NATIONAL UNIVERSITY OF SINGAPORE
DEPARTMENT OF MATHEMATICS
MA2214 COMBINATORIAL ANALYSIS

TUTORIAL 2: SUGGESTED SOLUTIONS **SEMESTER II, AY 2010/2011**

1. (i) Define $A = \{\{a,b\} | a,b \in [50]\}$ and $B = \{(x,y) | x,y \in [50], x < y\}$. Note that $\{a,b\}$ is unordered but (x,y) is ordered. Define the following bijection $f : A \to B$ by

$$f(\{a,b\}) = \begin{cases} (a,b) & \text{if } a < b, \\ (b,a) & \text{if } a > b. \end{cases}$$

- f is well-defined: Need to show that if $a_1 = a_2 \in A$, then $f(a_1) = f(a_2)$.

  Now in A, we have $\{a,b\} = \{b,a\}$. If $a < b$, then f is well-defined since $f(\{a,b\}) = f(\{b,a\}) = (a,b)$. Otherwise $a > b$ because a and b are distinct, then $f(\{a,b\}) = f(\{b,a\}) = (b,a)$ and f is also well-defined.
- f is onto: Need to show that if $b \in B$, there exists $a \in A$ such that $f(a) = b$.

  If $(x,y) \in B$, then $1 \leq x < y \leq 50$ and so $(x,y) = f(\{x,y\})$ for $\{x,y\} \in A$. So f is onto.
- f is 1-to-1: Need to show that if $f(a_1) = f(a_2)$, then $a_1 = a_2$.

  If $(x,y) = (x_2,y_2)$, then $x = x_2$ and $y = y_2$. Since $(x,y) = f(\{x,y\})$ and $(x_2,y_2) = f(\{x_2,y_2\})$, we have $\{x,y\} = \{x_2,y_2\}$. So f is 1-to-1.

Hence f is a bijection and $|A| = |B|$. The question actually asks for a subset of A that satisfies $|a-b| = 5$. This means either $a - b = 5$ or $a - b = -5$. In either case, f will map $\{a,b\}$ to $(x, x+5)$ where $x = \min\{a,b\}$.

In conclusion, the question is asking for the size of the following set

$$B^* = \{(x,x+5) | x, x+5 \in [50]\} = \{(x,x+5) | 1 \leq x \leq 45\}.$$

So $|B^*| = 45$.

(ii) Using the same idea we see that the set we are enumerating is

$$\begin{aligned} B^{**} &= \{(x,x+k) | x,x+k \in [50], 1 \leq k \leq 5\} \\ &= \bigcup_{k=1}^{5} \{(x,x+k) | 1 \leq x \leq 50-k\} \\ \Longrightarrow |B^{**}| &= \sum_{k=1}^{5} \left| \{(x,x+k) | 1 \leq x \leq 50-k\} \right| \qquad \text{(addition principle)} \\ &= 45+46+47+48+49 = 235. \end{aligned}$$

2.

3.

4.

5.

6. Part (iii). (Using part (ii))

$$\sum_{k=r}^{n}\binom{n}{k}\binom{k}{r} = \sum_{k=r}^{n}\binom{n}{r}\binom{n-r}{k-r}$$
$$= \binom{n}{r}\sum_{j=0}^{n-r}\binom{n-r}{j} \qquad (\text{set } k-r=j \implies k=j+r)$$
$$= \binom{n}{r}2^{n-r} \qquad \text{by the binomial theorem.}$$

Combinatorial interpretation:
We want to choose k committee members from n people and from the k committee members choose r to be in the executive committee, with the restriction that n and r are fixed but there can be any number of committee members. Since executive committee members must also be committee members, k ranges from r to n. By addition principle, this equals to the LHS.

For the RHS, we first choose r executive committee members from the n persons. It remains to choose any number of ordinary committee members from the remaining $n-r$ people. Each of them can either be an ordinary committee member or not a committee member, hence the number of ways is 2^{n-r}. By the product principle, the required number is $\binom{n}{r}2^{n-r}$.

Part (v). Combinatorial interpretation, see the textbook.