NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2002-2003

MA 1505 MATHEMATICS I

November 2002 Time allowed: 2 hours

INSTRUCTION TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided below.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of TEN (10) questions and comprises FORTY-ONE (41) printed pages.
3. Answer **ALL** questions. Write your answer for each question **only** in the spaces provided inside the booklet following that question.
4. The marks for each question are indicated at the beginning of the question.
5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Matriculation Number:

For official use only. Do not write below this line.

Question	1	2	3	4	5	6	7	8	9	10
Marks										

Total:

Question 1 (a) [5 marks]

Find

$$\lim_{x\to\infty} \frac{x^5}{5x^5 - 3x^2 + 2}.$$

Answer.

Question 1 (b) [5 marks]

Find

$$\lim_{x \to 1} \frac{\sin\left\{(\ln \sqrt{x})^3\right\}}{(x-1)^3}.$$

Answer.

Question 2 (a) [5 marks]

Let L denote the tangent line to the curve $y = x \sin \frac{1}{x}$ at the point $\left(\frac{1}{3\pi}, 0\right)$. Find the y-coordinate of the point of intersection of L and the y-axis.

Answer.

Question 2 (b) [5 marks]

Given a function f that can be differentiated four times at the point $x = 0$ and that $f(0), f'(0), f''(0)$ and $f'''(0)$ are all non-zero. Find the values of the constants a, b, c and k such that

$$\lim_{x \to 0} \frac{f(x) - \{af(0) + bxf'(0) + cx^2 f''(0)\}}{kx^3 f'''(0)} = 1$$

holds.

Answer.

Question 3 (a) [5 marks]

Find the value of

$$\int_0^{\frac{\pi}{2}} x^2 \sin(2x)\, dx.$$

Answer.

Question 3 (b) [5 marks]

Find the value of

$$\int_{\frac{1}{2\pi}}^{\frac{2}{\pi}} \frac{1}{x^2}\left(\sin\frac{1}{x}\right)^2\left(\cos\frac{1}{x}\right)dx.$$

Answer.

Question 4 (a) [5 marks]

By using the Ratio Test, or otherwise, determine whether the series

$$\sum_{n=1}^{\infty} \frac{n!}{5^n}$$

is convergent or divergent. Show clearly all your steps.

Answer.

Question 4 (b) [5 marks]

Determine whether the series

$$\sum_{n=1}^{\infty} n\left(\frac{1}{n} - \sin\frac{1}{n}\right)$$

is convergent or divergent. Give reasons to justify your answer.

Answer.

Question 5 (a) [5 marks]

Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{6^n} (3x - 8)^n .$$

Answer.

Question 5 (b) [5 marks]

Let
$$f(x) = \frac{5x - 13}{x^2 - 5x + 6}.$$
By using the Taylor Series of $f(x)$ centered at $a = 1$, or otherwise, find the value of $f^{(4)}(1)$.

Answer.

Question 6 (a) [5 marks]

Let

$$f(x) = \begin{cases} -\frac{\pi}{2} - \frac{x}{2}, & -\pi < x < 0; \\ \frac{\pi}{2} - \frac{x}{2}, & 0 < x < \pi. \end{cases}$$

Find the Fourier Series for $f(x)$.

You may use the following formulae: $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$.

Answer.

Question 6 (b) [5 marks]

Let $f(x) = \sin x$ for $0 < x < \pi$. Let $a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ be the Fourier Cosine Series which represents $f(x)$. Find the value of the coefficient a_4.

You may use the formulae given in Question 6 (a).

Answer.

Question 7 (a) [5 marks]

Solve the differential equation

$$\frac{dy}{dx} - \left(1 + \frac{1}{x}\right) y = 2x + x^2, \qquad x > 0.$$

Answer.

Question 7 (b) [5 marks]

By using the method of variation of parameters, or otherwise, solve the differential equation

$$y'' + y = \operatorname{cosec} x.$$

You may use the following formulae: $u' = \frac{-ry_2}{w}$, $v' = \frac{ry_1}{w}$, where $w = y_1y_2' - y_1'y_2$.

Answer.

Question 8 (a) [5 marks]

A certain kind of insect has constant birth and death rates per capita. If the birth rate is known to be B% per month, and if the population doubles in a time T, find a formula for the death rate per capita. Also find a formula for the time, expressed in terms of T, needed for the population to triple.

Answer.

Question 8 (b) [5 marks]

Solve the differential equation

$$y'' - 2y' + 5y = 0, \qquad y(0) = y'(0) = 1.$$

Answer.

Question 9 (a) [5 marks]

The population of zebras as a function of time in a certain national park, denoted by N, follows a logistic model given by $\frac{dN}{dt} = BN - SN^2$, where B and S are two positive constants. At time $t = 0$, the zebra population is Z_0, but in the long run the zebra population tends to Z_∞, with $Z_\infty < Z_0$. Find a formula for N in terms of Z_0, Z_∞, B and the time t.

Answer.

Question 9 (b) [5 marks]

Solve the differential equation

$$(1 + y)\, y' + (\tan x)\, y^2 = 0, \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

Answer.

Question 10 (a) [5 marks]

Find the Laplace transform of the function

$$f(t) = \left(2 + 3e^t\right)^2.$$

You may use the following formulae: $L(1) = \frac{1}{s}$, $L\left(e^{at}\right) = \frac{1}{s-a}$.

Answer.

Question 10 (b) [5 marks]

Find the function $y(t)$ which satisfies

$$y(t) = t^2 + \int_0^t y(u) \sin(t-u)\, du.$$

You may use the following formulae: $L(t^n) = \frac{n!}{s^{n+1}}$, $L(\sin t) = \frac{1}{s^2+1}$.

Answer.

END OF PAPER