Engineering Electromagnetics

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LECTURE 0

Chen Xudong

Dept. of Electrical and Computer Engineering
National University of Singapore

Introduction: Waves and Phasors

1. Review of Complex Numbers

Complex Number

$$z = x + jy$$

Where *x* and *y* are the *real* (Re) and *imaginary* (Im) parts of *z*, respectively

$$\boxed{j = \sqrt{-1}}$$

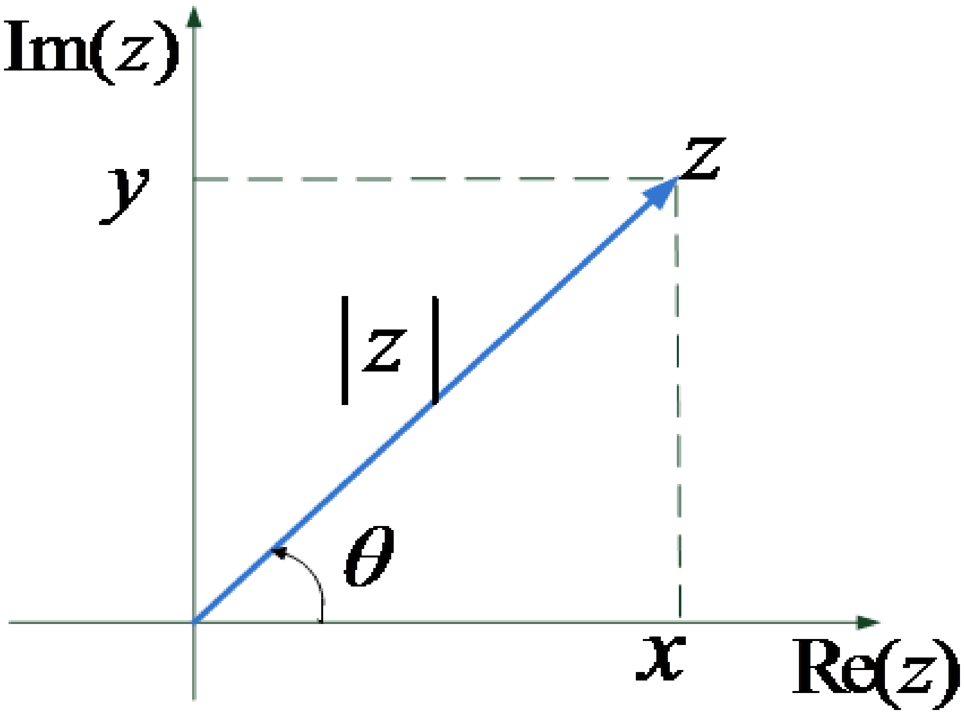
Polar Form of *z* is

$$z = |z| e^{j\theta} = |z| \angle \theta$$

Euler's identity is

$$\boxed{e^{j\theta} = \cos\theta + j\sin\theta}$$

Where |*z*| and θ are the *modulus* and *argument* of *z*, respectively



$$x = |z|\cos\theta$$
$$y = |z|\sin\theta$$
$$|z| = \sqrt[+]{x^2 + y^2}$$
$$\theta = \tan^{-1}(y/x)$$

Relationship between rectangular and polar representation of a complex number

Ensure that θ is in the proper quadrant

Complex Conjugate

$$z^* = (x + jy)^* = x - jy = |z| e^{-j\theta} = |z| \angle -\theta$$

The vector of *z and the vector of *z* are symmetric about the *x* axis**

Magnitude of *z* $\quad |z| = |z^*| = \sqrt{z\, z^*}$

Equality

If two complex numbers z_1 and z_2 are given by

$$z_1 = x_1 + jy_1 = |z_1| e^{j\theta_1} \qquad z_2 = x_2 + jy_2 = |z_2| e^{j\theta_2}$$

then $z_1 = z_2$ if and only if

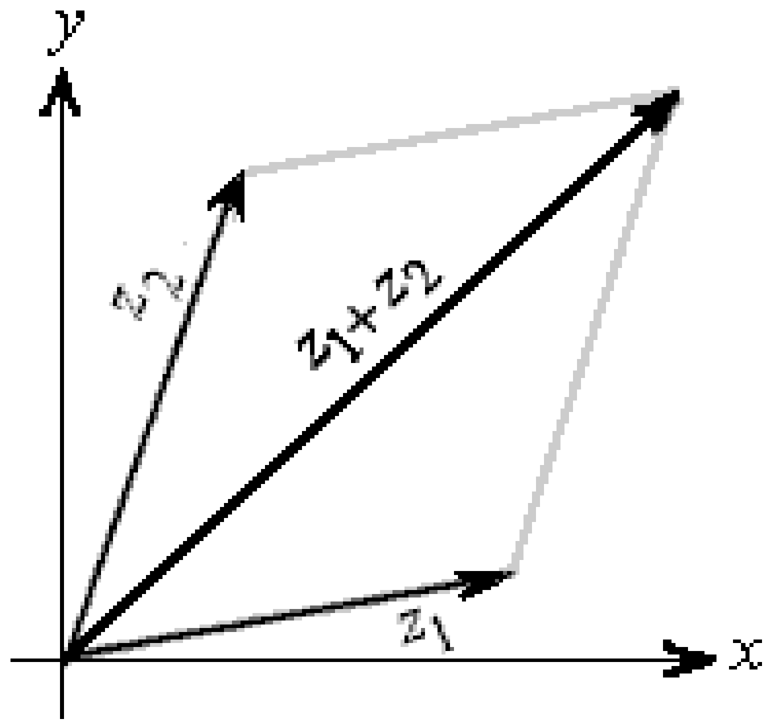
$$x_1 = x_2 \quad \text{and} \quad y_1 = y_2$$

or equivalently

$$\theta_1 = \theta_2 \quad \text{and} \quad |z_1| = |z_2|$$

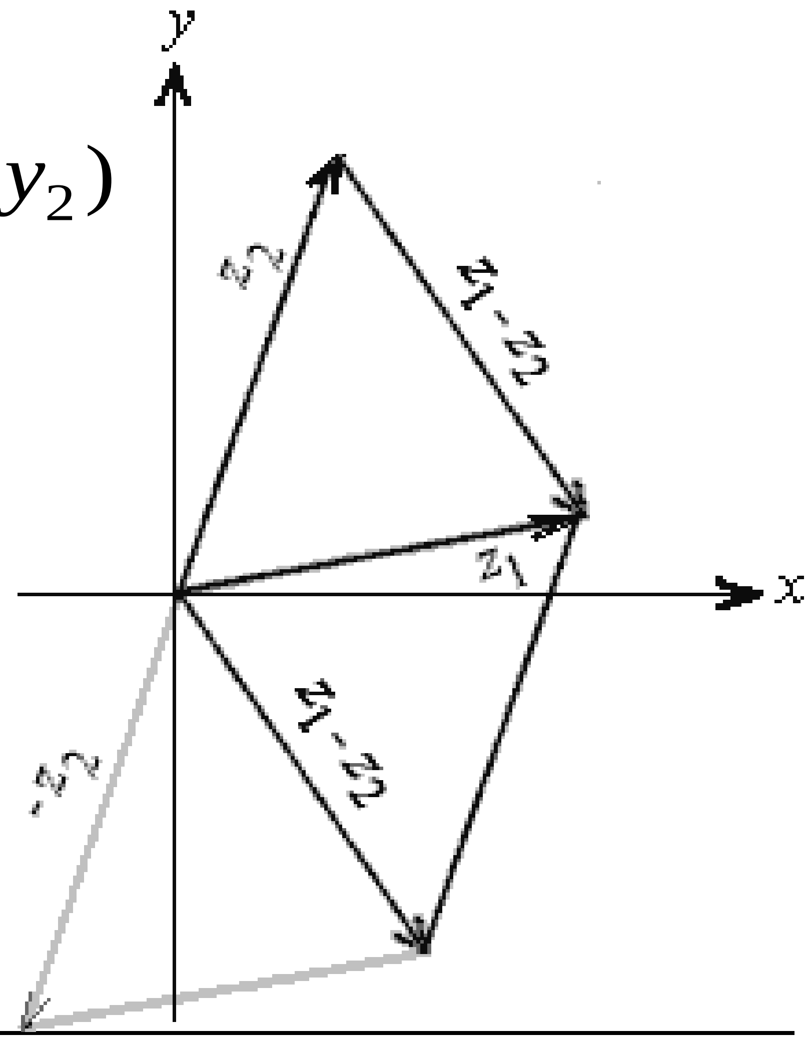
Addition

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$



Subtraction

$$z_1 - z_2 = z_1 + (-z_2)$$
$$= (x_1 - x_2) + j(y_1 - y_2)$$



In terms of vector:

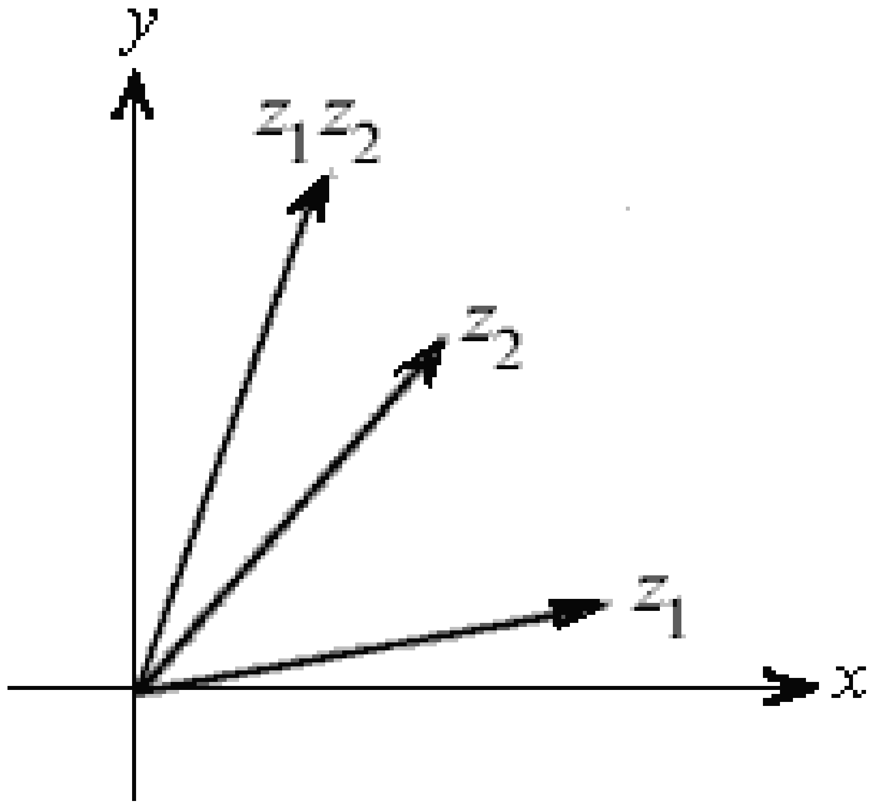
(End point) – (Starting point)

Multiplication

$$z_1 z_2 = (x_1 + jy_1)(x_2 + jy_2)$$
$$= (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1)$$

Or

$$z_1 z_2 = |z_1| e^{j\theta_1} \cdot |z_2| e^{j\theta_2}$$
$$= |z_1||z_2| e^{j(\theta_1 + \theta_2)}$$
$$= |z_1||z_2| [\cos(\theta_1 + \theta_2) + j\sin(\theta_1 + \theta_2)]$$



The geometric interpretation of multiplication of complex numbers $z_1 z_2$ is stretching (or squeezing) and rotation of vectors in the plane:

If you have two complex numbers z_1 and z_2, you can draw a vector z_1, multiply its length by the $|z_2|$, and rotate the resulting vector counterclockwise through the angle $\mathrm{Arg}(z_2)$. If $|z_2| > 1$, we deal with stretching. If $|z_2| < 1$, it is a case of squeezing.

Division

For $z_2 \neq 0$

$$\frac{z_1}{z_2} = \frac{x_1 + jy_1}{x_2 + jy_2} = \frac{x_1 + jy_1}{x_2 + jy_2} \cdot \frac{x_2 - jy_2}{x_2 - jy_2}$$

$$= \frac{(x_1x_2 + y_1y_2) + j(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$$

Or $\displaystyle \frac{z_1}{z_2} = \frac{|z_1| e^{j\theta_1}}{|z_2| e^{j\theta_2}} = \frac{|z_1|}{|z_2|} e^{j(\theta_1 - \theta_2)}$

$$= \frac{|z_1|}{|z_2|}[\cos(\theta_1 - \theta_2) + j\sin(\theta_1 - \theta_2)]$$

Powers

For any positive integer *n*

$$z^n = (|z|e^{j\theta})^n$$

$$= |z|^n e^{jn\theta} = |z|^n (\cos n\theta + j \sin n\theta)$$

$$z^{1/2} = \pm |z|^{1/2} e^{j\theta/2}$$

$$= \pm |z|^{1/2} [\cos(\theta/2) + j \sin(\theta/2)]$$

Useful Relations

$$-1 = e^{j\pi} = e^{-j\pi} = 1\angle 180^\circ$$

$$j = e^{j\pi/2} = 1\angle 90^\circ$$

$$-j = -e^{j\pi/2} = e^{-j\pi/2} = 1\angle -90^\circ$$

$$\sqrt{j} = (e^{j\pi/2})^{1/2} = \pm e^{j\pi/4} = \frac{\pm(1+j)}{\sqrt{2}}$$

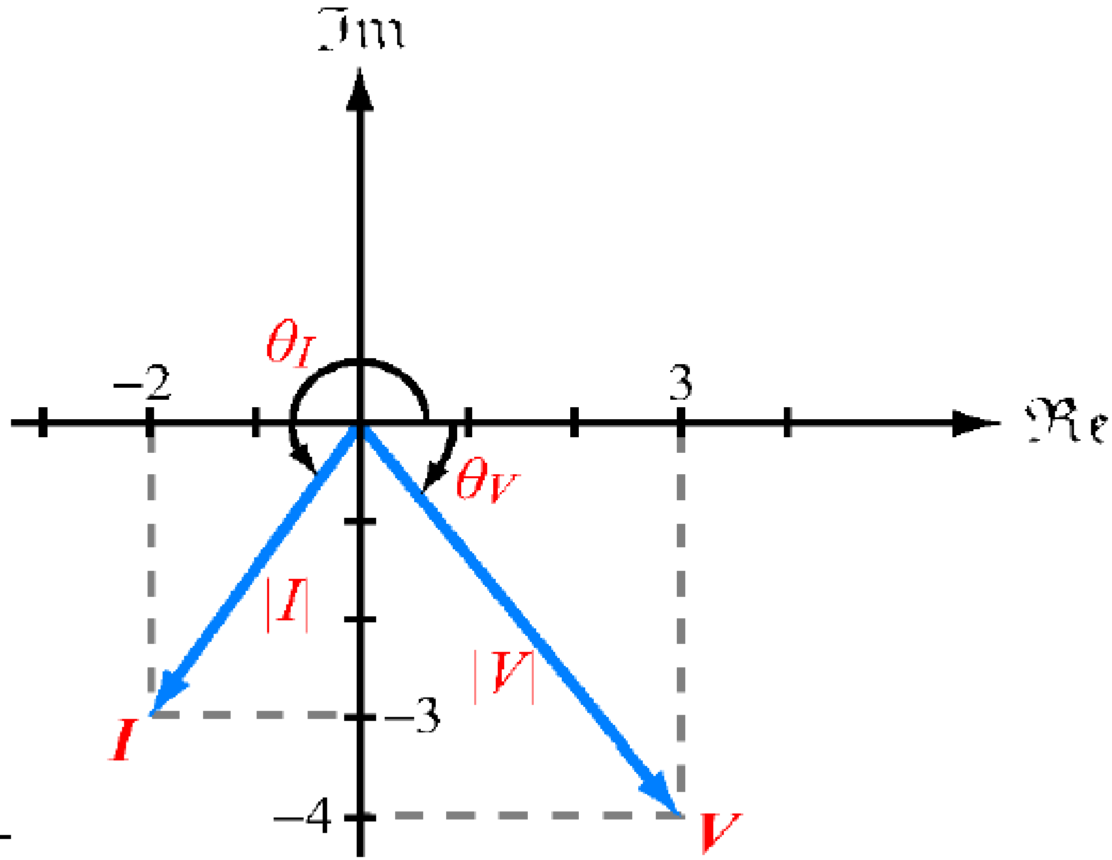
$$\sqrt{-j} = \pm e^{-j\pi/4} = \frac{\pm(1-j)}{\sqrt{2}}$$

Example 1-3 Given two complex numbers

$V = 3 - 4j$

$I = -(2 + 3j)$

(a) Express V and I in polar form, and find (b) VI (c) VI^* (d) V/I and (e) $\sqrt{I}$



Solution

(a)

$$|V| = \sqrt[+]{VV^*}$$

$$= \sqrt[+]{(3-4j)(3+4j)} = \sqrt[+]{9+16} = 5$$

$$\theta_V = \tan^{-1}(-4/3) = -53.1^\circ$$

$$V = |V|e^{j\theta_V} = 5e^{-53.1^\circ} = 5\angle -53.1^\circ$$

$$|I| = \sqrt[+]{2^2+3^2} = \sqrt[+]{13} = 3.61$$

Since $I = (-2 - j3)$ is in the third quadrant in the complex plane

$$\theta_I = 180^\circ + \tan^{-1}(\frac{3}{2}) = 236.3^\circ$$

$$I = 3.61\angle 236.3^\circ$$

(b)

$$VI = 5\angle -53.1^\circ \times 3.61\angle 236.3^\circ$$

$$= 18.05e^{j(236.3^\circ - 53.1^\circ)} = 18.05e^{j183.2^\circ}$$

(c)

$$VI^* = 5\angle -53.1^\circ \times 3.61\angle -236.3^\circ$$

$$= 18.05e^{j(-236.3^\circ - 53.1^\circ)} = 18.05e^{-j289.4^\circ}$$

$$= 18.05e^{j70.6^\circ}$$

(d) $$\frac{V}{I} = \frac{5\angle -53.1^\circ}{3.61\angle 236.3^\circ} = 1.39\angle -289.4^\circ = 1.39\angle 70.6^\circ$$

(e) $$\sqrt{I} = \sqrt{3.61e^{j236.3^\circ}} = \pm\sqrt{3.61}e^{j236.3^\circ/2} = \pm 1.90e^{j118.15^\circ}$$

Exercise 1.7 Express the following complex functions in polar form:

$$z_1 = (4 - j3)^2,$$
$$z_2 = (4 - j3)^{1/2}.$$

Solution:

$$\begin{aligned} z_1 &= (4 - j3)^2 \\ &= \left[(4^2 + 3^2)^{1/2} \angle -\tan^{-1} 3/4\right]^2 \\ &= [5\angle -36.87^\circ]^2 = 25\angle -73.7^\circ \,. \end{aligned}$$

$$\begin{aligned} z_2 &= (4 - j3)^{1/2} \\ &= \left[(4^2 + 3^2)^{1/2} \angle -j\tan^{-1} 3/4\right]^{1/2} \\ &= [5\angle -36.87^\circ]^{1/2} = \pm\sqrt{5}\angle -18.4^\circ \,. \end{aligned}$$

2. Introduction of phasor

Phasor analysis is a useful mathematical tool for solving problems involving linear systems in which the excitation is a periodic function. For example, we have learned in circuit theory for alternating current (AC) that sinusoidal varying voltages and currents can be expressed in phasor forms:

$$V(t) = \mathrm{Re}\left\{\tilde{V}e^{j\omega t}\right\}$$

$$I(t) = \mathrm{Re}\left\{\tilde{I}e^{j\omega t}\right\}$$

$\tilde{V}$ and $\tilde{I}$ are called phasors of $V(t)$ and $I(t)$.

More genearlly, any sinusoidally time-varying (a.k.a. time-harmomic) function $Z(t)$ can be expressed as

$$Z(t) = \mathrm{Re}\left\{\tilde{Z}e^{j\omega t}\right\}$$

where $\tilde{Z}$ is a time-independent function called the phasor of the instantaneous function $Z(t)$.

The benefit of using the phasor form is that:

$$\frac{\partial^n}{\partial t^n} Z(t) = \frac{\partial^n}{\partial t^n}\left\{\mathrm{Re}\left(\tilde{Z}e^{j\omega t}\right)\right\} = \mathrm{Re}\left\{(j\omega)^n \tilde{Z}e^{j\omega t}\right\}$$

The last term can be rewritten as $\mathrm{Re}\left\{\left[(j\omega)^n \tilde{Z}\right]e^{j\omega t}\right\}$

The differentiation with respect to time can be replaced by multiplication of the phasor form with the factor $\boldsymbol{j\omega}$.

Time-domain sinusoidal functions $z(t)$ and their cosine-reference phasor-domain counterparts $\widetilde{Z}$, where $z(t) = \mathfrak{Re}[\widetilde{Z}e^{j\omega t}]$

Note:

1. In the table, the coefficient A is real
2. $\sin(\omega t) = \cos(\omega t - \pi/2)$
3. By Euler's identity, the real part of a complex number is a cosine function. Thus, the phasor is, by definition, automatically cosine referenced.

$z(t)$		$\widetilde{Z}$
$A\cos\omega t$	$\leftrightarrow$	A
$A\cos(\omega t+\phi_0)$	$\leftrightarrow$	$Ae^{j\phi_0}$
$A\cos(\omega t+\beta x+\phi_0)$	$\leftrightarrow$	$Ae^{j(\beta x+\phi_0)}$
$Ae^{-\alpha x}\cos(\omega t+\beta x+\phi_0)$	$\leftrightarrow$	$Ae^{-\alpha x}e^{j(\beta x+\phi_0)}$
$A\sin\omega t$	$\leftrightarrow$	$Ae^{-j\pi/2}$
$A\sin(\omega t+\phi_0)$	$\leftrightarrow$	$Ae^{j(\phi_0-\pi/2)}$
$\frac{d}{dt}(z(t))$	$\leftrightarrow$	$j\omega\widetilde{Z}$
$\frac{d}{dt}[A\cos(\omega t+\phi_0)]$	$\leftrightarrow$	$j\omega Ae^{j\phi_0}$
$\int z(t)\,dt$	$\leftrightarrow$	$\frac{1}{j\omega}\widetilde{Z}$
$\int A\sin(\omega t+\phi_0)\,dt$	$\leftrightarrow$	$\frac{1}{j\omega}Ae^{j(\phi_0-\pi/2)}$

3. Wave Propagation: Direction and Speed

Consider a wave,

$$E(z,t) = A\cos\left(\frac{2\pi}{T}t - \frac{2\pi}{\lambda}z\right)$$

time period **wavelength**

Define

$$\omega = \frac{2\pi}{T} \text{ (angular frequency) and } k = \frac{2\pi}{\lambda} \text{ (wavenumber)}$$

Will be discussed in detailed in later lectures

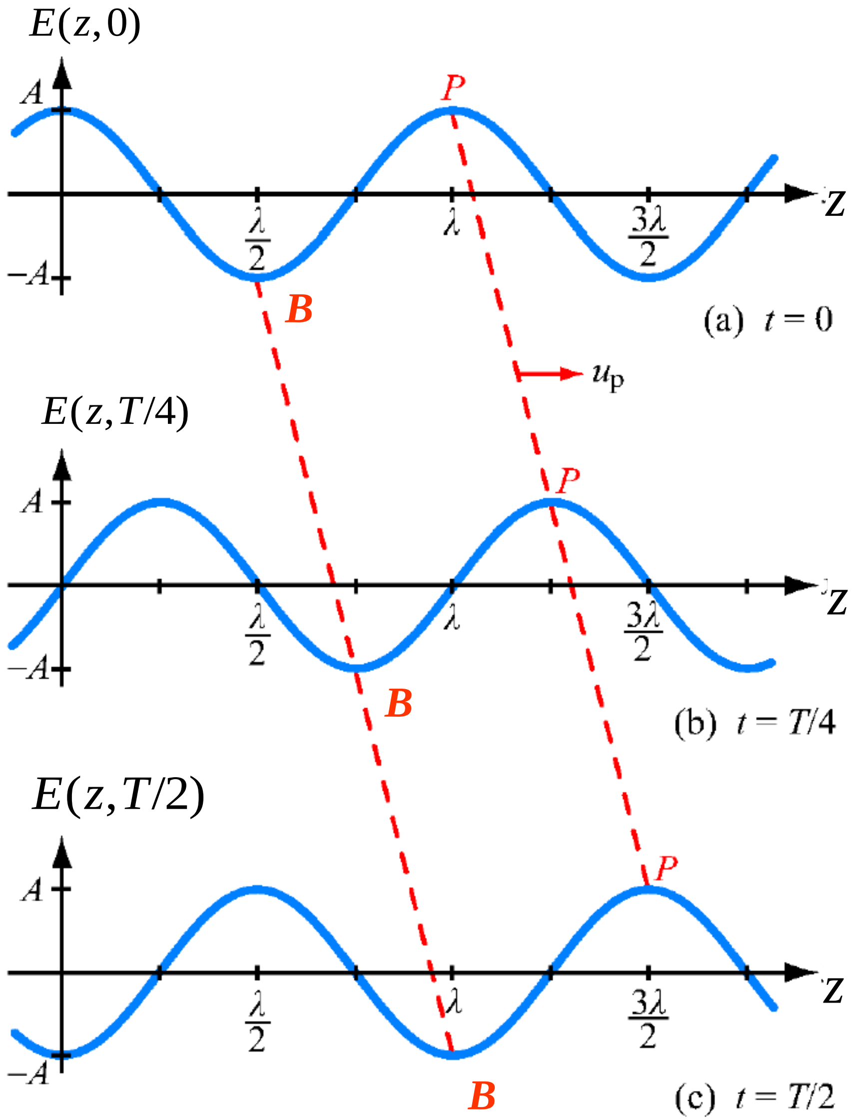
The field can be written as

$$E(z,t) = A\cos(\omega t - kz)$$

The corresponding phasor: $\tilde{E}(z) = Ae^{-jkz}$

For convenience, from here onwards we drop off the ~ on the top since the instantaneous and phasor forms can be easily distinguished.

We take three time (t = 0, T/4, T/2) snapshots of the wave profile



$$A\cos\left(\frac{2\pi}{T}t-\frac{2\pi}{\lambda}z\right)$$

or

$$A\cos(\omega t-kz)$$

Note: The wave travels a distance of one wavelength (λ) per time period (T)

To determine the moving direction and speed of a wave, actually we examine a fixed point in the wave, for example, the peak, the trough, or the zero point. The moving direction and speed of a wave is the same as those of the chosen fixed point, regardless of what fixed point we choose.

Mathematically, we examine a point with $A\cos(\omega t - kz)$ being a constant, i.e.,

$$\omega t - kz = \text{Constant}$$

Taking differentiation on both sides, we have

$$\omega dt - k dz = 0$$

$$\Rightarrow u_p = \frac{dz}{dt} = \frac{\omega}{k} > 0 \qquad \text{(m/s)}$$

The wave propagates in +z direction, with phase velocity: ω/k

By the same argument, we have

$$E(z) = Be^{+jkz} \rightarrow E(z,t) = \mathrm{Re}\left\{Be^{+jkz}e^{j\omega t}\right\}$$

$$= B\cos(\omega t + kz)$$

$$u_p = \frac{dz}{dt} = -\frac{\omega}{k} < 0 \qquad \text{(m/s)}$$

The wave propagates in $-z$ direction, with phase velocity: ω/k

To summarize:

$$\boxed{\begin{array}{l} Ae^{-jkz} \text{ is a wave propagating in the } +z \text{ direction.} \\ Be^{+jkz} \text{ is a wave propagating in the } -z \text{ direction.} \end{array}}$$

- **Textbooks:**
  - *Fundamentals of Applied Electromagnetics*,

    F. T. Ulaby, E. Michielssen, U. Ravaioli,

    Pearson Education, 2010, 6th edition

Suggested reading [textbook]:

- Section 1-4: Traveling Waves
- Section 1-6: Review of Complex Numbers
- Section 1-7: Review of Phasors