Engineering Electromagnetics

EE2011, Part CXD

LECTURE 2

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Transmission Lines – Smith Chart & Impedance Matching

1 Smith Chart

Smith chart is a graphical plot of the normalized impedance in the complex reflection-coefficient plane.

Smith chart is convenient for transmission line and circuit calculations. It is also a useful tool in impedance matching circuit design. It can provide an approximate solution quickly.

However, we should keep in mind that with the aid of computer and calculator, we can solve every T-line problem without resourcing to Smith Chart.

Recall that (See Lecture Note 1):

$$\Gamma(\ell) = \frac{Z(\ell) - Z_0}{Z(\ell) + Z_0}$$

It is easy to write

$$Z(\ell) = Z_0 \frac{1 + \Gamma(\ell)}{1 - \Gamma(\ell)}$$

Define the normalized impedance $z(l)$ as

$$z(\ell) = \frac{Z(\ell)}{Z_0}$$

For convenience, dropping the ℓ dependence

Reflection coefficient is a complex number,

$$\Gamma = \Gamma_{re} + j\Gamma_{im}$$

$$z = \frac{1 + \Gamma}{1 - \Gamma} = \frac{(1 + \Gamma_{re}) + j\Gamma_{im}}{(1 - \Gamma_{re}) - j\Gamma_{im}}$$

The normalized impedance z is also a complex number:

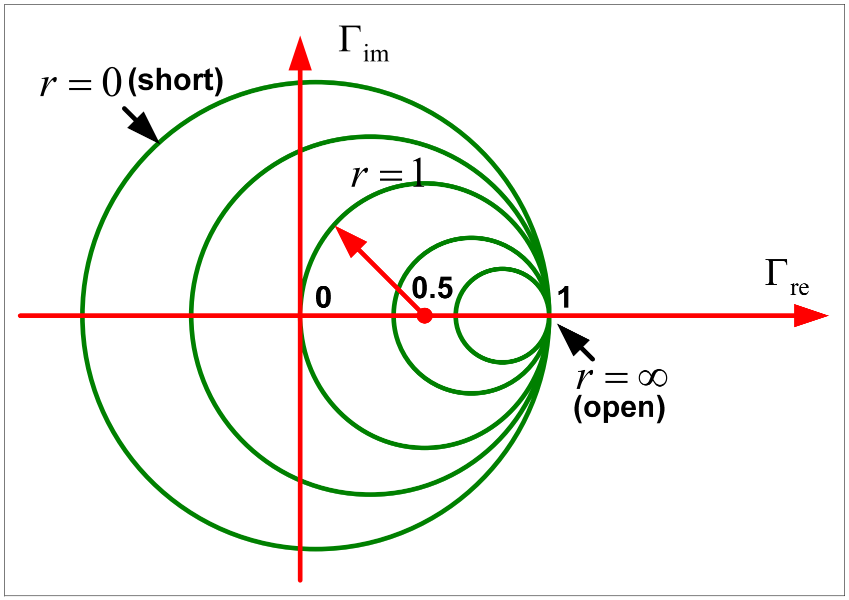
$$z = r + jx$$

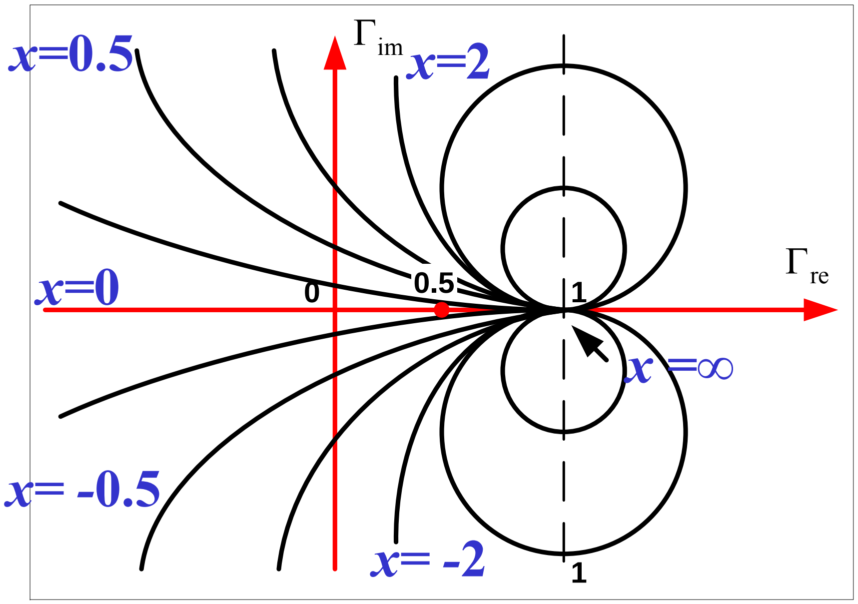
From the last two equations, equating the real and imaginary part respectively, we have

$$r = \frac{1 - \Gamma_{re}{}^{2} - \Gamma_{im}{}^{2}}{(1 - \Gamma_{re})^{2} + \Gamma_{im}{}^{2}}$$

$$x = \frac{2\Gamma_{im}{}^{2}}{(1 - \Gamma_{re})^{2} + \Gamma_{im}{}^{2}}$$

The last two equations of r and x define two families of circles in the complex plane of reflection coefficient Γ.

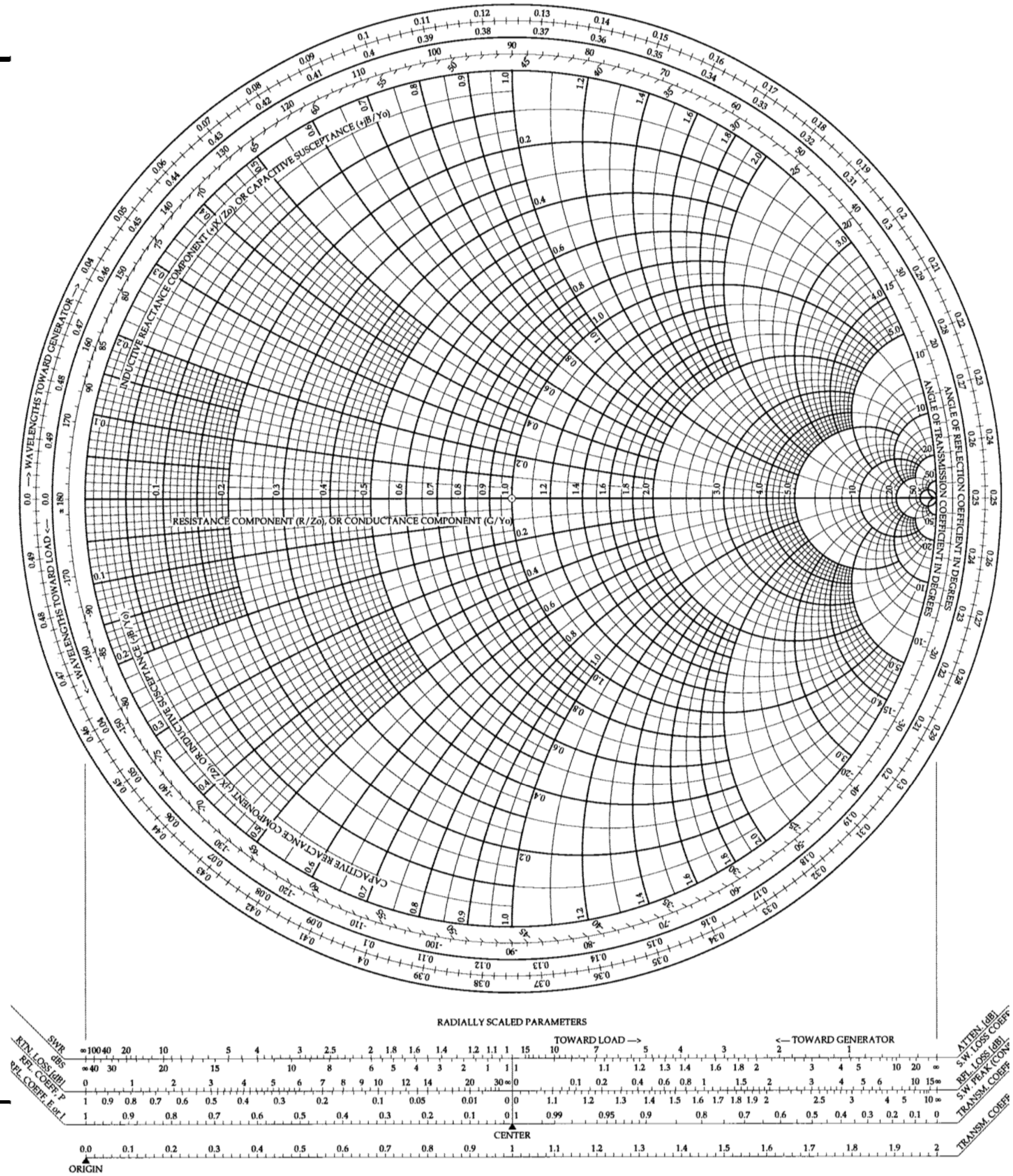




In particular, the x-circle is anti-symmetric about the real axis.

The Smith char is the superposition of these two families of circles together in the complex plane of reflection coefficient Γ.

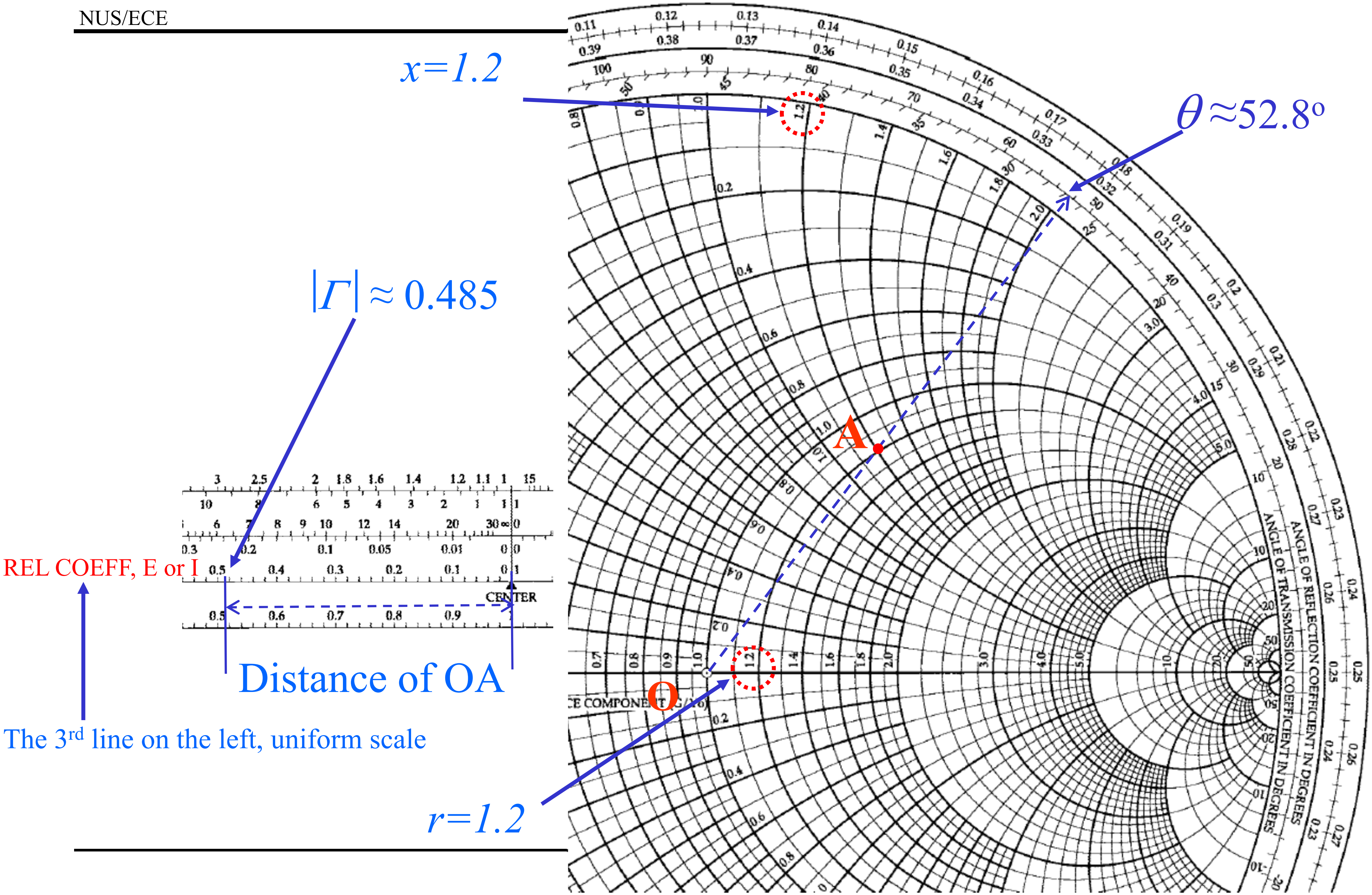
The Smith Chart



A point in the Smith chart gives the values of the normalized impedance $z=r+jx$ and the complex reflection coefficient $\Gamma=|\Gamma|e^{j\theta}$ at the same point on a transmission line.

For r and x:
Read from the r-circles and x-circles.

For $|\Gamma|$ and θ:
Read the modulus and angle from the complex plane.



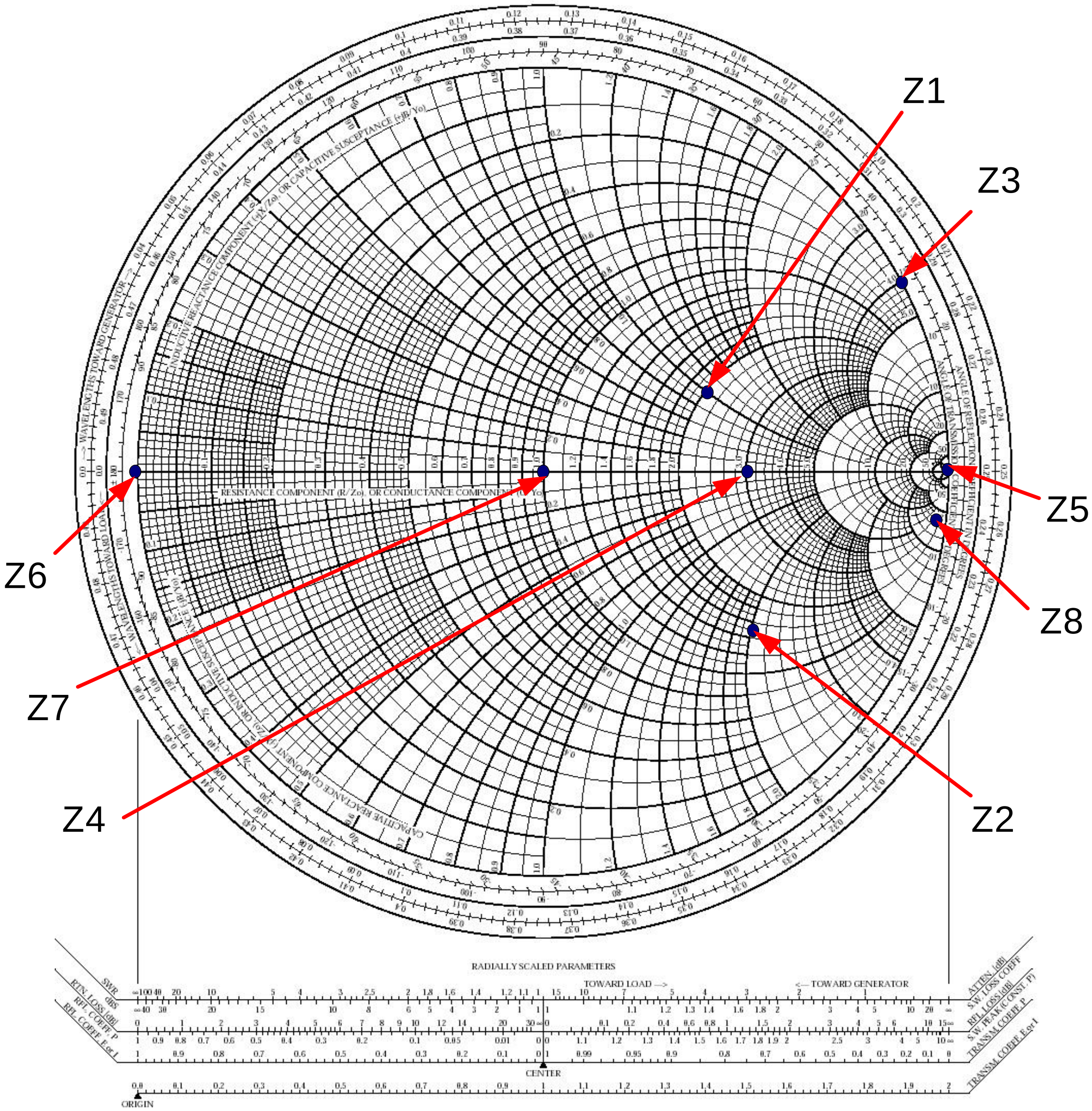
x=1.2
θ≈52.8°
|Γ| ≈ 0.485
A
O
REL COEFF, E or I
CENTER
Distance of OA
The 3rd line on the left, uniform scale
r=1.2
ANGLE OF REFLECTION COEFFICIENT IN DEGREES
ANGLE OF TRANSMISSION COEFFICIENT IN DEGREES

Example 1

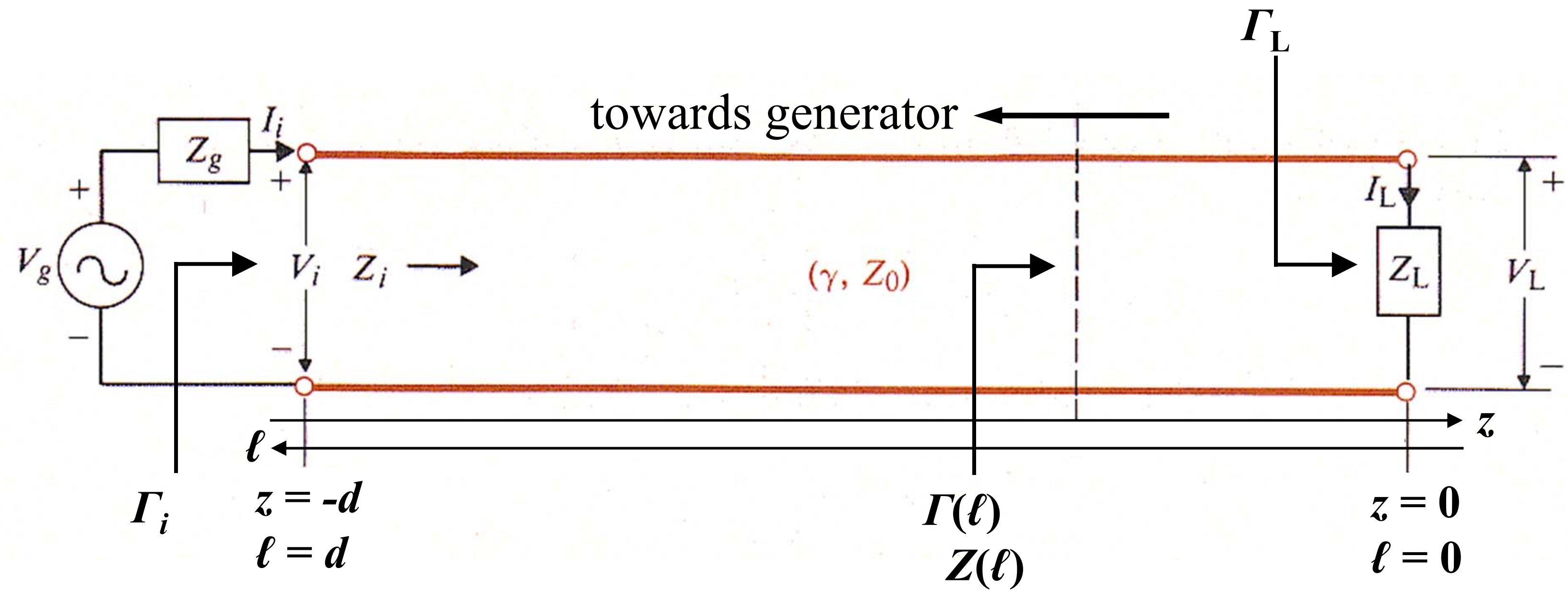
Plot the following impedances on to the Smith chart, where $Z_0 = 50\Omega$

Z (Ω)	z	Γ
$Z_1 = 100 + j50$	$z_1 = 2 + j$	$\Gamma_1 = 0.45\ \angle\ 27°$
$Z_2 = 75 - j100$	$z_2 = 1.5 - j2$	$\Gamma_2 = 0.65\ \angle -38°$
$Z_3 = j200$	$z_3 = j4$	$\Gamma_3 = 1\angle 28°$
$Z_4 = 150$	$z_4 = 3$	$\Gamma_4 = 0.5\ \angle 0°$
$Z_5 = \infty$	$z_5 = \infty$	$\Gamma_5 = 1\ \angle 0°$
$Z_6 = 0$	$z_6 = 0$	$\Gamma_6 = 1\ \angle 180°$
$Z_7 = 50$	$z_7 = 1$	$\Gamma_7 = 0$
$Z_8 = 184 - j900$	$z_8 = 3.68 - j18$	$\Gamma_8 = 0.97\ \angle -6°$

Solutions


Z1
Z3
Z5
Z6
Z8
Z7
Z4
Z2
RADIALLY SCALED PARAMETERS
TOWARD LOAD —>
<— TOWARD GENERATOR
CENTER
ORIGIN

2 Smith chart and transmission lines



Recall that on a transmission line:

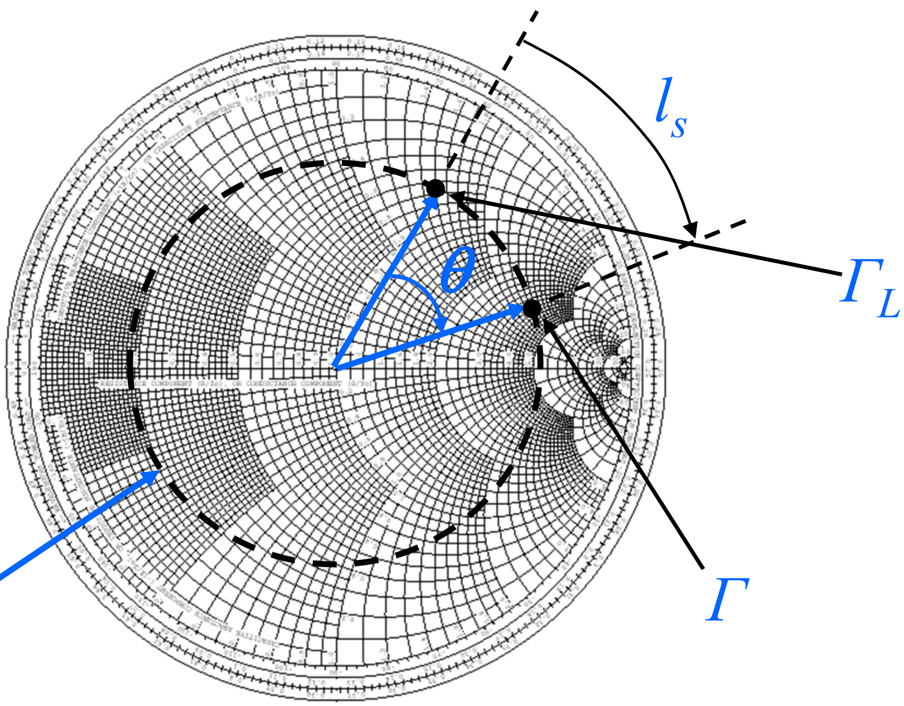
$$\Gamma(\ell) = \Gamma = \Gamma_L e^{-j2k\ell}$$

$$|\Gamma| = |\Gamma_L|$$

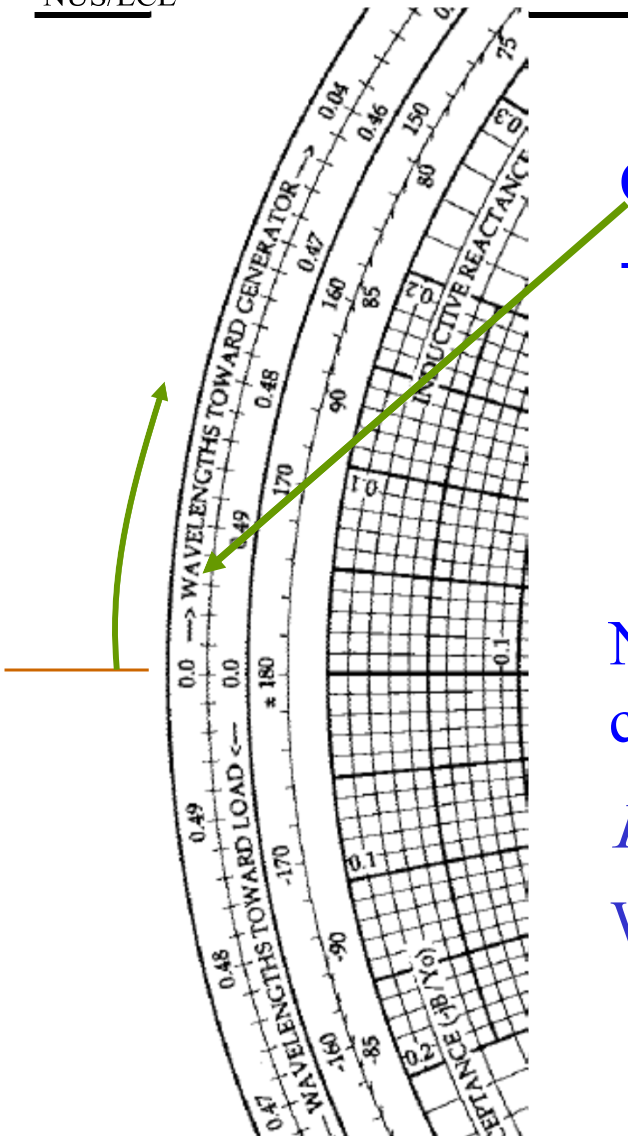
Note: when *l* increases, the modules does not change, the angle decrease

Hence, Γ can be obtained from Γ_L by moving clockwise along a constant circle on the Smith chart with a radius $|\Gamma_L|$ through an angle $-2k\ell$, clockwise since the angle decreases, which is equivalent to ℓ/λ wavelengths measured on **WAVELENGTHS TOWARDS THE GENERATOR** on the periphery of the Smith chart.

$l_s = \ell/\lambda$
$\theta = -2k\ell$



All the points on this circle has the same $|\Gamma|$.



Outmost scale on the periphery (in wavelengths):
-Wavelengths towards generator (WTG scale), clockwise sense

Note also that a complete turn around the Smith chart corresponds to a total length of $\lambda/2$. Because:

$$\Gamma(\ell_2) = \Gamma(\ell_1) e^{-j2k(\ell_2-\ell_1)}$$

When $\ell_2 - \ell_1 = 0.5\lambda$, the phase turned from $\Gamma(\ell_1)$ to $\Gamma(\ell_2)$ is :

$$2k(\ell_2 - \ell_1) = 2\frac{2\pi}{\lambda} \times 0.5\lambda = 2\pi = 360°$$

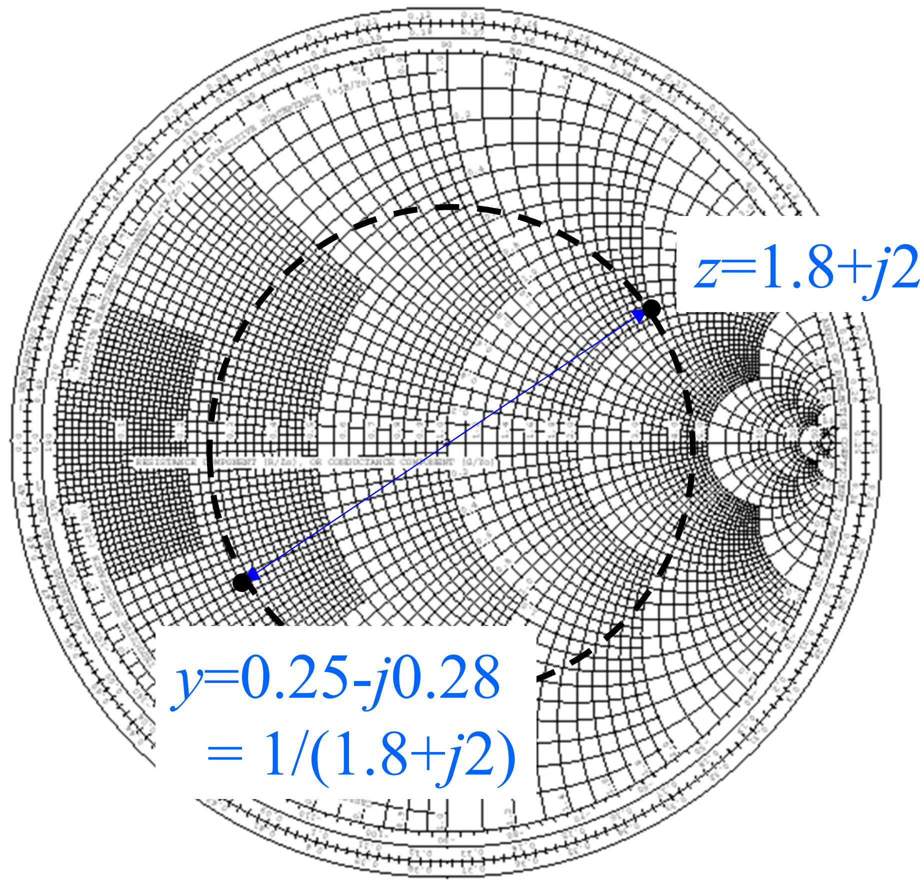
Impedance ↔ admittance

Any point reflected through the centre point converts an impedance to an admittance and vice versa.

$$\boxed{z \leftrightarrow y = 1/z}$$

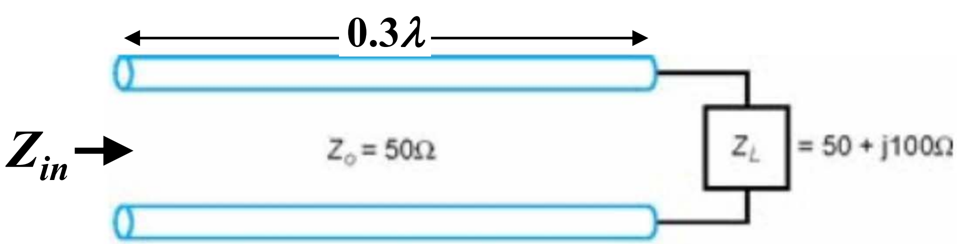
$$z = r + jx$$

$$y = g + jb$$



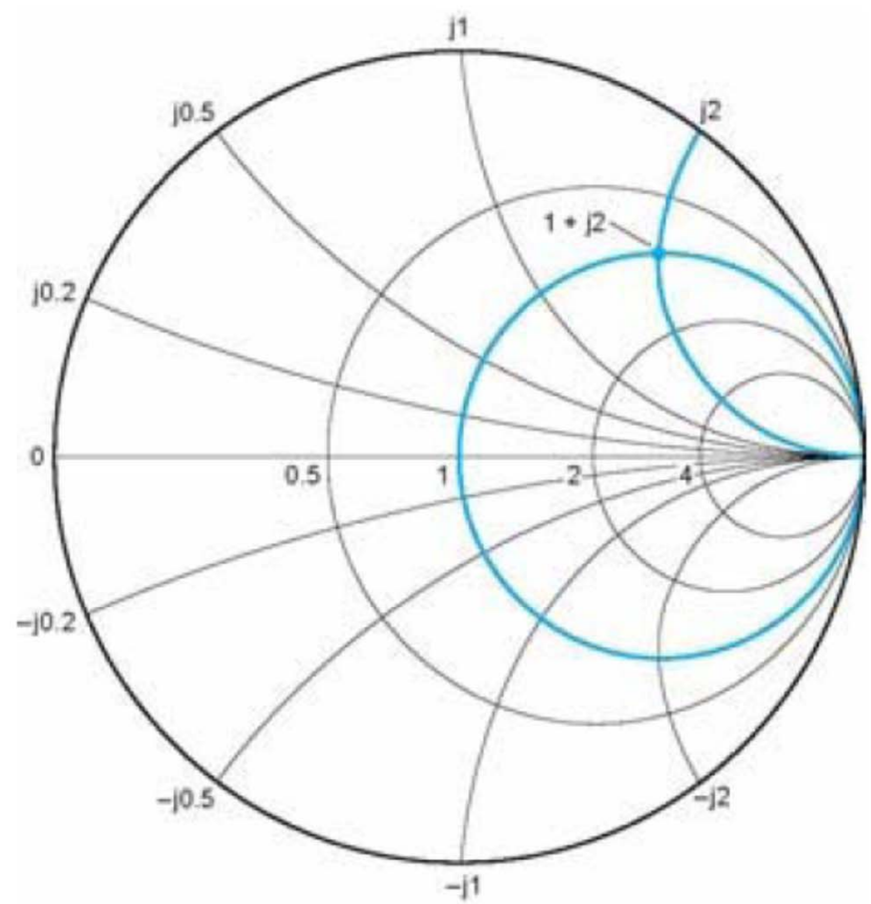
Example 2

Use Smith chart to find the input impedance Z_{in} looking at the input of a transmission line.



(a) Actual circuit

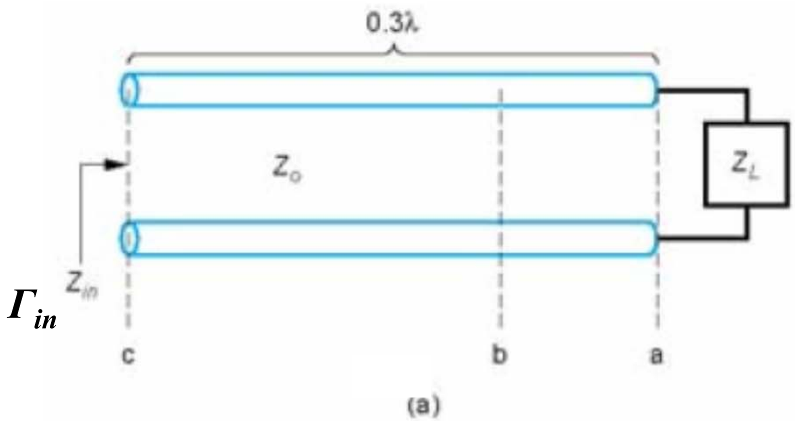
(b) Normalized circuit



First find:

$$z_L = (50 + j100)/50 = 1 + j2$$

Example 2 (cont'd):



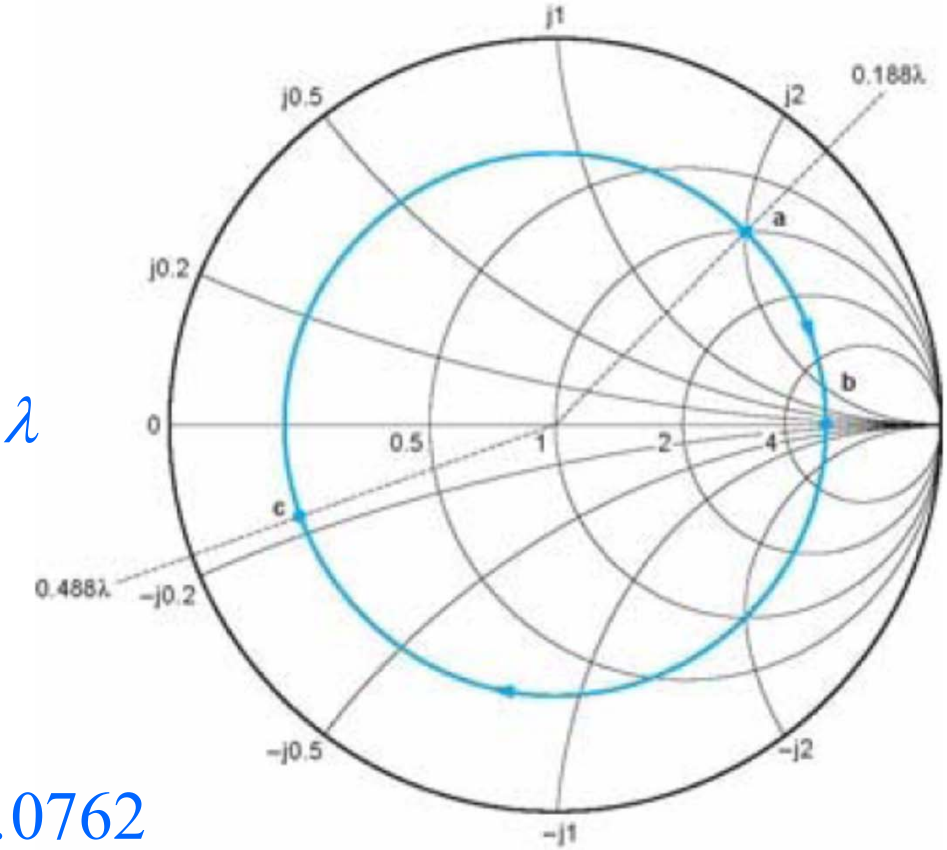
z_L corresponds to point a, with 0.188λ

$0.188\lambda + 0.3\lambda = 0.488\lambda$,

reaching point c.

At point c, read: $z_{in} = 0.1775 - j0.0762$

$Z_{in} = z_{in} Z_0 = 8.8765 - j3.8118 \ \ \Omega$



2 Impedance Matching

Meaning of impedance matching

Impedance matching is to eliminate the **reflected** voltage or current on a transmission line.

Reasons for impedance matching:

1. Maximize power transfer to the load
2. The input impedance remains constant at the value Z_0. Therefore, the input impedance is independent of the length of transmission line.
3. VSWR = 1. Therefore there are no voltage peaks on the transmission line.

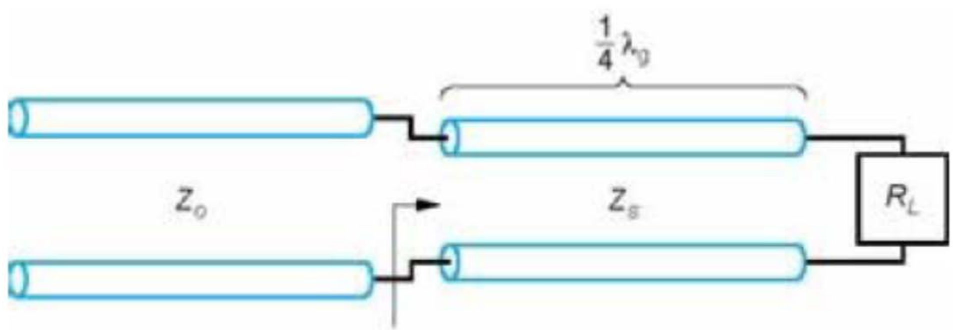
Two matching techniques:

1. Quarter-wave transformer
2. Single-stub matching network

2.1 Quarter-wave transformer

For a transmission line of length $d=\lambda/4$, characteristic impedance = Z_s, and terminated in an impedance R_L,

$$Z_i = Z(\ell = \lambda/4) = Z_s \frac{R_L + jZ_s \tan(\pi/2)}{Z_s + jR_L \tan(\pi/2)} = \frac{Z_s^2}{R_L}$$



$\boldsymbol{Z_i}$

We can change Z_s to achieve a desired Z_i.

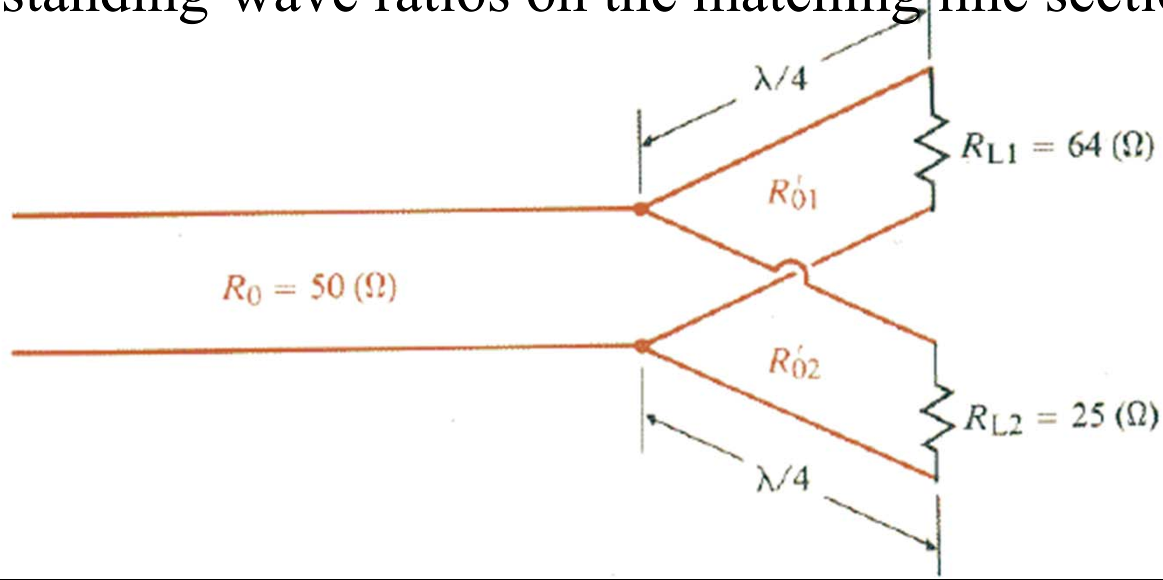
$$Z_S = \sqrt{Z_i R_L}$$

Example 3

A signal generator needs to feed equal power through a lossless 50 Ω transmission line to two separate resistive loads of 64 Ω and 25 Ω at a frequency of 10 MHz. Quarter-wave transformers are used to match the loads to the 50 Ω line, as shown below.

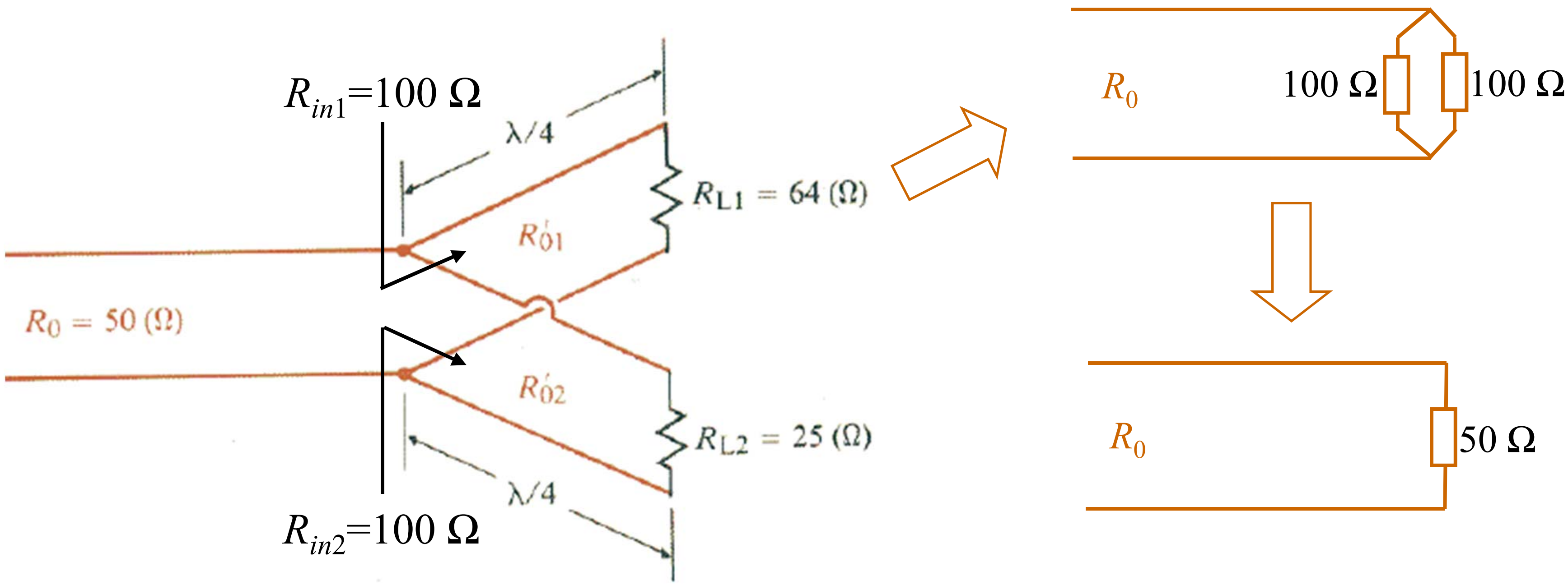
(a) Determine the required characteristic impedances and the physical lengths of the quarter-wavelength lines assuming the phase velocities of the waves traveling on the them is 0.5c.

(b) Find the standing-wave ratios on the matching line sections.



Solutions

(a) As the two quarter-wave transformers are connected in parallel to the 50-Ω line, if equal powers are required to the two loads, the input impedances of the two branches looking at the junction from the 50-Ω line must be equal to 100 Ω so that when they add together in parallel, the total impedance is 50 Ω.



Solutions (cont'd):

Therefore,
$$Z_{in1} = R_{in1} = 100\ \Omega$$
$$Z_{in2} = R_{in2} = 100\ \Omega$$

The characteristic impedances R'_{01} and R'_{02} can be found by:

$$R'_{01} = \sqrt{R_{in1}R_{L1}} = \sqrt{100 \times 64} = 80\ \Omega$$
$$R'_{02} = \sqrt{R_{in2}R_{L2}} = \sqrt{100 \times 25} = 50\ \Omega$$

$$u_p = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_0\varepsilon_0\varepsilon_r}} = \frac{c}{2} \Rightarrow \varepsilon_r = 4$$

$$\lambda = \text{wavelength along the transformers} = \frac{\lambda_0}{\sqrt{\varepsilon_r}} = \frac{c/f}{\sqrt{\varepsilon_r}} = \frac{30\ \text{m}}{2} = 15\ \text{m}$$

Physical length of the transformers = $\lambda/4$ = 3.75 m

Solutions (cont'd):

(b) Under matched conditions, there are no standing waves on the main transmission line, i.e. $S = 1$. The standing wave ratios on the two matching line sections are as follows:

Matching section No. 1:

$$\Gamma_{L1} = \frac{R_{L1} - R_{01}^{'}}{R_{L1} + R_{01}^{'}} = \frac{64-80}{64+80} = -0.11$$

$$S_1 = \frac{1+\left|\Gamma_{L1}\right|}{1-\left|\Gamma_{L1}\right|} = \frac{1+0.11}{1-0.11} = 1.25$$

Matching section No. 2:

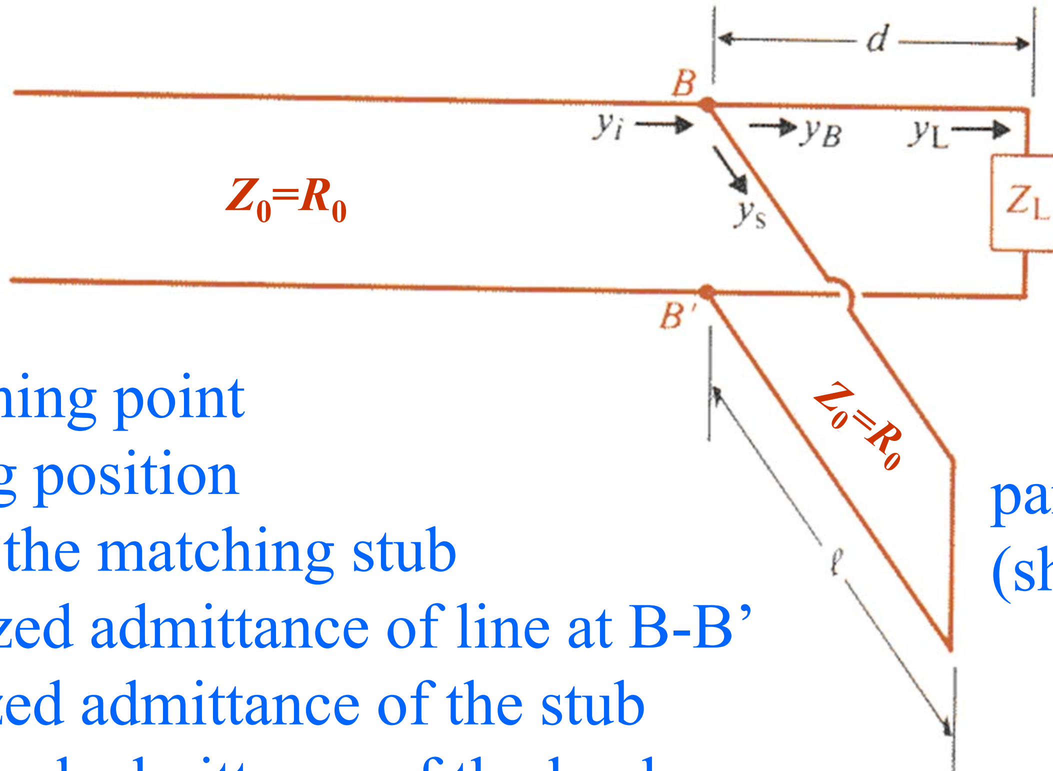
$$\Gamma_{L2} = \frac{R_{L2} - R_{02}^{'}}{R_{L2} + R_{02}^{'}} = \frac{25-50}{25+50} = -0.33$$

$$S_2 = \frac{1+\left|\Gamma_{L2}\right|}{1-\left|\Gamma_{L2}\right|} = \frac{1+0.33}{1-0.33} = 1.99$$

2.2 Single-stub matching network (Optional)

What is a stub?

A stub is a short section of transmission line (shorted or open at one end) whose input impedance can be changed by varying its length.



load to be matched

parallel stub (short-circuited)

B-B' = matching point

d = matching position

ℓ = length of the matching stub

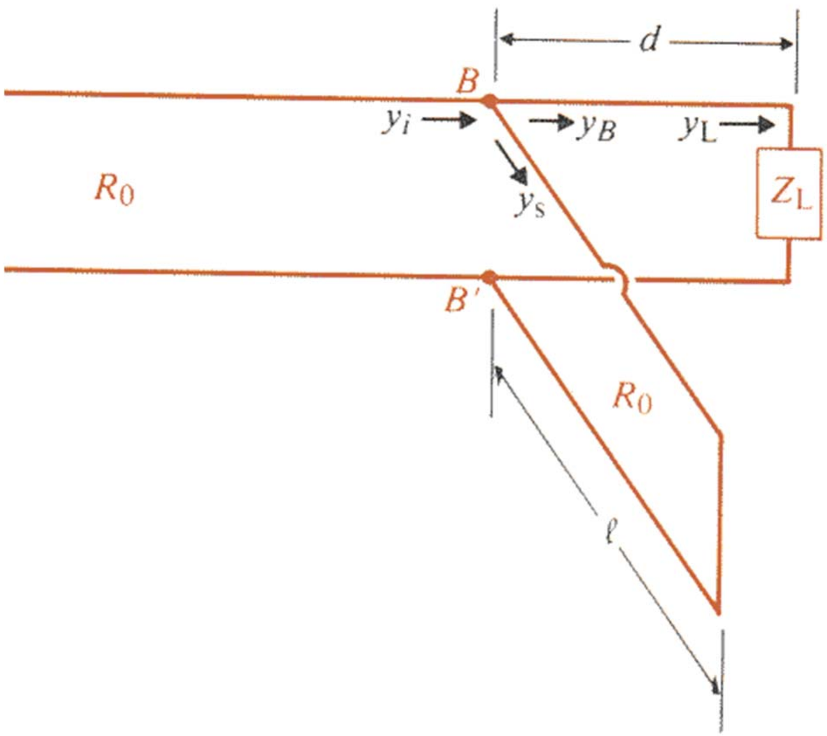
y_B = normalized admittance of line at B-B'

y_s = normalized admittance of the stub

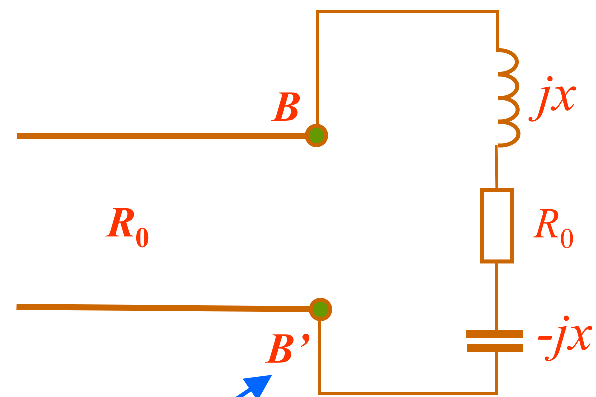
y_L = normalized admittance of the load

When a transmission line is matched at the matching point,

$$y_i = y_B + y_s = 1 \quad \text{or} \quad Y_i = Y_B + Y_s = Y_0$$



Matched
⇒



After matched, there is no reflection on the line to the left of B-B'. But there are reflections on the line to the right of B-B' and on the stub.

Use of admittance

For parallel stub matching, the stub is connected in parallel with the transmission line. Hence it will be more convenient to use admittance rather than impedance.

In Smith chart, when a impedance z is known, the corresponding admittance, $y = 1/z$, can be obtained by a reflection through the centre of the Smith chart. When every impedance point on the Smith chart is reflected in this way, we transform the **impedance Smith chart** to an **admittance Smith chart** in which every point now represents a normalized admittance. For parallel-stub matching, we work in the admittance Smith chart.

Reminder…

On plotting into the Smith chart, all values have to be normalized by the characteristic impedance Z_0 (or the characteristic admittance $Y_0 = 1/\ Z_0$) first. Normalized values are usually represented by small letters while un-normalized values by CAPITAL LETTERS. For example:

Normalized quantity	Un-normalized quantity
$z_L = Z_L / Z_0$	Z_L
$y_L = Y_L / Y_0$	Y_L
$z_{in} = Z_{in} / Z_0$	Z_{in}
$y_{in} = Y_L / Y_0$	Y_{in}

Method to determine d and ℓ

For a short-circuited stub:

$$y_s = \frac{1}{j\tan(k\ell)} = -j\cot(k\ell):$$ Always purely imaginary, regardless of ℓ

When matched,

$$y_i = y_B + y_s = 1$$

So that the real part of y_B must be 1,

Choose d such that:

$$y_B = \frac{1}{z(\ell = d)} = \frac{Z_0 + jZ_L\tan(kd)}{Z_L + jZ_0\tan(kd)} = 1 + jb_B$$

Choose ℓ such that y_s cancels the imaginary part of y_B:

$$y_s = \frac{1}{j\tan(k\ell)} = -j\cot(k\ell) = -jb_B$$

Steps in single-stub matching (using normalized *z* and *y*):

1. Convert the load impedance z_L to an equivalent admittance $y_L = 1/z_L$.
2. Use a line of length d and a characteristic impedance Z_0 (characteristic admittance $Y_0 = 1/Z_0$) to transform y_L to $y_B = 1 + jb_B$ at B-B'.
3. Connect a parallel stub of length ℓ and characteristic impedance Z_0 at B-B' with an input admittance y_s=-jb_B=-$j\cot(2\pi\ell/\lambda)$.
4. Then, the total admittance at B-B' is:

$$\boxed{y_i = y_B + y_s = (1 + jb_B) - jb_B = 1 \quad \Rightarrow \quad \text{matched}}$$

The detailed matching steps in the Smith chart will be explained by using the example shown below.

Example 4

A 50 Ω lossless transmission line is connected to a load impedance Z_L=35-j47.5Ω. Find the position *d* and length *l* of a short-circuit stub required to match the load at a frequency of 200 MHz. Assume that the transmission line is a coaxial line filled with a dielectric material for which $\varepsilon_r = 9$.

Solutions

Given $Z_0 = R_0 = 50\,\Omega$ and $Z_L = 35 - j47.5\,\Omega$. $\Rightarrow$ $z_L = Z_L / Z_0 = 0.7 - j0.95$.

- Enter z_L at point P_1.
- Draw a $|\Gamma|$-circle centred at O with radius $\overline{OP_1}$.
- Draw straight line from P_1 through O to point P_2' on the perimeter, intersecting the $|\Gamma|$-circle at P_2, which represents y_L . Note 0.109 at P_2' on the "wavelengths toward generator" scale.

- Note the two points of intersection of the $|\Gamma|$-circle with the $g=1$ circle:
  - At P_3: $y_{B1}=1+j1.2=1+jb_{B1}$
  - At P_4: $y_{B2}=1-j1.2=1+jb_{B2}$
- Solutions for the position of the stub:
  - For P_3 (from P_2' to P_3') $d_1=(0.168-0.109)\lambda=0.059\lambda$
  - For P_4 (from P_2' to P_4') $d_2=(0.332-0.109)\lambda=0.223\lambda$
- Solutions for the length of the short-circuited stub to provide $y_s=-jb_B$:
  - For P_3, $y_s=-jb_{B1}=-j1.2$ (from P_{sc} to P_3"): $l_1=(0.361-0.250)\lambda=0.111\lambda$
  - For P_4, $y_s=-jb_{B2}=j1.2$ (from P_{sc} to P_4"): $l_2=(0.139+0.250)\lambda=0.389\lambda$

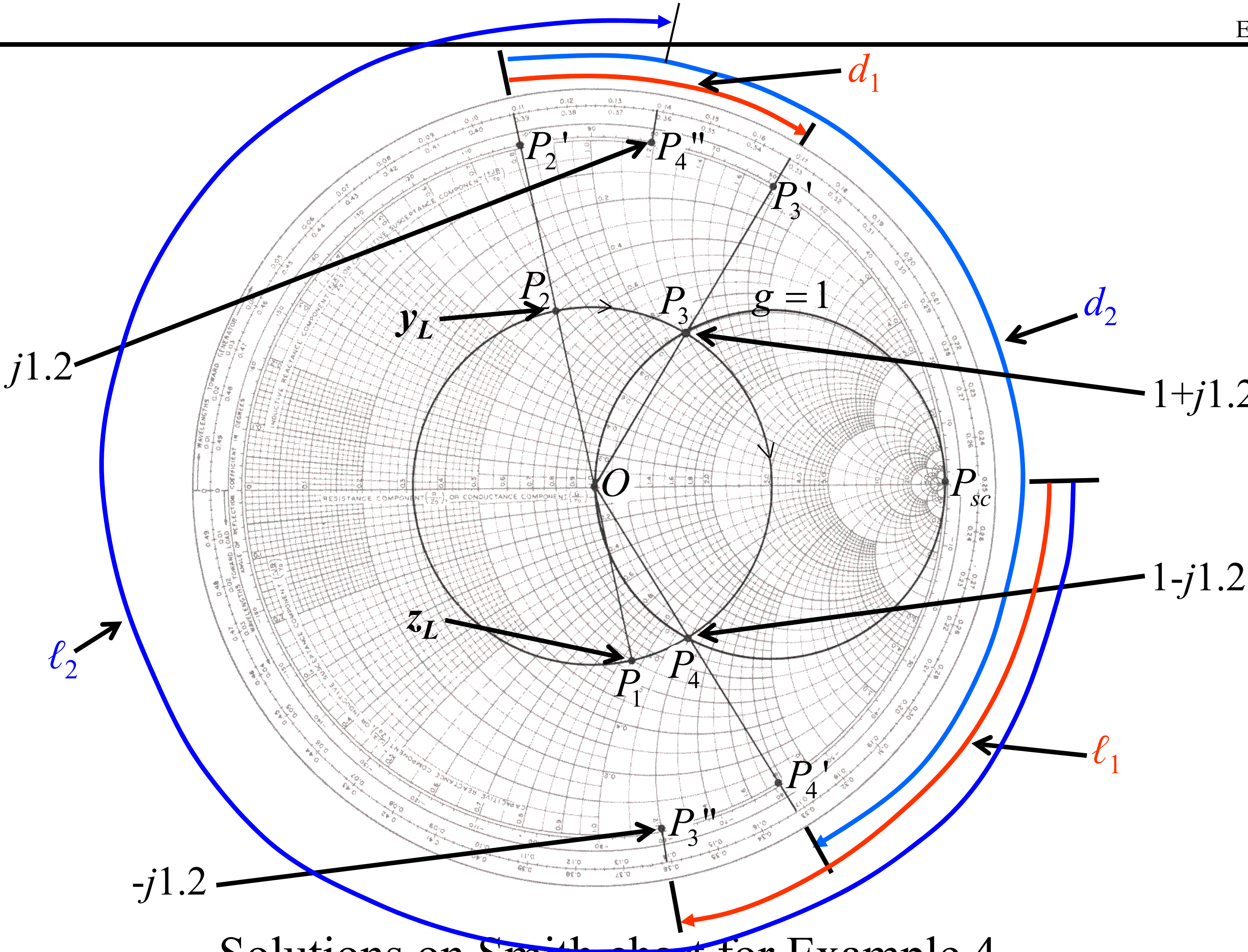
To compute the physical lengths of the transmission line sections, we need to calculate the <u>wavelength on the transmission line</u>. Therefore

$$\lambda=\frac{u_p}{f}=\frac{1/\sqrt{\mu\varepsilon}}{f}=\frac{c/\sqrt{\varepsilon_r}}{f}\approx 0.5\text{ m}.$$

Thus:

$d_1=0.059\lambda=29.5$ mm	$l_1=0.111\lambda=55.5$ mm
$d_2=0.223\lambda=111.5$ mm	$l_2=0.389\lambda=194.5$ mm

Either of these two sets of solutions would match the load. In fact, there is a whole range of possible solutions. For example, when calculating d_1, instead of going straight from P_2' to P_3', we could have started at P_2', rotated clockwise around the Smith chart n times (representing an additional length of $n\lambda/2$) and continued on to P_3', yielding $d_1 = 0.059\lambda + n\lambda/2$, $n = 0, 1, 2, \ldots$ The same argument applies for d_2, l_1 and l_2.



Solutions on Smith chart for Example 4

- **Textbooks:**
  - *Fundamentals of Applied Electromagnetics*,

    F. T. Ulaby, E. Michielssen, U. Ravaioli,

    Pearson Education, 2010, 6th edition

Suggested reading [textbook]:

Section 2-10: The Smith Chart

Section 2-11.2: Single-Stub Matching