Question:

How to prove that $P = \frac{1}{T_p}\int_{t_o}^{t_o+T_p} \left|x_p(t)\right|^2 dt = \sum_{k=-\infty}^{\infty} \left|X_k\right|^2$ on page 2.6 in Chapter 2 of the lecture notes?

Answer:

The Fourier series expansion of $x_p(t)$ is given by

$$x_p(t) = \sum_{k=-\infty}^{\infty} X_k \exp\left(j2\pi \frac{k}{T_p} t \right) \quad \cdots\cdots \quad (1)$$

Since $x_p(t)$ is periodic with period T_p, its average power P can be computed by averaging over one period, that is

$$P = \frac{1}{T_p}\int_0^{T_p} \left|x_p(t)\right|^2 dt \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (2).$$

Substituting (1) into (2), we get

$$\begin{aligned}
P &= \frac{1}{T_p}\int_0^{T_p} \left| \sum_{k=-\infty}^{\infty} X_k \exp\left(j2\pi \frac{k}{T_p} t \right) \right|^2 dt \\
&= \frac{1}{T_p}\int_0^{T_p} \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} X_k X_m^* \exp\left(j2\pi \frac{k}{T_p} t \right) \exp\left(-j2\pi \frac{m}{T_p} t \right) dt \\
&= \frac{1}{T_p} \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} X_k X_m^* \int_0^{T_p} \exp\left(j2\pi \frac{k-m}{T_p} t \right) dt \\
&= \frac{1}{T_p} \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} X_k X_m^* \left[\frac{\exp\left(j2\pi \frac{k-m}{T_p} t \right)}{j2\pi \frac{k-m}{T_p}} \right]_0^{T_p} \\
&= \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} X_k X_m^* \underbrace{\left[\frac{\exp\left(j2\pi (k-m) \right) - 1}{j2\pi (k-m)} \right]}_{= \begin{cases} 1; & m=k \\ 0; & m \neq k \end{cases}} = \sum_{k=-\infty}^{\infty} X_k X_k^* \\
&= \sum_{k=-\infty}^{\infty} \left|X_k\right|^2
\end{aligned}$$

Remarks: Another proof which makes use of the Fourier transform is given on page 3.7 in Chapter 3 of the lecture notes.
