Differentiation & Integration

If $\sum c_n (x-a)^n$ has radius of convergence h, it defines a function f :

$$\boxed{f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \ a-h < x < a+h.}$$

(i) The function f has derivatives of all orders in $(a-h, a+h)$.
The derivatives can be obtained by differentiating the power series term-by-term:

$$\boxed{f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}} \qquad \frac{d}{dx}\left(c_n (x-a)^n\right) = n c_n (x-a)^{n-1}$$

$$\boxed{f''(x) = \sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2}}$$

The differentiated series converges for $a-h < x < a+h$.

Example

$$\boxed{f(x) = \frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots, \quad -1 < x < 1}$$

Differentiating

$$\boxed{f'(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 \cdots + nx^{n-1} + \cdots, \quad -1 < x < 1}$$

$$\boxed{f''(x) = \frac{2}{(1-x)^3} = 2 + 6x + \cdots + n(n-1)x^{n-2} + \cdots, \quad -1 < x < 1}$$

Differentiation & Integration

(ii) The function f has anti-derivatives in $(a-h, a+h)$. The anti-derivatives can be obtained by integrating the power series term-by-term:

$$\boxed{\int f(x)\,dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C}$$

The integrated series converges for $a-h < x < a+h$.

$$\int c_n (x-a)^n\,dx = c_n \frac{(x-a)^{n+1}}{n+1} + C$$

Geometric Series

$$\frac{1}{1-r} = 1 + r + r^2 + r^3 + \cdots, \qquad -1 < t < 1$$

Put $r = -t$

$$\frac{1}{1+t} = \frac{1}{1-(-t)} = 1 - t + t^2 - t^3 + \cdots, \qquad -1 < t < 1$$

Example

$$\frac{1}{1+t}=\frac{1}{1-(-t)}=1-t+t^2-t^3+\cdots, \qquad -1<t<1$$

$$\frac{d}{dt}\left(\ln(1+t)\right)=\frac{1}{1+t}$$

$$\frac{1}{1+t}=1-t+t^2-t^3+\cdots$$

$$\int\frac{1}{1+t}dt=\int 1-t+t^2-t^3+\cdots\,dt$$

$$\ln(1+t)=t-\frac{t^2}{2}+\frac{t^3}{3}-\frac{t^4}{4}+\cdots$$

Is the answer correct ??

Is the working correct ??

Example

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$$

$$\int \frac{1}{1+t}\,dt = \int 1 - t + t^2 - t^3 + \cdots\,dt$$

$$\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots + C$$

Need to find C.

We put $t = 0$:

$$\ln(1+0) = 0 - 0 + 0 - 0 + \cdots + C$$

$$0 = C$$

$$\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots$$

Example

$$\frac{1}{1+t} = \frac{1}{1-(-t)} = 1 - t + t^2 - t^3 + \cdots, \qquad -1 < t < 1$$

$$\int_0^x \frac{1}{1+t}\, dt = \int_0^x 1 - t + t^2 - t^3 + \cdots\, dt$$

$$\left[\ln(1+t)\right]_0^x = \left[t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots \right]_0^x$$

$$\ln(1+x) - \ln(1+0) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots - (0 - 0 + 0 - 0 + \cdots)$$

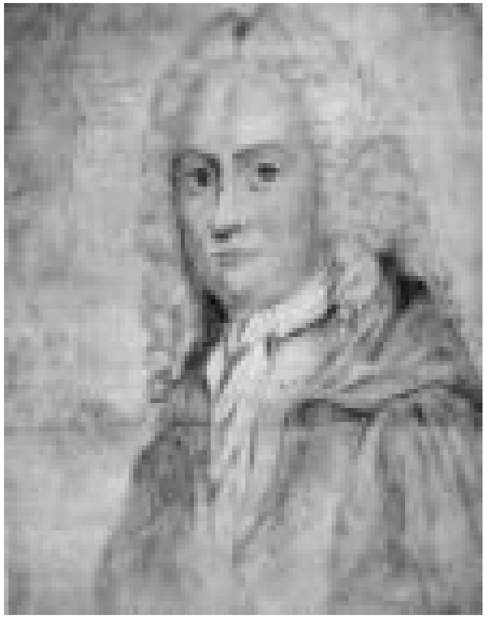
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Taylor Series

Taylor Series

Taylor
(1685-1731)

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$



Maclaurin
(1698-1746)

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n$$

Taylor Series - Definition

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point.

The ***Taylor series*** of f at a is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

We use Taylor series to expand a function into power series.

The ***Taylor series*** of f at a is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

We use Taylor series to expand a function into power series.

For a given function $f(x)$, to find the ***Taylor series*** at a, you just need to find the values of:

$$f(a),\ f'(a),\ f''(a),\ \cdots,\ f^{(n)}(a),\ \cdots$$

and then substitute them into the Taylor series formula.

The ***Taylor series*** of f at a is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

Special case: $a = 0$ (Taylor series of f at 0)

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

$$= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

Maclaurin series

We use Taylor series to expand a function into power series.

Special case: $a = 0$ (Taylor series of f at 0)

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

$$= f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \cdots + \frac{f^{(n)}(0)}{n!} x^n + \cdots$$

For a given function $f(x)$, to find the ***Taylor series*** at 0, you just need to find the values of:

$$f(0),\ f'(0),\ f''(0),\ \cdots,\ f^{(n)}(0),\ \cdots$$

and then substitute them into the Taylor series formula.

Taylor Series - Example

Find the Taylor series of e^x at $x = 0$.

Let $f(x) = e^x$, then we have $f(0) = e^0 = 1.$

$$f'(x) = e^x \qquad f'(0) = e^0 = 1$$

$$f''(x) = e^x \qquad f''(0) = e^0 = 1$$

$$\vdots \qquad \vdots$$

$$f^{(n)}(x) = e^x \qquad f^{(n)}(0) = e^0 = 1$$

Thus, $e^x = f(0) + f'(0)x + \dfrac{f''(0)}{2!}x^2 + \cdots + \dfrac{f^{(n)}(0)}{n!}x^n + \cdots$

$$= 1 + 1 \cdot x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

The radius of convergence of this series is ∞.

Taylor Series - Example

The Taylor series of $\sin x$ and $\cos x$ at $x = 0$ are

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Note : $\dfrac{d}{dx}(\sin x) = \cos x$

Differentiate the Taylor series of $\sin x$,

you get the Taylor series of $\cos x$.

To find the Taylor series of $\sin x$ at $x = 0$

$$\begin{aligned} f(x) &= \sin x \\ f^{(1)}(x) &= \cos x \\ f^{(2)}(x) &= -\sin x \\ f^{(3)}(x) &= -\cos x \\ f^{(4)}(x) &= \sin x \\ &\vdots \end{aligned}$$

$$\begin{aligned} f(0) &= 0 \\ f^{(1)}(0) &= 1 \\ f^{(2)}(0) &= 0 \\ f^{(3)}(0) &= -1 \\ f^{(4)}(0) &= 0 \\ &\vdots \end{aligned}$$

Observe that

$$f^{(2k)}(0) = 0 \quad \text{and} \quad f^{(2k+1)}(0) = (-1)^k.$$

Thus the Taylor series of $\sin x$ at $x = 0$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$x^{\text{even power}} = x^{2k}$$

$$x^{\text{odd power}} = x^{2k+1}$$

Find the Taylor series of $\ln(1+x)$ at $x=0$

$$f(x)=\ln(1+x)$$

$$f^{(1)}(x)=\frac{1}{1+x}$$

$$f^{(2)}(x)=-\frac{1}{(1+x)^2}$$

$$f^{(3)}(x)=\frac{2}{(1+x)^3}$$

$$f^{(4)}(x)=-\frac{6}{(1+x)^4}$$

$$\vdots$$

$$f(0)=0$$

$$f^{(1)}(0)=1$$

$$f^{(2)}(0)=-1$$

$$f^{(3)}(0)=2$$

$$f^{(4)}(0)=-6$$

$$\vdots$$

Very troublesome !!!

Any shortcut ???

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k$$

$$=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\cdots$$

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$$

Find the Taylor series of $\ln(1+x)$ at $x=0$

$$f(x)=\ln(1+x)$$

$$\frac{1}{1-r}=1+r+r^2+r^3+\cdots, \qquad -1<x<1$$

$$f'(x)=\frac{1}{1+x}$$

Put $r=-x$

$$\frac{1}{1+x}=\frac{1}{1-(-x)}=1-x+x^2-x^3+\cdots, \qquad -1<x<1$$

$$\frac{1}{1+x}=1-x+x^2-x^3+\cdots$$

$$\int\frac{1}{1+x}dx=\int 1-x+x^2-x^3+\cdots\,dx$$

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots+C$$

Put $x=0:\quad \ln(1+0)=0-0+0-0+\cdots+C$

$$0=C$$

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$$

Find the Taylor series of $\tan^{-1} x$ at $x=0$

$$f(x)=\tan^{-1} x$$

$$\frac{1}{1-r}=1+r+r^2+r^3+\cdots, \quad -1<x<1$$

$$f'(x)=\frac{1}{1+x^2}$$

Put $r=-x^2$

$$\frac{1}{1+x^2}=\frac{1}{1-(-x^2)}=1-x^2+x^4-x^6+\cdots, \quad -1<x<1$$

$$\frac{1}{1+x^2}=1-x^2+x^4-x^6+\cdots$$

$$\int \frac{1}{1+x^2}\,dx=\int 1-x^2+x^4-x^6+\cdots\,dx$$

$$\tan^{-1} x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots+C$$

Put $x=0$: $\quad \tan^{-1} 0=0-0+0-0+\cdots+C$

$$0=C$$

$$\tan^{-1} x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$$

Some Standard Taylor Series

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots, \ -\infty < x < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \ -\infty < x < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots, \ -\infty < x < \infty$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \ -1 \le x \le 1$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad -1 < x < 1$$

Find the Taylor series of $\frac{1}{2x+1}$ at $x=-2$

The ***Taylor series*** of f at $x=a$ is

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k = f(a)+f'(a)(x-a)+\cdots+$$

$$\frac{f^{(n)}(a)}{n!}(x-a)^n+\cdots$$

The ***Taylor series*** of f at $x=2$ is

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(2)}{k!}(x-2)^k = f(2)+f'(2)(x-2)+\cdots+$$

$$\frac{f^{(n)}(2)}{n!}(x-2)^n+\cdots$$

Find the Taylor series of $\dfrac{1}{2x+1}$ at $x=-2$

The ***Taylor series*** of f at $x=2$ is

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(2)}{k!}(x-2)^k = f(2)+f'(2)(x-2)+\cdots+$$

$$\frac{f^{(n)}(2)}{n!}(x-2)^n+\cdots$$

$$\frac{1}{2x+1}=\frac{1}{2(x+2)-3}$$

$$=\left(-\frac{1}{3}\right)\cdot\frac{1}{1-\frac{2}{3}(x+2)}$$

In terms of $(x+2)$

Geometric Series

$$\frac{1}{1-r}=1+r+r^2+r^3+\cdots=\sum_{n=0}^{\infty} r^n, \qquad |r|<1$$

Put $r=-x$

$$\frac{1}{1+x}=\frac{1}{1-(-x)}=1-x+x^2-x^3+\cdots, \quad |-x|<1$$

Put $r=-x^2$

$$\frac{1}{1+x^2}=\frac{1}{1-(-x^2)}=1-x^2+x^4-x^6+\cdots, \quad |-x^2|<1$$

$$\frac{1}{2x+1}=\frac{1}{2(x+2)-3}$$

$$=\left(-\frac{1}{3}\right)\cdot\frac{1}{1-\frac{2}{3}(x+2)}$$

Put $r=\frac{2}{3}(x+2)$

$$\frac{1}{1-\frac{2}{3}(x+2)}=1+\frac{2}{3}(x+2)+\left[\frac{2}{3}(x+2)\right]^2+\cdots$$

Find the Taylor series of $\frac{1}{2x+1}$ at $x=-2$

$$\frac{1}{1-r}=1+r+r^2+r^3+\cdots=\sum_{n=0}^{\infty}r^n, \qquad |r|<1$$

$$\frac{1}{2x+1}=\frac{1}{2(x+2)-3}$$

$$=\left(-\frac{1}{3}\right)\cdot\frac{1}{1-\frac{2}{3}(x+2)}$$

Put $r=\frac{2}{3}(x+2)$

$$\frac{1}{1-\frac{2}{3}(x+2)}=1+\frac{2}{3}(x+2)+\left[\frac{2}{3}(x+2)\right]^2+\cdots$$

$$=\sum_{n=0}^{\infty}\left[\frac{2}{3}(x+2)\right]^n, \qquad \left|\frac{2}{3}(x+2)\right|<1$$

Put $r = \frac{2}{3}(x+2)$

$$\frac{1}{1-\frac{2}{3}(x+2)} = 1 + \frac{2}{3}(x+2) + \left[\frac{2}{3}(x+2)\right]^2 + \cdots$$

$$= \sum_{n=0}^{\infty}\left[\frac{2}{3}(x+2)\right]^n, \qquad \left|\frac{2}{3}(x+2)\right| < 1$$

$$\frac{1}{2x+1} = \frac{1}{2(x+2)-3}$$

$$= \left(-\frac{1}{3}\right)\cdot\frac{1}{1-\frac{2}{3}(x+2)}$$

$$= \left(-\frac{1}{3}\right)\sum_{n=0}^{\infty}\left(\frac{2}{3}(x+2)\right)^n = \sum_{n=0}^{\infty}\left(-\frac{2^n}{3^{n+1}}\right)(x+2)^n$$

$$\left|\frac{2}{3}(x+2)\right| < 1 \quad \Leftrightarrow \quad |x+2| < \frac{3}{2}$$

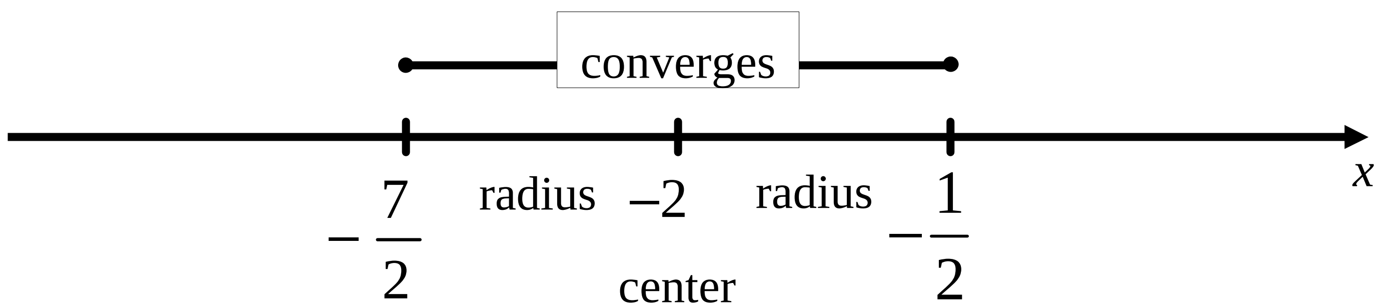
The radius of convergence of this series is $\frac{3}{2}$.

$$|x+2| < \frac{3}{2}$$

$$-\frac{3}{2} < x+2 < \frac{3}{2}$$

$$-2-\frac{3}{2} < x < -2+\frac{3}{2}$$

$$-\frac{7}{2} < x < -\frac{1}{2}$$



Centre at $x = -2$

$$\left|\frac{2}{3}(x+2)\right| < 1 \quad \Leftrightarrow \quad |x+2| < \frac{3}{2}$$

The radius of convergence of this series is $\frac{3}{2}$.

Pause and Think !!!

Let $f(x)=\tan^{-1}\dfrac{1+x}{1-x}$ where $-\dfrac{1}{2}\le x\le\dfrac{1}{2}$.

Find the value of $f^{(2009)}(0)$.

Pause and Think !!!

Let $f(x) = \dfrac{1}{x^2 + x + 1}$.

Let $f(x) = \sum_{n=0}^{\infty} c_n x^n$ be the Maclaurin series representation for $f(x)$.

Find the value of $c_{2008} - c_{2009} + c_{2010}$.

Taylor Polynomials

The n-th order Taylor polynomial of f at a is

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$= f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

It provides the best polynomial approximation of degree n.

Taylor Polynomials - Example

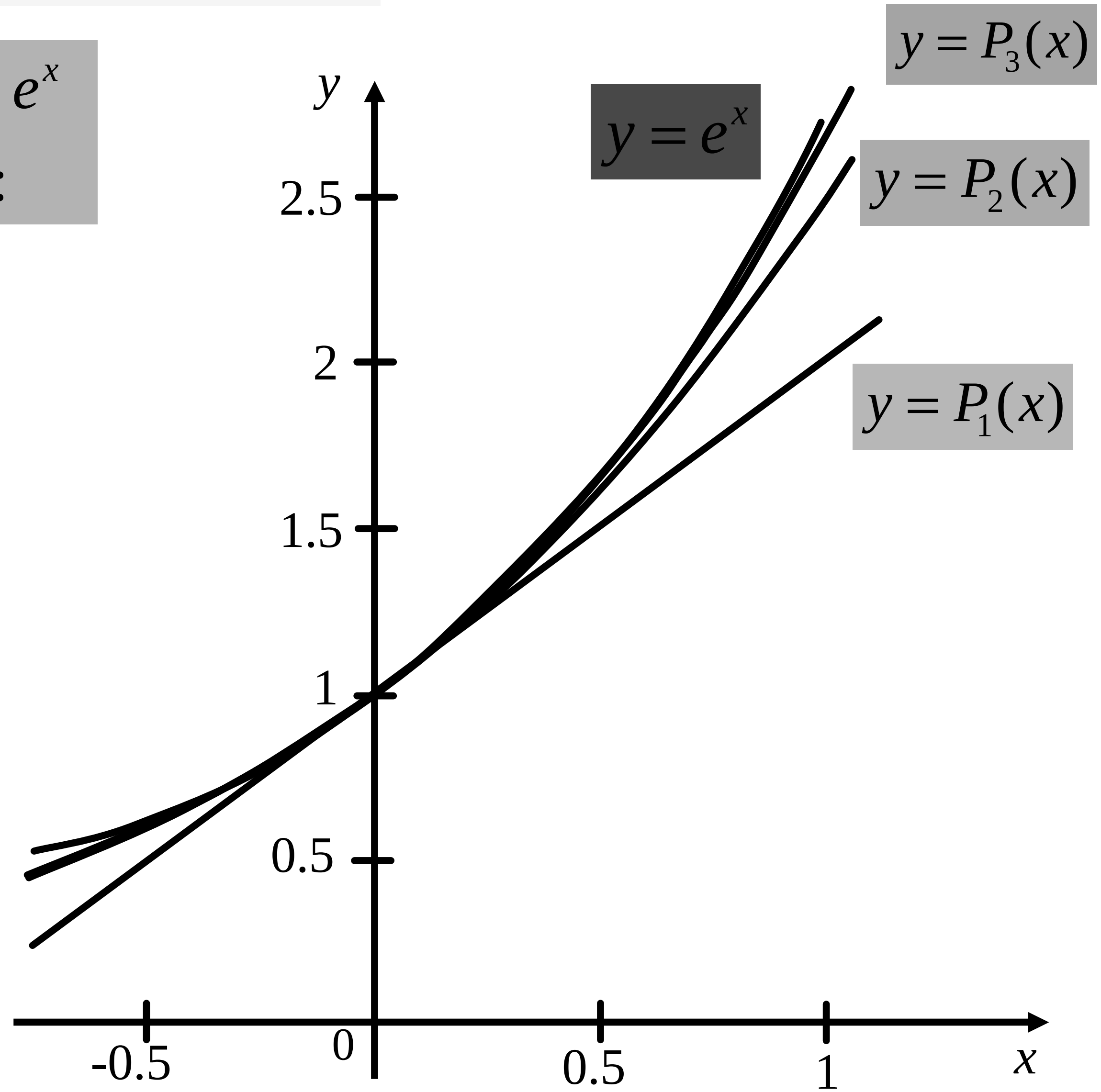
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots, \quad -\infty < x < \infty$$

The Taylor polynomials of e^x at $x = 0$ of order $1,\ 2$ and 3:

$$P_1(x) = 1 + x$$

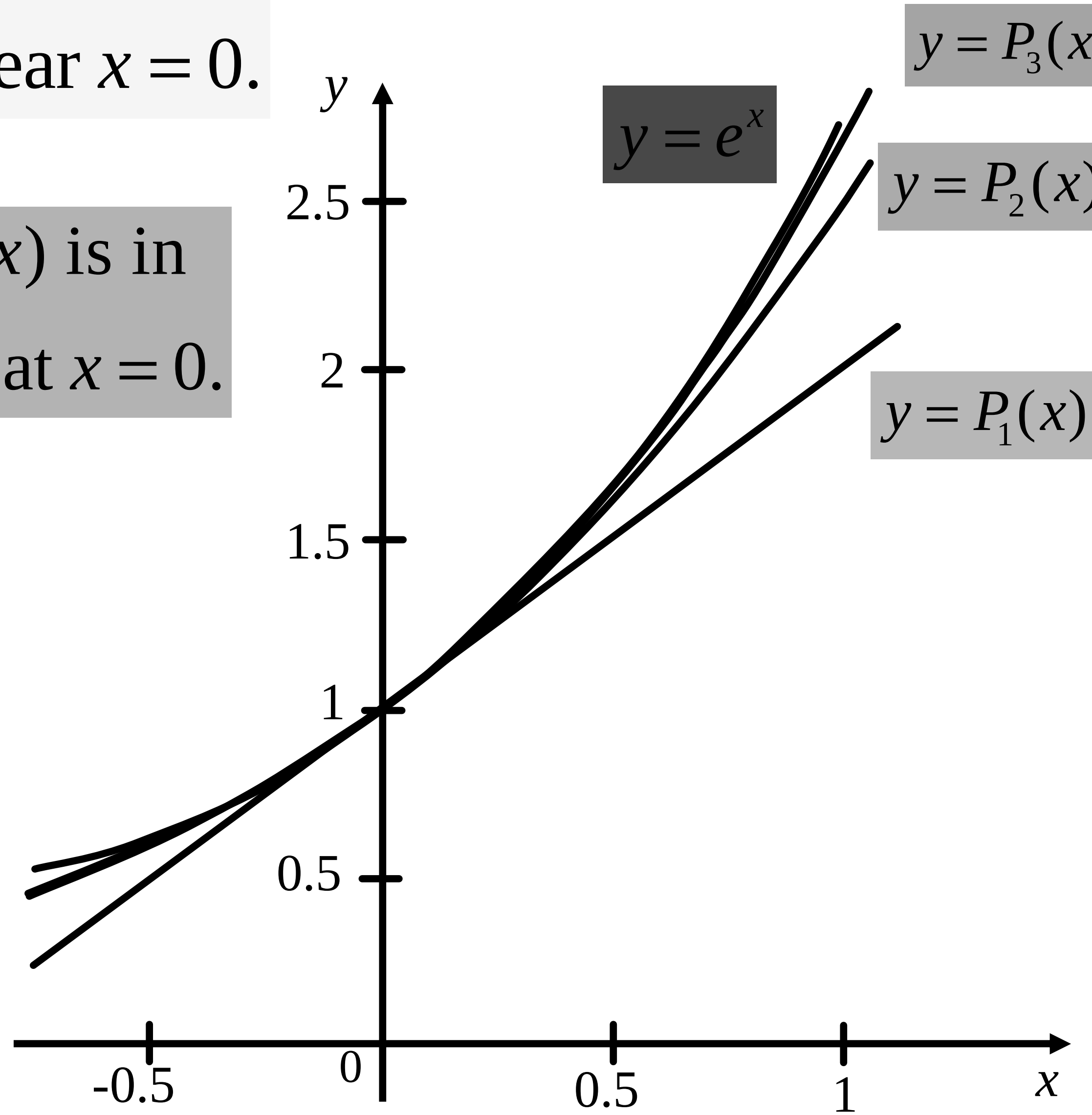
$$P_2(x) = 1 + x + \frac{x^2}{2!}$$

$$P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$



In the diagram, notice the very close agreement between e^x and its Taylor polynomials near $x = 0$.

Note that the graph of $P_1(x)$ is in fact the tangent line of e^x at $x = 0$.

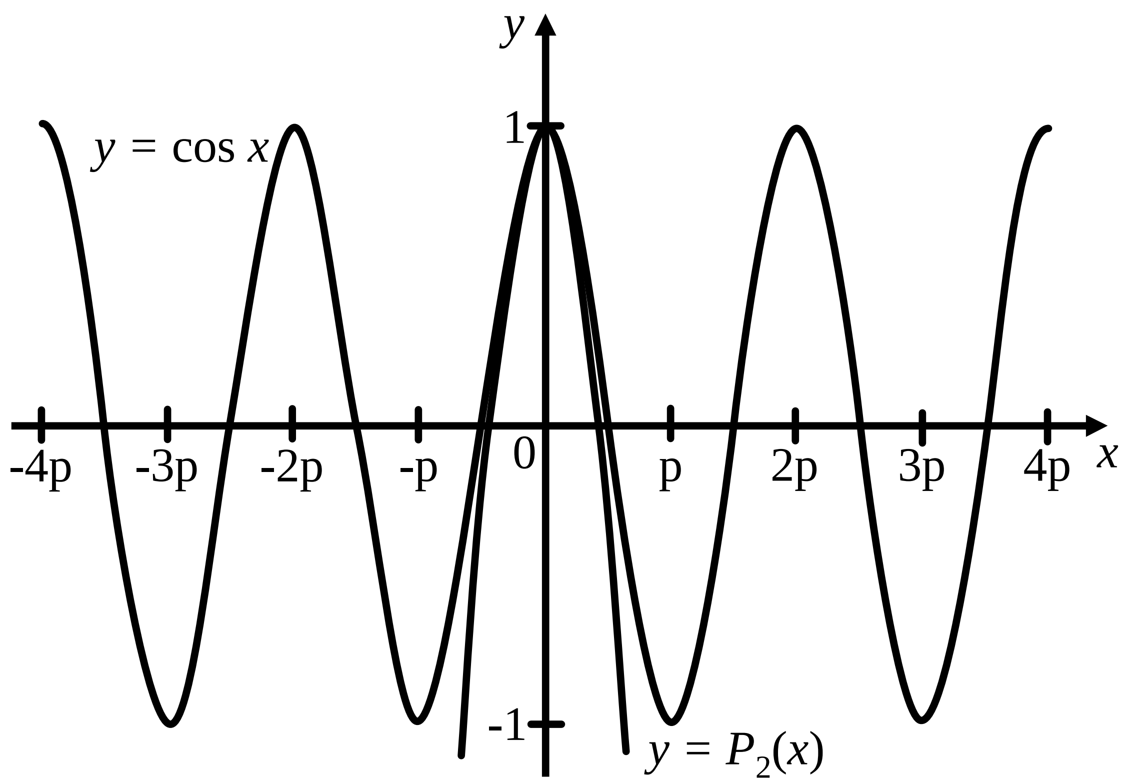


Taylor Polynomials - Example

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots, \quad -\infty < x < \infty$$

The Taylor polynomials of $\cos x$ at $x = 0$ of order 2:

$$P_2(x) = 1 - \frac{x^2}{2!}$$

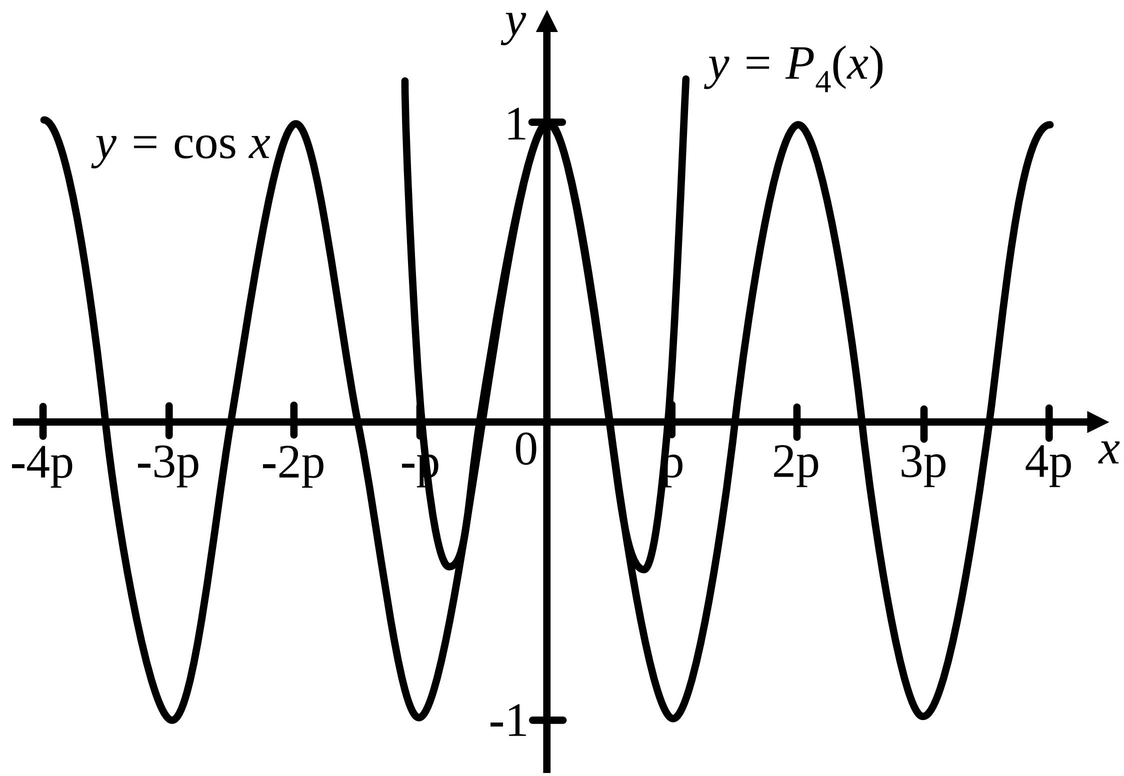


Taylor Polynomials - Example

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots, \quad -\infty < x < \infty$$

The Taylor polynomials of $\cos x$ at $x = 0$ of order 4:

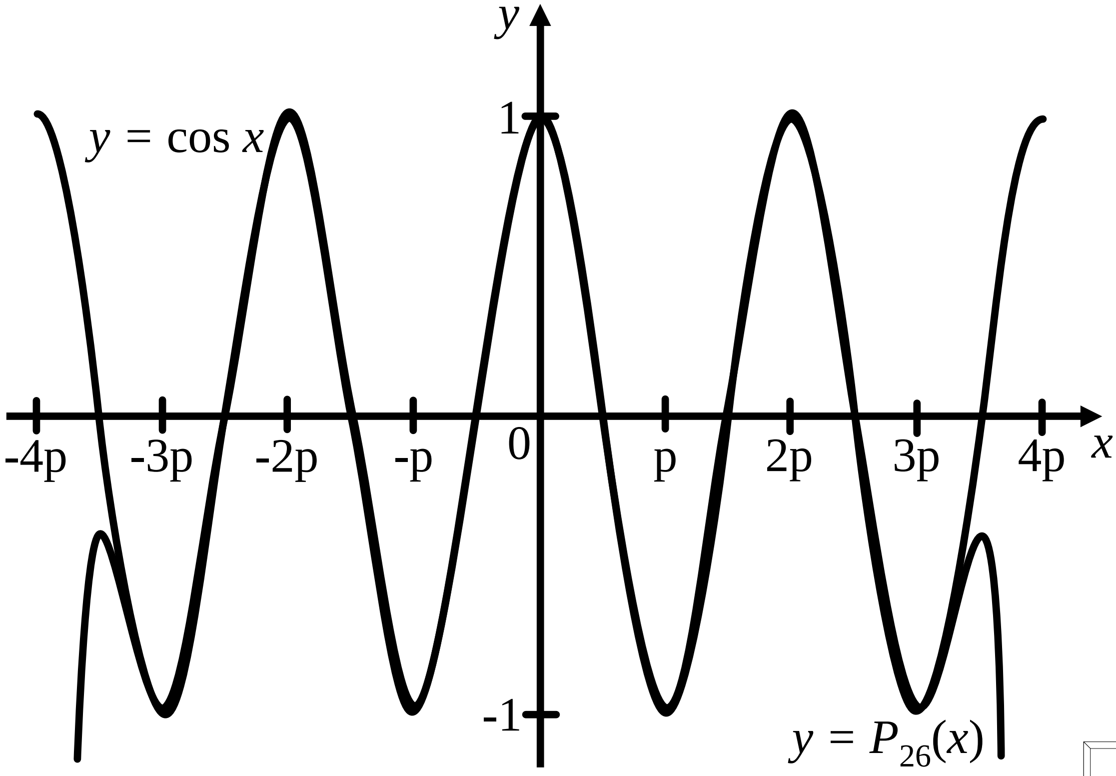
$$P_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$



Taylor Polynomials - Example

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots, \quad -\infty < x < \infty$$

The Taylor polynomials of $\cos x$ at $x = 0$ of order 26:



An Application of Taylor Polynomials

Suppose you are at the top of a lighthouse, height H above sea level.

How far out to sea can you see?

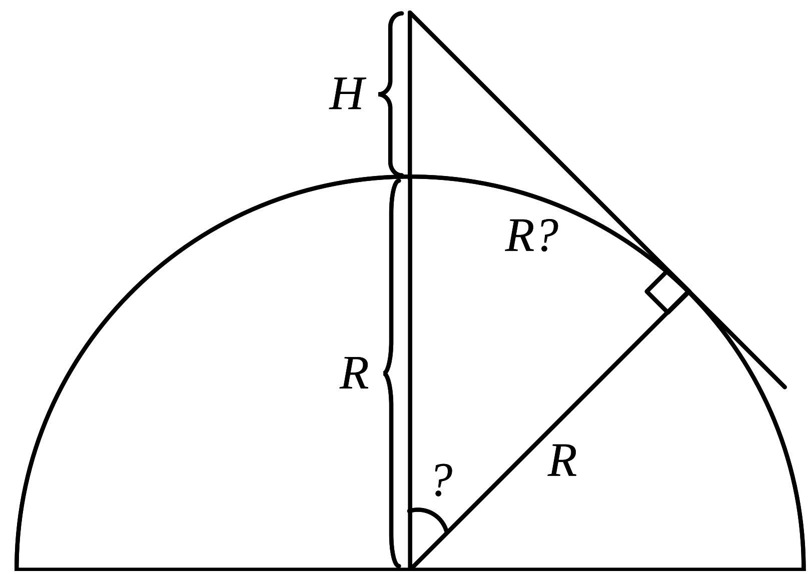
The most distant spots are called the HORIZON.

To find Rq.

Take $R = 6370$ km and $H = 0.1$ km

$$\cos q = \frac{R}{R+H} = \frac{\frac{R}{R}}{\frac{R}{R}+\frac{H}{R}} = \frac{1}{1+\frac{H}{R}}$$

$$\cos q = \frac{1}{1+\frac{H}{R}}$$



To find Rq.

$$\cos q = \frac{1}{1+\frac{H}{R}}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots, \quad -\infty < x < \infty$$

$$\cos q = 1 - \frac{q^2}{2}$$

$$\frac{1}{1-r} = 1 + r + r^2 + r^3 + \cdots, \qquad -1 < x < 1$$

Put $r = -\frac{H}{R}$

$$\frac{1}{1+\frac{H}{R}} = 1 - \frac{H}{R}$$

$$1 - \frac{q^2}{2} \approx 1 - \frac{H}{R}$$

$$Rq^2 = 2H$$

$$(Rq)^2 = 2HR$$

Multiply by R on both sides

$$Rq \approx \sqrt{2HR}$$

$$\approx \sqrt{2(0.1)6370}$$

$$\approx 35.7 \text{ km}$$

Take $R = 6370$ km and $H = 0.1$ km

End