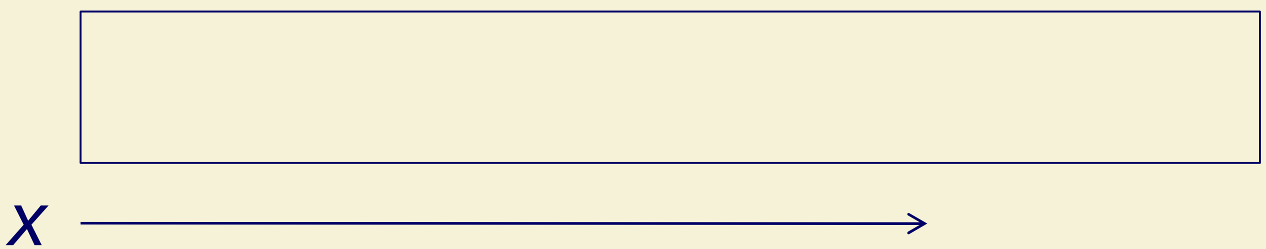
MA1506
Mathematics II

Partial Differential Equations

8.1 PDE

Math models of physical phenomena often involve d.e. with more than one variable

Eg: heat conduction in a pipe



$H(x,t)$: heat is function of position and time

8.1 PDE definition

A partial differential equation is an equation containing a function *u(x,y, …)* of two or more independent variables, *x, y, …* and its partial derivatives.

Examples

$$u_{xy} - 2x + y = 0 \qquad u(x,y)$$

$$w_{xy} + x(w_z)^2 = yz \qquad w(x,y,z)$$

8.1 PDE solutions

- A solution of a p.d.e. is any function that satisfies the p.d.e.
- Usually one or more family of solutions, called the general solution
- A particular solution is a specific function from that family.

$$u(x, y) = x^2 y - \frac{1}{2} x y^2 + F(x) + G(y)$$

is a general solution of $u_{xy} - 2x + y = 0$

8.1 PDE solutions

$$u(x,y) = x^2 y - \frac{1}{2} x y^2 + F(x) + G(y)$$

General solution

$$u_x = 2xy - \frac{1}{2} y^2 + F'(x)$$

$$u_{xy} = 2x - y$$

Set some functions *F(x)* and *G(y)*

Particular solution

$$u(x,y) = x^2 y - \frac{1}{2} x y^2 + 3\sin x + 4y^5 - 6$$

8.1 PDE solutions

$$u_{xy} = 2x - y$$

Plus boundary conditions:

$$u(x,0) = x^3, u(0,y) = \sin(3y)$$

$$u(x,y) = x^2 y - \frac{1}{2} x y^2 + F(x) + G(y)$$

$$u(x,0) = x^3 = F(x) + G(0)$$

$$u(0,y) = \sin(3y) = F(0) + G(y)$$

$$u(x,y) = x^2 y - \frac{1}{2} x y^2 + x^3 + \sin(3y)$$

Particular solution

8.1 PDE solutions

In general there are many solutions of pde

Laplace Equation: $u_{xx} + u_{yy} = 0$

Has solutions:

$$u(x, y) = x^2 - y^2$$

$$u(x, y) = e^x \cos(y)$$

$$u(x, y) = \ln(x^2 + y^2)$$

8.1 Order of PDE

The order of the pde is the higest derivative present

Examples of order 2:

$$u_{xy} - 2x + y = 0$$

$$w_{xy} + x(w_z)^2 = yz$$

8.1 Linearity and Homgeneity

An order 1 linear pde (in two variables) has the form:

$$Au_x + Bu_y + Cu = Z$$

An order 2 linear pde (in two variables) has the form:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = Z$$

- where *A, B, C, D, E, F, Z* are constants or functions of x, y but not u
- Homogeneous if *Z = 0*

8.1 Linearity and Homgeneity

p.d.e.	**order**	**linear**	**homogeneous**
$4u_{xx} - u_t = 0$	2	yes	yes
$x^2 R_{yyy} = y^3 R_{xx}$	3	yes	yes
$tu_{tx} + 2u_x = x^2$	2	yes	no
$4u_{xx} - uu_t = 0$	2	no	n.a.
$(u_x)^2 + (u_y)^2 = 2$	1	no	n.a.

8.1 Superposition

If u_1 and u_2 are any solutions of a linear homogeneous pde, then

$$u = c_1 u_1 + c_2 u_2$$

with c_1 and c_2 are constants is also a solution

8.1 PDE solutions

In general there are many solutions of pde

Laplace Equation: $u_{xx} + u_{yy} = 0$

Has solutions: $u(x,y) = x^2 - y^2$

$$u(x,y) = e^x \cos(y)$$

$$u(x,y) = \ln(x^2 + y^2)$$

By superposition, this is also a solution

$$u(x,y) = 3(x^2 - y^2) - 7e^x \cos(y) + 10\ln(x^2 + y^2)$$

8.2 Solving PDE

We will learn two main techniques

1. *Reducing PDE to ODE*
2. *Separation of Variables*

8.2 Reducing PDE to ODE

Example: Absence of one partial derivative

$$\boxed{u_{xx} - u = 0}$$

Treat "*y*" as constant: $u''(x) - u(x) = 0$

Aux. eq: $\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$

$$u(x) = Ae^{x} + Be^{-x}$$

A, *B* are "constants wrt *x*", could be functions of *x*

$$u(x, y) = A(y)e^{x} + B(y)e^{-x}.$$

8.2 Reducing PDE to ODE

Example: Common "inner" derivative

$$\boxed{u_{xy} = -u_x}$$

Treat : $u_x = p \Rightarrow p_y = -p$ Separable ode

$$\frac{dp}{dy} = -p \Rightarrow \int \frac{dp}{p} = \int -1dy \Rightarrow \ln|p| = -y + c$$

So $p = Ke^{-y} \Rightarrow u(x,y) = \int K(x)e^{-y}dx$

"constant wrt y"

$$= e^{-y}\int K(x)dx + g(y)$$

8.2 Separation of Variables (PDE)

Observe that is the solution in two variables can be separated, then

$$\boxed{u(x,y) = X(x)Y(y)}$$

$$u_x(x,y) = X'(x)Y(y) \qquad u_y(x,y) = X(x)Y'(y)$$

$$u_{xy}(x,y) = X'(x)Y'(y)$$

$$u_{xx}(x,y) = X''(x)Y(y) \qquad u_{yy}(x,y) = X(x)Y''(y)$$

8.2 Separation of Variables (PDE)

Example: $\boxed{u_x = f(x)g(y)u_y}$

Assuming $u(x, y) = X(x)Y(y)$

$$X'(x)Y(y) = f(x)g(y)X(x)Y'(y)$$

$$\frac{1}{f(x)} \cdot \frac{X'(x)}{X(x)} = g(y)\frac{Y'(y)}{Y(y)}$$

LHS is function of *x* , RHS is function of *y*,
Hence LHS=RHS = constant

8.2 Separation of Variables (PDE)

Two ode:

$$\frac{1}{f(x)} \cdot \frac{X'(x)}{X(x)} = k \Rightarrow X'(x) = kf(x)X(x)$$

$$g(y)\frac{Y'(y)}{Y(y)} = k \Rightarrow Y'(y) = \frac{k}{g(y)}Y(y)$$

Both ode can be solved as separable equations in ode.

8.2 Separation of Variables (PDE)

Example: $$\boxed{u_x + xu_y = 0}$$

Assuming $u(x, y) = X(x)Y(y)$

$$X'(x)Y(y) + xX(x)Y'(y) = 0$$

$$\frac{1}{x} \cdot \frac{X'(x)}{X(x)} = -\frac{Y'(y)}{Y(y)} = k$$

8.2 Separation of Variables (PDE)

Two ode:

$$\int \frac{X'(x)}{X(x)} = \int kx \Rightarrow \ln(X(x)) = \frac{k}{2}x^2 + c$$

$$\Rightarrow X(x) = Ae^{\frac{k}{2}x^2}$$

$$\int \frac{Y'(y)}{Y(y)} = \int -k \Rightarrow Y(y) = Be^{-ky}$$

Finally $u(x,y) = X(x)Y(y) = Ce^{\frac{k}{2}x^2 - ky}$