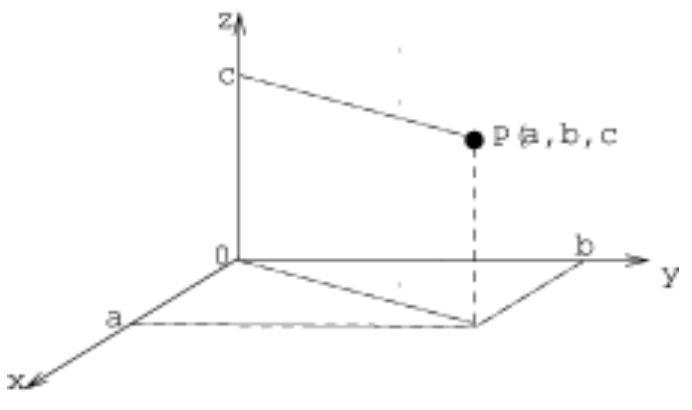
Chapter 6. Three Dimensional Space

6.1 The Coordinate System of the 3D Space

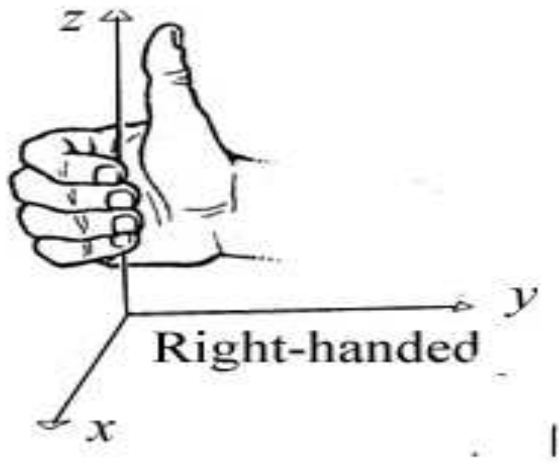
For three dimensional space, we first fix a coordinate system by choosing a point called the **origin**, and three lines, called the coordinate axes, so that each line is perpendicular to the other two. These lines are called the x-. y- and z-axes.



Associated with a point P in three dimensional space is an ordered triple (a, b, c) where a, b and c are the projections of P on the x-, y- and z-axes respectively.

This is the **Cartesian coordinate system** for three dimensional space. We also call this space the xyz-space.

By convention, we use the **right-handed coordinate system**. A right-handed coordinate system fix the orientation of the axes as follow:



If we rotate the x-axis counterclockwise toward the y-axis, then a right-handed screw will move in the positive z direction.

6.2 Vectors in xyz-Space

A vector is measurable quantity with a *magnitude* and a *direction*. It is geometrically represented by an arrow in the xyz-space with an initial point and a terminal point. The direction of the arrow gives the direction of the vector; and the length of the arrow gives the magnitude of the vector.

6.2.1 Terminologies and notations

(1) Let P and Q be points in the xyz-space with coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively.

Then the vector $\overrightarrow{PQ}$ is algebraically given by

$$\overrightarrow{PQ} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix}.$$

The vector $\overrightarrow{OP} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ is called the position vector of P.

(2) The zero vector in the xyz-space is $\mathbf{O} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(3) The sum of $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ is

$$\mathbf{v}_1 + \mathbf{v}_2 = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}.$$

[Note that $\mathbf{v}_1 + \mathbf{O} = \mathbf{O} + \mathbf{v}_1 = \mathbf{v}_1$.]

(4) The negative of $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ is $-\mathbf{v}_1 = \begin{bmatrix} -x_1 \\ -y_1 \\ -z_1 \end{bmatrix}$.

[Note that $\mathbf{v}_1 - \mathbf{v}_1 = -\mathbf{v}_1 + \mathbf{v}_1 = \mathbf{O}$.]

(5) The difference $\mathbf{v}_1 - \mathbf{v}_2$ is

$$\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{v}_1 + (-\mathbf{v}_2) = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} -x_2 \\ -y_2 \\ -z_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \end{bmatrix}.$$

(6) If c is a real number, the scalar $c\mathbf{v}_1$ of $\mathbf{v}_1$ by c is

$$c\mathbf{v}_1 = \begin{bmatrix} cx_1 \\ cy_1 \\ cz_1 \end{bmatrix}.$$

If $c > 0$, then $c\mathbf{v}_1$ is in the same direction as $\mathbf{v}_1$.

If $d < 0$, then $d\mathbf{v}_1$ is in the opposite direction as $\mathbf{v}_1$.

(7) The magnitude of $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ is

$$||\mathbf{v}_1|| = \sqrt{x_1^2 + y_1^2 + z_1^2}.$$

[Note that $||c\mathbf{v}_1|| = |c|\ ||\mathbf{v}_1||$ for a real number c.]

6.2.2 Example

Let P_1, P_2, Q_1 and Q_2 be the points $(3, 2, -1)$, $(0, 0, 0)$, $(5, 5, 4)$ and $(2, 3, 5)$ respectively.

$$\overrightarrow{P_1Q_1} = \begin{bmatrix} 5-3 \\ 5-2 \\ 4-(-1) \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

$$\overrightarrow{P_2Q_2} = \begin{bmatrix} 2-0 \\ 3-0 \\ 5-0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}.$$

Hence

$$\overrightarrow{P_1Q_1} = \overrightarrow{P_2Q_2}.$$

The magnitude of $\overrightarrow{P_1Q_1}$ is

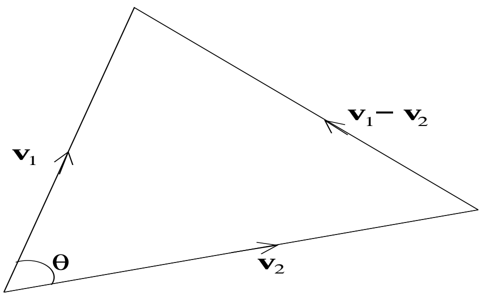
$$||\overrightarrow{P_1Q_1}|| = \sqrt{(2)^2 + (3)^2 + (5)^2} = \sqrt{38}.$$

So the magnitude of $5\overrightarrow{P_1Q_1}$ is

$$5||\overrightarrow{P_1Q_1}|| = 5\sqrt{38}.$$

6.2.3 Angle between two vectors

The angle between the nonzero vectors $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ is the angle θ, $(0 \leq \theta \leq 180^0)$ as shown below.



Applying the law of cosines to this triangle, we obtain

$$||\mathbf{v}_1 - \mathbf{v}_2||^2 = ||\mathbf{v}_1||^2 + ||\mathbf{v}_2||^2 - 2||\mathbf{v}_1||\,||\mathbf{v}_2||\cos\theta. \quad (1)$$

Now LHS of (1) $||\mathbf{v}_1 - \mathbf{v}_2||^2$ is given by

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$$

$$= x_1^2 + x_2^2 + y_1^2 + y_2^2 + z_1^2 + z_2^2 - 2(x_1x_2 + y_1y_2 + z_1z_2)$$

$$= ||\mathbf{v}_1||^2 + ||\mathbf{v}_2||^2 - 2(x_1x_2 + y_1y_2 + z_1z_2).$$

If we substitute this expression in (1) and solve for $\cos\theta$, we obtain

$$\cos\theta = \frac{x_1\ x_2 + y_1\ y_2 + z_1\ z_2}{||\mathbf{v}_1||\ ||\mathbf{v}_2||} \qquad (2)$$

6.2.4 Scalar or dot product

The **scalar product** or **dot product** of the vectors $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ is defined by

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = x_1x_2 + y_1y_2 + z_1z_2.$$

Thus we can rewrite (2), where $\mathbf{v}_1$ and $\mathbf{v}_2$ are nonzero vectors, as

$$\cos\theta = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{||\mathbf{v}_1||\ ||\mathbf{v}_2||}, \quad (0 \le \theta \le 180^0)$$

and notice that

$$\mathbf{v}_1 \text{ and } \mathbf{v}_2 \text{ are perpendicular} \iff \mathbf{v}_1 \cdot \mathbf{v}_2 = 0.$$

6.2.5 Example

If $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$, then

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = (2)(-1) + (4)(2) + (5)(3) = 21.$$

Also

$$||\mathbf{v}_1|| = \sqrt{2^2 + 4^2 + 5^2} = \sqrt{45},$$

$$||\mathbf{v}_2|| = \sqrt{(-1)^2 + 2^2 + 3^2} = \sqrt{14}.$$

Hence

$$\cos\theta = \frac{21}{\sqrt{45}\sqrt{14}} = \frac{\sqrt{7}}{\sqrt{10}}.$$

Thus θ is approximately $33°13'$.

The vectors $\mathbf{w}_1 = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$ and $\mathbf{w}_2 = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$ are perpendicular since their dot product

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = (2)(4) + (-5)(2) + (1)(2) = 0.$$

6.2.6 Properties of scalar product

If $\mathbf{v_1}$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are vectors in xyz-space and c is a real number, then

(a) $\mathbf{v}_1 \cdot \mathbf{v}_1 = ||\mathbf{v}_1||^2 \geq 0$.

(b) $\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_2 \cdot \mathbf{v}_1$.

(c) $(\mathbf{v}_1 + \mathbf{v}_2) \cdot \mathbf{v}_3 = \mathbf{v}_1 \cdot \mathbf{v}_3 + \mathbf{v}_2 \cdot \mathbf{v}_3$.

(d) $(c\mathbf{v}_1) \cdot \mathbf{v}_2 = \mathbf{v}_1 \cdot (c\mathbf{v}_2) = c(\mathbf{v}_1 \cdot \mathbf{v}_2)$.

6.2.7 Unit vector

A **unit vector** in xyz-space is a vector of magnitude or length 1. The vectors

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are unit vectors along the positive x-, y- and z-axes respectively.

Notice that every vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ can be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

For example,

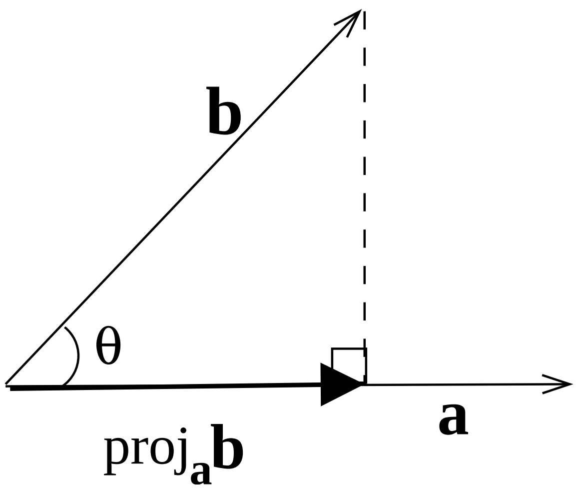
$$\mathbf{w} = \begin{bmatrix} 4 \\ -5 \\ 22 \end{bmatrix} = 4\mathbf{i} - 5\mathbf{j} + 22\mathbf{k}.$$

The unit vector with the same direction as $\mathbf{w}$ is

$$\begin{aligned} \frac{1}{||\mathbf{w}||}\mathbf{w} &= \frac{1}{\sqrt{4^2 + 5^2 + 22^2}}(4\mathbf{i} - 5\mathbf{j} + 22\mathbf{k}) \\ &= \frac{4}{\sqrt{525}}\mathbf{i} - \frac{5}{\sqrt{525}}\mathbf{j} + \frac{22}{\sqrt{525}}\mathbf{k}. \end{aligned}$$

6.2.8 Projection

The **projection** of a vector $\mathbf{b}$ onto a vector $\mathbf{a}$, denoted by $\mathrm{proj}_{\mathbf{a}}\mathbf{b}$ is illustrated below.



From the definition of the scalar product, we have

$$\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}||||\mathbf{b}|| \cos\theta.$$

Therefore the length of the projection of **b** onto **a** is

$$||\mathrm{proj}_{\mathbf{a}}\mathbf{b}|| = ||\mathbf{b}|| \cos\theta = \frac{(\mathbf{a} \cdot \mathbf{b})}{||\mathbf{a}||}.$$

So

$$\mathrm{proj}_{\mathbf{a}}\mathbf{b} = (||\mathrm{proj}_{\mathbf{a}}\mathbf{b}||) \cdot (\text{unit vector along } \mathbf{a})$$
$$= \frac{(\mathbf{a} \cdot \mathbf{b})}{||\mathbf{a}||} \left(\frac{\mathbf{a}}{||\mathbf{a}||} \right) = \frac{(\mathbf{a} \cdot \mathbf{b})}{||\mathbf{a}||^2}\, \mathbf{a}.$$

6.2.9 Example

Find the projection of $\mathbf{a} = 2\mathbf{i} + 5\mathbf{j}$ onto the vector $\mathbf{b} = \mathbf{i} + \mathbf{j}$.

Solution: The length of the projection of $\mathbf{a}$ onto $\mathbf{b}$ is

$$\frac{(\mathbf{a} \cdot \mathbf{b})}{||\mathbf{b}||} = \frac{(2\mathbf{i} + 5\mathbf{j}) \cdot (\mathbf{i} + \mathbf{j})}{\sqrt{1^2 + 1^2}} = \frac{7}{\sqrt{2}}.$$

A unit vector along $\mathbf{b}$ is

$$\frac{\mathbf{i} + \mathbf{j}}{||\mathbf{i} + \mathbf{j}||} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{2}}.$$

Hence the projection of $\mathbf{a}$ onto $\mathbf{b}$ is

$$\frac{7}{\sqrt{2}} \; \frac{\mathbf{i} + \mathbf{j}}{\sqrt{2}} = \frac{7}{2}\mathbf{i} + \frac{7}{2}\mathbf{j}.$$

6.3 Vector Product

If $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$,
then their **vector product or cross product** is

the vector

$$\begin{aligned} \mathbf{v}_1 \times \mathbf{v}_2 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} \\ &= (y_1 z_2 - y_2 z_1)\mathbf{i} - (x_1 z_2 - x_2 z_1)\mathbf{j} + (x_1 y_2 - x_2 y_1)\mathbf{k}. \end{aligned}$$

For example, if $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 3 \\ -1 \\ -3 \end{bmatrix}$,
then their vector product is the vector

$$\mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 2 \\ 3 & -1 & -3 \end{vmatrix} = -\mathbf{i} + 12\mathbf{j} - 5\mathbf{k}.$$

6.3.1 Properties of vector product

Let $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ be vectors in xyz-space, and let c be a real number. Then

(a) $\mathbf{v}_1 \times \mathbf{v}_2 = -\mathbf{v}_2 \times \mathbf{v}_1$.

(b) $\mathbf{v}_1 \times (\mathbf{v}_2 + \mathbf{v}_3) = \mathbf{v}_1 \times \mathbf{v}_2 + \mathbf{v}_1 \times \mathbf{v}_3$.

(c) $(\mathbf{v}_1 + \mathbf{v}_2) \times \mathbf{v}_3 = \mathbf{v}_1 \times \mathbf{v}_3 + \mathbf{v}_2 \times \mathbf{v}_3$.

(d) $c\,(\mathbf{v}_1 \times \mathbf{v}_2) = (c\mathbf{v}_1) \times \mathbf{v}_2 = \mathbf{v}_1 \times (c\mathbf{v}_2)$.

(e) $\mathbf{v}_1 \times \mathbf{v}_1 = \mathbf{O}$.

(f) $\mathbf{O} \times \mathbf{v}_1 = \mathbf{v}_1 \times \mathbf{O} = \mathbf{O}$.

6.3.2 Direction of $\mathbf{v}_1 \times \mathbf{v}_2$

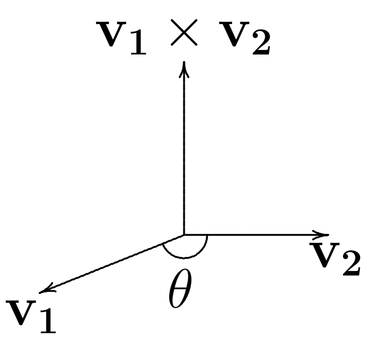
Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be two (non-parallel) vectors which determine a plane Π. i.e. Π is the plane that contains both $\mathbf{v}_1$ and $\mathbf{v}_2$.

Using the definition of vector product, we can check that

$$(\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_1 = 0 \quad \text{and} \quad (\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_2 = 0$$

i.e. $\mathrm{v}_1 \times \mathrm{v}_2$ is perpendicular to $\mathbf{v}_1$ and $\mathbf{v}_2$.

Hence $\mathbf{v}_1 \times \mathbf{v}_2$ is perpendicular to the plane Π.



6.3.3 Magnitude of $\mathbf{v}_1 \times \mathbf{v}_2$

Let θ be the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$.

We have

$$||\mathbf{v}_1 \times \mathbf{v}_2|| = ||\mathbf{v}_1|| \, ||\mathbf{v}_2|| \sin\theta.$$

6.4 Lines in 3D Space

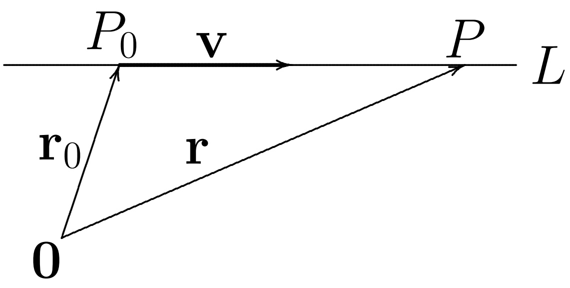
6.4.1 Vector equation of a line

Let L be a line passing through a point P_0 with position vector $\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$ and parallel to a vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. Then any point P on L has position vector

$$\begin{aligned}\overrightarrow{OP} &= \mathbf{r} = \mathbf{r}_0 + t\mathbf{v} \\ &= (x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}) + t(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \qquad (3)\end{aligned}$$

for some $t \in \mathbf{R}$.

(3) is called a **vector equation** of the line L.



6.4.2 Parametric equation of a line

Writing

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

the vector equation (3) becomes

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}) + t(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}).$$

Equating the three components, we get

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$

These are called the **parametric equations** of the line L due to the parameter t in the equations.

6.4.3 Example

The points A and B have position vectors

$$-3\mathbf{i} + 2\mathbf{j} - 3\mathbf{k} \quad \text{and} \quad \mathbf{i} - \mathbf{j} + 4\mathbf{k}$$

respectively. Write down the parametric equations of the line passing through A and B.

Solution: The position vectors of A and B are

$$\overrightarrow{OA} = -3\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}, \quad \overrightarrow{OB} = \mathbf{i} - \mathbf{j} + 4\mathbf{k}$$

respectively. So the line is parallel to the vector

$$\overrightarrow{AB} = (\mathbf{i} - \mathbf{j} + 4\mathbf{k}) - (-3\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) = 4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}.$$

If we use the position vector of A as $\mathbf{r}_0$, the vector equation is given by

$$\mathbf{r} = (-3\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) + t(4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}). \quad (4)$$

Alternatively, if we use the position vector of B as $\mathbf{r}_0$, the vector equation is given by

$$\mathbf{r} = (\mathbf{i} - \mathbf{j} + 4\mathbf{k}) + s(4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}). \quad (5)$$

To get the parametric equations of the line, let

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

and substitute in the LHS of equation (4) or (5).

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (-3 + 4t)\mathbf{i} + (2 - 3t)\mathbf{j} + (-3 + 7t)\mathbf{k}.$$

Hence

$$x = -3 + 4t, \quad y = 2 - 3t, \quad z = -3 + 7t.$$

6.4.4 Example.

Given the following lines whose vector equations are

$L_1 : \ \mathbf{r} = \mathbf{i} + t_1(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}),$

$L_2: \ \mathbf{r} = (2\mathbf{i} + \mathbf{j}) + t_2\left(3\mathbf{i} + \frac{9}{2}\mathbf{j} + \frac{9}{2}\mathbf{k}\right)$ and

$L_3: \ \mathbf{r} = (2\mathbf{i} + \mathbf{j}) + t_3(3\mathbf{i} + \mathbf{j})$.

(a) Find the position vector of the point of intersection of L_1 and L_2.

(b) Show that L_1 and L_3 are skew, i.e. do not intersect each other.

Solution:

(a) Eliminating $\mathbf{r}$ from the vector equations of L_1 and L_2, we get

$$\mathbf{i} + t_1(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = (2\mathbf{i} + \mathbf{j}) + t_2\left(3\mathbf{i} + \frac{9}{2}\mathbf{j} + \frac{9}{2}\mathbf{k}\right).$$

Hence it follows that

$$t_1 = 1 + 3t_2, \quad 2t_1 = 1 + \frac{9}{2}t_2, \quad 3t_1 = \frac{9}{2}t_2$$

from which we obtain

$$t_1 = -1, \quad t_2 = -2/3.$$

Putting $t_1 = -1$ into the vector equation of L_1, we obtain

$$\mathbf{r} = \mathbf{i} + (-1)(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = -2\mathbf{j} - 3\mathbf{k}.$$

So the position vector of the point of intersection P of the two lines:

$$\overrightarrow{OP} = -2\mathbf{j} - 3\mathbf{k}.$$

(b) Eliminating $\mathbf{r}$ from the vector equations of L_1 and L_3, we get

$$\mathbf{i} + t_1(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = (2\mathbf{i} + \mathbf{j}) + t_3(3\mathbf{i} + \mathbf{j}).$$

Hence it follows that

$$t_1 = 1 + 3t_3, \quad 2t_1 = 1 + t_3, \quad 3t_1 = 0$$

Solving the first two equations above gives $t_1 = 2/5$ but the last equation says $t_1 = 0$, thus there is a contradiction. So there is no solution to the equations and we conclude that L_1 and L_3 do not intersect.

6.4.5 Example

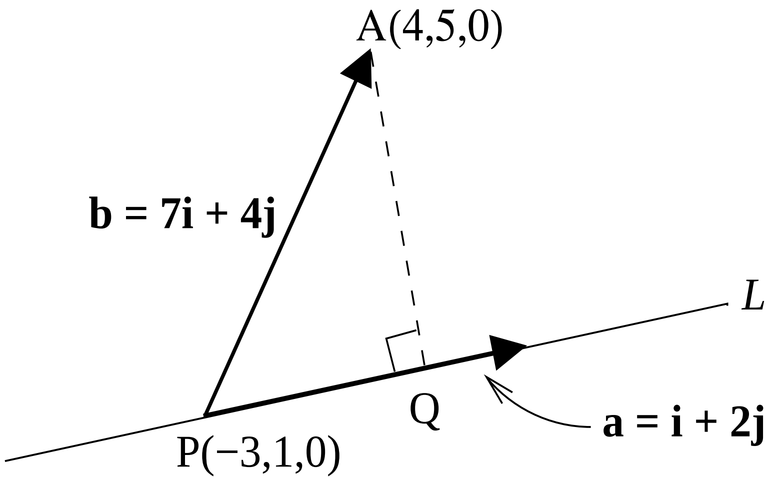
Find the shortest distance from the point A with position vector $4\mathbf{i} + 5\mathbf{j}$ to the line L whose vector equation is

$$\mathbf{r} = (-3\mathbf{i} + \mathbf{j}) + t(\mathbf{i} + 2\mathbf{j}).$$

Solution: L passes through the point $P(-3, 1, 0)$

and is parallel to the vector $\mathbf{a} = \mathbf{i} + 2\mathbf{j}$. Let $\mathbf{b}$ be the vector

$\overrightarrow{PA} = \overrightarrow{OA} - \overrightarrow{OP} = (4\mathbf{i} + 5\mathbf{j}) - (-3\mathbf{i} + \mathbf{j}) = 7\mathbf{i} + 4\mathbf{j}$.



From Section 6.2.8, the length of the projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$$|PQ| = \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}||} = \frac{(\mathbf{i} + 2\mathbf{j}) \cdot (7\mathbf{i} + 4\mathbf{j})}{\sqrt{1^2 + 2^2}} = \frac{15}{\sqrt{5}}.$$

Now the shortest distance from A to L is given by $|AQ|$.

Applying Pythagoras theorem on the right triangle

APQ,

$$|AP|^2 = |PQ|^2 + |AQ|^2,$$

we get

$$\begin{aligned} |AQ| &= \sqrt{||\mathbf{b}||^2 - \left(\frac{15}{\sqrt{5}}\right)^2} \\ &= \sqrt{7^2 + 4^2 - \frac{15^2}{5}} \\ &= 2\sqrt{5}. \end{aligned}$$

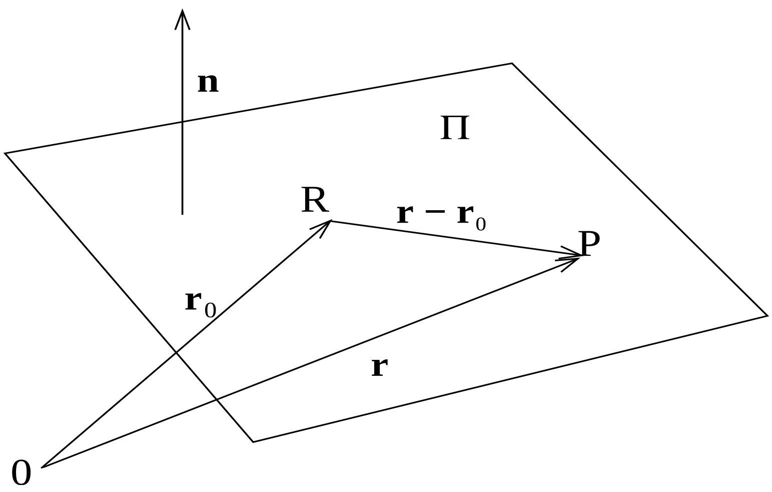
6.5 Planes in 3D Space

Suppose we wish to find the vector equation of a plane Π passing through a given point R with position vector $\mathbf{r_0}$ relative to the origin O and such that Π has $\mathbf{n}$ as a normal vector to it. Let P be a general point in the plane with position vector $\mathbf{r}$. Then

$\overrightarrow{RP} = \mathbf{r} - \mathbf{r_0}$ is a vector lying in the plane, and perpendicular to the normal vector $\mathbf{n}$.

Hence

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0.$$



6.5.1 Cartesian Equation of a plane

Let us write

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

$$\mathbf{r_0} = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k},$$

and

$$\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k},$$

so that

$$\mathbf{r} - \mathbf{r_0} = (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}$$

and

$$(\mathbf{r} - \mathbf{r_0}) \cdot \mathbf{n} = a(x - x_0) + b(y - y_0) + c(z - z_0).$$

Therefore, the vector equation of the plane can be written in the form

$$ax + by + cz = d, \text{ where } d = ax_0 + by_0 + cz_0.$$

The Cartesian equation of a plane passing through a point (x_0, y_0, z_0) and with normal vector $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is

$$ax + by + cz = ax_0 + by_0 + cz_0.$$

6.5.2 Example

Find the equation of the plane passing through the point $(0, 2, -1)$ normal to the vector $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Solution: The required equation is

$$3x + 2y - z = 3(0) + 2(2) - (-1), \quad \text{or}$$

$$3x + 2y - z = 5.$$

6.5.3 Example

Find the vector equation of the plane passing through the points $A(0, 0, 1)$, $B(2, 0, 0)$ and $C(0, 3, 0)$.

Solution: The following vector is perpendicular to the plane:

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}.$$

The plane passes through $(0,0,1)$. So an equation of the plane is

$$3x + 2y + 6z = 3(0) + 2(0) + 6(1),$$

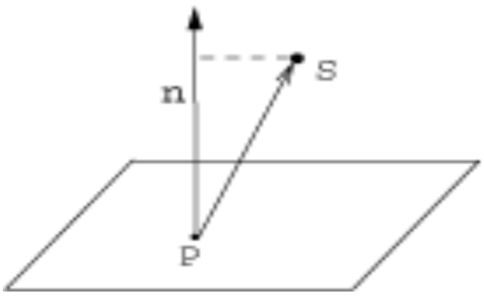
or

$$3x + 2y + 6z = 6.$$

The plane also passes through $(2,0,0)$, so we will get the same equation

$$3x + 2y + 6z = 3(2) + 2(0) + 6(0) = 6.$$

6.5.4 Distance from a point to a plane



The shortest distance from a point $S\ (x_0,\ y_0,\ z_0)$ to a plane $\Pi \quad : \quad ax + by + cz = d$, is given by

$$\frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}} \qquad (6)$$

Indeed for any point $P(x_1,\ y_1,\ z_1)$ on the plane Π, the length of the projection of $\overrightarrow{PS}$ onto a normal vector $\mathbf{n}$ of the plane gives the distance from S to Π.

Since $P(x_1, y_1, z_1)$ lies on Π, so $ax_1 + by_1 + cz_1 = d$.

Now

$$\overrightarrow{PS} = \overrightarrow{OS} - \overrightarrow{OP} = (x_0 - x_1)\mathbf{i} + (y_0 - y_1)\mathbf{j} + (z_0 - z_1)\mathbf{k}.$$

A normal vector to the plane Π is $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

Hence the distance from $S(x_0, y_0, z_0)$ to Π is

$$\begin{aligned}
& \left|\left|\mathrm{Proj}_{\mathbf{n}} \overrightarrow{PS}\right|\right| \\
=& \frac{|\overrightarrow{PS} \cdot \mathbf{n}|}{||\mathbf{n}||} \qquad \text{(Section 6.2.8)} \\
=& \frac{|[(x_0 - x_1)\mathbf{i} + (y_0 - y_1)\mathbf{j} + (z_0 - z_1)\mathbf{k}] \cdot (a\mathbf{i} + b\mathbf{j} + c\mathbf{k})|}{\sqrt{a^2 + b^2 + c^2}} \\
=& \frac{|a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)|}{\sqrt{a^2 + b^2 + c^2}} \\
=& \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}
\end{aligned}$$

since $ax_1 + by_1 + cz_1 = d$.

6.5.5 Example

Find the distance of the point $(2, -3, 4)$ to the plane

$x + 2y + 3z = 13$.

Solution: Using (6), we have $(x_0,\ y_0,\ z_0) = (2,-3,4)$ and $a=1,\ b=2,\ c=3.$

So the distance is

$$\frac{|1(2)+2(-3)+3(4)-13|}{\sqrt{1^2+2^2+3^2}} = \frac{5}{\sqrt{14}}$$

6.6 Vector Functions of One Variable

Let $f(t)$, $g(t)$ and $h(t)$ be real-valued functions of a real variable t. A **vector function**

$$\mathbf{r}(t) = \begin{bmatrix} f(t) \\ g(t) \\ h(t) \end{bmatrix} = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

is a function such that the images (output) are *vectors* (instead of scalars). The three functions $f(t)$, $g(t)$ and $h(t)$ are called the **component functions** of $\mathbf{r}(t)$.

6.6.1 Example

Consider the vector function

$$\mathbf{r}(t) = t\mathbf{i} + (t^2+1)\mathbf{j} + (2-7t)\mathbf{k}.$$

Then

$$\mathbf{r}(2) = 2\mathbf{i} + 5\mathbf{j} - 12\mathbf{k}.$$

6.6.2 Limits and continuity

We define the **limit** of $\mathbf{r}(t)$ as follows:

$$\lim_{t\to a} \mathbf{r}(t) = \left(\lim_{t\to a} f(t)\right)\mathbf{i} + \left(\lim_{t\to a} g(t)\right)\mathbf{j} + \left(\lim_{t\to a} h(t)\right)\mathbf{k}$$

provided the limit of each component function exists.

We say that $\mathbf{r}(t)$ is **continuous** at a point $t = a$ if

$$\lim_{t\to a}\mathbf{r}(t) = \mathbf{r}(a) = f(a)\mathbf{i} + g(a)\mathbf{j} + h(a)\mathbf{k}.$$

Equivalently, a vector function $\mathbf{r}(t)$ is continuous at a point a exactly when each of the component functions of $\mathbf{r}(t)$ is continuous at a, i.e. $f(t)$, $g(t)$ and $h(t)$ are continuous at a.

6.6.3 Example

Given vector function

$$\mathbf{r}(t) = t\mathbf{i} + (t^2 + 1)\mathbf{j} + (2 - 7t)\mathbf{k}.$$

We have

$$\lim_{t\to a} \mathbf{r}(t) = \left(\lim_{t\to a} t\right)\mathbf{i} + \left(\lim_{t\to a}(t^2 + 1)\right)\mathbf{j} + \left(\lim_{t\to a}(2 - 7t)\right)\mathbf{k}$$
$$= a\mathbf{i} + (a^2 + 1)\mathbf{j} + (2 - 7a)\mathbf{k} = \mathbf{r}(a)$$

for all real numbers a. Hence $\mathbf{r}(t)$ is continuous at every $t = a$.

6.6.4 Derivatives of vector functions

The **derivative** of a vector function $\mathbf{r}(t)$ is

$$\frac{d\mathbf{r}}{dt} = (\mathbf{r})'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} \quad (7)$$

provided the limit exists.

If

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k},$$

where f, g and h are differentiable functions, then the derivative is

$$(\mathbf{r})'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}. \quad (8)$$

6.6.5 Example

Consider the vector function

$$\mathbf{r}(t) = t\mathbf{i} + (t^2+1)\mathbf{j} + (2-7t)\mathbf{k}.$$

Then by (8), since

$$\frac{d}{dt}(t) = 1, \quad \frac{d}{dt}(t^2+1) = 2t, \quad \frac{d}{dt}(2-7t) = -7,$$

we have

$$\mathbf{r}'(t) = \mathbf{i} + (2t)\mathbf{j} - 7\mathbf{k}.$$

6.6.6 Definite integral of a vector function

The definite integral of a continuous vector function

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

on the interval $[a, b]$ is

$$\int_a^b \mathbf{r}(t)\,dt = \int_a^b f(t)\,dt\,\mathbf{i} + \int_a^b g(t)\,dt\,\mathbf{j} + \int_a^b h(t)\,dt\,\mathbf{k}.$$

For example,

$$\int_0^2 (2t\mathbf{i} + 3t^2\mathbf{j})dt = \left[t^2\right]_{t=0}^{t=2}\mathbf{i} + \left[t^3\right]_{t=0}^{t=2}\mathbf{j} = 4\mathbf{i} + 8\mathbf{j}.$$

6.7 Space curves

A curve in xyz-space can be represented by some continuous function

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

such that a point P lies on the curve if its position vector $\overrightarrow{OP}$ is the image of the vector function, i.e.,

$$\overrightarrow{OP} = \mathbf{r}(t_0) \text{ for some } t_0 \in \mathbf{R}.$$

We call

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

the **vector equation** of the curve and

$$x = f(t), \quad y = g(t), \quad z = h(t)$$

the **parametric equation** of the curve.

6.7.1 Example

The vector equation

$$\mathbf{r}(t) = (1+t)\mathbf{i} + (2+t)\mathbf{j} + (3+t)\mathbf{k}$$

represents the straight line in the xyz-space that passes through the point $(1,2,3)$ and is parallel to the vector $\mathbf{i}+\mathbf{j}+\mathbf{k}$.

6.7.2 Smooth curves

A vector function $\mathbf{r}(t)$ represents a **smooth curve** on an interval I if $\mathbf{r}'(t)$ is continuous and $\mathbf{r}'(t)$ is never zero, except perhaps at the endpoints of I. Geometrically, a smooth curve is one that does not have any sharp corner. A **piecewise smooth curve** is made up of a finite number of smooth pieces.

6.7.3 Example

Consider the vector function

$$\mathbf{r}(t) = t\mathbf{i} + (t^2 + 1)\mathbf{j} + (2 - 7t)\mathbf{k}.$$

We have

$$\mathbf{r}'(t) = \mathbf{i} + (2t)\mathbf{j} - 7\mathbf{k} \neq \mathbf{0}$$

for all t.

So $\mathbf{r}(t)$ represents a smooth curve.

6.7.4 Example

The following vector function represents a piecewise smooth curve:

$$\mathbf{r}(t) = \begin{cases} t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k} & \text{if } 0 \leq t \leq 1 \\ (2t-1)\mathbf{i} + t^2\mathbf{j} + (t^2 + t - 1)\mathbf{k} & \text{if } 1 < t \leq 2. \end{cases}$$

6.7.5 Tangent vector and tangent line to a curve

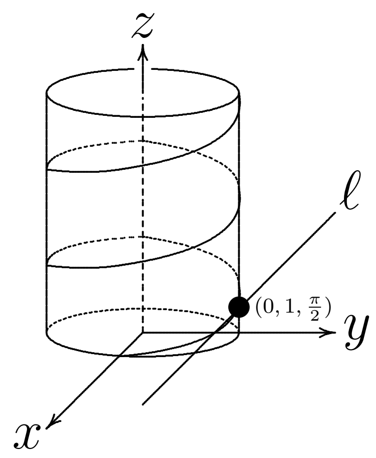
The **tangent line** to a curve $\mathbf{r}(t)$ at a point P whose position vector is $\mathbf{r}(t_0)$ is defined to be the line through P parallel to the tangent vector $\mathbf{r}'(t_0)$ (here it is assumed that $\mathbf{r}'(t_0) \neq \mathbf{0}$). The **unit tangent vector** to the curve at $t = t_0$ is

$$\frac{\mathbf{r}'(t_0)}{||\mathbf{r}'(t_0)||}.$$

6.7.6 Example

Consider the circular helix

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}.$$



$$\mathbf{r}(\frac{\pi}{2}) = (\cos\frac{\pi}{2})\mathbf{i} + (\sin\frac{\pi}{2})\mathbf{j} + \frac{\pi}{2}\mathbf{k} = 0\mathbf{i} + 1\mathbf{j} + \frac{\pi}{2}\mathbf{k} = \mathbf{j} + \frac{\pi}{2}\mathbf{k}.$$

Therefore the point $(0, 1, \frac{\pi}{2})$ (corresponding to $t = \frac{\pi}{2}$) lies on the helix.

Now we have

$$\mathbf{r}'(t) = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k} \neq \mathbf{0} \quad \text{for all } t \in \mathbf{R}.$$

Thus

$$\mathbf{r}'(\frac{\pi}{2}) = (-1)\mathbf{i} + (0)\mathbf{j} + (1)\mathbf{k} = -\mathbf{i} + \mathbf{k}$$

is the tangent vector to the circular helix at $(0, 1, \frac{\pi}{2})$, the point on the helix corresponding to $t = \frac{\pi}{2}$. The unit tangent vector to the curve at $(0, 1, \frac{\pi}{2})$ is

$$\frac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{k}).$$

The tangent line ℓ to the helix at $(0, 1, \frac{\pi}{2})$ is parallel to

$$\mathbf{r}'(\frac{\pi}{2}) = (-1)\mathbf{i} + (0)\mathbf{j} + (1)\mathbf{k}.$$

Therefore the parametric equations of the tangent line at the point $(0, 1, \frac{\pi}{2})$, are

$$x = -t, \quad y = 1, \quad z = \frac{\pi}{2} + t.$$

6.7.7 Arc length of a space curve

Suppose that a curve has the vector equation

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k},$$

or alternatively, parametric equations

$$x = f(t), \quad y = g(t), \quad z = h(t),$$

where $f'(t)$, $g'(t)$ and $h'(t)$ are continuous functions.

If this **curve is traversed exactly once** as t increases from a to b, then its arc length is

$$\begin{aligned} L &= \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\ \ dt \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ \ dt. \end{aligned}$$

A more compact formula of both arc length formulas is

$$L = \int_a^b ||\mathbf{r}'(t)|| \ dt.$$

6.7.8 Example

Recall the circular helix of Example 6.7.6:

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k},$$

$$\mathbf{r}'(t) = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k}.$$

Hence we can find the arc length from $t = 0$ to $t = 2\pi$ as follows:

$$||\mathbf{r}'(t)|| = \sqrt{(-\sin \ t)^2 + (\cos \ t)^2 + 1^2} = \sqrt{2},$$

$$L = \int_0^{2\pi} ||\mathbf{r}'(t)|| \ dt = \int_0^{2\pi} \sqrt{2} \ dt = 2\sqrt{2}\pi.$$