NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2001-2002

MA2214 Combinatorial Analysis

April/May 2002 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **ELEVEN (11)** questions and comprises **FOUR (4)** printed pages.

2. Answer **ALL** questions in **Section A**. The marks for questions in Section A are not necessarily the same; marks for each question are indicated at the beginning of the question.

3. Answer not more than **TWO (2)** questions from **Section B**. Each question in Section B carries 20 marks.

4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A

Answer **all** *the questions in this section. Section A carries a total of 60 marks.*

Question 1 [10 marks]

7 boys and 5 girls are to be seated during a class function. In how many ways can they be seated if

(i) they are seated in a row and 2 particular boys must be separated by exactly 3 girls?

(ii) they are seated in a row and the girls form a block?

(iii) they are seated around a round table and no girls are adjacent?

(iv) they are seated in 2 rows of 6 with exactly 4 boys in the front row?

Question 2 [8 marks]

Let $S = \{1, 2, ..., n+1\}$ where $n \geq 2$, and let

$$T = \{(x, y, z) \in S^3 \mid x < z \text{ and } y < z\}.$$

Show by counting $|T|$ in two different ways that

$$\sum_{k=1}^{n} k^2 = |T| = \binom{n+1}{2} + 2\binom{n+1}{3}.$$

Question 3 [6 marks]

Show that for $m \leq k \leq n$,

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}.$$

Hence or otherwise, prove that for $m \leq n$,

$$\sum_{k=m}^{n} \binom{n}{k}\binom{k}{m} = 2^{n-m}\binom{n}{m}.$$

Question 4 [6 marks]

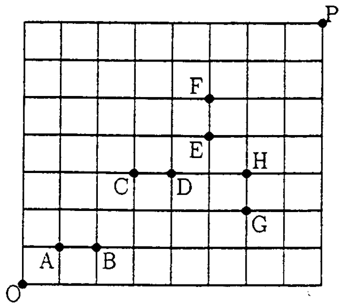
Let a_1, a_2, a_3, a_4, a_5 be any five positive integers, and let π be any permutation of $\{1, 2, 3, 4, 5\}$. Show by the Pigeonhole Principle or otherwise that the product

$$(a_1 - a_{\pi(1)})(a_2 - a_{\pi(2)}) \cdots (a_5 - a_{\pi(5)})$$

is always even.

Question 5 [10 marks]

Find the number of shortest routes from O to P that passes through an even number of the segments { AB, CD, EF, GH }.



Question 6 [6 marks]

I have 100 twenty-cents coins which I wish to distribute to my wife and two sons. How many ways can this distribution be done if my wife gets at least four dollars and each of my two sons gets between \$3.20 and \$8.60?

Question 7 [6 marks]

Find the number of ways to distribute 10 distinct books to 3 students such that no student receives exactly 2 books.

Question 8 [8 marks]

Given $a_0 = 0$ and $a_1 = 1$, solve

$$a_n + 4a_{n-1} + 4a_{n-2} = (-2)^n.$$

SECTION B

Answer not more than **two** *questions in this section. Each question in this section carries 20 marks.*

Question 9 [20 marks]

(a) Prove the following identity, where $n \in \mathbb{N}$.

$$\sum_{r=0}^{n} r\binom{2n}{r} = n \cdot 2^{2n-1}.$$

(b) Show that given any set of 11 integers, there are 4 numbers in the set whose sum is divisible by 4.

Question 10 [20 marks]

Let (a_n) be a sequence of numbers satisfying the recurrence relation

$$a_n = \frac{pa_{n-1} + q}{ra_{n-1} + s},$$

where p, q, r and s are constants with $r \neq 0$.

(i) Show that

$$ra_n + s = p + s + \frac{qr - ps}{ra_{n-1} + s} \qquad (1)$$

(ii) By the substitution

$$ra_n + s = \frac{b_{n+1}}{b_n},$$

show that (1) can be reduced to the second order linear homogeneous recurrence relation for (b_n):

$$b_{n+1} - (p + s)b_n + (ps - qr)b_{n-1} = 0.$$

(iii) Hence, or otherwise solve

$$a_n = \frac{3a_{n-1} + 1}{a_{n-1} + 3}$$

given that $a_0 = 5$.

Question 11 [20 marks]

After majoring in mathematics and graduating from the University, you joined the Police Force as a Police Officer. On a rainy day, a reckless driver was speeding and knocked down an old lady who was crossing the road. The driver drove away after the accident and the Police is now after him. Three witnesses who saw the car license number reported to the Police what they saw.

Witness 1: "The first and last digits are prime numbers."
Witness 2: "The second digit is 5."
Witness 3: "The sum of all the digits is 18."

Car license numbers in your country is a 5 digit natural number with leading zero allowed. To ensure that every potential suspect is investigated, your superior asked you how many numbers are there that satisfies **at least one** of the three witness' statements. What is your answer?

END OF PAPER