Chapter 10
Surface Integrals

Overview

- Parametric Surfaces
  - Tangent Planes and Normal Vectors
- Surface Integrals
  - Surface Integrals of Scalar Functions
  - Surface Integrals
  - Surface Integrals of Vector Fields
  - Orientation of Surfaces

Overview

- Curl and Divergence
  - Curl
  - Divergence
  - Del Operator
  - Curl and Conservative Fields
  - Stokes' Theorem

- Divergence Theorem (Gauss' Theorem)

Parametric Surfaces

Parametric Representation of Curves :

Plane Curve : $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, \quad a \le t \le b$

Space Curve : $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \quad a \le t \le b$

one parameter t

Representation of Surfaces

By a two variable function : $z = f(x, y)$

Parametric Representation of Surfaces ?

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

two parameters u and v

Parametric Surfaces

Parametric curves in space:

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \quad a \le t \le b.$$

Parametric surfaces in space:

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k} \qquad \text{------} \quad (1)$$

where u and v are two independent parameters.

The equations

$$x = x(u,v), \quad y = y(u,v) \quad \text{and} \quad z = z(u,v)$$

are called the ***parametric equations*** of the surface.

Parametric Surfaces

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

Single variable vector function

domain comes from the real line

$$t \longrightarrow \mathbf{r}(t) \longrightarrow \text{a vector}$$

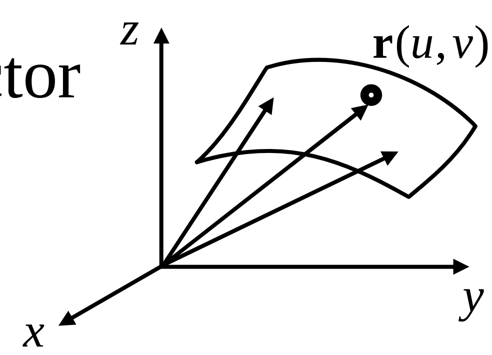
The collection of all output vectors form a space curve

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

Two variables vector function

domain comes from 2D-plane

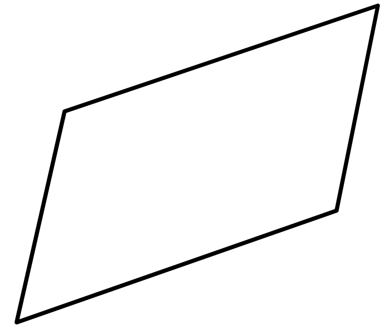
$$(u,v) \longrightarrow \mathbf{r}(u,v) \longrightarrow \text{a vector}$$



The collection of all output vectors form a surface

Example

Plane $3x+2y-4z=6$



$$z=\frac{1}{4}(3x+2y-6)$$

Parametric Representation

$$\mathbf{r}(u,v)=x(u,v)\mathbf{i}+y(u,v)\mathbf{j}+z(u,v)\mathbf{k}$$

$$x(u,v)=u,\quad y(u,v)=v,\quad z(u,v)=\frac{1}{4}(3u+2v-6)$$

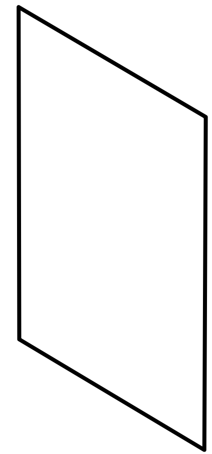
$$\boxed{\mathbf{r}(u,v)=u\mathbf{i}+v\mathbf{j}+\frac{1}{4}(3u+2v-6)\mathbf{k}}$$

Example (One Variable is absent)

Plane $2y + x = 7$

$$x = (7 - 2y)$$

z is missing



Parametric Representation

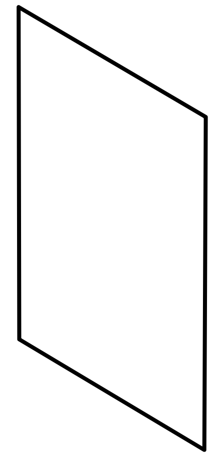
$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

$$z(u,v) = u, \quad y(u,v) = v, \quad x = (7 - 2v)$$

$$\boxed{\mathbf{r}(u,v) = (7 - 2v)\mathbf{i} + v\mathbf{j} + u\mathbf{k}}$$

Example (One Variable is absent)

Plane $2y + x = 7$



$$y = \frac{1}{2}(7 - x)$$

z is missing

Parametric Representation

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

$$z(u,v) = u, \quad x = v, \quad y(u,v) = \frac{1}{2}(7 - v)$$

$$\boxed{\mathbf{r}(u,v) = v\mathbf{i} + \frac{1}{2}(7 - v)\mathbf{j} + u\mathbf{k}}$$

Example: Two Variables are absent

$$\text{Plane :} \quad z = 7$$

x and y are missing

Parametric Representation

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

Let $x(u,v) = u$ and $y(u,v) = v$

$$z(u,v) = 7$$

$$\boxed{\mathbf{r}(u,v) = u\mathbf{i} + v\mathbf{j} + 7\mathbf{k}}$$

Example: Surfaces of the form $z = f(x,y)$

Parametric Representation

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

$z = f(x,y)$

Surface of the form $z = f(x,y)$

Take $x = u, \quad y = v, \quad z = f(u,v)$

$$\mathbf{r}(u,v) = u\mathbf{i} + v\mathbf{j} + f(u,v)\mathbf{k}$$

or

$$\mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + f(x,y)\mathbf{k}$$

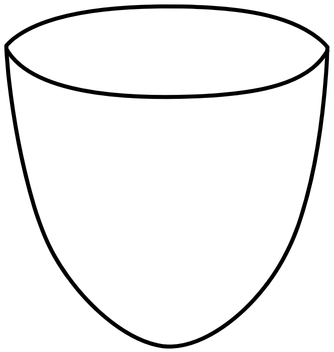
Example: Surfaces of the form $z = f(x,y)$

A natural parametric representation of S is

$$\mathbf{r}(u,v) = u\mathbf{i} + v\mathbf{j} + f(u,v)\mathbf{k}.$$

The paraboloid $z = x^2 + y^2$.

$$\mathbf{r}(u,v) = u\mathbf{i} + v\mathbf{j} + (u^2 + v^2)\mathbf{k}.$$



The upper cone $z = \sqrt{x^2 + y^2}$.

$$\mathbf{r}(u,v) = u\mathbf{i} + v\mathbf{j} + \sqrt{u^2 + v^2}\,\mathbf{k}.$$

Example: Surfaces of the form $z = f(x,y)$

Parametric Representation

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

$z = f(x, y)$

Surface of the form $z = f(x, y)$

Take $x = u,\ \ y = v,\ \ z = f(u,v)$

$$\mathbf{r}(u,v) = u\mathbf{i} + v\mathbf{j} + f(u,v)\mathbf{k}$$

or

$$\mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + f(x,y)\mathbf{k}$$

Similarly for surfaces of the form

$y = g(x,z)$ and $x = h(y,z)$

Example: Spheres

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

We have a ***standard*** parametric representation for a sphere $x^2 + y^2 + z^2 = a^2$ of radius a centered at the origin :

$$\mathbf{r}(u,v) = (a\sin u\cos v)\mathbf{i} + (a\sin u\sin v)\mathbf{j} + (a\cos u)\mathbf{k}$$

When

$$0 \le u \le \pi, \quad 0 \le v \le 2\pi,$$

the representation gives the full sphere.

When

$$0 \le u \le \frac{\pi}{2}, \quad 0 \le v \le 2\pi,$$

the representation gives the upper hemisphere.

Example: Spheres

$x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$

We have a ***standard*** parametric representation for a sphere $x^2+y^2+z^2=a^2$ of radius a centered at the origin :

$$\mathbf{r}(u,v)=(a\sin u\cos v)\mathbf{i}+(a\sin u\sin v)\mathbf{j}+(a\cos u)\mathbf{k}$$

$x=a\sin u\cos v$ $\quad$ $y=a\sin u\sin v$ $\quad$ $z=a\cos u$

$$\begin{aligned}x^2+y^2&=(a\sin u\cos v)^2+(a\sin u\sin v)^2\\&=a^2\sin^2 u\cos^2 v+a^2\sin^2 u\sin^2 v\\&=a^2\sin^2 u(\cos^2 v+\sin^2 v)\\&=a^2\sin^2 u\end{aligned}$$

$$\begin{aligned}x^2+y^2+z^2&=a^2\sin^2 u+(a\cos u)^2\\&=a^2\sin^2 u+a^2\cos^2 u\\&=a^2\end{aligned}$$

We have a ***standard*** parametric representation for a sphere $x^2 + y^2 + z^2 = a^2$ of radius a centered at the origin :

$$\mathbf{r}(u,v) = (a\sin u\cos v)\mathbf{i} + (a\sin u\sin v)\mathbf{j} + (a\cos u)\mathbf{k}$$

When

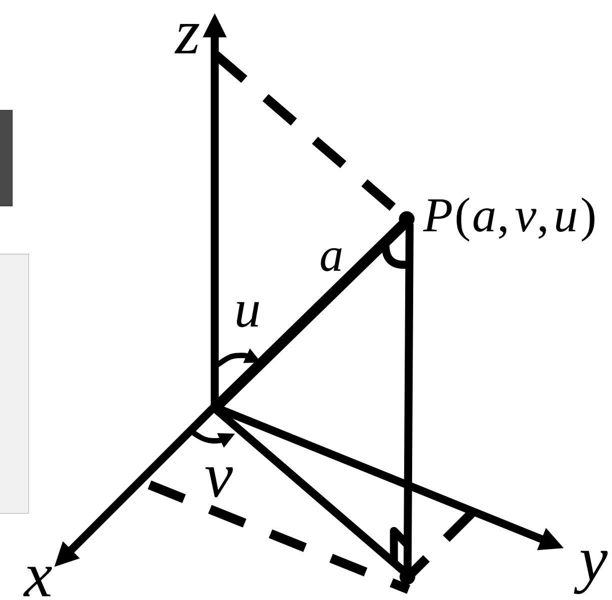
$$0 \le u \le \frac{\pi}{2}, \quad 0 \le v \le 2\pi,$$

the representation gives the upper hemisphere.

Note : As $z = a\cos u$, z depends on only u.

For $0 \le u \le \dfrac{\pi}{2}$, we have $\cos u \ge 0$, hence, $z \ge 0$.

Thus, we get the upper hemisphere.



We have a ***standard*** parametric representation for a sphere $x^2+y^2+z^2=a^2$ of radius a centered at the origin :

$$\mathbf{r}(u,v)=(a\sin u\cos v)\mathbf{i}+(a\sin u\sin v)\mathbf{j}+(a\cos u)\mathbf{k}$$

When

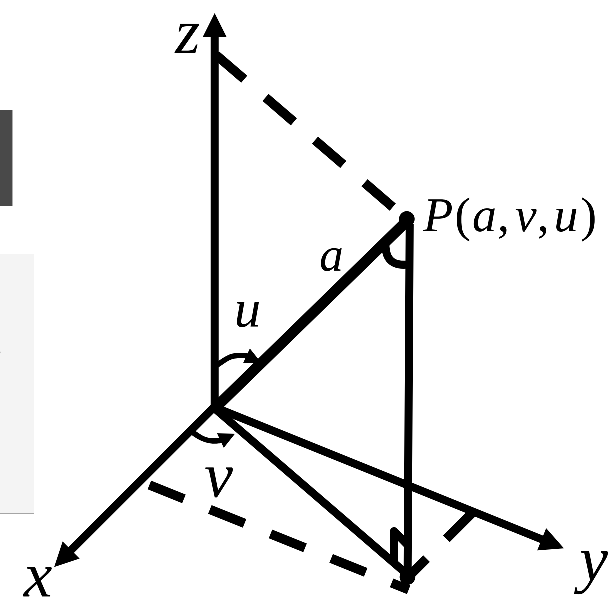
$$\frac{\pi}{2}\le u\le\pi,\quad 0\le v\le 2\pi,$$

the representation gives the lower hemisphere

Note : As $z=a\cos u$, z depends on only u.

For $\dfrac{\pi}{2}\le u\le\pi$, we have $\cos u\le 0$, hence, $z\le 0$.

Thus, we get the lower hemisphere.



We have a ***standard*** parametric representation for a sphere $x^2+y^2+z^2=a^2$ of radius a centered at the origin :

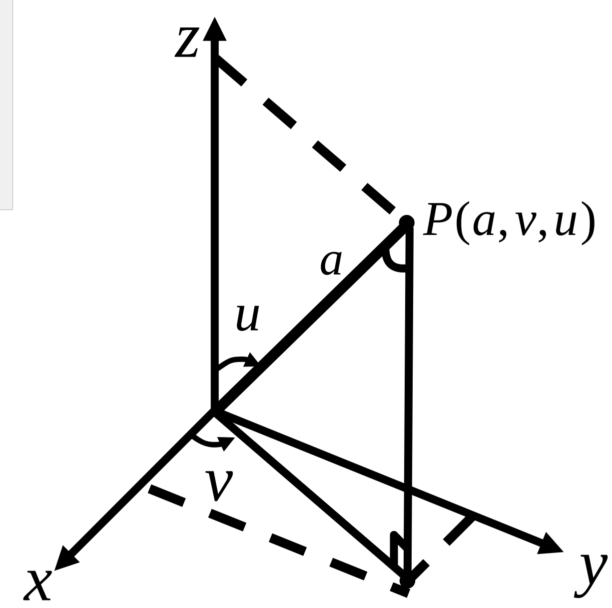
$$\mathbf{r}(u,v)=(a\sin u\cos v)\mathbf{i}+(a\sin u\sin v)\mathbf{j}+(a\cos u)\mathbf{k}$$

Note : As $z=a\cos u$, z depends on only u.

When

$$0\le u\le\pi,\quad 0\le v\le 2\pi,$$

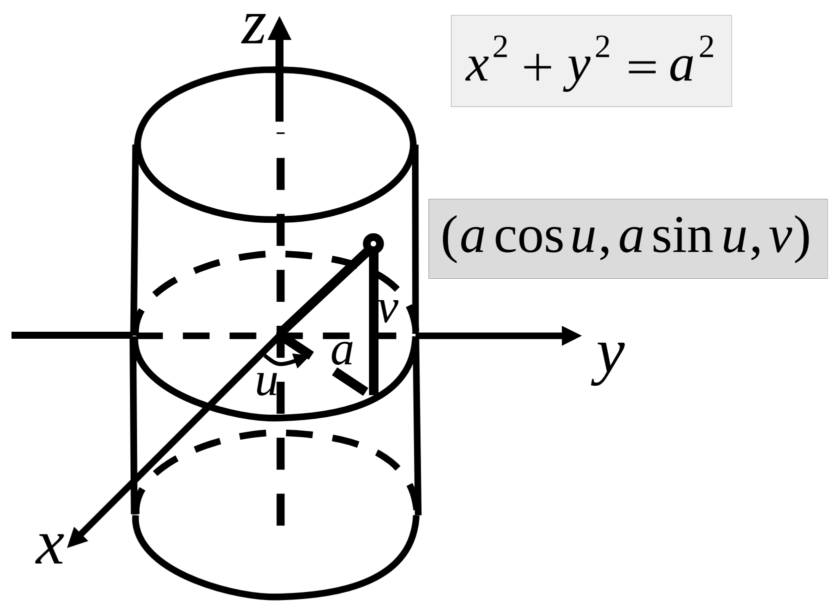
the representation gives the full sphere.



Example: Circular Cylinder

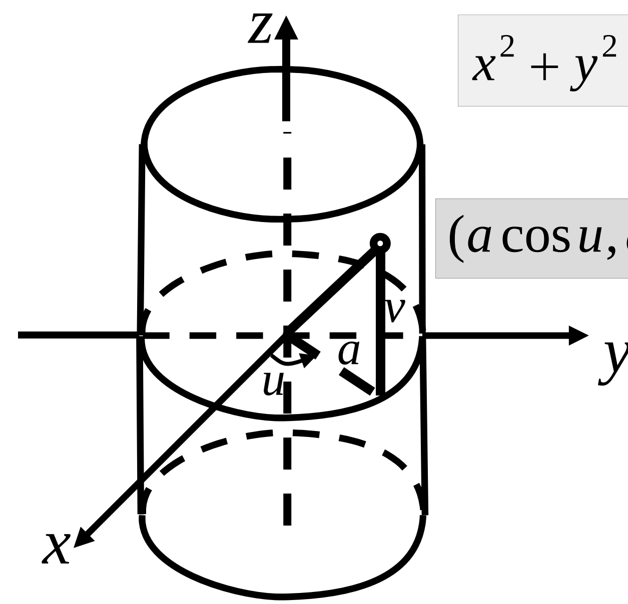
We have a ***standard*** parametric representation for circular cyclinder $x^2 + y^2 = a^2$ about the z-axis:

$$\mathbf{r}(u,v) = (a\cos u)\mathbf{i} + (a\sin u)\mathbf{j} + v\mathbf{k}$$



Here u measures the angle from the positive x-axis (about the z-axis) while v measures the height from the xy-plane along the cylinder.

Example: Circular Cylinder



$x^2 + y^2 = a^2$

$(a\cos u, a\sin u, v)$

Circular cyclinder $x^2 + y^2 = a^2$ about the z-axis:

$$\mathbf{r}(u,v) = (a\cos u)\mathbf{i} + (a\sin u)\mathbf{j} + v\mathbf{k}$$

Similarly for $x^2 + z^2 = a^2$ and $y^2 + z^2 = a^2$ (cylinders about y- and x-axes respectively), we have respectively

$$\mathbf{r}(u,v) = (a\cos u)\mathbf{i} + v\mathbf{j} + (a\sin u)\mathbf{k}$$

and

$$\mathbf{r}(u,v) = v\mathbf{i} + (a\cos u)\mathbf{j} + (a\sin u)\mathbf{k}$$

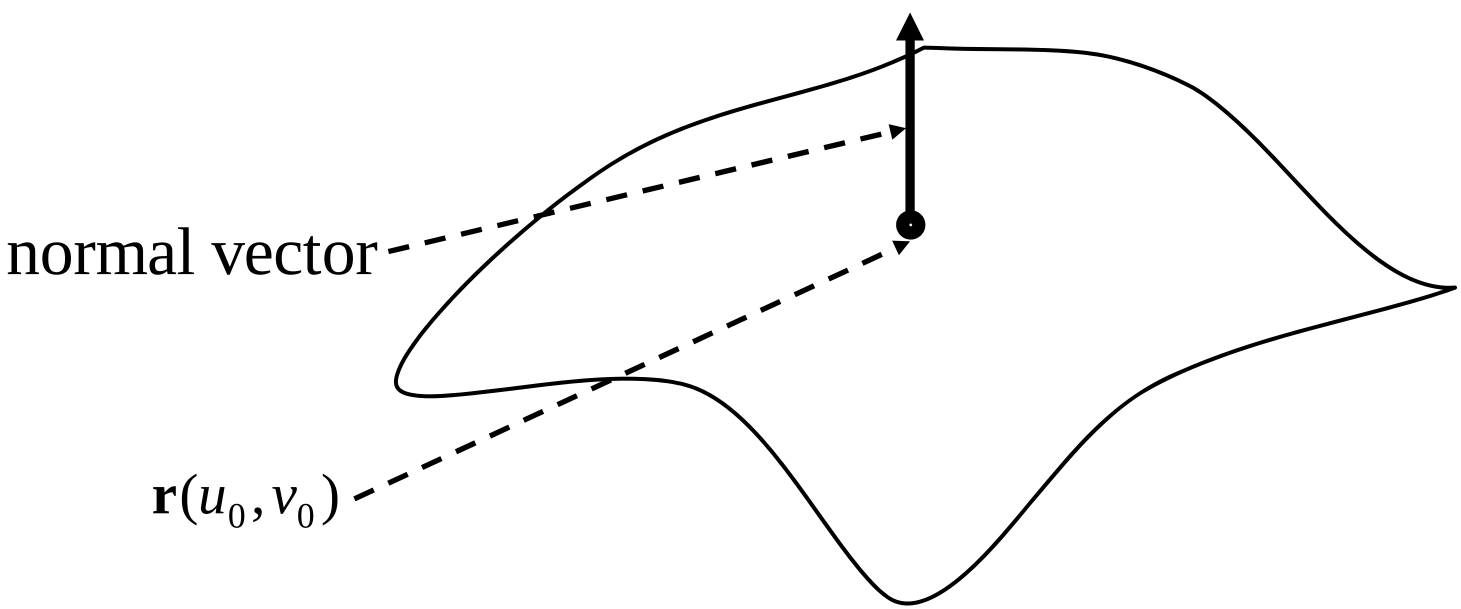
Tangent Planes and Normal Vectors

Given ***surface*** S whose parametric representation is

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}, \qquad \text{------} \qquad (2)$$

at a point P_0 with position vector $\mathbf{r}_0 = \mathbf{r}(u_0, v_0)$.

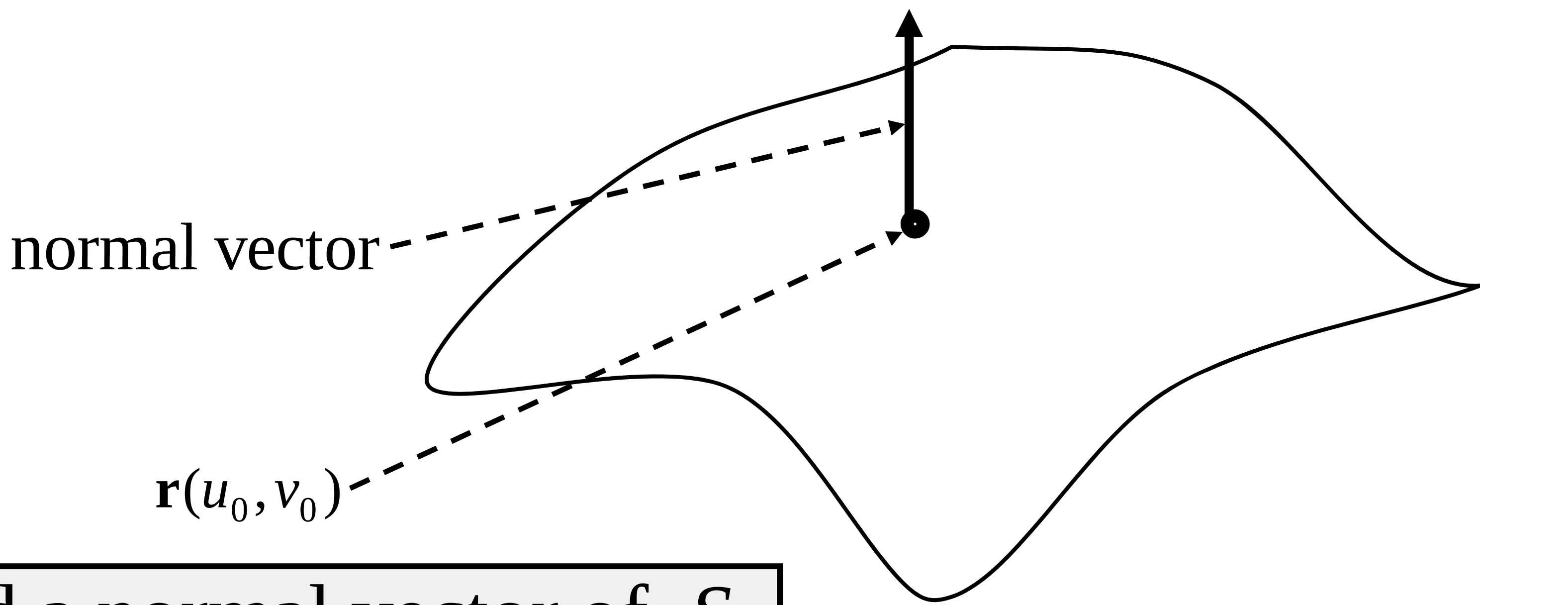
To find : the equation of the ***tangent plane*** to S at P_0.



Tangent Plane and Normal Vectors

Surface S is given by

$$S : \mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$



How to find a normal vector of S at a given point from $\mathbf{r}(u,v)$??

Answer :

Normal Vectors

Surface S is given by

$$S : \mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

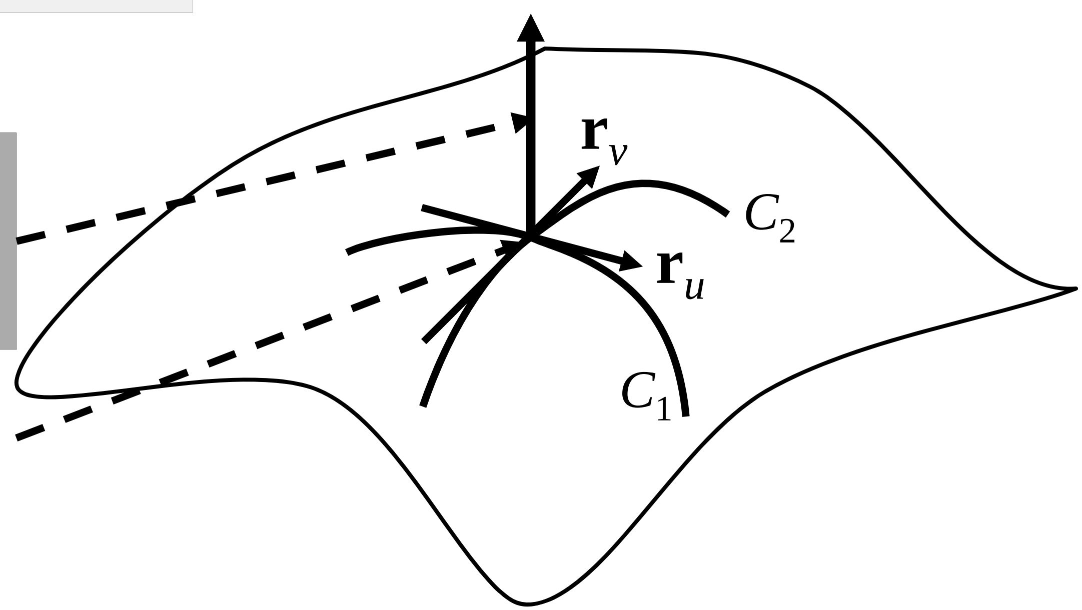
$$\mathbf{r}_u = x_u\mathbf{i} + y_u\mathbf{j} + z_u\mathbf{k}$$

$$\mathbf{r}_v = x_v\mathbf{i} + y_v\mathbf{j} + z_v\mathbf{k}$$

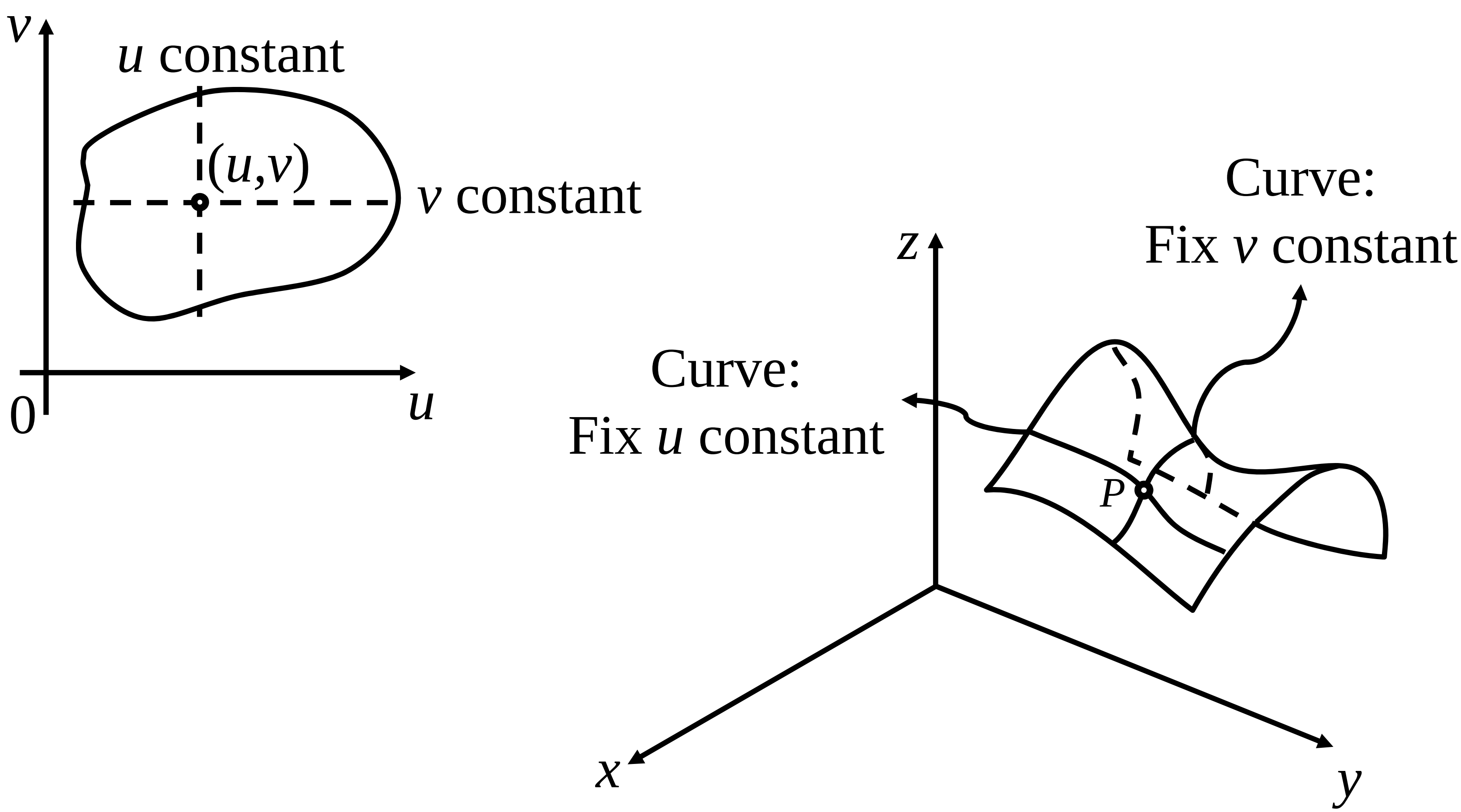
$\mathbf{r}_u \times \mathbf{r}_v$

normal vector

$\mathbf{r}(u_0, v_0)$

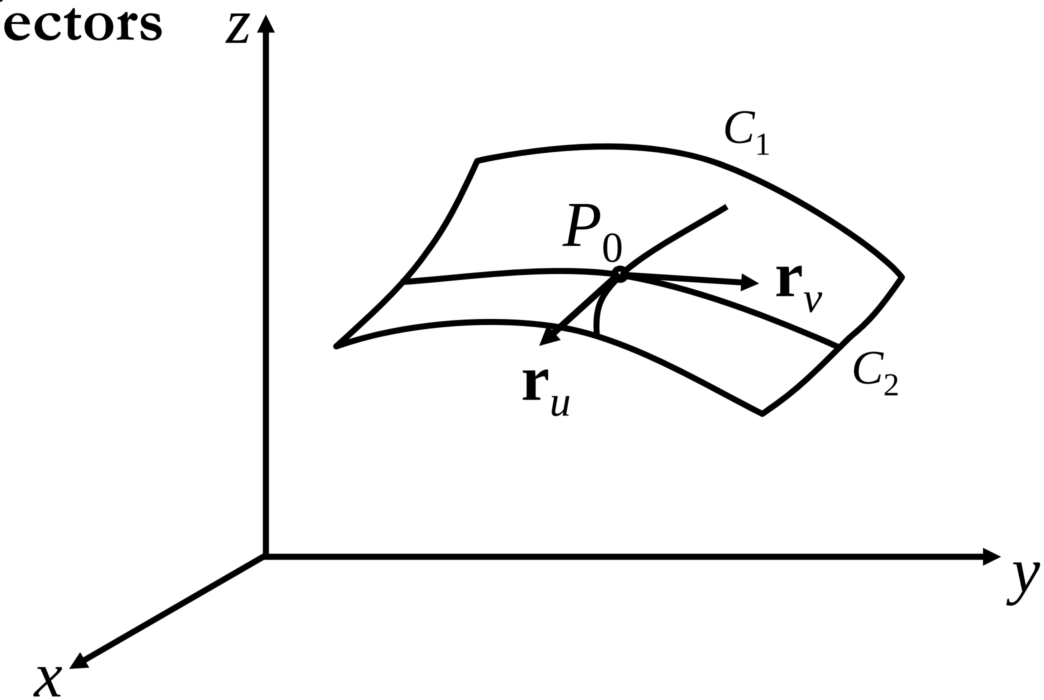


Parametric Surfaces



$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

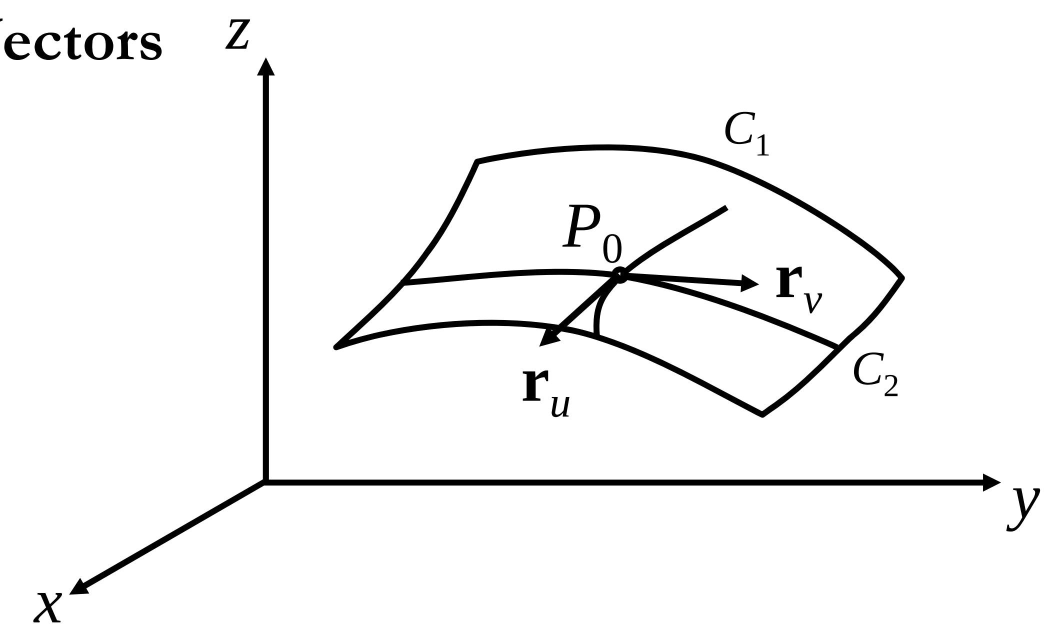
Tangent Planes and Normal Vectors



$$\text{Fix } v = v_0,$$
$$C_1 : \mathbf{r}(u, v_0) = x(u, v_0)\mathbf{i} + y(u, v_0)\mathbf{j} + z(u, v_0)\mathbf{k}.$$

$$\text{Fix } u = u_0,$$
$$C_2 : \mathbf{r}(u_0, v) = x(u_0, v)\mathbf{i} + y(u_0, v)\mathbf{j} + z(u_0, v)\mathbf{k}.$$

Tangent Planes and Normal Vectors



Fix $v = v_0$,

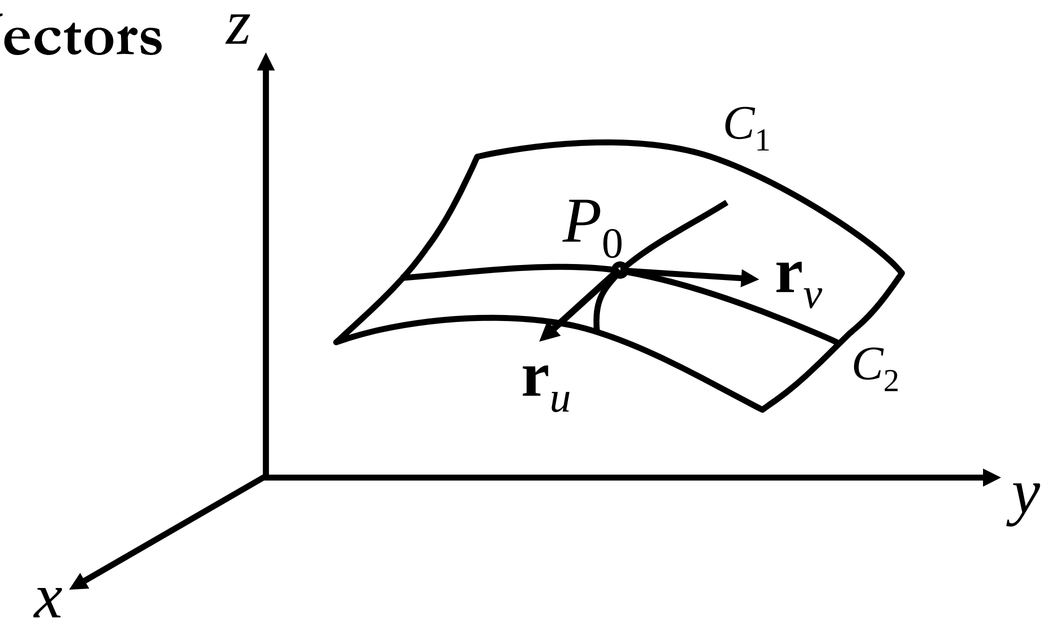
$$C_1 : \mathbf{r}(u, v_0) = x(u, v_0)\mathbf{i} + y(u, v_0)\mathbf{j} + z(u, v_0)\mathbf{k}.$$

The tangent vector to C_1 at P_0 is given by $\left.\dfrac{d}{du}\mathbf{r}(u, v_0)\right|_{u=u_0}$,

which is simply

$$\mathbf{r}_u \equiv \frac{\partial x}{\partial u}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\mathbf{k}.$$

Tangent Planes and Normal Vectors



Fix $u = u_0$,

$$C_2 : \mathbf{r}(u_0, v) = x(u_0, v)\mathbf{i} + y(u_0, v)\mathbf{j} + z(u_0, v)\mathbf{k}.$$

The tangent vector to C_2 at P_0 is given by

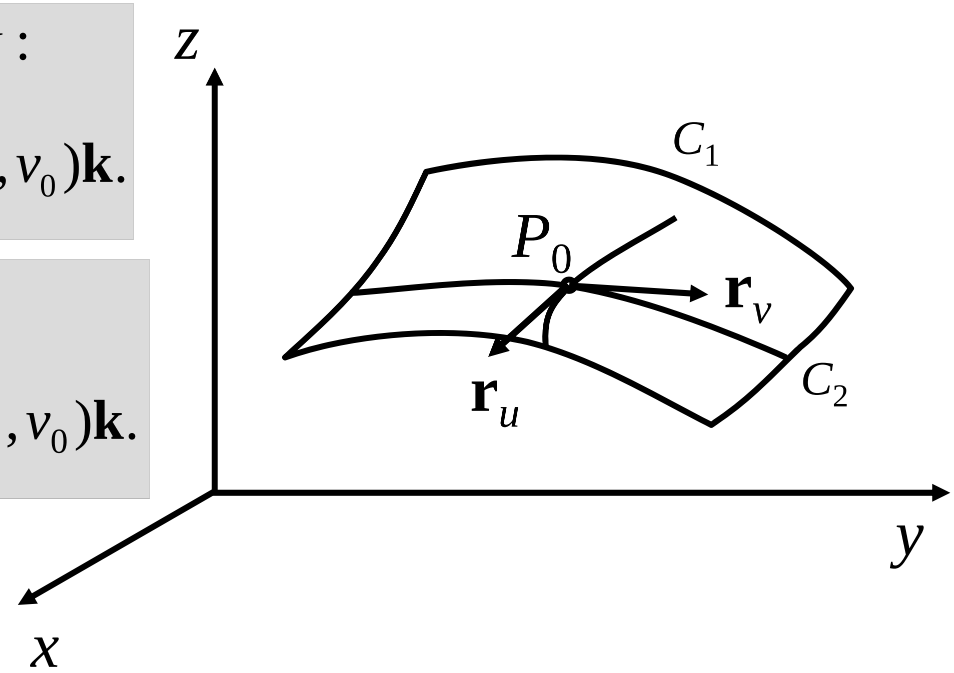
$$\mathbf{r}_v \equiv \frac{\partial x}{\partial v}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\mathbf{k}.$$

The tangent vector to C_1 at P_0 is given by :

$$\mathbf{r}_u \equiv \frac{\partial x}{\partial u}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\mathbf{k}.$$

The tangent vector to C_2 at P_0 is given by

$$\mathbf{r}_v \equiv \frac{\partial x}{\partial v}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\mathbf{k}.$$



Both vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ lie in the tangent plane to S at P_0. Therefore the ***cross product*** $\mathbf{r}_u \times \mathbf{r}_v$, assuming it is nonzero, provides a ***normal*** vector to the tangent plane to S at P_0.

Therefore,

$$(\mathbf{r} - \mathbf{r}_0) \cdot (\mathbf{r}_u \times \mathbf{r}_v) = 0$$

is the ***equation*** of the ***tangent plane*** at P_0.

Example

Find the equation of the tangent plane to the surface with parametric representation

$$\mathbf{r}(u,v) = u\mathbf{i} + v^2\mathbf{j} + (u^2 - v)\mathbf{k}$$

at the point $(1,4,-1)$.

$$\mathbf{r}_u = \mathbf{i} + 0\mathbf{j} + 2u\mathbf{k}$$

$$\mathbf{r}_v = 0\mathbf{i} + 2v\mathbf{j} - \mathbf{k}$$

$$\mathbf{r}_u \times \mathbf{r}_v = -4uv\mathbf{i} + \mathbf{j} + 2v\mathbf{k}$$

The point $(1,4,-1)$ corresponds to $\mathbf{r}(u,v) = \mathbf{i} + 4\mathbf{j} - \mathbf{k}$. so we have

$$\begin{cases} u = 1 \\ v^2 = 4 \\ u^2 - v = -1 \end{cases}$$

which implies $(u,v) = (1,2)$.

Example

Find the equation of the tangent plane to the surface with parametric representation

$$\mathbf{r}(u,v) = u\mathbf{i} + v^2\mathbf{j} + (u^2 - v)\mathbf{k}$$

at the point (1,4,–1).

$$\mathbf{r}_u \times \mathbf{r}_v = -4uv\mathbf{i} + \mathbf{j} + 2v\mathbf{k}$$

At the point $(1,4,-1)$, we have $u = 1$ and $v = 2$.

Thus, at $(1,-4,1)$,

$$\mathbf{r}_u \times \mathbf{r}_v = -4uv\mathbf{i} + \mathbf{j} + 2v\mathbf{k} = -8\mathbf{i} + \mathbf{j} + 4\mathbf{k}.$$

The equation of the tangent plane is:

$$(\mathbf{r} - \mathbf{r}_0) \cdot (\mathbf{r}_u \times \mathbf{r}_v) = 0$$

$$[(x-1)\mathbf{i} + (y-4)\mathbf{j} + (z+1)\mathbf{k}] \cdot (-8\mathbf{i} + \mathbf{j} + 4\mathbf{k}) = 0.$$

Final Answer : $-8x + y + 4z + 8 = 0.$

Example

If S has Cartesian equation $z = f(x, y)$, then a parametric representation of S is

$$\mathbf{r}(u,v) = u\mathbf{i} + v\mathbf{j} + f(u,v)\mathbf{k}.$$

Thus, $\mathbf{r}_u = \mathbf{i} + 0\mathbf{j} + f_u\mathbf{k}$ and $\mathbf{r}_v = 0\mathbf{i} + \mathbf{j} + f_v\mathbf{k}$.

So the normal vector

$$\mathbf{r}_u \times \mathbf{r}_v = -f_u\mathbf{i} - f_v\mathbf{j} + \mathbf{k}.$$

Surface Integrals

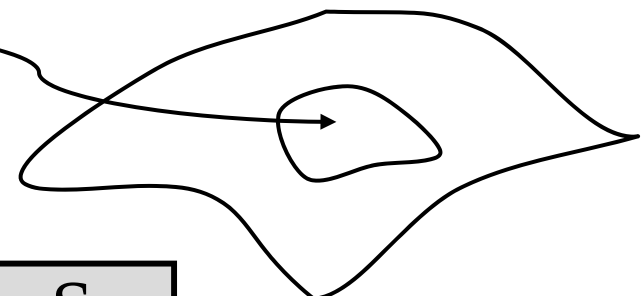
Surface Integrals of Scalar Functions

$f(x, y, z)$: a (scalar) function defined on surface S.

$$\iint_S f(x, y, z)\, dS \qquad \iint_R f(x, y)\, dA$$

Surface Integral   Double Integral

S is a bounded surface



Physical Meaning

$f(x, y, z) =$ density function of a surface S

Surface integral gives the mass of the surface

$f(x, y, z) =$ constant function 1

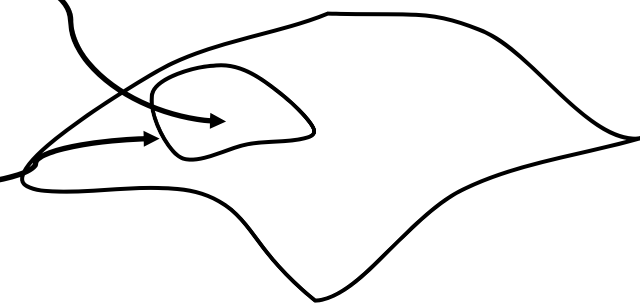
Surface integral gives the area of the surface

Surface Integrals of Scalar Functions

S is a bounded surface

$$S : \mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

(u,v) come from a bounded domain D



The surface integral of f over S is

$$\underbrace{\iint_S f(x,y,z)\, dS}_{\boxed{\text{Surface Integral}}} = \underbrace{\iint_D f(\mathbf{r}(u,v))\, \|\mathbf{r}_u \times \mathbf{r}_v\|\, dA}_{\boxed{\text{Double Integral}}}$$

$\boxed{dudv}$

Surface Integrals (Procedure)

Surface Integral of Scalar Functions $\iint_S f(x, y, z)\, dS$

1. Find the parametric equation of S:

$$S : \mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k} \qquad (u, v) \in D$$

2. Find the domain D in terms of ranges of u and v

3. Substitution: $f(\mathbf{r}(u, v)) = f(x(u, v), y(u, v), z(u, v))$

4. Find normal vector of $\mathbf{r}_u \times \mathbf{r}_v$: $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix}$

5. Compute $\iint_D f(\mathbf{r}(u, v)) \| \mathbf{r}_u \times \mathbf{r}_v \| \, dA$ Double Integral

Surface Integrals of Scalar Functions

Let

Surface S: $\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$,

D: corresponding domain for (u,v),

$f(x,y,z)$: a function defined on S.

The surface integral of a scalar function f over S is

$$\iint_S f(x,y,z)\, dS = \iint_D f(\mathbf{r}(u,v)) \| \mathbf{r}_u \times \mathbf{r}_v \| \, dA.$$

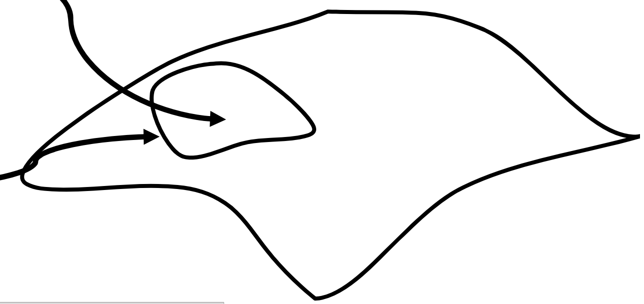
Note that $dS = \| \mathbf{r}_u \times \mathbf{r}_v \| \, dA = \| \mathbf{r}_u \times \mathbf{r}_v \| \, du \, dv$

Surface Integrals of Scalar Functions

S is a bounded surface

$$S : \mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

(u,v) come from a bounded domain D



The surface integral of f over S is

$$\underbrace{\iint_S f(x,y,z)\, dS}_{\boxed{\text{Surface Integral}}} = \underbrace{\iint_D f(\mathbf{r}(u,v)) \| \mathbf{r}_u \times \mathbf{r}_v \| \, dA}_{\boxed{\text{Double Integral}}}$$

$\boxed{dudv}$

$$\iint_S 1\, dS = \text{surface area of } S$$

$$= \iint_D \sqrt{f_x^{\,2} + f_y^{\,2} + 1}\, dA = \iint_D \| \mathbf{r}_u \times \mathbf{r}_v \| \, dA$$

Example

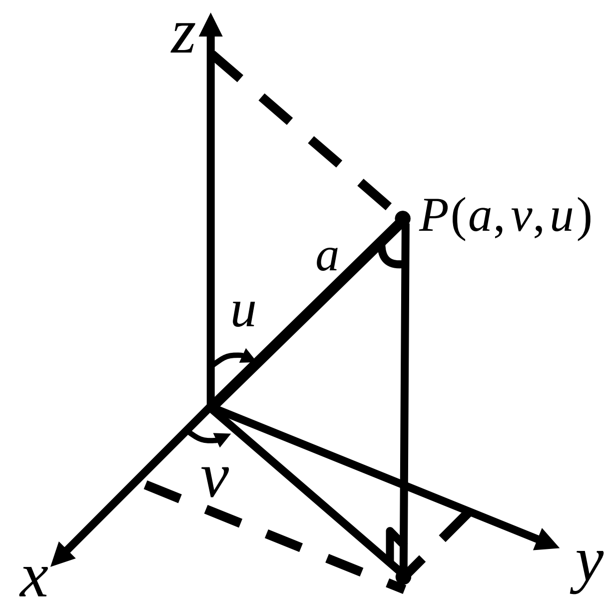
Evaluate $\iint_S (xz + yz)\, dS$ where S is part of the sphere $x^2 + y^2 + z^2 = 9$ in the first octant.

A parametric representation of the sphere is given by

$$\mathbf{r}(u, v) = 3\sin u \cos v\mathbf{i} + 3\sin u \sin v\mathbf{j} + 3\cos u\mathbf{k}.$$

To represent the first octant, the domain D is given by

$$0 \le u \le \frac{\pi}{2} \quad \text{and} \quad 0 \le v \le \frac{\pi}{2}.$$



Example

$$\mathbf{r}(u,v) = 3\sin u\cos v\mathbf{i} + 3\sin u\sin v\mathbf{j} + 3\cos u\mathbf{k}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3\cos u\cos v & 3\cos u\sin v & -3\sin u \\ -3\sin u\sin v & 3\sin u\cos v & 0 \end{vmatrix}$$

$$= 9\sin^2 u\cos v\mathbf{i} + 9\sin^2 u\sin v\mathbf{j} + 9\sin u\cos u\mathbf{k}$$

Thus, $\|\mathbf{r}_u \times \mathbf{r}_v\| = 9\sin u.$

Evaluate $\iint_S (xz + yz)\, dS$ where S is part of the sphere $x^2 + y^2 + z^2 = 9$ in the first octant.

$$\mathbf{r}(u,v) = 3\sin u \cos v\mathbf{i} + 3\sin u \sin v\mathbf{j} + 3\cos u\mathbf{k}$$

Domain D : $\quad 0 \le u \le \dfrac{\pi}{2}$ and $0 \le v \le \dfrac{\pi}{2}$.

Thus, $\|\mathbf{r}_u \times \mathbf{r}_v\| = 9\sin u$.

$$\begin{aligned}
\iint_S (xz + yz)\, dS &= \iint_D (9\sin u \cos u \cos v + 9\sin u \cos u \sin v) \|\mathbf{r}_u \times \mathbf{r}_v\|\ dA \\
&= \int_0^{\pi/2} \int_0^{\pi/2} 81\sin^2 u \cos u(\cos v + \sin v)\ du\ dv \\
&= 81\int_0^{\pi/2} \sin^2 u \cos u\ du \int_0^{\pi/2} (\cos v + \sin v)\ dv \\
&= 81\left[\frac{1}{3}\sin^3 u\right]_0^{\pi/2} = 54.
\end{aligned}$$

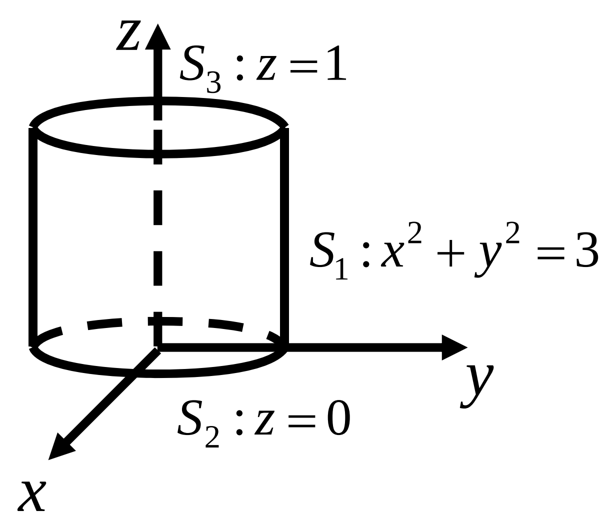
Evaluate $\iint_S z\, dS$, where S is the closed surface bounded laterally by S_1 : the cylinder $x^2 + y^2 = 3$; bounded below by S_2 : the xy - plane and bounded on top by S_3 : the horizontal plane $z = 1$.

Need to find the three surface integrals:

$$\iint_{S_1} z\ dS$$

$$\iint_{S_2} z\ dS$$

and $$\iint_{S_3} z\ dS.$$

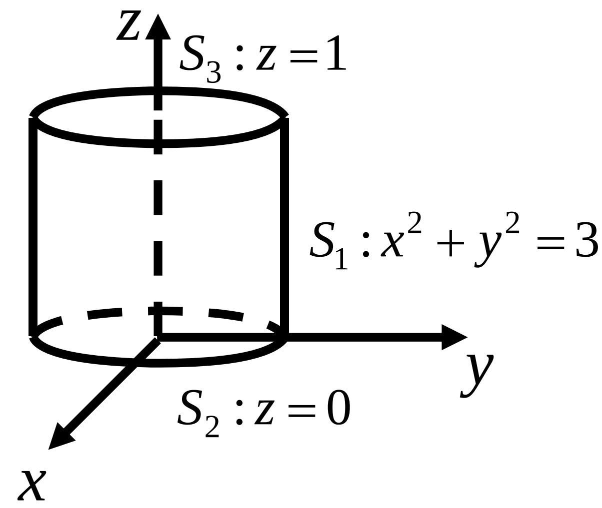


The surface integral is the sum of three surface integrals:

$$\iint_S z\ dS = \iint_{S_1} z\ dS + \iint_{S_2} z\ dS + \iint_{S_3} z\ dS.$$

To find $\iint_{S_1} z\, dS$

S_1 : lateral surface of the cylinder $x^2 + y^2 = 3$



$$S_1 : \quad \mathbf{r}(u,v) = \sqrt{3}\cos u\mathbf{i} + \sqrt{3}\sin u\mathbf{j} + v\mathbf{k}$$

$$D: \quad 0 \le u \le 2\pi \text{ and } 0 \le v \le 1$$

$$\mathbf{r}_u \times \mathbf{r}_v = \sqrt{3}\cos u\mathbf{i} + \sqrt{3}\sin u\mathbf{j} + 0\mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{3}$$

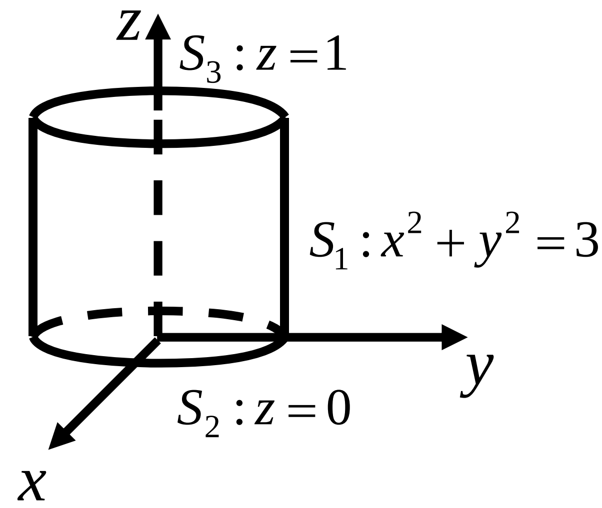
$$\iint_{S_1} z\, dS = \iint_D v \|\mathbf{r}_u \times \mathbf{r}_v\|\, dA$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{3}v\, dv\, du$$

$$= \int_0^{2\pi} \frac{\sqrt{3}}{2}\, du = \sqrt{3}\pi$$

To find $\iint_{S_2} z \, dS$

S_2: the xy-plane



S_2 is on the xy-plane.

Note that $z = 0$ on the xy-plane.

Thus, the integrand of $\iint_{S_2} z \, dS$ is zero.

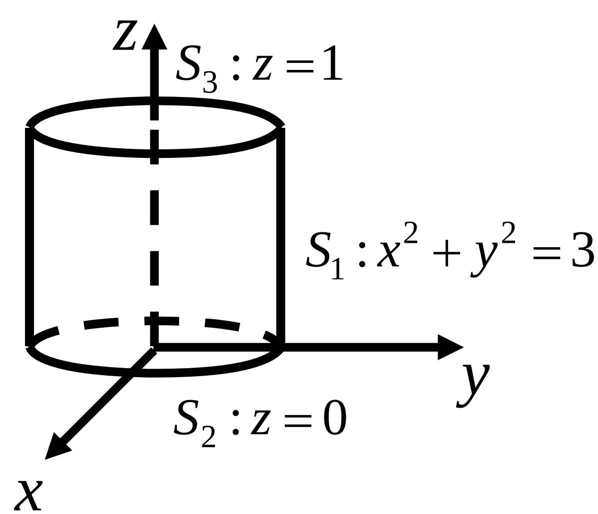
Hence, $\iint_{S_2} z \, dS = 0.$

To find $\iint_{S_3} z\,dS$

S_3 : the horizontal plane $z = 1$

The surface S_3 is on the horizontal plane $z = 1$.

$$\begin{aligned}\iint_{S_3} z\,dS &= \iint_{S_3} dS \\ &= \text{area of } S_3 \\ &= \pi(\sqrt{3})^2 \\ &= 3\pi.\end{aligned}$$

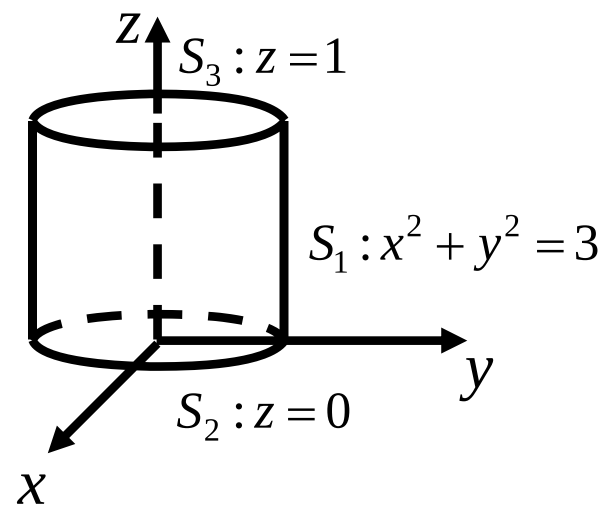


$$\begin{aligned}\iint_S 1\,dS &= \text{surface area of } S \\ &= \iint_D \sqrt{f_x^{\,2} + f_y^{\,2} + 1}\,dA = \iint_D \|\mathbf{r}_u \times \mathbf{r}_v\|\,dA\end{aligned}$$

$$\iint_{S_1} z\,dS = \sqrt{3}\pi$$

$$\iint_{S_2} z\,dS = 0$$

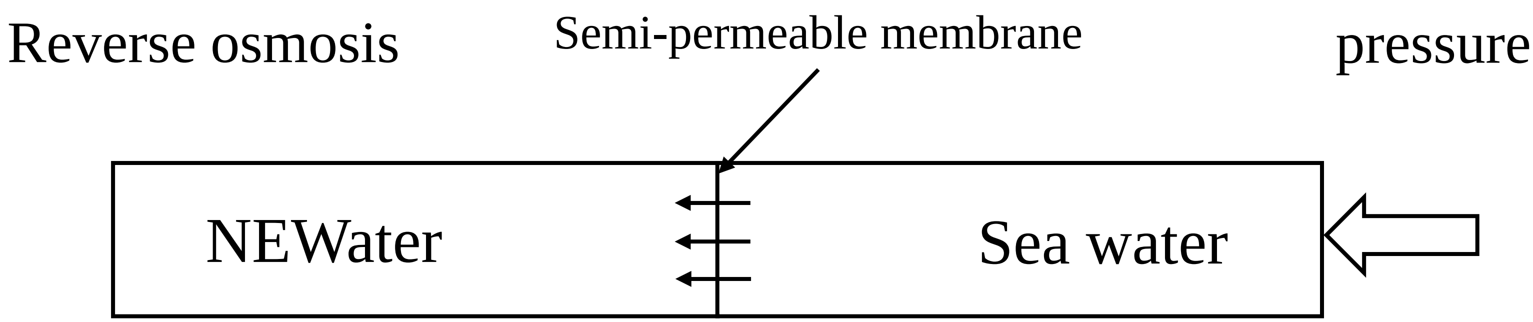
$$\iint_{S_3} z\,dS = 3\pi$$



Consequently,

$$\iint_S z\,dS = \iint_{S_1} z\,dS + \iint_{S_2} z\,dS + \iint_{S_3} z\,dS = (3+\sqrt{3})\pi.$$

Volume Flow Rate



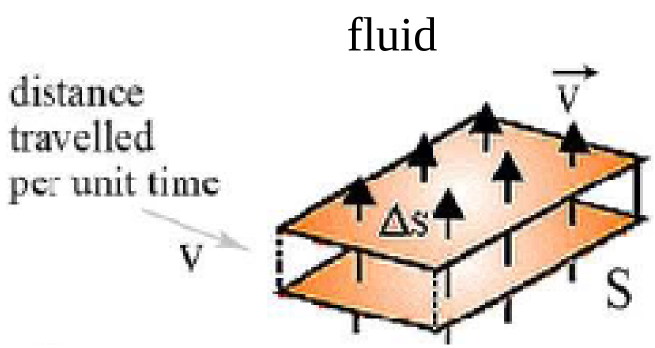
$$\text{Volume flow rate through the membrane}$$

$$\text{Volume / sec} = \text{lateral distance / sec} \times \text{cross-sectional area}$$

$$\text{Volume rate} = \text{velocity of water flow} \times \text{cross-sectional area}$$

Volume Flow Rate

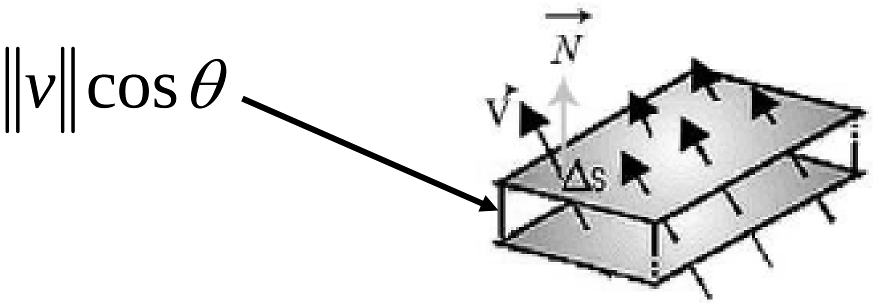
(i) The fluid velocity is constant over flat surface S and its direction is perpendicular to S.



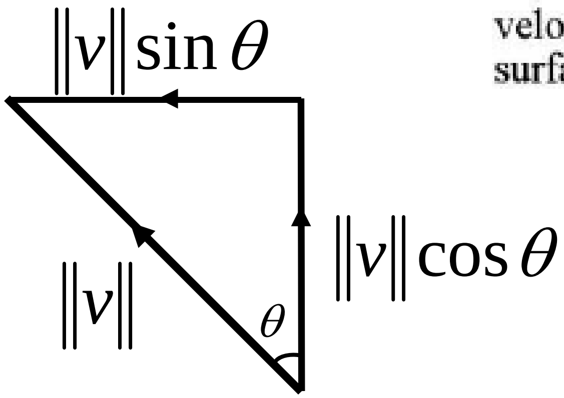
$$\text{The volume flow rate } w = \|v\|\,\Delta s$$

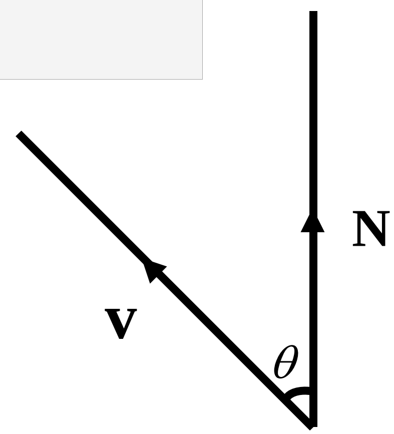
Volume Flow Rate

(ii) The fluid velocity is constant over flat surface S but its direction is not perpendicular to S.



velocity is constant
surface is flat





$$\boxed{\begin{aligned}\mathbf{v}\cdot\mathbf{N} &= \ \|\mathbf{v}\|\ \|\mathbf{N}\|\cos\theta \\ &= \ \|\mathbf{v}\|\cos\theta\end{aligned}}$$

N : unit normal vector to the surface

$$\text{The volume rate } w = (\|\mathbf{v}\|\cos\theta)\times\Delta s$$
$$= (\mathbf{v}\cdot\mathbf{N})\times\Delta s$$

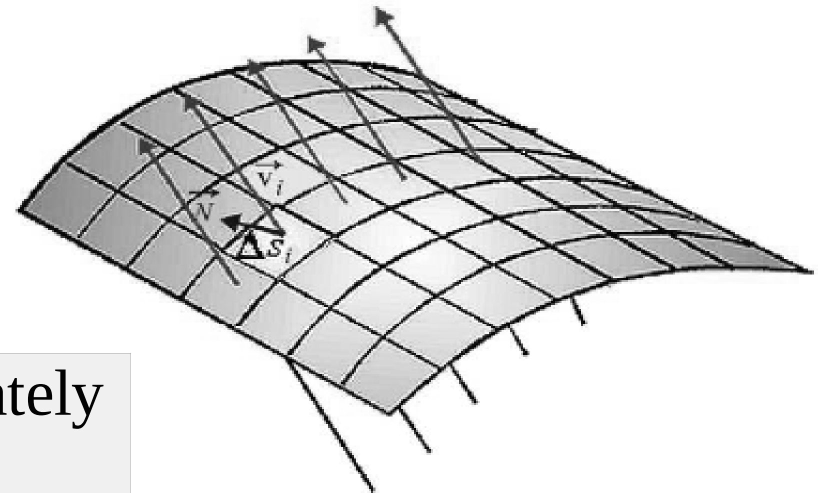
Volume Flow Rate

(iii) The fluid velocity is changing over curved surface S.

In a particular segment,

we have

$$w_i \approx \mathbf{v}(x_i, y_i, z_i) \cdot \mathbf{N}_i \Delta s_i.$$



General case (velocity is changing on a curved surface)

Thus, the total flow rate is approximately

$$w \approx \sum_{i=1}^{n} \mathbf{v}(x_i, y_i, z_i) \cdot \mathbf{N}_i \Delta s_i \quad \text{------} \quad (3)$$

Let n goes to infinity, the RHS of (3) becomes an integral

$$\iint_S \mathbf{v}(x, y, z) \cdot \mathbf{N} ds$$

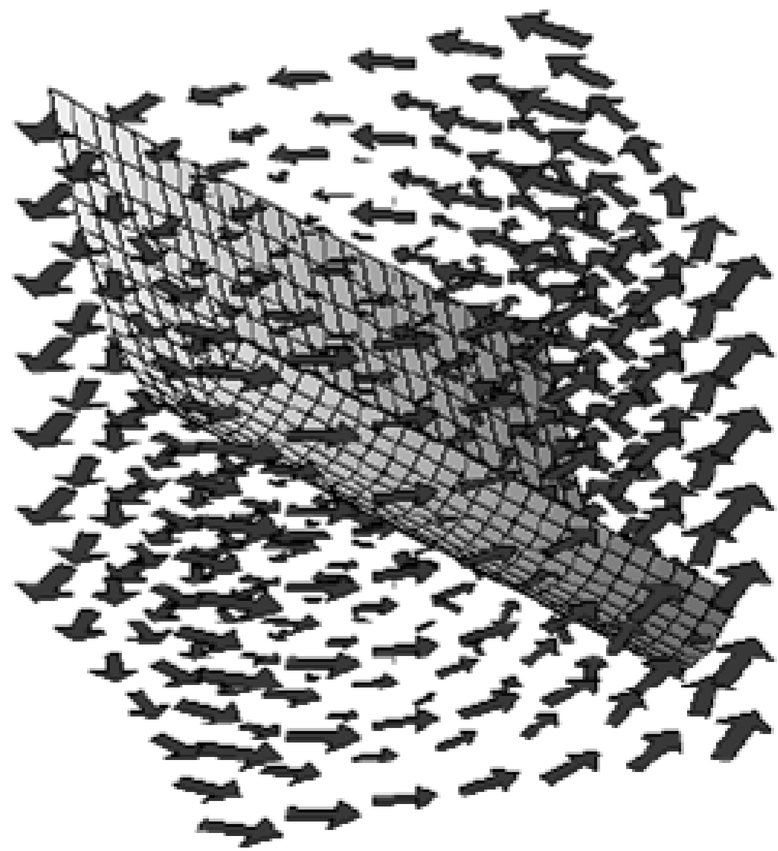
which represents the actual total volume flow rate.

This integral is called a ***surface integral of the vector field*** **v**.

Flux

The Flux of a vector field F through the surface is a measure of the rate of change of the amount of the flow through the surface

Fluid flow
Heat flow
Electric flow
Magnetic flow etc



Surface Integrals of Vector Fields

Let S : surface with unit normal vector $\mathbf{n}$,

and $\mathbf{F}$: continuous ***vector field*** defined on S.

The ***surface integral*** of $\mathbf{F}$ over S is given by

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS \text{ , where } \mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\| \mathbf{r}_u \times \mathbf{r}_v \|}$$

or simply

$$\iint_S \mathbf{F} \cdot d\mathbf{S}.$$

This integral is also called the ***flux*** of $\mathbf{F}$ over S as it is related to the volume flow rate of fluid.

Surface Integrals of Vector Fields

If S is given by the parametric representation $\mathbf{r}(u,v)$ with domain D, then

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}\, dS$$

$$= \iint_D \left[\mathbf{F}(\mathbf{r}(u,v))\cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}\right] \|\mathbf{r}_u \times \mathbf{r}_v\|\, dA$$

$$= \iint_D \mathbf{F}(\mathbf{r}(u,v))\cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

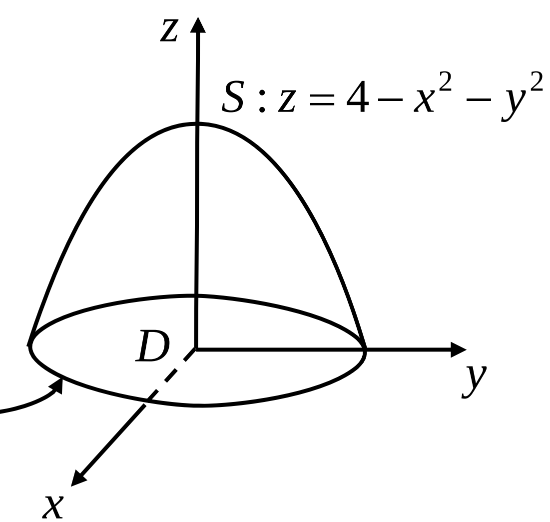
$$\boxed{\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u,v))\cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA}$$

Evaluate $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + xy\mathbf{k}$ and S is the part of the paraboloid $z = 4 - x^2 - y^2$ above the xy-plane.

$z = f(x, y)$

Since S has Cartesian equation $z = 4 - x^2 - y^2$, the parametric representation is

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + (4 - u^2 - v^2)\mathbf{k}.$$



D is the projection onto xy-plane, which is the disk of radius 2.

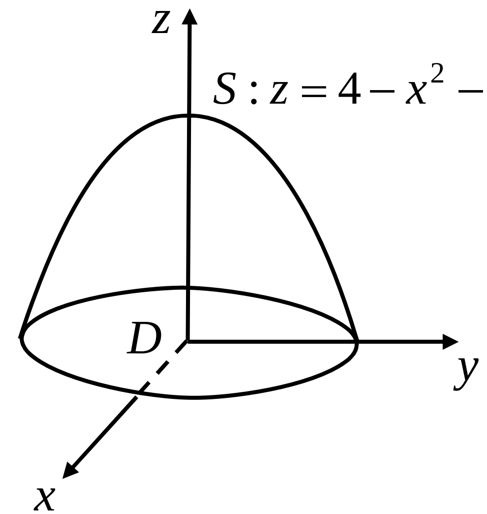
Evaluate $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + xy\mathbf{k}$ and S is the part of the paraboloid $z = 4 - x^2 - y^2$ above the xy-plane.

$$\mathbf{r}(u,v) = u\mathbf{i} + v\mathbf{j} + (4-u^2-v^2)\mathbf{k}$$

D : the disk of radius 2.

$\mathbf{r}_u = \mathbf{i} + 0\mathbf{j} - 2u\mathbf{k}$ and $\mathbf{r}_v = 0\mathbf{i} + \mathbf{j} - 2v\mathbf{k}$

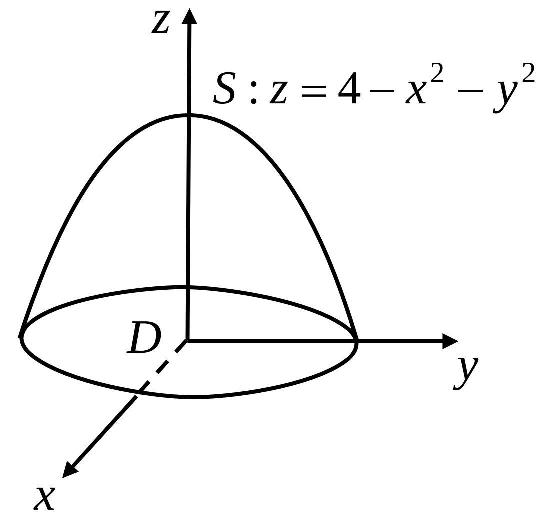
$$\mathbf{r}_u \times \mathbf{r}_v = 2u\mathbf{i} + 2v\mathbf{j} + \mathbf{k}$$



$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_D \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u \times \mathbf{r}_v)\, dA \\ &= \iint_D (u\mathbf{i} + v\mathbf{j} + uv\mathbf{k})\cdot(2u\mathbf{i} + 2v\mathbf{j} + \mathbf{k})\, dA \\ &= \iint_D (2u^2 + 2v^2 + uv)\, dA\end{aligned}$$

D : the disk of radius 2.

Note : the region D is a circular disk, we compute the double integral in polar coordinates.



$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\ dA$$
$$= \iint_D (u\mathbf{i} + v\mathbf{j} + uv\mathbf{k}) \cdot (2u\mathbf{i} + 2v\mathbf{j} + \mathbf{k})\ dA$$
$$= \iint_D (2u^2 + 2v^2 + uv)\ dA$$

$$= \int_0^{2\pi} \int_0^2 (2r^2 + r^2 \cos\theta \sin\theta) r\ dr\ d\theta$$
$$= 16\pi.$$

Let $\mathbf{F}(x, y, z) = y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is the sphere $x^2 + y^2 + z^2 = 1$.

A parametric representation of the unit sphere is given by

$$\mathbf{r}(u, v) = \sin u \cos v\mathbf{i} + \sin u \sin v\mathbf{j} + \cos u\mathbf{k},$$

with D given by $0 \le u \le \pi$ and $0 \le v \le 2\pi$.

$$\mathbf{r}_u \times \mathbf{r}_v = \sin^2 u \cos v\mathbf{i} + \sin^2 u \sin v\mathbf{j} + \sin u \cos u\mathbf{k}$$

$$\mathbf{F}(\mathbf{r}(u, v)) = \sin u \sin v\mathbf{i} + \sin u \cos v\mathbf{j} + \cos u\mathbf{k}$$

$$\mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) = 2\sin^3 u \sin v \cos v + \sin u \cos^2 u$$

Let $\mathbf{F}(x, y, z) = y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is the sphere $x^2 + y^2 + z^2 = 1$.

$$\mathbf{r}(u, v) = \sin u \cos v\mathbf{i} + \sin u \sin v\mathbf{j} + \cos u\mathbf{k},$$

$$D: \quad 0 \le u \le \pi \text{ and } 0 \le v \le 2\pi$$

$$\mathbf{r}_u \times \mathbf{r}_v = \sin^2 u \cos v\mathbf{i} + \sin^2 u \sin v\mathbf{j} + \sin u \cos u\mathbf{k}$$

$$\mathbf{F}(\mathbf{r}(u, v)) = \sin u \sin v\mathbf{i} + \sin u \cos v\mathbf{j} + \cos u\mathbf{k}$$

$$\mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) = 2\sin^3 u \sin v \cos v + \sin u \cos^2 u$$

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi} \int_0^{\pi} (2\sin^3 u \sin v \cos v + \sin u \cos^2 u)\, du\, dv \\
&= \int_0^{\pi} \sin^3 u\, du \int_0^{2\pi} \sin 2v\, dv + \int_0^{\pi} \sin u \cos^2 u\, du \int_0^{2\pi} dv \\
&= \frac{4\pi}{3}.
\end{aligned}$$

Orientation of Surfaces

If S is a surface given in parametric form by $\mathbf{r} = \mathbf{r}(u,v)$, then the normal vector $\mathbf{r}_u \times \mathbf{r}_v$ automatically defines an orientation of S.

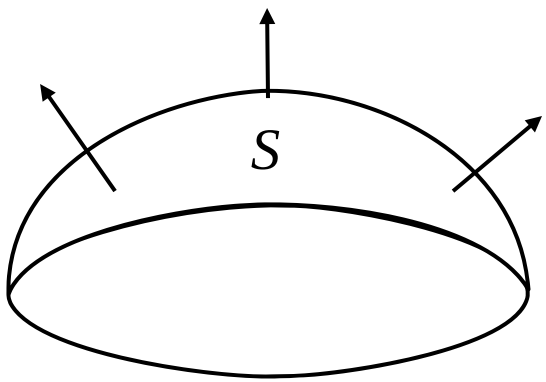
The opposite orientation is given by $\mathbf{r}_v \times \mathbf{r}_u = -\mathbf{r}_u \times \mathbf{r}_v$ and the corresponding oriented surface is denoted by $-S$.

$$\boxed{\iint_{-S} \mathbf{F} \cdot d\mathbf{S} = -\iint_{S} \mathbf{F} \cdot d\mathbf{S}}$$

Orientation of Surfaces

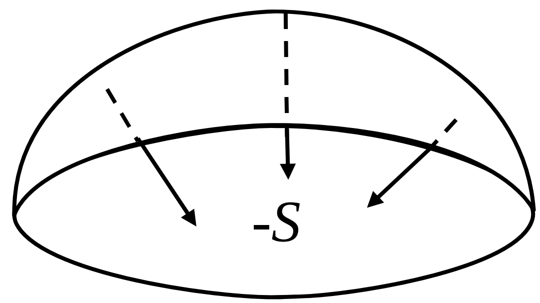
Two possible orientations of surface

------ depends on the choice of normal vectors



One Orientation

Upward Normal Vector
Outer Normal Vector



The Other Orientation

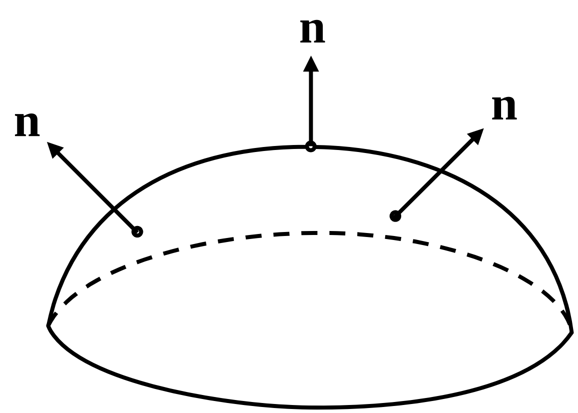
Downward Normal Vector
Inner Normal Vector

$S:\mathbf{r}(u,v)$ $\mathbf{r}_u \times \mathbf{r}_v$ supply S with an orientation

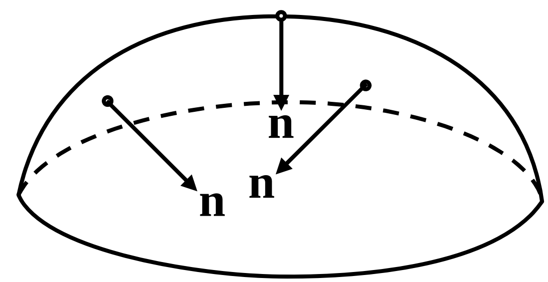
$\mathbf{r}_v \times \mathbf{r}_u = -\mathbf{r}_u \times \mathbf{r}_v$ the orientation of $-S$

$$\boxed{\iint_{-S} \mathbf{F}\cdot d\mathbf{S} = -\iint_{S} \mathbf{F}\cdot d\mathbf{S}}$$

Orientation of Surfaces



S



$-S$

If S has Cartesian equation $z = f(x, y)$, then a parametric representation of S is

$$\mathbf{r}(u,v) = u\mathbf{i} + v\mathbf{j} + f(u,v)\mathbf{k}.$$

Thus, $\mathbf{r}_u = \mathbf{i} + 0\mathbf{j} + f_u\mathbf{k}$ and $\mathbf{r}_v = 0\mathbf{i} + \mathbf{j} + f_v\mathbf{k}$.

So the normal vector

$$\mathbf{r}_u \times \mathbf{r}_v = -f_u\mathbf{i} - f_v\mathbf{j} + \mathbf{k}.$$

$S : \mathbf{r}(u,v)$ $\quad$ $\mathbf{r}_u \times \mathbf{r}_v$ $\quad$ supply S with an orientation

---

PAUSE and THINK !!

How to check $\mathbf{r}_u \times \mathbf{r}_v$ is

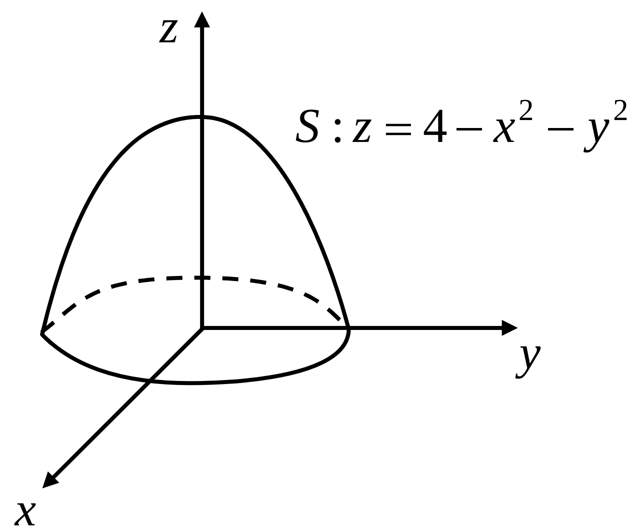
Upward Normal Vector

or

Downward Normal Vector ???

Example

Let S to be part of the paraboloid $z = 4 - x^2 - y^2$ above the xy-plane. We have seen the parametric representation $\mathbf{r}(u,v) = u\mathbf{i} + v\mathbf{j} + (4 - u^2 - v^2)\mathbf{k}$ and the normal vector $\mathbf{r}_u \times \mathbf{r}_v = 2u\mathbf{i} + 2v\mathbf{j} + \mathbf{k}$.



$$\mathbf{r}(u,v) = u\mathbf{i} + v\mathbf{j} + (4 - u^2 - v^2)\mathbf{k}$$

Normal Vector $\mathbf{r}_u \times \mathbf{r}_v = 2u\mathbf{i} + 2v\mathbf{j} + \mathbf{k}$ upward or downward?

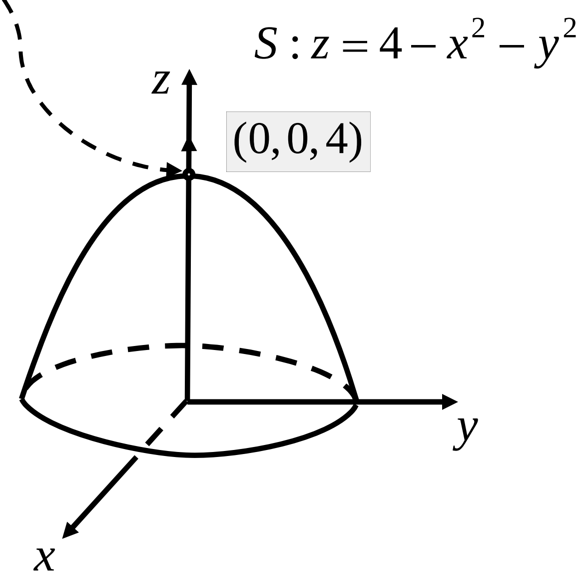
Consider the point $(0,0,4)$ on the paraboloid.

$u = 0, v = 0$

$$\mathbf{r}(u,v) = u\mathbf{i} + v\mathbf{j} + (4 - u^2 - v^2)\mathbf{k}$$

Put $u = 0$ and $v = 0$: $\mathbf{r}_u \times \mathbf{r}_v = \mathbf{k}$

Thus, $\mathbf{r}_u \times \mathbf{r}_v$ is pointing "upwards".



Orientation of the paraboloid is given by the

upward normal vector

Let S to be the unit sphere $x^2+y^2+z^2=1$. We have seen the parametric representation

$$\mathbf{r}(u,v)=\sin u\cos v\mathbf{i}+\sin u\sin v\mathbf{j}+\cos u\mathbf{k}$$

and the normal vector

$$\boxed{\mathbf{r}_u\times\mathbf{r}_v=\sin^2 u\cos v\mathbf{i}+\sin^2 u\sin v\mathbf{j}+\sin u\cos u\mathbf{k}.}$$

Consider the point $(1,0,0)$ on the sphere.

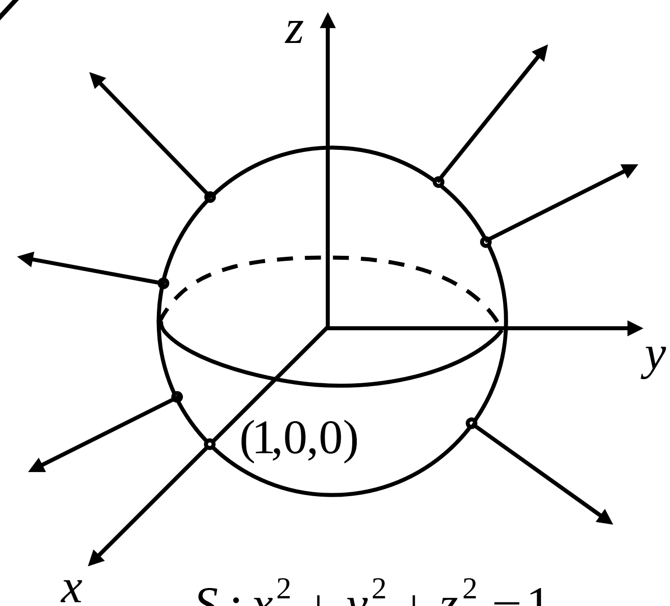
This point corresponds to $u=\dfrac{\pi}{2}$ and $v=0$.

Substitute $u=\dfrac{\pi}{2}$ and $v=0$ into $\mathbf{r}_u\times\mathbf{r}_v$

At (1,0,0), $\mathbf{r}_u\times\mathbf{r}_v=\mathbf{i}$,

which is pointing "outwards".

Hence, the orientation of the sphere is given by the ***outward normal vector***.



$S: x^2+y^2+z^2=1$

Curl and Divergence

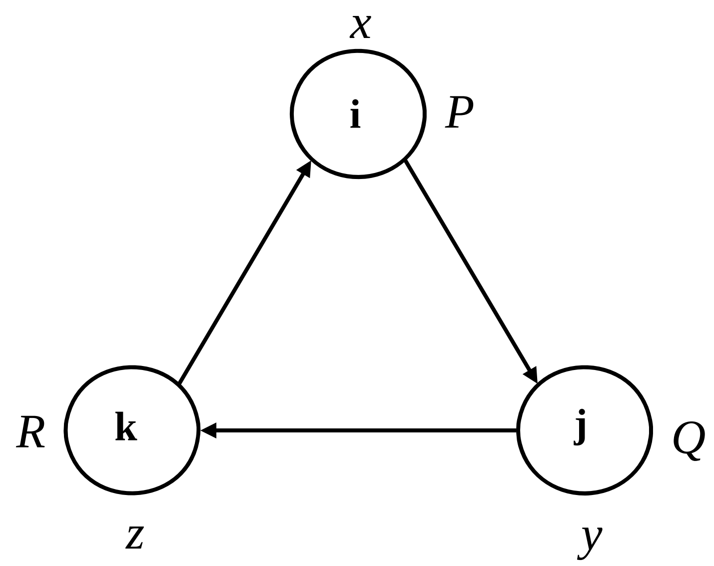
Curl

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a vector field in the xyz-space.

Then ***curl*** of $\mathbf{F}$ is defined by

$$\operatorname{curl} \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

is a ***vector*** field.



Divergence

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a vector field in the xyz-space.

Then ***divergence*** of $\mathbf{F}$ is defined by

$$\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

is a ***scalar*** field.

Curl and Divergence

Operations on vector fields

3 variable vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$

$$\text{Curl} : \text{curl}\,\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$$

vector field $\longrightarrow$ $\boxed{\text{curl}}$ $\longrightarrow$ vector field

---

$$\text{Divergence} : \text{div}\,\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

vector field $\longrightarrow$ $\boxed{\text{div}}$ $\longrightarrow$ scalar function

Del Operator

The curl and divergence operators can be expresses in terms of the del operator:

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$$

Del Operator

(i) Taking the cross product of ∇ with a vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$$

$$\boxed{\nabla \times \mathbf{F} = \operatorname{curl} \mathbf{F}}$$

Del Operator

(ii) Taking the dot product of ∇ with a vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$,

$$\nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \right) \cdot \left(P\mathbf{i} + Q\mathbf{j} + R\mathbf{k} \right)$$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$\boxed{\nabla \cdot \mathbf{F} = \operatorname{div} \mathbf{F}}$$

Example

$$\text{Let } \mathbf{F}(x, y, z) = x^2yz\mathbf{i} + xy^2z\mathbf{j} + xyz^2\mathbf{k}.$$

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$$

$$\begin{aligned}
\text{curl } \mathbf{F} &= \nabla \times \mathbf{F} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2yz & xy^2z & xyz^2 \end{vmatrix} \\
&= (xz^2 - xy^2)\mathbf{i} + (x^2y - yz^2)\mathbf{j} + (y^2z - x^2z)\mathbf{k}.
\end{aligned}$$

Example

$$\text{Let } \mathbf{F}(x, y, z) = x^2yz\mathbf{i} + xy^2z\mathbf{j} + xyz^2\mathbf{k}.$$

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$$

$$\begin{aligned}
\operatorname{div}\mathbf{F} &= \nabla \cdot \mathbf{F} \\
&= \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(xy^2z) + \frac{\partial}{\partial z}(xyz^2) \\
&= 6xyz.
\end{aligned}$$

Example

Show that $\operatorname{curl}(\nabla f) = \mathbf{0}$,

i.e., $\nabla \times (\nabla f) = \mathbf{0}$.

Note that

$$f_{xy} = f_{yx}, \quad f_{xz} = f_{zx} \quad \text{and} \quad f_{yz} = f_{yz}.$$

$$\text{Show that} \quad \text{curl}\,(\nabla f) = \mathbf{0},$$

$$\text{i.e., } \nabla \times (\nabla f) = \mathbf{0}.$$

$$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F}$$

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

$$\text{curl}\,(\nabla f)$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\mathbf{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\mathbf{k}$$

$$= \mathbf{0}.$$

Curl and Conservation Fields

Let $\mathbf{F}$ be a vector field in the *xyz*-space.

If curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservation field.

The converse is also true.

$$\boxed{\mathbf{F} \text{ is conservative} \quad \Leftrightarrow \quad \nabla \times \mathbf{F} = \mathbf{0}}$$

$$\boxed{\text{curl } \mathbf{F} = \nabla \times \mathbf{F}}$$

Curl and Conservative Fields

$$\text{curl}\,\mathbf{F} = 0 \quad \Leftrightarrow \mathbf{F} \quad \text{is conservative} \quad (\text{i.e., } \mathbf{F} = \nabla f\,)$$

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$.

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

$$\text{curl}\,\mathbf{F} = 0 \quad \Rightarrow \quad \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z},\ \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x},\ \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

$$\Rightarrow \mathbf{F} \text{ is conservative}$$

Example

Let $\mathbf{F}(x, y, z) = x^2yz\mathbf{i} + xy^2z\mathbf{j} + xyz^2\mathbf{k}$.

$$\begin{aligned}\text{curl } \mathbf{F} &= (xz^2 - xy^2)\mathbf{i} + (x^2y - yz^2)\mathbf{j} + (y^2z - x^2z)\mathbf{k} \\ &\neq \mathbf{0}\end{aligned}$$

Thus, $\mathbf{F}$ is not conservative.

Example

Find the ***curl*** of the velocity vector fields defined by

(a) $\mathbf{F}_1 = x\mathbf{i} + y\mathbf{j}$, (b) $\mathbf{F}_2 = -y\mathbf{i} + x\mathbf{j}$, (c) $\mathbf{F}_3 = \cos y\ \mathbf{i} + \sin x\ \mathbf{j}$.

Solution:

(a) $\text{curl } \mathbf{F}_1 = \mathbf{0}$

(b) $\text{curl } \mathbf{F}_2 = 2\mathbf{k}$

(c) $\text{curl } \mathbf{F}_3 = (\cos x + \sin y)\mathbf{k}$

Example

Find the ***divergence*** of the velocity vector fields defined by

(a) $\mathbf{F}_1 = x\mathbf{i} + y\mathbf{j}$, (b) $\mathbf{F}_2 = -y\mathbf{i} + x\mathbf{j}$, (c) $\mathbf{F}_3 = -x^2\mathbf{i} + y^2\mathbf{j}$.

Solution:

(a) $\text{div}\,\mathbf{F}_1 = 2$

(b) $\text{div}\,\mathbf{F}_2 = 0$

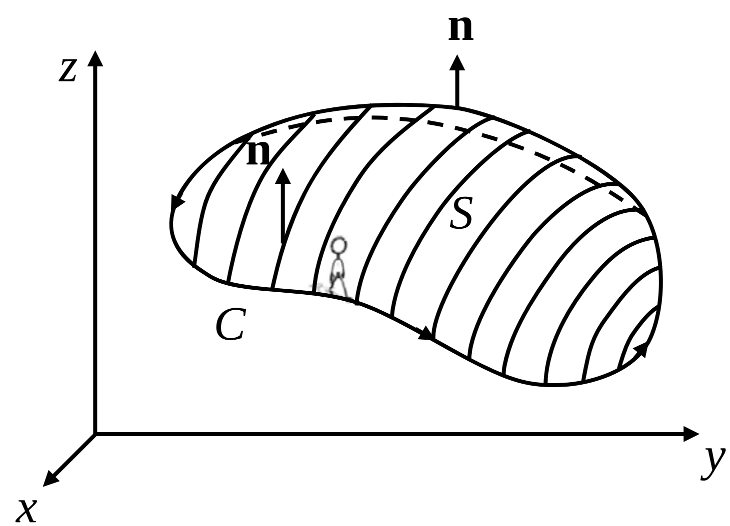
(c) $\text{div}\,\mathbf{F}_3 = 2(y - x)$

Stokes' Theorem

Let S be an oriented piecewise-smooth surface that is bounded by a closed, piecewise-smooth boundary curve C. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on S. Then

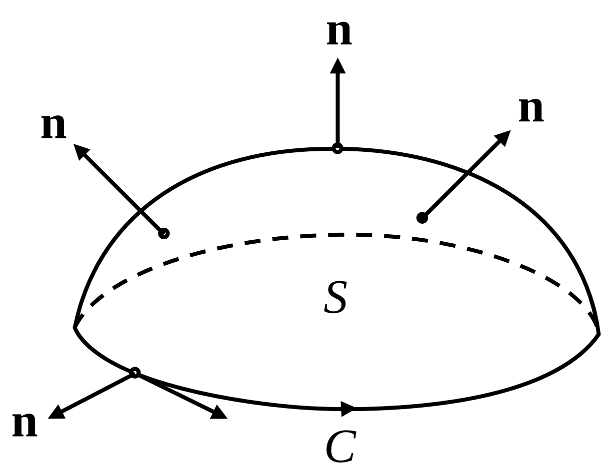
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}.$$

George Gabriel Stokes
1819 - 1903



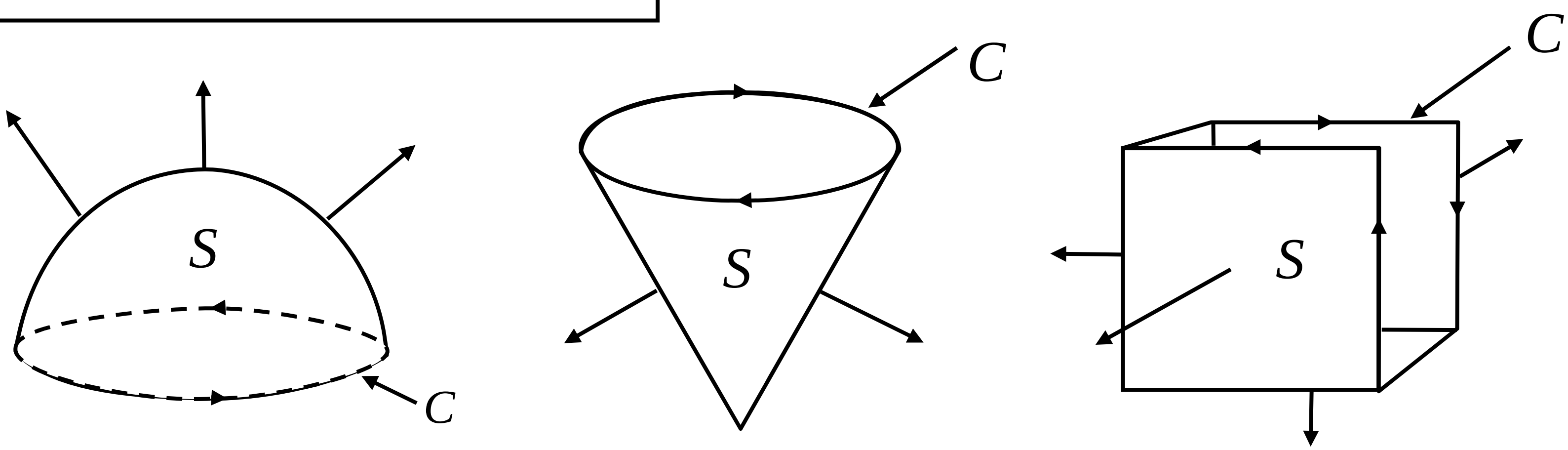
Stokes' Theorem - Note

The orientation of C must be consistent with that of S : when you walk in the direction (orientation) around C with your head pointing in the direction of the normal vector of S, the surface S should be on your left.



Orientation - Boundary Curve of Surface

$$\boxed{\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\operatorname{curl} \mathbf{F}) \cdot d\mathbf{S}}$$



Orientation of C must be consistent with orientation of S: Traverse in the direction (orientation) of C, with head in the direction (orientation) of the normal vector of S. The corresponding "face" of S must be on the left hand side.

Stokes' Theorem

$$\text{Let } \mathbf{F} = \mathbf{P}\mathbf{i} + Q\mathbf{j} + R\mathbf{k}.$$

$$\text{curl } \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P\ dx + Q\ dy + R\ dz$$

Stokes' Theorem

Let $\mathbf{F} = \mathbf{Pi} + Q\mathbf{j} + R\mathbf{k}$.

$\mathbf{F}$: vector field whose components have continuous partial derivatives on surface S.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\ dx + Q\ dy + R\ dz$$

$$\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

line integral of vector field

$$\boxed{\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\operatorname{curl} \mathbf{F}) \cdot d\mathbf{S}}$$

surface integral of vector field

Make sure the orientations of C and S are correct.

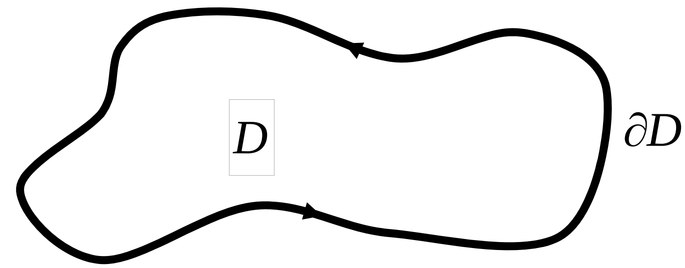
S : oriented piecewise-smooth surface with a "boundary curve" C.

Green’s Theorem

$$\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$$

We may write the line integral as

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C P\ dx + Q\ dy.$$



positive orientation of ∂D

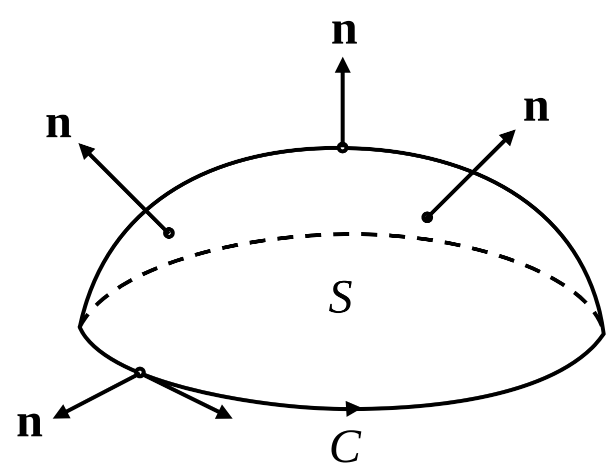
By Green's Theorem,

$$\oint_{\partial D} P\ dx + Q\ dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Note : Green's Theorem is for two variables

Stokes' Theorem

Let $\mathbf{F} = \mathbf{P}\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$.



We may write the line integral as

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\ dx + Q\ dy + R\ dz.$$

By Stokes' Theorem,

$$\int_C P\ dx + Q\ dy + R\ dz =$$

$$\iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k} \cdot d\mathbf{S}$$

Note : Stoke's Theorem is for three variables

Stokes' Theorem

Recall that

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\ dA$$

By Stokes' Theorem,

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} \\ &= \iint_D (\text{curl } \mathbf{F})(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\ dA \end{aligned}$$

Example

Use Stokes' Theorem to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = yz\mathbf{i} - xz\mathbf{j} + xy\mathbf{k}$ and C is the curve of intersection of the plane $y + z = 3$ and the cylinder $x^2 + y^2 = 4$. (C is oriented in the **counterclockwise** sense when viewed from above.)

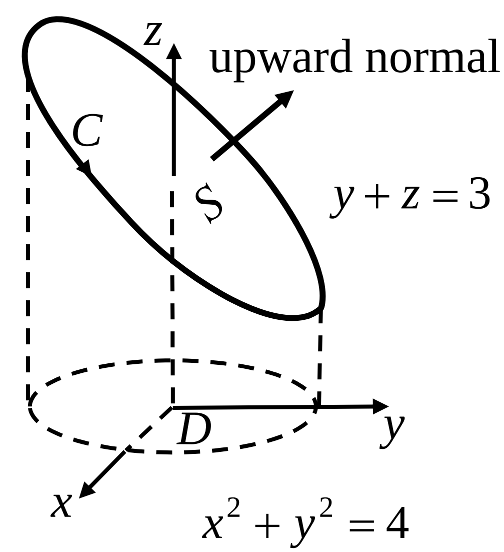
Let S be the (bounded) surface enclosed by C on the plane $y + z = 3$.

$$z = y - 3$$

S has parametric representation

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + (3 - v)\mathbf{k}$$

and the region D is the disk of radius 2.



D: disk of radius 2

$\mathbf{r}(u,v) = u\mathbf{i} + v\mathbf{j} + (3-v)\mathbf{k}$ $\mathbf{r}_u = \mathbf{i}$ $\mathbf{r}_v = \mathbf{j} - \mathbf{k}$

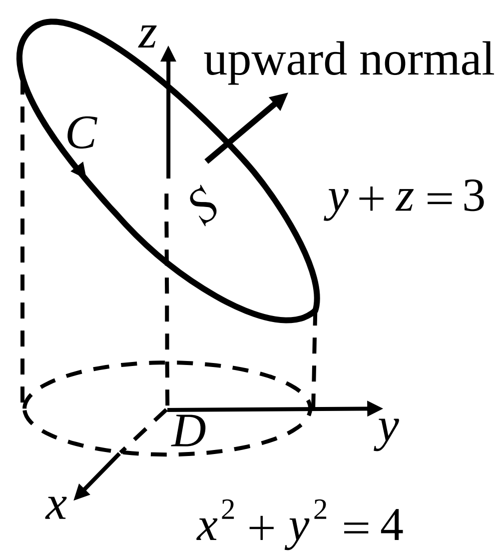
Note that : $\mathbf{r}_u \times \mathbf{r}_v = \mathbf{j} + \mathbf{k}$ upward normal vector of S.

This gives the orientation of S which agrees with that of C.

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F}$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & -xz & xy \end{vmatrix} = 2x\mathbf{i} - 2z\mathbf{k}.$$

By Stokes' Theorem,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$$
$$= \iint_D (2u\mathbf{i} - 2(3-v)\mathbf{k}) \cdot (\mathbf{j} + \mathbf{k}) \, dA$$
$$= \iint_D (-6 + 2v) \, dA$$



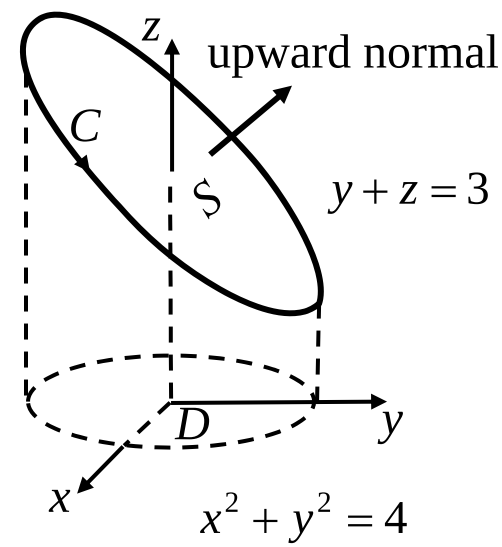
D: disk of radius 2

By Stokes' Theorem,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F}\cdot d\mathbf{S}$$

$$= \iint_D (2u\mathbf{i} - 2(3-v)\mathbf{k})\cdot(\mathbf{j}+\mathbf{k})\ dA$$

$$= \iint_D (-6+2v)\ dA$$



D: disk of radius 2

Since D is the disk of radius 2, we may use polar coordinates:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\int_0^2 (-6+2r\sin\theta)r\ dr\ d\theta$$

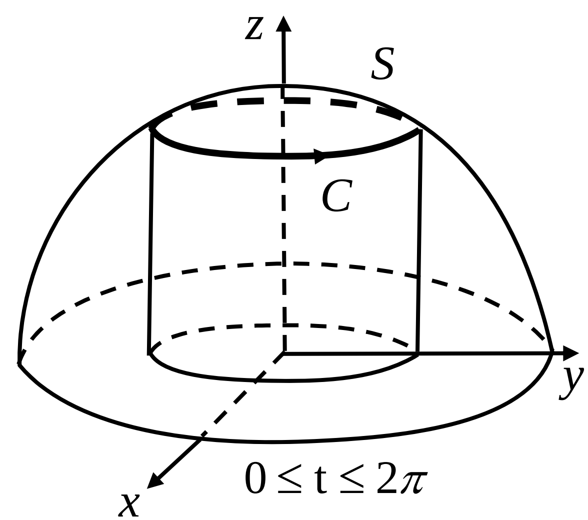
$$= \int_0^{2\pi}\left(-12+\frac{16}{3}\sin\theta\right)\ d\theta$$

$$= -24\pi.$$

Example

Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = y^2 z\mathbf{i} + x\mathbf{j} + (x + y)\mathbf{k}$ and S is the part of the upper hemisphere $z = \sqrt{9 - x^2 - y^2}$ that lies within the cylinder $x^2 + y^2 = 5$ and the orientation of S is given by the upward normal vector.

The boundary of S is C which is the intersection of hemisphere and cylinder (cirlce in horizontal plane)



S: part of the upper hemisphere $z = \sqrt{9 - x^2 - y^2}$

that lies within the cylinder $x^2 + y^2 = 5$.

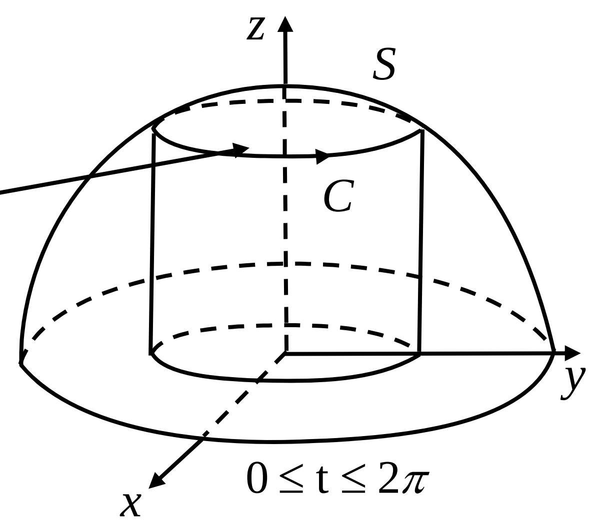
Not easy to have a parametric representation for S.

C : a cirlce in the horizontal plane

Easy to give a parametric representation for C.

Parametric equation of C:

$\mathbf{r}(t) = \sqrt{5}\cos t\ \mathbf{i} + \sqrt{5}\sin t\ \mathbf{j} + a\ \mathbf{k}$



Solving $x^2 + y^2 = 5$ and $z = \sqrt{9 - x^2 - y^2}$,

we get $z = 2$

$\iint_S (\text{curl}\,\mathbf{F}) \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r}$

Use Stoke's Theorem

find line integral instead.

S is the part of the upper hemisphere $z = \sqrt{9 - x^2 - y^2}$ that lies within the cylinder $x^2 + y^2 = 5$ and the orientation of S is given by the upward normal vector.

Parametric equation of C:

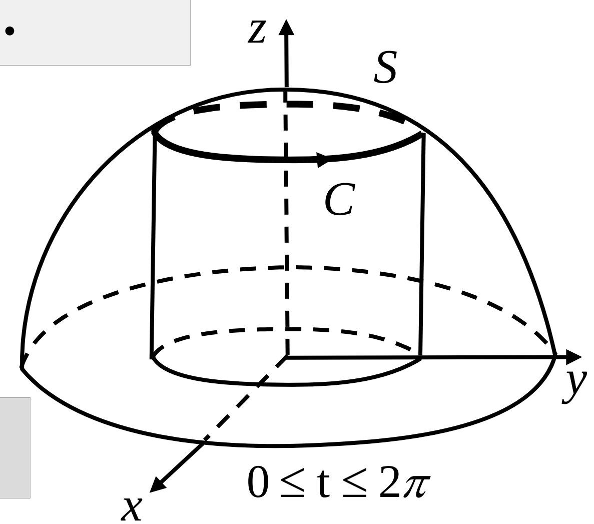
$$\mathbf{r}(t) = \sqrt{5}\cos t\ \mathbf{i} + \sqrt{5}\sin t\ \mathbf{j} + 2\ \mathbf{k}$$

With this vector equation, the curve traverses in anticlockwise direction when viewed from top. This agrees with the given orientation of S.

$$\mathbf{r}'(t) = -\sqrt{5}\sin t\ \mathbf{i} + \sqrt{5}\cos t\ \mathbf{j} + 0\mathbf{k}$$

$$\mathbf{F}(x, y, z) = y^2 z\mathbf{i} + x\mathbf{j} + (x + y)\mathbf{k}$$

$$\mathbf{F}(\mathbf{r}(t)) = 10\sin^2 t\ \mathbf{i} + \sqrt{5}\cos t\ \mathbf{j} + \sqrt{5}(\cos t + \sin t)\mathbf{k}$$



Parametric equation of C:

$$\mathbf{r}(t) = \sqrt{5}\cos t\ \mathbf{i} + \sqrt{5}\sin t\ \mathbf{j} + 2\ \mathbf{k}$$

$$0 \le t \le 2\pi$$

$$\mathbf{r}'(t) = -\sqrt{5}\sin t\ \mathbf{i} + \sqrt{5}\cos t\ \mathbf{j} + 0\mathbf{k}$$

$$\mathbf{F}(\mathbf{r}(t)) = 10\sin^2 t\ \mathbf{i} + \sqrt{5}\cos t\ \mathbf{j} + \sqrt{5}(\cos t + \sin t)\mathbf{k}$$

By Stokes' Theorem,

$$\begin{aligned}\iint_S \operatorname{curl} \mathbf{F}\cdot d\mathbf{S} &= \int_C \mathbf{F}\cdot d\mathbf{r} \\ &= \int_C \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\ dt \\ &= \int_0^{2\pi} (-10\sqrt{5}\sin^3 t + 5\cos^2 t)dt \\ &= 5\pi.\end{aligned}$$

PAUSE and THINK !!

What is the difference between the two examples ???

Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = yz\mathbf{i} - xz\mathbf{j} + xy\mathbf{k}$ and C is the curve of intersection of the plane $y + z = 3$ and the cylinder $x^2 + y^2 = 4$. (C is oriented in the clockwise sense when viewed from above.)

By Stokes' Theorem, $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_D (2u\mathbf{i} - 2(3-v)\mathbf{k}) \cdot (\mathbf{j} + \mathbf{k}) \, dA$

Use Stokes' Theorem to evaluate $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = y^2 z\mathbf{i} + x\mathbf{j} + (x + y)\mathbf{k}$ and S is the part of the upper hemisphere $z = \sqrt{9 - x^2 - y^2}$ that lies within the cylinder $x^2 + y^2 = 5$ and the orientation of S is given by the upward normal vector.

By Stokes' Theorem,

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt = \int_0^{2\pi} (-10\sqrt{5} \sin^3 t + 5\cos^2 t) dt = 5\pi$$

Divergence Theorem (Gauss' Theorem)

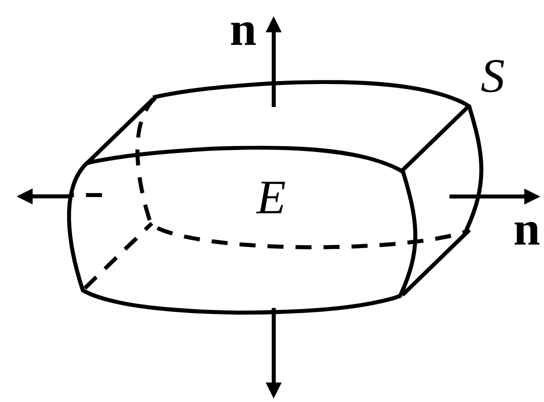
Divergence Theorem

Divergence Theorem

Let E be a solid region and let S be the boundary of E, given with the ***outward orientation****. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives in E. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV$$

*The outward orientation of the boundary surface of a solid region E is the one for which the normal vector point outward from E.



Divergence Theorem

surface integral of vector field

$$\boxed{\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV}$$

triple integral of scalar function

take the outer normal orientation

$\mathbf{F}$: a vector field whose component functions have continuous partial derivatives in E.

E : a bounded solid region

S : the boundary surface of E

i.e., S must be a closed surface

Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = x^2\mathbf{i} + (xy + x\cos z)\mathbf{j} + e^{xy}\mathbf{k}$ and S is the surface of the cubic region E bounded by the three coordinate planes $x = 0, y = 0, z = 0$ and the three planes $x = 1, y = 1, z = 1$. The orientation of S is given by the outward normal vector.

The cubic region E can be described as

$$E: 0 \le x \le 1, \quad 0 \le y \le 1, \quad 0 \le z \le 1.$$

By the Divergence Theorem, we have

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E \text{div } \mathbf{F} \; dV \\ &= \iiint_E 3x \; dV \\ &= 3\int_0^1 \int_0^1 \int_0^1 x \; dx \; dy \; dz = \frac{3}{2}.\end{aligned}$$

Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F}(x, y, z) = (x + y)\mathbf{i} + (y + z)\mathbf{j} + (z + x)\mathbf{k}$ and S is the unit sphere $x^2 + y^2 + z^2 = 1$.

Note that : S is a closed surface

By the Divergence Theorem,

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E \operatorname{div} \mathbf{F}\, dV \quad \text{where } E = \text{unit ball} \\ &= \iiint_E 3\, dV \\ &= 3 \times \text{volume of the unit ball} \\ &= 4\pi\end{aligned}$$

$$\text{volume of the unit ball} = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi$$

End