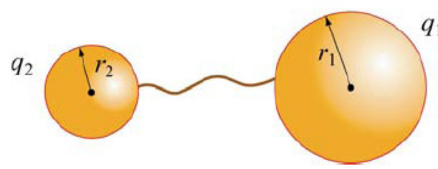
Electrical & Computer Engineering Department
National University of Singapore

# EE2011 Engineering Electromagnetics
## Tutorial 4: Electric Fields

Q1(a) Two spheres (each carrying charge $q_k$ where k = 1 or 2) are connected by a metallic wire as shown in Figure 1(a). Given that the radius $r_k$ of each sphere is much smaller than the distance between the two spheres, show that $\frac{E_1}{E_2} = \frac{r_2}{r_1}$ where $E_k$ is the electric field normal to the surface of sphere k.

common potential because of wire connection




$$\frac{q_1}{4\pi\varepsilon_o r_1} = \frac{q_2}{4\pi\varepsilon_o r_2} \quad \Rightarrow \quad \frac{q_1}{q_2} = \frac{r_1}{r_2}$$

$$\therefore \quad \frac{E_1}{E_2} = \frac{\dfrac{q_1}{4\pi\varepsilon_o r_1^2}}{\dfrac{q_2}{4\pi\varepsilon_o r_2^2}} = \frac{q_1}{q_2}\frac{r_2^2}{r_1^2} = \frac{r_2}{r_1}$$

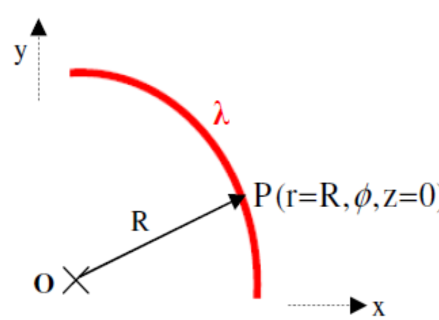
implication**:** stronger electric field expected at sharper corners

Q1(b) A flexible rod (which has been uniformly charged) is bent into a quarter-circular arc. If the rod has a linear charge density of λ, determine the electric field intensity at O which is at a distance of R from the arc.

consider elemental charge at P

$$d\vec{E} = -\frac{\lambda(R\,d\phi)}{4\pi\varepsilon_o R^2}\,\hat{u}_r$$

need to decompose $\hat{u}_r$ into x- and y-components which involve constant unit vectors $\hat{u}_x$ and $\hat{u}_y$

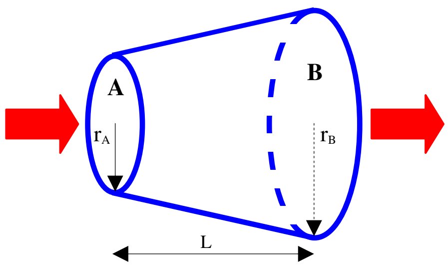


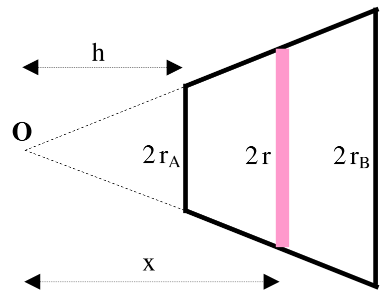

$$\vec{E} = -\int_0^{\frac{1}{2}\pi} \frac{\lambda}{4\pi\varepsilon_o R}\left(\cos\phi\,\hat{u}_x + \sin\phi\,\hat{u}_y\right)d\phi$$

$$= -\frac{\lambda}{4\pi\varepsilon_o R}\left\{\hat{u}_x\int_0^{\frac{1}{2}\pi}\cos\phi\,d\phi + \hat{u}_y\int_0^{\frac{1}{2}\pi}\sin\phi\,d\phi\right\}$$

$$= -\frac{\lambda}{4\pi\varepsilon_o R}\left(\hat{u}_x + \hat{u}_y\right)$$

Q1(c) Depicted in Figure 1(c) is a length L of truncated cone where $r_A$ and $r_B$ are the radii of the circular cross-sections at A and B respectively. Show that the resistance for current flowing from A to B is given by $\frac{\rho L}{\pi r_A r_B}$ where $\rho$ is the resistivity of the cone.




elemental resistance of dx strip at x from O: $\quad dR = \frac{\rho\, dx}{\pi r^2}$

need to integrate from $x = h$ to $x = h+L$: $\quad R = \frac{\rho}{\pi}\int_h^{h+L} \frac{dx}{r^2} = \frac{\rho L}{\pi r_A r_B}$

based on geometrical relationships $\frac{r}{x} = \frac{r_A}{h} = \frac{r_B}{h+L}$

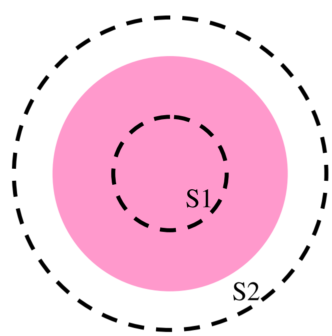
Q2. A sphere (of radius a) has a volume charge density of $\sigma(0<r<a) = \frac{\sigma_0 r}{a}$ where $\sigma_0$ is a constant and r is the distance from the center of the sphere.

(a) Derive expressions for the electric field inside the sphere (where $r < a$) and outside the sphere (where $r > a$).

(b) The charged sphere is placed concentrically inside a metallic spherical shell (of inner radius b and outer radius c). Derive expressions for the electric field in the exterior region (where $r > c$) for the following cases:

(i) when the spherical shell is left unearthed

(ii) after the spherical shell has subsequently been earthed.




check for spherical symmetry $\rightarrow$ can apply Gauss's Law

LHS = $\oiint \varepsilon \vec{E} \bullet d\vec{A} = \varepsilon E_r \oiint dA = E_r 4\pi\varepsilon_o r^2$

RHS = $\iiint \sigma(r)\, dV = \int_0^r \frac{\sigma_0 r}{a} r^2 dr \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi = \frac{\pi\sigma_0 r^4}{a}$

equate LHS and RHS expressions for S1 within sphere

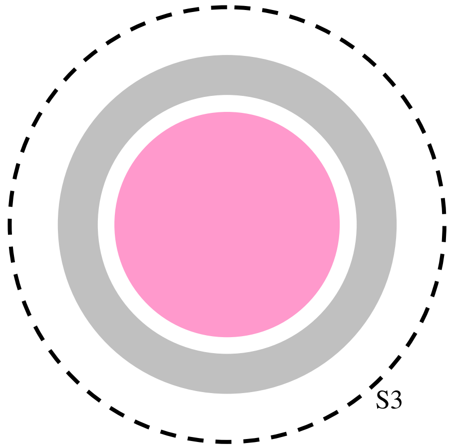
$$\vec{E}(r<a) = \frac{\sigma_0 r^2}{4\varepsilon_o a}\hat{u}_r$$

change upper bound of RHS for S2 outside sphere to obtain $Q = 4\pi\int_0^a \frac{\sigma_0 r}{a} r^2 dr = \pi\sigma_0 a^3$

$$\vec{E}(r>a) = \frac{\sigma_0 a^3}{4\varepsilon_o r^2}\hat{u}_r$$

add metallic spherical shell (to shield charged sphere)

$-Q$ induced on $r = b$ surface of spherical shell




(i) <u>without earthing</u>

$+Q$ residing on $r = c$ surface of spherical shell

no change in total charge for RHS expression

$$\vec{E}(r>c) = \frac{\sigma_0 a^3}{4\varepsilon_o r^2}\hat{u}_r \quad \text{before earthing}$$

(ii) <u>with earthing</u>

zero charge on $r = c$ surface of spherical shell

need to include in RHS expression additional charge of $-Q$ (from earth)

$$\vec{E}(r>c) = 0 \qquad \text{after earthing}$$

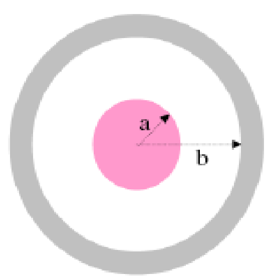
Q3. Figure 3 depicts a sphere (of radius a) which has been placed concentrically inside a spherical shell (of inner radius b and outer radius c). There are no free charges in the interior space (where $a < r < b$) between the two conductors.

For such a structure (with spherical symmetry), the electric potential V in the interior space is governed by the following second-order differential equation:

$$\frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr} = 0$$

(a) Solve this differential equation for V in the interior space given that the sphere is held at a potential $V_0$ while the shell has been earthed.

(b) Hence, derive an expression for the capacitance of this structure.




re-write differential equation as $\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = 0$

integrate twice $V = c_1 - \frac{c_2}{r}$ where $c_1$ and $c_2$ are constants apply boundary conditions:

$$c_1 - \frac{c_2}{b} = 0 \qquad \text{at inner surface of surrounding shell}$$

$$c_1 - \frac{c_2}{a} = V_0 \qquad \text{at surface of enclosed sphere}$$

$$\Rightarrow \quad V = \frac{V_0 a}{b-a}\left(\frac{b}{r}-1\right) \qquad \text{for } a < r < b$$

differentiate to derive field $\quad E_r = -\frac{dV}{dr} = \frac{V_0 ab}{b-a}\frac{1}{r^2}$

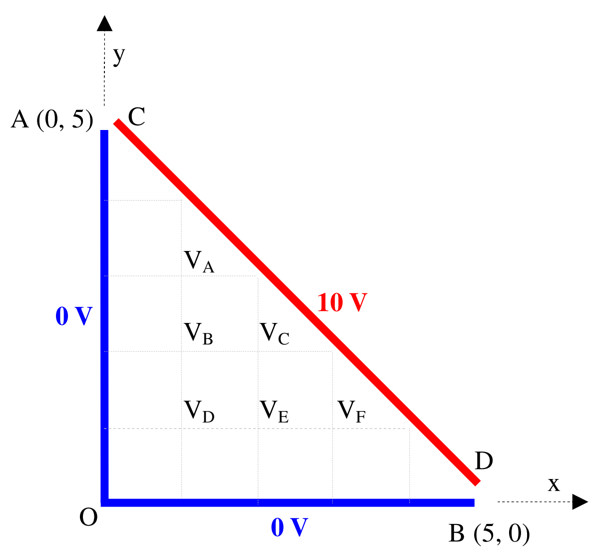
require surface charge density $\quad \sigma_S = \varepsilon_0 E_r = \frac{V_0 \varepsilon_0}{b-a}\frac{b}{a}$ at sphere's surface

obtain total charge $\quad Q = \sigma_S 4\pi a^2 = \frac{4\pi\varepsilon_0 V_0 ab}{b-a}$ at sphere's surface

divide Q expression by $\Delta V$ $\quad C = \frac{Q}{V_0} = \frac{4\pi\varepsilon_0 ab}{b-a}$

Q4. Figure 4 depicts the two-dimensional cross-section of a long prism-like structure. The L-shaped side AOB has been earthed whereas the hypotenuse side CD is held at a potential of 10 V.

Apply an appropriate numerical technique to estimate the potentials at the grid nodes identified by (1, 2) and (2, 2) where O is the origin of the two-dimensional Cartesian coordinate system represented in Figure 4 by faint dashed lines.




total of 6 nodes but require only 4 independent parameters

reflective symmetry $\rightarrow$ $\quad V_E = V_B$ and $V_F = V_A$

choose initial values: $\quad V_A = V_B = 5$, $V_C = 7.5$ and $V_D = 2.5$

apply iterative formulas: $V_A = \frac{1}{4}(10 + 10 + V_B + 0)$

$$V_B = \tfrac{1}{4}(V_A + V_C + V_D + 0)$$

$$V_C = \tfrac{1}{4}(10 + 10 + V_E + V_B) = \tfrac{1}{2}(10 + V_B)$$

$$V_D = \tfrac{1}{4}(V_B + V_E + 0 + 0) = \tfrac{1}{2} V_B$$

| # | $V_A$ at (1, 3) | $V_B$ at (1, 2) | $V_C$ at (2, 2) | $V_D$ at (1, 1) |
|---|---|---|---|---|
| 0 | 5.00 | 5.00 | 7.50 | 2.50 |
| 1 | 6.25 | 4.06 | 7.03 | 1.89 |
| 2 | 6.02 | 3.77 | 6.89 | 1.84 |
| 3 | 5.94 | 3.68 | 6.84 | 1.83 |
| 4 | 5.92 | 3.65 | 6.83 | 1.82 |
| 5 | 5.91 | 3.64 | 6.82 | 1.82 |
| 6 | 5.91 | 3.64 | 6.82 | 1.82 |

convergence of numerical results from 5$^{th}$ iteration onwards