MA1505 (Chapters 7 - 11)
A compilation of relevant exam questions
from old MA1506 Syllabus

Chapter 7

1. **2002/2003 Sem 2 Q4(a)**

Find the local (or relative) maxima, minima and saddle points of the function

$$f(x, y) = \frac{1}{2}x^2 + 3y^3 + 9y^2 - 3xy + 9y - 9x.$$

Ans: saddle point $(3, -2)$, local min $(12, 1)$

1. **2002/2003 Sem 2 Q4(a)**

   Set $f_x = x - 3y - 9 = 0 - - - (1)$

   and $f_y = 9y^2 + 18y - 3x + 9 = 0 - - - (2)$.

   From (1), $x = 3y + 9$. Substitute into (2), we get

$$9y^2 + 18y - 3(3y + 9) + 9 = 0 \Rightarrow 9y^2 + 9y - 18 = 0$$

   which gives $y = -2$ or $y = 1$.

$y = -2 \Rightarrow x = 3y + 9 = 3.$

$y = 1 \Rightarrow x = 3y + 9 = 12.$

So critical points are $(3, -2)$ and $(12, 1)$.

2nd derivative test:

$$D(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = 18y + 9.$$

$D(3, -2) = -36 + 9 < 0.$

$D(12, 1) = 18 + 9 > 0$ and $f_{xx}(12, 1) = 1 > 0.$

So $(3, -2)$ is a saddle point and $(12, 1)$ is a local minimum point.

2. **2004/2005 Sem 1 Q3**

Let $f(x, y) = x^2y - y^2 + 2\sqrt{y}$.

(a) Find the domain of $f(x, y)$.

(b) Find the maximum rate of change of $f(x, y)$ at the point $(2, 1)$ and the direction in which it occurs.

(c) Find a unit vector $\mathbf{u}$ such that $D_{\mathbf{u}}f(2, 1) = -3$.

Ans: (a) $\{(x, y)|x \in \mathbb{R}, y \geq 0\}$; (b) $4\mathbf{i} + 3\mathbf{j}$, 5; (c) $(-24/25, 7/25)$

2. **2005/2006 Sem 1 Q3**

(a) domain of $f(x, y)$: $\{(x, y) : \ x \in \mathbb{R}, \ y \geq 0\}$.

(b) Let $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ be a unit vector.

$$\begin{aligned} D_{\mathbf{u}}f(2,1) &= f_x(2,1) \times a + f_y(2,1) \times b \\ &= (4\mathbf{i} + 3\mathbf{j}) \bullet (a\mathbf{i} + b\mathbf{j}) \\ &= \|4\mathbf{i} + 3\mathbf{j}\| \ \|\mathbf{u}\| \cos\theta \end{aligned}$$

where θ is the angle between $4\mathbf{i} + 3\mathbf{j}$ and $\mathbf{u}$.

Since $\|\mathbf{u}\| = 1$ and the largest value of $\cos\theta$ is 1, the maximum rate of change is $\|4\mathbf{i} + 3\mathbf{j}\| = 5$.

This maximum rate occurs when $\theta = 0$, this means that $\mathbf{u}$ is in the same direction as $4\mathbf{i} + 3\mathbf{j}$.

(c) Let $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$. Then

$$D_{\mathbf{u}}f(2,1) = f_x(2,1) \times a + f_y(2,1) \times b \Rightarrow 4a + 3b = -3.$$

Since $\mathbf{u}$ is a unit vector, $a^2 + b^2 = 1$.

On solving, we get $(a, b) = (0, -1)$ or $(-24/25, 7/25)$.

Chapter 8

1. **2002/2003 Sem 2 Q6**

   (a) Evaluate $\displaystyle\iint_R (3 - x^2 - 2y^2)\, dA$, where R is the region in the xy-plane given by $x^2 + y^2 \leq 1$.

   (b) Evaluate $\displaystyle\int_0^6 \left[\int_{x/3}^{2} x\ \sqrt[3]{y^3+1}\ dy\right]\, dx$.

   Ans: (a) $9\pi/4$; (b) $\frac{9}{8}(9\sqrt[3]{9} - 1)$

1. **2002/2003 Sem 2 Q6**

   (a) Use polar coordinates to describe R:

$$0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi.$$

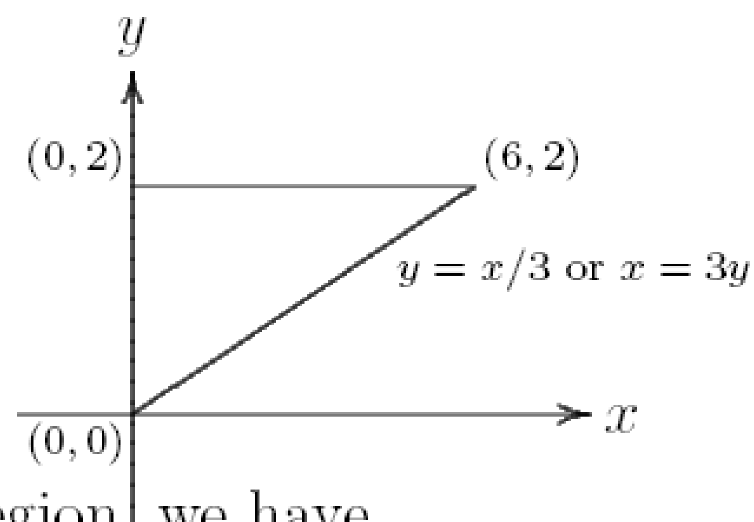
   So

$$\begin{aligned}
\iint_R (3 - x^2 - 2y^2)dA &= \int_0^{2\pi} \int_0^1 (3 - r^2\cos^2\theta - 2r^2\sin^2\theta) r\,dr\,d\theta \\
&= \int_0^{2\pi} \int_0^1 [3 - r^2(1 + \sin^2\theta)] r\,dr\,d\theta \\
&= \int_0^{2\pi} \int_0^1 3r - r^3 \left(\frac{3}{2} - \frac{1}{2}\cos 2\theta\right) dr\,d\theta
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{2\pi} \left[\frac{3}{2}r^2 - \frac{3}{8}r^4 + \frac{1}{8}r^3 \cos^2\theta\right]_0^1 d\theta \\
&= \int_0^{2\pi} \left[\frac{9}{8} + \frac{1}{8}\cos^2\theta\right] d\theta \\
&= \left[\frac{9}{8}\theta + \frac{1}{16}\sin^2\theta\right]_0^{2\pi} = 9\pi/4
\end{aligned}
$$

(b) The given iterated integral is over a type A region:

$$R : \frac{x}{3} \leq y \leq 2, \quad 0 \leq x \leq 6$$



Convert it to a type B region, we have

$$R : 0 \leq x \leq 3y, \quad 0 \leq y \leq 2$$

$$
\begin{aligned}
& \int_0^6 \left[\int_{x/3}^2 x\sqrt[3]{y^3+1}\,dy\right] dx \\
= & \int_0^2 \left[\int_0^{3y} x\sqrt[3]{y^3+1}\,dx\right] dy \\
= & \int_0^2 \left[\frac{x^2}{2}\sqrt[3]{y^3+1}\right]_0^{3y} dy \\
= & \int_0^2 \frac{9y^2}{2}\sqrt[3]{y^3+1}\,dy
\end{aligned}
$$

Change of variable $u = y^3 \Rightarrow \dfrac{du}{dy} = 3y^2$.

So the integral becomes

$$
\begin{aligned}
& \frac{3}{2}\int_{u=0}^{u=8} (u+1)^{1/3} du \\
= \; & \frac{3}{2}\left[\frac{3}{4}(u+1)^{4/3}\right]_{u=0}^{u=8} du \\
= \; & \frac{3}{2}\left[\frac{3}{4}(9)^{4/3} - \frac{3}{4}\right] \\
= \; & \frac{9}{8}\left[9\sqrt[3]{9} - 1\right]
\end{aligned}
$$

3. **2005/2006 Sem 1 Q5**

Let P be the plane given by $z = k$. Assume that $0 < k < 1$, so that P intersects the unit sphere centered at the origin in some curve C at height k. Let S denote the part of the sphere lying above the plane P, which has boundary C.

(a) Find a parametrization for the curve C, and describe the projection of S onto the xy-plane. Your answers will depend on k.

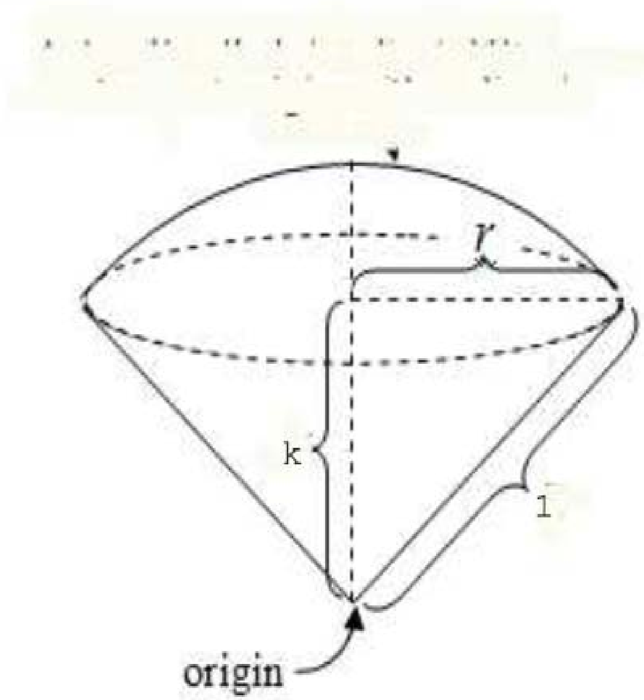
(b) Write down and evaluate an integral which calculates the surface area of S in terms of k.

(c) Find the value of k for which the surface area of S is equal to π.

Ans: (a)$r(t) = \sqrt{1-k^2}\cos t\mathbf{i} + \sqrt{1-k^2}\sin t\mathbf{j} + k\mathbf{k}$, circle of radius $\sqrt{1-k^2}$;
(b) $2\pi(1-k)$; (c) $1/2$;

3. **2005/2006 Sem 1 Q5**

(a) The horizontal plane intersect the unit sphere at a circle. The radius r of this circle can be computed as $\sqrt{1-k^2}$ using Pythagoras Theorem as shown in diagram.



So a parametrization for C: $\mathbf{r}(t) = \sqrt{1-k^2}\cos t\mathbf{i} + \sqrt{1-k^2}\sin t\mathbf{j} + k\mathbf{k}$.

The projection of S onto the xy-plane is the circle of radius $\sqrt{1-k^2}$.

(b) $f(x, y) = \sqrt{1-x^2-y^2}$, $\quad f_x = \frac{-x}{\sqrt{1-x^2-y^2}}$, $\quad f_y = \frac{-y}{\sqrt{1-x^2-y^2}}$.

Surface area $A = \iint_R \sqrt{f_x^2 + f_y^2 + 1}dA$ where R is the disk $x^2 + y^2 \leq 1 - k^2$.

In polar coordinates,

$A = \int_0^{2\pi} \int_0^{\sqrt{1-k^2}} \frac{1}{\sqrt{1-r^2}} r\ dr\ dA = 2\pi \int_0^{\sqrt{1-k^2}} \frac{r}{\sqrt{1-r^2}} dr = 2\pi(1-k)$.

(c) By solving $2\pi(1-k) = \pi$, we get $k = 1/2$.

Chapter 9

1. **2002/2003 Sem 2 Q8(b)**

   Let
   $$\mathbf{F} = (e^y - ze^x)\mathbf{i} + xe^y\mathbf{j} - e^x\mathbf{k}.$$
   Find f so that
   $$\nabla f = \mathbf{F}.$$
   Evaluate
   $$\int_C \mathbf{F} \bullet d\mathbf{r}$$
   on C, the line segment on the z-axis from (0,0;0) to (0,0,1).

   Ans: -1

1. **2002/2003 Sem 2 Q8(b)**

Equating ∇f with $\mathbf{F}$ and compare components:

$f_x = e^y - ze^x \quad (1)$

$f_y = xe^y \qquad\qquad (2)$

$f_z = -e^x \qquad\qquad (3)$

$$
\begin{aligned}
(2) \quad &\Rightarrow \quad f = xe^y + h(x, z) - - - (4) \\
&\Rightarrow \quad f_z = h_z \\
&\Rightarrow \quad h_z = -e^x \quad \text{from } (3) \\
&\Rightarrow \quad h(x, z) = -ze^x + g(x)
\end{aligned}
$$

$$\begin{aligned}
(4) \quad \Rightarrow \quad & f = xe^y - ze^x + g(x) \\
\Rightarrow \quad & f_x = e^y - ze^x + g'(x) \\
\Rightarrow \quad & e^y - ze^x = e^y - ze^x + g'(x) \quad \text{from (1)} \\
\Rightarrow \quad & g'(x) = 0 \\
\Rightarrow \quad & g(x) = c
\end{aligned}$$

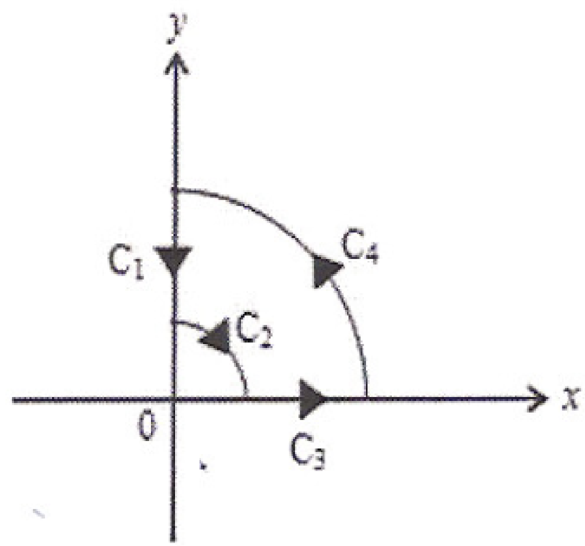
So $f(x, y, z) = xe^y - ze^x + c$.

Since $\mathbf{F} = \nabla f$, the line integral $\int_C \mathbf{F} \bullet d\mathbf{r}$ is independent of path. Thus

$$\int_C \mathbf{F} \bullet d\mathbf{r} = f(0,0,1) - f(0,0,0) = (0e^0 - 1e^0 + c) - (0e^0 - 0e^0 + c) = -1$$

2. **2004/2005 Sem 2 Q4**

A closed curve with <u>positive</u> orientation is made up of four curves C_1, C_2, C_3 and C_4 as shown in the diagram below.



C_2 is the portion of the unit circle $x^2 + y^2 = 1$ that lies in the first quadrant. C_4 is the portion of the circle $x^2 + y^2 = 9$ that lies in the first quadrant. Evaluate the following line integrals:

(a) $\displaystyle\oint_{C_1+C_2+C_3+C_4} \left(\frac{-y}{x^2+y^2} + y^2\right) dx + \left(\frac{x}{x^2+y^2} - x^2\right) dy,$

(b) $\displaystyle\int_{C_1+C_2+C_3} \left(\frac{-y}{x^2+y^2} + y^2\right) dx + \left(\frac{x}{x^2+y^2} - x^2\right) dy.$

Ans: (a) $-104/3$; (b) $\frac{4}{3} - \frac{\pi}{2}$

2. **2004/2005 Sem 2 Q4**

(a) By Green's theorem,

$$\oint_C \left(\frac{-y}{x^2+y^2}+y^2\right)dx+\left(\frac{x}{x^2+y^2}-x^2\right)dy$$

$$=\iint_D \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}-x^2\right)-\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}+y^2\right)dA$$

$$=\iint_D \left(\frac{y^2-x^2}{(x^2+y^2)^2}-2x\right)-\left(\frac{y^2-x^2}{(x^2+y^2)^2}+2y\right)dA$$

$$=\iint_D -2(x+y)\,dA.$$

The region D is given in polar coordinates by:

$$0 \leq \theta \leq \frac{\pi}{2}, \quad 1 \leq r \leq 3.$$

Converting to polar coordinates, we have

$$\iint_D -2(x+y)\ dA = \int_0^{\pi/2} \int_1^3 -2(r\cos\theta + r\sin\theta)r\ dr d\theta = -\frac{104}{3}.$$

(b) Note that $\int_{C_1+C_2+C_3} = \oint_{C_1+C_2+C_3+C_4} - \int_{C_4}$. So we only need to compute the line integral

$$\int_{C_4} \left(\frac{-y}{x^2+y^2} + y^2 \right) dx + \left(\frac{x}{x^2+y^2} - x^2 \right) dy,$$

and then subtract the result from the answer to (a). We can rewrite this line integral as $\int_{C_4} \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}$ is the vector field

$$\mathbf{F} = \left(\frac{-y}{x^2+y^2} + y^2 \right)\mathbf{i} + \left(\frac{x}{x^2+y^2} - x^2 \right)\mathbf{j}.$$

First, we need to parametrize the curve C_4. The easiest way to do this is to take $x = 3\cos\theta$, $y = 3\sin\theta$, $0 \le \theta \le \frac{\pi}{2}$. The corresponding vector equation of C_4 is:

$$\mathbf{r}(\theta) = 3\cos\theta\ \mathbf{i} + 3\sin\theta\ \mathbf{j}, \quad 0 \le \theta \le \frac{\pi}{2},$$

$$\mathbf{r}'(\theta) = -3\sin\theta\ \mathbf{i} + 3\cos\theta\mathbf{j}.$$

$$\mathbf{F}(\mathbf{r}(\theta)) = \left(\frac{-3\sin\theta}{9\cos^2\theta + 9\sin^2\theta} + 9\sin^2\theta\right)\mathbf{i} + \left(\frac{3\cos\theta}{9\cos^2\theta + 9\sin^2\theta} - 9\cos^2\theta\right)\mathbf{j}$$

$$= \left(\frac{-\sin\theta}{3} + 9\sin^2\theta\right)\mathbf{i} + \left(\frac{\cos\theta}{3} - 9\cos^2\theta\right)\mathbf{j}.$$

We have

$$\int_{C_4} \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi/2} \mathbf{F}(\mathbf{r}(\theta))\cdot\mathbf{r}'(\theta)\, d\theta = \int_0^{\pi/2} \left(\sin^2\theta - 27\sin^3\theta + \cos^2\theta - 27\cos^3\theta\right) d\theta$$

$$= \int_0^{\pi/2} 1 - 27(\sin^3\theta + \cos^3\theta)\, d\theta = \frac{\pi}{2} - 36.$$

Using the result of (a), we have

$$\int_{C_1+C_2+C_3} \mathbf{F} \cdot d\mathbf{r} = -\frac{104}{3} - \left(\frac{\pi}{2} - 36\right) = \frac{4}{3} - \frac{\pi}{2}.$$

3. 2005/2006 Sem 1 Q1

(a) Find a potential function for the gradient vector field

$$\mathbf{F} = e^x\,\mathbf{i} + \frac{z}{y}\,\mathbf{j} + \ln y\,\mathbf{k},$$

and evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is given by the vector function

$$\mathbf{r}(t) = t\,\mathbf{i} + (t^2+1)\,\mathbf{j} + (t^3+2)\,\mathbf{k},$$

for $0 \leq t \leq 1$.

Ans: (a) $e + 3\ln 2 - 1$

3. **2005/2006 Sem 1 Q1**

(a) Equating ∇f with $\mathbf{F}$ and compare components:

$$f_x = e^x \qquad (1)$$

$$f_y = z/y \qquad (2)$$

$$f_z = \ln y \qquad (3)$$

$$
\begin{aligned}
(1) \quad &\Rightarrow \quad f = e^x + h(y,z) - - - (4) \\
&\Rightarrow \quad f_z = h_z \\
&\Rightarrow \quad h_z = \ln y \quad \text{from (3)} \\
&\Rightarrow \quad h(y,z) = z\ln y + g(y) \\
(4) \quad &\Rightarrow \quad f = e^x + z\ln y + g(y) \\
&\Rightarrow \quad f_y = z/y + g'(y) \\
&\Rightarrow \quad z/y = z/y + g'(y) \quad \text{from (2)} \\
&\Rightarrow \quad g'(y) = 0 \\
&\Rightarrow \quad g(y) = c
\end{aligned}
$$

So $f(x,y,z) = e^x + z \ln y + c$.

$\mathbf{r}(0) = 0\ \mathbf{i} + 1\ \mathbf{j} + 2\ \mathbf{k}$ and $\mathbf{r}(1) = 1\ \mathbf{i} + 2\ \mathbf{j} + 3\ \mathbf{k}$.

By Fundamental Theorem of line integral,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(1,2,3) - f(0,1,2) = e + 3\ln 2 - 1$$

4. **2005/2006 Sem 2 Q1**

Use Green's Theorem to evaluate

$$\oint_C (1 + 10xy + y^2)dx + (6xy + 5x^2)dy,$$

where C is the positively oriented triangle with vertices at $(0,0)$, $(a,0)$ and $(0,a)$ with $a > 0$.

Ans: $\frac{2}{3}a^3$

4. **2005/2006 Sem 2 Q1**

By Green's Theorem,

$$
\begin{aligned}
& \oint_C (1 + 10xy + y^2)dx + (6xy + 5x^2)dy \\
= & \iint_D \frac{\partial}{\partial x}(6xy + 5x^2) - \frac{\partial}{\partial y}(1 + 10xy + y^2)dA \\
= & \iint_D (6y + 10x) - (10x - 2y)dA \\
= & \iint_D 4y dA
\end{aligned}
$$

where D is the triangular region bounded by C.

Draw the diagram of D and we see the three sides of D are x-axis (from 0 to a), y-axis (from 0 to a) and the line $x + y = a$. This last equation can be obtained from the two points $(0, a)$ and $(a, 0)$.

So we can describe D as Type A region:

$$0 \leq y \leq a - x, \quad 0 \leq x \leq a.$$

Continuing with the computation of the integral:

$$\begin{aligned}\iint_D 4y\,dA &= \int_0^a \int_0^{a-x} 4y\,dy\,dx \\ &= \int_0^a 2(a-x)^2\,dx \\ &= \frac{2}{3}a^3\end{aligned}$$

Chapter 10

1. **2002/2003 Sem 2 Q8(a)**

If $\mathbf{r}$ is the position vector of any point in R^3 and $\mathbf{a} = \mathbf{i} + \mathbf{j} + 0\mathbf{k}$, compute

$$\operatorname{div}(\mathbf{a} \times \mathbf{r}).$$

Ans: 0

1. **2002/2003 Sem 2 Q8(a)**

$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{a} = 1\mathbf{i} + 1\mathbf{j} + 0\mathbf{k}$. Then

$$\mathbf{a} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ x & y & z \end{vmatrix} = z\mathbf{i} - z\mathbf{j} + (y - x)\mathbf{k}$$

$$\operatorname{div}(\mathbf{a} \times \mathbf{r}) = \frac{\partial z}{\partial x} + \frac{\partial(-z)}{\partial y} + \frac{\partial(y - x)}{\partial z} = 0.$$

3. **2003/2004 Sem 2 Q8**

Let $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$.

(i) Find curl $\mathbf{F}$.

(ii) Use Stoke's Theorem to evaluate the line integral $\int_C \mathbf{F} \bullet d\mathbf{r}$ where C is the triangle with vertices $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ oriented counter-clockwise when viewed from above.

Ans: $-y\mathbf{i} - z\mathbf{j} - x\mathbf{k}$; (ii) $-1/2$

3. **2003/2004 Sem 2 Q8**

(i) $$\mathrm{curl}(F) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix} = -y\mathbf{i} - z\mathbf{j} - x\mathbf{k}.$$

(ii) Stoke's Theorem:

$$\oint_C \mathbf{F} \bullet d\mathbf{r} = \iint_S \mathrm{curl}\mathbf{F} \bullet d\mathbf{S}$$

where S can be taken as the plane containing $(1,0,0), (0,1,0), (0,0,1)$. So an equation of S can be found to be $x + y + z = 1$ or $z = 1 - x - y$.

A parametric equation of S is given by

$$r(u, v) = u\mathbf{i} + v\mathbf{j} + (1 - u - v)\mathbf{k}, \quad (u, v) \in D$$

where D is the projection of the portion of S in first octant onto the xy-plane.

So D is given by $0 \leq v \leq 1-u, \quad 0 \leq u \leq 1$.

We check that $\mathbf{r}_u \times \mathbf{r}_v = \mathbf{i}+\mathbf{j}+\mathbf{k}$ gives an upward pointing normal vector, which agrees with the orientation of the boundary curve C (counter-clockwise).

Now curl $\mathbf{F}(\mathbf{r}) = -v\mathbf{i} - (1-u-v)\mathbf{j} - u\mathbf{k}$.

$$
\begin{aligned}
\iint_S \text{curl } \mathbf{F} \bullet d\mathbf{S} &= \iint_D [-v\mathbf{i} - (1-u-v)\mathbf{j} - u\mathbf{k}] \bullet [\mathbf{i}+\mathbf{j}+\mathbf{k}] dA \\
&= \int_0^1 \int_0^{1-u} (-v-(1-u-v)-u) dv du \\
&= \int_0^1 \int_0^{1-u} -1 dv du = -1/2
\end{aligned}
$$

4. **2004/2005 Sem 2 Q5**

Let S be the portion of the unit sphere $x^2 + y^2 + z^2 = 1$ in the first octant and let C be the boundary of S. The orientation of C is counterclockwise when looking down at the surface S. Find a vector field $\mathbf{G}(x, y, z)$ such that

$$\oint_C x^2\, dx + 2xy\, dy + xz\, dz = \iint_S \mathbf{G}(x, y, z) \bullet d\mathbf{S},$$

and evaluate the surface integral $\iint_S \mathbf{G}(x, y, z) \bullet d\mathbf{S}$ directly.

Ans: $-z\mathbf{j} + 2y\mathbf{k}$; $1/3$

4. **2004/2005 Sem 2 Q5**

By Stokes's theorem, $\mathbf{G} = \text{curl } \mathbf{F}$, where $\mathbf{F} = x^2\,\mathbf{i} + 2xy\,\mathbf{j} + xz\,\mathbf{k}$, and a straightforward computation shows that $\mathbf{G} = 0\,\mathbf{i} - z\,\mathbf{j} + 2y\,\mathbf{k}$. In order to evaluate the surface integral $\iint_S \mathbf{G} \cdot d\mathbf{S}$ directly, we first need to parametrize S.

$$\mathbf{r}(u,v) = \sin u \cos v\,\mathbf{i} + \sin u \sin v\,\mathbf{j} + \cos u\,\mathbf{k}, \quad 0 \le u \le \frac{\pi}{2}, \quad 0 \le v \le \frac{\pi}{2}.$$

To find the normal vector, compute

$$\mathbf{r}_u \times \mathbf{r}_v = \sin^2 u \cos v\,\mathbf{i} + \sin^2 u \sin v\,\mathbf{j} + \sin u \cos u\,\mathbf{k}.$$

Note that on the domain of $\mathbf{r}(u, v)$, $\mathbf{r}_u \times \mathbf{r}_v$ has nonnegative $\mathbf{k}$-component, so this is the upward-pointing normal vector. Since the orientation on C is counterclockwise, this normal vector gives the correct orientation on S. By definition,

$$\iint_S \mathbf{G} \cdot d\mathbf{S} = \iint_D \mathbf{G}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA,$$

where D is the region

$$0 \leq u \leq \frac{\pi}{2}, \quad 0 \leq v \leq \frac{\pi}{2}.$$

We have $\mathbf{G}(\mathbf{r}(u,v)) = -\cos u\ \mathbf{j} + 2\sin u \sin v\ \mathbf{k}$. Hence

$$\iint_D \mathbf{G}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\ dA$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \Big(-\cos u\,\mathbf{j} + 2\sin u \sin v\,\mathbf{k} \Big) \cdot \Big(\sin^2 u \cos v\,\mathbf{i} + \sin^2 u \sin v\,\mathbf{j} + \sin u \cos u\,\mathbf{k} \Big)\, du dv$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \big(-\sin^2 u \cos u \sin v + 2\sin^2 u \cos u \sin v \big)\ du dv$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \big(\sin^2 u \cos u \sin v \big)\ du dv = \frac{1}{3}.$$

6. **2005/2006 Sem 1 Q4**

Let $\mathbf{F} = y\,\mathbf{i} + xz\,\mathbf{j} + z^2\,\mathbf{k}$.

(a) Find curl $\mathbf{F}$.

(b) Use Stokes' theorem to evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the triangle with vertices $(-1, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 3)$, oriented *clockwise* when viewed from above.

Ans: (a) $-x\mathbf{i} + (z-1)\mathbf{j}$; (b) 1

6. **2005/2006 Sem 1 Q4**

(a) $$\operatorname{curl}(F) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & xz & z^2 \end{vmatrix} = -x\mathbf{i} + 0\mathbf{j} + (z-1)\mathbf{k}.$$

(b) Plane with boundary C is $-6x + 3y + 2z = 6$.

Parametrization: $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + (3 + 3u - \frac{3}{2}v)\mathbf{k}$

Correctly oriented normal vector: $-\mathbf{r}_u \times \mathbf{r}_v = 3\mathbf{i} - \frac{3}{2}\mathbf{j} - \mathbf{k}$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_R (-3u - 3u + \frac{3}{2}v - 2)dA$$

(where R is the projection of S onto xy plane)

$$= \int_{-1}^{0} \int_{0}^{2+2u} (-6u + \frac{3}{2}v - 2)dv\ du = 1.$$

8. **2005/2006 Sem 2 Q2**

Let S be the surface $x^2 + y^2 = 9, \quad 0 \le z \le 3$ oriented with outward normal vector. Compute the surface integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S},$$

where $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Ans: 54π

8. **2005/2006 Sem 2 Q2**

S is a cylinder with parametric equation:

$$\mathbf{r}(u, v) = 3\cos u\mathbf{i} + 3\sin u\mathbf{j} + v\mathbf{k}$$

where $0 \leq u \leq 2\pi$ and $0 \leq v \leq 3$.

$\mathbf{r}_u = -3\sin u\mathbf{i} + 3\cos u\mathbf{j} + 0\mathbf{k}$ and $\mathbf{r}_v = 0\mathbf{i} + 0\mathbf{j} + \mathbf{k}$.

$$\mathbf{r}_u \times \mathbf{r}_v = 3\cos u\mathbf{i} + 3\sin u\mathbf{j} + 0\mathbf{k}.$$

By testing the point $u = 0, v = 0$, we get $\mathbf{r}_u \times \mathbf{r}_v = 3\mathbf{i}+0\mathbf{j}+0\mathbf{k}$ which is a outer normal vector.

$$
\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^3 \int_0^{2\pi} (3\cos u\mathbf{i} + 3\sin u\mathbf{j} + v\mathbf{k}) \cdot (3\cos u\mathbf{i} + 3\sin u\mathbf{j} + 0\mathbf{k}) du dv \\
&= \int_0^3 \int_0^{2\pi} \left(9\cos^2 u + 9\sin^2 u\right) du dv \\
&= \int_0^3 \int_0^{2\pi} 9 du dv \\
&= 9 \times 2\pi \times 3 = 54\pi
\end{aligned}
$$

9. **2005/2006 Sem 2 Q3(i)**

Let $\mathbf{F}(x, y, z) = e^x\mathbf{i}+\cos y\mathbf{j}+2z\mathbf{k}$ and C the curve of intersection of the plane $2y+z=5$ and the cylinder $x^2+4y^2=4$, oriented counterclockwise when viewed from above.

Use Stoke's Theorem to evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$.

Ans: 0

9. **2005/2006 Sem 2 Q3(i)**

Check that the curl of $\mathbf{F}(x, y, z)$:

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & \cos y & 2z \end{vmatrix} = \mathbf{0}.$$

By Stoke's Theorem,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = 0.$$

Chapter 11

1. 2002/2003 Sem 2 Q10(a)

   Use the method of separation of variable to obtain solutions $u(x, y)$ of the equation

   $$u_x + u_y = 2(x + y)u.$$

   Ans: $u(x, y) = Ce^{kx-ky+x^2+y^2}$

Chapter 11

1. **2002/2003 Sem 2 Q10(a)**

   Let $u(x,y) = X(x)Y(y)$ so that $u_x = X'Y$ and $u_y = XY'$.

   Hence $u_x + u_y = 2(x+y)u$ becomes

   $$X'Y + XY' = 2(x+y)XY$$

   or

   $$\frac{X'}{X} - 2x = -\frac{Y'}{Y} + 2y = k$$

$$\frac{X'}{X} = k + 2x \Rightarrow \int \frac{dX}{X} = \int (k + 2x)dx \Rightarrow \ln|X| = kx + x^2 + c \Rightarrow X = Ae^{(k+x)x}$$

and

$$\frac{Y'}{Y} = -k + 2y \Rightarrow \int \frac{dY}{Y} = \int (-k + 2y)dy \Rightarrow \ln|Y| = -ky + y^2 + d \Rightarrow Y = Be^{(-k+y)y}$$

So

$$u(x, y) = Ce^{(kx-ky+x^2+y^2)}$$

where $C = AB$.

3. **2005/2006 Sem 2 Q4**

Use the method of separation of variables to find $u(x,y)$ that satisfies the partial differential equation

$$u_{xy} + \frac{\sin y}{x+2}u = 0,$$

given that $u\left(2, \frac{\pi}{2}\right) = 10$ and $u\left(7, \frac{\pi}{2}\right) = 15$.

Ans: $u(x,y) = 5(x+2)^{1/2}e^{\frac{1}{2}\cos y}$

3. **2005/2006 Sem 2 Q4**

Let $u(x, y) = X(x)Y(y)$. Then the p.d.e can be rewritten as:

$$\begin{aligned} u_{xy} \ + \ \frac{\sin y}{x+2}u \ &= \ 0 \\ \Rightarrow \quad X'(x)Y'(y) + \frac{\sin y}{x+2}X(x)Y(y) = 0 \\ \Rightarrow \quad X'(x)Y'(y) = -\frac{\sin y}{x+2}X(x)Y(y) \\ \Rightarrow \quad (x+2)\frac{X'}{X} = -\sin y\frac{Y}{Y'} \end{aligned}$$

The two sides of the last equation above gives two o.d.e's

$$\frac{X'}{X} = \frac{k}{(x+2)} \quad \text{and} \quad \frac{Y'}{Y} = -\frac{1}{k}\sin y.$$

Solve each of the two o.d.e's separately:

$$\int \frac{dX}{X} = \int \frac{k}{(x+2)} dx \Rightarrow \ln|X| = k\ln(x+2) + A \Rightarrow X = c_1(x+2)^k$$

and

$$\int \frac{dY}{Y} = \int -\frac{1}{k}\sin y dy \Rightarrow \ln|Y| = \frac{1}{k}\cos y + B \Rightarrow Y = c_2 e^{\frac{1}{k}\cos y}.$$

So $u(x,y) = C(x+2)^k e^{\frac{1}{k}\cos y}$.

To solve for C and k,

$$u(2, \frac{\pi}{2}) = C \cdot 4^k e^{\frac{1}{k}\cos\frac{\pi}{2}} = 10 \Rightarrow C \cdot 4^k = 10 - - - (1)$$

$$u(7, \frac{\pi}{2}) = C \cdot 9^k e^{\frac{1}{k}\cos\frac{\pi}{2}} = 15 \Rightarrow C \cdot 9^k = 15 - - - (2)$$

On solving (1) and (2) we get $C = 5$ and $k = 1/2$.

i.e. $u(x, y) = 5(x + 2)^{1/2} e^{2\cos y}$