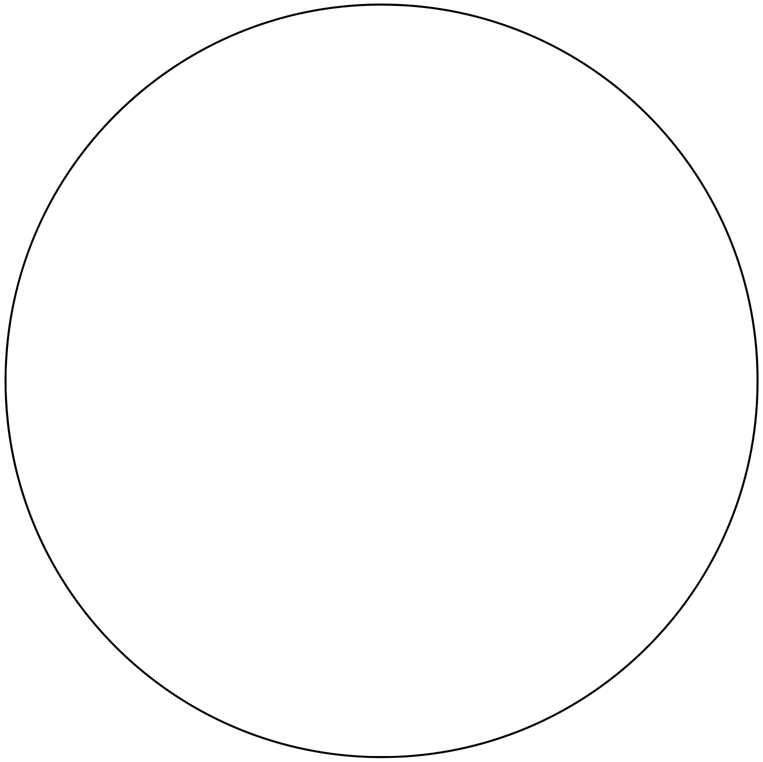
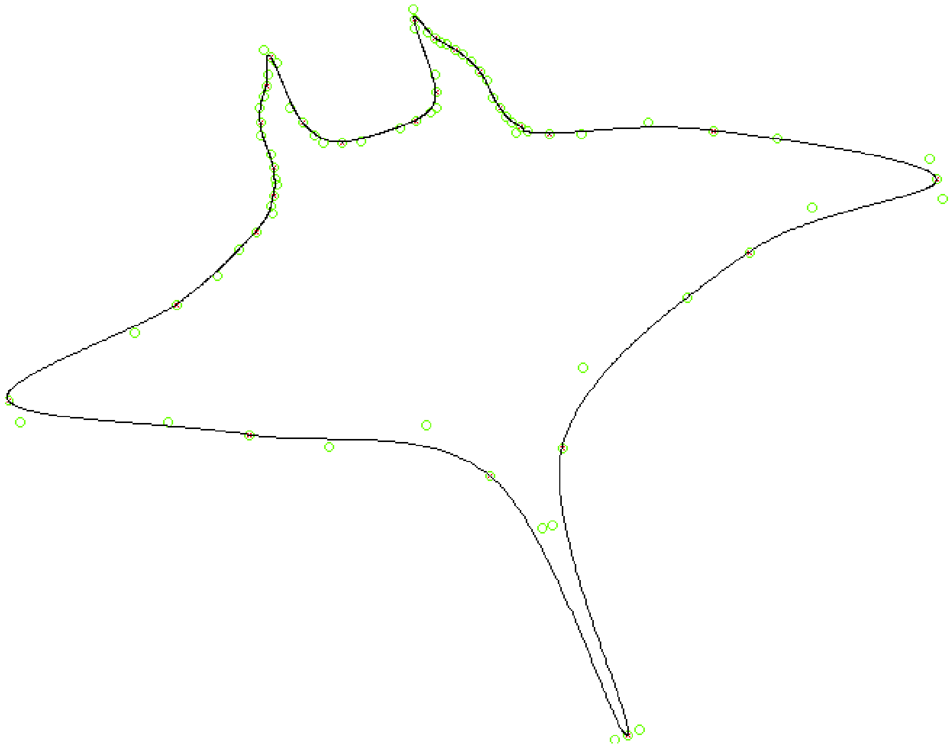
Parametric Curves and Surfaces

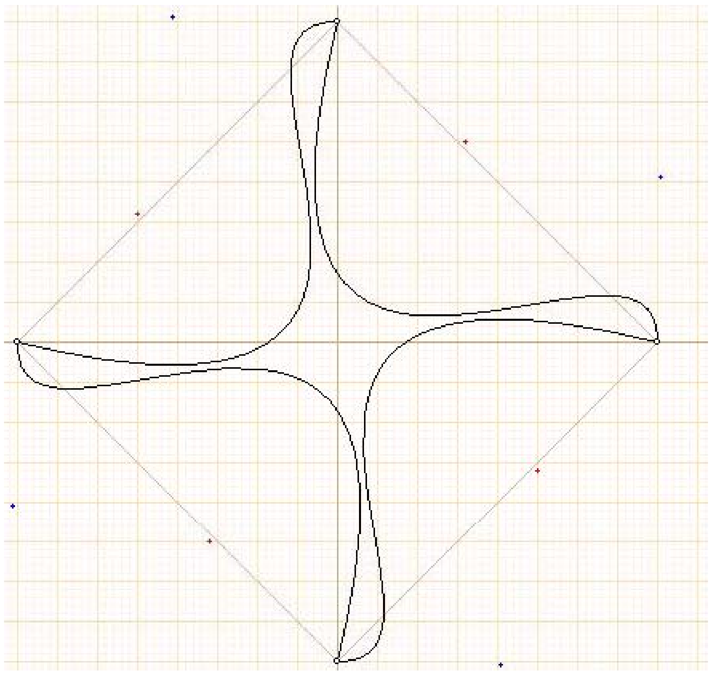
How Do You Specify This? (1D)

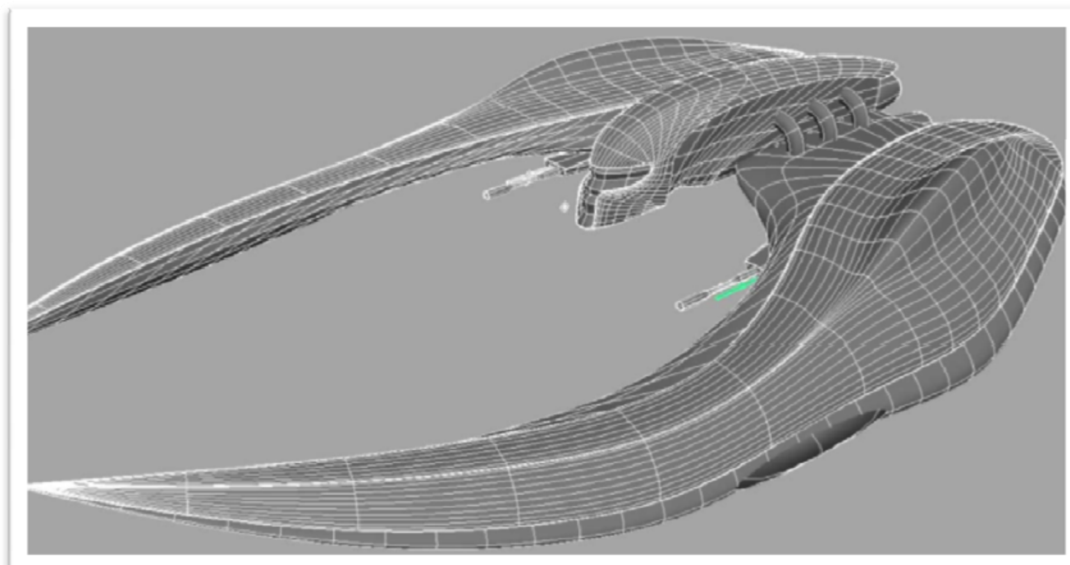
- There are many ways
  - Namely, different representations

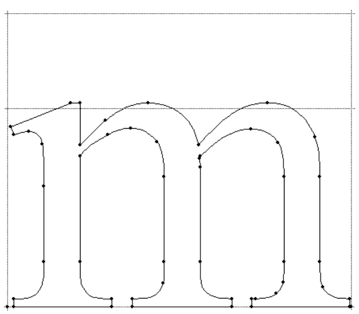


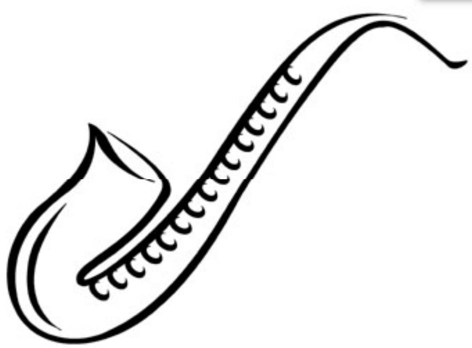


Curves and Surfaces



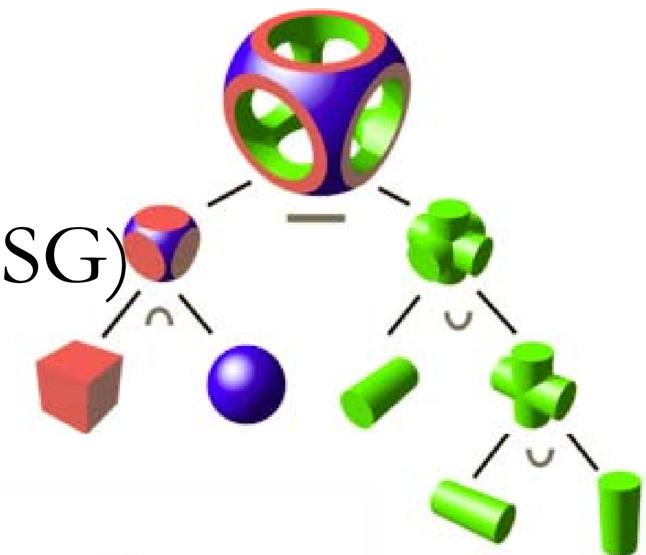


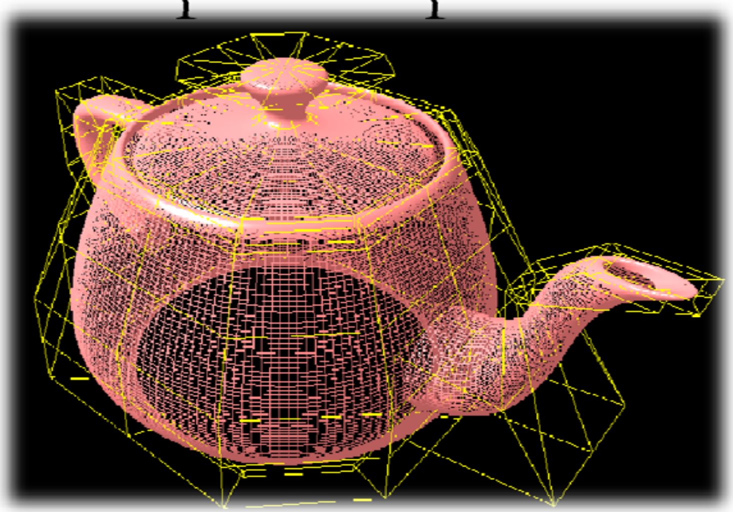


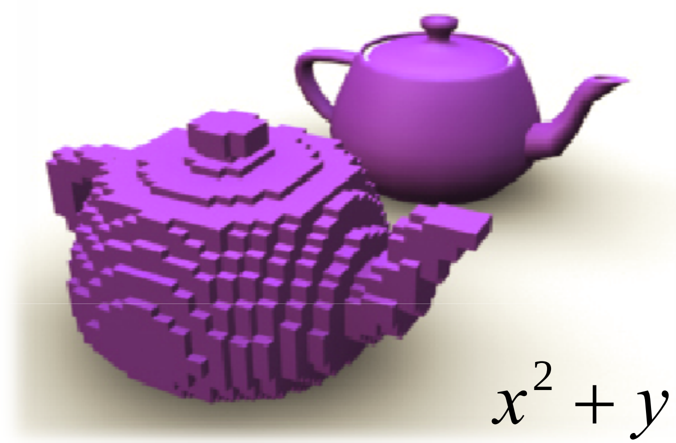


Various Object **<u>Representations</u>**

- Polygonal meshes
- Parametric patches
- Constructive solid geometry (CSG)
- Spatial subdivision techniques
- Implicit representation







$$x^2 + y^2 + z^2 = 1$$

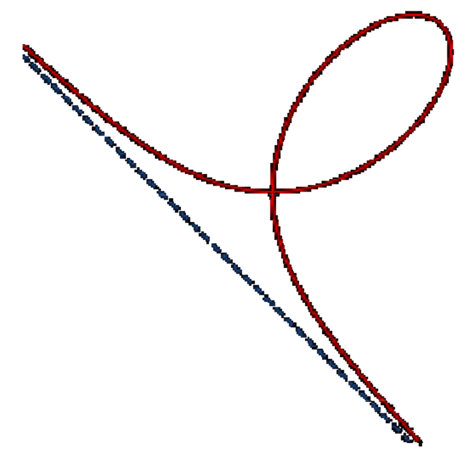
How to define a curve?

- What we learned in school:

$$x^2 + y^2 = 1$$

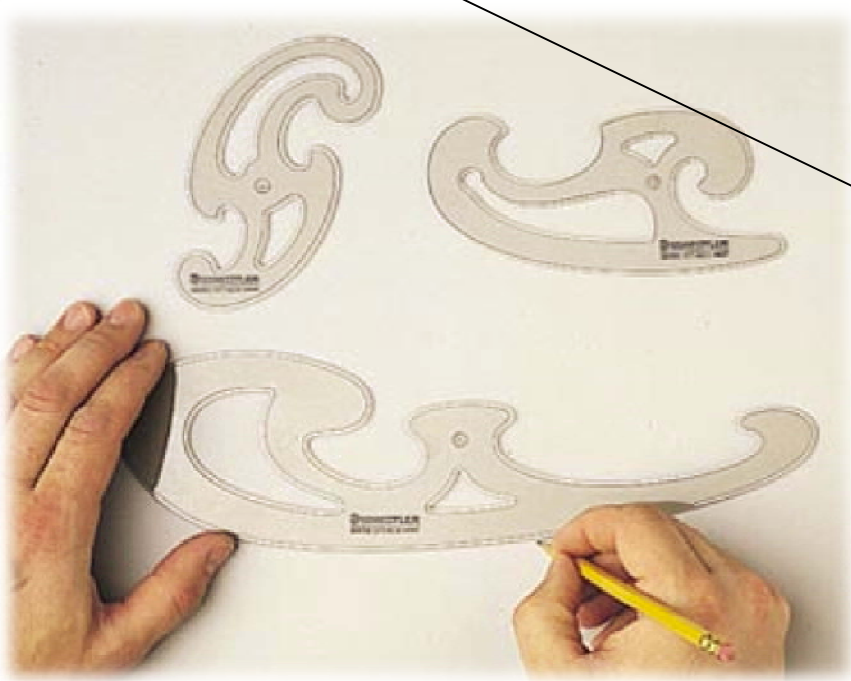
- Folium of Descartes

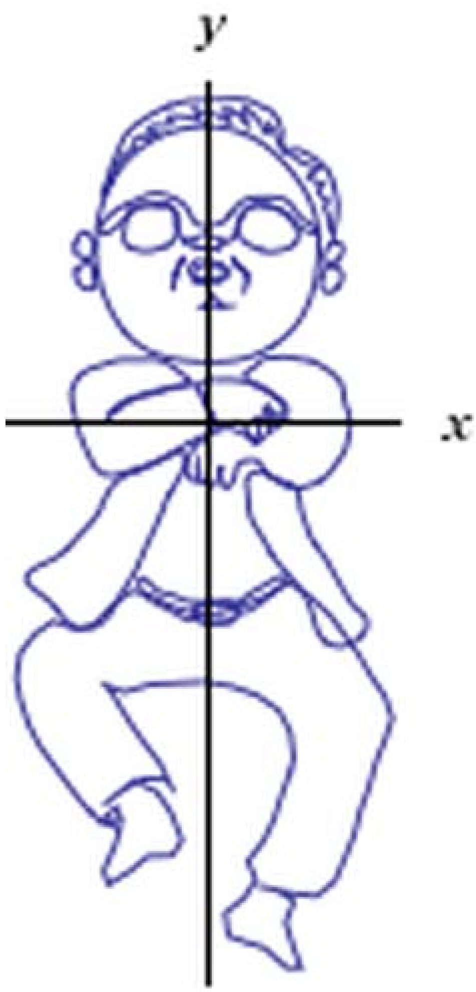
$$x^3 + y^3 + 3axy = 0$$



Difficulty for “Normal” People

- Let’s ask a fashion designer
  - “What is the equation for this curve?”



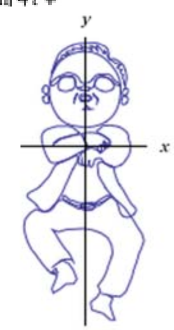


(plotted for t from 0 to 72π)

Equations:

Parametric equations:

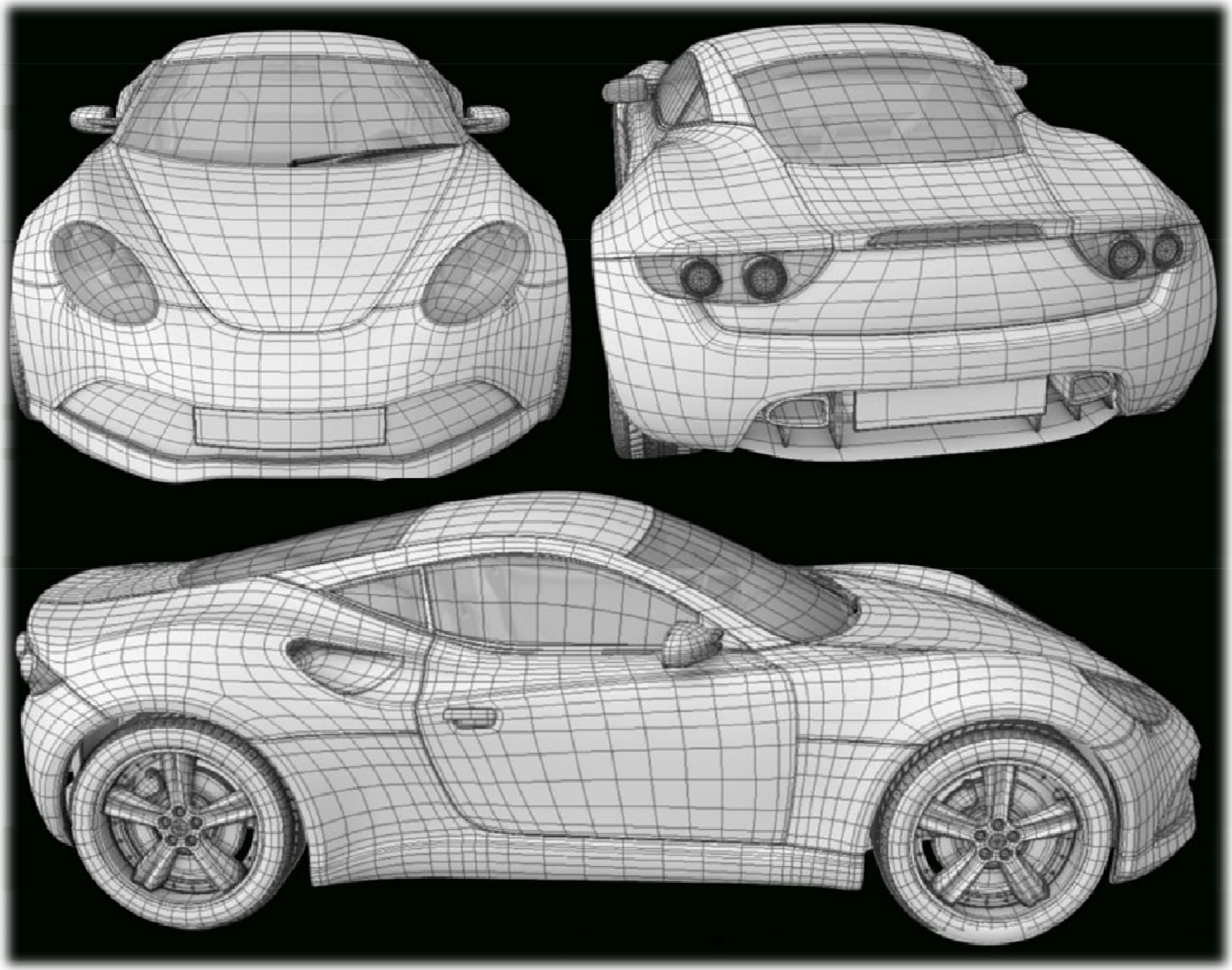
$$x(t) = \Big(\Big(\Big(-\frac{25}{37}\sin\Big(\frac{58}{37} - 4t\Big) + \frac{809}{54}\sin\Big(t + \frac{11}{7}\Big) + \frac{9}{19}\sin\Big(2t + \frac{108}{23}\Big) + \frac{17}{23}\sin\Big(3t + \frac{49}{31}\Big) +$$
$$\frac{11}{12}\sin\Big(5t + \frac{107}{68}\Big) - \frac{17}{27}\Big)\theta(71\pi - t)\,\theta(t - 67\pi) + \Big(-\frac{6}{49}\sin\Big(\frac{39}{25} - 8t\Big) - \frac{3}{32}\sin\Big(\frac{39}{25} - 7$$
$$t\Big) - \frac{2}{11}\sin\Big(\frac{64}{41} - 6t\Big) - \frac{107}{68}\sin\Big(\frac{36}{23} - 2t\Big) - \frac{228}{31}\sin\Big(\frac{69}{44} - t\Big) + \frac{17}{36}\sin\Big(3t + \frac{11}{7}\Big) + \frac{11}{49}$$
$$\sin\Big(4t + \frac{113}{24}\Big) + \frac{1}{63}\sin\Big(5t + \frac{29}{19}\Big) + \frac{1091}{31}\Big)\theta(67\pi - t)\,\theta(t - 63\pi) + \Big(\frac{35}{8}\sin\Big(t + \frac{11}{7}\Big) +$$
$$\frac{15}{23}\sin\Big(2t + \frac{52}{33}\Big) + \frac{11}{30}\sin\Big(3t + \frac{11}{7}\Big) + \frac{1}{18}\sin\Big(4t + \frac{63}{40}\Big) - \frac{556}{17}\Big)\theta(63\pi - t)\,\theta(t - 59\pi) +$$
$$\Big(-\frac{104}{31}\sin\Big(\frac{27}{41} - 2t\Big) + \frac{523}{45}\sin\Big(t + \frac{29}{25}\Big) + \frac{40}{37}\sin\Big(3t + \frac{25}{41}\Big) + \frac{21}{17}\sin\Big(4t + \frac{121}{26}\Big) + \frac{148}{27}\Big)$$
$$\theta(59\pi - t)\,\theta(t - 55\pi) + \Big(-\frac{9}{20}\sin\Big(\frac{13}{11} - 18t\Big) - \frac{9}{26}\sin\Big(\frac{88}{63} - 17t\Big) - \frac{15}{41}\sin\Big(\frac{81}{161} - 15t\Big) -$$
$$\frac{7}{15}\sin\Big(\frac{25}{18} - 14t\Big) - \frac{59}{89}\sin\Big(\frac{13}{34} - 12t\Big) - \frac{87}{38}\sin\Big(\frac{9}{10} - 6t\Big) - \frac{117}{22}\sin\Big(\frac{23}{42} - 5t\Big) + \frac{582}{13}$$
$$\sin\Big(t + \frac{64}{37}\Big) + \frac{88}{25}\sin\Big(2t + \frac{229}{49}\Big) + \frac{1458}{47}\sin\Big(3t + \frac{48}{23}\Big) + \frac{167}{50}\sin\Big(4t + \frac{151}{50}\Big) + \frac{135}{22}\sin\Big($$
$$7t + \frac{93}{32}\Big) + \frac{143}{102}\sin\Big(8t + \frac{100}{33}\Big) + \frac{2}{21}\sin\Big(9t + \frac{169}{47}\Big) + \frac{41}{124}\sin\Big(10t + \frac{84}{19}\Big) + \frac{7}{39}\sin\Big(11$$
$$t + \frac{147}{73}\Big) + \frac{15}{17}\sin\Big(13t + \frac{69}{17}\Big) + \frac{6}{29}\sin\Big(16t + \frac{2}{25}\Big) + \frac{7}{43}\sin\Big(19t + \frac{57}{16}\Big) + \frac{5}{36}\sin\Big(20t +$$
$$\frac{20}{53}\Big) + \frac{225}{41}\Big)\theta(55\pi - t)\,\theta(t - 51\pi) + \Big(-\frac{22}{27}\sin\Big(\frac{21}{19} - 33t\Big) - \frac{23}{28}\sin\Big(\frac{56}{75} - 32t\Big) - \frac{9}{14}\sin\Big($$
$$\frac{18}{29} - 31t\Big) - \frac{32}{33}\sin\Big(\frac{145}{109} - 29t\Big) - \frac{11}{23}\sin\Big(\frac{27}{38} - 28t\Big) - \frac{91}{114}\sin\Big(\frac{26}{31} - 26t\Big) - \frac{47}{40}\sin\Big($$
$$\frac{73}{55} - 24t\Big) - \frac{54}{53}\sin\Big(\frac{8}{21} - 21t\Big) - \frac{38}{25}\sin\Big(\frac{65}{43} - 19t\Big) - \frac{108}{65}\sin\Big(\frac{11}{17} - 14t\Big) - \frac{11}{27}\sin\Big($$
$$\frac{289}{193} - 9t\Big) + \frac{3415}{31}\sin\Big(t + \frac{65}{34}\Big) + \frac{916}{35}\sin\Big(2t + \frac{168}{41}\Big) + \frac{85}{16}\sin\Big(3t + \frac{37}{41}\Big) + \frac{92}{35}\sin\Big(4t +$$
$$\frac{107}{32}\Big) + \frac{107}{27}\sin\Big(5t + \frac{14}{31}\Big) + \frac{47}{25}\sin\Big(6t + \frac{7}{11}\Big) + \frac{35}{12}\sin\Big(7t + \frac{49}{26}\Big) + \frac{26}{23}\sin\Big(8t +$$
$$\frac{48}{25}\sin\Big(10t + \frac{26}{103}\Big) + \frac{35}{38}\sin\Big(11t + \frac{23}{32}\Big) + \frac{15}{22}\sin\Big(12t + \frac{160}{41}\Big) + \frac{285}{286}\sin\Big(13t +$$
$$\frac{8}{19}\sin\Big(15t + \frac{336}{79}\Big) + \frac{48}{31}\sin\Big(16t + \frac{10}{3}\Big) + \frac{122}{87}\sin\Big(17t + \frac{37}{23}\Big) + \frac{68}{31}\sin\Big(18t +$$
$$\sin\Big(20t + \frac{58}{21}\Big) + \frac{49}{44}\sin(22t + 4) + \frac{22}{29}\sin\Big(23t + \frac{68}{53}\Big) + \frac{1}{2}\sin\Big(25t + \frac{86}{23}\Big) + \frac{17}{37}$$
$$\frac{148}{33}\Big) + \frac{21}{29}\sin\Big(30t + \frac{137}{30}\Big) + \frac{12}{29}\sin\Big(34t + \frac{202}{47}\Big) + \frac{15}{34}\sin\Big(35t + \frac{10}{29}\Big) + \frac{861}{37}\Big)\theta($$
$$\theta(t - 47\pi) + \Big(-\frac{5}{13}\sin\Big(\frac{9}{14} - 8t\Big) - \frac{33}{25}\sin\Big(\frac{105}{106} - 6t\Big) - \frac{12}{61}\sin\Big(\frac{47}{46} - 5t\Big) - \frac{17}{35}$$
$$4t\Big) - \frac{617}{35}\sin\Big(\frac{58}{41} - 2t\Big) + \frac{97}{24}\sin\Big(t + \frac{49}{11}\Big) + \frac{93}{43}\sin\Big(3t + \frac{53}{24}\Big) + \frac{52}{69}\sin\Big(7t +$$
$$\theta(47\pi - t)\,\theta(t - 43\pi) + \Big(-\frac{10}{31}\sin\Big(\frac{55}{39} - 6t\Big) + \frac{708}{23}\sin\Big(t + \frac{53}{28}\Big) + \frac{10}{17}\sin\Big(2t + \frac{30}{70}$$
$$\frac{201}{200}\sin\Big(3t + \frac{275}{92}\Big) + \frac{8}{15}\sin\Big(4t + \frac{129}{29}\Big) + \frac{2}{23}\sin\Big(5t + \frac{110}{41}\Big) + \frac{18}{89}\sin\Big(7t + \frac{73}{30}\Big) + \frac{949}{15}\Big)$$
$$\theta(43\pi - t)\,\theta(t - 39\pi) + \Big(-\frac{22}{41}\sin\Big(\frac{42}{41} - 5t\Big) - \frac{9}{19}\sin\Big(\frac{11}{31} - 3t\Big) + \frac{1168}{41}\sin\Big(t + \frac{59}{37}\Big) + \frac{13}{21}$$
$$\sin\Big(2t + \frac{382}{83}\Big) + \frac{15}{29}\sin\Big(4t + \frac{105}{29}\Big) + \frac{7}{17}\sin\Big(6t + \frac{67}{19}\Big) - \frac{2875}{47}\Big)\theta(39\pi - t)\,\theta(t - 35\pi) +$$
$$\Big(-\frac{15}{19}\sin\Big(\frac{1}{26} - 14t\Big) - \frac{111}{31}\sin\Big(\frac{7}{31} - 4t\Big) + \frac{3698}{39}\sin\Big(t + \frac{31}{20}\Big) + \frac{135}{22}\sin\Big(2t + \frac{1}{35}\Big) + \frac{136}{11}$$
$$\sin\Big(3t + \frac{71}{46}\Big) + \frac{57}{13}\sin\Big(5t + \frac{29}{19}\Big) + \frac{22}{25}\sin\Big(6t + \frac{80}{31}\Big) + \frac{29}{40}\sin\Big(7t + \frac{212}{47}\Big) + \frac{31}{26}\sin\Big(8t +$$
$$\frac{1}{[illegible]}\Big) + \frac{55}{[illegible]}\sin\Big(9t + \frac{51}{[illegible]}\Big) + \frac{46}{[illegible]}\sin\Big(10t + \frac{3}{[illegible]}\Big) + \frac{27}{[illegible]}\sin\Big(11t + \frac{48}{[illegible]}\Big) + \frac{5}{[illegible]}\sin\Big(12t + \frac{49}{[illegible]}\Big) +$$



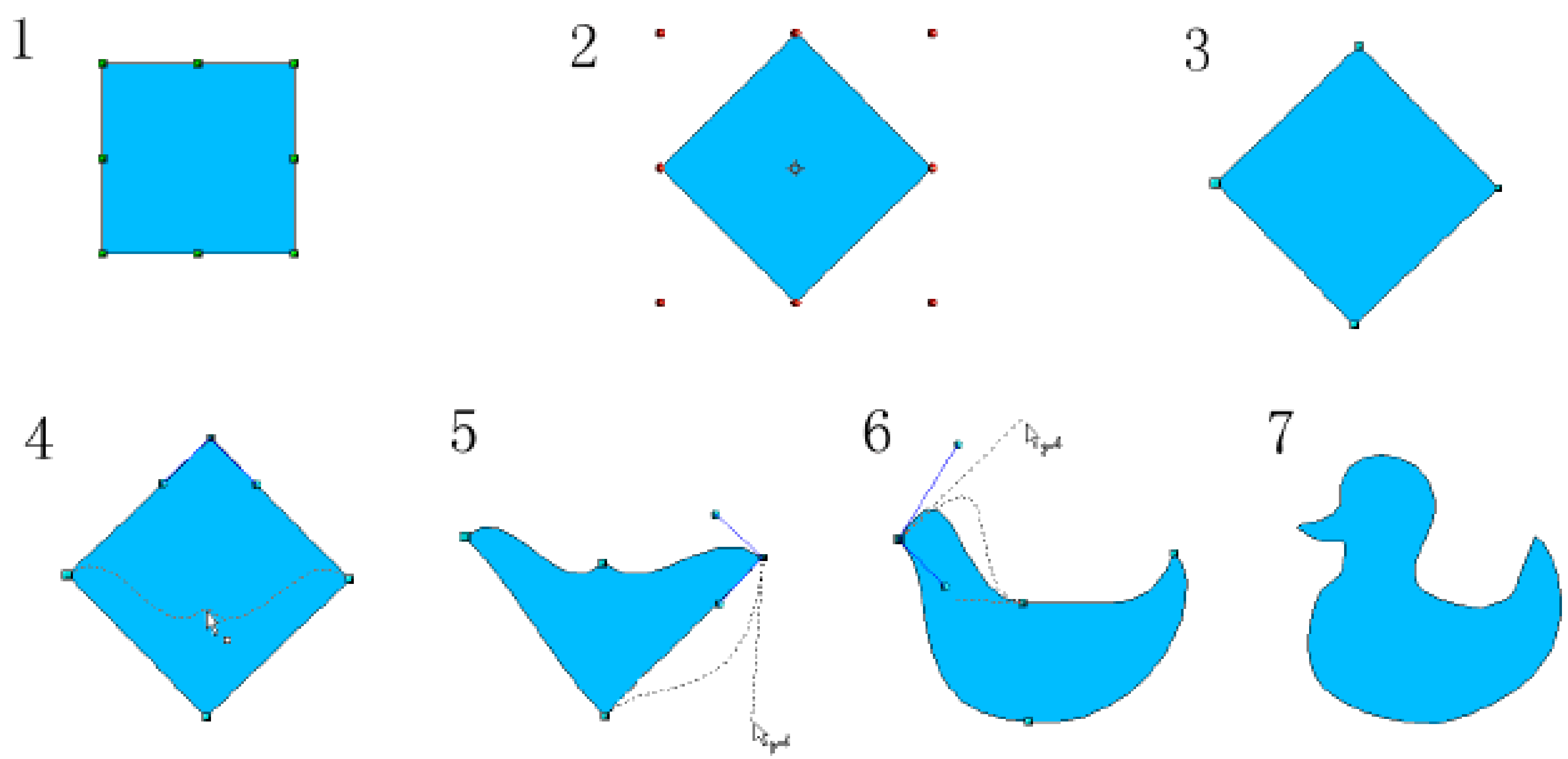
(plotted for t from 0 to 72π)

$$y(t) = \Big(\Big(\Big(\frac{16}{35}\sin\Big(t + \frac{11}{7}\Big) + \frac{56}{27}\sin\Big(2t + \frac{96}{61}\Big) + \frac{22}{27}\sin\Big(3t + \frac{52}{33}\Big) + \frac{74}{47}\sin\Big(4t + \frac{52}{33}\Big) + \frac{13}{32}$$
$$\sin\Big(5t + \frac{67}{42}\Big) + \frac{4093}{22}\Big)\theta(71\pi - t)\,\theta(t - 67\pi) + \Big(-\frac{3}{32}\sin\Big(\frac{61}{39} - 8t\Big) + \frac{341}{32}\sin\Big(t + \frac{11}{7}\Big) +$$
$$\frac{11}{14}\sin\Big(2t + \frac{47}{10}\Big) + \frac{23}{12}\sin\Big(3t + \frac{52}{33}\Big) + \frac{3}{20}\sin\Big(4t + \frac{53}{33}\Big) + \frac{58}{117}\sin\Big(5t + \frac{52}{33}\Big) + \frac{3}{38}\sin\Big(6$$
$$t + \frac{61}{13}\Big) + \frac{8}{41}\sin\Big(7t + \frac{58}{37}\Big) + \frac{3055}{19}\Big)\theta(67\pi - t)\,\theta(t - 63\pi) + \Big(\frac{111}{10}\sin\Big(t + \frac{11}{7}\Big) + \frac{10}{29}\sin\Big($$
$$2t + \frac{202}{43}\Big) + \frac{67}{50}\sin\Big(3t + \frac{11}{7}\Big) + \frac{1}{32}\sin\Big(4t + \frac{639}{137}\Big) + \frac{5035}{31}\Big)\theta(63\pi - t)\,\theta(t - 59\pi) +$$
$$\Big(-\frac{35}{71}\sin\Big(\frac{4}{17} - 4t\Big) - \frac{35}{34}\sin\Big(\frac{53}{52} - 3t\Big) + \frac{175}{36}\sin\Big(t + \frac{123}{44}\Big) + \frac{83}{27}\sin\Big(2t + \frac{133}{32}\Big) +$$
$$\frac{13847}{111}\Big)\theta(59\pi - t)\,\theta(t - 55\pi) + \Big(-\frac{38}{39}\sin\Big(\frac{19}{14} - 17t\Big) - \frac{13}{27}\sin\Big(\frac{31}{22} - 16t\Big) - \frac{49}{39}\sin\Big(\frac{31}{33} -$$
$$13t\Big) - \frac{55}{56}\sin\Big(\frac{4}{3} - 11t\Big) - \frac{76}{51}\sin\Big(\frac{6}{29} - 8t\Big) + \frac{127}{35}\sin\Big(t + \frac{190}{67}\Big) + \frac{1061}{78}\sin\Big(2t + \frac{61}{33}\Big) +$$
$$\frac{953}{143}\sin\Big(3t + \frac{53}{16}\Big) + \frac{48}{29}\sin\Big(4t + \frac{33}{20}\Big) + \frac{127}{26}\sin\Big(5t + \frac{317}{72}\Big) + \frac{281}{36}\sin\Big(6t + \frac{107}{39}\Big) + \frac{48}{35}$$
$$\sin\Big(7t + \frac{41}{12}\Big) + \frac{1}{8}\sin\Big(9t + \frac{3}{34}\Big) + \frac{380}{381}\sin\Big(10t + \frac{47}{26}\Big) + \frac{27}{31}\sin\Big(12t + \frac{86}{35}\Big) + \frac{11}{32}\sin\Big(14$$
$$t + \frac{10}{19}\Big) + \frac{15}{22}\sin\Big(15t + \frac{11}{27}\Big) + \frac{2}{3}\sin\Big(18t + \frac{17}{8}\Big) + \frac{2}{29}\sin\Big(19t + \frac{181}{40}\Big) + \frac{5}{27}\sin\Big(20t +$$
$$\frac{20}{13}\Big) - \frac{14371}{74}\Big)\theta(55\pi - t)\,\theta(t - 51\pi) + \Big(-\frac{14}{27}\sin\Big(\frac{31}{23} - 35t\Big) - \frac{9}{17}\sin\Big(\frac{23}{28} - 34t\Big) - \frac{5}{14}$$
$$\sin\Big(\frac{18}{13} - 28t\Big) - \frac{9}{23}\sin\Big(\frac{11}{20} - 27t\Big) - \frac{14}{29}\sin\Big(\frac{33}{46} - 26t\Big) - \frac{59}{24}\sin\Big(\frac{44}{31} - 17t\Big) - \frac{137}{73}\sin\Big($$
$$\frac{15}{22} - 16t\Big) - \frac{7}{23}\sin\Big(\frac{7}{5} - 12t\Big) - \frac{137}{73}\sin\Big(\frac{11}{19} - 10t\Big) - \frac{44}{31}\sin\Big(\frac{2}{33} - 9t\Big) - \frac{2708}{65}\sin\Big(\frac{16}{21} -$$
$$2t\Big) - \frac{818}{31}\sin\Big(\frac{29}{33} - t\Big) + \frac{435}{22}\sin\Big(3t + \frac{21}{20}\Big) + \frac{113}{24}\sin\Big(4t + \frac{124}{27}\Big) + \frac{641}{95}\sin\Big(5t + \frac{51}{31}\Big) +$$
$$\sin\Big(6t + \frac{39}{11}\Big) + \frac{303}{76}\sin\Big(7t + \frac{275}{118}\Big) + \frac{140}{29}\sin\Big(8t + \frac{127}{29}\Big) + \frac{51}{28}\sin\Big(11t + \frac{50}{29}\Big) + \frac{43}{40}$$
$$\Big(13t + \frac{67}{43}\Big) + \frac{59}{46}\sin\Big(14t + \frac{173}{99}\Big) + \frac{67}{35}\sin\Big(15t + \frac{31}{24}\Big) + \frac{45}{22}\sin\Big(18t + \frac{67}{16}\Big) + \frac{24}{19}\sin\Big($$
$$t + \frac{20}{9}\Big) + \frac{3}{17}\sin\Big(20t + \frac{44}{27}\Big) + \frac{18}{31}\sin\Big(21t + \frac{49}{34}\Big) + \frac{37}{62}\sin\Big(22t + \frac{11}{15}\Big) + \frac{13}{31}\sin\Big(23t +$$
$$\frac{15}{7}\Big) + \frac{16}{37}\sin\Big(24t + \frac{22}{7}\Big) + \frac{4}{35}\sin\Big(25t + \frac{26}{33}\Big) + \frac{4}{43}\sin\Big(29t + \frac{79}{17}\Big) + \frac{11}{37}\sin\Big(30t +$$
$$\frac{33}{5}\Big) + \frac{8}{35}\sin\Big(31t + \frac{259}{97}\Big) + \frac{9}{20}\sin\Big(32t + \frac{27}{10}\Big) + \frac{22}{37}\sin\Big(33t + \frac{26}{15}\Big) + \frac{5589}{19}\Big)\theta(51\pi - t)$$
$$- 47\pi) + \Big(-\frac{101}{30}\sin\Big(\frac{23}{15} - 3t\Big) + \frac{177}{19}\sin\Big(t + \frac{63}{38}\Big) + \frac{73}{19}\sin\Big(2t + \frac{58}{15}\Big) + \frac{50}{33}\sin\Big(4t +$$
$$\Big) + \frac{5}{17}\sin\Big(5t + \frac{541}{135}\Big) + \frac{29}{35}\sin\Big(6t + \frac{125}{29}\Big) + \frac{31}{52}\sin\Big(7t + \frac{181}{41}\Big) + \frac{7}{12}\sin\Big(8t + \frac{122}{35}\Big) +$$
$$\frac{86}{25}\Big)\theta(47\pi - t)\,\theta(t - 43\pi) + \Big(-\frac{2}{25}\sin\Big(\frac{13}{34} - 7t\Big) - \frac{3}{26}\sin\Big(\frac{9}{8} - 4t\Big) + \frac{430}{17}\sin\Big(t + \frac{6}{31}\Big) +$$
$$\frac{24}{37}\sin\Big(2t + \frac{137}{35}\Big) + \frac{50}{37}\sin\Big(3t + \frac{26}{23}\Big) + \frac{4}{45}\sin\Big(5t + \frac{268}{161}\Big) + \frac{7}{20}\sin\Big(6t + \frac{19}{6}\Big) + \frac{8835}{43}\Big)\theta($$
$$43\pi - t)\,\theta(t - 39\pi) + \Big(-\frac{11}{50}\sin\Big(\frac{46}{55} - 5t\Big) + \frac{345}{13}\sin\Big(t + \frac{1}{10}\Big) + \frac{67}{41}\sin\Big(2t + \frac{90}{29}\Big) + \frac{5}{2}\sin\Big($$
$$3t + \frac{2}{19}\Big) + \frac{39}{79}\sin\Big(4t + \frac{565}{141}\Big) + \frac{23}{42}\sin\Big(6t + \frac{67}{22}\Big) + \frac{6008}{29}\Big)\theta(39\pi - t)\,\theta(t - 35\pi) + \Big(-\frac{21}{53}$$
$$\sin\Big(\frac{74}{73} - 15t\Big) - \frac{15}{22}\sin\Big(\frac{78}{67} - 14t\Big) - \frac{291}{104}\sin\Big(\frac{1}{14} - 7t\Big) - \frac{4561}{456}\sin\Big(\frac{2}{17} - t\Big) + \frac{107}{44}\sin\Big($$
$$2t + \frac{46}{29}\Big) + \frac{45}{22}\sin\Big(3t + \frac{128}{51}\Big) + \frac{553}{32}\sin\Big(4t + \frac{144}{31}\Big) + \frac{25}{16}\sin\Big(5t + \frac{127}{37}\Big) + \frac{53}{30}\sin\Big(6t +$$
$$\frac{129}{[illegible]}\Big) + \frac{1}{[illegible]}\sin\Big(8t + \frac{60}{[illegible]}\Big) + \frac{36}{[illegible]}\sin\Big(9t + \frac{107}{[illegible]}\Big) + \frac{61}{[illegible]}\sin\Big(10t + \frac{141}{[illegible]}\Big) + \frac{26}{[illegible]}\sin\Big(11t +$$

How Do You Specify This? (2D)



Editing Curves



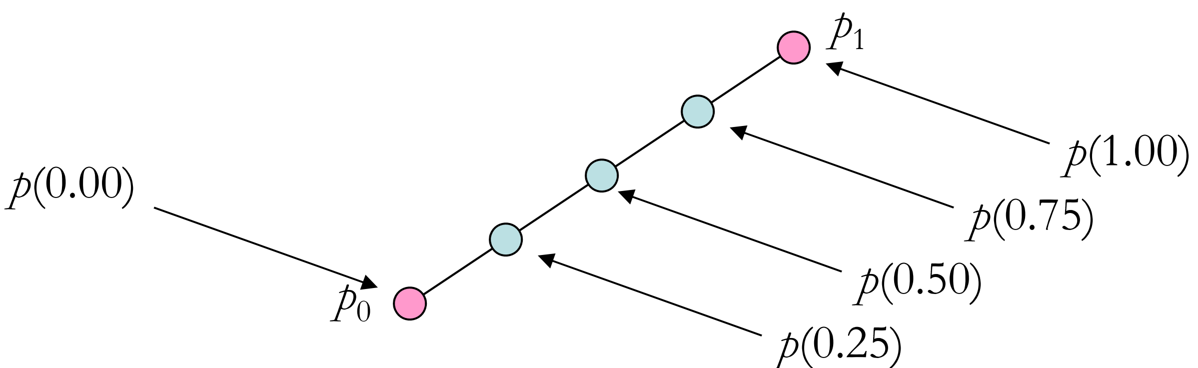
Parametric Curves and Surfaces

- Introduction on the simplest ‘curve’
- About **explicit** and **implicit** representations
- One of the parametric (explicit) curves:
  - the Bezier curve
- Bezier surfaces

Parametric(Explicit)
vs
Implicit Curves

Straight Line Segments

- A straight line segment is the line joining two end points p_0 and p_1 :



- Any point on the line segment can be expressed in the form of

$$p(t) = (1-t)\, p_0 + t\, p_1$$

- This is called the ***linear interpolation*** of p_0 and p_1

The Parametric (Explicit) Form of a Curve

- Let $p_0 = (x_0, y_0)$ and $p_1 = (x_1, y_1)$
- We can write the point as $p(t) = (x(t), y(t))$
  - for $x(t) = (1-t)x_0 + tx_1$ and $y(t)=(1-t)y_0+ty_1$
- This is called the ***parametric form*** (or the ***explicit form***) of a curve if we can express the point with a ***parameter*** t
- Usually t is from 0 to 1
- The two end points are called the ***control points***
  - We can use different control points to represent different line segments

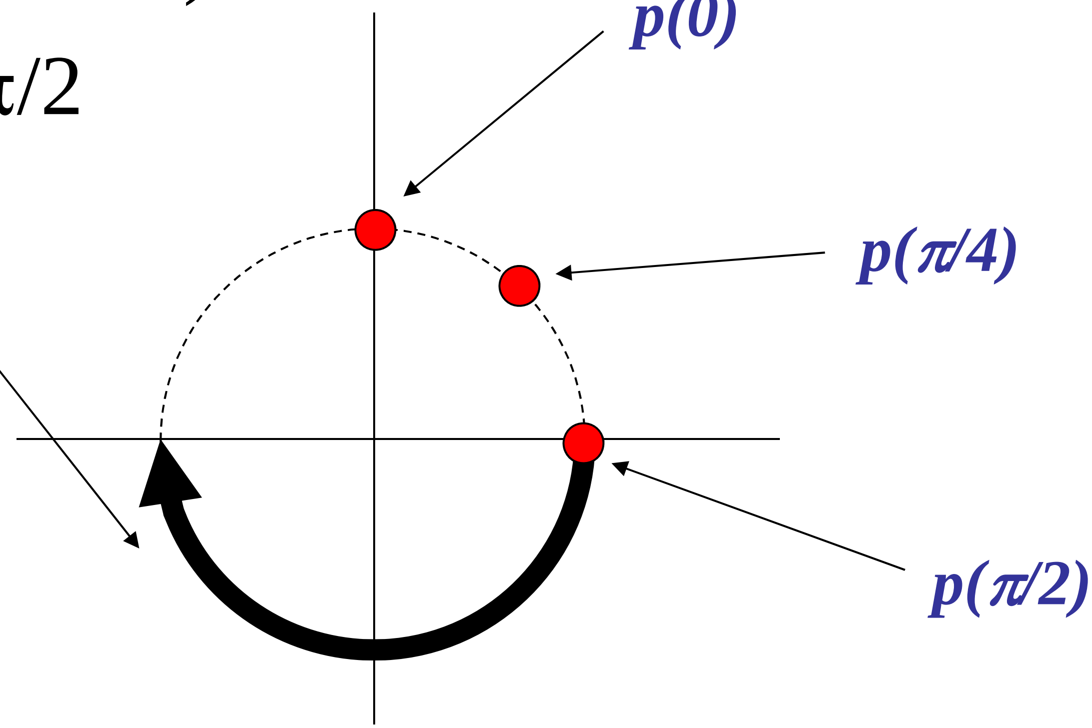
Parametric Curves

A function in t

- A 2D parametric curve can be expressed as

$$p(t) = (\ x(t), y(t)\ )$$

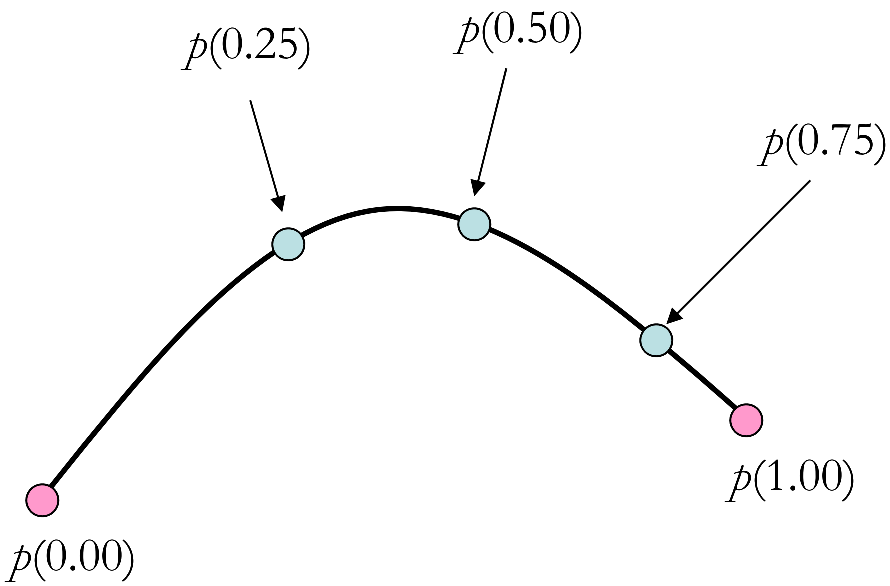
- e.g. $p(t) = (\ \sin t, \cos t\ )$
- $p(t) : \pi/2 < t < 3\pi/2$

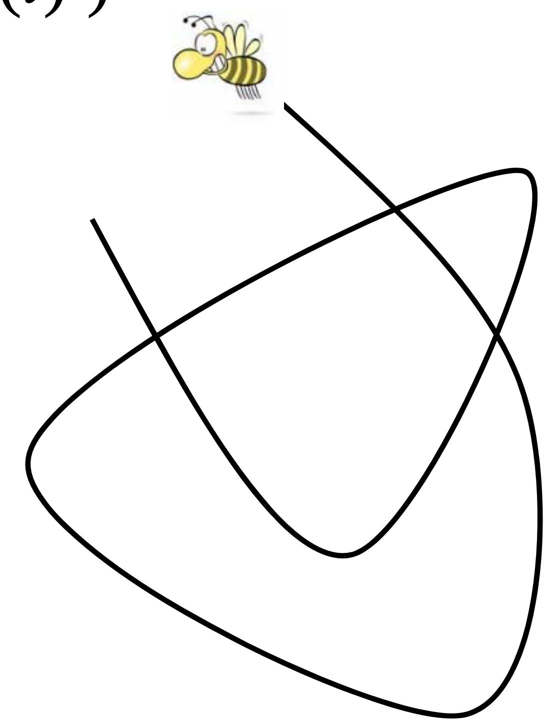


Parametric Curves

- A 2D parametric curve can be expressed as

$$p(t) = (\ x(t), y(t)\ )$$





Parametric Curves

- A 2D parametric curve can be expressed as

$$p(t) = (\ x(t), y(t)\ )$$

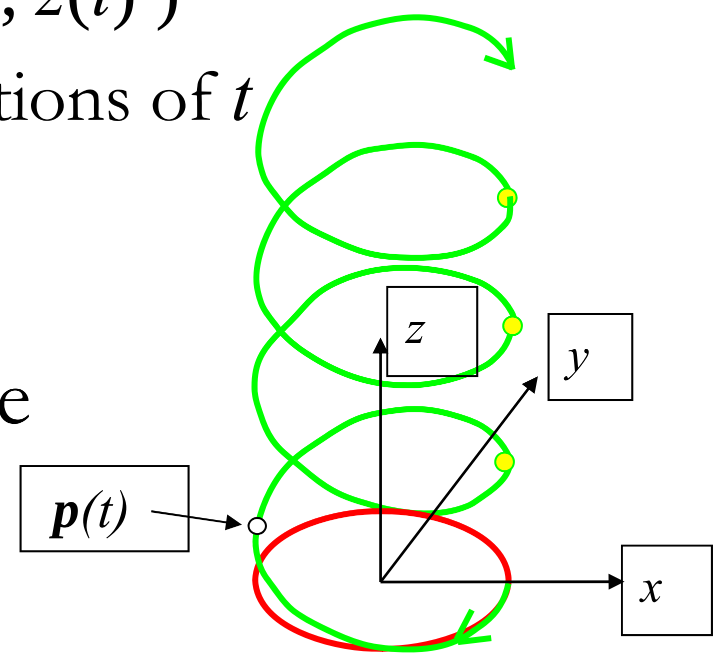
- A 3D parametric curve can be expressed as

$$p(t) = (\ x(t), y(t), z(t)\ )$$

where $x(t)$, $y(t)$ and $z(t)$ are functions of t

- e.g. $p(t) = (\ \sin t, \cos t, t)$

- We limit those functions to be **polynomials** <u>in this course</u>.



Degree of a Curve

- The maximum degree in these functions are called the ***degree*** of the parametric curve, for example:
  - a quadratic curve is a curve with degree 2
  - a cubic curve is a curve with degree 3
- E.g. the line segment is a degree 1 "curve"

The Implicit Form of a Curve

- On the other hand, the line segment can be expressed in other form, namely the ***implicit form***

$$f(x,y,z) = 0$$

(e.g. the line segment is a part of the whole line $ax+by+cz+d=0$)

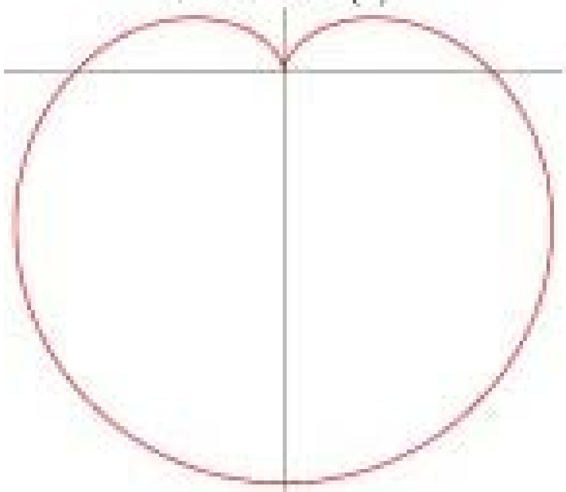
- Most of the time, a curve can be expressed in both implicit form and explicit form, i.e. difference **representations**
- For example

$f(x,y) = x^2+y^2-1=0$ **vs** $p(t) = (\sin t, \cos t)$ for $t = 0$ to 2π

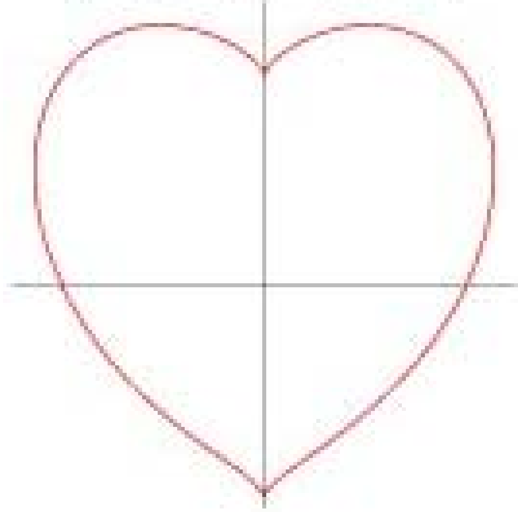
Implicit Form

Parametric Curve

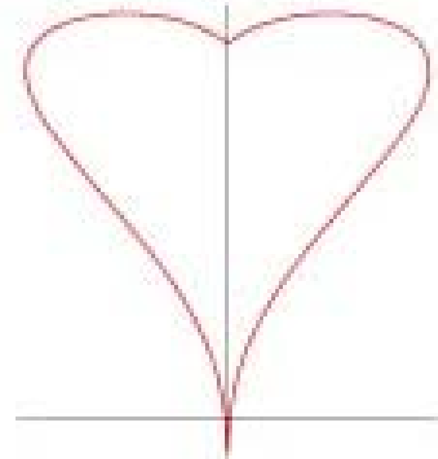
$$r = 1 - \sin(\theta)$$



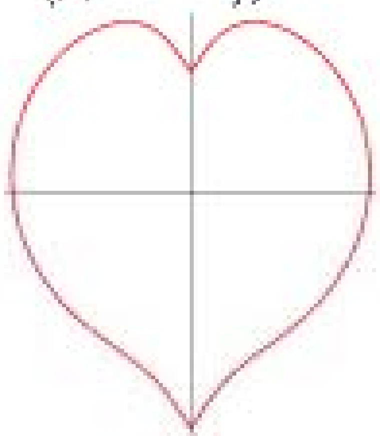
$$\left(x^2 + y^2 - 1\right)^3 - x^2 y^3 = 0$$



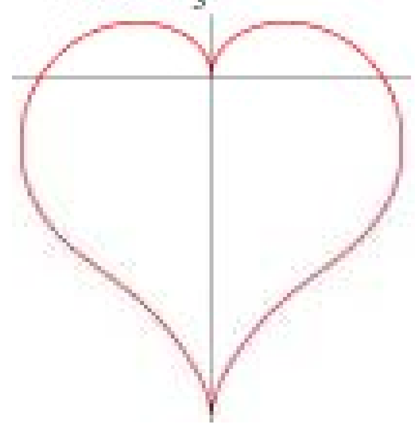
$$x = \sin(t)\cos(t)\log(|t|)$$
$$y = |t|^{0.3}\sqrt{\cos(t)}$$



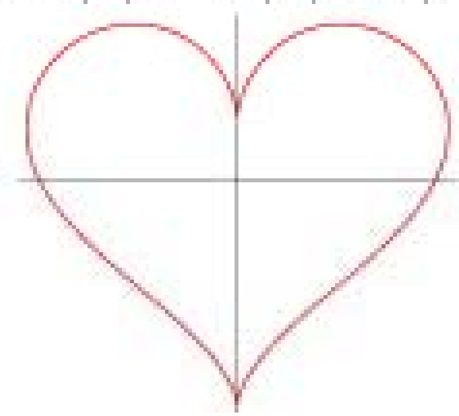
$$\left(y - \frac{2\left(|x| + x^2 - 6\right)}{3\left(|x| + x^2 + 2\right)}\right)^2 + x^2 = 36$$



$$r = \frac{\sin(t)\sqrt{|\cos(t)|}}{\sin(t) + \frac{7}{5}} - 2\sin(t) + 2$$



$$x = 16\sin^3(t)$$
$$y = 13\cos(t) -$$
$$5\cos(2t) - 2\cos(3t) - \cos(4t)$$



Curves Representations

- Curves
  - Explicit (Parametric)
    - Bezier Curves
    - B-Splines
    - NURBS, etc…
  - Implicit

Bezier Curves

[An Example of Parametric Curves]

[An Example of Parametric Curves]

Control Points

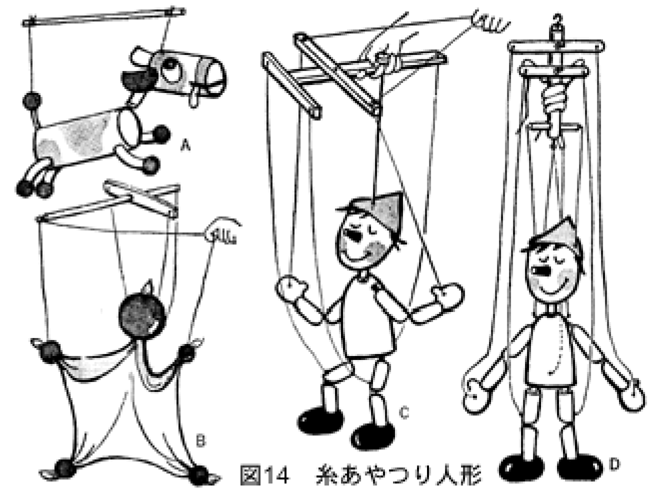
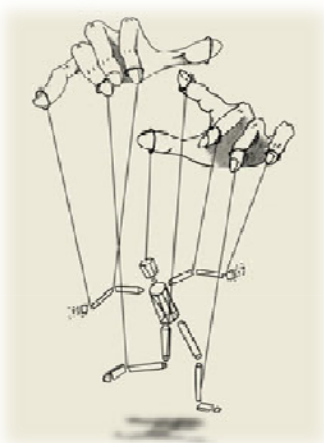
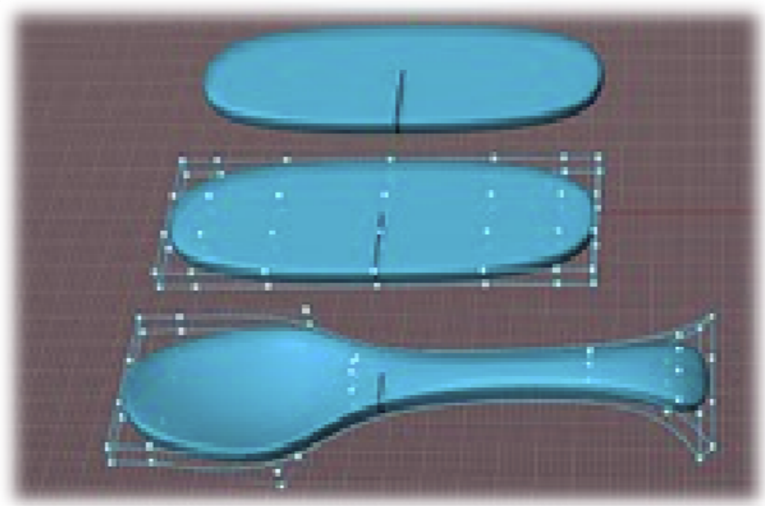


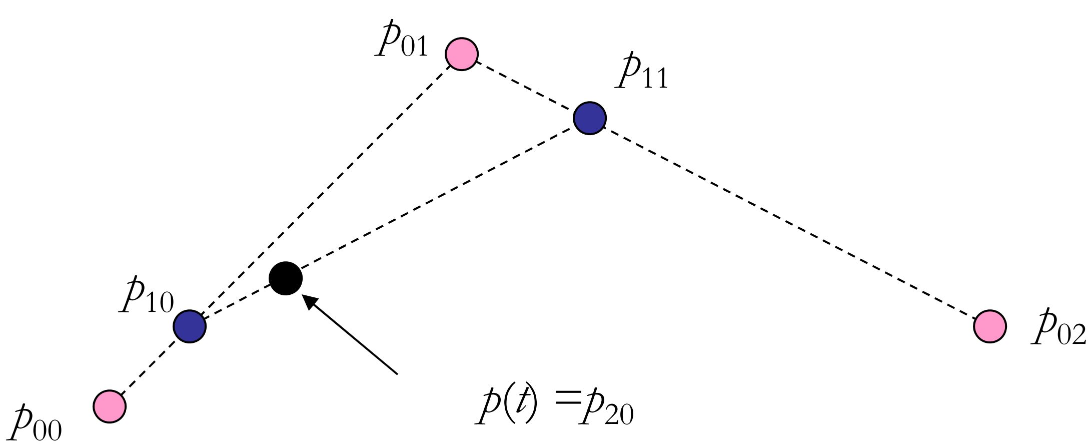
図14　糸あやつり人形

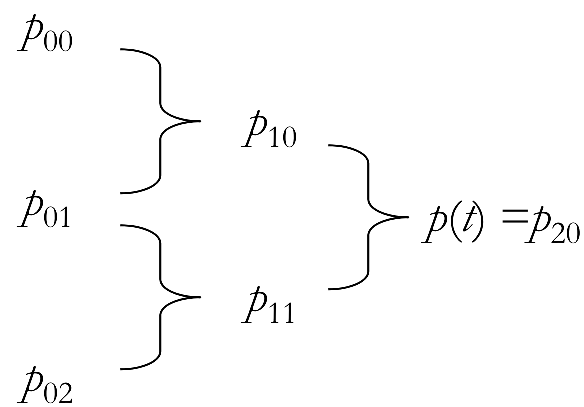


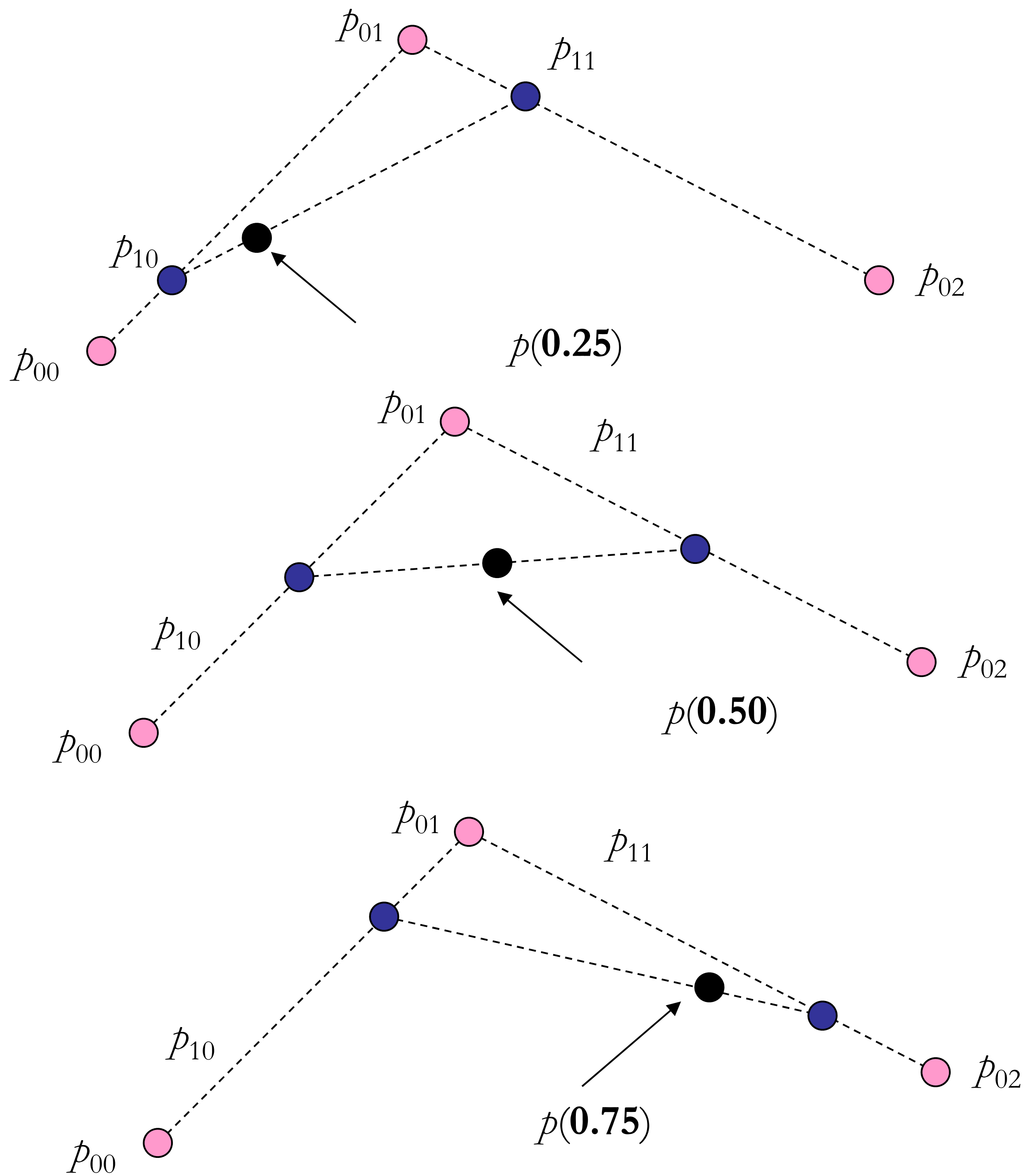


Defining a Quadratic Bezier Curve

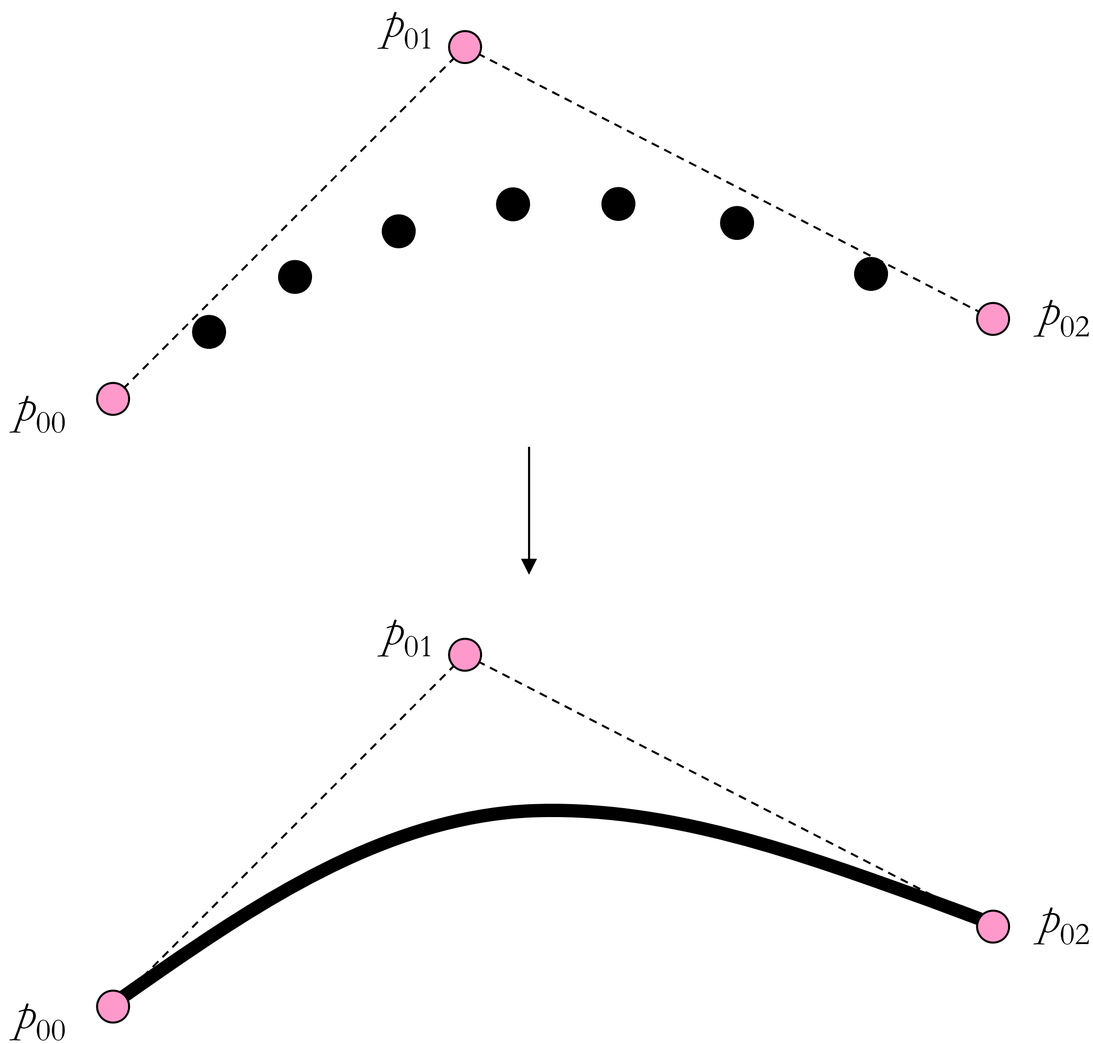
- ***Input***: Given 3 control points p_{00} , p_{01} and p_{02}
- $p_{10}(t) = (1\text{-}t)\, p_{00} + t\, p_{01}$
- $p_{11}(t) = (1\text{-}t)\, p_{01} + t\, p_{02}$
- $p(t) = p_{20}(t) = (1\text{-}t)\, p_{10}(t) + t\, p_{11}(t)$






p_{01}
p_{11}
p_{10}
p_{02}
p_{00}
$p(\mathbf{0.25})$
p_{01}
p_{11}
p_{10}
p_{02}
$p(\mathbf{0.50})$
p_{00}
p_{01}
p_{11}
p_{10}
p_{02}
$p(\mathbf{0.75})$
p_{00}

A Quadratic Bezier Curve



A Quadratic Bezier Curve

- Recall
  - $p_{10}(t) = (1\text{-}t)\, p_{00} + t\, p_{01}$
  - $p_{11}(t) = (1\text{-}t)\, p_{01} + t\, p_{02}$
  - $p(t) = p_{20}(t) = (1\text{-}t)\, p_{10}(t) + t\, p_{11}(t)$
- So

$$p(t) = (1\text{-}t)^2\, p_{00} + 2\, t\, (1\text{-}t)\, p_{01} + t^2\, p_{02}$$

- or

$$p(t) = \sum_{i=0}^{2} b_{2,i}(t)\, p_{0i} \qquad b_{2,i}(t) = \binom{2}{i} (1-t)^{2-i}\, t^i$$

Degree n Bezier Curves

- Given $n+1$ control points, p_{0i} for $i = 0$ to n
- A degree n Bezier Curve is defined as

$$p(t) = \sum_{i=0}^{n} b_{n,i}(t)\, p_{0i}$$

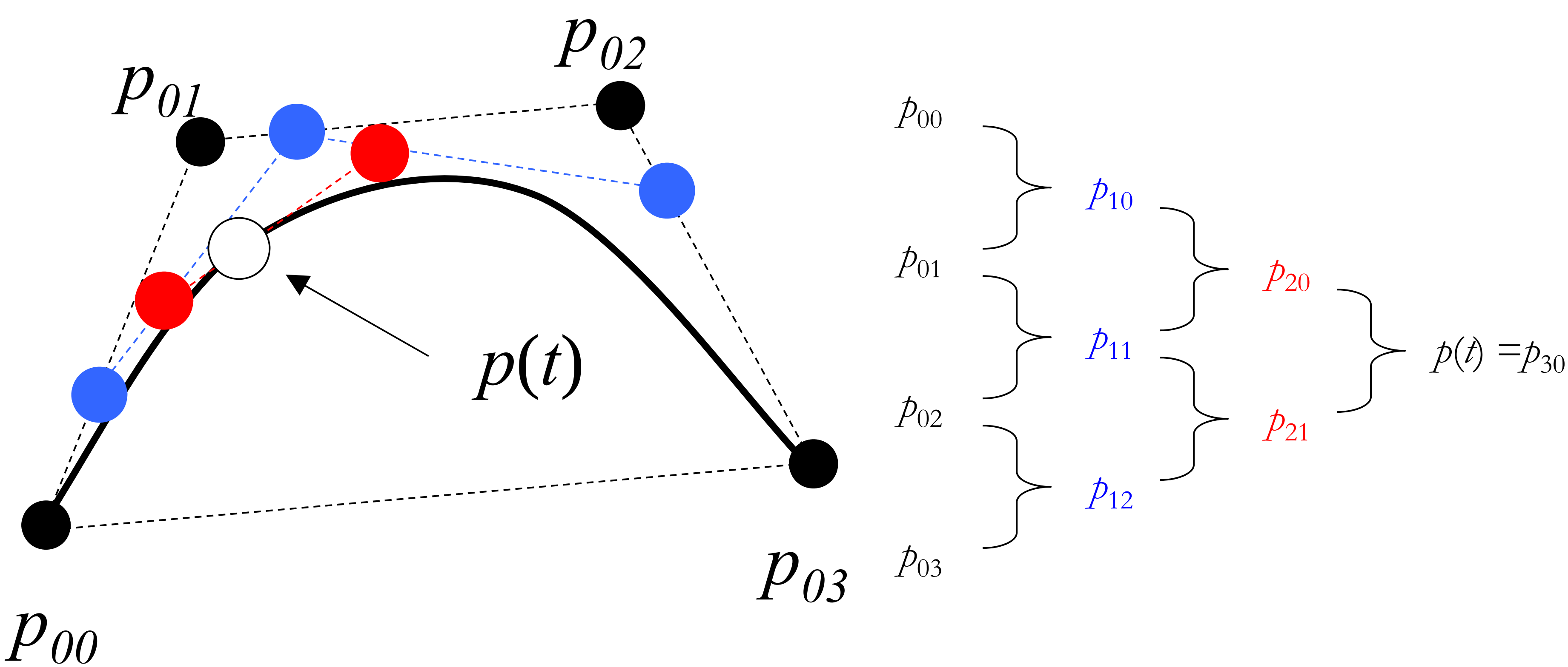
- for

$$b_{n,i}(t) = \binom{n}{i} (1-t)^{n-i} t^{i}$$

- the function $b_{n,i}(t)$ is called the ***basis function***
- the number $n+1$ is the ***order*** of the curve (order = degree + 1)

Another Example: a Cubic Bezier Curve

- Degree of the curve = 3
- Given the 4 control points p_{00}, p_{01}, p_{02}, and p_{03}



Properties of Bezier Curves

Properties of a Bezier Curve

- Tangent Vectors
- Control Points Manipulation of the Curves
- Connecting two Bezier Curves
- Convex Hull Properties
- Drawing a Bezier Curve, Subdivision Method
- OpenGL Implementation

Tangent Vectors (for any parametric curve)

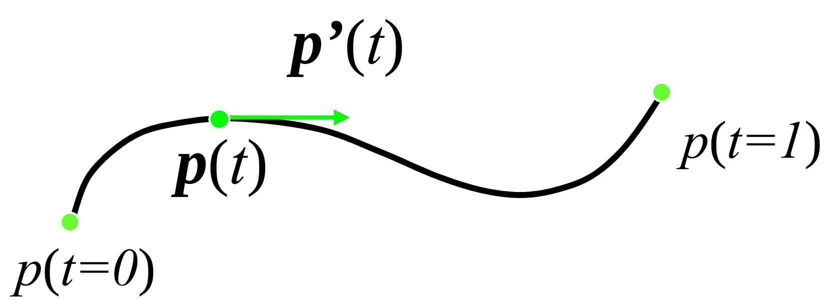
- Given a parametric curve

$$p(t) = (\ x(t), y(t), z(t)\ )$$

- The tangent vector of the curve is the derivative

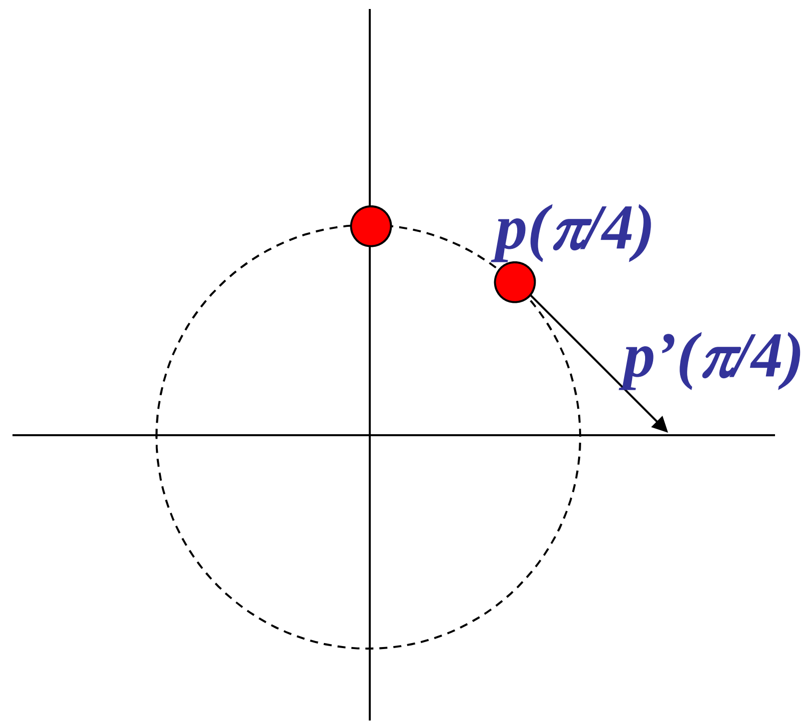
$$p'(t) = \mathrm{d}p(t)/\mathrm{dt} = (\ \mathrm{d}x(t)/\mathrm{dt}, \mathrm{d}y(t)/\mathrm{dt}, \mathrm{d}z(t)/\mathrm{dt}\ )$$

- Also can be viewed as the velocity at $p(t)$



For example

- $p(t) = (\sin(t), \cos(t))$
- $p'(t) = \mathbf{d}\, p(t)/\mathbf{d}t = (\cos(t), -\sin(t))$

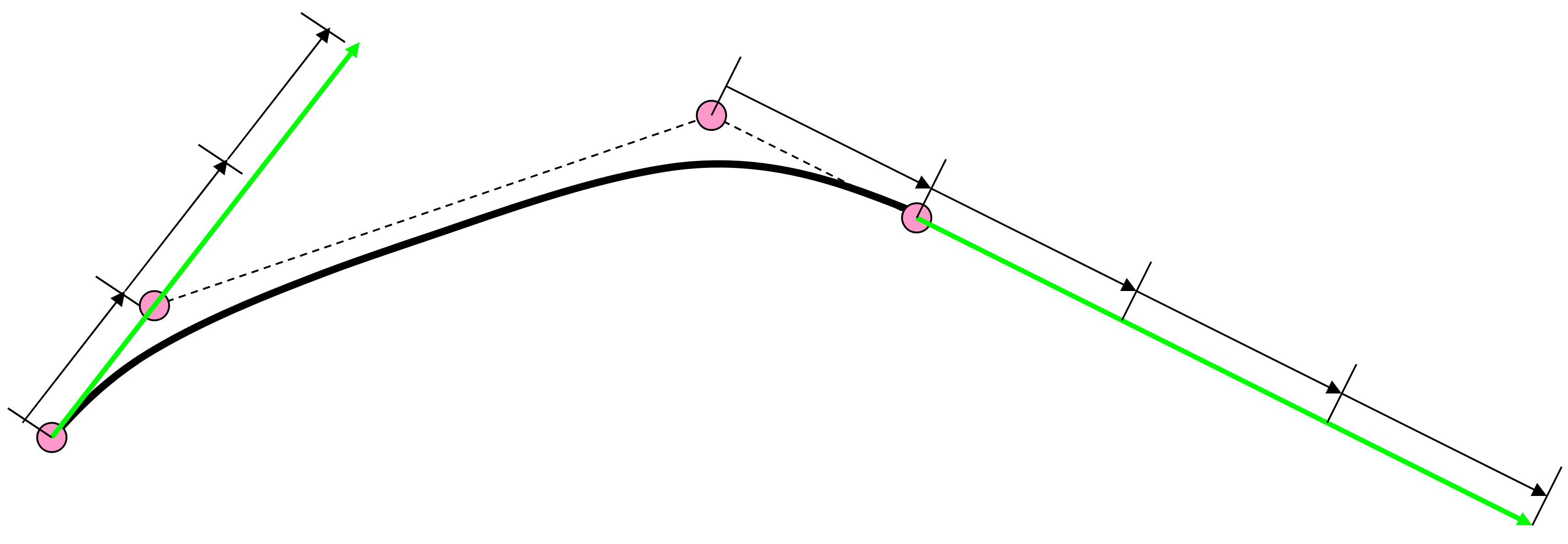


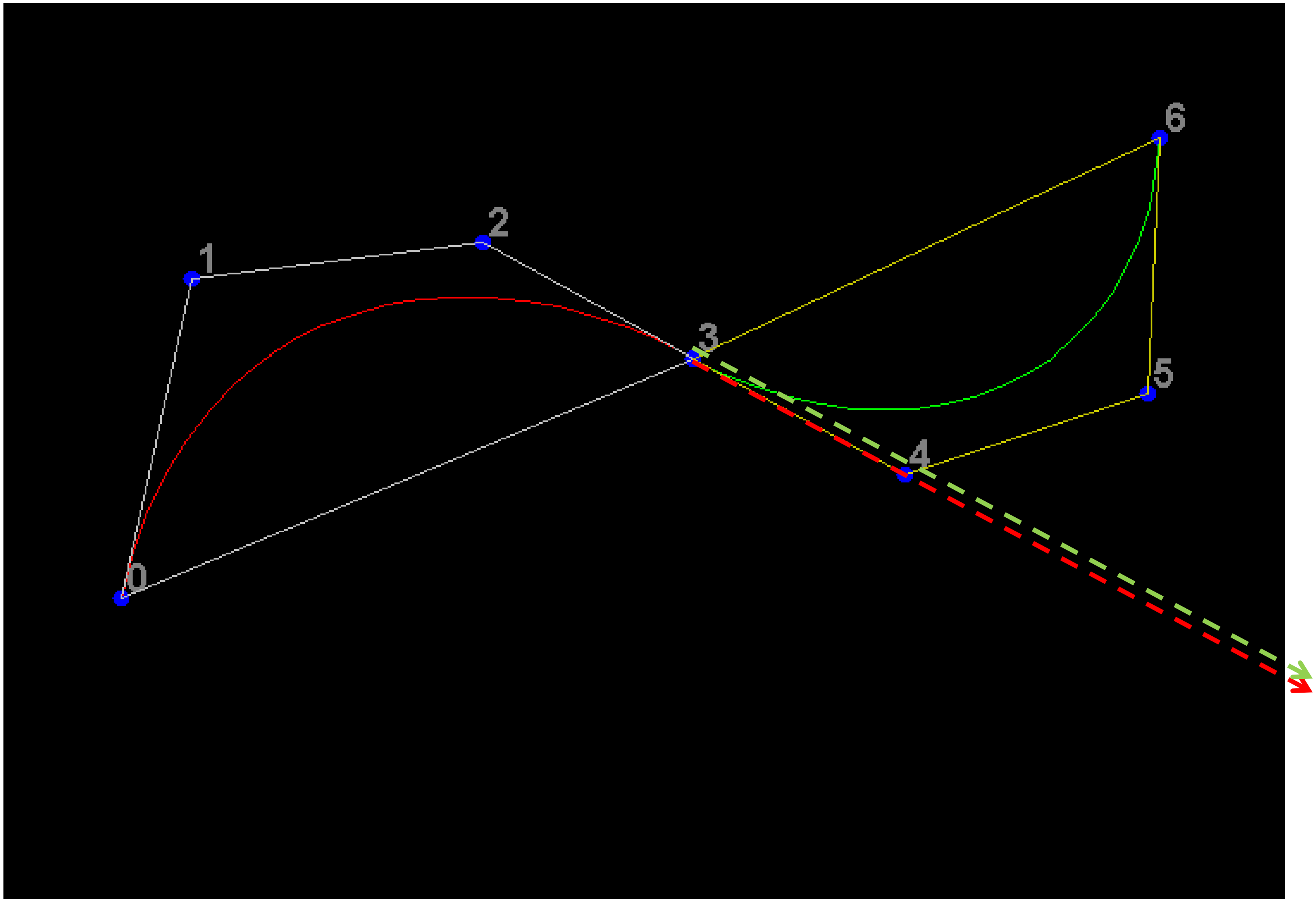
Tangent Vector of a Bezier Curve

- If $p(t)$ is a cubic Bezier curve, after differentiation,

$$p'(0) = 3(p_{01} - p_{00}) = 3(p_{00}p_{01}), \text{ and}$$

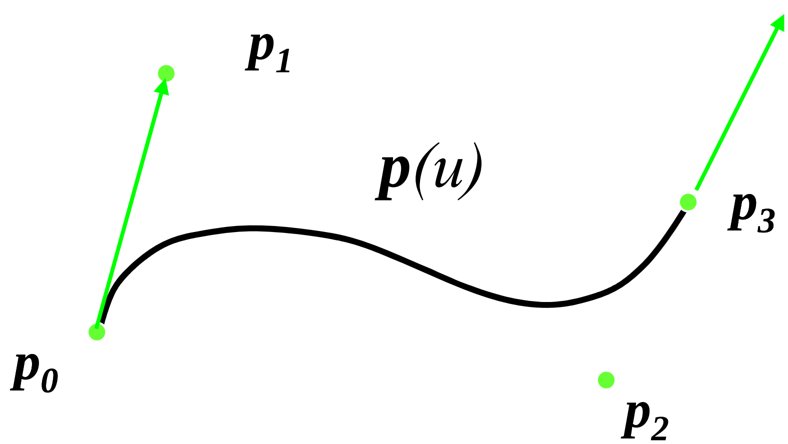
$$p'(1) = 3(p_{0n} - p_{0n-1}) = 3(p_{0n-1}p_{0n})$$

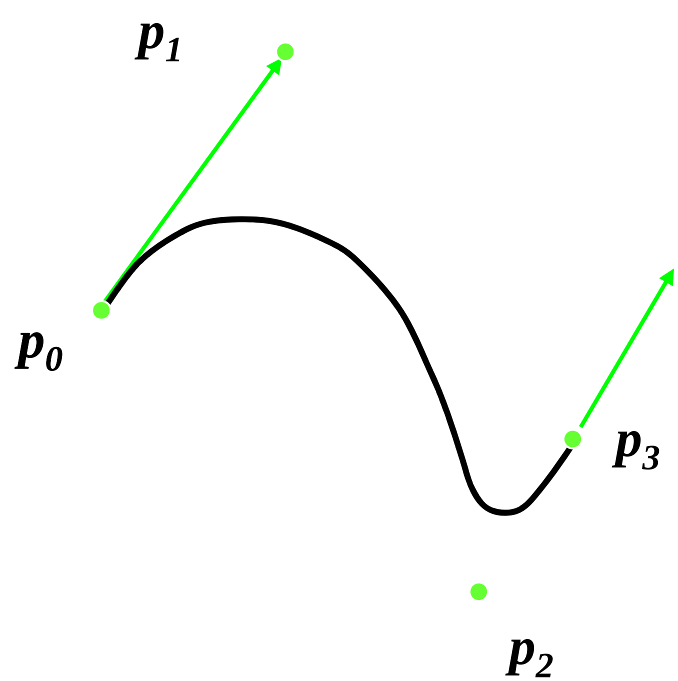



0
1
2
3
4
5
6

Control Point Manipulation

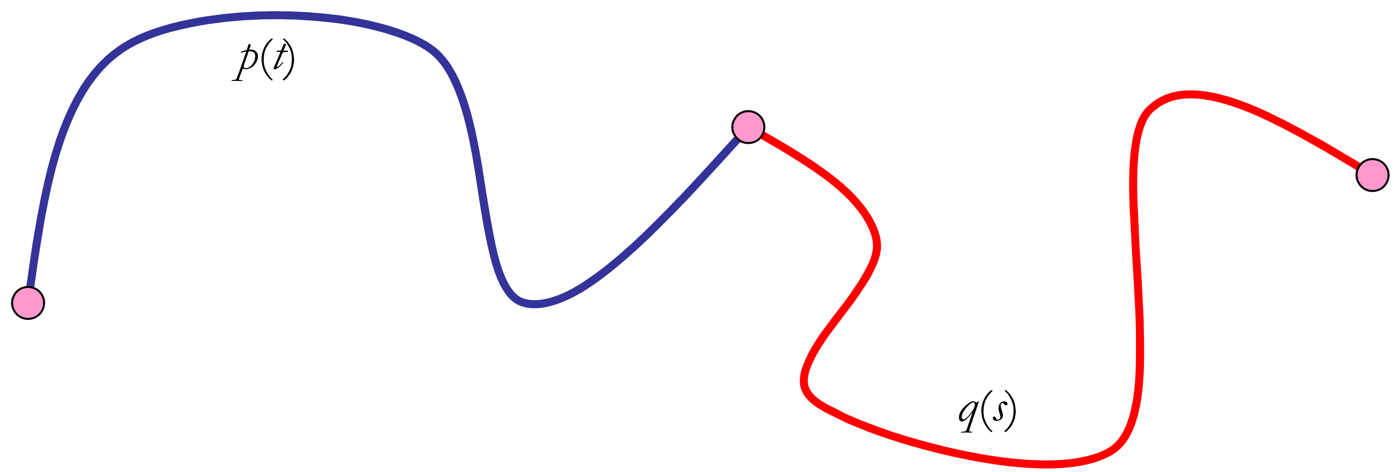
- Easy control on the curve by the moving the control points





Connecting Two Bezier Curves

- Given two curves $p(t)$ and $q(s)$
- They have G^0 (or C^0) continuity if $p(1) = q(0)$



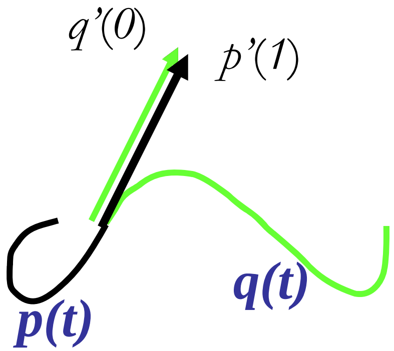
- For Bezier curves, set $p_{0n} = q_{00}$

Connecting Two Bezier Curves

- C^1 continuity:

$$p'(1) = q'(0)$$

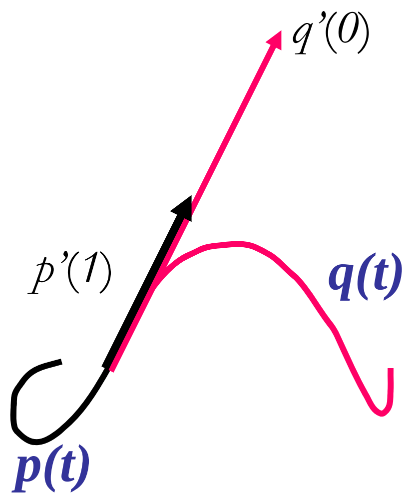
  – The two tangent vectors are the same



- G^1 continuity:

$$p'(1) = \alpha q'(0)$$

  – The two tangent vectors have the same direction



- To achieve C^1 continuity:

$$p_{0n-1}p_{0n} = q_{00}\,q_{01}$$

- To achieve G^1 continuity:

$$p_{0n-1}p_{0n} = \alpha\, q_{00}\,q_{01}$$

C^n continuity

- C^n continuity:

$$dp^n(1)/dt^n = dq^n(0)/dt^n$$

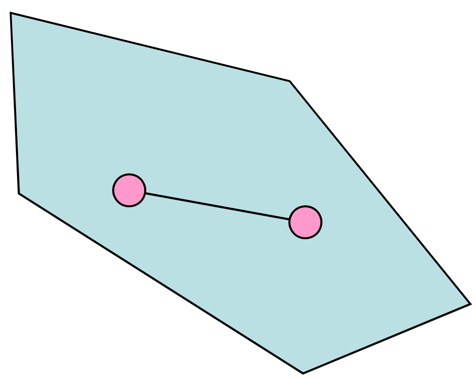
- G^n continuity:

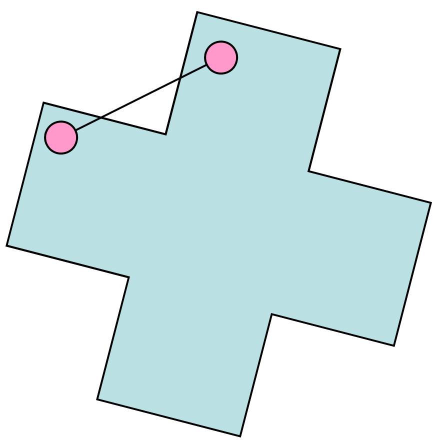
$$dp^n(1)/dt^n = \alpha\, dq^n(0)/dt^n$$

- For example, C^2 continuity is the acceleration continuity
- Two Bezier curves with degree d have C^k continuity automatically, with $k > d + 1$

Convex Objects

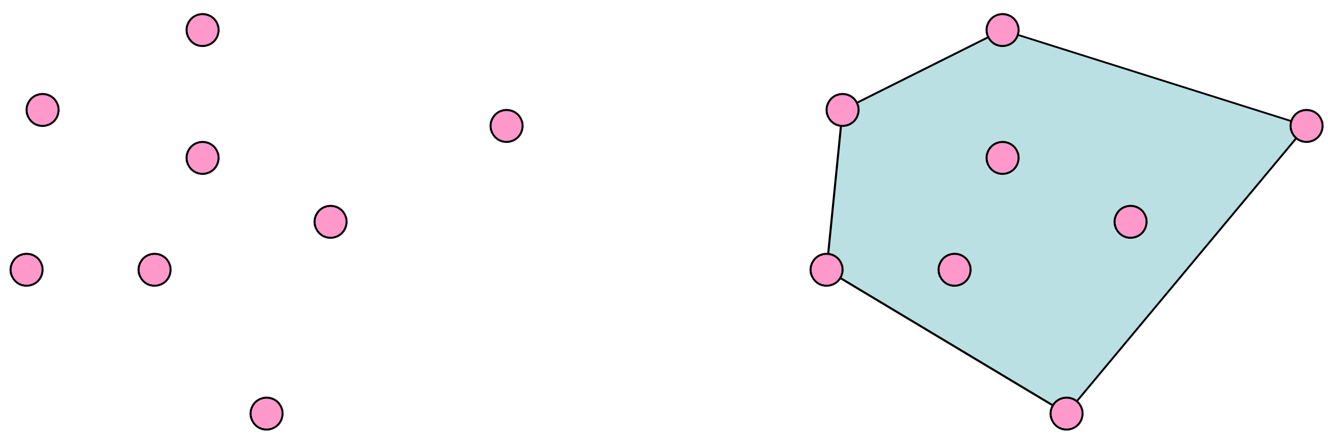
- Convexity: an object is ***convex*** if..
  - Pick any two point *x*, *y* from the object
  - any point *p* between *x* and *y* are inside the object
- This definition can be applied to any dimension





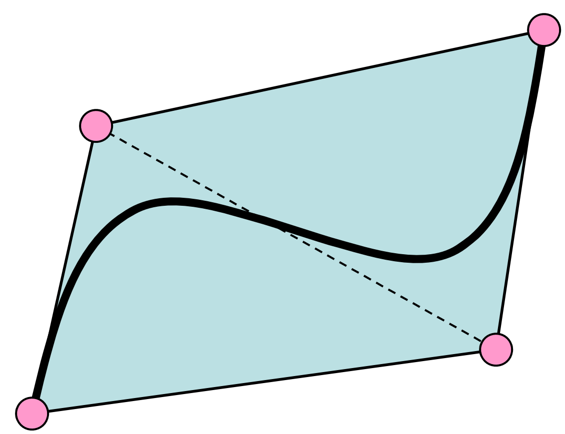
Convex Hull of a Set of Points

- Given a set of points . . .
- the **<u>convex hull</u>** of them is the smallest convex object that enclose all the points



Convex Hull Properties of Bezier Curves

- Given a Bezier curve with its control points
- The whole curve will be inside the convex hull of the control points
- This is useful for
  - Screen clipping
  - Intersection testing

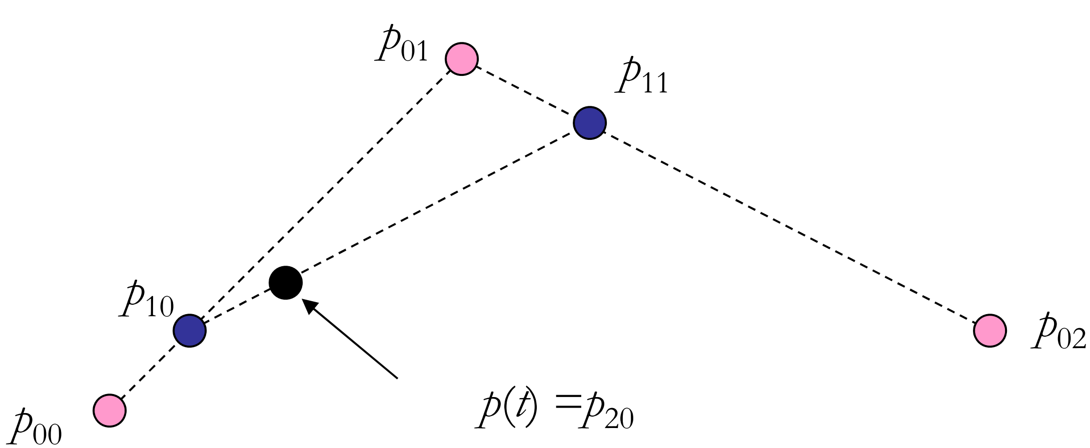


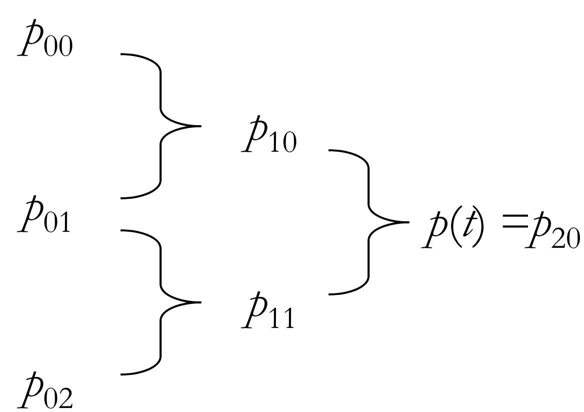
How to Draw/Compute a Bezier Curve?

<u>Iterative</u> vs <u>Divide and Conquer</u>

How to find a certain point $p(t)$

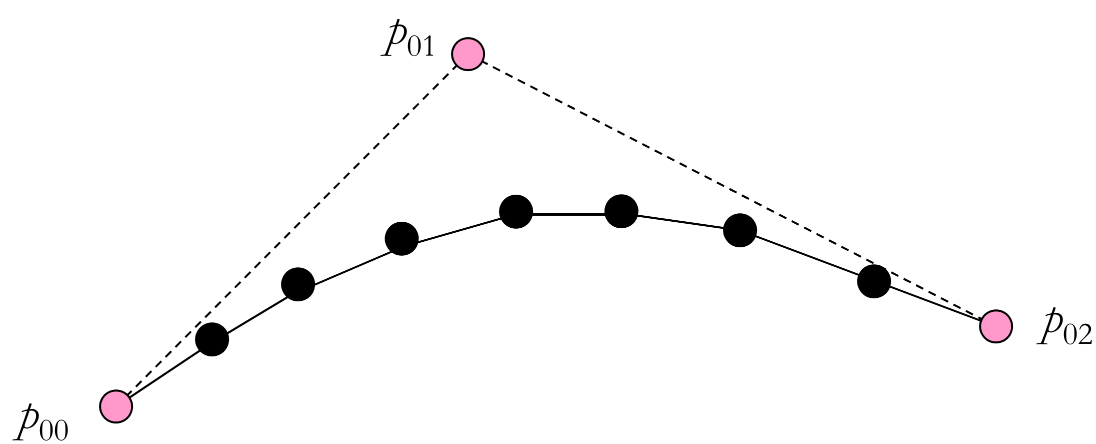
- Given a number t_0, how to compute the exact position of $p(t)$?
  I. Evaluate the equation
  II. By recursive subdivision





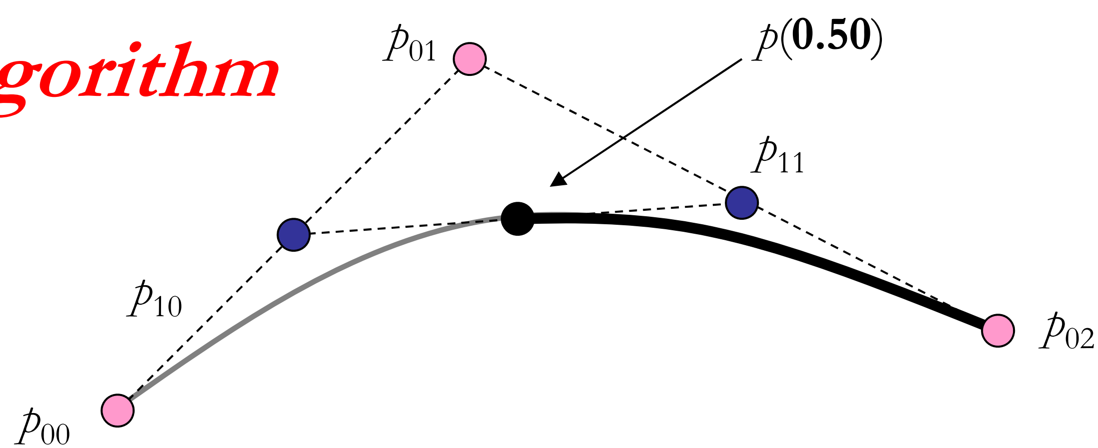
How to Draw a Bezier Curve

I. Simplest/iterative method:
   - Compute $p_i = p(i/m)$ for $i = 1$ to m
   - Connect each pair of p_i and p_{i+1} with a straight line

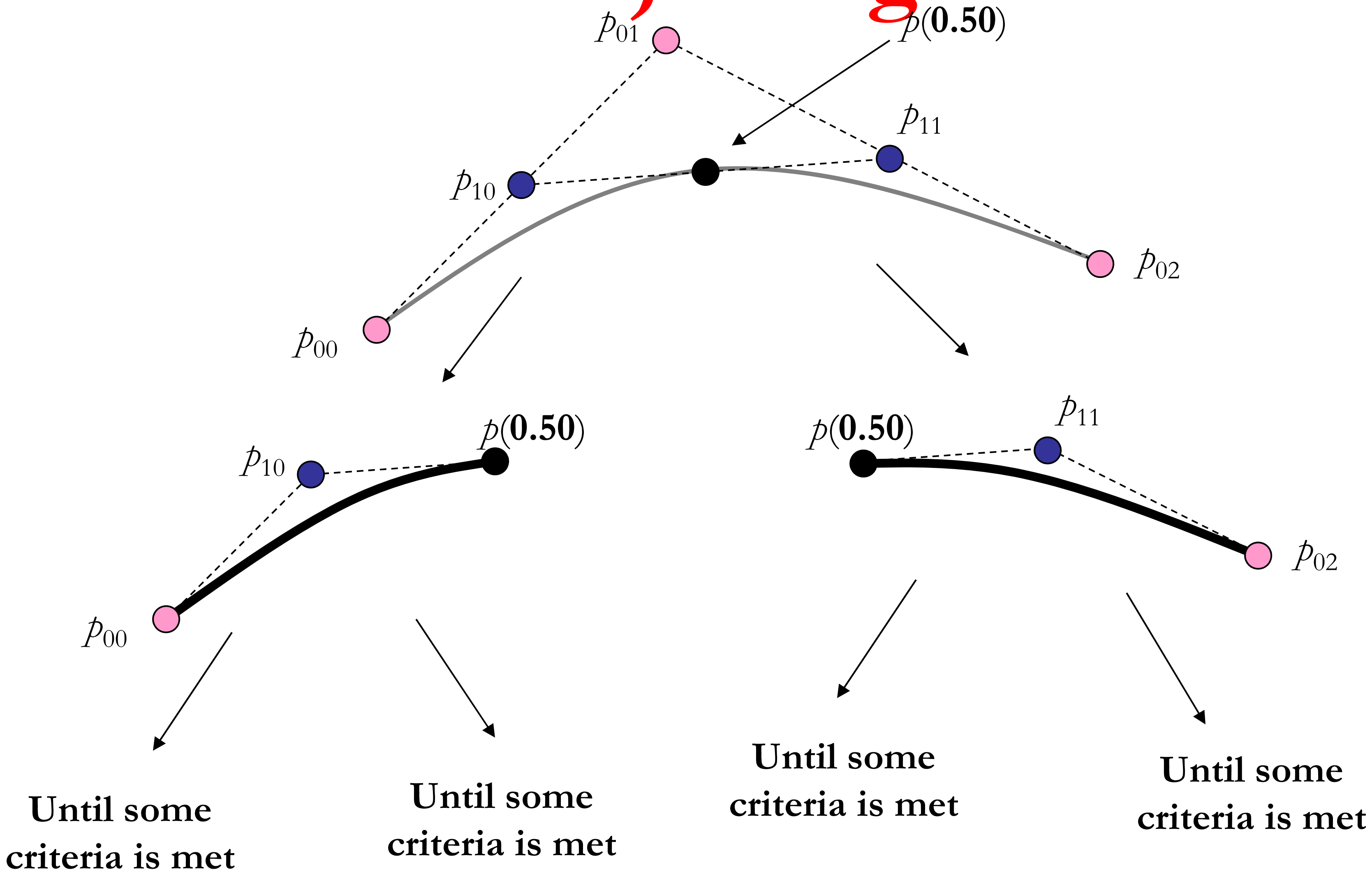


II. Divide and Conquer Method

- A divide and conquer method for drawing a Bezier Curve
- The three points $p(t)$, p_{11} and p_{02} form the same curve on the second half
- Recursion until some conditions are satisfied
- ***De Casteljau Algorithm***

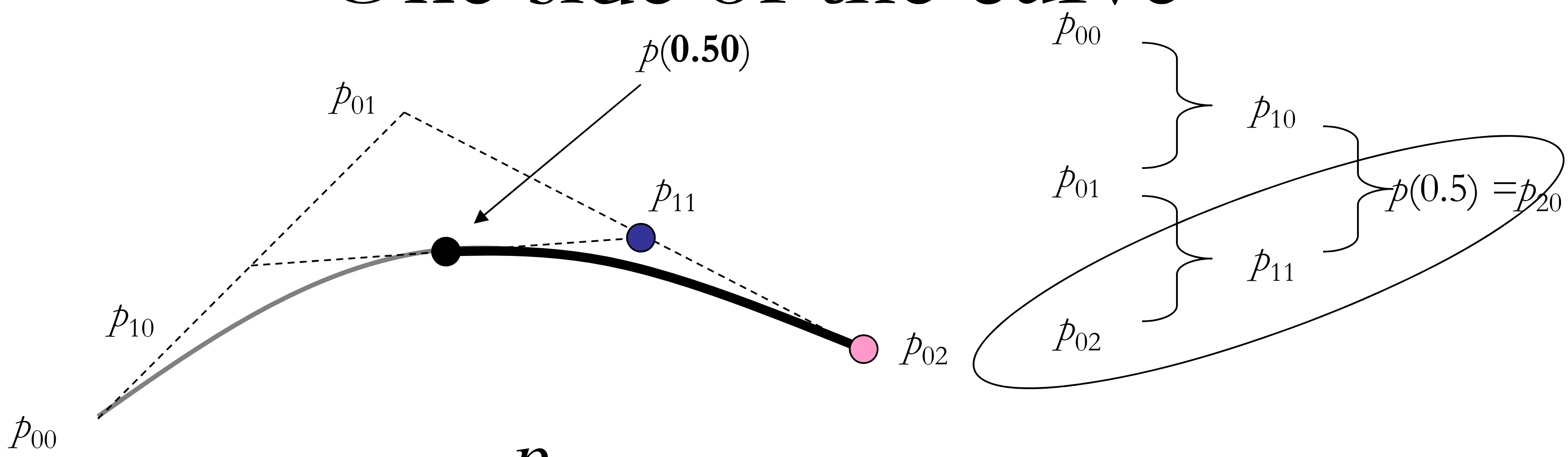


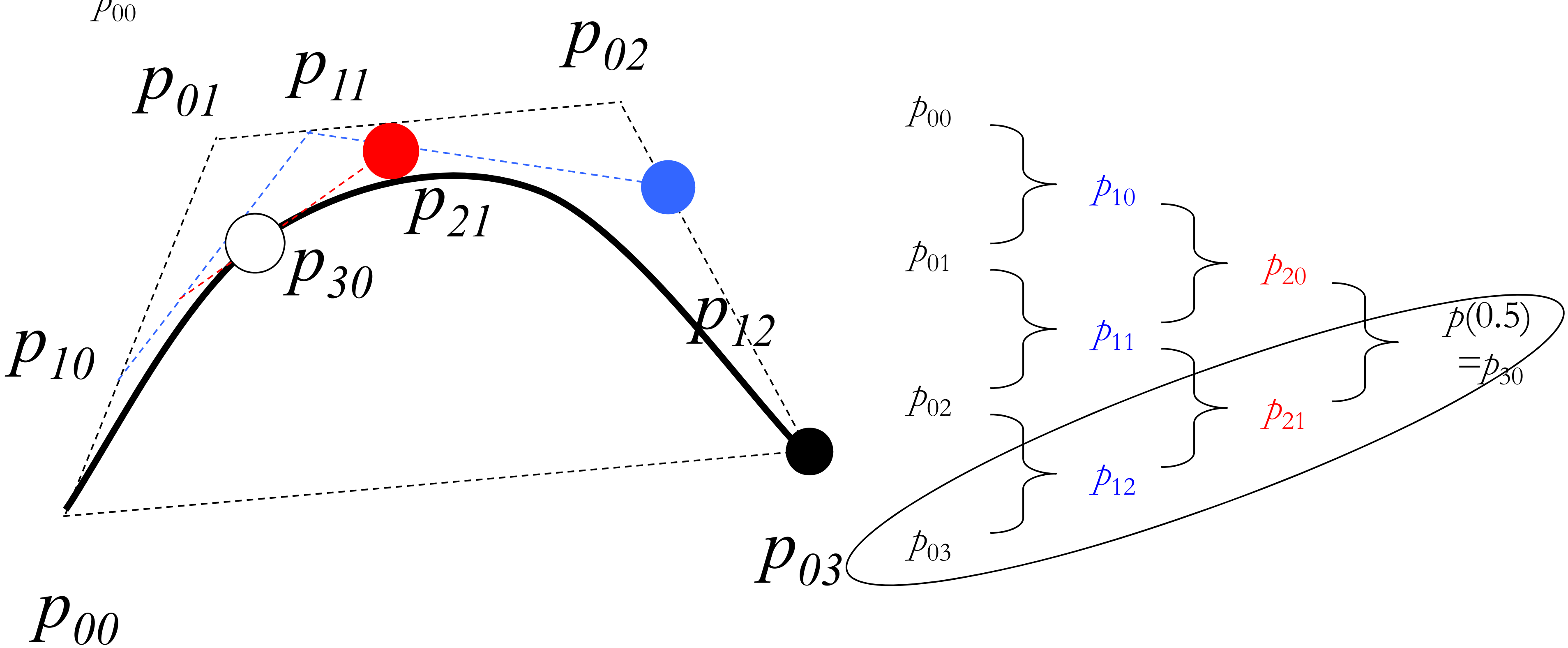
De Casteljau Algorithm



DEMO: http://demonstrations.wolfram.com/BezierCurveByDeCasteljausAlgorithm/

One side of the curve





Bezier Curves in OpenGL

Bezier Curve in OpenGL

- OpenGL supports Bezier curves through mechanisms called ***evaluators*** that are used to compute the blending functions, $b_i(u)$, of any degree.

- ***Evaluators*** do not require uniform spacing of u. Bezier curves can then be rendered at any precision.

- 1D Bezier curves can also be used to define paths in time for animation

```
e.g.   /*  define and enable 1D evaluator for Bezier
           cubic curve  */
point  ctrlpts[ ]  =  { ... ... ... ... ... } ;

glMaplf  (GL_MAP1_VERTEX_3, 0.0, 1.0, 3, 4, ctrlpts);
glEnable (GL_MAP1_VERTEX_3);

/* GL_MAP1_VERTEX_3 specifies data type for ctrlpts ,
   range of t = [ 0.0, 1.0],  3 is the number of values
   between control points , (order = degree +1) = 4  */


/*  With evaluator enabled, draw line segments for
    Bezier curve  */
 glBegin (GL_LINE_STRIP);
        for ( i = 0; i <= 30;  i ++)
             glEvalCoord1f ( (Glfloat)  i/30.0);
 glEnd ( )
```

Conclusion of Curves

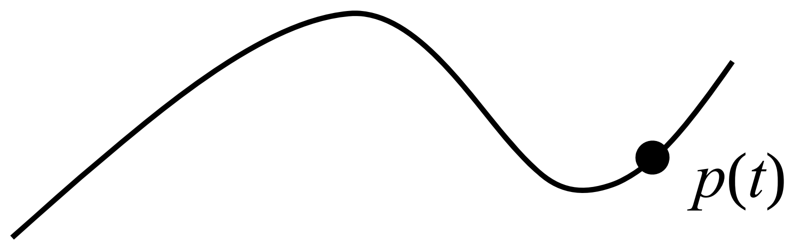
- The Beizer Curves are only **<u>one</u>** type of parametric curves
  – e.g. Rational Beizer Curves, B-splines, NURBS, etc
- A complicated curve can be formed by joining several low degree Bezier curves
- Good for drawing purpose, or modeling the path of an object

Bezier Surfaces

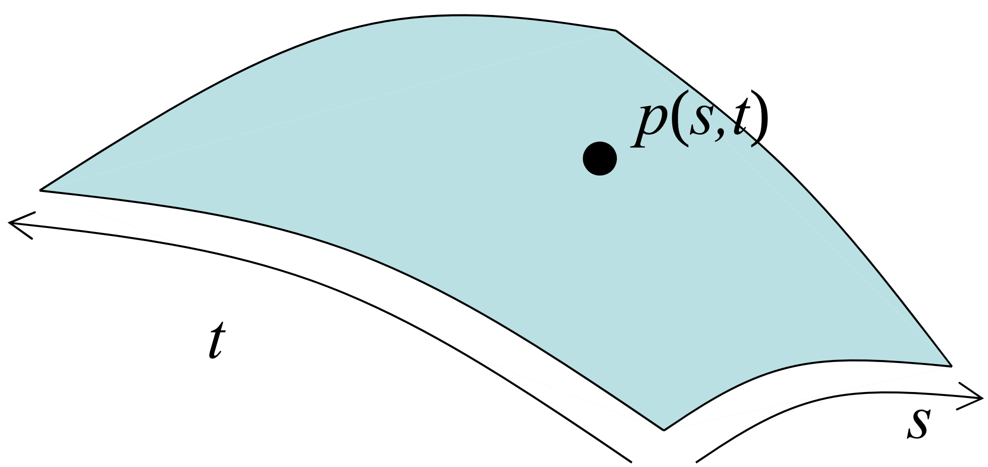
Extending Bezier Curves to Bezier Surfaces

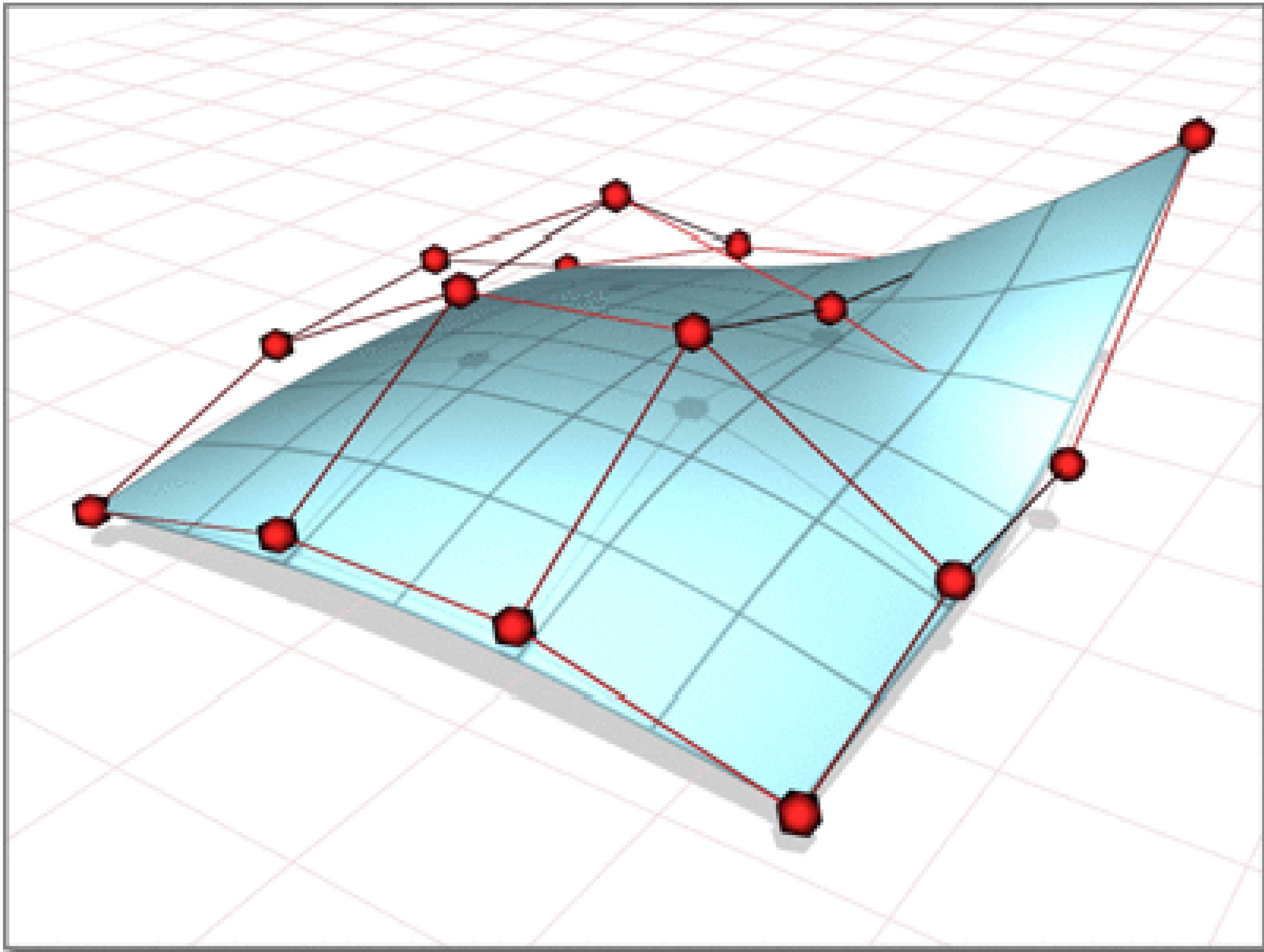
Parametric Surfaces

- A parametric curve is specified by a parameter *t*

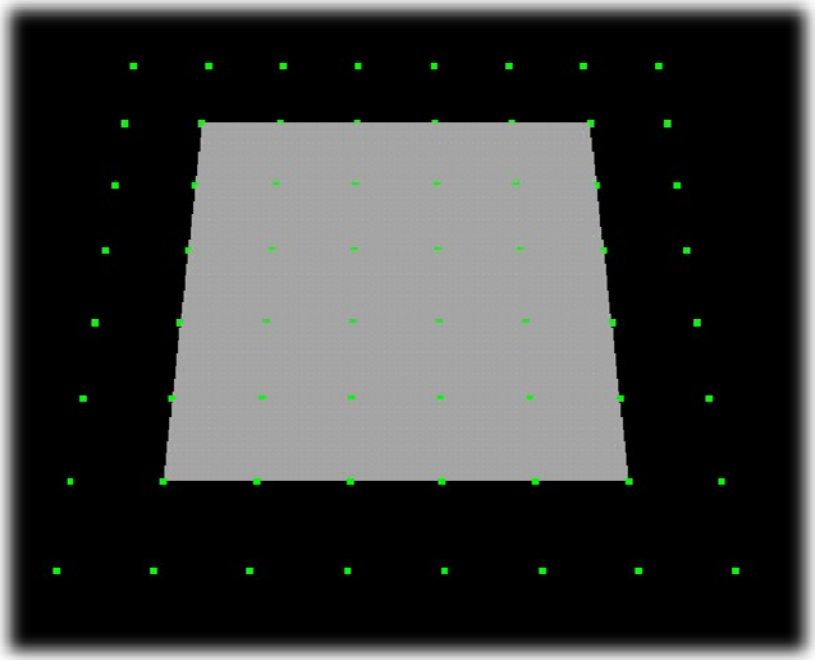


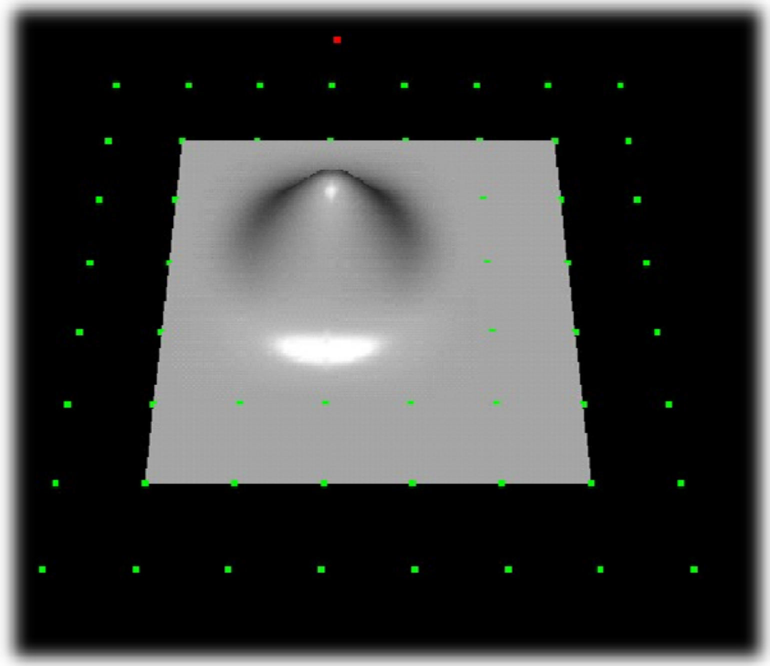
- A parametric surface is parametised by two parameters *s,t*





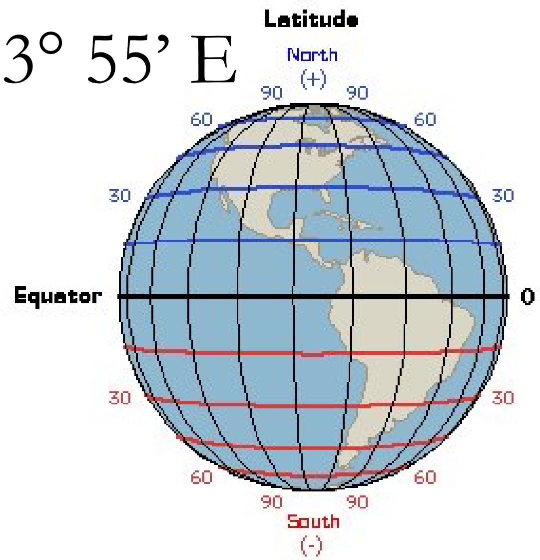
Control Points of a Surface

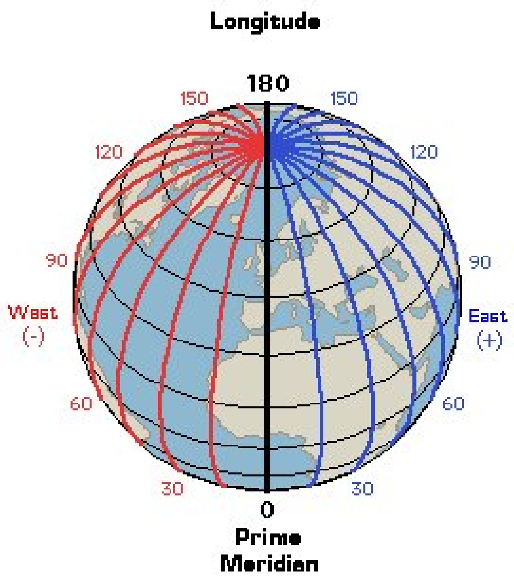




Parametrization

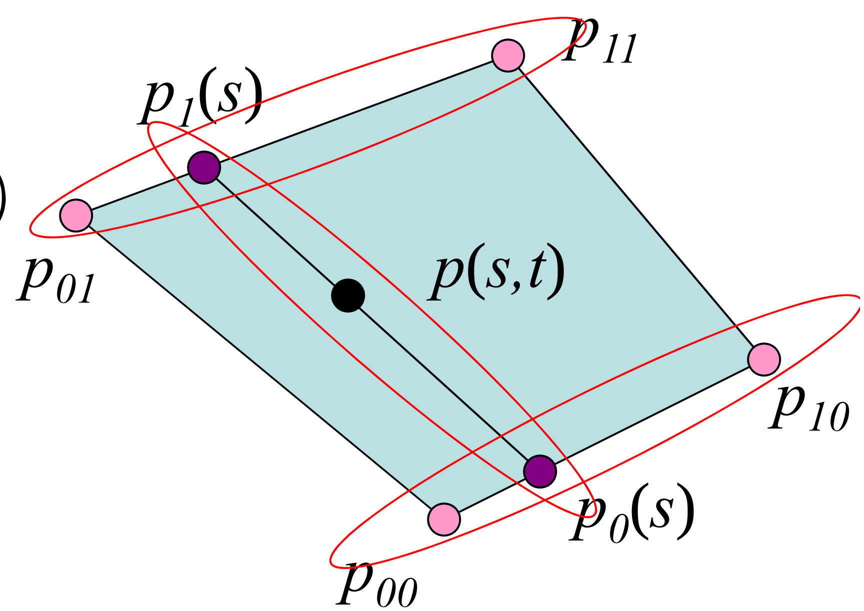
- A point on a surface is parametrized by two numbers
- Singapore:
  - Latitude = s = 1° 22’ N
  - Longitude = t = 103° 55’ E





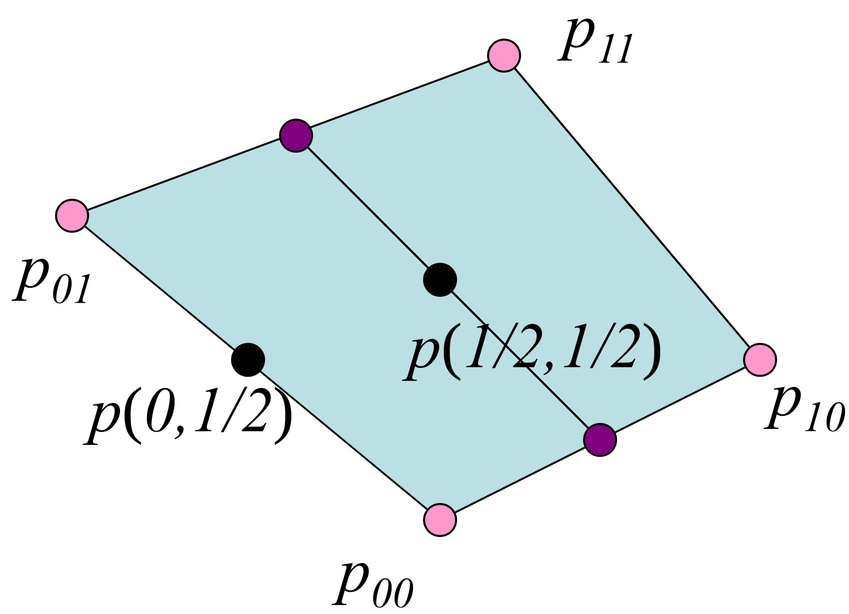
Parametric Surfaces

- For example, a parametric quadrilateral
- Given four vertices p_{00}, p_{01}, p_{10}, and p_{11}
- Assuming they are coplanar
- $p_0(s) = (1-s)\,p_{00} + s\,p_{10}$
- $p_1(s) = (1-s)\,p_{01} + s\,p_{11}$
- $p(s,t) = (1-t)\,p_0(s) + t\,p_1(s)$



Parametric Surfaces

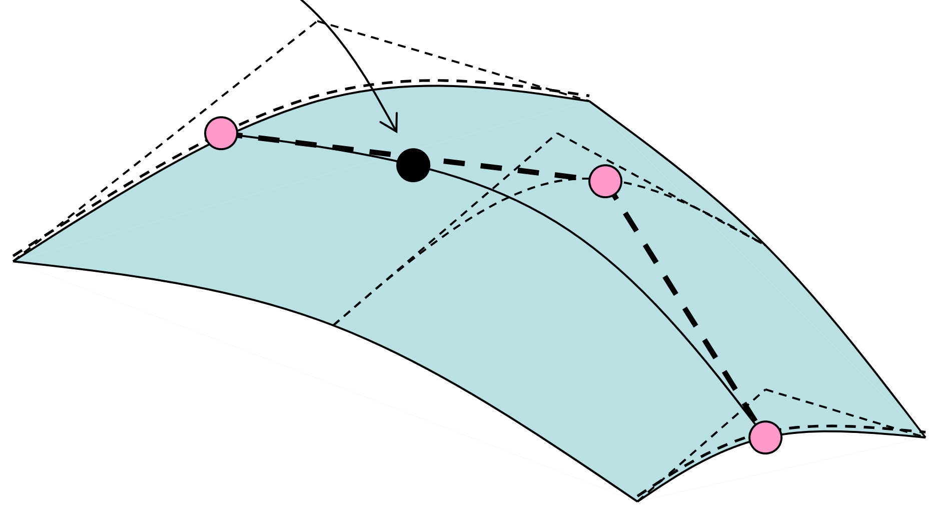
- For example
  - $p(0,0) = p_{00}$
  - $p(0,1/2) = (p_{00} + p_{01})/2$
  - $p(1/2,1/2)$: mid point of every corner



2 x 2 Bezier Surfaces

- A first quadratic Bezier curve $p_0(s)$ with control points p_{00} , p_{01} and p_{02}
- Second and third quadratic Bezier curves $p_i(s)$ with control points p_{i0} , p_{i1} and p_{i2} for i = 1,2

$$p(s,t) = \sum_{i=0}^{2} b_{2,i}(t)\, p_i(s)$$

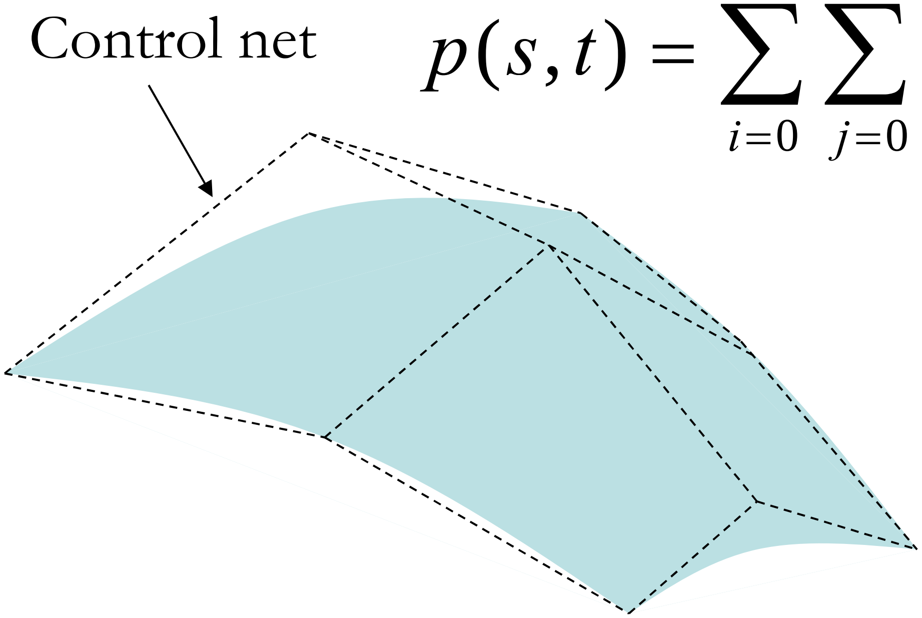


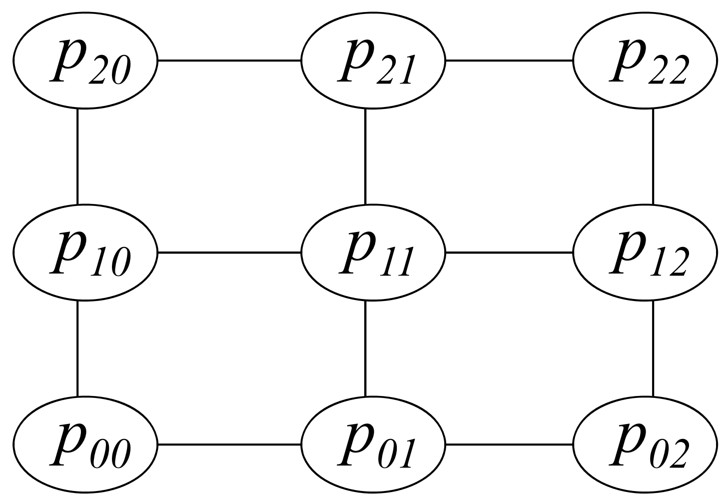
2 × 2 Bezier Surfaces

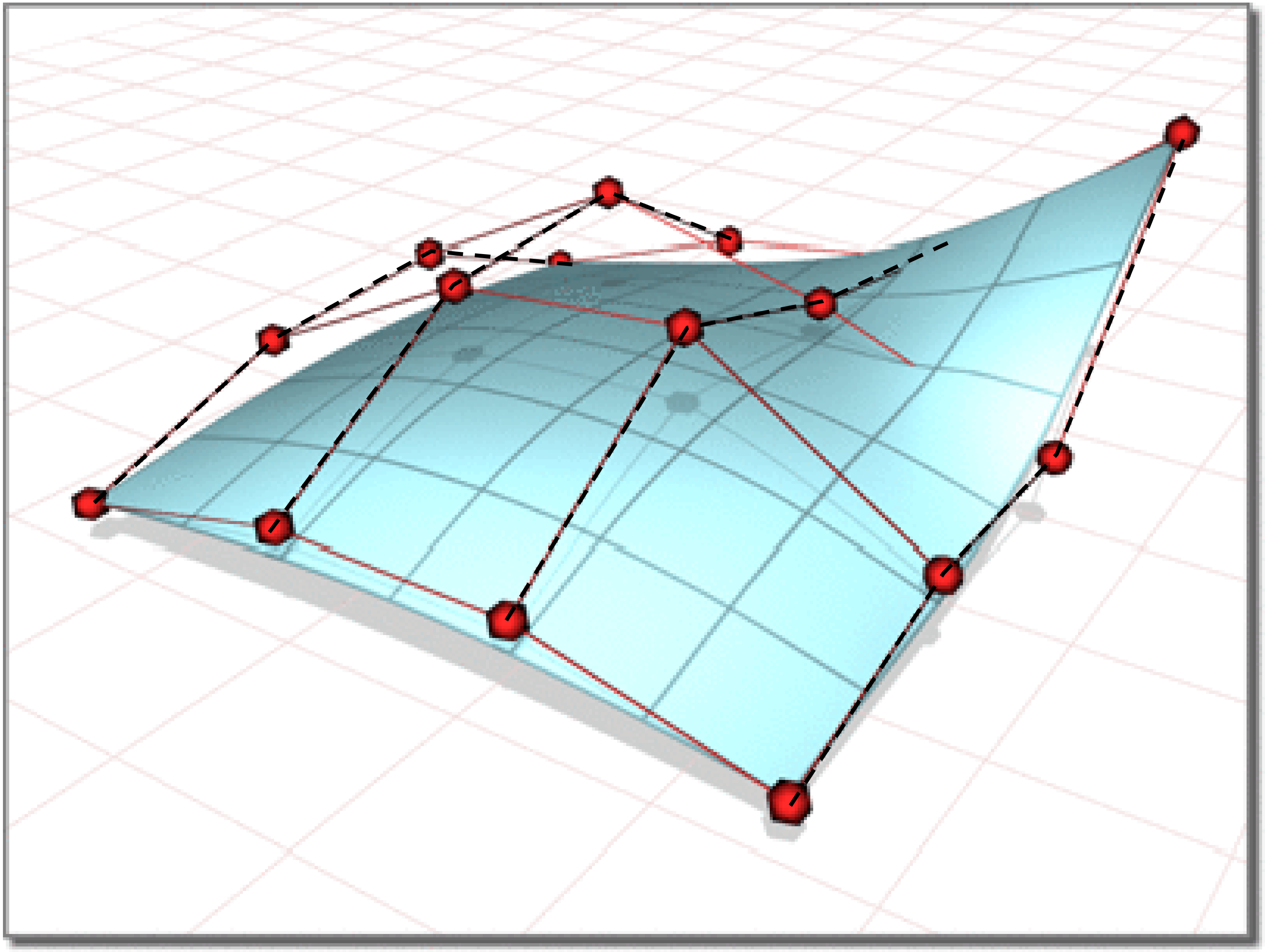
- for i = 0 to 2

$$p_i(s) = \sum_{j=0}^{2} b_{2,j}(s)\, p_{ij} \qquad\qquad p(s,t) = \sum_{i=0}^{2} b_{2,i}(t)\, p_i(s)$$

$$p(s,t) = \sum_{i=0}^{2} \sum_{j=0}^{2} b_{2,i}(t)\, b_{2,j}(s)\, p_{ij}$$





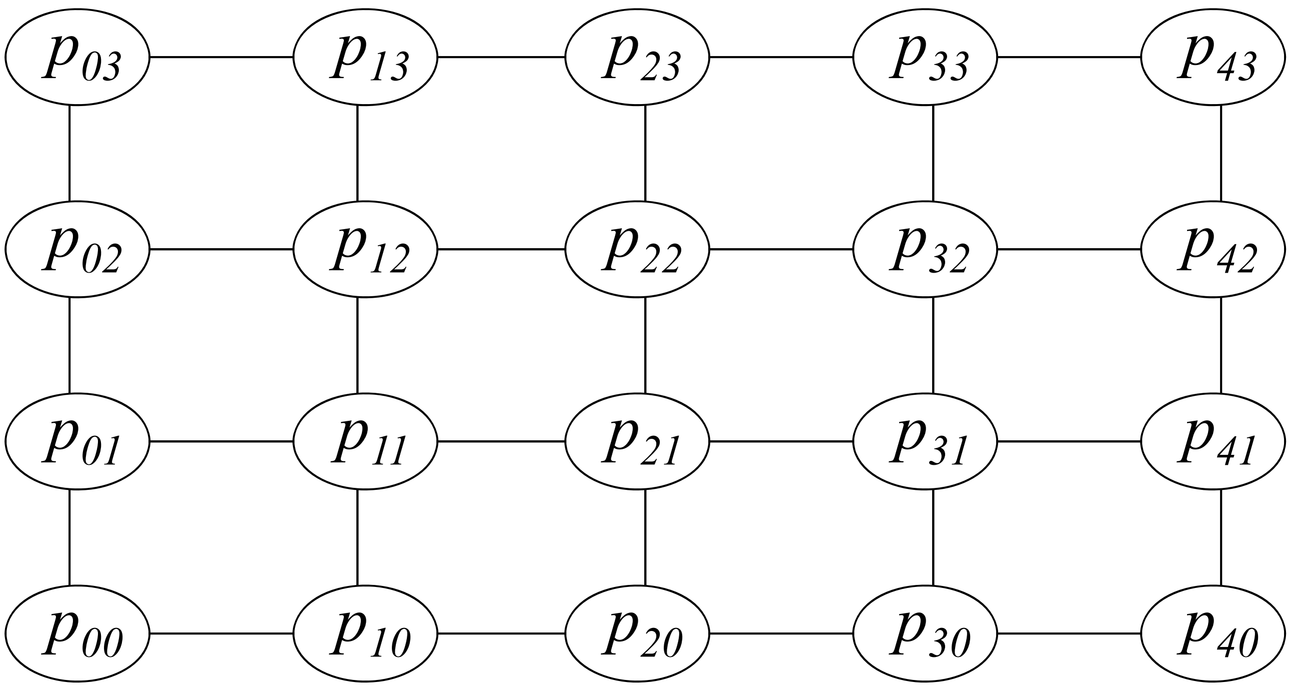


n × m Bezier Curves

- Use m+1 degree n Bezier Curves :

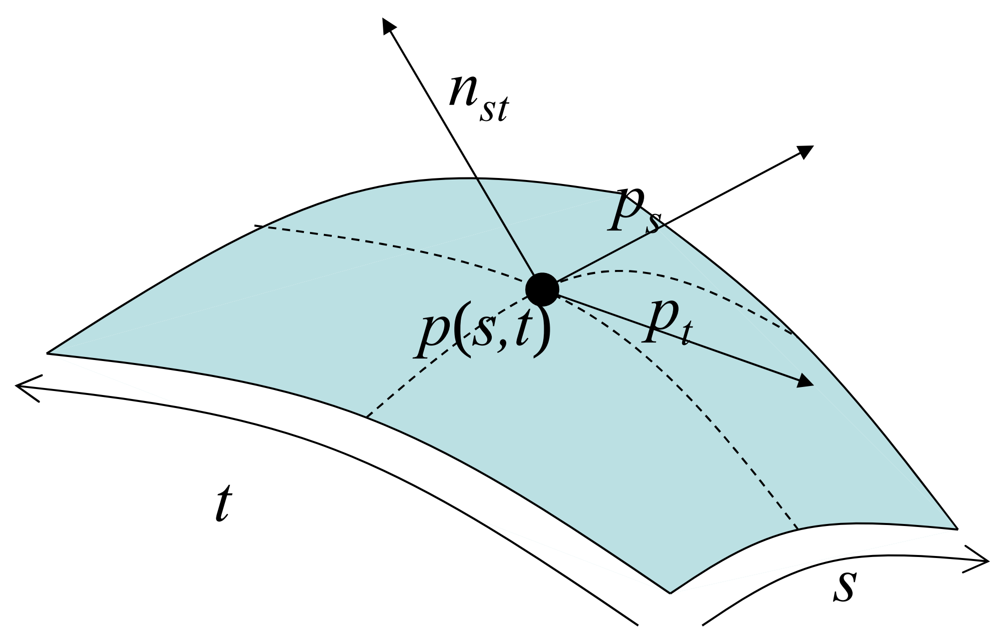
$$p(s,t) = \sum_{i=0}^{m} \sum_{j=0}^{n} b_{m,i}(t) b_{n,j}(s) p_{ij}$$

- e.g. a 3 × 4 Bezier control net:

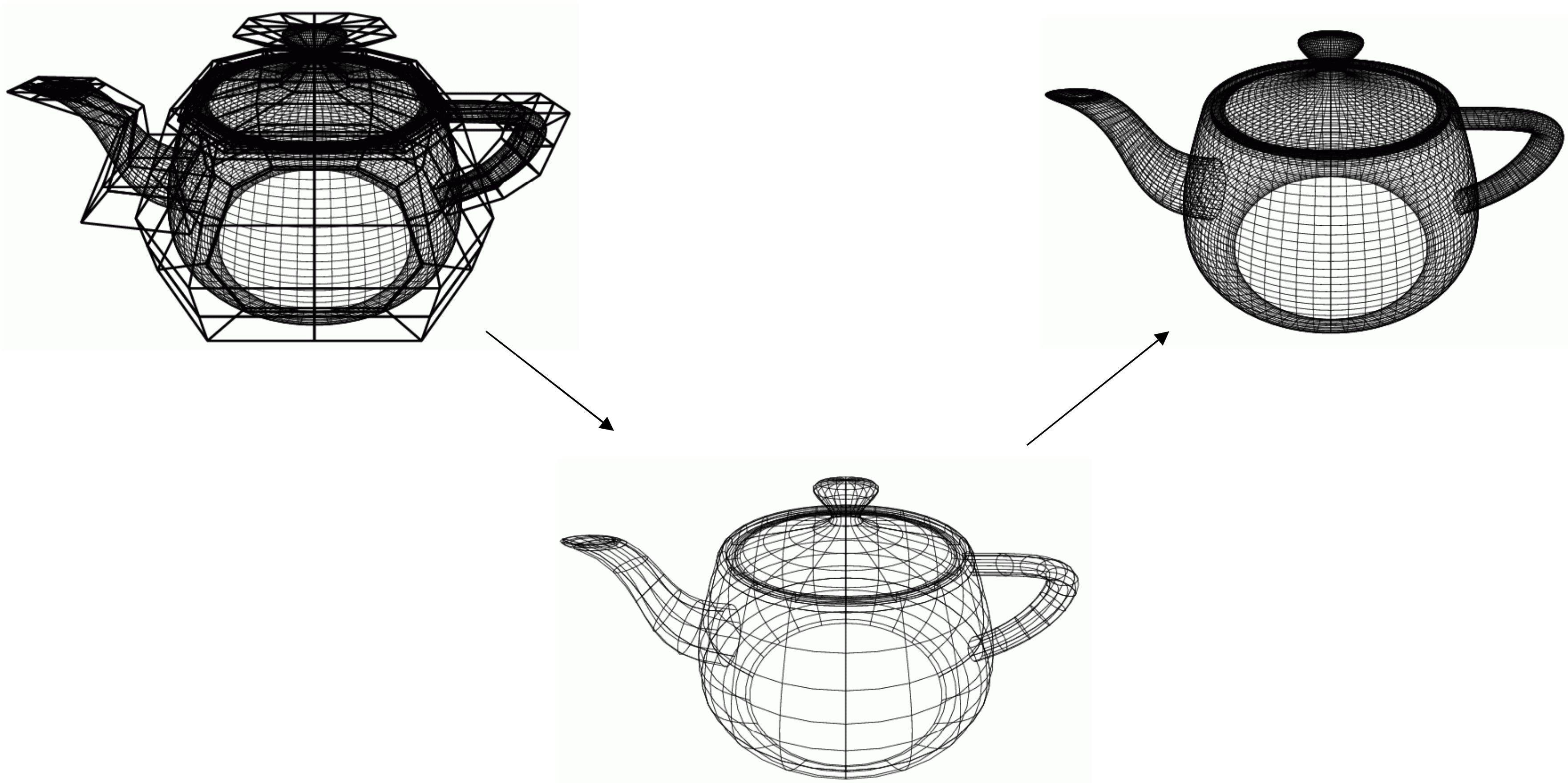


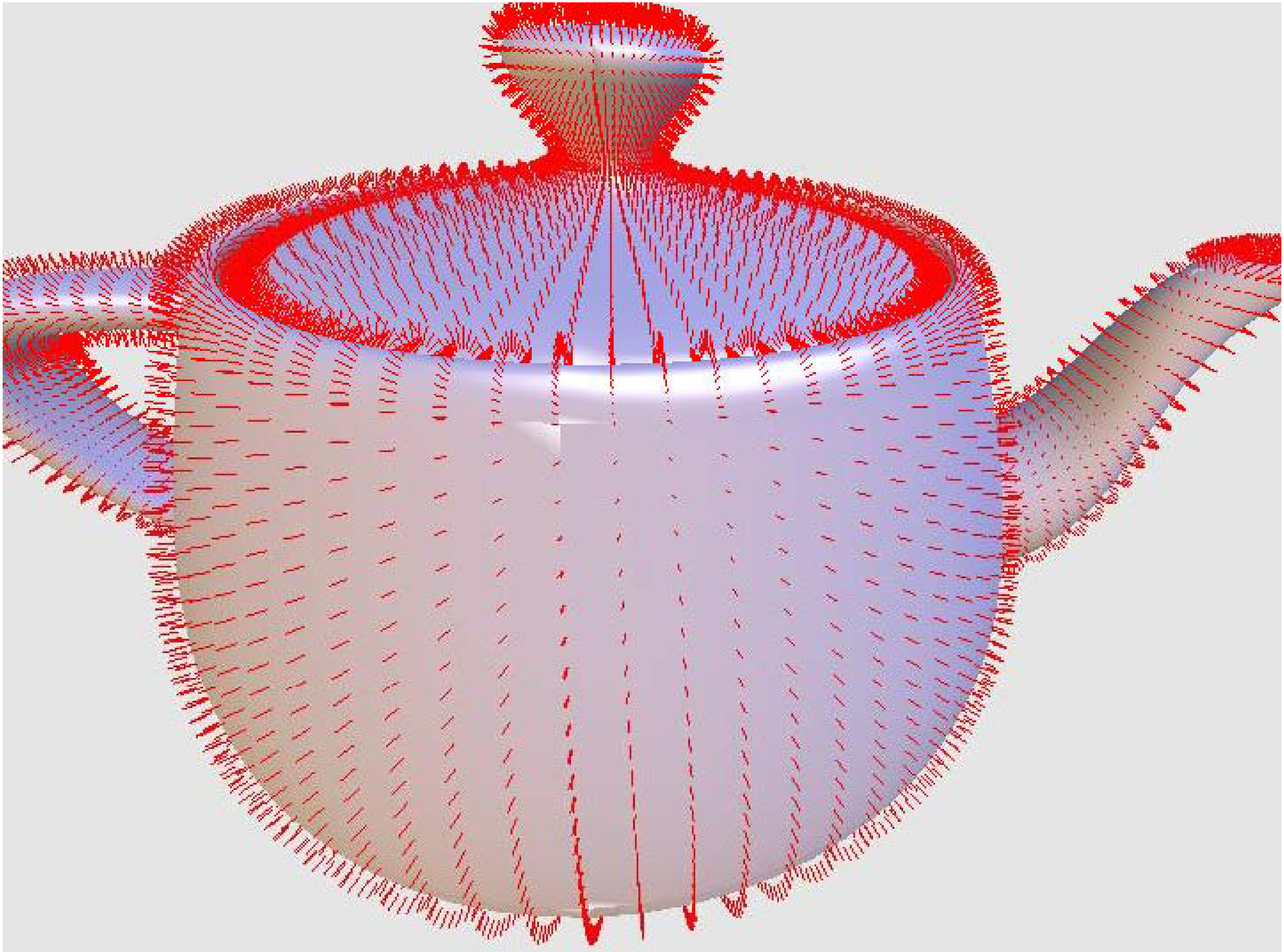
Tangent and Normal Vectors

- Treat s as a constant, $p(s,t)$ is a curve in parameter t, its tangent vector p_t is $\dfrac{\partial p(s,t)}{\partial t}$
- Treat t as a constant, $p(s,t)$ is a curve in parameter s, its tangent vector p_s is $\dfrac{\partial p(s,t)}{\partial s}$
- The normal vector at $p(s,t)$ is $n_{st} = p_t \times p_s$



Example: The Utah Teapot






View Select Object
View Select Curve
View Select Key
Panels
Done

Conclusion of Bezier Surfaces

- Good for
  - Triangulation (Polygonization)
  - Normal computation
  - CAD/CAM, or CG movies modeling
- Bad for
  - Ray tracing

Annoucement

- Lab 4 next week
  - Let's get twisted
- NO tutorial next week
  - Because of good Friday
- Lab 3 voting
  - Please post your artwork onto FB