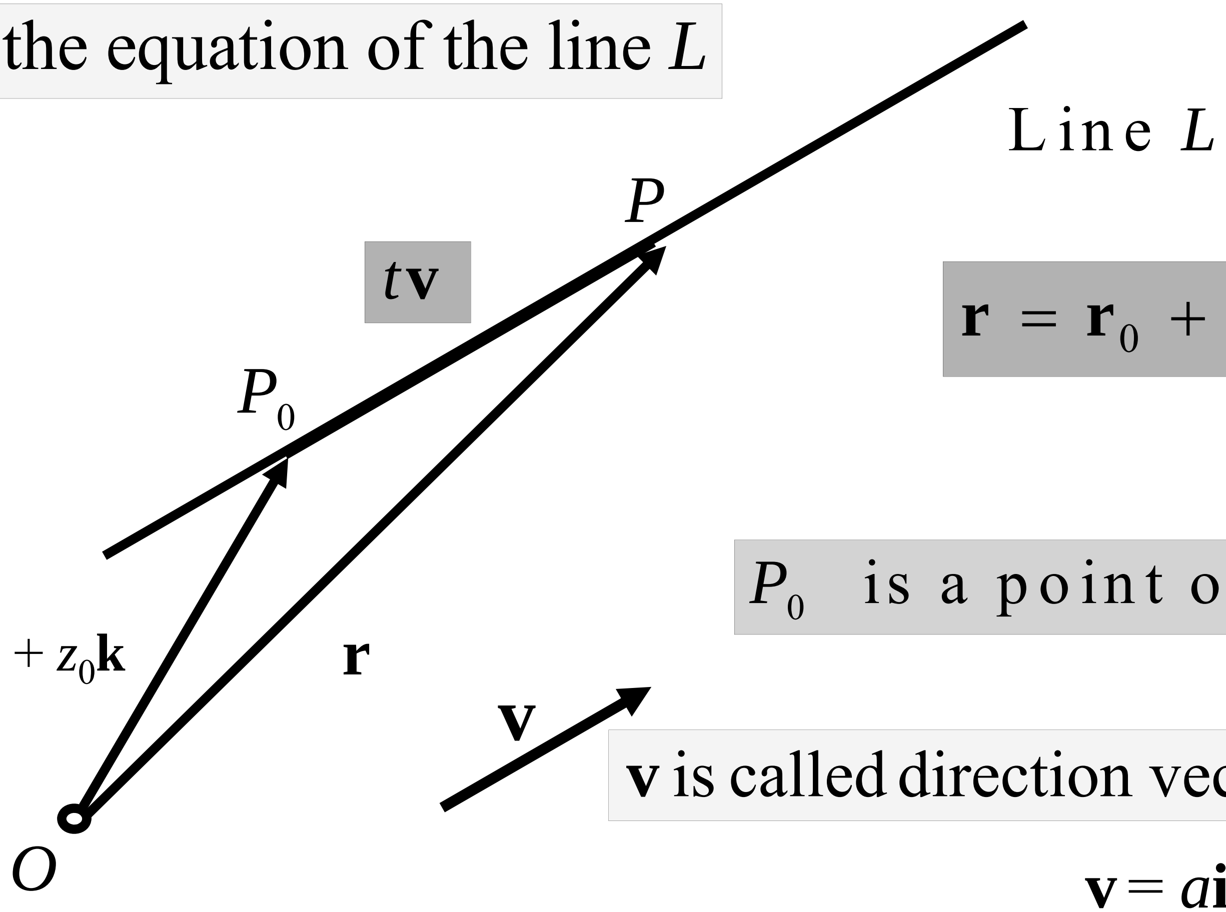
Let $P(x, y, z)$ be any point on L with position vector $\mathbf{r}$.



If you walk in the direction parallel to $\mathbf{v}$, then you will be always on the line L.

In this way, you can reach any point on the line L.

Find the equation of the line L



$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$

$\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$

P_0 is a point on the line L

$\mathbf{v}$ is called direction vector of the line

$\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

Then a ***vector equation*** of L is

$$\mathbf{r} = (x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}) + t(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})$$

or

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}.$$

Parametric Equation of a Line

Write $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$$= (x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}) + t(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \ .$$

Equating, we have

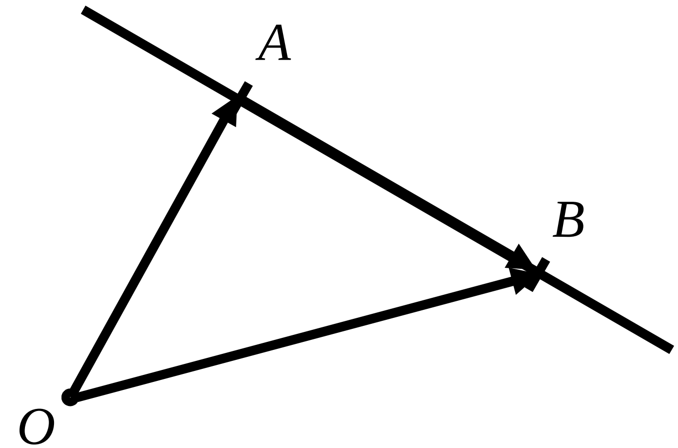
$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$

which are the parametric equations of the line L.

Example

The points A and B have position vectors $-3\mathbf{i}+2\mathbf{j}-3\mathbf{k}$ and $\mathbf{i}-\mathbf{j}+4\mathbf{k}$ respectively. Write down the parametric equations of the line passing through A and B.

$$\begin{aligned}\overrightarrow{AB} &= (\mathbf{i}-\mathbf{j}+4\mathbf{k})-(-3\mathbf{i}+2\mathbf{j}-3\mathbf{k})\\ &= 4\mathbf{i}-3\mathbf{j}+7\mathbf{k}\end{aligned}$$



We may take $\mathbf{v}=\overrightarrow{AB}$ as the direction vector of line AB.

The vector equation is

$$\mathbf{r}=(-3\mathbf{i}+2\mathbf{j}-3\mathbf{k})+t(4\mathbf{i}-3\mathbf{j}+7\mathbf{k}).$$

Example

The vector equation is

$$\mathbf{r} = (-3\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}\ ) + t(4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}).$$

Write $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

Hence the parametric equations of the line passing through A and B are

$$\begin{cases} x = -3 + 4t \\ y = 2 - 3t \\ z = -3 + 7t. \end{cases}$$

Example

Find the position vector of the point of intersection of L_1 and L_2.

$$L_1 : \mathbf{r} = \mathbf{i} + t_1(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}),$$

$$L_2 : \mathbf{r} = (2\mathbf{i} + \mathbf{j}) + t_2(3\mathbf{i} + \frac{9}{2}\mathbf{j} + \frac{9}{2}\mathbf{k}).$$

Eliminating $\mathbf{r}$ from the vector equations of L_1 and L_2, we get

$$\mathbf{i} + t_1(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = (2\mathbf{i} + \mathbf{j}) + t_2(3\mathbf{i} + \frac{9}{2}\mathbf{j} + \frac{9}{2}\mathbf{k}).$$

Hence it follows that

$$1 + t_1 = 2 + 3t_2, \quad 2t_1 = 1 + \frac{9}{2}t_2, \quad 3t_1 = \frac{9}{2}t_2$$

from which we obtain $t_1 = -1$ and $t_2 = -\dfrac{2}{3}$.

Find the position vector of the point of intersection of L_1 and L_2.

$$L_1 : \mathbf{r} = \mathbf{i} + t_1(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}),$$

$$L_2 : \mathbf{r} = (2\mathbf{i} + \mathbf{j}) + t_2(3\mathbf{i} + \frac{9}{2}\mathbf{j} + \frac{9}{2}\mathbf{k}).$$

Putting $t_1 = -1$ into the vector equation of L_1, we obtain

$$\mathbf{r} = \mathbf{i} + (-1)(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = -2\mathbf{j} - 3\mathbf{k}.$$

So the position vector of the point of intersection P of the two lines is

$$\overrightarrow{OP} = -2\mathbf{j} - 3\mathbf{k}.$$

Skew lines

are lines on

different

parallel planes

Example

Show that L_1 and L_3 are skew, i.e., do not intersect each other.

$$L_1: \quad \mathbf{r} = \mathbf{i} \ + t_1(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}),$$
$$L_3: \quad \mathbf{r} = (2\mathbf{i} + \mathbf{j}) \ + t_3(3\mathbf{i} + \mathbf{j}).$$

Eliminating $\mathbf{r}$ from the vector equations of L_1 and L_3, we get

$$\mathbf{i} + t_1(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = (2\mathbf{i} + \mathbf{j}) + t_3(3\mathbf{i} + \mathbf{j}).$$

Hence it follows that

$$1 + t_1 = 2 + 3t_2, \quad 2t_1 = 1 + t_3, \quad 3t_1 = 0.$$

Solving the first two equations above gives $t_1 = \dfrac{2}{5}$ but the last equation says $t_1 = 0,$ thus there is a contradiction.

So there is no solution to the equations and we conclude that L_1 and L_3 do not intersect.

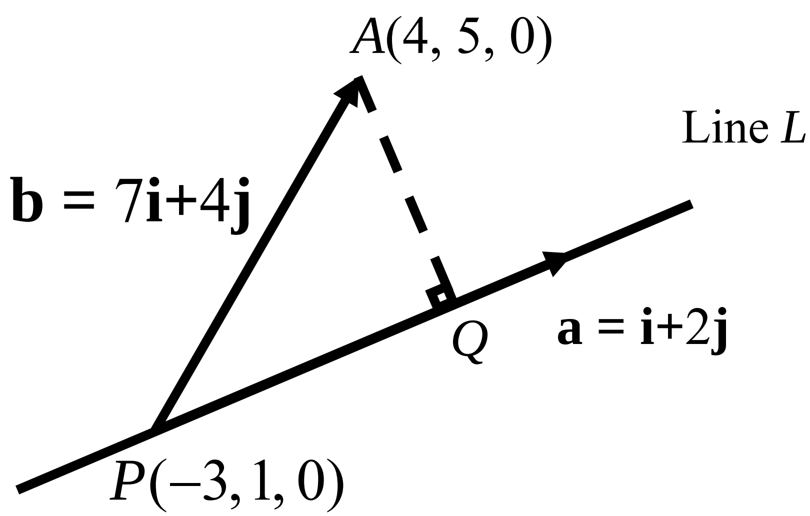
Example

Find the shortest distance from the point A with position vector $4\mathbf{i}+5\mathbf{j}$ to the line L whose vector equation is

$$\mathbf{r}=(-3\mathbf{i}+\mathbf{j})+t(\mathbf{i}+2\mathbf{j}).$$

Need to find: $|AQ|$.

$$\overrightarrow{PA}=\overrightarrow{OA}-\overrightarrow{OP}$$
$$=(4\mathbf{i}+5\mathbf{j})-(-3\mathbf{i}+\mathbf{j})=7\mathbf{i}+4\mathbf{j}.$$



Note : PQ is the length of projection of $\overrightarrow{PA}$ on $\mathbf{a}$

With the length of PQ, we may find the length of AQ

Example

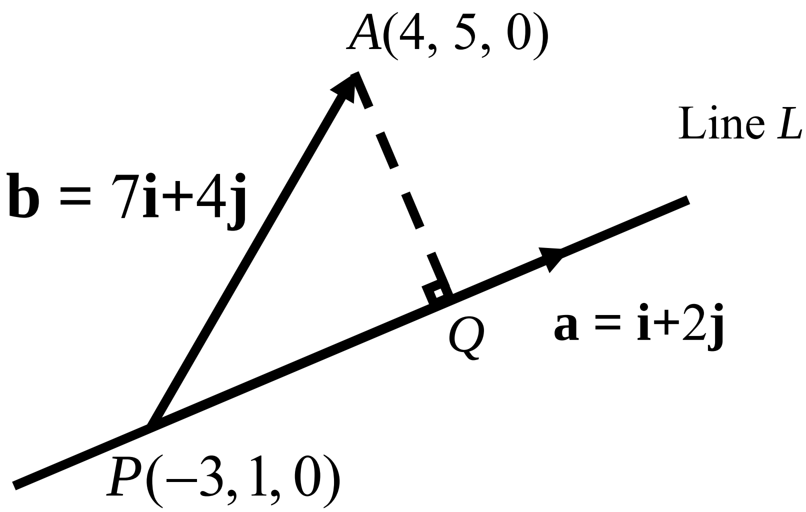
Find the shortest distance from the point A with position vector $4\mathbf{i}+5\mathbf{j}$ to the line L whose vector equation is

$$\mathbf{r}=(-3\mathbf{i}+\mathbf{j})+t(\mathbf{i}+2\mathbf{j}).$$

Need to find: $|AQ|$.

$$\overrightarrow{PA}=\overrightarrow{OA}-\overrightarrow{OP}$$
$$=(4\mathbf{i}+5\mathbf{j})-(-3\mathbf{i}+\mathbf{j})=7\mathbf{i}+4\mathbf{j}.$$

$$|PQ| = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}=\frac{(\mathbf{i}+2\mathbf{j})\cdot(7\mathbf{i}+4\mathbf{j})}{\sqrt{1^2+2^2}}=\frac{15}{\sqrt{5}}$$

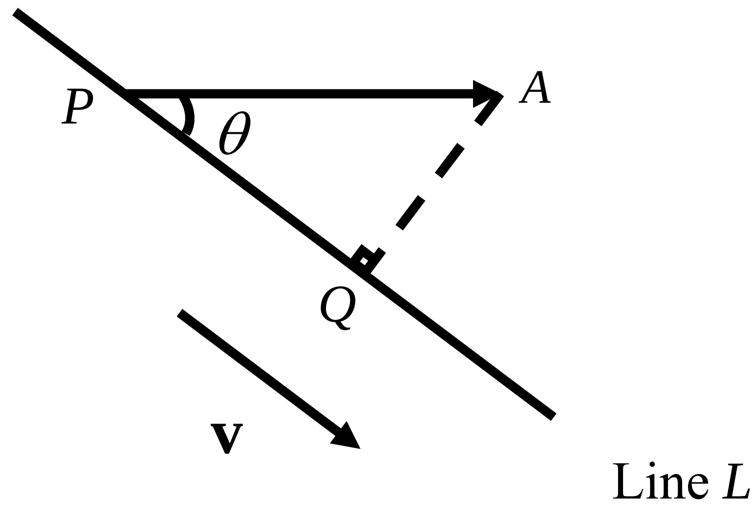


Now the shortest distance from A to L is given by

$$|AQ| = \sqrt{|\mathbf{b}|^2-|PQ|^2}=2\sqrt{5}\text{ units.}$$

$$\frac{AQ}{PA} = \sin\theta$$

$$AQ = PA\sin\theta \text{ ------ } (1)$$



Using vector product :

$$\left\|\overrightarrow{PA} \times \mathbf{v}\right\| = \left\|\overrightarrow{PA}\right\| \; \|\mathbf{v}\| \; \sin\theta$$

$$\frac{\left\|\overrightarrow{PA} \times \mathbf{v}\right\|}{\|\mathbf{v}\|} = \left\|\overrightarrow{PA}\right\| \; \sin\theta \text{ ------ } (2)$$

By (1) and (2), the shortest distance from point A to the line L is given by :

$$\boxed{AQ = \frac{\| \overrightarrow{PA} \times \mathbf{v} \|}{\| \mathbf{v} \|}}$$

Planes

Planes

A plane π in space is determined by

(i) a point it contains and

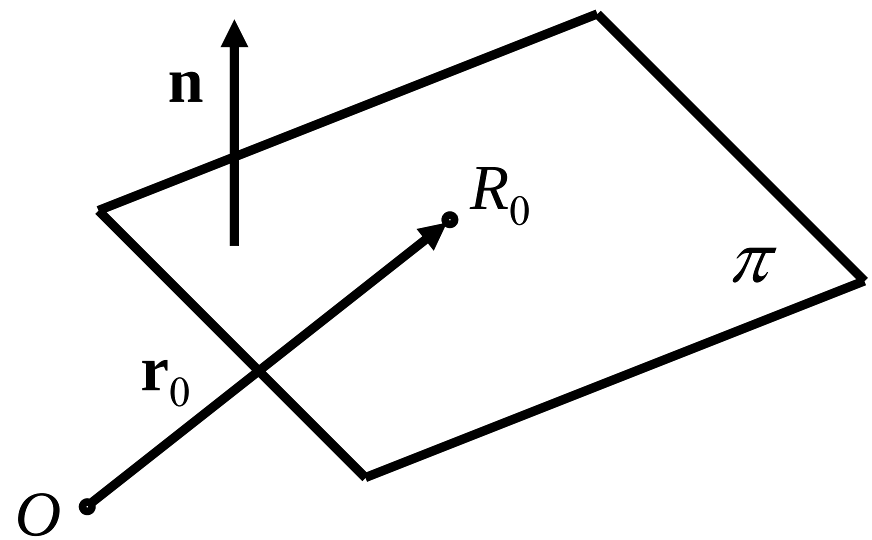
(ii) its '***tilt***' (defined by a ***normal*** to π)

Planes - Problem

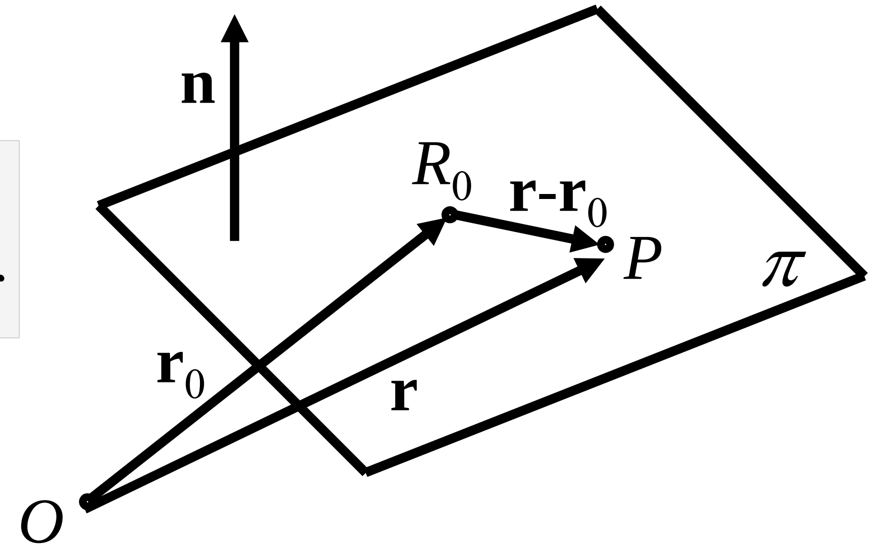
Given a point $R_0(x_0, y_0, z_0)$ in π

and a normal $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ to π,

how to find an equation for π.



Planes in 3D space



Note that :

$\mathbf{r}-\mathbf{r}_0$ and $\mathbf{n}$ are perpendicular

Let $P(x,y,z)$ be a point in π with $\overrightarrow{OP}=\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, then

$$(\mathbf{r}-\mathbf{r}_0)\cdot\mathbf{n}=\mathbf{0}$$

$$\mathbf{r}\cdot\mathbf{n}=\mathbf{r}_0\cdot\mathbf{n}$$

$$\mathbf{r}\cdot\mathbf{n}=d$$

where $d=\mathbf{r}_0\cdot\mathbf{n}$

$$\mathbf{r}_0=x_0\mathbf{i}+y_0\mathbf{j}+z_0\mathbf{k}$$

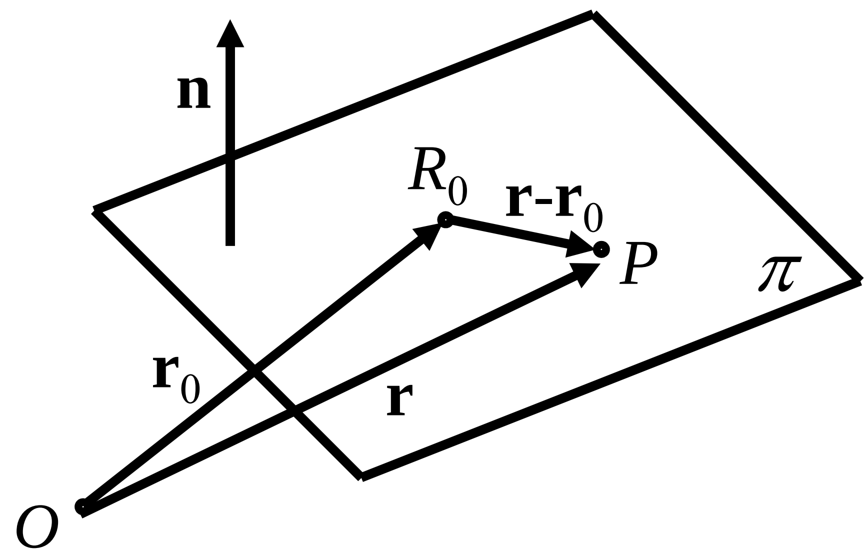
$$\mathbf{n}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$$

$$d=ax_0+by_0+cz_0$$

Cartesian Equation of Planes

Let $P(x, y, z)$ be a point in π with $\overrightarrow{OP} = \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0$$



Write
$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$
$$\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$$
$$\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}.$$

$$\mathbf{r} - \mathbf{r}_0 = (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}$$

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = a(x - x_0) + b(y - y_0) + c(z - z_0)$$

Thus $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

$$ax - ax_0 + by - by_0 + cz - cz_0 = 0$$

$$\boxed{ax + by + cz = d, \text{ where } d = ax_0 + by_0 + cz_0.}$$

Equation for Plane

Write $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$

$$\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$$

$$\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}.$$

Vector Equation:

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = \mathbf{0}, \text{ where } \mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

Vector Equation:

$$\mathbf{r} \cdot \mathbf{n} = d, \text{ where } d = ax_0 + by_0 + cz_0$$

Cartesian Equation :

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Cartesian Equation simplified :

$$ax + by + cz = d, \text{ where } d = ax_0 + by_0 + cz_0.$$

Equation for Plane - Example

Find the equation of the plane passing through the point $(0, 2, -1)$ normal to the vector $3\mathbf{i}+2\mathbf{j}-\mathbf{k}$.

Cartesian Equation simplified:

$$ax+by+cz=d, \text{ where } d=ax_0+by_0+cz_0.$$

$$\mathbf{r}_0 = x_0\mathbf{i}+y_0\mathbf{j}+z_0\mathbf{k}$$
$$\mathbf{n} = a\mathbf{i}+b\mathbf{j}+c\mathbf{k}.$$

$$\mathbf{r}_0 = 0\mathbf{i}+2\mathbf{j}-1\mathbf{k}$$
$$\mathbf{n} = 3\mathbf{i}+2\mathbf{j}-1\mathbf{k}.$$

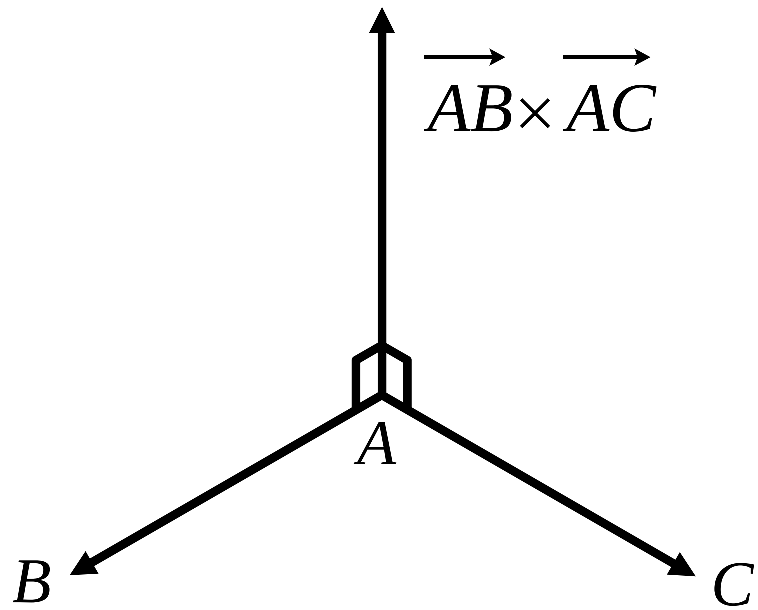
The required equation is

$$3x+2y-z=3(0)+2(2)-(-1),$$

or

$$3x+2y-z=5.$$

Planes in 3D space



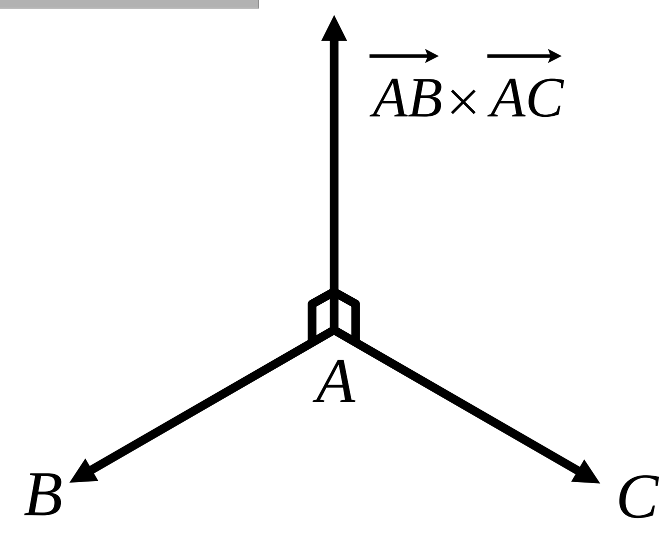
$\overrightarrow{AB}\times\overrightarrow{AC}$ is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$, and so is a normal vector to the plane containing points A, B and C.

Equation for Plane - Example

Find the vector equation of the plane passing through the points $A(0, 0, 1)$, $B(2, 0, 0)$ and $C(0, 3, 0)$.

The following vector is perpendicular to the plane:

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}.$$



Equation for Plane - Example

Find the vector equation of the plane passing through the points $A(0, 0, 1)$, $B(2, 0, 0)$ and $C(0, 3, 0)$.

The following vector is perpendicular to the plane:

$$\overrightarrow{AB} \times \overrightarrow{AC} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}.$$

Take $\mathbf{n} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$

$$\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$$
$$\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}.$$

The plane passes through (0, 0, 1).

So an equation of the plane is

$$3x + 2y + 6z = 3(0) + 2(0) + 6(1)$$

or

$$3x + 2y + 6z = 6.$$

Cartesian Equation simplified:

$$ax + by + cz = d, \text{ where } d = ax_0 + by_0 + cz_0.$$

Equation for Plane - Example

Find the vector equation of the plane passing through the points $A(0, 0, 1)$, $B(2, 0, 0)$ and $C(0, 3, 0)$.

The following vector is perpendicular to the plane:

$$\overrightarrow{AB} \times \overrightarrow{AC} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}.$$

Take $\mathbf{n} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$

$$\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$$

$$\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}.$$

The plane also passes through $(2,\ 0,\ 0)$.

We can also use the point $(2,\ 0,\ 0)$ to find the equation of plane

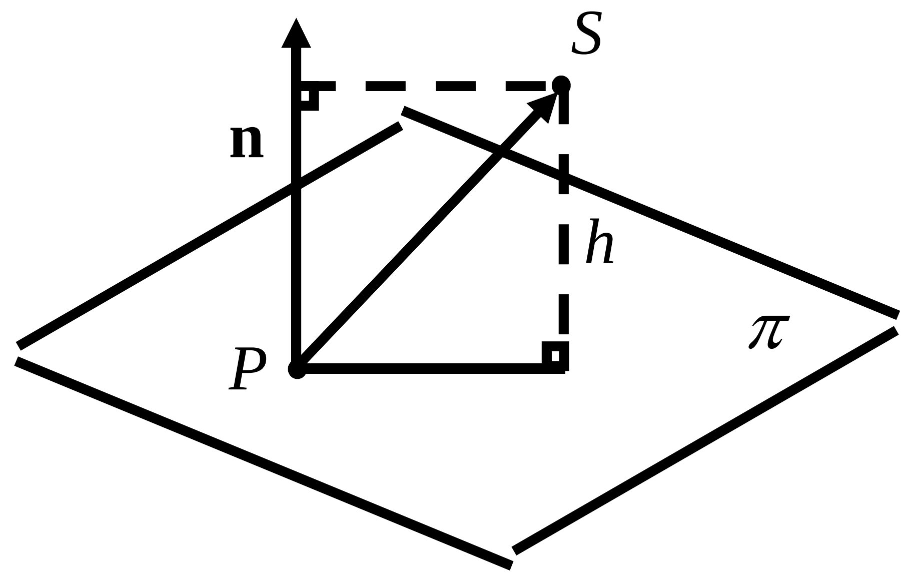
$$3x + 2y + 6z = 3(2) + 2(0) + 6(0) = 6.$$

Cartesian Equation simplified:

$$ax + by + cz = d, \text{ where } d = ax_0 + by_0 + cz_0.$$

Distance from Point to Plane

Let $P(x_1, y_1, z_1)$ be a point in π with normal $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

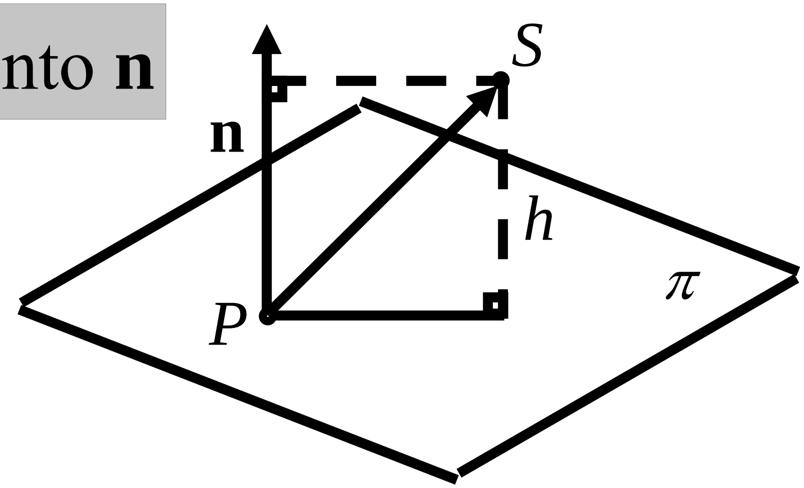


Find distance h from point $S(x_0, y_0, z_0)$ to plane π.

Find distance h from point $S(x_0, y_0, z_0)$ to plane π.

Let $P(x_1, y_1, z_1)$ be a point in π with normal $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

Note that h is the projection of $\overrightarrow{PS}$ onto $\mathbf{n}$



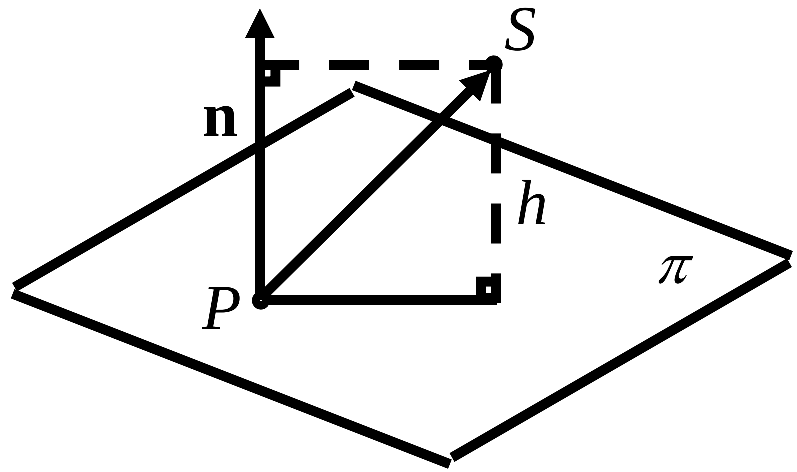
$$\overrightarrow{PS} = (x_0 - x_1)\mathbf{i} + (y_0 - y_1)\mathbf{j} + (z_0 - z_1)\mathbf{k}$$

$$h = \frac{\| \overrightarrow{PS} \cdot \mathbf{n} \|}{\| \mathbf{n} \|}$$

$$= \frac{\| [(x_0 - x_1)\mathbf{i} + (y_0 - y_1)\mathbf{j} + (z_0 - z_1)\mathbf{k}] \cdot (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{| a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1) |}{\sqrt{a^2 + b^2 + c^2}}$$

Find distance h from point $S(x_0, y_0, z_0)$ to plane π.



$$h = \frac{|a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|ax_0 + by_0 + cz_0 - (ax_1 + by_1 + cz_1)|}{\sqrt{a^2 + b^2 + c^2}}$$

Since equation of π is $ax + by + cz = d$ and point P lies on Π, we have

$$ax_1 + by_1 + cz_1 = d$$

Thus,

$$h = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}} \qquad (6)$$

Example

Find the distance of the point $(2, -3, 4)$ to the plane $x + 2y + 3z = 13$.

$$h = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}} \qquad (6)$$

$(x_0, y_0, z_0) = (2, \ -3, \ 4)$ and $a = 1, \ b = 2, \ c = 3.$

Using (6), we have

$$h = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|1(2) + 2(-3) + 3(4) - 13|}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{5}{\sqrt{14}} \text{ units}$$

Vector Functions of One Variable

Vector Function

Let $\mathbf{r}(t) = \begin{bmatrix} f(t) \\ g(t) \\ h(t) \end{bmatrix} = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k},$

where f, g and h are real-valued functions of a ***real*** variable t.

$\mathbf{r}(t)$ is a vector function and

f, g and h are component functions of $\mathbf{r}(t)$.

Vector Function - Example

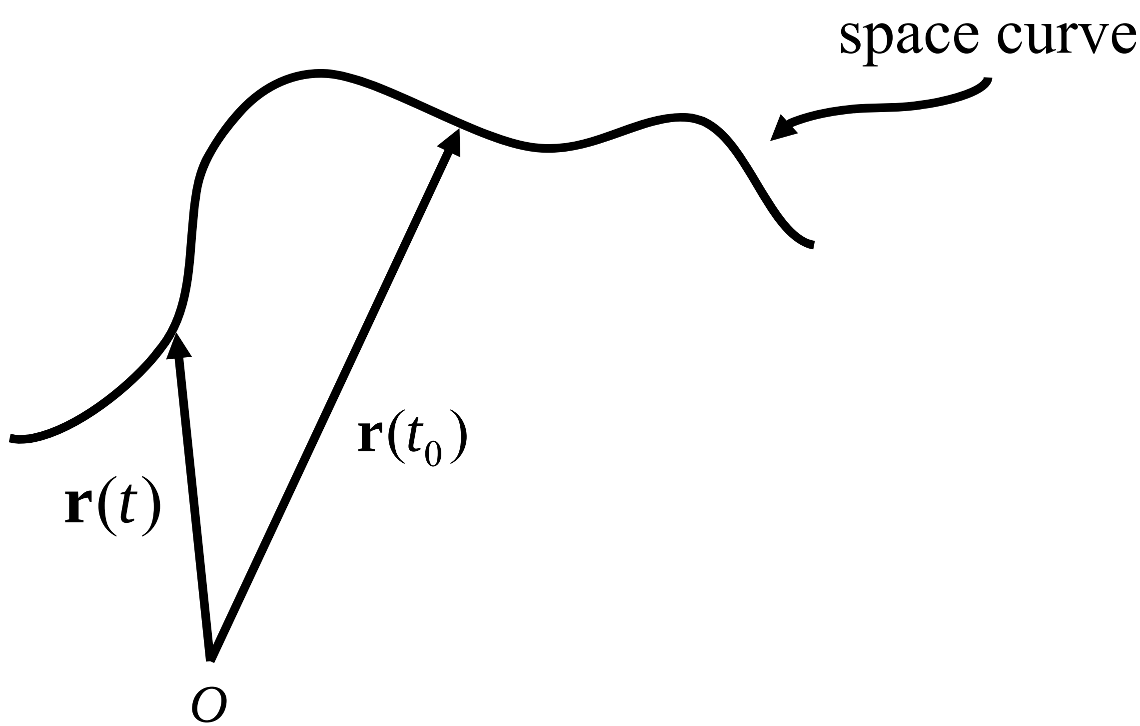
Consider the vector function

$$\mathbf{r}(t) = t\mathbf{i} + (t^2 + 1)\mathbf{j} + (2 - 7t)\mathbf{k}.$$

$$\mathbf{r}(2) = 2\mathbf{i} + 5\mathbf{j} - 12\mathbf{k}$$

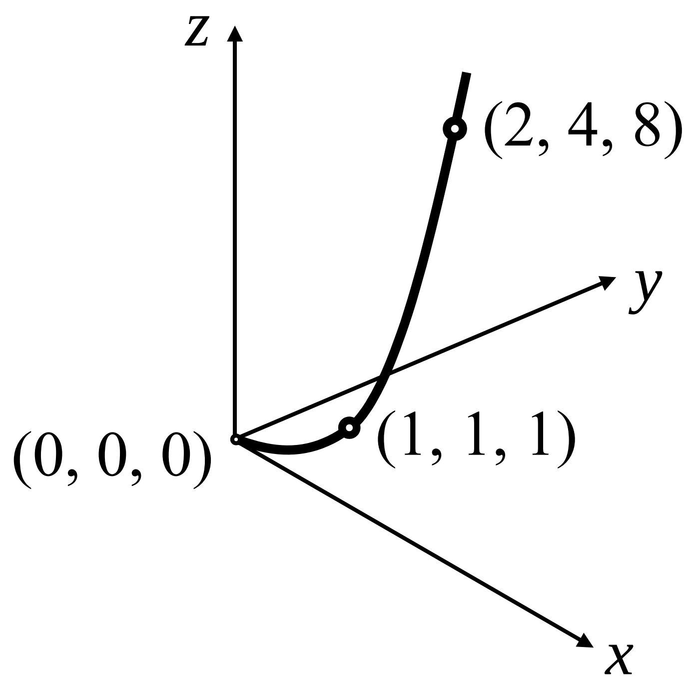
Space Curve

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$



Vector Functions of One Variable

Sketch the curve of $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}, t \geq 0.$



Limits and Continuity

We define the ***limit*** of $\mathbf{r}(t)$ as follows:

$$\lim_{t\to a}\mathbf{r}(t)=\left(\lim_{t\to a} f(t)\right)\mathbf{i}+\left(\lim_{t\to a} g(t)\right)\mathbf{j}+\left(\lim_{t\to a} h(t)\right)\mathbf{k}$$

provided the limit of each component function exists.

We say that $\mathbf{r}(t)$ is ***continuous*** at a point $t=a$ if

$$\lim_{t\to a}\mathbf{r}(t)=\mathbf{r}(a)=f(a)\mathbf{i}+g(a)\mathbf{j}+h(a)\mathbf{k}.$$

Equivalently, a vector function $\mathbf{r}(t)$ is continuous at a point a exactly when each of the component functions of $\mathbf{r}(t)$ is continuous at a, i.e., $f(t)$, $g(t)$ and $h(t)$ are continuous at a.

Limits and Continuity - Example

For the given vector function

$$\mathbf{r}(t) \;= t\mathbf{i} + (t^2+1)\mathbf{j} + (2-7t)\mathbf{k}.$$

We have

$$\lim_{t\to a}\mathbf{r}(t) = \left(\lim_{t\to a} t\right)\,\mathbf{i} + \left(\lim_{t\to a}(t^2+1)\right)\,\mathbf{j} + \left(\lim_{t\to a}(2-7t)\right)\mathbf{k}$$

$$= a\mathbf{i} + (a^2+1)\mathbf{j} + (2-7a)\mathbf{k}$$

$$= \mathbf{r}(a)$$

for all real numbers a.

Hence $\mathbf{r}(t)$ is continuous at every $t = a$.

Derivatives of Vector Functions

The ***derivative*** of a vector function $\mathbf{r}(t)$ is

$$\frac{d\mathbf{r}}{dt} = (\mathbf{r})'(t) = \lim_{h\to 0}\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h} \qquad (7)$$

provided the limit exists.

Let $f(x)$ be a function.

The derivative of f at a is defined to be

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

Derivatives of Vector Functions

If

$$\boxed{\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k},}$$

where f, g and h are differentiable functions, then the derivative is

$$\boxed{\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k} \qquad (8)}$$

Example

Consider the vector function

$$\mathbf{r}(t) = t\mathbf{i} + (t^2 + 1)\mathbf{j} + (2 - 7t)\mathbf{k}.$$

Since

$$\frac{d}{dt}(t) = 1, \quad \frac{d}{dt}(t^2 + 1) = 2t, \quad \frac{d}{dt}(2 - 7t) = -7$$

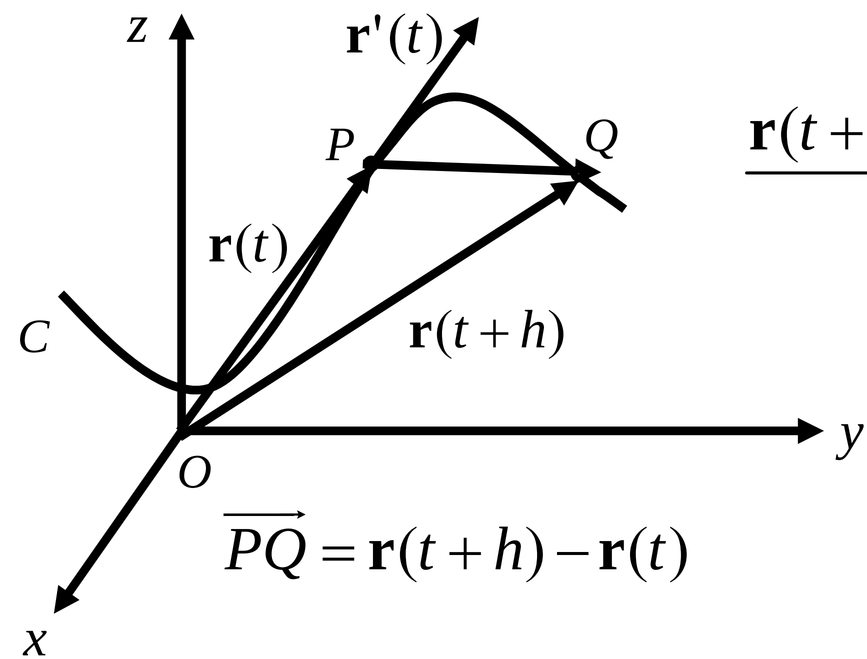
we have

$$\mathbf{r}'(t) = \mathbf{i} + (2t)\mathbf{j} - 7\mathbf{k}.$$

The ***derivative*** of a vector function $\mathbf{r}(t)$ is

$$\frac{d\mathbf{r}}{dt} = (\mathbf{r})'(t) = \lim_{h\to 0}\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h} \qquad (7)$$

provided the limit exists.



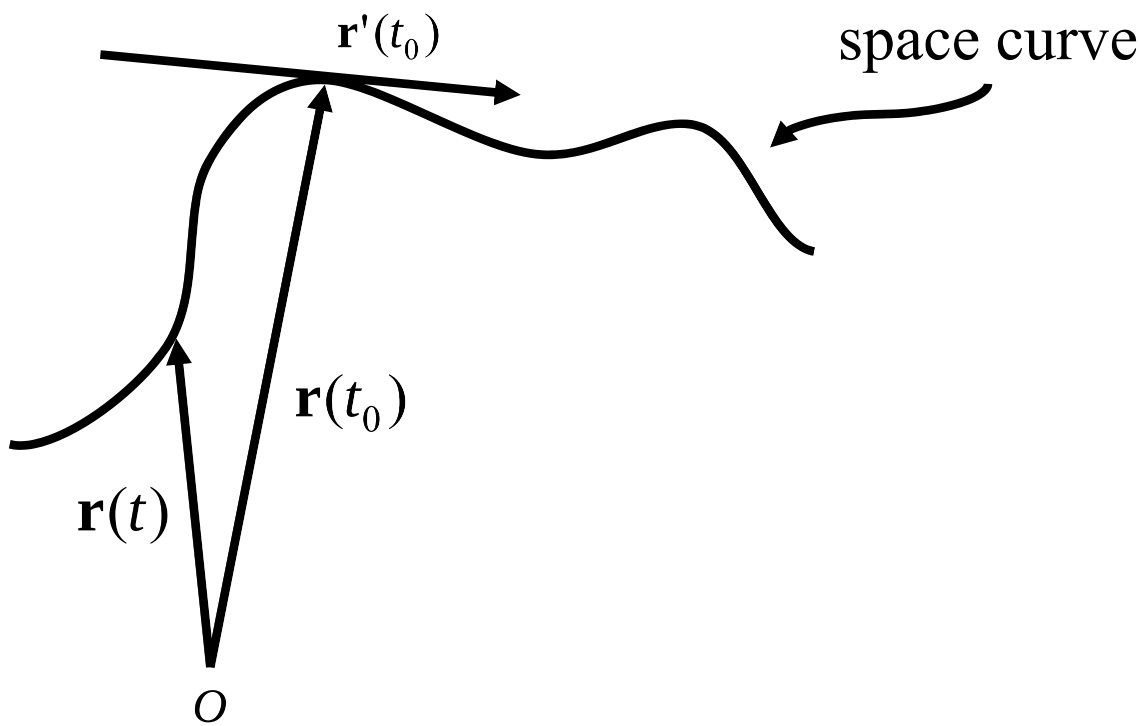
$$\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$$

$$\overrightarrow{PQ} = \mathbf{r}(t+h)-\mathbf{r}(t)$$

As $h \to 0$, $Q \to P$ along C and $\dfrac{\overrightarrow{PQ}}{h}$ becomes the ***tangent vector*** $\mathbf{r}'(t)$.

Space Curve

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$



Definite Integral of a Vector Function

The ***definite integral*** of a continuous vector function

$$\boxed{\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}}$$

on the interval $[a,b]$ is

$$\boxed{\int_a^b \mathbf{r}(t)\,dt = \int_a^b f(t)\,dt\;\mathbf{i} + \int_a^b g(t)\,dt\;\mathbf{j} + \int_a^b h(t)\,dt\;\mathbf{k}.}$$

Example

Find the definite integral of the vector function

$$\mathbf{r}(t) = 2t\mathbf{i} + 3t^2\mathbf{j}$$

on the interval $[0,2]$.

$$\begin{aligned}\int_0^2 (2t\mathbf{i} + 3t^2\mathbf{j})\,dt &= \int_0^2 2t\,dt\,\mathbf{i} + \int_0^2 3t^2\,dt\,\mathbf{j} \\ &= \left[t^2\right]_{t=0}^{t=2}\mathbf{i} + \left[t^3\right]_{t=0}^{t=2}\mathbf{j} \\ &= 4\mathbf{i} + 8\mathbf{j}.\end{aligned}$$

Space Curve

Space Curve

A curve in xyz-space can be represented by some continuous function

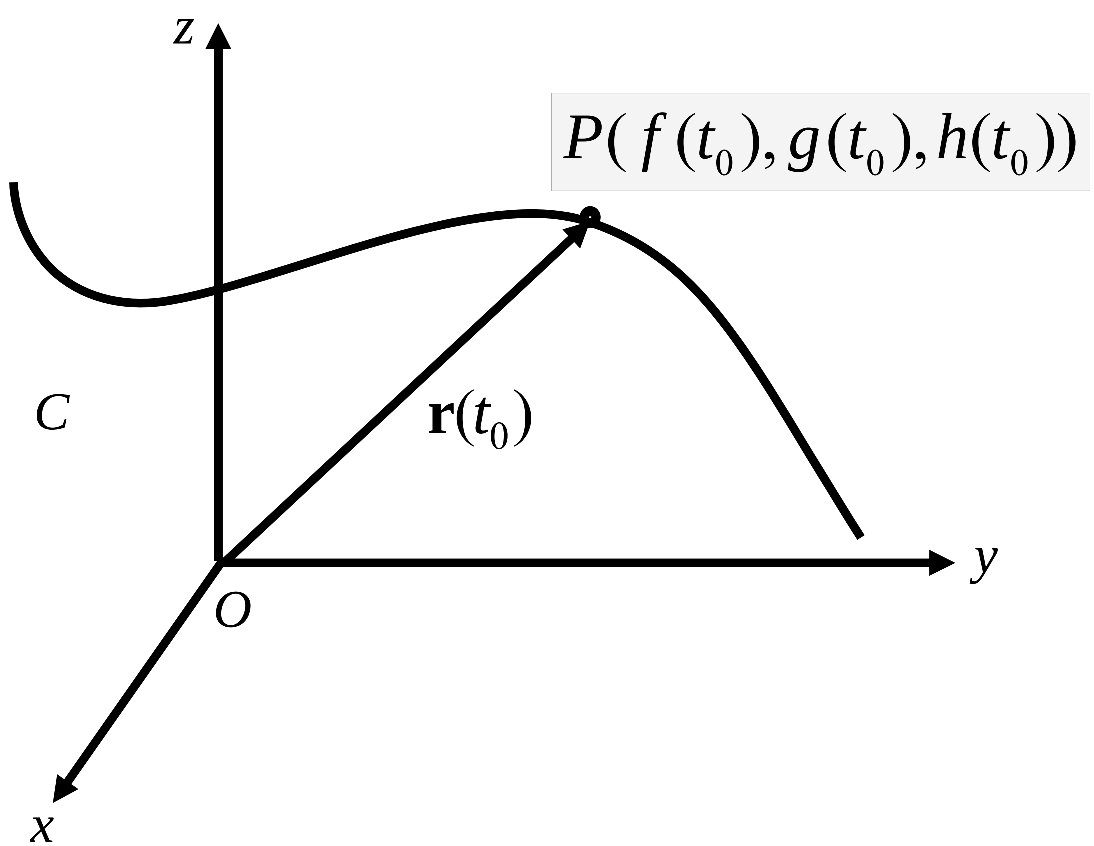
$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

such that a point P lies on the curve if its position vector $\overrightarrow{OP}$ is the image of the vector function, i.e.,

$$\overrightarrow{OP} = \mathbf{r}(t_0) \text{ for some } t_0 \in R.$$

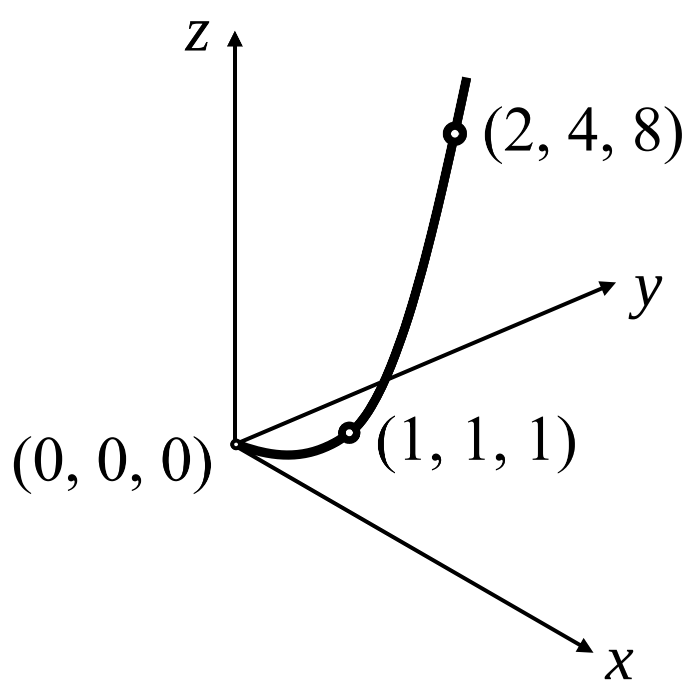
Space Curve

$$\boxed{\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}}$$



Vector Functions of One Variable

Sketch the curve of $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}, t \geq 0.$



Space Curve

We call

$$\boxed{\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}}$$

the ***vector equation*** of the curve and

$$\boxed{x = f(t), \quad y = g(t), \quad z = h(t)}$$

the ***parametric equation*** of the curve.

Space Curve - Example

The vector equation

$$\mathbf{r}(t) = (1+t)\mathbf{i} + (2+t)\mathbf{j} + (3+t)\mathbf{k}$$

represents the straight line in the *xyz*-space that passes through the point (1, 2, 3) and is parallel to the vector $\mathbf{i}+\mathbf{j}+\mathbf{k}$.

Note: $$\begin{aligned}\mathbf{r}(t) &= (1+t)\mathbf{i} + (2+t)\mathbf{j} + (3+t)\mathbf{k} \\ &= (\mathbf{i}+2\mathbf{j}+3\mathbf{k}) + t(\mathbf{i}+\mathbf{j}+\mathbf{k})\end{aligned}$$

Space Curve - Example

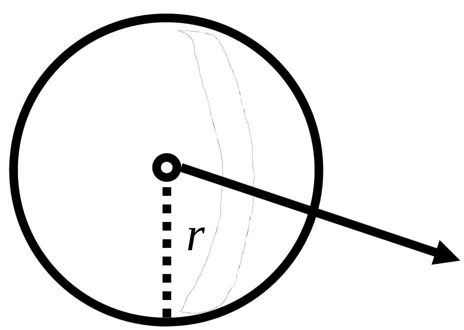
PAUSE AND THINK !!!

Sketch the space curve of

$$\mathbf{r}(t) = (r\cos t)\mathbf{i} + (r\sin t)\mathbf{j}.$$

$$x = r\cos t \quad \text{and} \quad y = r\sin t$$

$$\begin{aligned} x^2 + y^2 &= r^2\cos^2 t + r^2\sin^2 t \\ &= r^2(\cos^2 t + \sin^2 t) \\ &= r^2 \end{aligned}$$



(0,0)

Smooth Curves

A vector function $\mathbf{r}(t)$ is ***differentiable*** if $\mathbf{r}'(t)$ exists for each t in the domain.

The curve traced by $\mathbf{r}$ is said to be ***smooth*** if

(i) $\mathbf{r}(t)$ is continuous and

(ii) $\mathbf{r}'(t) \neq \mathbf{0}$ (Zero vector)

(i.e., $f'(t)$, $g'(t)$ and $h'(t)$ are all continuous and are not 0 simultaneously.)

The condition that $\mathbf{r}'(t) \neq \mathbf{0}$ is to make sure that the curve has a continuously turning tangent at every point (and thus has no sharp corners or cusps).

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(i.e., $f'(t)$, $g'(t)$ and $h'(t)$ are all continuous and are not 0 simultaneously.)

The condition that $\mathbf{r}'(t) \neq \mathbf{0}$ is to make sure that the curve has a continuously turning tangent at every point (and thus has no sharp corners or cusps).

$$\mathbf{r}(t) = (1 + t^3)\mathbf{i} + t^2\mathbf{j}$$

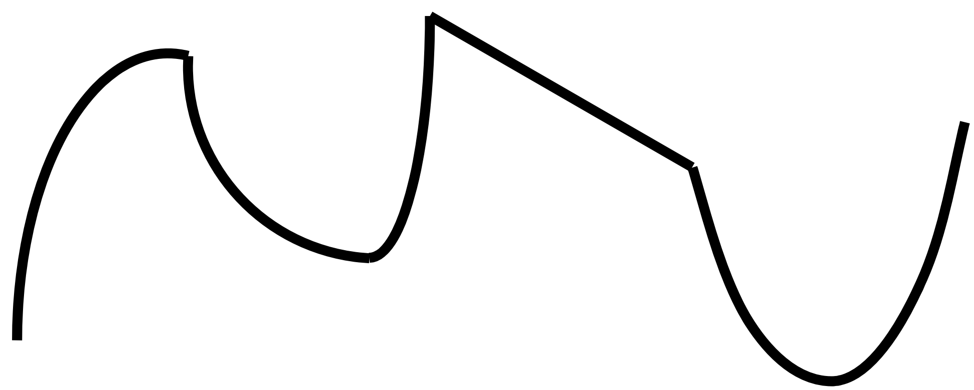
$$\mathbf{r}'(t) = 3t^2\mathbf{i} + 2t\mathbf{j}$$

Note : $\mathbf{r}'(0) = \mathbf{0}$

Space curve of $\mathbf{r}(t) = (1 + t^3)\mathbf{i} + t^2\mathbf{j}$ is not smooth

Smooth Curves

A ***piecewise smooth curve*** is made up of a finite number of smooth pieces.



Smooth Curves - Example

Consider the vector function

$$\boxed{\mathbf{r}(t) = t\mathbf{i} + (t^2 + 1)\mathbf{j} + (2 - 7t)\mathbf{k}.}$$

We have

$$\boxed{\mathbf{r}'(t) = \mathbf{i} + (2t)\mathbf{j} - 7\mathbf{k} \quad \neq \mathbf{0} \text{ for all } t.}$$

So $\mathbf{r}(t)$ represents a smooth curve.

Smooth Curves - Example

The following vector function represents a piecewise smooth curve:

$$\mathbf{r}(t) = \begin{cases} t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k} & \text{if } 0 \le t \le 1 \\ (2t-1)\mathbf{i} + t^2\mathbf{j} + (t^2 + t - 1)\mathbf{k} & \text{if } 1 < t \le 2. \end{cases}$$

$$\boxed{\text{Check that } : \ \mathbf{r}'(t) \neq \mathbf{0}}$$

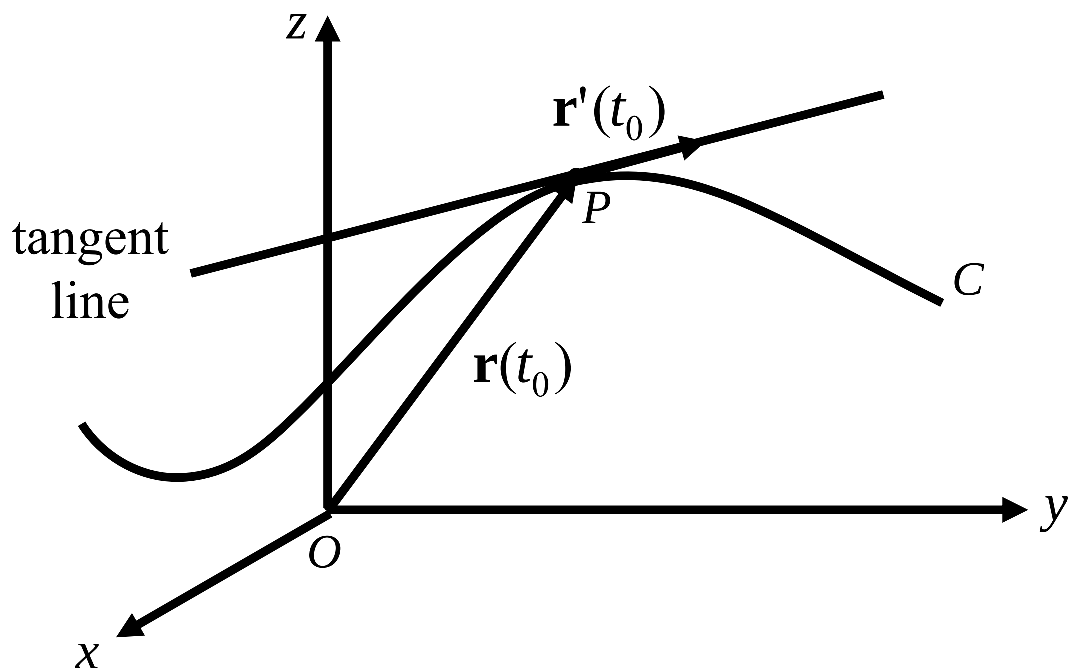
Tangent Vector and Line to a Curve

The ***tangent line*** to a curve $\mathbf{r}(t)$ at a point P whose position vector is $\mathbf{r}(t_0)$ is defined to be the line through P parallel to the tangent vector $\mathbf{r}'(t_0)$
(here it is assumed that $\mathbf{r}'(t_0) \neq \mathbf{0}$).

The unit tangent vector to the curve at $t = t_0$ is

$$\frac{\mathbf{r}'(t_0)}{\|\mathbf{r}'(t_0)\|}.$$

Tangent Vector and Line to a Curve



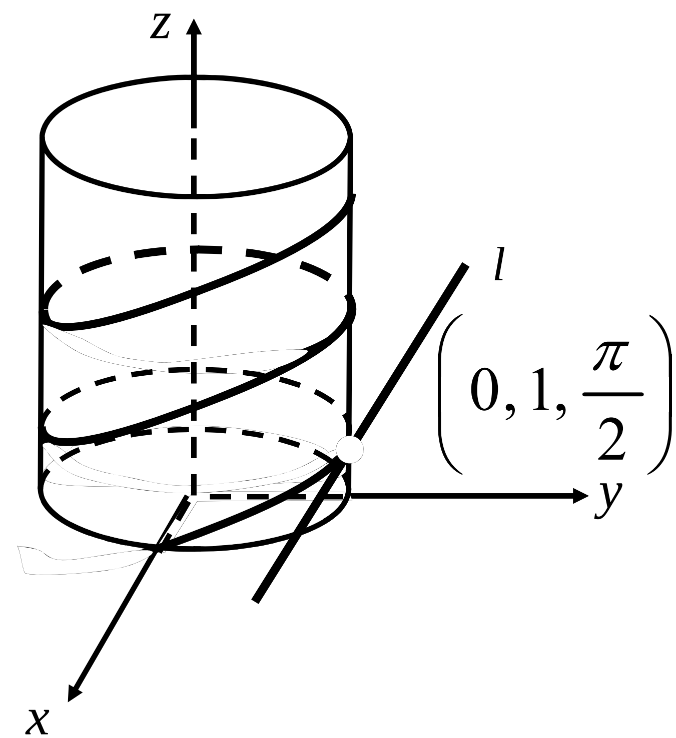
Example

Consider the circular helix

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}.$$

$$\mathbf{r}\left(\frac{\pi}{2}\right) = \left(\cos\frac{\pi}{2}\right)\mathbf{i} + \left(\sin\frac{\pi}{2}\right)\mathbf{j} + \frac{\pi}{2}\mathbf{k}$$

$$= 0\mathbf{i} + 1\mathbf{j} + \frac{\pi}{2}\mathbf{k} = \mathbf{j} + \frac{\pi}{2}\mathbf{k}$$



Thus, the point $\left(0, 1, \frac{\pi}{2}\right)$ lies on the helix.

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$$

$$\mathbf{r}'(t) = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k} \neq \mathbf{0}$$

So $\mathbf{r}(t)$ represents a smooth curve and

$$\mathbf{r}'\left(\frac{\pi}{2}\right) = (-1)\mathbf{i} + (0)\mathbf{j} + (1)\mathbf{k} = -\mathbf{i} + \mathbf{k}$$

is the tangent vector at $\left(0, 1, \frac{\pi}{2}\right)$

The ***unit tangent vector*** at $\left(0, 1, \frac{\pi}{2}\right)$ is $\frac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{k})$.

The tangent line at $\left(0, 1, \frac{\pi}{2}\right)$ is $\mathbf{r}(t) = (0\mathbf{i} + 1\mathbf{j} + \frac{\pi}{2}\mathbf{k}) + t(-\mathbf{i} + \mathbf{k})$

Parametric equations of the tangent line at $\left(0, 1, \frac{\pi}{2}\right)$ are

$$x = -t, \quad y = 1, \quad z = \frac{\pi}{2} + \mathrm{t}\,.$$

Arc Length of a Space Curve

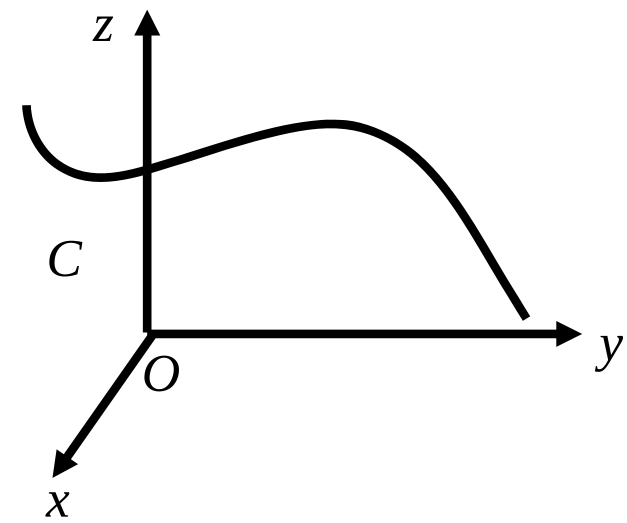
Suppose that a curve has the vector equation

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k},$$

or alternatively, parametric equations

$$x = f(t), \quad y = g(t), \quad z = h(t),$$

where $f'(t),\ g'(t),\ h'(t)$ are continuous functions.



If the curve is traversed exactly once as t increases from a to b, then its arc length is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\ dt$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt.$$

Arc Length of a Space Curve

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\ dt$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

A more compact formula of both arc length formulas is

$$L = \int_a^b \|\mathbf{r}'(t)\|\ dt$$

Recall : $\mathbf{v} = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$

$\|\mathbf{v}\| = \sqrt{x_0^2 + y_0^2 + z_0^2}$

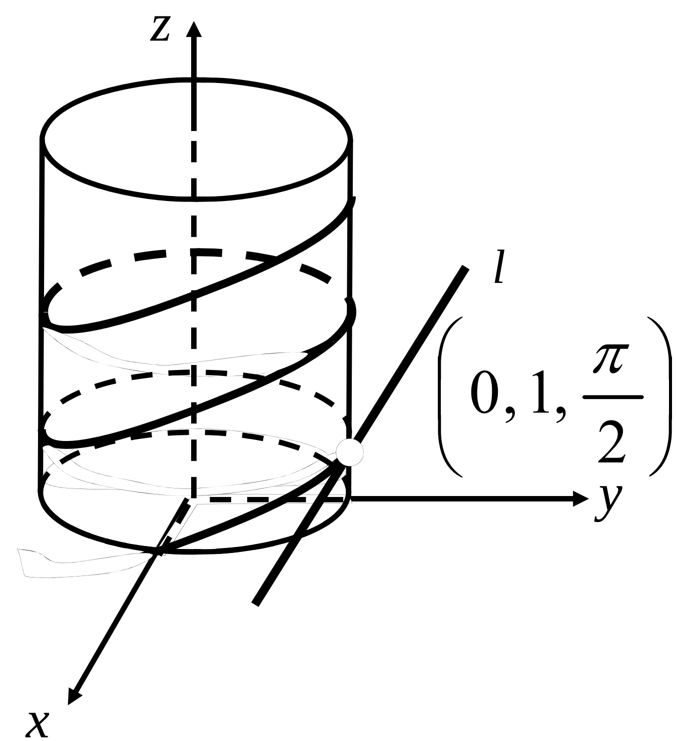
$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$

$\|\mathbf{r}'(t)\| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$

Example

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$$

$$\mathbf{r}'(t) = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k} \neq \mathbf{0}$$



We can find the arc length from $t = 0$ to $t = 2\pi$ as follows:

$$\|\mathbf{r}'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{2}$$

$$\begin{aligned} L &= \int_0^{2\pi} \|\mathbf{r}'(t)\|\ dt \\ &= \int_0^{2\pi} \sqrt{2}\ dt \\ &= 2\sqrt{2}\pi \text{ units.} \end{aligned}$$

End