MA1506 TUTORIAL 9

Question 1

Billionaire engineer Tan Ah Lian believes that she can get even richer by gambling. To this end, she goes to an Integrated Resort[1] and plays the following game [along with several other players]. The players and the croupier each flip coins. If a player's coin matches that of the croupier [both heads or both tails] then the player pays \$1. If they do not match, the house pays the player \$1. [This kind of game is designed to prevent cheating by either party.] Initially TAL has \$3. If at any point she loses all her money, she will be violently thrown out of the den with probability 1, and the game goes on without her. If at any point she wins a total of \$2, then she will also be booted out even more violently with probability 1, because the owner of the gambling den is her old bitter enemy Lim Ah Huat, who doesn't allow anyone, especially Tan Ah Lian, to make more than \$2 from him. What is the probability that TAL will be broke by the time 5 rounds of this game have been played? What is the probability that she will have been thrown out, by 5 rounds, for being too successful? Is Ah Huat a born loser?[Hint: over the course of the game, there are six possible amounts of money that TAL can have. Set up a 6 by 6 matrix to represent the Markov process of this game; that is, the first column represents the probabilities of reaching the six different possible amounts of money given that TAL has \$0, the second column represents the probabilities given that she has \$1, and so on. You can use the matrix calculator at http://wims.unice.fr/wims/ to work out the necessary power of this matrix.][Answers: 22%, 38%.]

Question 2

The Leontief model can be applied to the economies of entire countries, as follows. The economy of the Republic of Progensia consists of Agriculture, Manufacturing, and Energy, and the corresponding technological matrix [in the order AME for both columns and rows] is $\begin{bmatrix} 0.30 & 0.00 & 0.00 \\ 0.10 & 0.20 & 0.20 \\ 0.05 & 0.01 & 0.02 \end{bmatrix}$, so for example this means that each Progensian tael of Agricultural produce requires 0.30 taels' worth of Agricultural produce, 0.10 taels' worth of manufactured goods, and 0.05 taels' worth of energy [the tael being the Progensian currency]. Progensia's government hopes to export 140 million taels' worth of agricultural produce, 20 million taels' worth of manufactured goods, and 2 million taels' worth of energy this year. Find out how much agricultural produce, manufactured goods, and energy they have to produce in order to meet this target. [Answers: 200.2, 53.52, 12.04 million taels respectively.]

Question 3

Use the eigen-engine at the IDE website [http://www.aw-bc.com/ide/idefiles/media/JavaTools/eignengn.html] to find the eigenvalues and eigenvectors of the following matrices:

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

[1] Gambling den

Question 4

Find the eigenvectors and eigenvalues of the matrices in Question 4 *by hand*, that is, not by using any computer other than the one inside your head.

Question 5

In our study of Markov processes, we used matrices of a special kind, called a [left] stochastic matrix http://en.wikipedia.org/wiki/Stochastic_matrix. In such a matrix, the entries are numbers between 0 and 1, such that *the total of the entries in each column is 1.* This is because the entries represent probabilities, and probabilities in any given situation *always have to total to 1.* [Check this for our gambling game, above.] Prove that a 2×2 matrix of this sort must have at least one eigenvalue equal to 1. Why is this important?

Question 6

Use the wims website to find all of the eigenvectors of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. You should find that this matrix has exactly two non-parallel eigenvectors with non-zero eigenvalues. The third eigenvector is not parallel to either of these, but it has eigenvalue zero. Explain why this implies that this matrix *has rank 2*.

The two non-parallel eigenvectors with non-zero eigenvalues define a *2-dimensional* space, a *plane* [like any pair of non-parallel vectors in three-dimensional space]. Show that the vector $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ does not lie in this plane. Hence explain why the system of equations $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ has *no* solutions. Show that $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ *does* lie in this plane, and explain why this means that $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ has *infinitely many* solutions.