Indeterminate Forms

Indeterminate Forms

Let f and g be continuous at $x = a$.

Suppose $f(a) = 0$ and $f(b) = 0$.

Then the limit

$$\lim_{x \to a} \frac{f(x)}{g(x)} \quad \text{is of the form} \quad \frac{0}{0}$$

$$\text{because} \quad \frac{f(a)}{g(a)} = \frac{0}{0}$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$$

$\frac{0}{0}$ Indeterminate form

Use L'Hospital's Rule

Who is this person ???

L'Hospital's Rule

Suppose that

(1) f and g are differentiable in a neighborhood of x_0;

(2) $f(x_0) = g(x_0) = 0$;

(3) $g'(x_0) \neq 0$ except possibly at x_0.

then

$$\boxed{\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}}$$

L’Hospital’s Rule

$$\boxed{\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{f'(x)}{g'(x)}}$$

Note: Before applying L' Hospital's Rule

[redacted]

$$\lim_{x\to a}\frac{f(x)}{g(x)} \quad \text{is of the form} \quad \frac{0}{0}$$

L'Hospital's Rule

$$\boxed{\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{f'(x)}{g'(x)}}$$

In particular,

Suppose $f(a)=g(a)=0$, $f'(a)$ and $g'(a)$ exist, and $g'(a)\neq 0$.

$$\boxed{\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{f'(a)}{g'(a)}}$$

$$\text{(ii)} \quad \lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\frac{\sqrt{1+0}-1}{0}=\frac{0}{0}$$

Can use L'Hospital's rule

$$\text{(ii)} \quad \lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\lim_{x\to 0}\frac{\frac{d}{dx}\left((1+x)^{\frac{1}{2}}-1\right)}{\frac{d}{dx}(x)}$$

Applying L'H

$$=\lim_{x\to 0}\frac{\frac{1}{2}(1+x)^{-\frac{1}{2}}}{1}$$

$$=\lim_{x\to 0}\frac{1}{2\sqrt{x+1}}$$

$$=\frac{1}{2}$$

Question: Can you find

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}$$

without using L'Hospital's rule ???

(i) $\lim_{x\to 0}\frac{3x-\sin x}{x}=\frac{3(0)-\sin 0}{0}=\frac{0}{0}$

Can use L'Hospital's rule

$$\lim_{x\to 0}\frac{3x-\sin x}{x}=\lim_{x\to 0}\frac{\frac{d}{dx}(3x-\sin x)}{\frac{d}{dx}(x)}$$

Applying L'H

$$=\lim_{x\to 0}\frac{3-\cos x}{1}$$

$$=3-\cos 0$$

$$=3-1$$

$$=2$$

(iii) $\lim_{x\to 0}\frac{x-\sin x}{x^3}=\frac{0-\sin 0}{0}=\frac{0}{0}$ Can use L'Hospital's rule

Applying L'H $\lim_{x\to 0}\frac{x-\sin x}{x^3}=\lim_{x\to 0}\frac{1-\cos x}{3x^2}$ $\frac{1-\cos 0}{3(0)^2}=\frac{0}{0}$

Applying L'H $=\lim_{x\to 0}\frac{\sin x}{6x}$ $\frac{\sin 0}{6(0)}=\frac{0}{0}$

Applying L'H $=\lim_{x\to 0}\frac{\cos x}{6}$

$$=\frac{\cos 0}{6}$$

$$=\frac{1}{6}$$

Pause and Think !!!

Question: Can you find

$$\lim_{x\to 0}\frac{3x-\sin x}{x} \qquad \lim_{x\to 0}\frac{x-\sin x}{x^3}$$

without using L'Hospital's rule ???

(iv) $\lim_{x\to 0}\frac{1-\cos x}{x+x^2}=\frac{1-\cos 0}{0+0^2}=\frac{0}{0}$

Can use L'Hospital's rule

Applying L'H

$$\lim_{x\to 0}\frac{1-\cos x}{x+x^2}=\lim_{x\to 0}\frac{\sin x}{1+2x}$$

$$=\frac{\sin 0}{1+2(0)}$$

$$=0$$

Pause and Think !!!

Question: Can you find

$$\lim_{x \to 0} \frac{1 - \cos x}{x + x^2}$$

without using L'Hospital's rule ???

(v) $\lim_{x \to 0} \frac{\sin x}{x^2} = \frac{\sin 0}{0^2} = \frac{0}{0}$

Can use L'Hospital's rule

Applying L'H

$$\lim_{x \to 0} \frac{\sin x}{x^2} = \lim_{x \to 0} \frac{\cos x}{2x} = \infty$$

Other Indeterminate forms

Take $\dfrac{1}{0}$ same as ∞

$$\frac{\infty}{\infty}=\frac{\frac{1}{0}}{\frac{1}{0}}$$

$$=\frac{1}{0}\div\frac{1}{0}$$

$$=\frac{1}{0}\times\frac{0}{1}$$

$$=\frac{0}{0}$$

$\dfrac{\infty}{\infty}$ is also an indeterminate form

We may apply L' Hospital's Rule

L’Hospital’s Rule

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

Note: Before applying L' Hospital's Rule

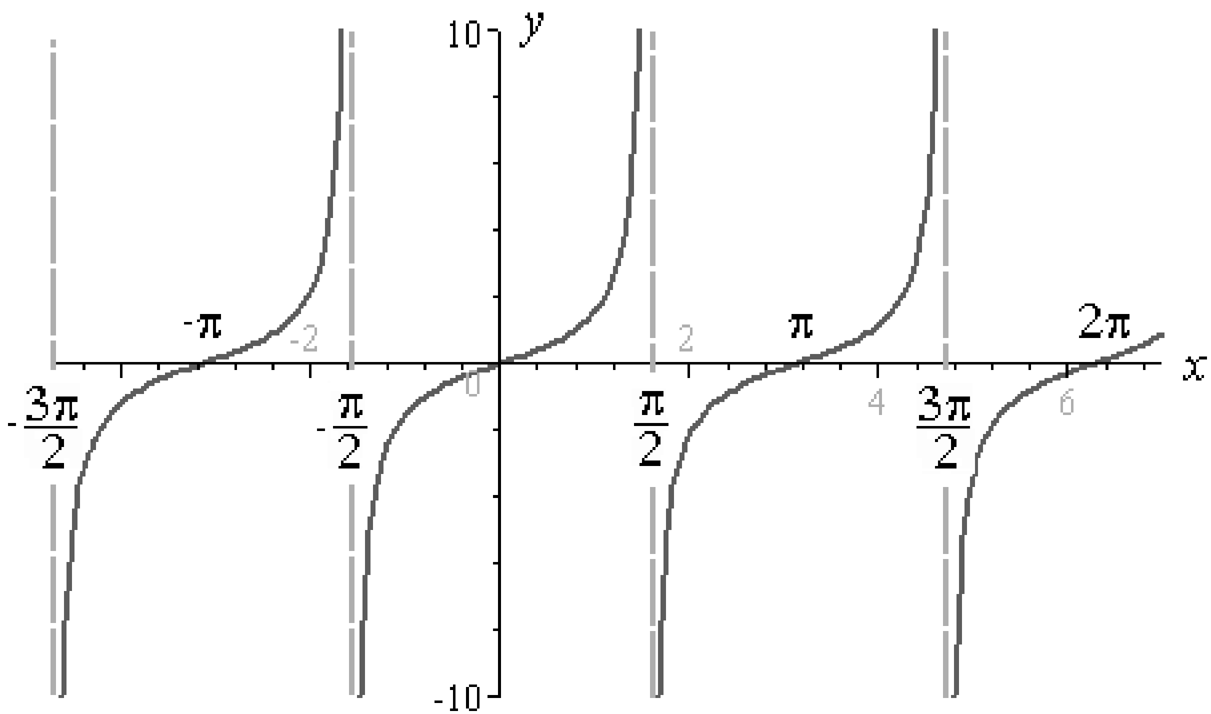
must check that

$\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$ is of the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$

Note : Only this two forms we may apply L'Hospital Rule!!

$$\lim_{x\to\frac{p}{2}^-}\frac{\tan x}{1+\tan x}=\frac{\infty}{\infty}$$

Can use L'Hospital's rule



Applying L'H

$$\lim_{x\to\frac{p}{2}^-}\frac{\tan x}{1+\tan x}=\lim_{x\to\frac{p}{2}^-}\frac{\sec^2 x}{\sec^2 x}=1$$

$$\lim_{x\to\infty}\frac{x-2x^2}{3x^2+5}$$ $\frac{\infty}{\infty}$ Can use L'Hospital's rule

---

Applying L'H $$\lim_{x\to\infty}\frac{x-2x^2}{3x^2+5}=\lim_{x\to\infty}\frac{1-4x}{6x}$$ $\frac{\infty}{\infty}$

$$=\lim_{x\to\infty}\frac{-4}{6}$$

$$=-\frac{2}{3}$$

Limit can be found without using L'Hospital's rule.

$$\lim_{x\to\infty}\frac{2x^2+3x+2}{5x^2-x+2} = \lim_{x\to\infty}\frac{\frac{2x^2}{x^2}+\frac{3x}{x^2}+\frac{2}{x^2}}{\frac{5x^2}{x^2}-\frac{x}{x^2}+\frac{2}{x^2}}$$

$$= \lim_{x\to\infty}\frac{2+\frac{3}{x}+\frac{2}{x^2}}{5-\frac{1}{x}+\frac{2}{x^2}}$$

divide the numerator and denominator by x^2.

$$= \frac{2+0+0}{5-0+0} = \frac{2}{5}$$

$$\lim_{x\to\infty}\frac{1}{x}=0$$

$$\lim_{x\to\infty}\frac{1}{x^2}=0$$

Other indeterminate forms

$(0 \cdot \infty)$ $(\infty - \infty)$ and ......

Take $\frac{1}{0}$ same as ∞

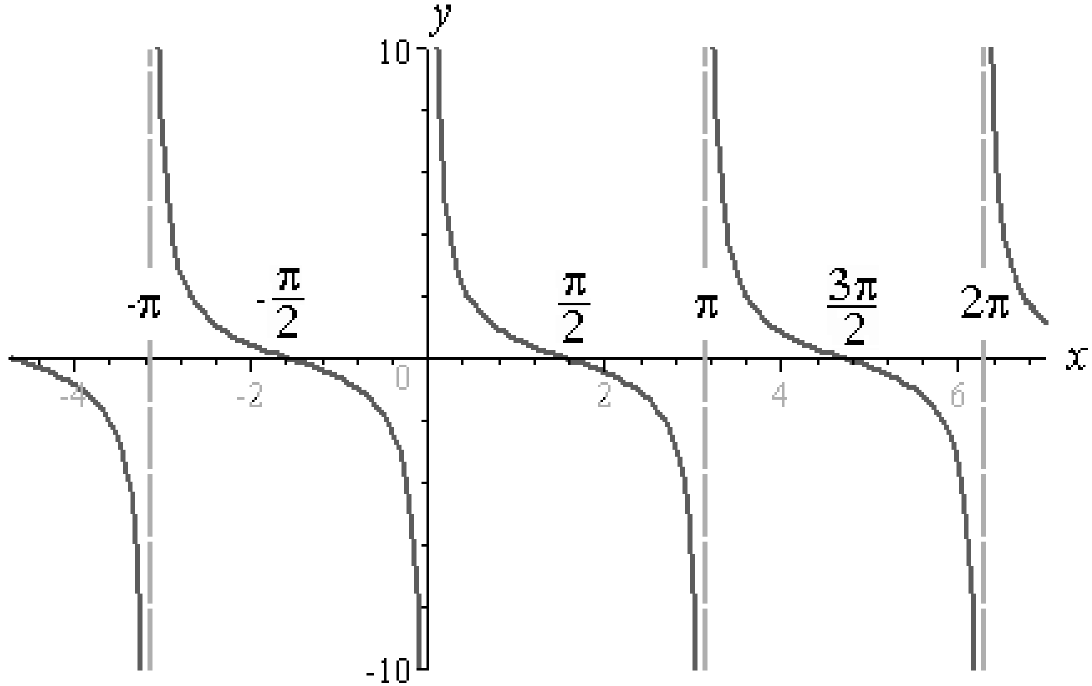
$$(0 \cdot \infty) = 0 \cdot \frac{1}{0}$$
$$= \frac{0}{0}$$

$(0 \cdot \infty)$ is also an indeterminate form

We may apply L' Hospital's Rule after rewriting

$0 \cdot \infty$ into either $\frac{0}{0}$ or $\frac{\infty}{\infty}$

$$\lim_{x\to 0^+} x\cot x \quad (0\cdot\infty)$$



$$\lim_{x\to 0^+} x\cot x = \lim_{x\to 0^+} \frac{x}{\tan x} \quad \frac{0}{0}$$

$$\cot x = \frac{1}{\tan x}$$

$$= \lim_{x\to 0^+} \frac{1}{\sec^2 x}$$

Applying L'H

$$\frac{1}{\sec^2 x} = \cos^2 x$$

$$= \lim_{x\to 0^+} \cos^2 x$$

$$= \cos^2 0 = 1$$

Other indeterminate forms $(\infty-\infty)$

We may apply L' Hospital's Rule after rewriting

$\infty-\infty$ into either $\frac{0}{0}$ or $\frac{\infty}{\infty}$

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right) \qquad (\infty-\infty)$$

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\lim_{x\to 0}\frac{x-\sin x}{x\sin x} \qquad \frac{0-\sin 0}{0\sin 0}=\frac{0}{0}$$

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\lim_{x\to 0}\frac{x-\sin x}{x\sin x}=0$$

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\lim_{x\to 0}\frac{x-\sin x}{x\sin x}$$

$$\frac{0-\sin 0}{0\sin 0}=\frac{0}{0}$$

Applying L'H $$=\lim_{x\to 0}\frac{1-\cos x}{x\cos x+\sin x}$$

$$\frac{1-\cos 0}{0\cos 0+\sin 0}=\frac{0}{0}$$

Applying L'H $$=\lim_{x\to 0}\frac{0+\sin x}{-x\sin x+\cos x+\cos x}$$

$$=\lim_{x\to 0}\frac{\sin x}{-x\sin x+2\cos x}$$

$$=\frac{\sin 0}{-0\sin 0+2\cos 0}$$

$$=0$$

Product rule

$$\frac{d}{dx}(x\sin x)=x\cos x+\sin x$$

Find $\lim\limits_{x \to 0} \dfrac{(e^x - 1 - x)^2}{x \sin^3 x}$ $\qquad \dfrac{0}{0}$

Apply L'Hospital's Rule needs Product rule

Product rule

$$\tfrac{d}{dx}(x \sin^3 x) = 3x \sin^2 x \cos x + \sin^3 x$$

If you need to apply L'Hospital's Rule

a few times, problem has more and more terms

Pause and Think !!!

Which of the following are indeterminate forms ???

0^0 ∞^∞ 0^∞ ∞^0 1^∞ ∞^1

Pause and Think !!!

How to check which of the following are indeterminate forms ???

0^0 ∞^∞ 0^∞ ∞^0 1^∞ ∞^1

Pause and Think !!!

What type of limit questions can have the following forms ???

0^0 ∞^∞ 0^∞ ∞^0 1^∞ ∞^1

Pause and Think !!!

How to find $\lim_{x \to a} f(x)^{g(x)}$????

Steps to find $\lim_{x\to a} f(x)^{g(x)}$

Step 1. Consider $\lim_{x\to a}\left(\ln f(x)^{g(x)}\right)$

$\ln a^b = b\ln a$

$$= \lim_{x\to a} g(x)\ln f(x)$$

$$= \vdots$$

$$= \vdots$$

$$= L$$

Need L'Hospital Rule to find limit L

Step 2. $\lim_{x\to a} f(x)^{g(x)} = e^L$

Pause and Think !!!

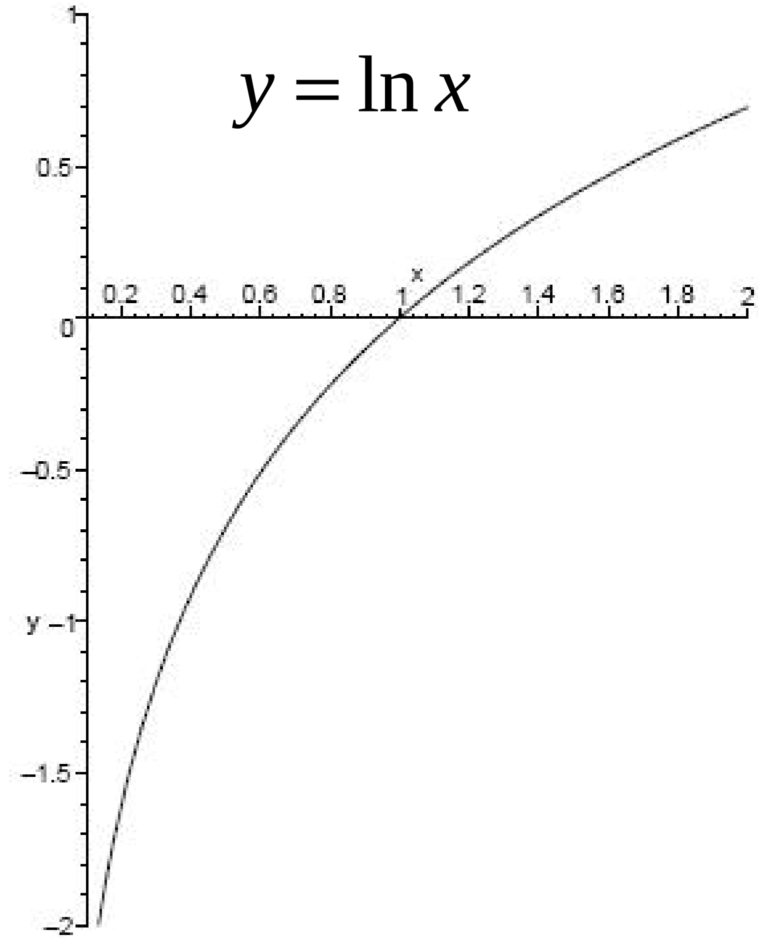
How to check which of the following are indeterminate forms ???

0^0 $\quad \infty^\infty$ $\quad 0^\infty$ $\quad \infty^0$ $\quad 1^\infty$ $\quad \infty^1$

How to check which of the following are indeterminate forms ???

$$\ln x^n = n \ln x$$

From the graph



How to check which of the following are indeterminate forms ???

How to check which of the following are indeterminate forms ???

Answer : Indeterminate forms

Indeterminate forms 0^0 ∞^0 1^∞

$$\lim_{x\to 0^+} x^x \quad 0^0$$

$$\lim_{x\to 0^+} \left(\frac{1}{x}\right)^{\tan x} \quad \infty^0$$

$$\lim_{x\to 0^+} \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}} \quad 1^\infty$$

$$\lim_{x \to 0^+} x^x$$ 0^0

Step 1. $$\lim_{x \to 0^+} \ln x^x = \lim_{x \to 0^+} x \ln x$$ $0 \cdot \infty$

$$= \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}}$$ rewrite to $\frac{\infty}{\infty}$

Applying L'H $$= \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

$$= \lim_{x \to 0^+} - x$$

$$= 0$$

Step 2. $$\lim_{x \to 0^+} x^x = e^0 = 1$$

Pause and Think !!!

Can we always use L'Hospital's Rule

for indeterminate forms to find limit ???????

$$\frac{d}{dx}\sqrt{x^2+1} = \frac{d}{dx}(x^2+1)^{\frac{1}{2}}$$

$$= \frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x)$$

$$= \frac{x}{\sqrt{x^2+1}}$$

$$\boxed{\frac{d}{dx}\sqrt{x^2+1} = \frac{x}{\sqrt{x^2+1}}}$$

Past Exam Question

- Find the value of $\lim_{x\to 0}\dfrac{\cos^2 8x - \cos^2 5x}{x^2}$.

$$\lim_{x\to 0}\frac{\cos^2 8x - \cos^2 5x}{x^2}$$

$$=\left(\lim_{x\to 0}\frac{\cos 8x - \cos 5x}{x^2}\right)\left[\lim_{x\to 0}(\cos 8x + \cos 5x)\right]$$

$$=2\lim_{x\to 0}\frac{-8\sin 8x + 5\sin 5x}{2x}$$

$$=\lim_{x\to 0}(-64\cos 8x + 25\cos 5x) = -39$$

Using $\lim_{y \to 0} \frac{\sin y}{y} = 1$, evaluate the following limits.

(a) $\lim_{x \to 0} \frac{\sin 3x}{\sin 2x}$ (b) $\lim_{x \to 0} \frac{\tan 4x}{\tan 3x}$

(a) $$\lim_{x\to 0}\frac{\sin 3x}{\sin 2x} = \lim_{x\to 0}\frac{\sin 3x}{3x}\times\frac{2x}{\sin 2x}\times\frac{3}{2}$$

$$= \frac{3}{2}\lim_{x\to 0}\frac{\sin 3x}{3x}\lim_{x\to 0}\frac{2x}{\sin 2x}$$

$$= \frac{3}{2}(1)(1)$$

$$= \frac{3}{2}$$

Note using the result $\lim_{y\to 0}\frac{\sin y}{y} = 1$, we have

$$\lim_{y\to 0}\frac{y}{\sin y} = 1 \quad \lim_{x\to 0}\frac{\sin 3x}{3x} = 1 \quad \lim_{x\to 0}\frac{2x}{\sin 2x} = 1$$

$$\text{(b)} \lim_{x \to 0} \frac{\tan 4x}{\tan 3x} = \lim_{x \to 0} (\tan 4x) \div (\tan 3x)$$

$$= \lim_{x \to 0} \left(\frac{\sin 4x}{\cos 4x} \right) \div \left(\frac{\sin 3x}{\cos 3x} \right)$$

$$\tan q = \frac{\sin q}{\cos q}$$

$$= \lim_{x \to 0} \left(\frac{\sin 4x}{\cos 4x} \right) \times \left(\frac{\cos 3x}{\sin 3x} \right)$$

$$= \lim_{x \to 0} \left(\frac{\sin 4x}{4x} \right) \times \frac{1}{\cos 4x} \times \left(\frac{3x}{\sin 3x} \right) \times \cos 3x \times \frac{4}{3}$$

$$= (1) \times \frac{1}{\cos 0} \times (1) \times \cos 0 \times \frac{4}{3}$$

$$\cos(0) = 1$$

$$= \frac{4}{3}$$

$$\lim_{x \to 0} \frac{\sin 4x}{4x} = 1 \qquad \lim_{x \to 0} \frac{3x}{\sin 3x} = 1$$

End