MA 1505 Mathematics I

Tutorial 3 Solutions

1. (a) Observe that $\sec^2 x > 0$ and $-4\sin^2 x \leq 0$ on $[-\pi/3,\ \pi/3]$.

$$\begin{aligned}
\text{Area} &= \int_{-\pi/3}^{\pi/3} \left[\frac{1}{2}\sec^2 x - (-4\sin^2 x)\right] dx \\
&= \left[\frac{1}{2}\tan x + \int (2 - 2\cos 2x)\, dx\right]_{-\pi/3}^{\pi/3} \\
&= \tan\frac{\pi}{3} + (2x - \sin 2x)\Big|_{-\pi/3}^{\pi/3} \\
&= \sqrt{3} + \frac{4}{3}\pi - 2\sin\frac{\pi}{3} = \frac{4}{3}\pi.
\end{aligned}$$

(b) The points of intersection: $x = x^2/4$ implies $x = 0$ or $x = 4$. Hence the points of intersection are $(0,0)$ and $(4,4)$.

Note that $y = x^2/4 \Leftrightarrow x = 2\sqrt{y}$.

$$\text{The required area } = \int_0^1 \left[2\sqrt{y} - (y)\right] dy = \left[\frac{4}{3}y^{3/2} - \frac{1}{2}y^2\right]_0^1 = \frac{4}{3} - \frac{1}{2} = \frac{5}{6}.$$

(c) We have that $\quad (2-x) - (4-x^2) = x^2 - x - 2 = (x+1)(x-2)$

is negative if and only if $x \in (-1, 2)$.

Hence

$$\begin{aligned}
\text{Area} &= \int_{-2}^{3} \left|(2-x) - (4-x^2)\right| dx \\
&= \left[\int_{-2}^{-1} + \int_{2}^{3}\right](x^2 - x - 2)\, dx + \int_{-1}^{2} -(x^2 - x - 2)\, dx \\
&= \left[\int_{-2}^{3} -2\int_{-1}^{2}\right](x^2 - x - 2)\, dx \\
&= \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x\right]_{-2}^{3} - 2\left[\frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x\right]_{-1}^{2} \\
&= \frac{1}{3}\left[(27+8) - 2(8+1)\right] - \frac{1}{2}\left[(9-4) - 2(4-1)\right] - 2\left[5 - 2(3)\right] \\
&= \frac{1}{3}17 + \frac{1}{2} + 2 = \frac{49}{6}.
\end{aligned}$$

2. (a) The parabola and the line meet at (x, y) with $3 = y^2 + 1$, i.e. at $(3, \pm\sqrt{2})$.

By formula,

$$\begin{aligned}
\text{Volume} &= \int_{-\sqrt{2}}^{\sqrt{2}} \pi\left[(y^2+1) - 3\right]^2 dy = \pi\int_{-\sqrt{2}}^{\sqrt{2}} \left[y^4 - 4y^2 + 4\right] dy \\
&= \pi\left[\frac{1}{5}y^5 - \frac{4}{3}y^3 + 4y\right]_{-\sqrt{2}}^{\sqrt{2}} \\
&= \pi\, 2\left[\frac{1}{5}4\sqrt{2} - \frac{4}{3}2\sqrt{2} + 4\sqrt{2}\right] = \frac{64}{15}\sqrt{2}\pi.
\end{aligned}$$

(b) The parabola and the line meet at (x, y) with $x^2 = 2x$, i.e. at $(0, 0)$ and $(2, 4)$.

Now $y = 2x \Leftrightarrow x = y/2$ and $y = x^2 \Leftrightarrow x = \sqrt{y}$, while $\sqrt{y} - (y/2) = \sqrt{y}\,(1 - \sqrt{y}/2)$ is positive for $y \in (0, 4)$.

So $x = \sqrt{y}$ is the outer curve and $x = y/2$ is the inner curve. Hence,

$$\begin{aligned}\text{volume } &= \text{volume of space enclosed by outer shell} - \text{volume of hole enclosed by inner shell} \\ &= \int_0^4 \pi \sqrt{y}^2\, dy - \int_0^4 \pi \left(\frac{y}{2}\right)^2 dy = \pi \frac{1}{2}\left[4^2 - 0^2\right] - \pi \frac{1}{4}\frac{1}{3}\left[4^3 - 0^3\right] = \frac{8}{3}\pi.\end{aligned}$$

3. (a) Let $u_n = (-1)^n \frac{(x+2)^n}{n}$.

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(x+2)^{n+1}}{n+1} \cdot \frac{n}{(x+2)^n}\right| = |x+2|.$$

By ratio test, the power series is convergence in $|x+2| < 1$.

So the radius of convergence is 1.

(b) Let $u_n = \frac{(3x-2)^n}{n}$.

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(3x-2)^{n+1}}{n+1} \cdot \frac{n}{(3x-2)^n}\right| = |3x-2|.$$

By ratio test, the power series is convergence in $|3x-2| < 1 \Rightarrow |x - \frac{2}{3}| < \frac{1}{3}$.

So the radius of convergence is $\dfrac{1}{3}$.

(c) Let $u_n = (-1)^n (4x+1)^n$.

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(4x+1)^{n+1}}{(4x+1)^n}\right| = |4x+1|.$$

By ratio test, the power series is convergence in $|4x+1| < 1 \Rightarrow |x + \frac{1}{4}| < \frac{1}{4}$.

So the radius of convergence is $\dfrac{1}{4}$.

(d) Let $u_n = \frac{(3x)^n}{n!}$.

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(3x)^{n+1}}{(n+1)!} \cdot \frac{n!}{(3x)^n}\right| = \lim_{n\to\infty} \left|\frac{3x}{n+1}\right| = 0.$$

Since the limit is less than 1 for any x, so the radius of convergence is ∞.

(e) Let $u_n = (nx)^n$.

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{((n+1)x)^{n+1}}{(nx)^n}\right| = \lim_{n\to\infty} \left|\left(1 + \frac{1}{n}\right)^n (n+1)x\right| = \infty$$

for all $x \neq 0$.

So the radius of convergence is 0.

(f) Let $u_n = \frac{(4x-5)^{2n+1}}{n^{3/2}}$.

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(4x-5)^{2n+3}}{(n+1)^{3/2}} \cdot \frac{n^{3/2}}{(4x-5)^{2n+1}}\right| = |4x-5|^2.$$

By ratio test, the power series is convergence in $|4x-5|^2 < 1 \Rightarrow |4x-5| < 1 \Rightarrow |x - \frac{5}{4}| < \frac{1}{4}$.

So the radius of convergence is $\dfrac{1}{4}$.

4. The first term of the geometric series is $a = 1$ and the common ratio is $r = -\dfrac{(x-3)}{2}$.

   So the sum of the series is

$$\frac{a}{1-r} = \frac{1}{1+(x-3)/2} = \frac{2}{x-1}$$

   provided $\left|\dfrac{(x-3)}{2}\right| < 1 \Rightarrow 1 < x < 5$.

5. (a)

$$\frac{x}{1-x} = x\left(\frac{1}{1-x}\right) = x\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+1}$$

   (b) Let $f(x) = \frac{1}{x^2}$.

   Then $f'(x) = -\dfrac{2}{x^3}$, $f''(x) = \dfrac{3!}{x^4}$, ... and in general $f^{(n)}(x) = (-1)^n \dfrac{(n+1)!}{x^{n+2}}$.

   So $f^{(n)}(1) = (-1)^n(n+1)!$.

   The Taylor series of f at $x = 1$ is thus:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(1)}{n!}(x-1)^n = \sum_{n=0}^{\infty}(-1)^n(n+1)(x-1)^n.$$

   (c)

$$\begin{aligned}\frac{x}{1+x} &= \frac{(1+x)-1}{1+x} = 1 - \frac{1}{1+x} = 1 - \frac{1}{1+(x+2)-2} \\ &= 1 + \frac{1}{1-(x+2)} = 1 + \sum_{n=0}^{\infty}(x+2)^n = 2 + \sum_{n=1}^{\infty}(x+2)^n\end{aligned}$$

6. We need to find order 2 Taylor polynomial at $x = 0$.

(i) Let $f(x) = e^{\sin x}$.

Then $f'(x) = \cos x e^{\sin x}$, $f''(x) = \cos^2 x e^{\sin x} - \sin x e^{\sin x}$.

So $f(0) = 1, \quad f'(0) = 1, \quad f''(0) = 1.$

The order 2 Taylor polynomial of f at $x = 0$ is thus:

$$P_2(x) = \frac{1}{2}x^2 + x + 1.$$

(ii) Let $f(x) = \ln \cos x$.

Then $f'(x) = -\dfrac{\sin x}{\cos x} = -\tan x$, $f''(x) = -\sec^2 x$.

So $f(0) = 0, \quad f'(0) = 0, \quad f''(0) = -1.$

The order 2 Taylor polynomial of f at $x = 0$ is thus:

$$P_2(x) = -\frac{1}{2}x^2.$$

7. (i) We have

$$xe^x = x\sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$$

Note that the radius of convergence of this power series is infinite. Therefore,we can integrate both sides from 0 to 1.

$$\int_0^1 xe^x \, dx = \sum_{n=0}^{\infty} \int_0^1 \frac{x^{n+1}}{n!} \, dx = \sum_{n=0}^{\infty} \frac{1}{n!(n+2)}.$$

On the other hand,

$$\int_0^1 xe^x \, dx = [xe^x]_0^1 - \int_0^1 e^x \, dx = e - (e-1) = 1.$$

So we conclude $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!(n+2)} = 1.$

(ii) We have

$$\frac{e^x - 1}{x} = \frac{\sum_{n=1}^{\infty} \frac{x^n}{n!}}{x} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}.$$

Note that the radius of convergence of this power series is infinite.
Differentiate both sides with respect to x, we have

$$\frac{xe^x - (e^x - 1)}{x^2} = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{(n+1)!} = \sum_{n=0}^{\infty} \frac{(n+1)x^n}{(n+2)!}.$$

The result now follows by setting $x = 1$.