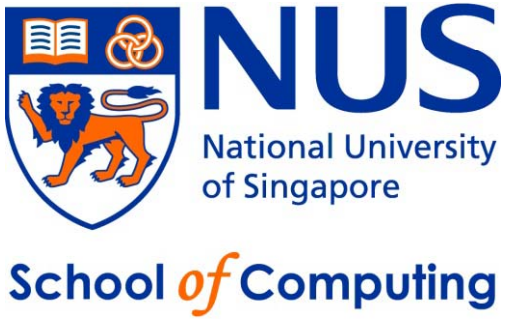
CS2020 – Data Structures and Algorithms Accelerated

Lecture 18 – Algorithms on DAG

stevenhalim@gmail.com

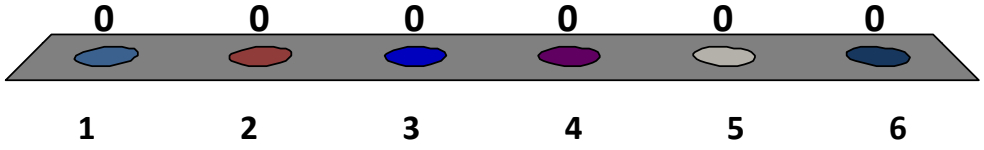


Outline

- What are we going to learn in this lecture?
  - Review + PS8 Reminder + PS9 Preview
  - DP on DAG
    - SSSP on DAG Revisited
      - Gentle introduction to Dynamic Programming (DP) technique
        - » Optimal sub structure
        - » Overlapping sub problems
    - SS Longest Paths (SSLP) on DAG
    - SSLP on DAG → Longest Increasing Subsequence (LIS)
    - Counting Paths on DAG

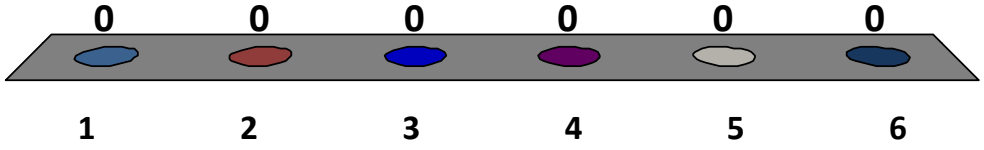
Given a general graph, edge weights are positive integers, we want to solve SSSP, we should use:

1. O(V + E) DFS
2. O(V + E) BFS
3. O(E log V) Kruskal's
4. O(E log V) Prim's
5. O(VE) Bellman Ford's
6. O((V + E) log V) Dijkstra's



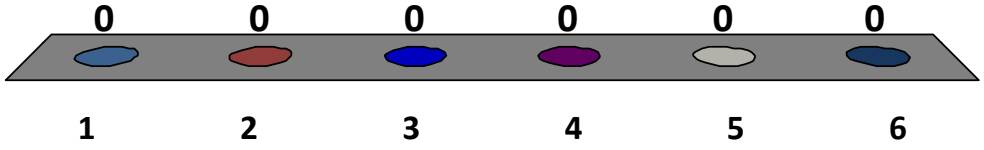
Given a weighted graph with one unique path between any two vertices, we want to solve SSSP, we should use:

1. O(V + E) DFS
2. O(V + E) BFS
3. O(E log V) Kruskal's
4. O(E log V) Prim's
5. O(VE) Bellman Ford's
6. O((V + E) log V) Dijkstra's



Given a general graph, all edge weights are 1, we want to solve SSSP, we should use:

1. O(V + E) DFS
2. O(V + E) BFS
3. O(E log V) Kruskal's
4. O(E log V) Prim's
5. O(VE) Bellman Ford's
6. O((V + E) log V) Dijkstra's

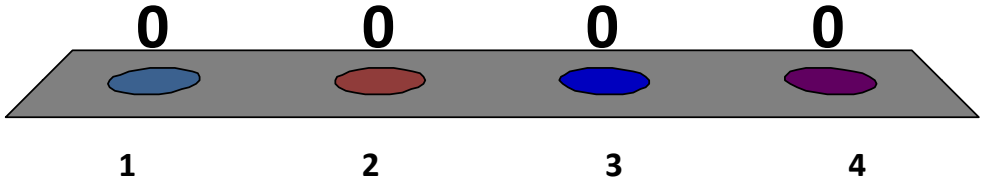


PS8 Reminder + PS9 Preview

- PS8 (the short one)
  - Deadline is tomorrow, Wed 30 March 2011, 2pm
  - Space for announcements/bug fixes, etc
- PS9 (just opened)
  - There are three DP-on-graph programming tasks
    - Two are DP-on-DAG problems as discussed today
    - One more is a problem related to APSP (discussed this Friday)
  - Deadline for PS9 is Wed 6 April 2011, 2pm

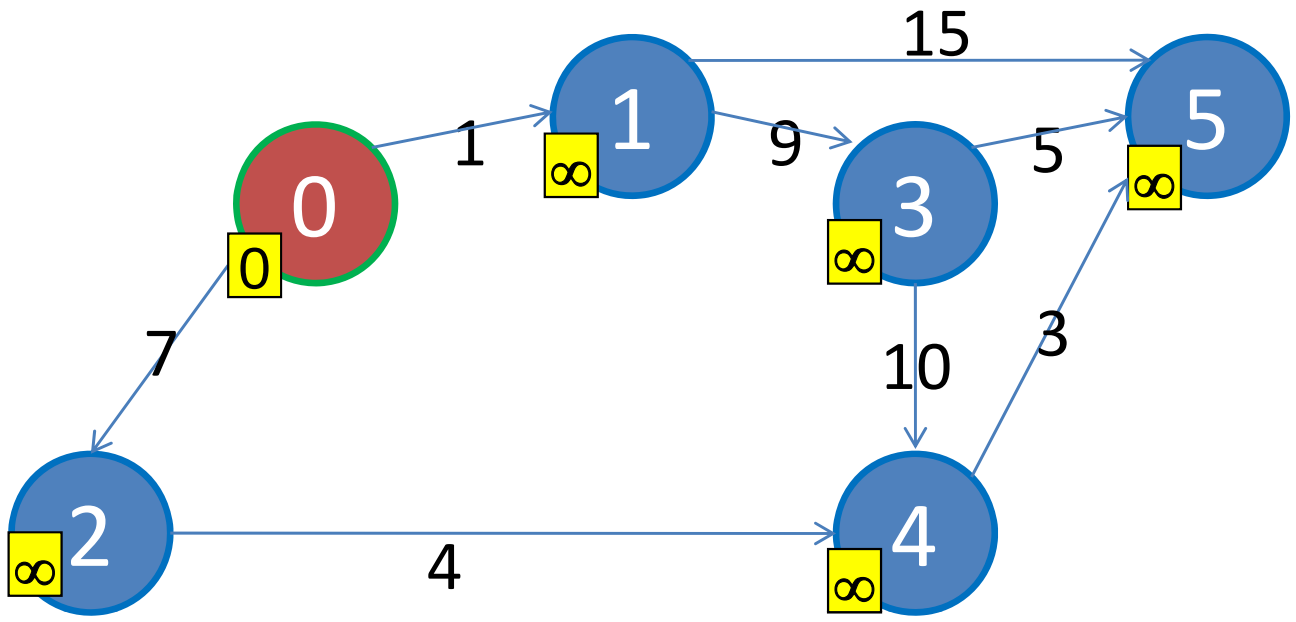
Now we come to the last topic of CS2020: Dynamic Programming ☺

1. I have no problem with recursion (I passed the recursion-heavy module CS1101S ☺)
2. I am not from CS1101S, but I am ready with lots of recursion, bring it on ☺
3. Hey, I have skimmed through this lecture note, where are the recursions?
4. I am afraid I will have problems with recursion :O



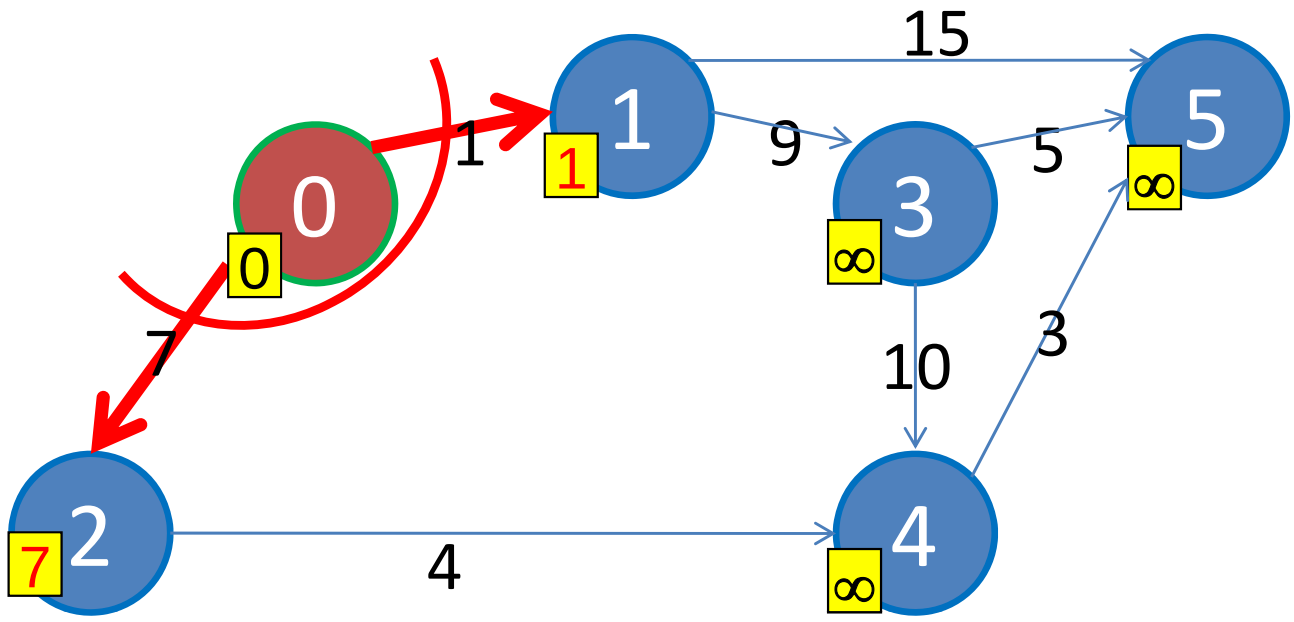
Review: SSSP in DAG (1)

- Topological Sort of this DAG is {0, 2, 1, 3, 4, 5}
  - Try relaxing the edges using the toposort above
    - With just one pass, all vertex will have the correct D[v]



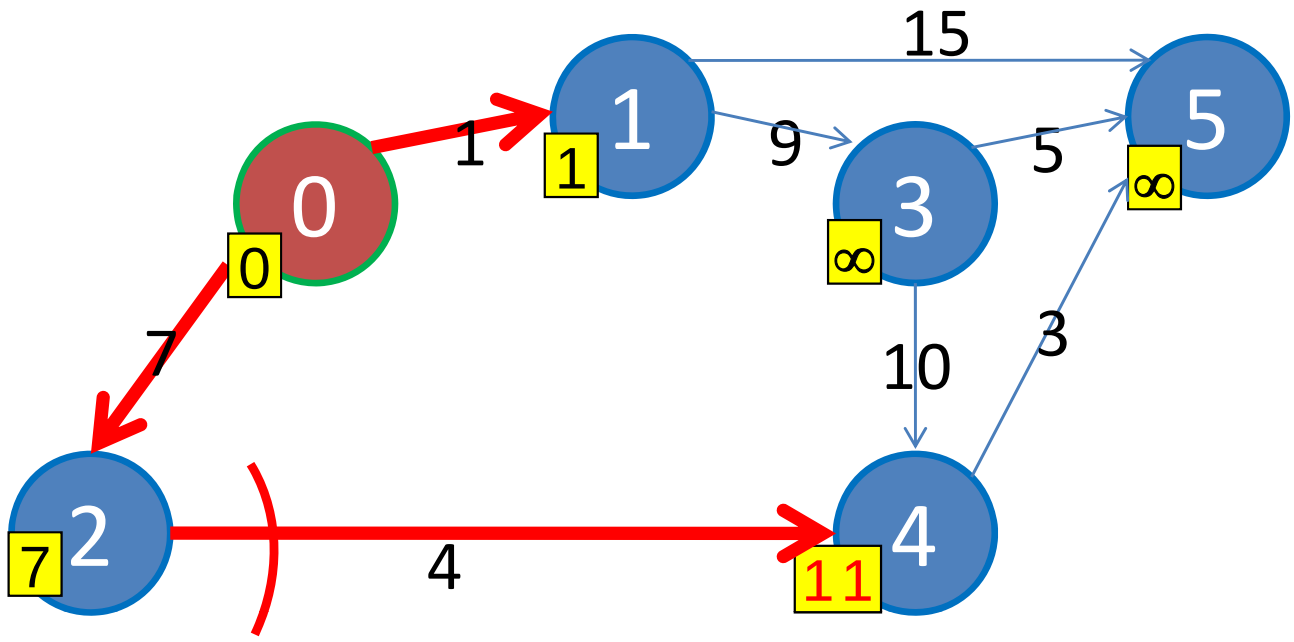
Review: SSSP in DAG (1)

- Topological Sort of this DAG is {0, 2, 1, 3, 4, 5}
  - Start from source (vertex 0)



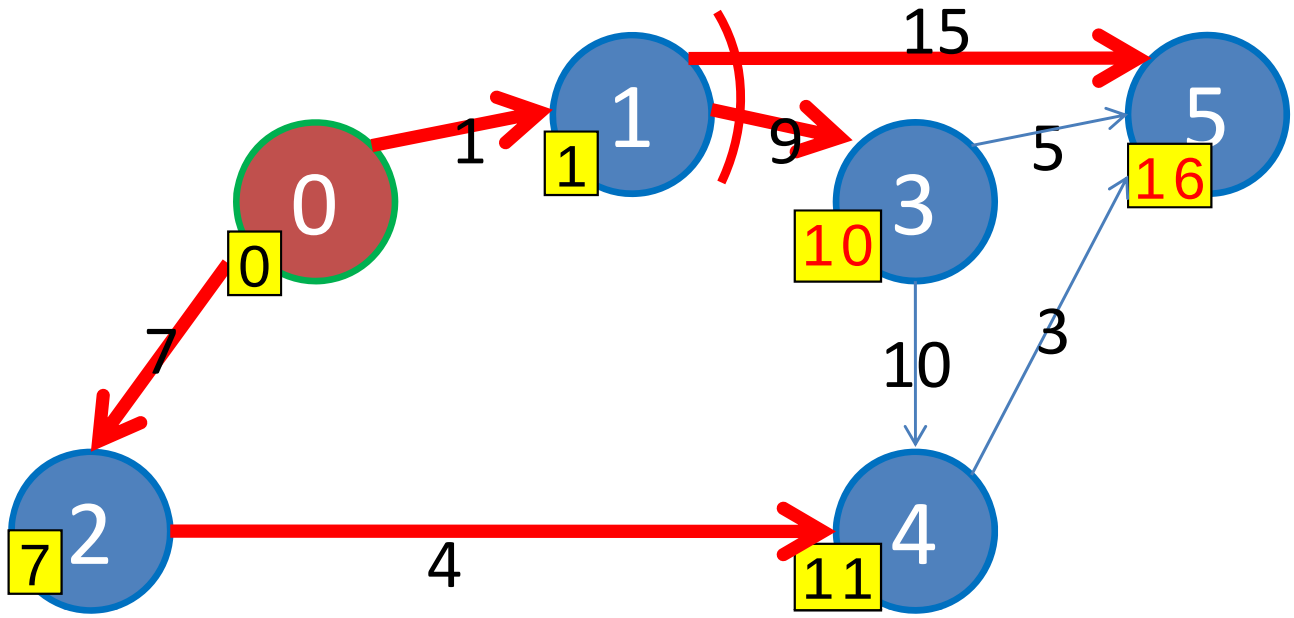
Review: SSSP in DAG (2)

- Topological Sort of this DAG is {0, 2, 1, 3, 4, 5}
  - Continue with vertex 2



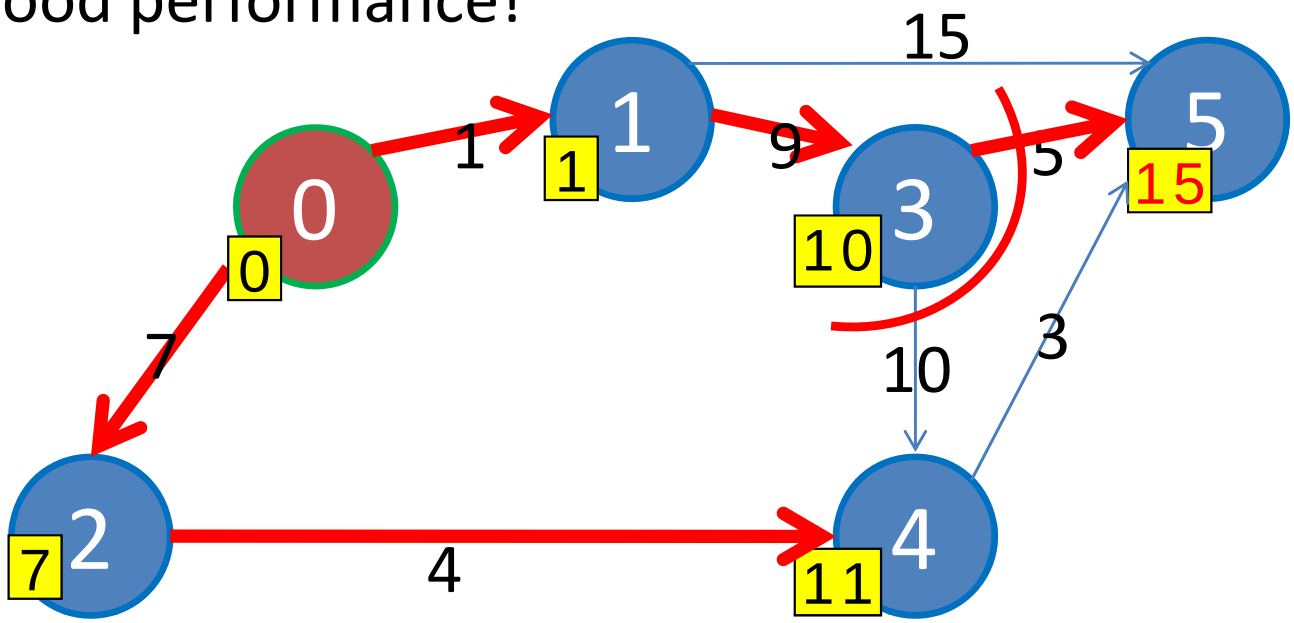
Review: SSSP in DAG (3)

- Topological Sort of this DAG is {0, 2, 1, 3, 4, 5}
  - Then vertex 1



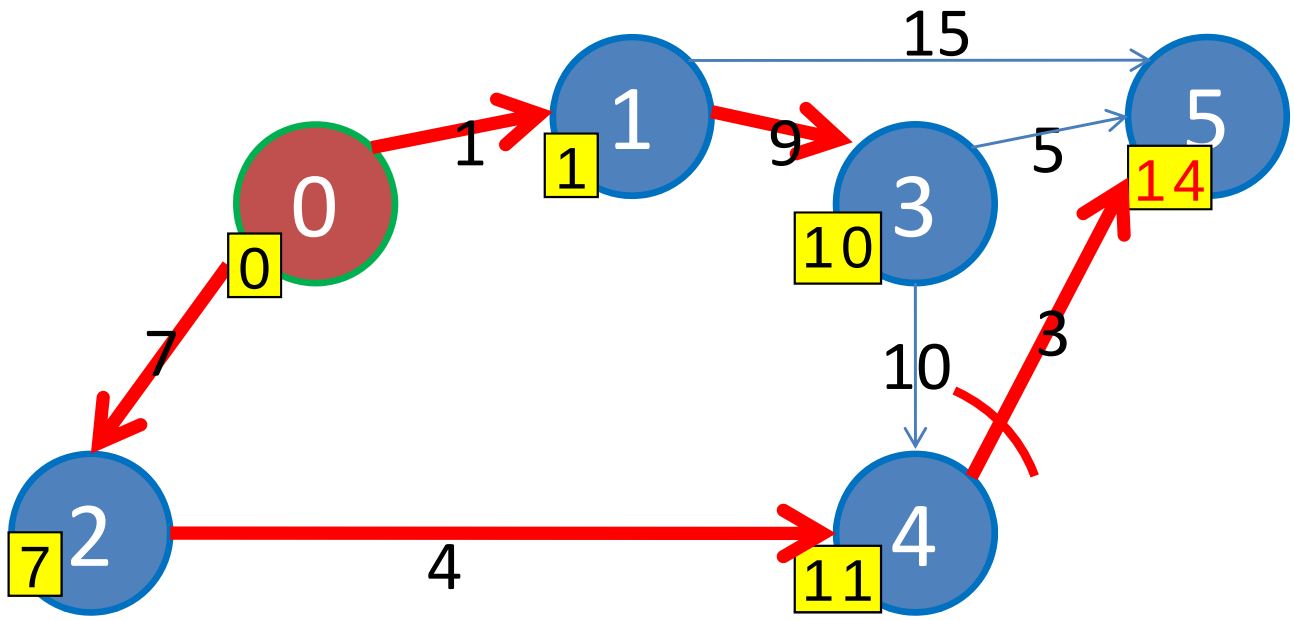
Review: SSSP in DAG (4)

- Topological Sort of this DAG is {0, 2, 1, 3, 4, 5}
  - Then vertex 3; Vertices that **have been processed so far**, i.e. {0, 2, 1, 3} already have the correct final shortest path values, we **do not have to re-trace** our steps → key for good performance!



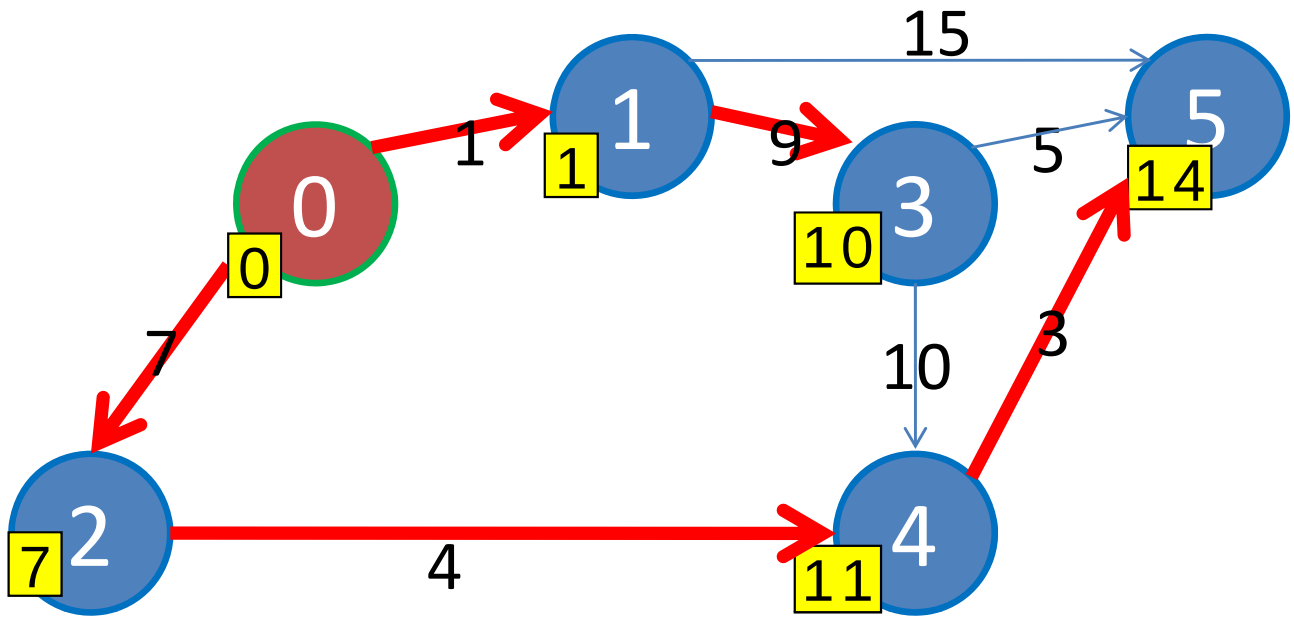
Review: SSSP in DAG (5)

- Topological Sort of this DAG is {0, 2, 1, 3, 4, 5}
  - Almost? done



Review: SSSP in DAG (5)

- Topological Sort of this DAG is {0, 2, 1, 3, 4, 5}
  - Final state
  - The **thick red edges** form the shortest paths spanning tree

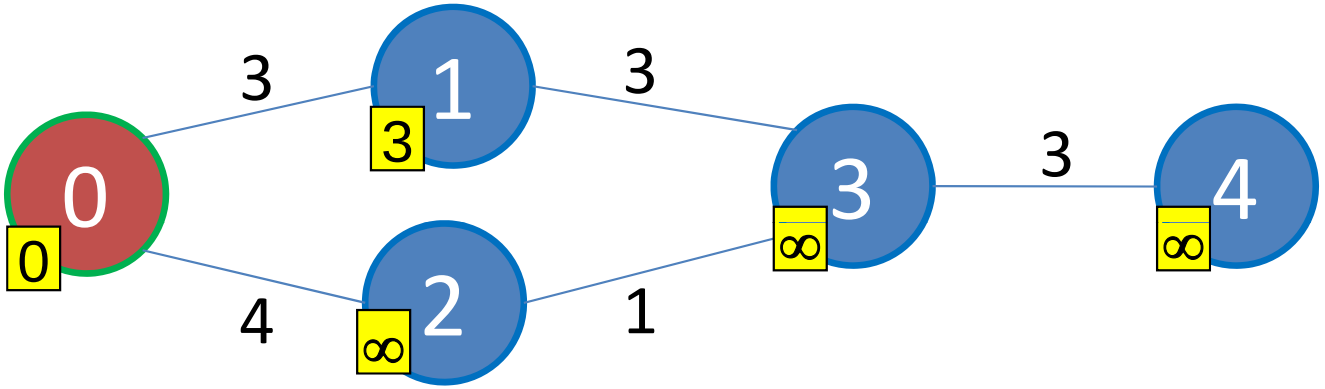


Analysis of SSSP on DAG

- Pre-processing step: topological sort
  - This can be done in O(V + E) using modified DFS
- Then, following this topological order (V items), relax a total of E edges
  - Again, it is O(V + E)
- In overall, SSSP on DAG can be solved in linear time: O(V + E)
  - Linear in terms of V and E

Why It Works? (1)

- On general graph, Bellman Ford's algorithm has to repeat this all-edges O(E) relaxation V-1 times
  - Thus Bellman Ford's runs in O(VE) time
    - Reason: there exist (non negative) cycles in general graph
    - After relax(u, v, w_u_v) is performed, there *may be* better other path *in the future* that reaches vertex **u** (the origin) so that this relax(u, v, w_u_v) has to be repeated…
    - We can only be sure after we have done this all-edges relaxation V-1 times (recall the proof of correctness of Bellman Ford's)



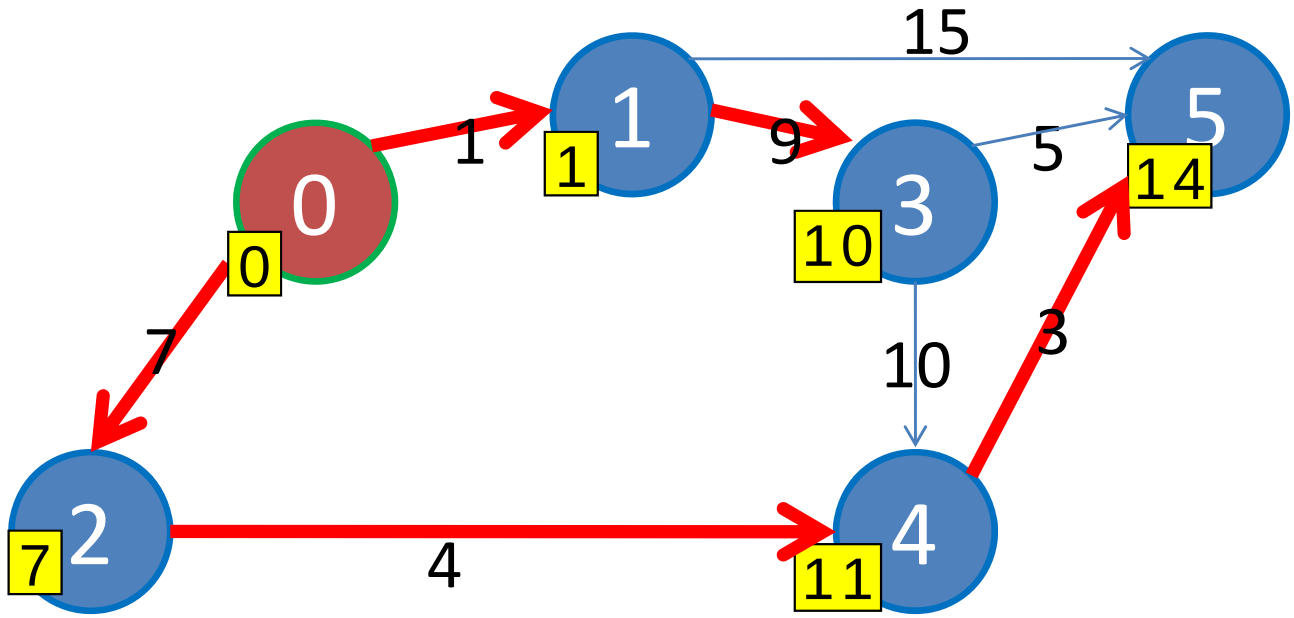
Assume edge ordering:
0-1
1-3
3-4
0-2
2-3

Why It Works? (2)

- On DAG, there is **no cycle** → we have topological order
  - Recall the meaning of topological order:
    - Linear ordering of vertices such that for every **edge(u, v)** in DAG, vertex **u** comes before **v** in the ordering
  - If the vertices are processed according to this topological order, then after relax(u, v, w_u_v) is performed, there *will never be* any better other path in the future that reaches vertex **u** so that this relax(u, v, w_u_v) has to be repeated…
    - There is no way vertex **v** can reach back to vertex **u** because vertex **v** appears later in the topological ordering and there is **no cycle** that allows **v** ~→ some other vertices ~→ **u**!
  - Thus SSSP on DAG can be solved in O(E) time
    - We do not have to repeat this V-1 times ☺

Where is the DP? (Part 1)

- Observe, for example, shortest path 0→2→4→5
  - Sub paths of this path, e.g. 0→2→4, or 0→2 are shortest paths too! (shown in DG7)
  - You may not realize it, but we are not doing re-computations on these (clearly) overlapping sub paths
    - Topological order is the correct order to avoid re-computations

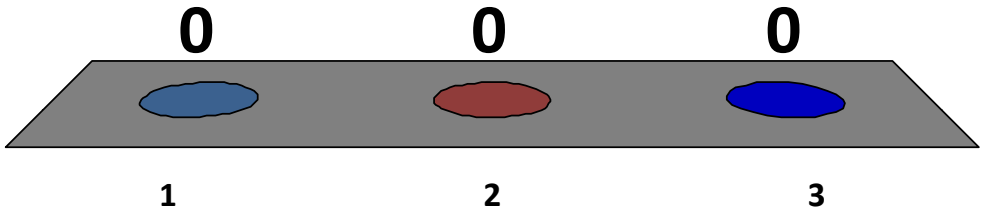


SS LONGEST PATHS ON DAG

If we can do SSSP efficiently, can we do SS **longest (simple) paths** on **general graph** too?

1. Yes, why not?
2. This is a trick question, I know that the longest (simple) path problem is NP-complete ~ something hard (although I am not sure why it is hard…)
3. Generally no, longest (simple) paths is not an easy problem because

_____

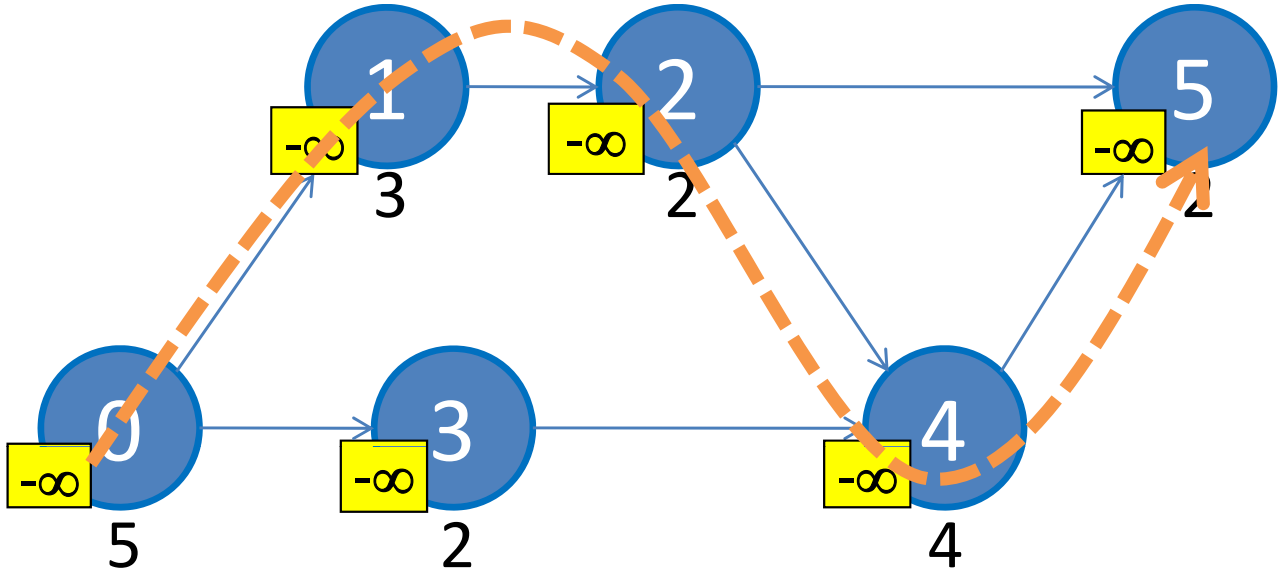


Longest Paths on DAG (1)

- Program Evaluation and Review Technique (PERT)
  - PERT is a project management technique
  - It involves breaking a large project into a number of tasks, estimating the time required to perform each task, and determining which tasks can not be started until others have been completed
    - This is similar to module pre-requisites!
    - This is a DAG!
  - The project is then summarized in chart form
  - See the next few slides for an example

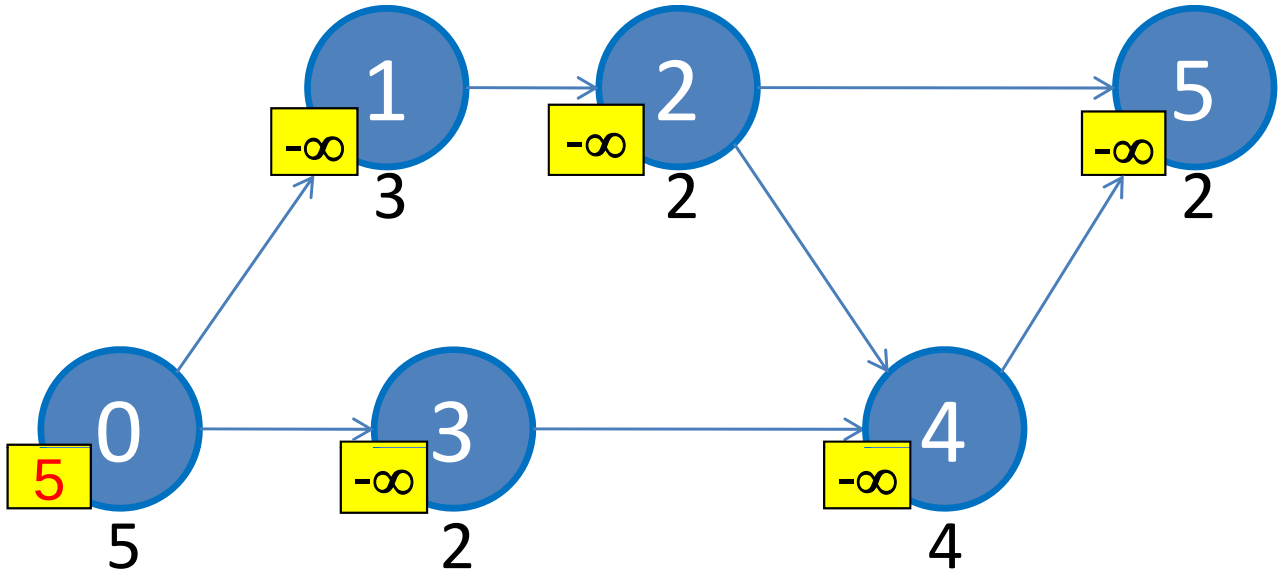
Longest Paths on DAG (2)

- Problem source: UVa 452 – Project Scheduling
  - Verify that this graph is a DAG!
    - Notice that the weight is on vertices, e.g. weight(0) = 5
  - The shortest way to complete this project is the…
    - **longest path** found in the DAG… (a bit counter intuitive)



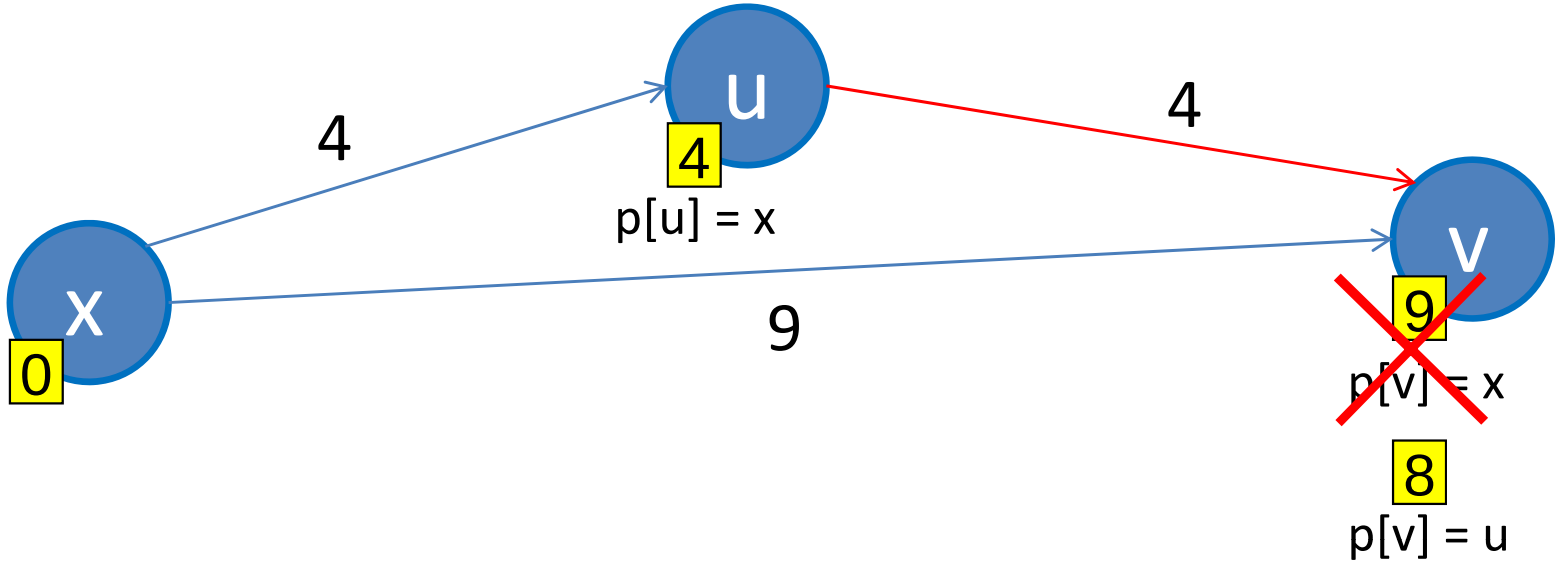
Longest Paths on DAG (3)

- First, find one topological order: {0, 3, 1, 2, 4, 5}
  - Can be found with O(V + E) modified DFS as in Lecture14
  - Initially, set D[0] = weight(0) = 5
- Then "stretch" (antonym of "relax") the vertices according to this topological order



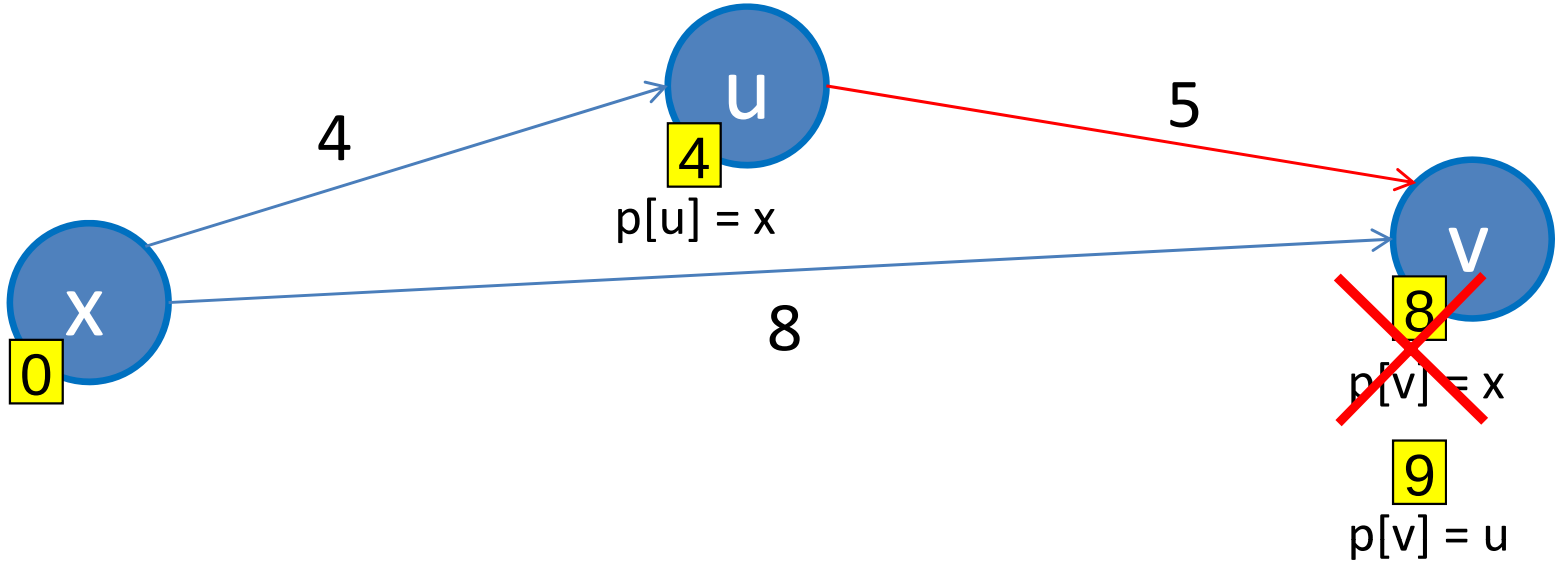
Review: “Relax” Operation

```
relax(u, v, w_u_v)
  if D[v] > D[u] + w_u_v // if SP can be shortened
    D[v] ← D[u] + w_u_v // relax this edge
    p[v] ← u // remember/update the predecessor
```



“Stretch” Operation

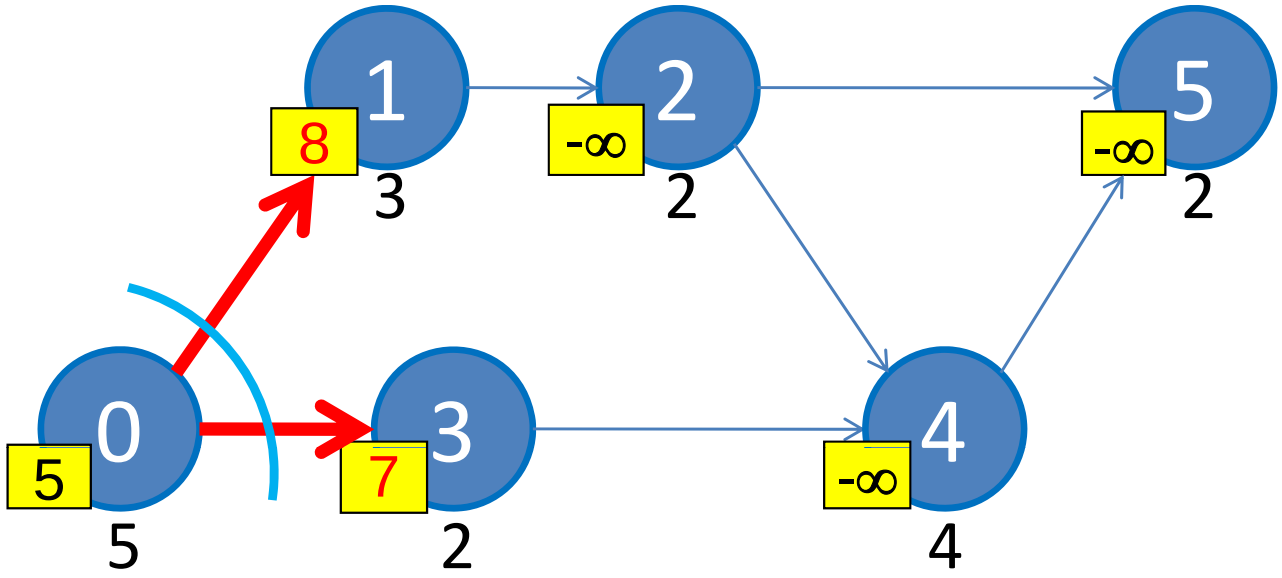
```
stretch(u, v, w_u_v)
  if D[v] < D[u] + w_u_v // if LP can be lengthened
    D[v] ← D[u] + w_u_v // stretch this edge
    p[v] ← u // remember/update the predecessor
```



Longest Paths on DAG (4)

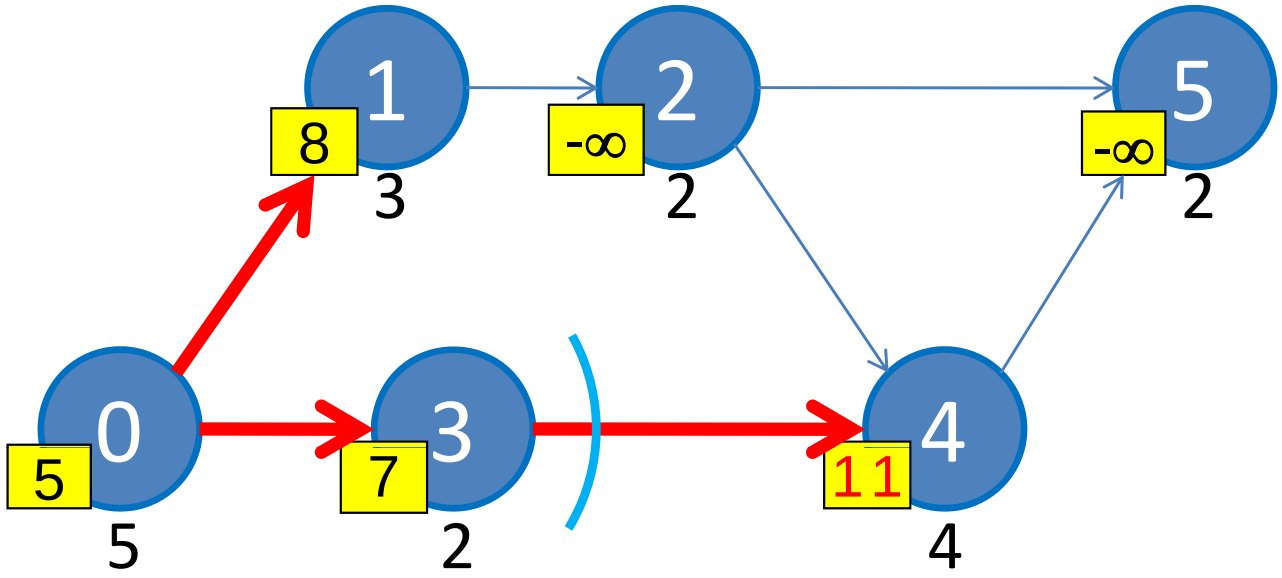
- The topological order: {0, 3, 1, 2, 4, 5}
  - Now relax outgoing edges from vertex 0
  - Use weight of destination vertex as the “edge weight”

    e.g. stretch(0, 1, weight(1)); stretch(0, 3, weight(3))



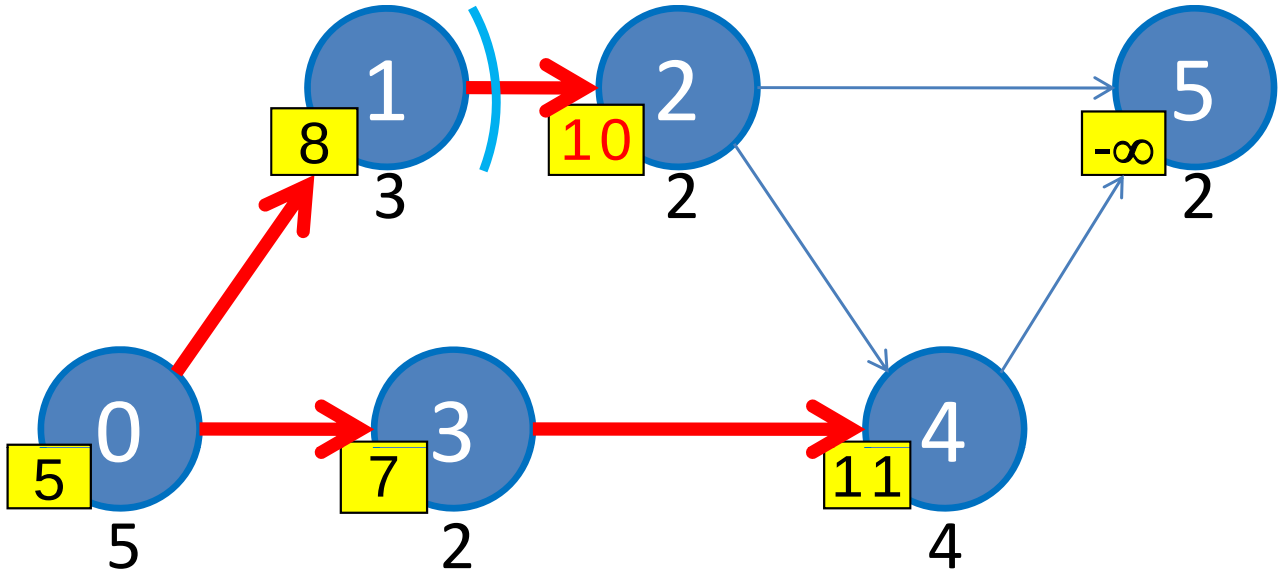
Longest Paths on DAG (5)

- The topological order: {0, 3, 1, 2, 4, 5}
  - Continue with vertex 3



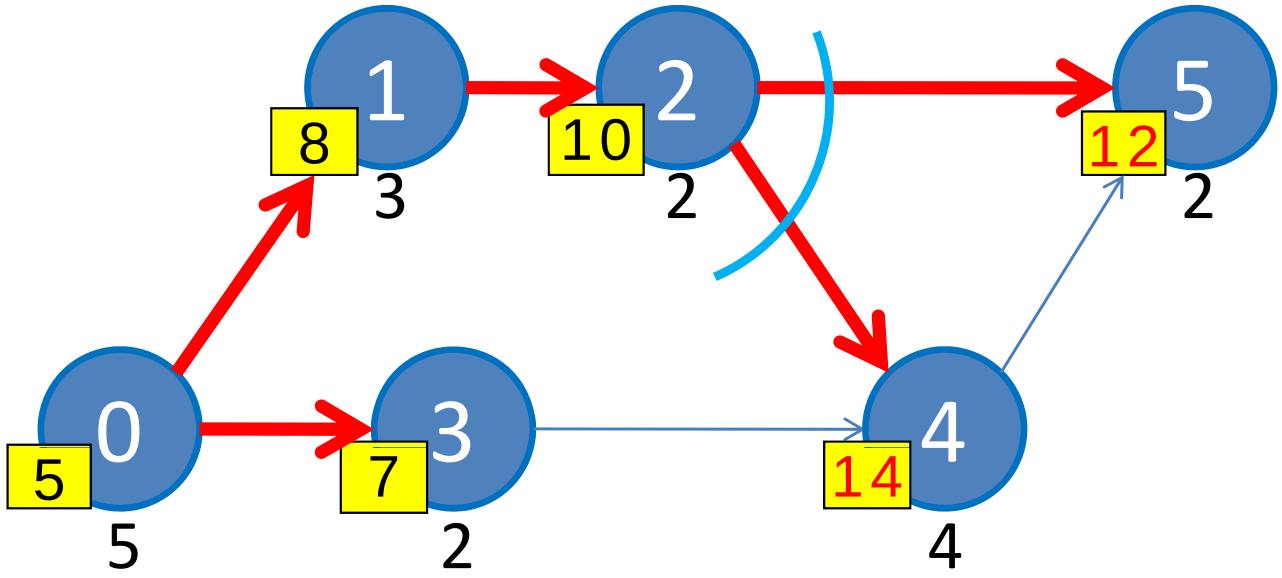
Longest Paths on DAG (6)

- The topological order: {0, 3, 1, 2, 4, 5}
  - Continue with vertex 1



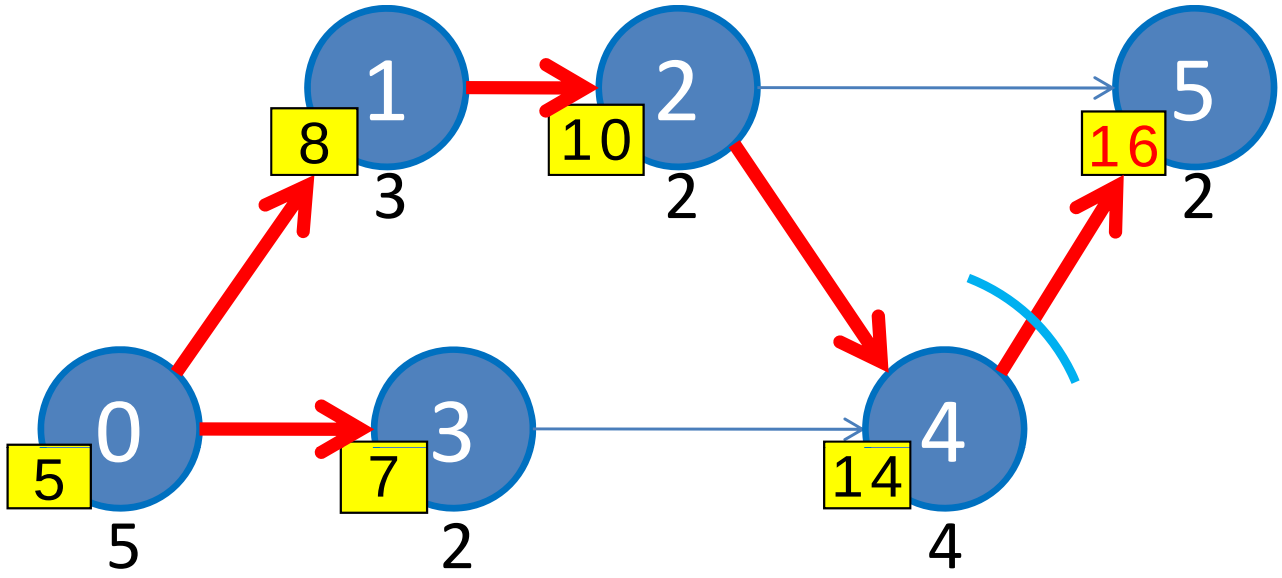
Longest Paths on DAG (7)

- The topological order: {0, 3, 1, 2, 4, 5}
  - Continue with vertex 2
  - Notice (again) that the vertices that have been processed so far, i.e. {0, 3, 1, 2} already have the correct final longest path values ☺, we do not have to re-trace our steps



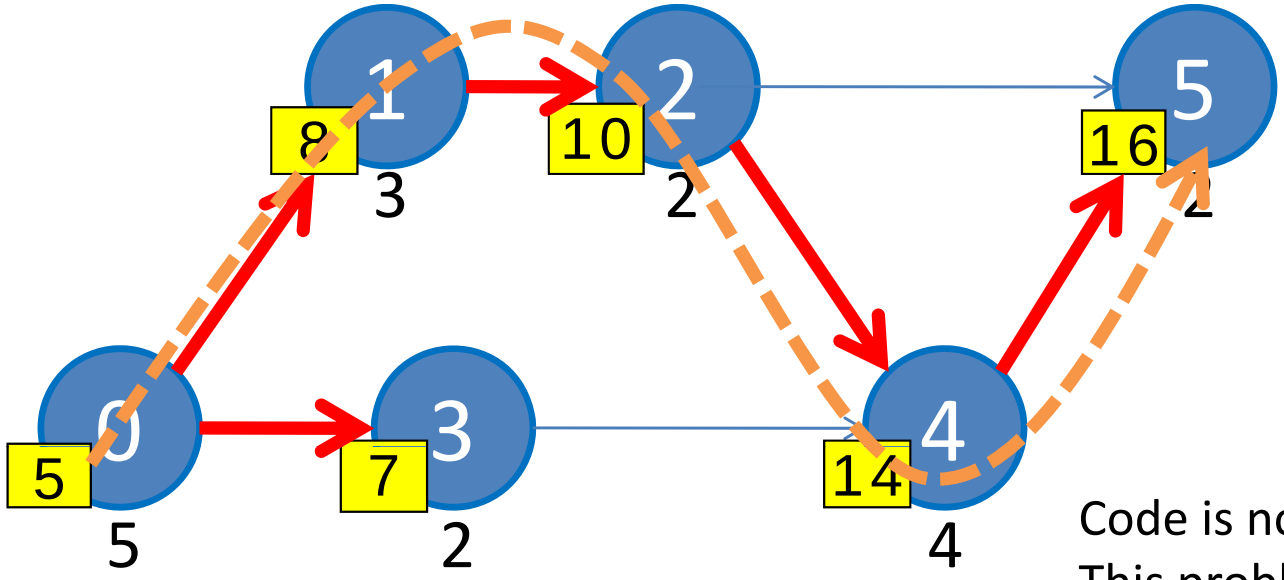
Longest Paths on DAG (8)

- The topological order: {0, 3, 1, 2, 4, 5}
  - Question (to be discussed in DG)
  - Can we stop here, i.e. the second last vertex?
    - Proof or provide counter example!



Longest Paths on DAG (9)

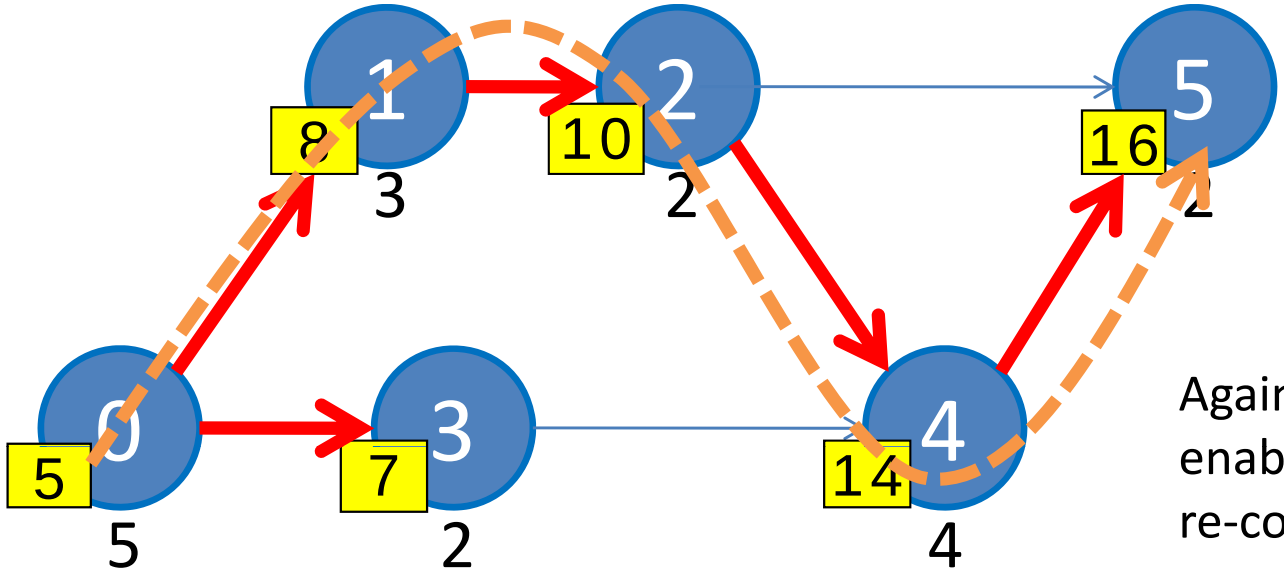
- The topological order: {0, 3, 1, 2, 4, 5}
  - Final solution, again the **thick red edges** are the LP Sp Tree
  - Scan the whole D[v], find the largest one
    - In this example D[5] = 16 is the largest
    - Use predecessor information (the **thick red edges**) to reconstruct the longest path: 0→1→2→4→5



Code is not given.
This problem is part of PS9 ☺

Where is the DP? (Part 2)

- These two major ingredients for DP technique are also present in the SSLP on DAG problem:
  - Optimal sub-structure
    - Sub paths of longest paths **<u>on DAG</u>** are longest paths too
  - Overlapping sub-problem
    - Longest path 0->1->2->4->5 contains longest path 0->1->2->4, etc



Again, topological order enable us to avoid re-computations

Analysis of Longest Paths on DAG

(The same as SSSP on DAG)

- Pre-processing step: topological sort
  - This can be done in O(V + E) using modified DFS
- Then, following this topological order (V items), "stretch" a total of E edges
  - Again, it is O(V + E)
- In overall, longest paths on DAG can be solved in linear time: O(V + E)
  - Linear in terms of V and E

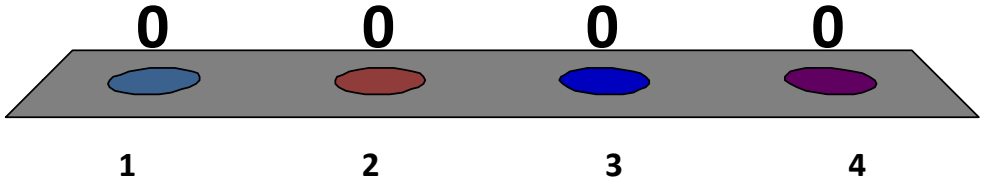
Longest Paths ← → LIS :O

- There is one more classical CS problem that can be modelled as longest paths in (implicit) DAG
  - The Longest Increasing Subsequence (LIS)
- While we are at this topic, let's discuss it as well ☺
  - In the next few slides, we will see LIS, the implicit DAG in LIS, and the solution

Have you heard about LIS?

10 minutes break after this

1. This is my first time, tell me!
2. Yes, I know the $O(n^2)$ algorithmic solution for LIS
3. Yes, I have solved/coded some $O(n^2)$ LIS solution before
4. Yes, I have solved/coded some O(n log k) LIS solution before (if you say "why there is a log factor?", do not select this)



A sister problem that is very related to SSLP on DAG

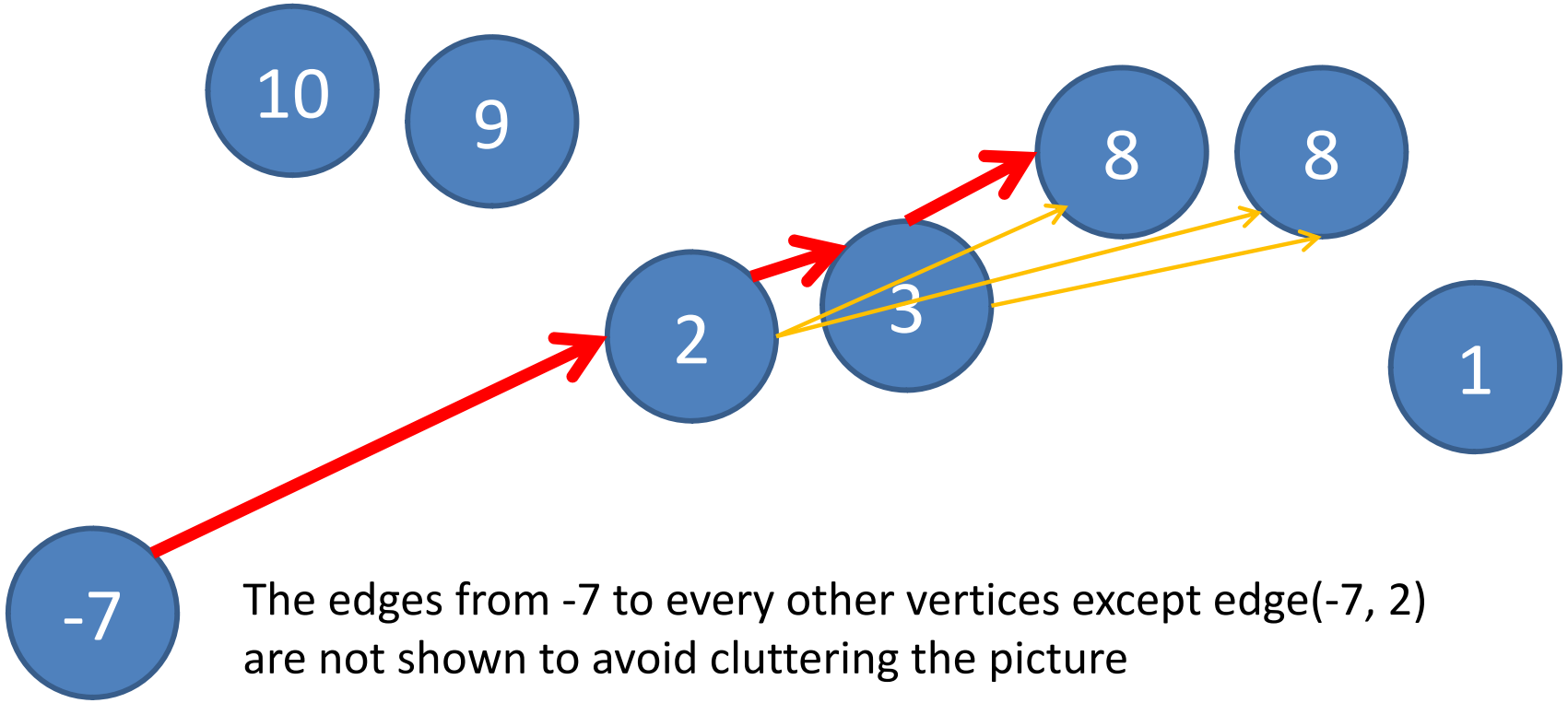
LONGEST INCREASING SUBSEQUENCE (LIS)

Longest Increasing Subsequence (1)

- Problem Description (Abbreviated as LIS):
  - As implied by its name….
    Given a sequence {X[0], X[1], …, X[N-1]},
    determine the Longest Increasing Subsequence
    - Subsequence is not necessarily contiguous
  - Example: N = 8, sequence X = {-7, 10, 9, 2, 3, 8, 8, 1}
    - LIS is {-7, 2, 3, 8} of length 4
  - Variants:
    - Longest Decreasing Subsequence
    - Longest Non Decreasing^ Subsequence

Longest Increasing Subsequence (2)

- There is **an implicit DAG** in this sequence X
  - See the implicit DAG of sequence X = {-7, 10, 9, 2, 3, 8, 8, 1}
    - As discussed in Recitation of Week09,
      we do not have to store implicit graph in a graph DS



The edges from -7 to every other vertices except edge(-7, 2)
are not shown to avoid cluttering the picture

Longest Increasing Subsequence (3)

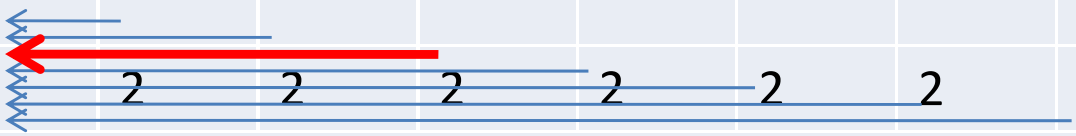
- Let D[i] = the best LIS ending at index (vertex) i
  - i.e. the longest path value from source (index 0) to i
- The topological order is obviously {0, 1, 2, …, N - 1}
  - "Stretch" all index (vertex) one by one using this order

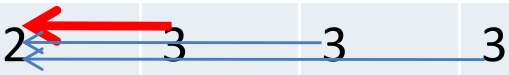
```
D[0] = 1 // base case (source vertex)
for i = 0 to N – 2 // this is O(N^2)
  for j = i + 1 to N - 1
    if X[i] < X[j] // implicit edge!
      stretch(i, j, 1) // edge weight is 1
```

- The answer is max(D[i]), $i \in [0 .. N - 1]$

Longest Increasing Subsequence (4)

Index	0	1	2	3	4	5	6	7
X	-7	10	9	2	3	8	8	1
D (initial)	1							
D (i = 0)	1	2	2	2	2	2	2	2
D (i = 1-2)	no change during these two iterations							
D (i = 3)	1	2	2	2	3	3	3	2
D (i = 4)	1	2	2	2	3	4	4	2
D (i = 5-7)	no more change							





- This LIS problem can be solved in $O(n^2)$, analysis:
  1. Use the fact that there are two nested loops of size n, or
  2. Use the analysis of the longest paths on (implicit) DAG where there are V = n vertices and E = n^2 edges

Longest Increasing Subsequence (5)

- LIS is actually solvable in O(n log **k**)
  - Where **k** is the length of LIS
- How?
  - Utilize the fact that LIS is sorted
    - Use binary search
  - A greedy solution
  - Not important for CS2020
    - But it is for CS3233
  - Discuss this in DG8 if you want to

Where is the DP? (Part 3)

- This LIS problem is more naturally solved in "pure Dynamic Programming (DP)" fashion
  - Let **LIS(i)** be the value of the longest LIS starting from index i until N - 1
    - This can be written as a function with one parameter, index i
  - We can write the solution using this recurrence relations:
    - LIS(N - 1) = 1 // at last position, we cannot extend the LIS anymore
    - LIS(i) = max(LIS(j) + 1), for all $j \in [i + 1 .. N - 1]$ where X[i] < X[j]
  - To avoid recomputations, memoize the LIS value of each index/vertex **i**
    - This term "memoize" will be explained soon

Where is the DP? (Part 4)

- This can be written using (Java) recursive function
  - Notice that this version is very slow due to recomputations

```
private static int LIS(int i) {
  if (i == N - 1) return 1;

  int ans = 0;
  for (int j = i + 1; j < N; j++)
    if (X.get(i) < X.get(j))
      ans = Math.max(ans, LIS(j) + 1);
  return ans;
}
```

Turn Recursion into Memoization

initialize memo table in the main method

```
return_value recursive_function() {
   if already calculated, simply return the result
   calculate the result using recursion
   save the result in the memo table
   return the result
}
```

Where is the DP? (Part 5)

- A better version (see LISDPDemo.java):

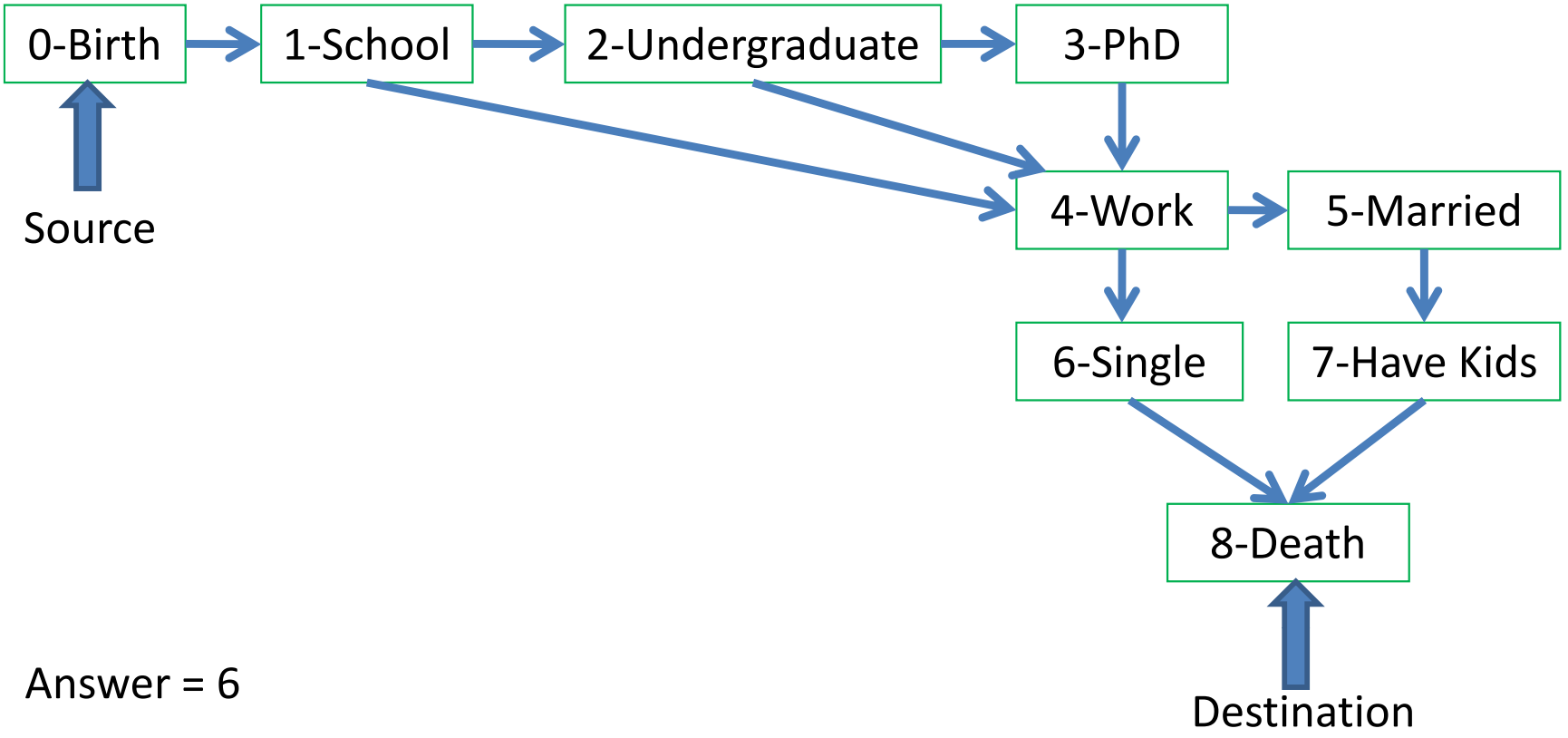
```
private static int LIS(int i) {
  if (i == N - 1) return 1;
  if (memo.get(i) != -1) return memo.get(i);

  int ans = 0;
  for (int j = i + 1; j < N; j++)
    if (X.get(i) < X.get(j))
      ans = Math.max(ans, LIS(j) + 1);
  memo.set(i, ans);
  return ans;
}
// values in memo are set to -1 in main method
```

Final discussion for today, again about DAG ☺

COUNTING PATHS ON DAG

Counting Paths in DAG

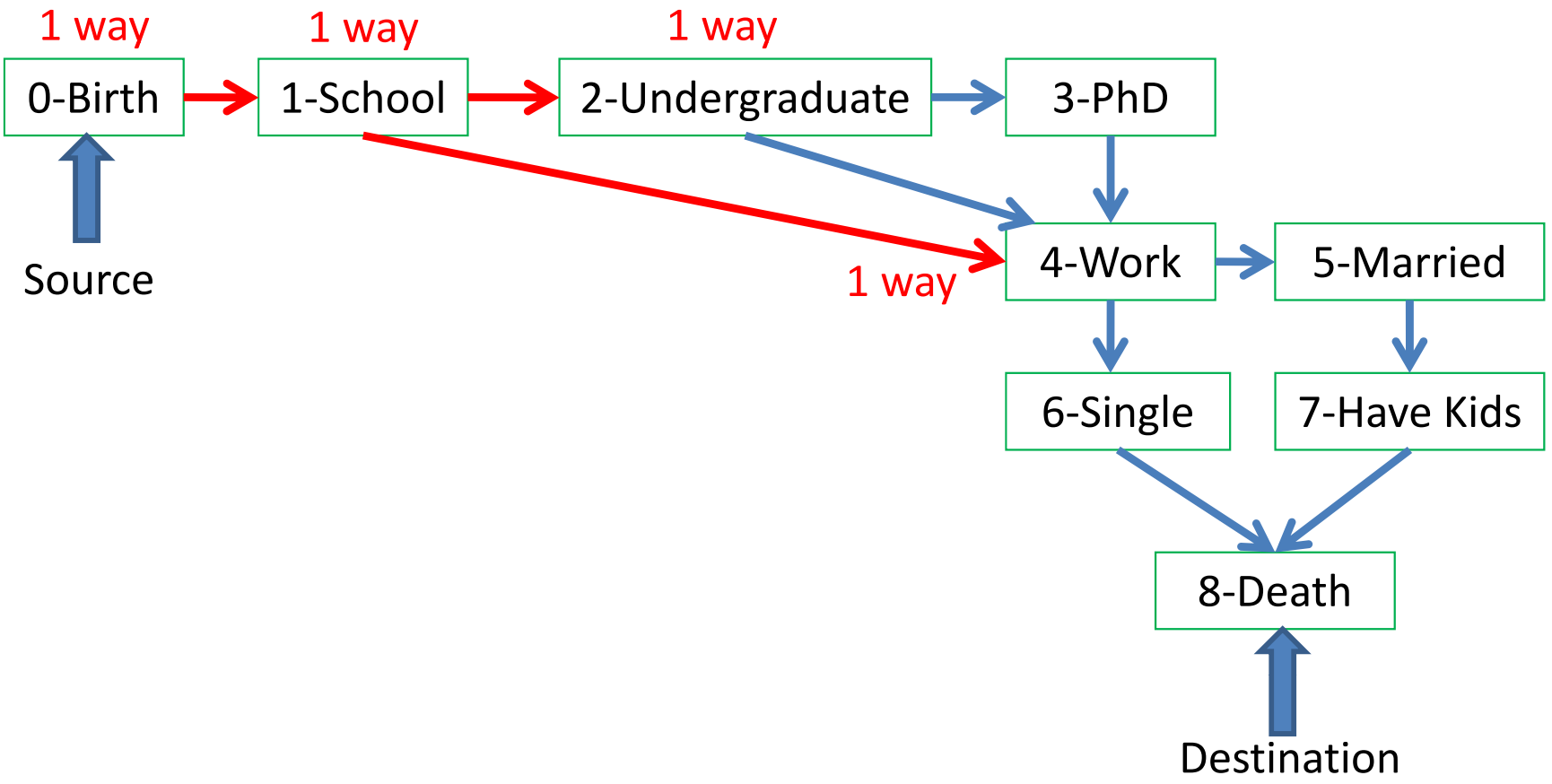
- Given some real-life time line (obviously a DAG)
  - How many different possible lives that you can live (from birth/vertex 0 to death/vertex 8)?



Answer = 6

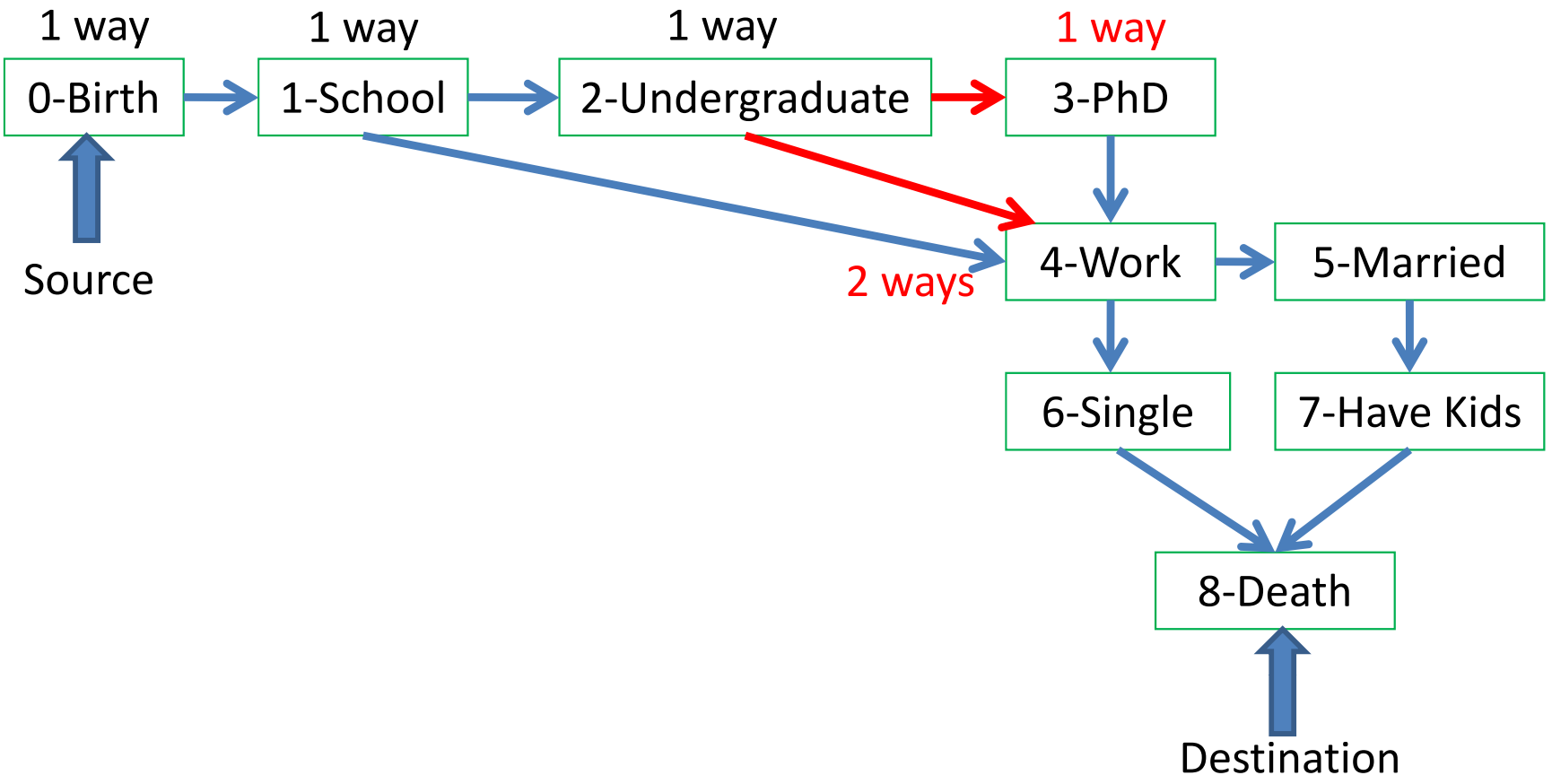
Toposort Way (1)

- Find the toposort first: {0, 1, 2, 3, 4, 6, 5, 7, 8}
  - numPaths[0] = 1, propagate to vertex 1, and then 2, 4



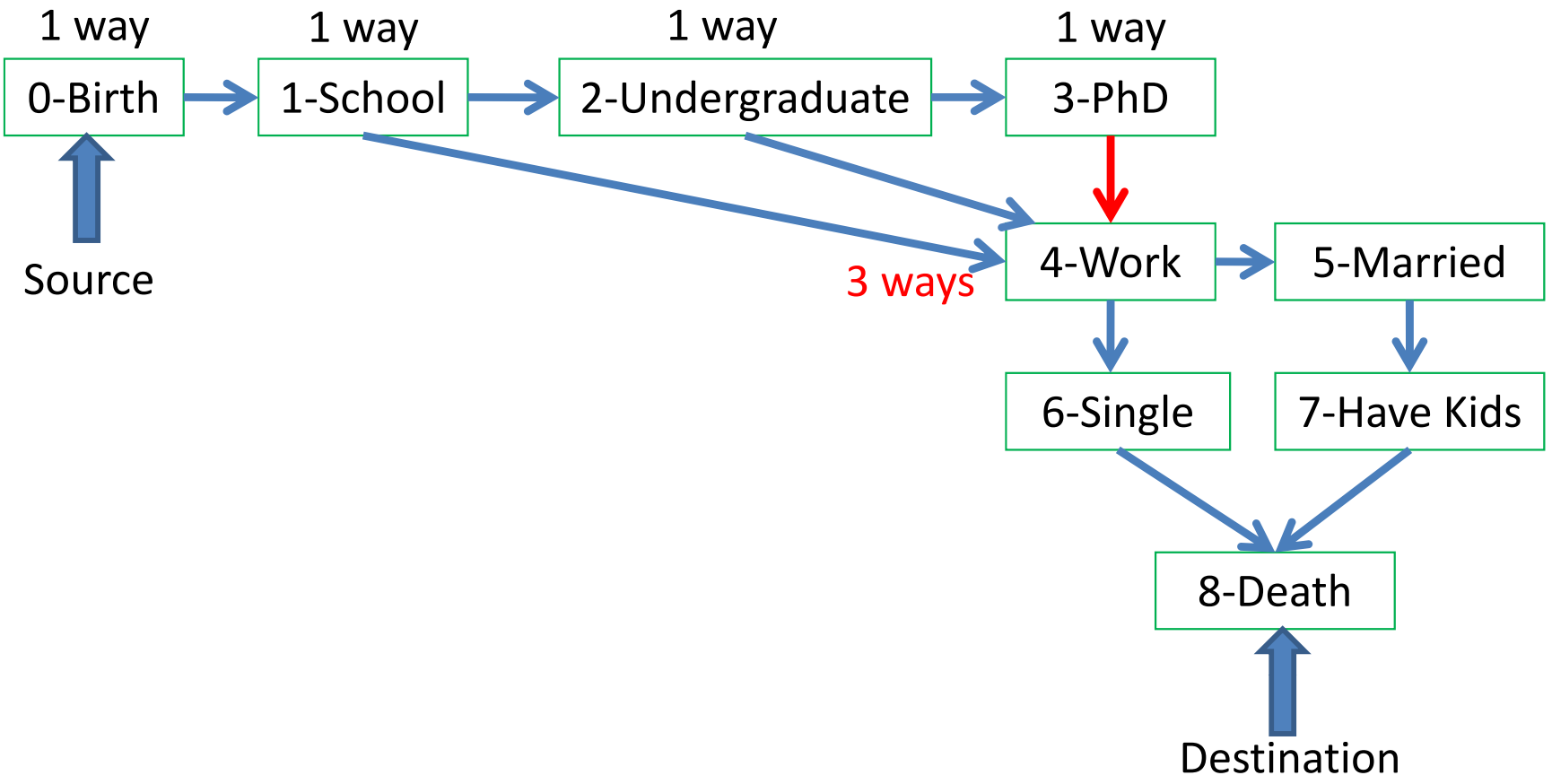
Toposort Way (2)

- Find the toposort first: {0, 1, 2, 3, 4, 6, 5, 7, 8}
  - numPaths[2] = 1, propagate to vertex 3 (0→1) and 4 (1→2)



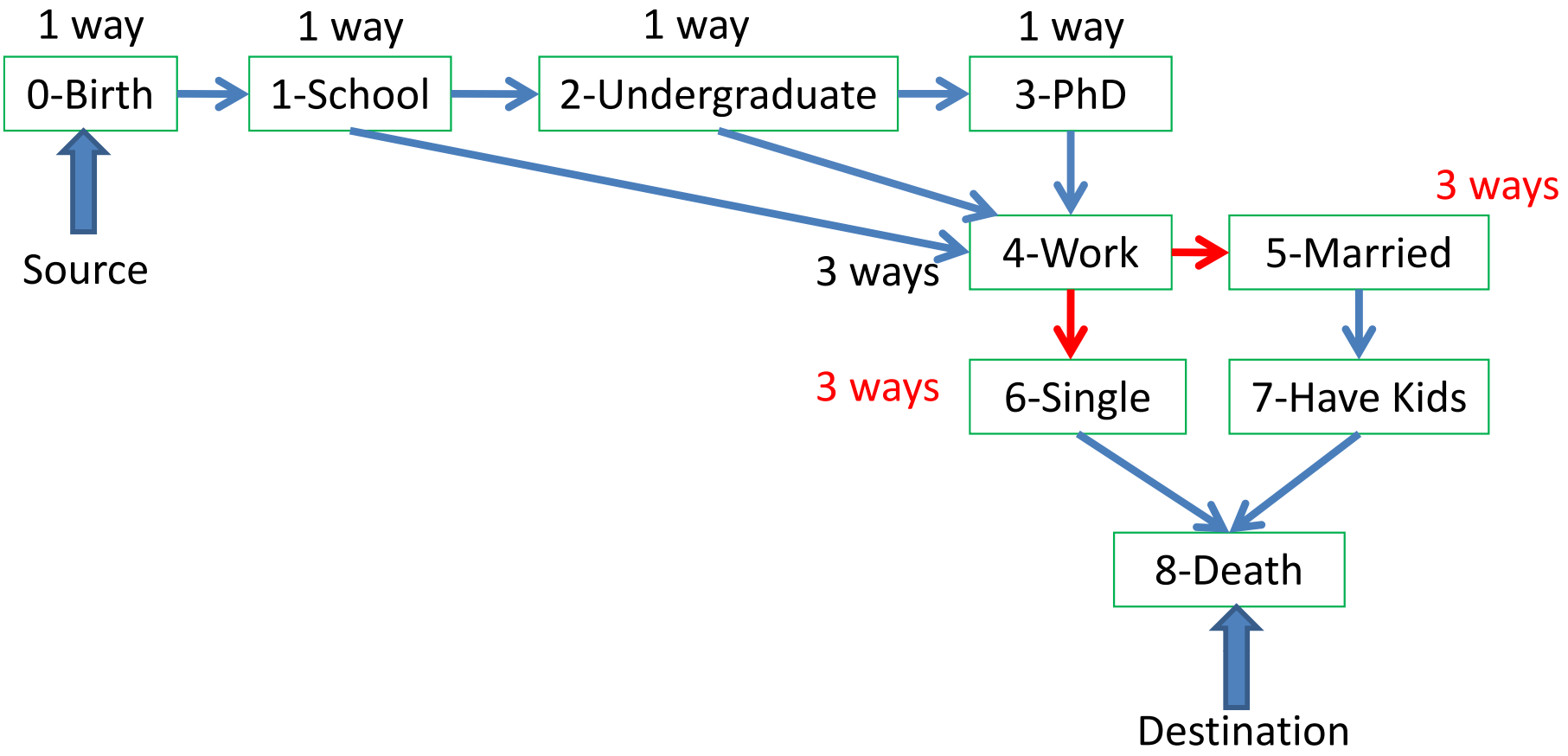
Toposort Way (3)

- Find the toposort first: {0, 1, 2, 3, 4, 6, 5, 7, 8}
  - numPaths[3] = 1, propagate to vertex 4 (2→3)



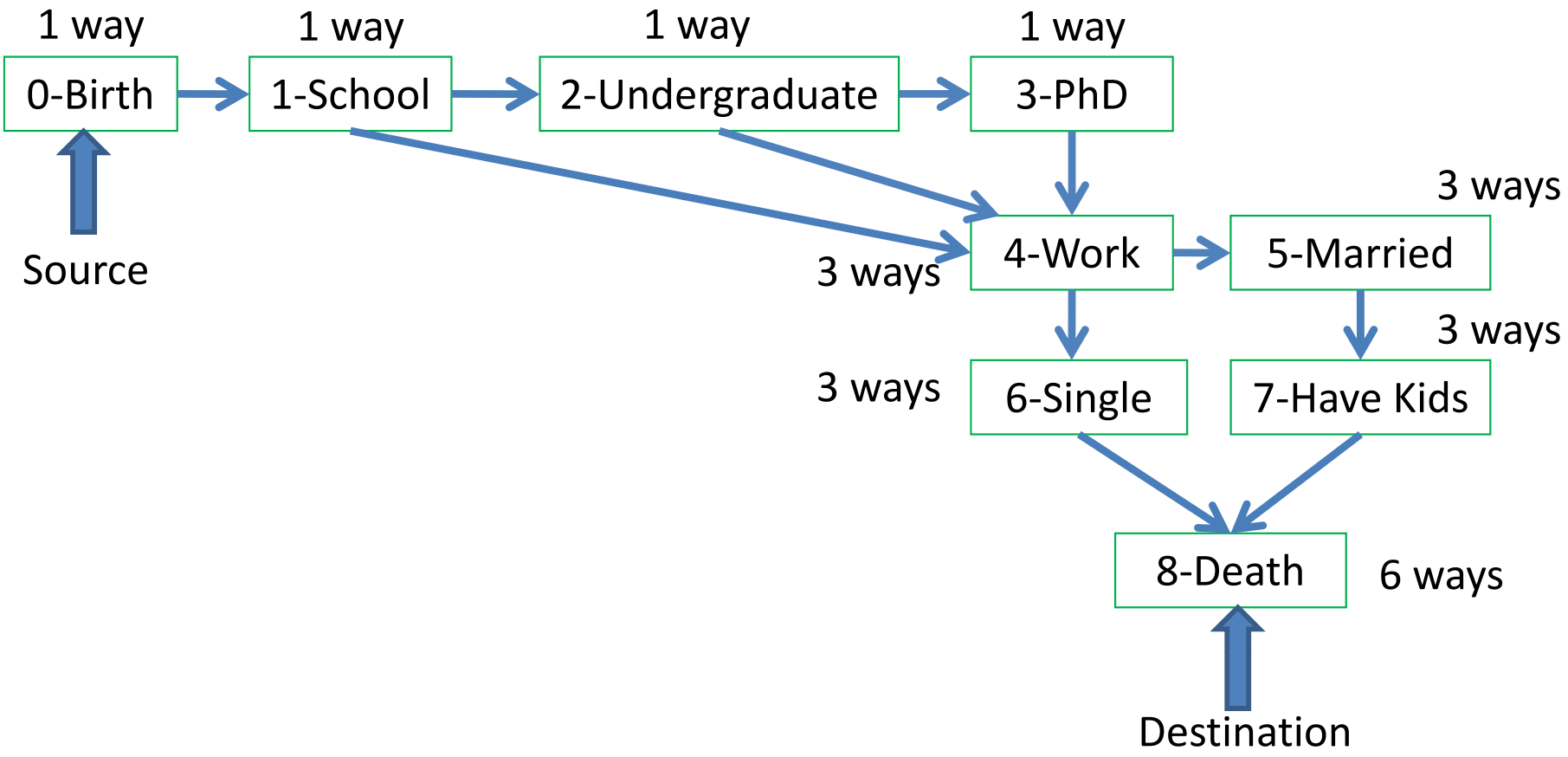
Toposort Way (4)

- Find the toposort first: {0, 1, 2, 3, 4, 6, 5, 7, 8}
  - numPaths[4] = 3, propagate to vertex 5 (0→3) and 6 (0→3)



Toposort Way (5)

- Find the toposort first: {0, 1, 2, 3, 4, 6, 5, 7, 8}
  - >> >> Fast forward…, this is the final state



Where is the DP? (Part 6)

- We can solve "counting paths in DAG" using pure DP
  - That is: using functions/loops, parameters, and "tables"
  - Let **numPaths(i)** be the number of paths starting from vertex **i** to destination **t**
  - We can write the solution using this recurrence relations:
    - numPaths(t) = 1 // at destination t, obviously only one path
    - numPaths(i) = ∑ numPaths(j), for all j adjacent to i
  - To avoid recomputations,
    memoize the number of paths for each vertex **I**
  - Code is not given, they are left as exercise for PS9 ☺
    - See the LISDPDemo.java example shown earlier

Summary / Transition to DP Topics

- In this lecture, we link graph topic (DAG) to DP
  - SSSP on DAG (revisited)
  - SSLP on DAG ←→ LIS
  - Counting Paths on DAG
  - In the next 3 lectures, we will use more implicit DAGs (on more structured problems) and use more DP terminologies than graph terminologies ☺
    - Ingredients:
      - Optimal sub-structure and Overlapping sub-problem
    - Terminologies:
      - Vertices → States; Edges → Transitions
      - |V| → Space Complexity; |E| → Time Complexity