Outline

Power Series

Definition 11.1

Let k be a non-negative integer and a be a real number. Define the generalized k-th falling factorial by

$$(a)_{\underline{k}} = a(a-1)(a-2)\cdots(a-k+1).$$

- $(a)_{\underline{0}} = 1$
- We similarly define the generalized binomial coefficients by $\binom{a}{k} = \frac{(a)_{\underline{k}}}{k!}$
- $(a)_{\underline{k}}$ and $\binom{a}{k}$ coincide with our previous definitions when a is a non-negative integer.

Power Series

Theorem 11.2 (Newton)

Let a be a real number and $|x| < 1$. Then

$$(1+x)^a = \sum_{n=0}^{\infty} \binom{a}{n} x^n.$$

- When a is a non-negative integer, $\binom{a}{n} = 0$ if $n > a$ and we get a finite sum.
- For other a, this series converges for $|x| < 1$.

Power Series

Example 11.3

$$\sqrt{1+x} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^n = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots.$$

- $\sqrt{1+0.02} \approx 1.009950494$
- Using $x = 0.02$ in the RHS with four terms gives 1.009950500

Power Series

Example 11.4

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} \binom{-1}{n} (-x)^n = \sum_{n=0}^{\infty} x^n.$$

- Well known geometric series which converges for $|x| < 1$
- $(-1)_{\underline{n}} = (-1)(-2)\cdots(-n) = (-1)^n n!$

Power Series

Example 11.5

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} \binom{-2}{n} (-x)^n = \sum_{n=0}^{\infty} (n+1)x^n.$$

- $(-2)_{\underline{n}} = (-2)(-3)\cdots(-n)(-n-1) = (-1)^n(n+1)!$
- We can write the RHS as $\sum_{n=1}^{\infty} nx^{n-1}$
- Derivative of the geometric series.

Formal Power Series

A generating function is a clothesline on which we hang up a sequence of numbers for display.
– Herb Wilf in **generatingfunctionology**

Definition 11.6

Let a_n be an infinite sequence of numbers, then the (ordinary) generating function of a_n is the formal power series of the form

$$\sum_{n \geq 0} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots .$$

Formal Power Series

Definition 11.7

Let $A(x) = \sum_n a_n x^n$ and $B(x) = \sum_n b_n x^n$ be formal power series then we define formally

- $(A+B)(x) = \sum_n (a_n + b_n)x^n.$
- $A(x)B(x) = (A \cdot B)(x) = \sum_n \left(\sum_{i=0}^{n} a_i b_{n-i} \right) x^n.$
- $A'(x) = \sum_{n \geq 1} n a_n x^{n-1} = \sum_{n \geq 0} (n+1) a_{n+1} x^n.$
- $\int A(x) = \sum_{n \geq 0} \frac{a_n}{n+1} x^{n+1} + C.$

Formal Power Series

Definition 11.8

Let $A(x) = \sum_n a_n x^n$ and $B(x) = \sum_n b_n x^n$ be formal power series such that $A(x)B(x) = 1$, then we write $B(x) = \frac{1}{A(x)}$ and call it the inverse of $A(x)$.

Example 11.9

- *Let $A(x) = 1 - x$, i.e $a_0 = 1, a_1 = -1, a_i = 0$ for all $i \geq 2$*
- *$B(x) = \sum_n x^n$, i.e $b_i = 1$ for all i*
- *$A(x)B(x) = C(x) = 1$ since $c_0 = a_0 b_0 = 1$*
- *and $c_n = \sum_{i=0}^{n} a_i b_{n-i} = \sum_{i=0}^{n} a_i = 0$ for $n \geq 1$.*
- *Hence $B(x) = \frac{1}{1-x}$*

Formal Power Series

Theorem 11.10 (Leibniz's Rule)

$(A(x)B(x))' = A'(x)B(x) + A(x)B'(x).$

- Proof: LHS $= \sum_n (n+1) \left(\sum_{i=0}^{n+1} a_i b_{n+1-i} \right) x^n$
- $A'(x)B(x) = \sum_n \left(\sum_{i=0}^{n} (i+1) a_{i+1} b_{n-i} \right) x^n$
- $A(x)B'(x) = \sum_n \left(\sum_{i=0}^{n} a_i b_{n+1-i} (n+1-i) \right) x^n$

$$\begin{aligned}
[x^n]RHS &= \sum_{i=0}^{n} (i+1) a_{i+1} b_{n-i} + \sum_{i=0}^{n} a_i b_{n+1-i} (n+1-i) \\
&= (n+1) a_{n+1} b_0 + \sum_{j=1}^{n} j a_j b_{n-j+1} + a_j b_{n+1-j} (n+1-j) \\
&\quad + a_0 b_{n+1} (n+1)
\end{aligned}$$

Formal Power Series

Example 11.5: Formal version

$$\frac{1}{(1-x)^2} = \sum_{n \geq 0} (n+1)x^n.$$

Proof: Direct multiplication

- $\frac{1}{(1-x)^2}$ is the inverse of $(1-x)^2$
- Let $A(x) = B(x) = \frac{1}{1-x} = \sum_n x^n$
- $A(x)B(x) = \sum_n \left(\sum_{i=0}^n a_i b_{n-i} x^n\right)$
- Since $a_n = b_n = 1$, $\sum_{i=0}^n a_i b_{n-i} = n+1$.

Formal Power Series

2nd proof of Example 11.5

$$\frac{1}{(1-x)^2} = \sum_{n\geq 0}(n+1)x^n.$$

Proof: Using formal differentiation

- If $\frac{1}{A(x)} = B(x) \implies \left(\frac{1}{A(x)}\right)' = B'(x)$
- Now $1 = A(x)B(x) \implies 0 = A'(x)B(x) + A(x)B'(x)$
- so $B'(x) = -A'(x)B^2(x) = \frac{-A'(x)}{A^2(x)}$
- Apply this result to $\frac{1}{1-x} = \sum_n x^n$

Formal Power Series

Theorem 11.11 (Quotient Rule for Formal Power Series)

If $C(x) = \frac{A(x)}{B(x)}$ *is a formal power series,*

$$C'(x) = \frac{A'(x)B(x) - A(x)B'(x)}{B^2(x)}.$$

Proof:

- Since
  $A(x) = C(x)B(x) \implies A'(x) = C'(x)B(x) + C(x)B'(x)$
- Hence $C'(x) = \frac{A'(x) - C(x)B'(x)}{B(x)}$.

Formal Power Series

Example 11.12

$$\frac{x}{1-kx} = \sum_{n\geq 1} k^{n-1}x^n,$$

is the generating function for the sequence $a_0 = 0$ *and* $a_n = k^{n-1}$ *for all* $n \geq 1$

- $\frac{1}{1-kx} = \sum_{n\geq 0}(kx)^n$
- Now multiply x to both sides
- $\frac{x}{1-kx} = \sum_{n\geq 0} k^n x^{n+1} = \sum_{n\geq 1} k^{n-1}x^n.$

Formal Power Series

Theorem 11.13

For positive integer m,

$$\frac{1}{(1-x)^m} = \sum_{n \geq 0} \binom{m+n-1}{n} x^n.$$

- By induction on m, $m = 1$ true by Example 11.4
- $\frac{1}{(1-x)^m} = \frac{1}{(1-x)^{m-1}} \frac{1}{1-x} = \sum_{n \geq 0} c_n x^n$ where

$$c_n = \sum_{i=0}^{n} \binom{m-1+i-1}{i} = \sum_{i=0}^{n} \binom{m-2+i}{m-2} = \binom{m-1+n}{m-1}$$

by Theorem 3.23.

Formal Power Series

Theorem 11.13

For positive integer m,

$$\frac{1}{(1-x)^m} = \sum_{n \geq 0} \binom{m+n-1}{n} x^n.$$

- Remark: Assuming Newton's Generalized Binomial theorem
- $(1-x)^{-m} = \sum_{n\geq 0} \binom{-m}{n}(-x)^n$.
- Now $(-m)_{\underline{n}} = (-1)^n (m+n-1)_{\underline{n}}$.

Example 11.14

Find the coefficient of x^{30} in the expansion

$$A(x) = (x^3 + x^4 + x^5 + \cdots)^6.$$

$$\begin{aligned} A(x) &= x^{18}\left(\sum_{n\geq 0} x^n\right)^6 = \frac{x^{18}}{(1-x)^6} \\ &= x^{18}\sum_{n\geq 0}\binom{5+n}{n}x^n, \text{ by Theorem 11.13} \\ \Longrightarrow [x^k]A(x) &= [x^k]\sum_{n\geq 0}\binom{5+n}{5}x^{18+n} = \binom{5+k-18}{5} \end{aligned}$$

Answer $\binom{17}{5}$.

Example 11.15

Find the generating function for the sequence $a_r = 3r + 5$.

- $\dfrac{x}{(1-x)^2} = \sum_{r \geq 0} r x^r$
- $\dfrac{1}{1-x} = \sum_{r \geq 0} x^r$
- Hence $A(x) = \sum_{r \geq 0} a_r x^r = \dfrac{3x}{(1-x)^2} + \dfrac{5}{1-x}$.

Example 11.16

Find the generating function for the sequence $a_r = r^2$.

- $\dfrac{x}{(1-x)^2} = \sum_{r\geq 0} r x^r$
- $\left(\dfrac{x}{(1-x)^2}\right)' = \sum_{r\geq 0} r^2 x^{r-1}$
- Hence the generating function is

$$x\left(\frac{(1-x)^2-(-2)x(1-x)}{(1-x)^4}\right) = \frac{x(1+x)}{(1-x)^3}.$$

Example 11.17

Let $a_0 = 1$ and suppose that $\binom{n+2}{2} = \sum_{i=0}^{n} a_i a_{n-i}$, for all integers $n \geq 1$. Find an explicit formula for a_n.

We first compute some explicit values

- $n = 0$, $1 = a_0 a_0$
- $n = 1$, $3 = a_0 a_1 + a_1 a_0 = 2a_1 \implies a_1 = \frac{3}{2}$
- $n = 2$, $6 = a_0 a_2 + a_1^2 + a_2 a_0 \implies a_2 = \frac{1}{2}(6 - \frac{9}{4}) = \frac{15}{8}$.

Observe that in terms of power series

$$\sum_n \binom{n+2}{2} x^n = \sum_n \binom{n+2}{n} x^n = \sum_n \left(\sum_{i=0}^{n} a_i a_{n-i} \right) x^n.$$

Example 11.17 cont'd:

- If we let $A(x) = \sum a_n x^n$, then from Theorem 11.13
- $(1-x)^{-3} = A(x)^2 \implies A(x) = (1-x)^{-\frac{3}{2}} = \sum_n \binom{-\frac{3}{2}}{n}(-x)^n$.

$$\begin{aligned} a_n &= (-1)^n \binom{-\frac{3}{2}}{n} \\ &= (-1)^n \frac{\frac{-3}{2}\frac{-5}{2}\cdots(\frac{-3}{2}-n+1)}{n!} \\ &= (-1)^n \frac{(-3)(-5)\cdots(-2n-1)}{2^n n!} \\ &= \frac{(2n+1)!}{2^{2n}(n!)^2}. \end{aligned}$$

Counting with Generating Functions

- Consider three distinct objects $\{a, b, c\}$. There are 1, 3, 3 and 1 ways to select 0, 1, 2, 3 objects respectively.
- Now consider the polynomial

$$(1+ax)(1+bx)(1+cx) = 1+(a+b+c)x+(ab+bc+ac)x^2+(abc)x^3.$$

- The coefficient of x^n corresponds to the selection of n objects.
- Set $a = b = c = 1$ since we are interested in the # of ways and not the actual choices. In general,

Example 11.18

The # ways to select r objects from $[n]$ is given by the sequence $a_r = \binom{n}{r}$ with the generating function $A(x) = \sum \binom{n}{r} x^r = (1 + x)^n$.

Counting with Generating Functions

- Consider the multiset $\{a, a, b\}$ also denoted by $\{2 \cdot a, 1 \cdot b\}$
- Next expand the polynomial

$$(1 + ax + a^2x^2)(1 + bx) = 1 + (a + b)x + (a^2 + ab)x^2 + (a^2b)x^3.$$

- The coefficient of x^n corresponds to the selection of n objects.
- Again set $a = b = 1$ since we are interested in the # of ways

Example 11.19

Let a_n be the # of ways to select n objects from a multiset of 3 objects in which two of them are identical. Then the generating function for a_n is $(1 + x + x^2)(1 + x)$.

Counting with Generating Functions

Example 11.20

A shop has 2 blue balloons, 2 red balloons, 1 yellow and 1 green balloon. If balloons of the same colour are identical, and you only want to buy four. How # different choices are there?

- Let S be the multiset $\{2 \cdot b, 2 \cdot r, 1 \cdot y, 1 \cdot g\}$
- Let a_n be the # of ways to choose n objects from S.
- Generating function for a_n is $(1 + x + x^2)(1 + x + x^2)(1 + x)(1 + x) = 1 + 4x + 8x^2 + 10x^3 + 8x^4 + 4x^5 + x^6$.
- So $a_4 = 8$
- Instead of expanding the whole generating function, note that $A(x) = (1 + 2x + 2x^2 + x^3)(1 + 2x + 2x^2 + x^3)$.
- Hence $[x^4]A(x) = 2 \times 1 + 2 \times 2 + 1 \times 2 = 8$.

Counting with Generating Functions

Theorem 11.21

Let a_n be the # of ways to select n objects from the multiset $S = \{k_1 \cdot b_1, k_2 \cdot b_2, \cdots, k_m \cdot b_m\}$,
then the generating function for a_n is
$(1 + x + \cdots + x^{k_1})(1 + x + \cdots + x^{k_2}) \cdots (1 + x + \cdots + x^{k_m})$.

Remark: In particular, we may rewrite the generating function as

$$\prod_{i=1}^{m} \frac{1 - x^{k_i+1}}{1 - x} = \frac{1 - x^{k_1+1}}{1 - x} \cdot \frac{1 - x^{k_2+1}}{1 - x} \cdots \frac{1 - x^{k_m+1}}{1 - x}.$$

Counting with Generating Functions

Suppose now that there are no limit to the objects $b_1, \ldots, b_m$ in S, then the generating function is

$$(1+x+x^2+\cdots)(1+x+x^2+\cdots)\cdots(1+x+x^2+\cdots) = \prod_{i=1}^{m} \frac{1}{1-x}.$$

Theorem 11.22

Let a_n be the $\#$ of ways to select n objects from the multiset $S = \{\infty \cdot b_1, \infty \cdot b_2, \cdots, \infty \cdot b_m\}$, then the generating function for a_n is

$$\frac{1}{(1-x)^m} = \sum_{n=0} \binom{m+n-1}{n} x^n = \sum_{n=0} \binom{m+n-1}{m-1} x^n.$$

Remark: a_n is exactly the $\#$ of weak compositions of n into m parts. See Example 5.6.

Counting with Generating Functions

Example 11.23

3 boys each tosses a dice. Find the # ways for them to get a total of 14.

Let a_n be the # of ways to get 14, then the generating function is

$$\begin{aligned} A(x) &= (x + x^2 + x^3 + x^4 + x^5 + x^6)^3 \\ &= x^3 \left(\frac{1 - x^6}{1 - x}\right)^3 \\ &= x^3 (1 - 3x^6 + 3x^{12} - x^{18}) \sum_{k=0} \binom{k+2}{2} x^k \end{aligned}$$

$$\implies [x^{14}]A(x) = \binom{11+2}{2} - 3\binom{5+2}{2} = 15.$$

Counting with Generating Functions

Example 11.24

A utility crew is painting n poles next to a road, starting from one end. For each pole, they randomly choose a color out of red, green or blue. At at certain time, they notice that their shift will end soon, so they decided to stop work and they choose one out of the remaining unpainted pole, and paint a smiling face on it. How many different looks can the poles have?

- we do not know when they decide to stop, but there is at least one unpainted pole.
- suppose they stop after the k-th pole, then the # of looks
$$a_n = \sum_{k=0}^{n-1} 3^k (n-k) = \sum_{k=0}^{n} 3^k (n-k)$$
- check $a_0 = 0$, $a_1 = 1$, $a_2 = 5$.

Counting with Generating Functions

Example 11.24 cont'd

- If $B(x) = \sum 3^n x^n$ and $C(x) = \sum n x^n$
- $A(x) = B(x)C(x) = \left(\frac{1}{1-3x}\right)\left(\frac{x}{(1-x)^2}\right) = \sum a_n x^n$
- Using partial fractions

$$\begin{aligned} A(x) &= \frac{3}{4(1-3x)} - \frac{1}{4(1-x)} - \frac{1}{2(1-x)^2} \\ &= \frac{3}{4}\sum_{n\geq 0} 3^n x^n - \frac{1}{4}\sum_{n\geq 0} x^n - \frac{1}{2}\sum_{n\geq 0}(n+1)x^n \\ &= \sum_{n\geq 0} \frac{1}{4}\left(3^{n+1} - 1 - 2(n+1)\right) x^n. \end{aligned}$$

Counting with Generating Functions

Example 11.25

A student makes a study plan for final exam. She splits the study period into two parts, in the first she devotes each day to either algebra or analysis, except on one day where she only study the chosen subject in the morning. In the second part, she studies combinatorics. Because she thinks it's easy, she will take a break for 2 days which need not be consecutive. How many different study plans are there?

- If first part has k days, she has $k2^k$ choices.
- She has $\binom{n-k}{2}$ choices in the second part.
- By product principle, the # is $\sum_{k=1}^{n} k2^k \binom{n-k}{2}$

Counting with Generating Functions

Example 11.25 cont'd

- $B(x) = \sum n2^n x^n = \frac{2x}{(1-2x)^2}$ and $C(x) = \sum \binom{n}{2} x^n = \frac{x^2}{(1-x)^3}$
- $A(x) = B(x)C(x) = \frac{2x^3}{(1-2x)^2(1-x)^3}$
- Using partial fractions

$$\begin{aligned} A(x) &= \frac{2}{(1-x)^3} + \frac{2}{(1-x)^2} + \frac{6}{(1-x)} + \frac{2}{(1-2x)^2} - \frac{12}{1-2x} \\ &= \sum_{n\geq 0}(n+2)(n+1)x^n + \sum_{n\geq 0} 2(n+1)x^n + \sum_{n\geq 0} 6x^n \\ &\quad + \sum_{n\geq 0} 2(n+1)2^n x^n - \sum_{n\geq 0} 12 \cdot 2^n x^n \end{aligned}$$

$$[x^n]A(x) = 2^{n+1}(n-5) + (n+1)(n+4) + 6$$

Counting with Generating Functions

Example 10.22 Third solution

Find the number of nonnegative integer solution to $x_1 + x_2 + x_3 = 15$, where $x_1 \leq 5, x_2 \leq 6, x_3 \leq 7$.

The generating function is

$$
\begin{aligned}
A(x) &= \frac{1-x^6}{1-x} \cdot \frac{1-x^7}{1-x} \cdot \frac{1-x^8}{1-x} \\
&= (1 - x^6 - x^7 - x^8 + x^{13} + x^{14} + x^{15}) \sum_n \binom{n+2}{2} x^n
\end{aligned}
$$

$$
[x^{15}]A(x) = \binom{17}{2} - \binom{11}{2} - \binom{10}{2} - \binom{9}{2} + \binom{4}{2} + \binom{3}{2} + \binom{2}{2}.
$$

Partitions

Example 11.26

Let a_n denote the # of partitions of n into parts of size 1, 2 or 3. Then the generating function $A(x) = \dfrac{1}{(1-x)(1-x^2)(1-x^3)}$.

$$(1+x+x^2+x^3+\ldots)(1+x^2+x^4+x^6+\ldots)(1+x^3+x^6+x^9+\ldots).$$
$$= 1+x+2x^2+3x^3+4x^4+5x^5+7x^6+8x^7+10x^8+12x^9+\ldots$$

- The three products contribute parts of sizes 1,2 and 3.
- $a_3 = 3$ since $3 = 2+1 = 1+1+1$.
- Note that the x^3 from the first product contributes 1+1+1 while the x^3 from the third product contributes 3.

Partitions

Example 11.27

From previous example, there are 8 partitions of 7 into parts of sizes 1,2 and 3.

	$\frac{1}{1-x}$	$\frac{1}{1-x^2}$	$\frac{1}{1-x^3}$	
1	x		x^6	$1+3+3$
2		x^4	x^3	$2+2+3$
3	x^4		x^3	$1+1+1+1+3$
4	x^2	x^2	x^3	$1+1+2+3$
5	x^1	x^6		$1+2+2+2$
6	x^3	x^4		$1+1+1+2+2$
7	x^5	x^2		$1+1+1+1+1+2$
8	x^7			$1+1+1+1+1+1+1$

Partitions

Theorem 11.28

The generating function for $p(n)$, the # of partitions of n, is given by

$$\sum_{n\geq 0} p(n)x^n = \prod_{n=1}^{\infty} \frac{1}{1-x^n}.$$

$$\begin{aligned} P(x) = 1 &+ x + 2\,x^2 + 3\,x^3 + 5\,x^4 + 7\,x^5 + 11\,x^6 + 15\,x^7 \\ &+ 22\,x^8 + 30\,x^9 + 42\,x^{10} + 56\,x^{11} + 77\,x^{12} + 101\,x^{13} \\ &+ 135\,x^{14} + 176\,x^{15} + 231\,x^{16} + 297\,x^{17} + 385\,x^{18} \\ &+ 490\,x^{19} + 627\,x^{20} + \cdots \end{aligned}$$

Partitions

Example 11.29

Let a_n denote the # of partitions of n into distinct parts of size 1, 2, 3 or 4. Then the generating function

$$A(x) = (1+x)(1+x^2)(1+x^3)(1+x^4).$$

$$A(x) = 1 + x + x^2 + 2\,x^3 + 2\,x^4 + 2\,x^5 + 2\,x^6 + 2\,x^7 + x^8 + x^9 + x^{10}$$

- Note that each part is used at most once.
- $a_6 = 2$ since only $4+2$ and $3+2+1$ are allowed.

Partitions

Theorem 11.30

Let $p_d(n)$ denote the # of partitions of n into distinct parts, then the generating function

$$\sum_{n\geq 0} p_d(n)x^n = \prod_{n=1}^{\infty} (1+x^n).$$

Theorem 11.31

Let $p_{odd}(n)$ denote the # of partitions of n into odd parts, then the generating function

$$\sum_{n\geq 0} p_{odd}(n)x^n = \prod_{n=1}^{\infty} \frac{1}{1-x^{2n-1}}.$$

Partitions

Theorem 11.32 (Euler)

 # partitions of n into odd parts is equal to # partitions of n into distinct parts.

$$\begin{aligned}
\sum_{n=1}^{\infty} p_{odd}(n)x^n &= \frac{1}{\prod(1-x^{2k-1})} \\
&= \frac{1}{\prod(1-x^{2k-1})} \times \frac{\prod(1-x^{2k})}{\prod(1-x^{2k})} \\
&= \frac{\prod(1-x^k)(1+x^k)}{\prod(1-x^k)} = (1+x)(1+x^2)(1+x^3)\dots \\
&= \sum_{n=1}^{\infty} p_d(n)x^n
\end{aligned}$$

Partitions

Theorem 11.33

 # partitions of n into parts, each of which appears at most twice, is equal to # partitions of n into parts whose sizes are not divisible by 3.

- Consider $n = 6$, $p(6) = 11$
- With the restriction that each part appears at most twice, we have only 7 partitions
- 6=5+1=4+2=4+1+1=3+3=3+2+1=2+2+1+1
- Consider partitions with sizes not divisible by 3
- 5+1=4+2=4+1+1=2+2+2=2+2+1+1= 2+1+1+1+1=1+1+1+1+1+1

Partitions

Theorem 11.33

\# partitions of n into parts, each of which appears at most twice, is equal to # partitions of n into parts whose sizes are not divisible by 3.

Proof:

$$\begin{aligned}\prod_{k=1}^{\infty}(1+x^k+x^{2k}) &= \prod_{k=1}^{\infty}(1+x^k+x^{2k}) \times \frac{(1-x^k)}{(1-x^k)} \\ &= \frac{\prod_{k=1}^{\infty}(1-x^{3k})}{\prod_{k=1}^{\infty}(1-x^k)} \\ &= \prod_{k=1, 3\nmid k}^{\infty} \frac{1}{(1-x^k)}\end{aligned}$$

Partitions

Example 11.34

Let a_n denote the # of ways to distribute n identical balls into 3 identical boxes such that no box is empty. Then the generating function $A(x) = \dfrac{x^3}{(1-x)(1-x^2)(1-x^3)}$.

- Equivalent to partition of n into exactly 3 parts, $p(n, 3)$
- By the conjugate map, equivalent to partitions into parts not exceeding 3 with at least one part of 3.
- $A(x) = (1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(x^3 + x^6 + \cdots)$

Euler's Pentagonal Number Theorem

Recall Theorem 7.15

Let n be a positive integer, then

$$p_{d,e}(n) - p_{d,o}(n) = \begin{cases} (-1)^j & \text{if } n = \frac{3j^2 \pm j}{2} \\ 0 & \text{otherwise} . \end{cases}$$

- How to model $p_{d,e}$, partitions of n into an even # of parts?
- Replace x^k by $-x^k$, then for even # of distinct parts $(-x^a)(-x^b)(-x^c)(-x^d) = x^{a+b+c+d}$
- for odd # of parts $(-x^a)(-x^b)(-x^c) = -x^{a+b+c}$.

Theorem 11.35 (Euler's Pentagonal Number Theorem)

$$\prod_{n=1}^{\infty}(1-x^n) = \sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{3n^2+n}{2}} = 1 - x - x^2 + x^5 + x^7 \ldots .$$

- The power series is the inverse of the generating function for $p(n)$
- $1 = \left(\sum_n p(n)x^n\right)\left(\sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{3n^2+n}{2}}\right)$
- Extracting coefficients of x^n gives $p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) + \ldots = 0.$

Sylvester's result

Theorem 11.36

The # of self-conjugate partitions of n is equal to # of partitions of n into distinct odd parts.

- The result can be proved by using a bijection
- The generating function version is the following

$$\prod_{n=1}^{\infty}(1+x^{2n-1}) = 1 + \sum_{n\geq 1} \frac{x^{n^2}}{\prod_{k=1}^{n}(1-x^{2k})}.$$

Angpao Problem

Example 11.37

During Chinese New Year, you received a total of $48 in angpao money. (A lean year.) You remembered receiving angpaos of $2, $4, $8 and $10, but you could not recall exactly how many angpaos you received. How many different ways could you have received your $48?

- Reduces to $y_1 + 2y_2 + 4y_3 + 5y_4 = 12, \quad y_i \geq 0$.
- Generating function $\frac{1}{(1-x)(1-x^2)(1-x^4)(1-x^5)}$
- Using a computer to compute up to x^{12}

$$A(x) = 1 + x + 2\,x^2 + 2\,x^3 + 4\,x^4 + 5\,x^5 + 7\,x^6 + 8\,x^7 + 11\,x^8 + 13\,x^9 + 17\,x^{10} + 19\,x^{11} + 24\,x^{12} + \cdots$$

McNugget Problem

Example 11.38

You can buy chicken McNuggets in boxes of 6, 9 or 20. Hence it is impossible to buy exactly 1, 2, 3 or even 19 pieces of McNuggets. What is the largest number of McNuggets that cannot be bought exactly?

- partition of n into parts which are multiples of 6, 9 or 20.
- Generating function $A(x) = \frac{1}{(1-x^6)(1-x^9)(1-x^{20})}$
- Using a computer to compute up to x^{50}

$$\begin{aligned} A(x) = 1 &+ x^6 + x^9 + x^{12} + x^{15} + 2\,x^{18} + x^{20} + x^{21} + 2\,x^{24} + x^{26} \\ &+ 2\,x^{27} + x^{29} + 2\,x^{30} + x^{32} + 2\,x^{33} + x^{35} + 3\,x^{36} + 2\,x^{38} \\ &+ 2\,x^{39} + x^{40} + x^{41} + 3\,x^{42} + 2\,x^{44} + 3\,x^{45} + x^{46} + 2\,x^{47} \\ &+ 3\,x^{48} + x^{49} + 2\,x^{50} + \cdots \end{aligned}$$

Happy McNugget Problem

Example 11.39

You can buy chicken McNuggets in boxes of 6, 9 or 20. If you can also buy a happy meal consisting of 4 McNuggets plus a toy. What is the largest number of McNuggets that cannot be bought exactly?

- Generating function $A(x) = \frac{1}{(1-x^4)(1-x^6)(1-x^9)(1-x^{20})}$
- Using a computer to compute up to x^{16}

$$\begin{aligned} A(x) = 1 + x^4 + x^6 + x^8 + x^9 + x^{10} \\ + 2\,x^{12} + x^{13} + x^{14} + x^{15} + 2\,x^{16} \end{aligned}$$

- Answer is 11.

Exponential Generating Function

Definition 11.40

Let f_n be an infinite sequence of numbers, then the exponential generating function of f_n is the formal power series of the form

$$\sum_{n\geq 0} f_n \frac{x^n}{n!} = f_0 + f_1 x + f_2 \frac{x^2}{2} + \cdots$$

Definition 11.41

Let $f_n = 1$ for every n, then the exponential generating function, denoted by e^x, is $e^x = \sum_{n\geq 0} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \cdots$

Exponential Generating Function

Example 11.42

Let $f_n = k^n$ for every n, then the exponential generating function is denoted by e^{kx},

$$e^{kx} = \sum_{n \geq 0} k^n \frac{x^n}{n!} = \sum_{n \geq 0} \frac{(kx)^n}{n!}.$$

Example 11.43

Let $f_n = n!$ for every n, then the exponential generating function is

$$\sum_{n \geq 0} n! \frac{x^n}{n!} = \sum_{n \geq 0} x^n = \frac{1}{1-x}.$$

Exponential Generating Function

Theorem 11.44

Let $F(x) = \sum f_n \frac{x^n}{n!}$ *and* $G(x) = \sum g_n \frac{x^n}{n!}$, *then*

$$(F \cdot G)(x) = \sum_{n \geq 0} \left(\sum_{i=0}^{n} \binom{n}{i} f_i g_{n-i} \right) \frac{x^n}{n!}.$$

Proof: By the multiplication of formal power series, coefficient of $(F \cdot G)(x)$ is

$$\sum_{i=0}^{n} \frac{f_i}{i!} \frac{g_{n-i}}{(n-i)!} = \frac{1}{n!} \sum_{i=0}^{n} \binom{n}{i} f_i g_{n-i}.$$

Exponential Generating Function

Example 11.45

Let $F(x) = e^{ax}$ *and* $G(x) = e^{bx}$, *then*

$$(F \cdot G)(x) = \sum_{n \geq 0} \left(\sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i} \right) \frac{x^n}{n!}$$
$$= \sum_{n \geq 0} ((a+b)^n) \frac{x^n}{n!} = e^{(a+b)x}.$$

Counting with Exponential Generating Functions

- Consider three distinct objects $\{a, b, c\}$ and the following polynomial as an exponential generating function.

$$\begin{aligned}&(1+ax)(1+bx)(1+cx)\\&=1+(a+b+c)x+(ab+bc+ac)x^2+(abc)x^3\\&=1+(a+b+c)x+2!(ab+bc+ac)\frac{x^2}{2!}+3!(abc)\frac{x^3}{3!}\end{aligned}$$

- The coefficient of $\frac{x^n}{n!}$ corresponds to the # permutation of n objects.

Counting with Exponential Generating Functions

Example 11.46

The # of permutations of r objects from $[n]$ is given by the sequence $(n)_{\underline{r}}$ with the exponential generating function

$$A(x) = \sum (n)_{\underline{r}} \frac{x^r}{r!} = \sum \binom{n}{r} x^r = (1+x)^n.$$

Remark: Compare this with the ordinary generating function for # of combinations of r objects from $[n]$.

Counting with Exponential Generating Functions

- Consider the multiset $\{a, a, b\}$ and the following polynomial as an exponential generating function.

$$\begin{aligned}
&(1 + ax + a^2\frac{x^2}{2!})(1 + bx) \\
&= 1 + (a+b)x + (ab + \frac{1}{2!}a^2)x^2 + \frac{1}{2!}(a^2b)x^3 \\
&= 1 + (a+b)x + (2!ab + a^2)\frac{x^2}{2!} + \frac{3!}{2!}(a^2b)\frac{x^3}{3!}
\end{aligned}$$

- The coefficient of $\frac{x^n}{n!}$ corresponds to the # of permutations of n objects with repetition.

Counting with Exponential Generating Functions

Theorem 11.47

Let a_n be the # of ways to arrange n objects from the multiset $S = \{k_1 \cdot b_1, k_2 \cdot b_2, \cdots, k_m \cdot b_m\}$,
then the exponential generating function for a_n is

$$(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{k_1}}{k_1!}) \cdots (1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{k_m}}{k_m!}).$$

Counting with Exponential Generating Functions

Suppose that there are no limit to the objects $b_1, \dots, b_k$ in S, then the exponential generating function is
$(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots)^k = e^{kx}$.

Theorem 11.48

Let a_n be the # of permutations of n objects from the multiset $S = \{\infty \cdot b_1, \infty \cdot b_2, \cdots, \infty \cdot b_k\}$, then the exponential generating function for a_n is

$$e^{kx} = \sum_{n=0} \frac{k^n x^n}{n!}.$$

Remark: $a_n = k^n$ is equivalent to the # of ways to distribute n distinct balls into k boxes. Can you find the bijection?

Counting with Exponential Generating Functions

Theorem 11.49

Let a_n be the # of ways to distribute n distinct balls to k distinct boxes, then the exponential generating function for a_n is

$$e^{kx} = \sum_{n=0} \frac{k^n x^n}{n!}.$$

- Equivalent to # of permutations of n objects from the multiset $S = \{\infty \cdot b_1, \infty \cdot b_2, \cdots, \infty \cdot b_k\}$
- For every permutation, if b_j appears in the r-th position, we will distribute the r-th ball to the j-th box.
- For example, a permutation $(n = 4)$ that looks like $b_3 b_2 b_2 b_1$ means ball 1 goes to box 3, ball 2 goes to box 2, ball 3 goes to box 2 and ball 4 goes to box 1.
- This is clearly a bijection.

Counting with Exponential Generating Functions

Example 11.50

In how many ways can 4 letters of PAPAYA be arranged?

- Let a_n be the number of permutations of n objects from the multiset $\{3 \cdot A, 2 \cdot P, 1 \cdot Y\}$.
- exponential generating function

$$(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!})(1 + x + \frac{x^2}{2!})(1 + x)$$

- Coefficient of x^4 is

$$4!(\frac{1}{1!2!1!} + \frac{1}{2!1!1!} + \frac{1}{2!2!0!} + \frac{1}{3!1!0!} + \frac{1}{3!0!1!})$$

Counting with Exponential Generating Functions

Example 11.51

Let a_n be the # of ternary sequences with the letters X, Y, Z such that Y and Z appears at least once. Find a_n.

- $a_1 = 0, a_2 = 2, a_3 = 12.$
- $a_n = 3^n - 2^{n+1} + 1.$

$$\begin{aligned}(1 + x + \frac{x^2}{2!} + \cdots)(x + \frac{x^2}{2!} + \cdots)^2 &= e^x(e^x - 1)^2 \\ &= e^{3x} - 2e^{2x} + e^x \\ &= \sum (3^n - 2 \cdot 2^n + 1)\frac{x^n}{n!}\end{aligned}$$

Counting with Exponential Generating Functions

Example 11.52

Let a_n be the # of ternary sequences with the letters X, Y, Z with exactly k X. Find a_n.

- Exponential generating function $\frac{x^k}{k!}(1 + x + \frac{x^2}{2!} + \cdots)^2$

$$\frac{x^k}{k!}e^{2x} = \sum_{r=0}^{\infty} \frac{2^r x^{k+r}}{k!r!} = \sum_{n=k}^{\infty} \frac{2^{n-k} x^n}{k!(n-k)!} \cdot \frac{n!}{n!}$$
$$= \sum_{n=k}^{\infty} 2^{n-k} \binom{n}{k} \frac{x^n}{n!}$$
$$= \sum_{n=0}^{\infty} 2^{n-k} \binom{n}{k} \frac{x^n}{n!}.$$

Counting with Exponential Generating Functions

Example 11.53

Let a_n be the # of ternary sequences with the letters X, Y, Z with odd number of X and even number of Y. Find a_n.

- $\frac{e^x+e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$
- $\frac{e^x-e^{-x}}{2} = \frac{x}{1!} + \frac{x^3}{3!} + \cdots$
- Exponential generating function $\frac{e^x+e^{-x}}{2}\frac{e^x-e^{-x}}{2}e^x$

$$\sum_{n\geq 0} a_n \frac{x^n}{n!} = \frac{1}{4}(e^{3x} - e^{-x})$$
$$= \sum_{n\geq 0} \frac{1}{4}(3^n - (-1)^n)\frac{x^n}{n!}.$$

Counting with Exponential Generating Functions

Example 11.54

Let a_n be the # of ways to distribute n distinct balls to three distinct boxes such that box 1 has an odd # of balls and box 2 has an even # of balls. Find a_n.

- By theorem 11.49 equivalent to # of permutations of n from a multiset with 3 type of objects.
- Equivalent to a ternary sequence.
- Hence exponential generating function $\frac{e^x+e^{-x}}{2}\frac{e^x-e^{-x}}{2}e^x$

Example 11.55

Let a_n be the # of ways to distribute n distinct balls to k distinct boxes such that no box is empty. Find a_n.

Exponential generating function

$$\begin{aligned}(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots)^k &= (e^x - 1)^k \\ &= \sum_{j=0}^{k} \binom{k}{j} (e^x)^{k-j}(-1)^j \\ &= \sum_{j=0}^{k} \binom{k}{j} \left(\sum_{n\geq 0} \frac{(k-j)^n x^n}{n!} \right) (-1)^j \\ &= \sum_{n\geq 0} \left(\sum_{j=0}^{k} \binom{k}{j} (-1)^j (k-j)^n \right) \frac{x^n}{n!}\end{aligned}$$

Recurrence Relations

Example 11.56

5 persons spread a rumour that a deadly flu virus is spreading in campus and started to wear face masks. Every hour, each person who believed in the rumour and wearing a face mask will convince two other persons to start wearing face masks. At the same time, one person each hour will stop believing in the rumour and removes his face mask. How many people will be wearing face masks after n hours?

- Let a_n be the $\#$ of people wearing face mask after n hours.
- $a_0 = 5$
- $a_n = 3a_{n-1} - 1$

Recurrence Relations

Example 11.56

$a_n = 3a_{n-1} - 1$, $a_0 = 5$.
Let $A(x) = \sum_{n\geq 0} a_n x^n$ be the generating function of a_n

$$
\begin{aligned}
a_n &= 3a_{n-1} - 1 \\
\implies a_n x^n &= 3a_{n-1}x^n - x^n \\
\implies \sum_{n\geq 1} a_n x^n &= \sum_{n\geq 1} 3a_{n-1}x^n - \sum_{n\geq 1} x^n \\
\implies A(x) - a_0 &= 3x \sum_{r\geq 0} a_r x^r - \frac{x}{1-x} \\
\implies A(x) - 5 &= 3xA(x) - \frac{x}{1-x}
\end{aligned}
$$

Recurrence Relations

Example 11.56 cont'd

$$\begin{aligned}
A(x) &= \frac{5}{1-3x} - \frac{x}{(1-x)(1-3x)} \\
&= \frac{5}{1-3x} - \left(\frac{P}{(1-x)} + \frac{Q}{(1-3x)}\right), \quad (x = P(1-3x) + Q(1-x)) \\
&= \frac{5}{1-3x} - \left(-\frac{1}{2(1-x)} + \frac{1}{2(1-3x)}\right) \\
&= \frac{9}{2(1-3x)} + \frac{1}{2(1-x)} = \frac{9}{2}\sum_{n\geq 0} 3^n x^n + \frac{1}{2}\sum_{n\geq 0} x^n
\end{aligned}$$

$$a_n = 3a_{n-1} - 1,\ a_0 = 5 \implies a_n = \frac{9 \cdot 3^n + 1}{2}.$$

Recurrence Relations

Example 11.57

How many ways are there to pave a $2 \times n$ rectangle with tiles of sizes 1×2, which may be laid horizontally or vertically.

- Let a_n be the # of ways
- $a_0 := 1$, $a_1 = 1$, $a_2 = 2$
- For a_n, the left most tile can be vertical or horizontal
- $a_n = a_{n-1} + a_{n-2}$ (addition principle)
- Fibonacci numbers shifted by 1.

Example 11.58

Find a formula for the Fibonacci numbers given by

$$f_n = f_{n-1} + f_{n-2}, \qquad f_0 = 0, f_1 = 1.$$

Let $F(x)$ be the generating function for f_n.

$$\begin{aligned} f_n x^n &= f_{n-1}x^n + f_{n-2}x^n \\ \sum_{n\geq 2} f_n x^n &= x\sum_{n\geq 2} f_{n-1}x^{n-1} + x^2 \sum_{n\geq 2} f_{n-2}x^{n-2} \\ F(x) - f_1 x - f_0 &= x(F(x) - f_0) + x^2 F(x) \\ F(x) &= \frac{x}{1-x-x^2} = \frac{-x}{(x-a)(x-b)} \end{aligned}$$

Fibonacci numbers

$$F(x) = \frac{x}{1-x-x^2} = \frac{-x}{(x-a)(x-b)} = \frac{1}{b-a}\left(\frac{a}{(x-a)} - \frac{b}{(x-b)}\right)$$
$$= \frac{1}{b-a}\left(-\frac{1}{(1-x/a)} + \frac{1}{(1-x/b)}\right)$$
$$= \frac{1}{b-a}\left(-\sum_{n\geq 0}(\frac{x}{a})^n + \sum_{n\geq 0}(\frac{x}{b})^n\right) = \frac{1}{\sqrt{5}}\sum_{n\geq 0}(\phi^n - \psi^n)x^n.$$

- $x^2 + x - 1 = 0 \implies (x+\frac{1}{2})^2 = \frac{5}{4} \implies a, b = \frac{-1\pm\sqrt{5}}{2}$
- $\frac{1}{b} = \frac{2}{-1+\sqrt{5}} \cdot \frac{-1-\sqrt{5}}{-1-\sqrt{5}} = \frac{1+\sqrt{5}}{2} = \phi$
- $\frac{1}{a} = \frac{1-\sqrt{5}}{2} = \psi$ and $b - a = \sqrt{5}$.

Example 11.59

Find a formula for a general recurrence relation of the form $a_n + c_1 a_{n-1} + c_2 a_{n-2} = 0$ *where* c_1, c_2 *are constants.*

Let $A(x)$ be the generating function for a_n.

$$\sum_{n\geq 2} (a_n x^n + c_1 a_{n-1} x^n + c_2 a_{n-2} x^n) = 0$$

$$(A(x) - a_1 x - a_0) + c_1 x (A(x) - a_0) + c_2 x^2 A(x) = 0$$

$$A(x) = \frac{a_1 x + a_0 + c_1 a_0}{1 + c_1 x + c_2 x^2} = \frac{P}{(1-\alpha x)} + \frac{Q}{(1-\beta x)}$$

$$\implies a_n = P(\alpha)^n + Q(\beta)^n, \;\; \alpha, \beta \text{ roots of } \lambda^2 + c_1 \lambda + c_2.$$

Remark:

$(1-\alpha x)(1-\beta x) = (1 + c_1 x + c_2 x^2) \implies \alpha\beta = c_2, -\alpha - \beta = c_1$

If $\lambda^2 + c_1\lambda + c_2$ has a repeated root.

$$A(x) = \frac{a_1x + a_0 + c_1a_0}{1 + c_1x + c_2x^2} = \frac{P}{(1-\alpha x)} + \frac{Q}{(1-\alpha x)^2}$$
$$\implies a_n = P(\alpha)^n + Q(n+1)(\alpha)^n.$$

In summary,

Theorem 11.60 (2nd Order linear homogeneous RR)

If a_n satisfies the recurrence relation $a_n + c_1a_{n-1} + c_2a_{n-2} = 0$, then a_n has a close form formula determined by the roots of the characteristic equation $\lambda^2 + c_1\lambda + c_2 = (\lambda - \alpha)(\lambda - \beta)$.
If $\alpha \neq \beta$, $a_n = P(\alpha)^n + Q(\beta)^n$
If $\alpha = \beta$, $a_n = (P + Qn)(\alpha)^n$.

Recurrence Relations

Theorem 11.61 (r-th Order linear homogeneous RR)

If a_n satisfies the recurrence relation $a_n + c_1 a_{n-1} + \cdots + c_r a_{n-r} = 0$, then a_n has a close form formula determined by the roots of the characteristic equation $\lambda^r + c_1\lambda^{r-1} + \cdots + c_{r-1}\lambda + c_r$.
If α_i with multiplicities m_i are the roots, then

$$a_n = (A_1 + A_2 n + \cdots + A_{m_1} n^{m_1 - 1})(\alpha_1)^n + \\ + (B_1 + B_2 n + \cdots + B_{m_2} n^{m_2 - 1})(\alpha_2)^n + \cdots$$

Recurrence Relations

Example 11.62

Solve the recurrence relation $a_n - 7a_{n-1} + 15a_{n-2} - 9a_{n-3} = 0$, given $a_0 = 1, a_1 = 2$ and $a_2 = 3$.

- $\lambda^3 - 7\lambda^2 + 15\lambda - 9 = (\lambda - 3)^2(\lambda - 1)$
- $a_n = (A + Bn)3^n + C(1)^n$
- $a_0 = 1 = A + C$
- $a_1 = 2 = 3A + 3B + C$
- $a_2 = 3 = 9A + 18B + C$
- $A = 1,\ B = -\frac{1}{3},\ C = 0$
- Hence $a_n = (3 - n)3^{n-1}$.

Recurrence Relations

Theorem 11.63 (r-th Order linear non-homogeneous RR)

If a_n satisfies the non-homogeneous recurrence relation

$$a_n + c_1 a_{n-1} + \cdots + c_r a_{n-r} = f(n),$$

then $a_n = a_n^h + a_n^p$,
where a_n^h is the general solution of the homogeneous relation

$$a_n + c_1 a_{n-1} + \cdots + c_r a_{n-r} = 0$$

and a_n^p is any particular solution of the non-homogeneous relation.

Recurrence Relations

Example 11.64

Solve the recurrence relation $a_n - 3a_{n-1} = 2 - 2n^2$, given $a_0 = 3$.

- $a_n = a_n^h + a_n^p$
- $a_n - 3a_{n-1} = 0 \implies \lambda - 3 = 0 \implies a_n^h = A \cdot 3^n$
- For a_n^p, we guess a solution of the same form as $2 - 2n^2$
- Assume $a_n^p = B + Cn + Dn^2$ satisfies the original RR.
- $(B + Cn + Dn^2) - 3(B + C(n-1) + D(n-1)^2) = 2 - 2n^2$
- Compare coeff, $D = 1$, $C = 3$ and $B = 2$.
- $a_n = A \cdot 3^n + 2 + 3n + n^2$, if $a_0 = 3 \implies A = 1$

Recurrence Relations

Example 11.64: Using GF

Solve the recurrence relation $a_n - 3a_{n-1} = 2 - 2n^2$, given $a_0 = 3$.

- $a_n x^n - 3x a_{n-1} x^{n-1} = 2x^n - 2n^2 x^n$
- $(A(x) - a_0) - 3xA(x) = \dfrac{2x}{1-x} - \dfrac{2x(1+x)}{1-x^3}$
- $A(x) = \frac{1}{1-3x} + \frac{2}{(1-x)^3}$
- $a_n = 3^n + 2\binom{n+2}{2}$

Example 11.65

Solve the recurrence relation $a_n - 3a_{n-1} + 2a_{n-2} = 2^n$, given $a_0 = 3, a_1 = 8$.

- $a_n = a_n^h + a_n^p$
- $a_n - 3a_{n-1} + 2a_{n-2} = 0 \implies \lambda^2 - 3\lambda + 2 = 0$
- $a_n^h = A \cdot 2^n + B \cdot 1^n = A \cdot 2^n + B$
- For a_n^p, we guess a solution of the same form as 2^n
- Assume $a_n^p = Cn2^n$, the additional factor n ensures a_n^p is distinct from a_n^h
- $(Cn2^n) - 3(C(n-1)2^{n-1}) + 2(C(n-2)2^{n-2}) = 2^n$
- $C = 2 \implies a_n = A \cdot 2^n + B + n \cdot 2^{n+1}$
- $a_0 = 3 = A + B$, $a_1 = 8 = 2A + B + 4$
- $a_n = 2 + 2^n + n2^{n+1}$.

Recurrence Relations

Example 11.65: Using GF

Solve the recurrence relation $a_n - 3a_{n-1} + 2a_{n-2} = 2^n$, given $a_0 = 3, a_1 = 8$.

- $a_n x^n - 3x a_{n-1} x^{n-1} + 2x^2 a_{n-2} x^{n-2} = (2x)^n$
- $(A(x) - 8x - 3) - 3x(A(x) - 3) + 2x^2 A(x) = \dfrac{4x^2}{1-2x}$
- $A(x) = \frac{2}{1-x} - \frac{1}{1-2x} + \frac{2}{(1-2x)^2}$
- $a_n = 2 - 2^n + 2(n+1)2^n$

Derangements

Example 11.66

A derangement is a permutation of $[n]$ *such that* j *is not in the* j*-th position. If* d_n *is the* $\#$ *of derangements of* $[n]$*, then* $d_n = (n-1)(d_{n-1} + d_{n-2})$*. Hence find an exact formula for* d_n*.*

Proof:

- For any derangement, choose k as the number that occupies the 1st position. There are $n-1$ choices.
- Either 1 is in the k-th position or 1 is not in the k-th position.
- In the first case, there problem reduces to derangement of $n-2$ objects, i.e. d_{n-2} ways.
- In the second case, if we identify 1 as k, then $\#$ of ways is equal to derangements of $n-1$ objects, i.e. d_{n-1}

Derangements

Exmaple 11.66 cont'd

Solve $d_n = (n-1)(d_{n-1} + d_{n-2})$, with $d_0 = 1$, $d_1 = 0$, $d_2 = 1$.

- Non-linear recurrence relation
- Use exponential generating function, $F(x) = \sum d_n \frac{x^n}{n!}$

$$d_n \frac{x^n}{n!} = \frac{n-1}{n} d_{n-1} \frac{x^n}{(n-1)!} + \frac{1}{n} d_{n-2} \frac{x^n}{(n-2)!}$$

$$x \sum_{n\geq 2} n d_n \frac{x^{n-1}}{n!} = x^2 \sum_{n\geq 2} (n-1) d_{n-1} \frac{x^{n-2}}{(n-1)!} + x^2 \sum_{n\geq 2} d_{n-2} \frac{x^n}{(n-2)!}$$

$$x(F' - d_1) = x^2 F' + x^2 F$$

Derangements

Exmaple 11.66 cont'd

Solve $d_n = (n-1)(d_{n-1} + d_{n-2})$, with $d_0 = 1,\ d_1 = 0,\ d_2 = 1$.

$$xF' = x^2F' + x^2F \implies \frac{F'}{F} = \frac{x}{1-x}$$

$$\implies \log(F) = \int \frac{x}{1-x} = -x - \log(1-x) + C$$

$$F(x) = \frac{Ae^{-x}}{1-x} \implies A = 1,\ \text{since } F(0) = d_0 = 1$$

$$= \sum_{n\geq 0}\left(\sum_{j=0}^{n} \frac{(-1)^j}{j!}\right)x^n = \sum_{n\geq 0}\left(\sum_{j=0}^{n} \frac{(-1)^j n!}{j!}\right)\frac{x^n}{n!}.$$

Outline

Arithmetic Progressions

Example 12.1

A sequence $3, 7, 11, 15, \ldots$ *is easily identified as an arithmetic progression because the successive differences is a constant sequence.*
i.e. $4 = 7 - 3 = 11 - 7 = 15 - 11 \cdots$

Definition 12.2

An arithmetic progression $a(0), a(1), a(2), \ldots$ *can be defined by the formula* $a(n) = b + dn$, *where* d *is the value of the successive differences, i.e.* $a(k+1) - a(k) = d$ *for every* k.

Arithmetic Progressions

Example 12.3

Consider the sequence $a(n)$ and its first successive differences $b(n)$.

$$a(n) : 5, 11, 19, 29, 41, \dots$$
$$b(n) : \ 6,\ 8,\ 10,\ 12, \dots$$

Since $b(n)$ is not constant, $a(n)$ is not an arithmetic progression.

But it is not difficult to see that $b(n)$ forms an arithmetic progression.
We say that $a(n)$ is an arithmetic progression of order 2.

Arithmetic Progressions

Definition 12.4

A sequence $a(0), a(1), a(2), \ldots$ is called an arithmetic progression of order k, if the k-th successive differences forms a constant sequence.

Example 12.5

The following is an example of an arithmetic progression of order 4.

$$\begin{array}{rccccccccccc}
a(n): & 5, & & 13, & & 25, & & 83, & & 277, & & 745, \\
\text{1st:} & & 8, & & 12, & & 58, & & 194, & & 468, & \\
\text{2nd:} & & & 4, & & 46, & & 136, & & 274, & & \ldots \\
\text{3rd:} & & & & 42, & & 90, & & 138, & & \ldots & \\
\text{4th:} & & & & & 48, & & 48, & & \ldots & &
\end{array}$$

Arithmetic Progressions

Theorem 12.6

An arithmetic progression of order k is given by the formula

$$a(n) = \sum_{i=0}^{k} f(i)\binom{n}{i},$$

where $f(i)$ is the first term of the i-th successive differences.

Example 12.1

$$\begin{array}{lccccccccc} a(n): & 3, & & 7, & & 11, & & 15, & & \dots, \\ \text{1st:} & & 4, & & 4, & & 4, & & \dots, & \end{array}$$

Hence $a(n) = f(0)\binom{n}{0} + f(1)\binom{n}{1} = 3 + 4n$

Arithmetic Progressions

Example 12.3

$a(n)$:	5,	11,	19,	29,	41,	$\dots,$
1st:	6,	8,	10,	12,	$\dots,$	
2nd:	2,	2,	2,	$\dots$		

$$\begin{aligned} a(n) &= f(0)\binom{n}{0} + f(1)\binom{n}{1} + f(2)\binom{n}{2} \\ &= 5 + 6n + 2\frac{n(n-1)}{2} \\ &= n^2 + 5n + 5 \end{aligned}$$

Arithmetic Progressions

Example 12.5

$$\begin{array}{rccccccccccc}
a(n): & 5, & & 13, & & 25, & & 83, & & 277, & & 745, \\
\text{1st:} & & 8, & & 12, & & 58, & & 194, & & 468, & \\
\text{2nd:} & & & 4, & & 46, & & 136, & & 274, & & \ldots \\
\text{3rd:} & & & & 42, & & 90, & & 138, & & \ldots & \\
\text{4th:} & & & & & 48, & & 48, & & \ldots & &
\end{array}$$

$$\begin{aligned}
a(n) &= 5\binom{n}{0} + 8\binom{n}{1} + 4\binom{n}{2} + 42\binom{n}{3} + 48\binom{n}{4} \\
&= 2n^4 - 5n^3 + 3n^2 + 8n + 5
\end{aligned}$$

Proof of Theorem 12.6

By induction.
Base case: True for arithmetic progressions (of order 1.)
Assume $a(n)$ is an arithmetic progression of order k.
Let $b(n) = a(n+1) - a(n)$ then $b(n)$ is an arithmetic progression of order $k-1$.
By our induction hypothesis,

$$b(n) = \sum_{\ell=0}^{k-1} g(\ell) \binom{n}{\ell}$$

where $g(\ell)$ is the first term of the ℓ successive differences
$\implies g(\ell) = f(\ell+1)$.

Proof of Theorem 12.6

$$
\begin{aligned}
a(n) &= a(n-1) + b(n-1) \\
&= a(n-2) + b(n-2) + b(n-1) \\
&= \cdots = a(0) + \sum_{j=0}^{n-1} b(j) \\
&= a(0) + \sum_{j=0}^{n-1} \sum_{\ell=0}^{k-1} f(\ell+1) \binom{j}{\ell} \\
&= a(0) + \sum_{\ell=0}^{k-1} f(\ell+1) \sum_{j=\ell}^{n-1} \binom{j}{\ell}
\end{aligned}
$$

Proof of Theorem 12.6

$$
\begin{aligned}
a(n) &= a(0) + \sum_{\ell=0}^{k-1} f(\ell+1) \sum_{j=\ell}^{n-1} \binom{j}{\ell} \\
&= a(0) + \sum_{\ell=0}^{k-1} f(\ell+1) \binom{n-1+1}{\ell+1} \\
&= f(0)\binom{n}{0} + \sum_{i=1}^{k} f(i)\binom{n}{i}
\end{aligned}
$$

Note: Chu's identity (Theorem 3.23) $\sum_{j=r}^{n} \binom{j}{r} = \binom{n+1}{r+1}$ was used.