Stanley's Twelvefold Way

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Richard Stanley, in his book *Enumerative Combinatorics*[1], describes the *Twelvefold Way*, which organizes the basic kinds of "balls and boxes" distribution problems that we consider in combinatorics. Imagine distributing n balls into m boxes, where the balls and boxes may be distinguishable (labeled) or indistinguishable (unlabeled)—there are four possible combinations. Then we can ask about the number of ways to distribute the balls into the boxes, which describes a function from the set of balls to the set of boxes. Three natural restrictions to put on such a function are injectivity, surjectivity, and... well, no restriction at all. These four restrictions, together with the four combinations of labeled/unlabeled, yield twelve possible kinds of functions, which Stanley calls the Twelvefold Way.

Before presenting the Twelvefold Way, here's a summary of some the notation I use in this note:

- $n^{\underline{k}}$ is the falling factorial; it equals $n(n-1)\cdots(n-k+1)$. The notation $(n)_k$ is also used for this.
- $\left\{ {n \atop k} \right\}$ is the Stirling number of the second kind, also written $S(n,k)$. It is the number of set partitions of $[n]$ into k blocks.
- $[P]$, where P is some logical statement, equals 1 if P is true and 0 if P is false.
- $p_k(n)$ is the number of integer partitions of n into exactly k parts.

Now let's see the Twelvefold way.

Labeled or not?		Type of functions					
n balls	m boxes	any		injective		surjective	
L	L	m^n	§1.1	$m^{\underline{n}}$	§1.2	$m!\left\{ {n \atop m} \right\}$	§1.3
U	L	$\binom{n+m-1}{m-1}$	§2.1	$\binom{m}{n}$	§2.2	$\binom{n-1}{m-1}$	§2.3
U	U	$\sum_{k\leqslant m} p_k(n)$	§3.1	$[n \leqslant m]$	§3.2	$p_m(n)$	§3.3
L	U	$\sum_{k\leqslant m} \left\{ {n \atop k} \right\}$	§4.1	$[n \leqslant m]$	§4.2	$\left\{ {n \atop m} \right\}$	§4.3

Table 1: The Twelvefold Way. In each cell is the number of ways to distribute the balls into the boxes and the section number where we explain the formula. The section numbers are clickable links if you are viewing this as a PDF.

[1] http://www.worldcat.org/isbn/0521663512 and in the KAIST library, call number QA164.8 S73

1 Labeled balls, labeled boxes: ordinary functions from $[n]$ to $[m]$

Since the balls and boxes are both labeled, distributing the balls into the boxes corresponds to ordinary functions from $[n]$ to $[m]$. Putting ball k into box j means the function sends k to j.

1.1 Labeled balls, labeled boxes, arbitrary functions

If there are no restrictions on the function from $[n]$ to $[m]$, it's easy to see that there are m^n functions: for each element in $[n]$, there are m choices for the image of that element: m^n total functions.

Click to return to the Twelvefold Way table.

1.2 Labeled balls, labeled boxes, injective functions

We have m choices for the image of $1 \in [n]$ (that is, the box into which we put the ball labeled 1), and since we are counting injective functions, we then have $m - 1$ choices for the image of 2, and so on—we have a falling factorial: there are $m^{\underline{n}}$ injective functions.

Click to return to the Twelvefold Way table.

1.3 Labeled balls, labeled boxes, surjective functions

If a function from $[n]$ to $[m]$ is surjective, then every element of $[m]$ has a nonempty preimage, and because we have a function, the collection of preimages forms a set partition of $[n]$. This means we can describe any such function by first choosing a set partition of $[n]$ of size m, then matching up those m blocks with elements of $[m]$; altogether, there are $\left\{ {n \atop m} \right\} m!$ such functions.

Click to return to the Twelvefold Way table.

2 Unlabeled balls, labeled boxes: compositions

Putting n indistinguishable balls into m distinguishable boxes is the same as forming a composition of n. Allowing empty boxes corresponds to weak compositions; demanding that every box have a ball corresponds to strong compositions. (Many people say "strict composition".)

2.1 Unlabeled balls, labeled boxes, arbitrary functions

As mentioned above, this situation describes weak compositions of n into m parts. The standard "stars and bars" argument shows us that there are $\binom{n+m-1}{m-1}$ such functions.

Click to return to the Twelvefold Way table.

2.2 Unlabeled balls, labeled boxes, injective functions

Demanding that a function be injective corresponds to requiring that at most one ball be in each box. We have n identical balls and can put at most one in each labeled box—such a function describes a subset of $[m]$ of size $[n]$ by simply looking where the balls have been placed, so there are $\binom{m}{n}$ such functions.

Click to return to the Twelvefold Way table.

2.3 Unlabeled balls, labeled boxes, surjective functions

If we need each box to have at least one ball in it, the corresponding compositions have no zero parts—they are strong compositions. We can begin by putting one ball in each box, and then distributing the remaining balls any way we like. This is the same as finding a weak composition of $n - m$ into m parts, so there are $\binom{n-1}{m-1}$ surjective functions.

Click to return to the Twelvefold Way table.

3 Unlabeled balls, unlabeled boxes: integer partitions

We have n total indistinguishable balls, and are putting them somehow into m boxes, which are also indistinguishable and hence are not ordered in any particular way. In other words, all these functions correspond to integer partitions of n into at most m parts.

Click to return to the Twelvefold Way table.

3.1 Unlabeled balls, unlabeled boxes, all functions

As described above, the number of functions here is the number of integer partitions of n into at most m parts. We use $p_k(n)$ for the number of partitions of n into *exactly* k parts, so we need to sum: there are $\sum_{k \leqslant m} p_k(n)$ functions.

Click to return to the Twelvefold Way table.

3.2 Unlabeled balls, unlabeled boxes, injective functions

We can put at most one ball into each box, and the balls and boxes are all indistinguishable. There's only one thing to do in this case: place a ball in a box, then place another ball in a different box... and so on, until you run out of balls or boxes. There's one way to do this if $n \leqslant m$, and zero ways otherwise: there are $[n \leqslant m]$ injective functions.

Click to return to the Twelvefold Way table.

3.3 Unlabeled balls, unlabeled boxes, surjective functions

In this context, a surjective function is one for which there is at least one ball in each box, so the number of surjective functions is the same as the number of partitions of n into exactly m parts, which we denote by $p_m(n)$.

Click to return to the Twelvefold Way table.

4 Labeled balls, unlabeled boxes: set partitions

Putting labeled balls into unlabeled boxes is the same as forming some kind of set partition of $[n]$. The balls are the elements of the set, and the boxes are the blocks of the set partition.

4.1 Labeled balls, unlabeled boxes, all functions

We have a set partition of $[n]$, and since we can use any number of boxes, the set partition has at most m blocks. As in subsection 3.1, we need to use a sum to express this: the total number of functions is $\sum_{k \leqslant m} \left\{ {n \atop k} \right\}$.

Click to return to the Twelvefold Way table.

4.2 Labeled balls, unlabeled boxes, injective functions

This is very similar to subsection 3.2: we can put at most one ball into each box, and since the boxes are unlabeled, having labels on the balls doesn't give us anything new—if you have enough boxes, there's one way to do this, otherwise, there are zero ways. There are $[n \leqslant m]$ injective functions.

Click to return to the Twelvefold Way table.

4.3 Labeled balls, unlabeled boxes, surjective functions

We know that distributing labeled balls into unlabeled boxes describes a set partition. Here, since we must use every box, we know that the set partition has exactly m blocks, so there are $\left\{ {n \atop m} \right\}$ surjective functions.

Click to return to the Twelvefold Way table.

5 Things to think about

Here are some questions that naturally occur to me, and would be good practice to work out:

- Add a fourth column to the table for bijective functions and fill it in.
- Another way to think about what it means to have a collection of n indistinguishable objects is to think of it as a multiset with n copies of one element. What if, instead of just having n copies of one element, we had a generic multiset with, say, n_i copies of element i, for $1 \leqslant i \leqslant k$? Choose a row in the Twelvefold Way table (except the top row, obviously, since both the balls and boxes are labeled) and find counting formulas for this more general situation.
- Why did I put the rows in the order I did? Some people might say that the natural order of the rows would start with LL and end with UU (or vice versa, maybe), but I have UU in the middle. What rule did I use to choose the order of the rows?
- Related to the above question, note that some of the "vertical transitions" are very simple: if you start in the "LL-surjective" cell, there are $m!\left\{ {n \atop m} \right\}$ functions. If you move up one cell and wrap around to the bottom, you get the same thing except without $m!$, because the boxes have gone from labeled to unlabeled. Can you come up with explanations for any other pairs of boxes that are next to one other?

Dan Drake (drake@kaist.edu)