MA1506

Mathematics II

Chapter 5

Matrices and their uses

Reference: Chapter 3 of Textbook

5.1 What is a Matrix?

$$\begin{aligned} 2x + 7y &= 3 \\ 4x + 8y &= 11. \end{aligned}$$

A system of linear algebraic eqns in 2 variables

No differentiation

Just constant multiples of *x* and *y*.

Linearity

$$\frac{d}{dx}(af + bg) = a\frac{df}{dx} + b\frac{dg}{dx}$$

$$\int (af + bg) = a \int f + b \int g$$

$$L(af + bg) = aL(f) + bL(g)$$

$f(x) = x$	$f(x) = x^2$	$f(x) = \sin x$
Linear	Nonlinear	Nonlinear

5.1 What is a Matrix?

$$
\begin{aligned}
2x + 7y &= 3 \\
4x + 8y &= 11.
\end{aligned}
$$

Rewritten as

$$
\underbrace{\begin{bmatrix} 2 & 7 \\ 4 & 8 \end{bmatrix}}_{\text{2x2 Matrix}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{Vectors}} = \underbrace{\begin{bmatrix} 3 \\ 11 \end{bmatrix}}_{\text{Vectors}}
$$

5.1 What is a Matrix?

3x3 Matrix $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

m x n Matrix: m rows, n columns

a_{ij} i-th row j-th column

Usually : $a_{ij} \neq a_{ji}$

$$A = (a_{ij})$$

5.2 Matrix Arithmetic

- Matrix addition
- Scalar multiplication
- Matrix multiplication

term by term

5.2 Matrix Addition

$$\left.\begin{aligned} A &= (a_{ij}) \\ B &= (b_{ij}) \end{aligned}\right\} \quad \textit{m x n} \text{ matrices}$$

Term by term addition

$$\boxed{A + B = (a_{ij} + b_{ij})}$$

$$\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 7 & 3 \\ 6 & 9 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ 10 & 17 \end{bmatrix}$$

5.2 Scalar multiplication

$A = (a_{ij})$ *m x n* matrix

c, a scalar (real or complex)

Term by term multiplication

$$\boxed{cA = (ca_{ij})}$$

$$3\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 12 & 24 \end{bmatrix}$$

5.2 Matrix Multiplication

$$A = (a_{ij})$$

m x n matrix

$$B = (b_{ij})$$

n x p matrix

$$AB = C$$

m x p matrix

$$C = (c_{ij})$$

Not term by term but row to column

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$$

Definition (Matrix Multiplication)

$$\boxed{c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}}$$

(i,j) entry

$$\begin{pmatrix} & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ & & & \end{pmatrix}\begin{pmatrix} & b_{1j} & \\ & b_{2j} & \\ & \vdots & \\ & b_{nj} & \end{pmatrix} = \begin{pmatrix} & & \\ & c_{ij} & \\ & & \end{pmatrix}$$

i th row

j th column

Example

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1\times1+2\times2+3\times(-1) & 1\times1+2\times3+3\times(-2) \\ 4\times1+5\times2+6\times(-1) & 4\times1+5\times3+6\times(-2) \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \\ 8 & 7 \end{pmatrix}$$

Examples

$$\begin{bmatrix} 2 & 7 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + 7y \\ 4x + 8y \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \end{bmatrix}.$$

In general, $AB \neq BA$ Non commutative

$$AB = \begin{bmatrix} 2 & 7 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 16 & -1 \\ 20 & 4 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} 14 & 31 \\ 0 & 6 \end{bmatrix}.$$

5.2 matrix transposition

$A = (a_{ij})$ *m x n* matrix

swap rows with columns

$$\boxed{A^T = (a_{ji}) \quad n \times m \text{ matrix}}$$

$$\begin{bmatrix} 1 & 7 & 9 \\ 6 & 8 & 2 \\ 4 & 10 & 12 \end{bmatrix}^T = \begin{bmatrix} 1 & 6 & 4 \\ 7 & 8 & 10 \\ 9 & 2 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 9 \end{bmatrix}^T = \begin{bmatrix} 1 & 6 \\ 2 & 8 \\ 4 & 9 \end{bmatrix}$$

5.2 matrix transposition

$$(A^T)^T = A$$

$$(A+B)^T = A^T + B^T$$

$$(cA)^T = cA^T$$

$$(AB)^T = B^T A^T$$

A: m x n , B: n x p $\rightarrow$ *AB: m x p*

A^T*: n x m , B*^{T}*: p x n* $\rightarrow$

5.2 Symmetric matrix

An n x n matrix is symmetric if

square

$$\boxed{A^T = A}$$

$$\begin{bmatrix} 1 & 7 & 9 \\ 7 & 8 & 2 \\ 9 & 2 & 12 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Diagonal

5.2 Anti-Symmetric matrix

An *n x n* matrix is anti-symmetric or skew symmetric if

$$\boxed{A^T = -A}$$

$$\begin{bmatrix} 0 & -7 & -9 \\ 7 & 0 & 2 \\ 9 & -2 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

5.2 Properties of Symmetric matrices

If *A* is symmetric and *B* is any square matrix

$$(B + B^T)^T = B^T + (B^T)^T = B^T + B$$

$$(BAB^T)^T = (B^T)^T A^T B^T = BAB^T$$

If *A* is anti-symmetric and *B* is any square matrix

$$(B - B^T)^T = B^T - (B^T)^T = -(B - B^T)$$

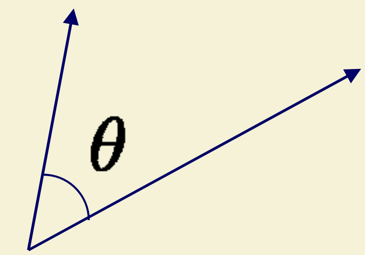
$$(BAB^T)^T = (B^T)^T A^T B^T = -BAB^T$$

5.2 Scalar product for vectors

$$\vec{u} \cdot \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$= u_1 v_1 + u_2 v_2 + u_3 v_3$$

$$= |\vec{u}||\vec{v}| \cos\theta$$

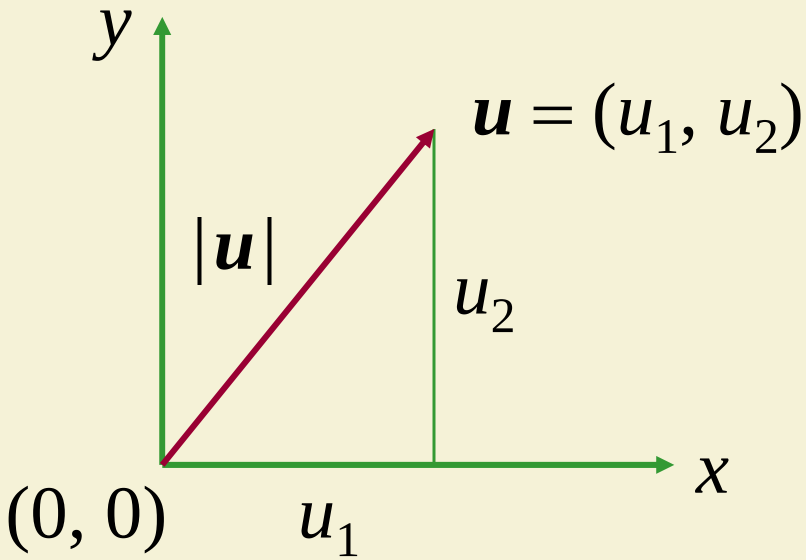


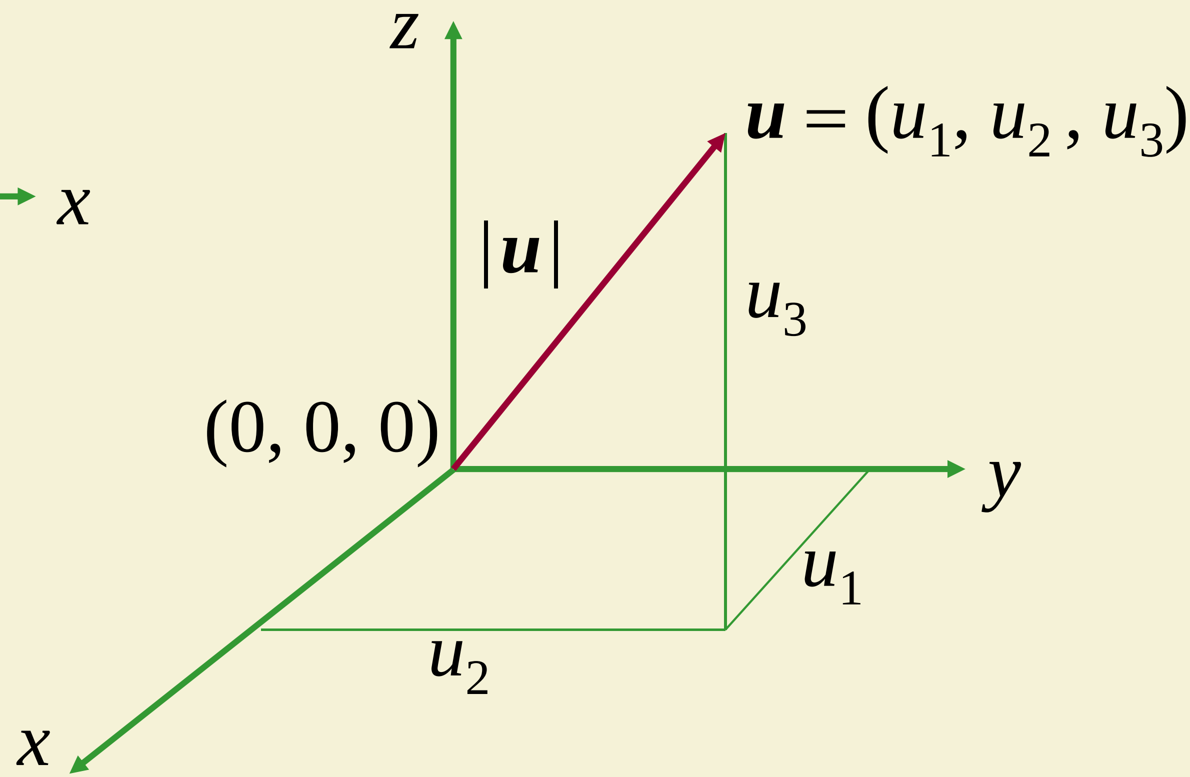
Geometrically

$$\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

5.2 length of vectors

$$\vec{u}\cdot\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = u_1^2 + u_2^2 + u_3^2$$



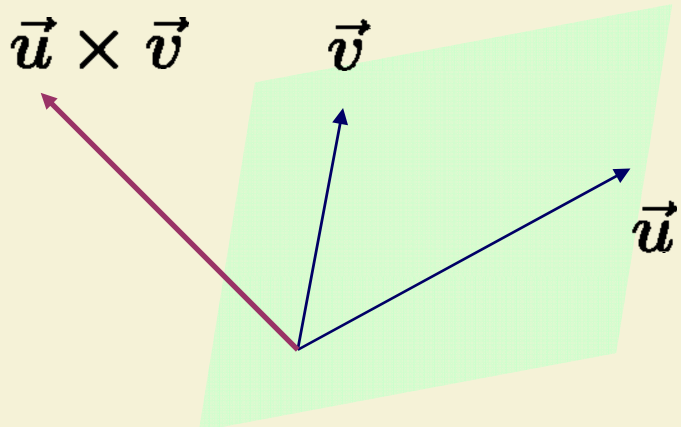


$$|\vec{u}| = \sqrt{\vec{u}\cdot\vec{u}}$$

5.2 Cross product in 3-D space

$$\vec{u} \times \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$= \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ -u_1 v_3 + u_3 v_1 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}.$$



$\vec{u} \times \vec{v}$ is the normal vector to the plane containing $\vec{u}$ and $\vec{v}$

$$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$$

5.2 Cross product in 3-D space

$$\vec{u} \times \vec{v}$$

$$= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$= \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ -u_1 v_3 + u_3 v_1 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

$$\boxed{\vec{u} \times \vec{v} = A\vec{v}}$$

$$= \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Anti-symmetric

5.2 Identity matrix

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

n x *n* identity matrix

sometimes denoted I_n

$$\boxed{AI = IA = A}$$

5.2 Orthogonal matrix

An *n x n* matrix, *B*, is orthogonal if

$$\boxed{BB^T = I}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$ is orthogonal

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$
$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \sin^2\theta + \cos^2\theta \end{bmatrix} = I$$

5.3 Model weather forecasting

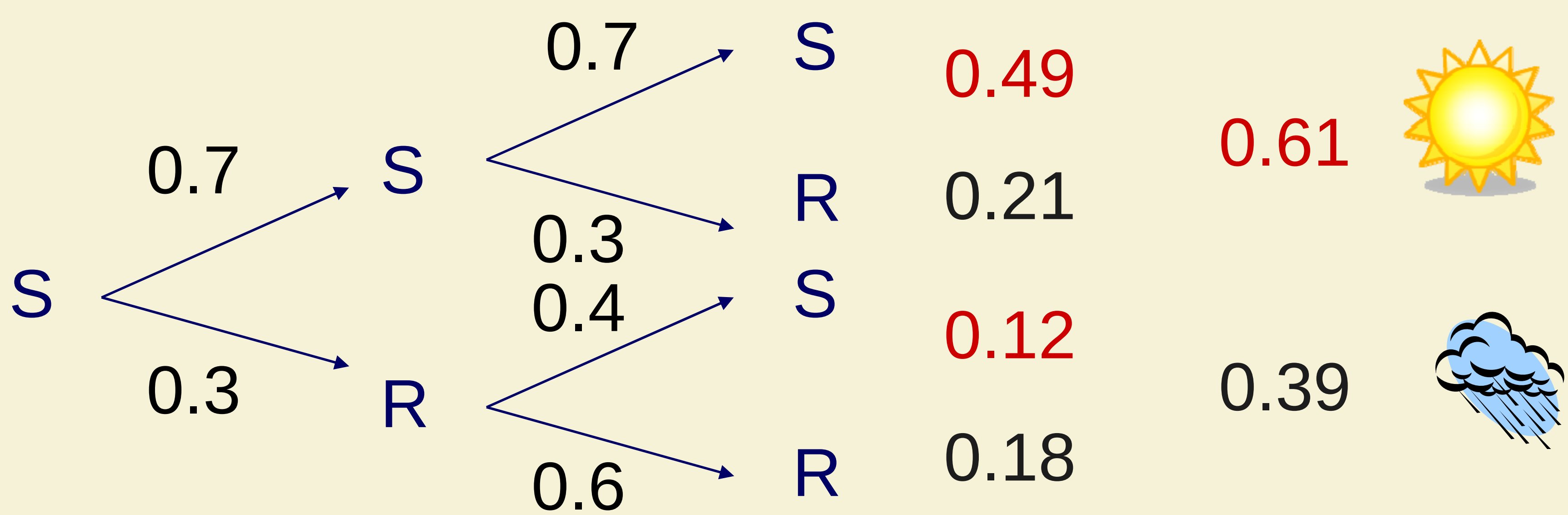
Today	Tomorrow	Probability
Rainy	Rainy	60%
	Sunny	40%
Sunny	Rainy	30%
	Sunny	70%

$$M = \begin{bmatrix} R \to R & S \to R \\ R \to S & S \to S \end{bmatrix} = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}.$$

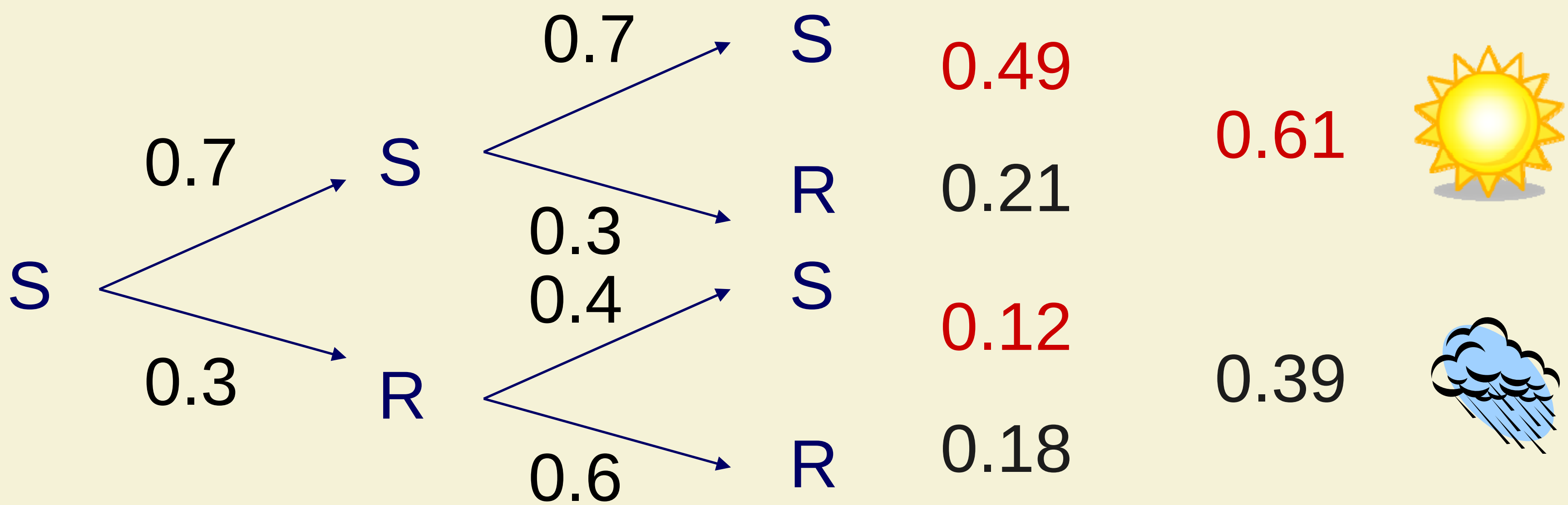
5.3 Model weather forecasting

$$M = \begin{bmatrix} R \to R & S \to R \\ R \to S & S \to S \end{bmatrix} = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}.$$

Today is Sunny, will it be rainy 2 days later?



5.3 Model weather forecasting



$$\begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} =$$

$$\begin{bmatrix} 0.6 \times 0.6 + 0.3 \times 0.4 & \boxed{0.3 \times 0.7 + 0.6 \times 0.3} \\ 0.4 \times 0.6 + 0.7 \times 0.4 & \boxed{0.7 \times 0.7 + 0.4 \times 0.3} \end{bmatrix}$$

5.3 Model weather forecasting

$$M = \begin{bmatrix} R \rightarrow R & S \rightarrow R \\ R \rightarrow S & S \rightarrow S \end{bmatrix} = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}.$$

Today is Rainy, will it be rainy 4 days later?

$$M^4 = \begin{bmatrix} R \rightarrow R_4 & S \rightarrow R_4 \\ R \rightarrow S_4 & S \rightarrow S_4 \end{bmatrix}$$

$$M^4 = \begin{bmatrix} 0.4332 & 0.4251 \\ 0.5668 & 0.5749 \end{bmatrix}$$

5.3 Markov Chains

Current : R S

$$M = \begin{bmatrix} R \to R & S \to R \\ R \to S & S \to S \end{bmatrix} = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}. \quad \begin{array}{l} \text{Next:} \\ R \\ S \end{array}$$

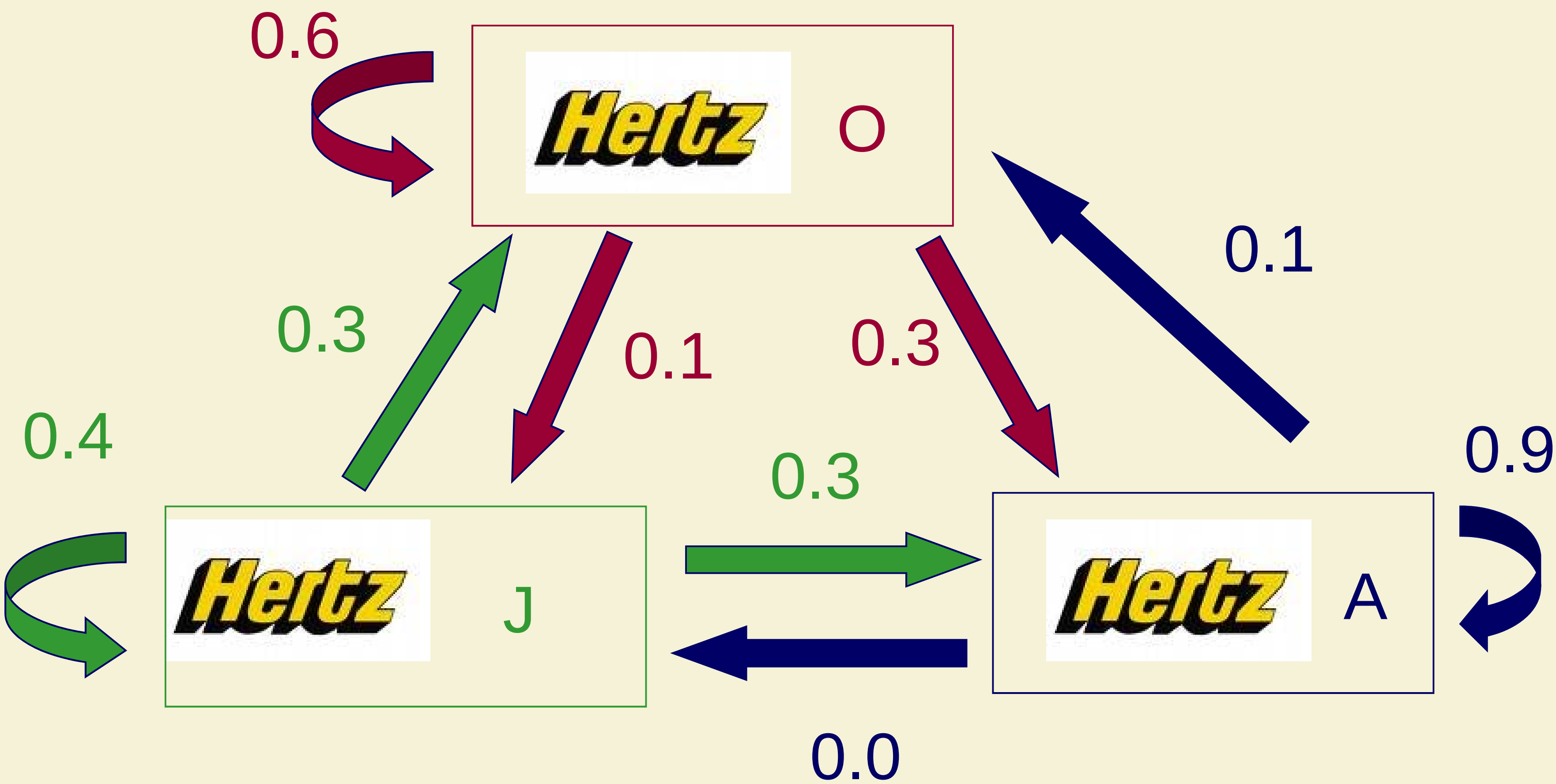
columns add to 1

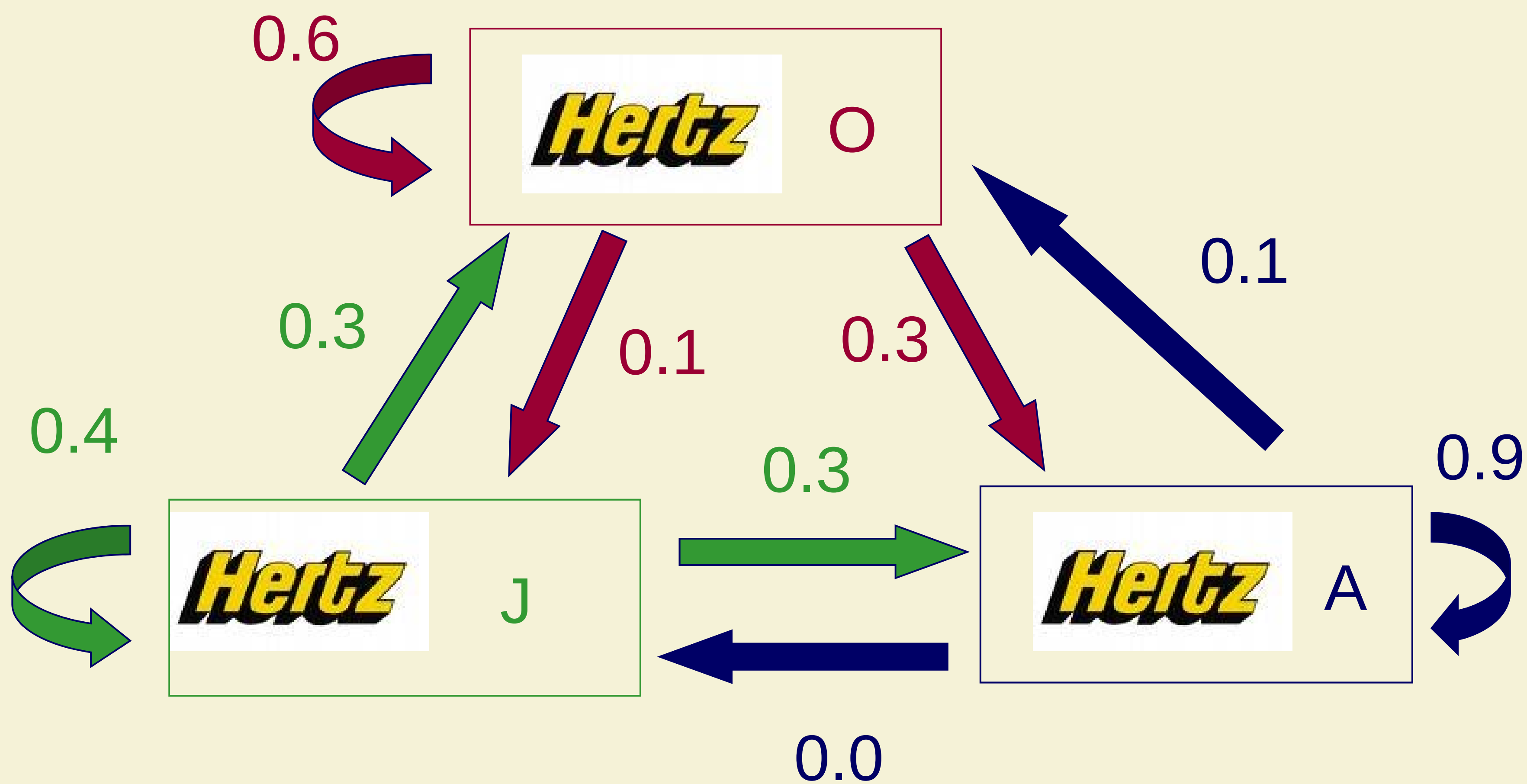
<u>Assumptions:</u>

- k states for each time period
- Probability of changing states depend only on current state

5.3 Example Markov Chains

A car rental agency has 3 offices and allows rental from and returns to any location





$$M = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{ccc|c} A & O & J & \\ \end{array}$$

$$\begin{array}{cc} & \begin{array}{ccc} \mathrm{A} & \mathrm{O} & \mathrm{J} \end{array} \\ M = & \left[\begin{array}{ccc} 0.9 & 0.3 & 0.3 \\ 0.1 & 0.6 & 0.3 \\ 0.0 & 0.1 & 0.4 \end{array}\right] \begin{array}{c} \mathrm{A} \\ \mathrm{O} \\ \mathrm{J} \end{array} \end{array}$$

$$M = \begin{bmatrix} 0.9 & 0.3 & 0.3 \\ 0.1 & 0.6 & 0.3 \\ 0.0 & 0.1 & 0.4 \end{bmatrix}$$

$$\Rightarrow \quad M^{10} = \begin{bmatrix} 0.7515 & 0.7455 & 0.7455 \\ 0.2133 & 0.2173 & 0.2173 \\ 0.0352 & 0.0372 & 0.0372 \end{bmatrix}$$

Any car from any location:

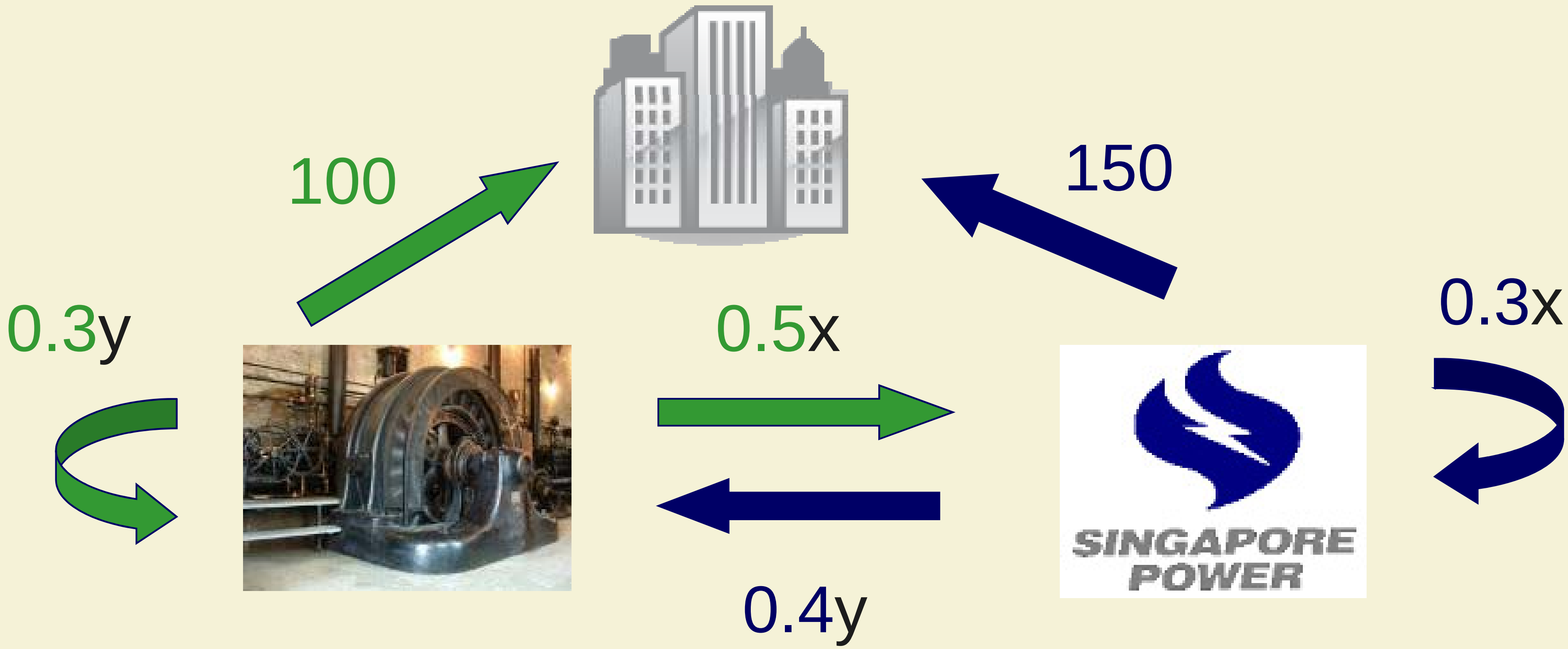
- 75% chance at A
- 21% chance at O
- 4% chance at J

5.4 Leontief Model of Manufacturing

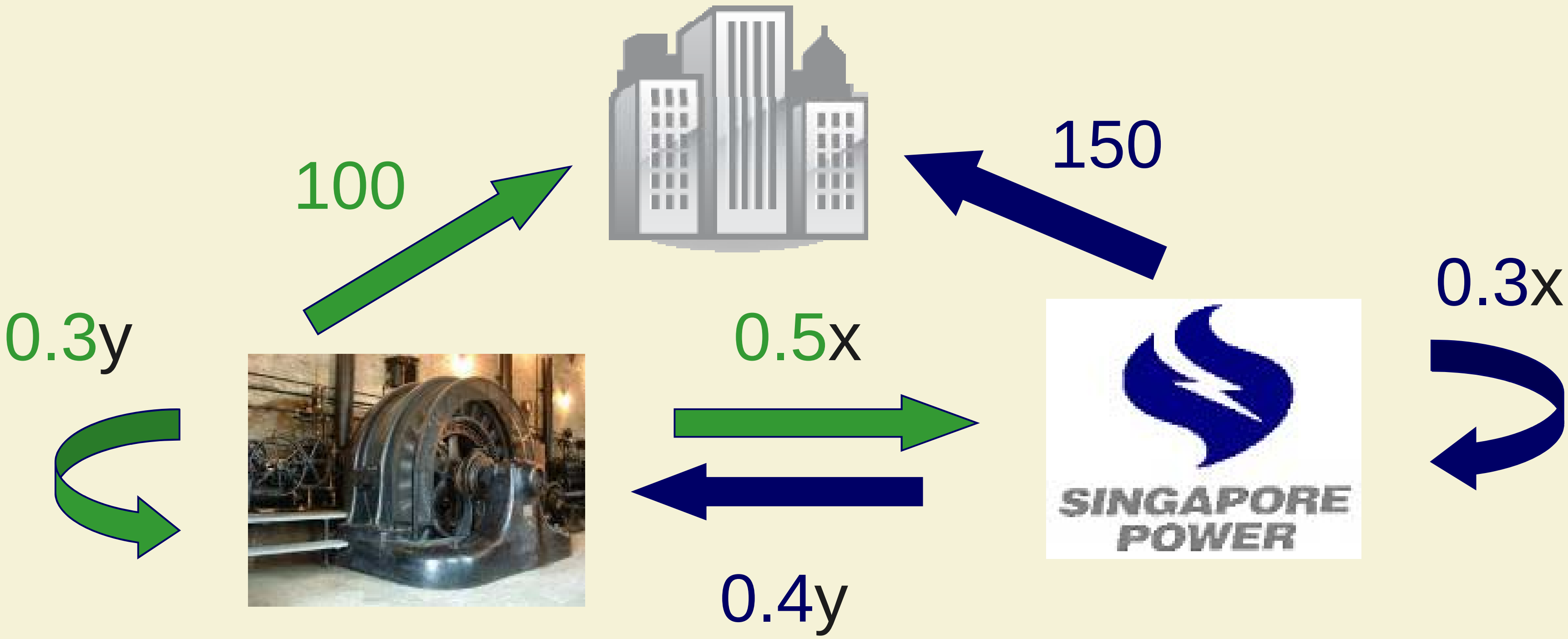
Economics of inter-dependent companies

x ($) of electricity produced,

y ($) generators manufactured



5.4 Leontief Model of Manufacturing



Output = Int Consumption + Ext Demand

$$
\begin{aligned}
x &= 0.3x + 0.4y + 150 \\
y &= 0.5x + 0.3y + 100
\end{aligned}
$$

5.4 Leontief Model of Manufacturing

$$\begin{aligned} x &= 0.3x + 0.4y + 150 \\ y &= 0.5x + 0.3y + 100 \end{aligned}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.3 & 0.4 \\ 0.5 & 0.3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 150 \\ 100 \end{bmatrix}$$

Technology matrix

$$\begin{aligned} \vec{u} &= T\vec{u} + \vec{c} \\ (I - T)\vec{u} &= \vec{c} \end{aligned}$$

$$I\vec{u} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

5.4 Leontief Model of Manufacturing

$$(I - T)\vec{u} \;=\; \vec{c}$$

Find ***S*** such that ***S(I-T) = I***

$$\Rightarrow \quad \vec{u} \;=\; S\vec{c}$$

$$\begin{bmatrix} 0.7 & -0.4 \\ -0.5 & 0.7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 150 \\ 100 \end{bmatrix}$$

$$S = \frac{1}{29} \begin{bmatrix} 70 & 40 \\ 50 & 70 \end{bmatrix}$$ Does the job

5.4 Leontief Model of Manufacturing

$$\frac{1}{29}\begin{bmatrix} 70 & 40 \\ 50 & 70 \end{bmatrix}\begin{bmatrix} 0.7 & -0.4 \\ -0.5 & 0.7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{29}\begin{bmatrix} 70 & 40 \\ 50 & 70 \end{bmatrix}\begin{bmatrix} 150 \\ 100 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 500 \\ 500 \end{bmatrix}.$$

$500 elec = $150 fuel + $200 factory +$150 sold

$500 gen = $150 parts + $250 elec + $100 sold