Geometric Series

For $a \neq 0$, the series

$$\boxed{a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}}$$

is called a geometric series, where a and r are fixed numbers, a is called the first term and r is the (common) ratio.

Geometric Series

For this series, the n-th partial sum s_n is given by

$$s_n = a + \cancel{ar} + \cancel{ar^2} + \cdots + \cancel{ar^{n-1}}$$

$$rs_n = \cancel{ar} + \cancel{ar^2} + ar^3 + \cdots + \cancel{ar^{n-1}} + ar^n.$$

$$s_n - rs_n = a - ar^n$$

$$s_n = a\frac{1-r^n}{1-r}$$

$$r \neq 1$$

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$$

(i) $r = 1$ $\quad a + a + a + a + \cdots$

Then $s_n = na \to \infty$ if $a > 0$ (or $-\infty$ if $a < 0$)

Thus, the series is ***divergent***.

(ii) $r = -1$ $\quad a - a + a - a + \cdots$

Then $\{s_n\}$ is $\quad a,\ 0,\ a,\ 0,\ \cdots$

Thus, the series is ***divergent***.

$$s_n = a\frac{1-r^n}{1-r}$$

(iii) If $|r|<1$, then $r^n \to 0$.

Thus, $$s_n \to \frac{a}{1-r}.$$

Hence, the sum of the series is $\frac{a}{1-r}$.

(iv) If $|r|>1$, then $r^n \to \infty$ (or $-\infty$), and the series diverges.

Convergence of Geometric Series

The geometric series

$$\boxed{a + ar + ar^2 + \cdots + ar^{n-1} + \cdots}$$

with $a \neq 0$ converges to the sum

$$\boxed{\frac{a}{1-r} \qquad \text{if } |r| < 1}$$

and

it diverges if $|r| \geq 1$.

Example

(i) $\dfrac{1}{9}+\dfrac{1}{27}+\dfrac{1}{81}+\cdots$ is a geometric series

first term $a=\dfrac{1}{9}$ and common ratio $r=\dfrac{1}{3}$.

It converges to $\dfrac{a}{1-r}=\dfrac{\frac{1}{9}}{1-\frac{1}{3}}$

$$=\frac{1}{6}.$$

Example

(ii) $4-2+1-\frac{1}{2}+\frac{1}{4}-\cdots$

first term $a=4$ and common ratio $r=-\frac{1}{2}$.

$$
\begin{aligned}
4+4\left(-\frac{1}{2}\right)+4\left(-\frac{1}{2}\right)^2+\cdots &= \frac{a}{1-r} \\
&= \frac{4}{1-\left(-\frac{1}{2}\right)} \\
&= \frac{8}{3}
\end{aligned}
$$

Some Rules on Series

If $\sum a_n = A$, and $\sum b_n = B$, then

(1) Sum rule: $\sum (a_n + b_n) = A + B$.

(2) Difference rule: $\sum (a_n - b_n) = A - B$.

(3) Constant multiple rule: $\sum (ka_n) = kA$.

Question

$$\text{Infinite Series}: \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

How to check a given infinite series is convergent ???

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$$\text{Infinite Series}: \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

How to check a given infinite series is convergent ???

Consider the ***partial sum*** $s_n = a_1 + a_2 + \cdots + a_n.$

If $\lim_{n\to\infty} s_n = L,$ then we have

$$a_1 + a_2 + \cdots + a_n + \cdots = L$$

$$\sum_{n=1}^{\infty} a_n = L$$

Ratio Test

Let $\sum a_n$ be a series, and let

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = r.$$

Then

(1) the series converges if $r < 1$.

(2) the series diverges if $r > 1$.

(3) no conclusion if $r = 1$.

Ratio Test - Example

(i) $\sum a_n$ where $a_1 = 1$ and $a_{n+1} = \dfrac{n}{2n+1} a_n$

To find a_2, put $n = 1$

$$a_{1+1} = \frac{1}{2(1)+1} a_1$$

$$a_2 = \frac{1}{3}$$

To find a_3, put $n = 2$

$$a_{2+1} = \frac{2}{2(2)+1} a_2$$

$$a_3 = \frac{2}{5} \cdot \frac{1}{3}$$

The series is

$$\sum a_n = 1 + \frac{1}{3} + \frac{1 \cdot 2}{3 \cdot 5} + \frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7} + \cdots.$$

(i) $\sum a_n$ where $a_1 = 1$ and $a_{n+1} = \frac{n}{2n+1} a_n$

The series is

$$\sum a_n = 1 + \frac{1}{3} + \frac{1\cdot 2}{3\cdot 5} + \frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7} + \cdots.$$

From $a_{n+1} = \frac{n}{2n+1} a_n$, we have $\frac{a_{n+1}}{a_n} = \frac{n}{2n+1}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n}{2n+1}$$

$$= \frac{\frac{n}{n}}{\frac{2n}{n} + \frac{1}{n}}$$

$$= \frac{1}{2+\frac{1}{n}} \rightarrow \frac{1}{2} \quad \text{as } n \rightarrow \infty.$$

By ratio test,

the given series is convergent.

$\frac{1}{n} \rightarrow 0$ as $n \rightarrow \infty$

Note

The factorial of a non-negative integer n, denoted by $n!$, is given by

$$n! = 1 \times 2 \times 3 \times \cdots \times n$$

$$5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$$

Note that :

$$(n+1)! = 1 \times 2 \times 3 \times \cdots \times n \times (n+1) = n! \times (n+1)$$

Thus, we have $\dfrac{(n+1)!}{n!} = \dfrac{n! \times (n+1)}{n!} = n+1.$

Ratio Test - Example

(ii) Determine if $\sum \frac{(n!)^2}{(2n)!}$ is convergent.

$$a_n = \frac{(n!)^2}{(2n)!} = \frac{n!n!}{(2n)!}$$

Replace n by $n+1$

$$a_{n+1} = \frac{(n+1)!(n+1)!}{(2n+2)!}$$

$$\frac{a_n}{a_{n+1}} = \frac{(n+1)!(n+1)!}{(2n+2)!}\frac{(2n)!}{n!n!}$$

$$= \frac{(n+1)\cdot n!(n+1)\cdot n!}{(2n+2)(2n+1)\cdot(2n)!}\frac{(2n)!}{n!n!}$$

Note that :

$(2n+2)! = (2n+2)(2n+1)\cdot(2n)!$

$$= \frac{(n+1)(n+1)}{(2n+2)(2n+1)}$$

$$= \frac{n+1}{2(2n+1)} = \frac{1+\frac{1}{n}}{2(2+\frac{1}{n})} \rightarrow \frac{1}{2(2)} = \frac{1}{4}.$$

By ratio test, the given series is convergent.

Ratio Test - Example

(iii) Determine if $\sum \frac{3^n}{2^n+5}$ is convergent.

$$a_n = \frac{3^n}{2^n+5}$$

Replace n by $n+1$

$$a_{n+1} = \frac{3^{n+1}}{2^{n+1}+5}$$

$$\frac{a_{n+1}}{a_n} = \frac{3^{n+1}}{2^{n+1}+5} \cdot \frac{2^n+5}{3^n}$$

$$= 3\,\frac{2^n+5}{2^{n+1}+5}$$

$$= 3 \cdot \frac{1+\frac{5}{2^n}}{2+\frac{5}{2^n}} \to \frac{3}{2}$$

Divide by 2^n

$$\frac{5}{2^n} \to 0 \quad \text{as} \quad n \to \infty$$

By ratio test, the given series is divergent.

Ratio Test - Example

(iv) Determine if the Harmonic series $\sum \frac{1}{n}$ is convergent.

$a_n = \frac{1}{n}$

Replace n by $n+1$

$a_{n+1} = \frac{1}{n+1}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n}{n+1}$$

$$= \frac{1}{1+\frac{1}{n}} \to 1.$$

We cannot draw conclusion from ratio test.

Ratio Test - Example

(v) Determine if $\sum \frac{1}{n^2}$ is convergent.

$$a_n = \frac{1}{n^2}$$

Replace n by $n+1$

$$a_{n+1} = \frac{1}{(n+1)^2}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n^2}{(n+1)^2}$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)^2} \to 1.$$

We cannot draw conclusion from ratio test.

Ratio Test - Example

(iv) Determine if $\sum \frac{1}{n}$ is convergent.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n}{n+1}$$

$$= \frac{1}{1+\frac{1}{n}} \to 1.$$

(v) Determine if $\sum \frac{1}{n^2}$ is convergent.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n^2}{(n+1)^2}$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)^2} \to 1.$$

We cannot draw conclusion from ratio test.

It can be shown that $\sum \frac{1}{n}$ is divergent.

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Ratio Test - Example

To show the Harmonic series is divergent.

$$\sum \frac{1}{n} = 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots$$

$$> 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots$$

$$> 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) + \cdots$$

Thus, the Harmonic series is divergent.

p-series

The ***p* - *series*** is the series

$$\sum \frac{1}{n^p}$$

for any non-negative real number *p*.

(i) It diverges if $0 \leq p \leq 1$.

(ii) It converges if $p > 1$.