

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2005-2006

**MA1505 MATHEMATICS I**

November 2005 Time allowed: 2 hours

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**INSTRUCTIONS TO CANDIDATES**

1. **Write down your matriculation number neatly in the space provided below.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **EIGHT (8)** questions and comprises **THIRTY THREE (33)** printed pages.
3. Answer **ALL** questions. For each question, write your answer in the box and your working in the space provided inside the booklet following that question.
4. The marks for each question are indicated at the beginning of the question.
5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

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**Matriculation Number:**

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**For official use only. Do not write below this line.**

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|----------|---|---|---|---|---|---|---|---|
| Question | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| Marks    |   |   |   |   |   |   |   |   |

**Question 1 (a)** [5 marks]

Given that  $y^2 - 4x = 4 - 4y$ . Find the value of  $\frac{dy}{dx}$  at the point  $(2, 2)$ .

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|--------------------|--|
| <b>Answer 1(a)</b> |  |
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*(Show your working below and on the next page.)*

*(More working space for Question 1(a))*

**Question 1 (b)** [5 marks]

Let  $f(x) = (\sin x)^{\sin x}$  for all  $x \in (0, \frac{\pi}{2})$ . Given that  $f$  has a critical point at  $c \in (0, \frac{\pi}{2})$ . Find the value of  $\sin c$ .

|                              |  |
|------------------------------|--|
| <b>Answer</b><br><b>1(b)</b> |  |
|------------------------------|--|

*(Show your working below and on the next page.)*

*(More working space for Question 1(b))*

**Question 2 (a)** [5 marks]

The region bounded by the graphs of  $y = \frac{1}{\sqrt{1+x^2}}$ ,  $y = \frac{1}{\sqrt{4+x^2}}$ ,  $x = 0$  and  $x = b$  where  $b$  denotes a positive constant is rotated about the  $x$ -axis to generate a solid of revolution. Let  $V(b)$  denote the volume of this solid of revolution. Find the value of  $\lim_{b \rightarrow \infty} V(b)$ .

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| <b>Answer 2(a)</b> |  |
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*(Show your working below and on the next page.)*

*(More working space for Question 2(a))*

**Question 2 (b)** [5 marks]

Find the value of

$$\frac{\int_{-\frac{\pi}{2}}^0 \cos^{10} x \, dx}{\int_{-\frac{\pi}{2}}^0 \cos^8 x \, dx}.$$

|                              |  |
|------------------------------|--|
| <b>Answer</b><br><b>2(b)</b> |  |
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*(Show your working below and on the next page.)*



*(More working space for Question 2(b))*

**Question 3 (a)** [5 marks]

Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{8^n + (-9)^n}{n+1} (x+2)^{2n}.$$

|                    |  |
|--------------------|--|
| <b>Answer 3(a)</b> |  |
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*(Show your working below and on the next page.)*

*(More working space for Question 3(a))*

**Question 3 (b)** [5 marks]

Let  $f(x) = \tan^{-1}\left(\frac{1+x}{1-x}\right)$  where  $-\frac{1}{2} \leq x \leq \frac{1}{2}$ . Find the value of

$$f^{(2005)}(0).$$

Give your answer in terms of factorials.

|                              |  |
|------------------------------|--|
| <b>Answer</b><br><b>3(b)</b> |  |
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*(Show your working below and on the next page.)*

*(More working space for Question 3(b))*

**Question 4 (a)** [5 marks]

Let  $f(x) = \cos \frac{x}{2}$  for all  $x \in (0, \pi)$ . Let

$$a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

be the Fourier Cosine Series which represents  $f(x)$ . Find the value of  $a_0 + a_1$ . Give your answer in terms of  $\pi$ .

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|--------------------|--|
| <b>Answer 4(a)</b> |  |
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*(Show your working below and on the next page.)*

*(More working space for Question 4(a))*

**Question 4 (b)** [5 marks]

Let  $f(x) = 2x + 1$  for all  $x \in (-\pi, \pi)$  and  $f(x) = f(x + 2\pi)$ . Let

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be the Fourier Series which represents  $f(x)$ . Find the value of  $a_0 + a_5 + b_5$ .

|                              |  |
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| <b>Answer</b><br><b>4(b)</b> |  |
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*(Show your working below and on the next page.)*



*(More working space for Question 4(b))*

**Question 5 (a)** [5 marks]

The number of bacteria in a certain bacterium culture is 1000 at a certain initial time. Two hours after the initial time there are 1200 of them. Assuming constant birth and death rates per capita, how many bacteria will we have 6 hours after the initial time?

|                    |  |
|--------------------|--|
| <b>Answer 5(a)</b> |  |
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*(Show your working below and on the next page.)*

*(More working space for Question 5(a))*

**Question 5 (b)** [5 marks]

Solve the differential equation

$$(y^2 - x^2) \frac{dy}{dx} + 2xy = 0$$

with  $x > 0$  and the initial condition  $y = 1$  when  $x = \sqrt{2}$ .

|                              |  |
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| <b>Answer</b><br><b>5(b)</b> |  |
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*(Show your working below and on the next page.)*

*(More working space for Question 5(b))*

**Question 6 (a)** [5 marks]

A water tank has a capacity of 120 litres. Initially the tank contains 90 litres of a salt solution with a concentration of 1 gram of salt per litre. A tap is then turned on and a salt solution with a concentration of 2 grams of salt per litre enters the tank at a rate of 4 litres per minute. At the same time when the tap is turned on, a drain is also turned on and the well-stirred mixture flows out of the tank at a rate of 3 litres per minute. How much salt in grams (round off your answer to the nearest integer) does the tank contain at the moment when it is full ?

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| <b>Answer 6(a)</b> |  |
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*(Show your working below and on the next page.)*

*(More working space for Question 6(a))*

**Question 6 (b)** [5 marks]

Solve the differential equation

$$y'' - y' - 2y = 0$$

with the initial conditions that  $y = 1$  and  $y' = 5$  when  $x = 0$ .

|                        |  |
|------------------------|--|
| <b>Answer<br/>6(b)</b> |  |
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*(Show your working below and on the next page.)*



*(More working space for Question 6(b))*

**Question 7 (a)** [5 marks]

Solve the differential equation

$$y'' - 5y' + 6y = 4e^{2x}$$

with the initial conditions that  $y = 0$  and  $y' = 1$  when  $x = 0$ .

|                    |  |
|--------------------|--|
| <b>Answer 7(a)</b> |  |
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*(Show your working below and on the next page.)*

*(More working space for Question 7(a))*

**Question 7 (b)** [5 marks]

Solve the differential equation

$$y'' + y = \tan^2 x$$

with the initial conditions that  $y = 1$  and  $y' = 1$  when  $x = 0$ .

|                              |  |
|------------------------------|--|
| <b>Answer</b><br><b>7(b)</b> |  |
|------------------------------|--|

*(Show your working below and on the next page.)*

*(More working space for Question 7(b))*

**Question 8 (a)** [5 marks]

Find the Laplace transform

$$L\left\{(\sin t - \cos t)^2\right\}.$$

|                    |  |
|--------------------|--|
| <b>Answer 8(a)</b> |  |
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*(Show your working below and on the next page.)*

*(More working space for Question 8(a))*

**Question 8 (b)** [5 marks]

Find the inverse Laplace transform

$$L^{-1} \left( e^{-s} \frac{s+3}{s^2+4s+4} \right).$$

|                              |  |
|------------------------------|--|
| <b>Answer</b><br><b>8(b)</b> |  |
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*(Show your working below and on the next page.)*



*(More working space for Question 8(b))*

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**END OF PAPER**

