2009/2010 SEMESTER 1 MID-TERM TEST

MA1505 MATHEMATICS I

29 September 2009

8:30pm to 9:30pm

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY:

1. This test paper consists of **TEN (10)** multiple choice questions and comprises **Twelve (12)** printed pages.

2. Answer all 10 questions. 1 mark for each correct answer. No penalty for wrong answers. Full mark is 10.

3. All answers (Choices A, B, C, D, E) are to be submitted using the pink form (FORM CC1).

4. Use **only 2B pencils** for FORM CC1.

5. On FORM CC1 (section B), **write** your **matriculation number** and **shade** the corresponding numbered circles **completely**. Your FORM CC1 will be graded by a computer and it will record a **ZERO** for your score if your matriculation number is not correct.

6. Write your full name in section A of FORM CC1.

7. Only circles for answers 1 to 10 are to be shaded.

8. For each answer, the circle corresponding to your choice should be **properly** and **completely** shaded. If you change your answer later, you must make sure that the original answer is properly erased.

9. For each answer, **do not shade more than one circle**. The answer for a question with more than one circle shaded will be marked wrong.

10. **Do not fold** FORM CC1.

11. Submit FORM CC1 before you leave the test hall.

Formulae List

1. The **Taylor series** of f at a is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

2.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

3.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

4.

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

5.

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$$

6.

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

1. Let $f(x) = (2 - \cos x)^{\frac{x}{\pi}}$. Find $f'(\pi)$.

(A) $\frac{1}{\pi}\ln 3$

(B) $\ln 3$

(C) $\frac{1}{\pi}\ln 27$

(D) $\frac{3}{\pi}\ln 2$

(E) None of the above

2. A curve (called a deltoid) has parametric equations

$$x = 2\cos t + \cos 2t$$

$$y = 2\sin t - \sin 2t,$$

where $0 \le t \le 2\pi$. Let L denote the tangent line to this curve at the point where $t = \frac{\pi}{4}$. Find the x-coordinate of the point of intersection of L with the line $y = -1$.

(A) $2 + \sqrt{2}$

(B) $2\sqrt{2} + 2$

(C) $2 - \sqrt{2}$

(D) $2\sqrt{2} - 2$

(E) None of the above

3. In a certain problem, two quantities x and y are related by the equation

$$y = 20x^2 - x^3 + 1505.$$

It is known that x is increasing at a rate of 3 units per second. Find the rate of change of y when x is equal to 10 units.

(A) Increasing at 300 units per second

(B) Increasing at 330 units per second

(C) Increasing at 200 units per second

(D) Increasing at 250 units per second

(E) None of the above

4. Let a be a positive constant. Let M and m denote the absolute maximum value and absolute minimum value respectively of the function

$$f(x) = x^2 + \frac{2a^3}{x},$$

in the domain $\left[\frac{a}{2}, \frac{4a}{3}\right]$. Find $\frac{M}{m}$.

(A) $\frac{59}{54}$

(B) $\frac{153}{118}$

(C) $\frac{21}{16}$

(D) $\frac{17}{12}$

(E) None of the above

5. Evaluate

$$\int_0^{\frac{\pi}{3}} |\cos^3 2x| \; dx$$

(A) $\frac{2\pi}{9} - \frac{5\sqrt{3}}{24}$

(B) $\frac{2}{3} - \frac{\sqrt{3}}{16}\pi$

(C) $\frac{2}{3} - \frac{11\sqrt{3}}{56}$

(D) $\frac{2}{3} - \frac{3\sqrt{3}}{16}$

(E) None of the above

6. Find the area of the finite region bounded by the curves

$y^2 + 4x = 0$ and $2x + y + 4 = 0$.

(A) $\frac{22}{3}$

(B) 9

(C) 7

(D) $\frac{25}{3}$

(E) None of the above

7. Find

$$\int \frac{1}{\sqrt{1+e^x}}dx.$$

(A) $\frac{1}{2}\ln\frac{\sqrt{1+e^x}+1}{\sqrt{1+e^x}-1}+C$

(B) $\frac{1}{2}\ln\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}+C$

(C) $\ln\frac{\sqrt{1+e^x}+1}{\sqrt{1+e^x}-1}+C$

(D) $\ln\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}+C$

(E) None of the above

8. A finite region R is bounded by the curves $y = 2 - x^2$ and $y = x^2$. Find the volume of the solid formed by revolving R one complete round about the x-axis.

(A) $\frac{16\pi}{3}$

(B) $\frac{64\pi}{15}$

(C) $\frac{15\pi}{8}$

(D) $\frac{3\pi}{2}$

(E) None of the above

9. Let $f(x) = \ln\left(1 + x + x^2 + x^3\right)$ and

$$\sum_{n=0}^{\infty} c_n x^n$$

be the Taylor series of f at $x = 0$. Then the value of $c_{2009} + c_{2010}$ is

(A) $\frac{1}{2009} + \frac{1}{2010}$

(B) $\frac{1}{2009} - \frac{1}{2010}$

(C) $-\frac{1}{2009} + \frac{1}{2010}$

(D) $-\frac{1}{2009} - \frac{1}{2010}$

(E) None of the above

10. Find the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \left(\frac{5^n + (-1)^n}{n^3} \right) (x-2)^n .$$

(A) 6

(B) $\frac{1}{3}$

(C) $\frac{1}{2}$

(D) 5

(E) None of the above

END OF PAPER

National University of Singapore

Department of Mathematics

2008-2009 Semester 1 MA1505 Mathematics I Mid-Term Test Answers

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Question	1	2	3	4	5	6	7	8	9	10
Answer	C	B	A	D	D	B	D	A	A	E

2009 Test Solutions

1). C

$$f(x) = (2-\cos x)^{\frac{x}{\pi}}$$

$$\ln f(x) = \frac{x}{\pi}\ln(2-\cos x)$$

$$\frac{f'(x)}{f(x)} = \frac{1}{\pi}\ln(2-\cos x) + \frac{x}{\pi}\,\frac{1}{2-\cos x}(\sin x)$$

$$f'(x) = f(x)\left\{\frac{1}{\pi}\ln(2-\cos x) + \frac{x\sin x}{\pi(2-\cos x)}\right\}$$

$$f'(\pi) = f(\pi)\left\{\frac{1}{\pi}\ln(2-\cos\pi) + \frac{\pi\sin\pi}{\pi(2-\cos\pi)}\right\}$$

$$= 3\left\{\frac{1}{\pi}\ln 3\right\}$$

$$= \frac{3}{\pi}\ln 3 = \underline{\underline{\frac{1}{\pi}\ln 27}}$$

2). B

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos t - 2\cos 2t}{-2\sin t - 2\sin 2t}$$

$$t = \frac{\pi}{4} \Rightarrow \frac{dy}{dx} = \frac{\sqrt{2}}{-\sqrt{2}-2}, \quad x = \sqrt{2}, \quad y = \sqrt{2}-1$$

$$-1-(\sqrt{2}-1) = \frac{\sqrt{2}}{-\sqrt{2}-2}(x-\sqrt{2})$$

$$\sqrt{2}+2 = x-\sqrt{2}$$

$$\underline{\underline{x = 2\sqrt{2}+2}}$$

3). A

$$y = 20x^2 - x^3 + 1505$$

$$\frac{dy}{dt} = 40x\frac{dx}{dt} - 3x^2\frac{dx}{dt}$$

$$x=10,\ \frac{dx}{dt} = 3 \Rightarrow \frac{dy}{dt} = 1200 - 900 = \underline{\underline{300}}$$

4). D

$$f(x) = x^2 + \frac{2a^3}{x}, \quad x \in \left[\frac{a}{2}, \frac{4a}{3}\right]$$

$$f'(x) = 2x - \frac{2a^3}{x^2} = \frac{2x^3 - 2a^3}{x^2} = \frac{2(x-a)(x^2+ax+a^2)}{x^2}$$

Only one critical point $x = a \in \left[\frac{a}{2}, \frac{4a}{3}\right]$

$$f\left(\frac{a}{2}\right) = \frac{a^2}{4} + 4a^2 = \frac{17}{4}a^2$$

$$f(a) = a^2 + \frac{2a^3}{a} = 3a^2$$

$$f\left(\frac{4a}{3}\right) = \frac{16a^2}{9} + \frac{3}{2}a^2 = \frac{59}{18}a^2$$

$$\therefore M = \frac{17}{4}a^2, \quad m = 3a^2$$

$$\frac{M}{m} = \underline{\underline{\frac{17}{12}}}$$

5). D.

$$\int_0^{\frac{\pi}{3}} |\cos^3 2x|\,dx$$

$$= \int_0^{\frac{\pi}{4}} |\cos^3 2x|\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} |\cos^3 2x|\,dx$$

$$= \int_0^{\frac{\pi}{4}} \cos^3 2x\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (-\cos^3 2x)\,dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \cos^2 2x\,d(\sin 2x) - \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos^2 2x\,d(\sin 2x)$$

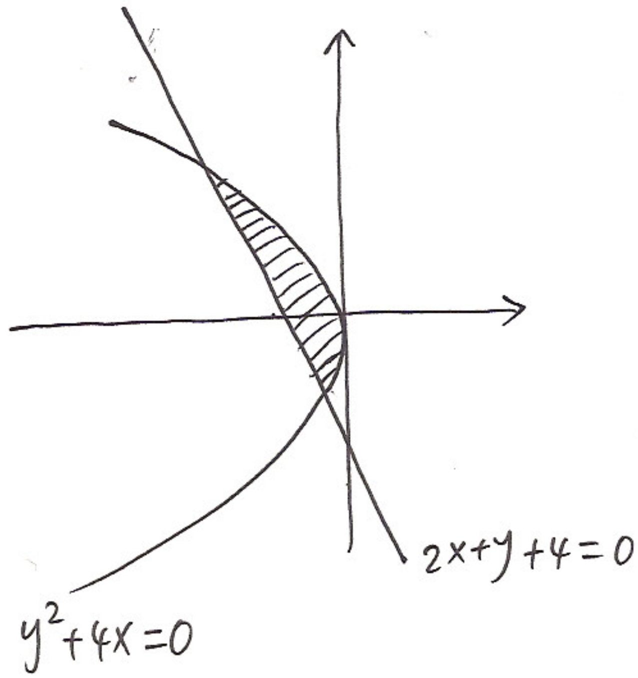
$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} (1-\sin^2 2x)\,d(\sin 2x) - \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (1-\sin^2 2x)\,d(\sin 2x)$$

$$= \frac{1}{2}\left[\sin 2x - \frac{1}{3}\sin^3 2x\right]_0^{\frac{\pi}{4}} - \frac{1}{2}\left[\sin 2x - \frac{1}{3}\sin^3 2x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \frac{1}{2}\left[1 - \frac{1}{3}\right] - \frac{1}{2}\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{8} - \left(1 - \frac{1}{3}\right)\right)$$

$$= \frac{2}{3} - \frac{3\sqrt{3}}{16}$$

6). B



$$\begin{cases} y^2+4x=0 \\ 2x+y+4=0 \end{cases} \Rightarrow y^2-2y-8=0$$

$$(y-4)(y+2)=0$$

$$\text{Area} = \int_{-2}^{4}\left[-\frac{1}{4}y^2-\left\{\frac{1}{2}(-y-4)\right\}\right]dy$$

$$= \left[-\frac{1}{12}y^3+\frac{1}{4}y^2+2y\right]_{-2}^{4}$$

$$= \left(-\frac{16}{3}+4+8\right)-\left(\frac{2}{3}+1-4\right)$$

$$= \underline{\underline{9}}$$

7). D

$$I = \int \frac{1}{\sqrt{1+e^x}} dx$$

Let $u = \sqrt{1+e^x}$

$$u^2 = 1+e^x \Rightarrow 2u\,du = e^x dx$$

$$\Rightarrow dx = \frac{2u\,du}{e^x} = \frac{2u\,du}{u^2-1}$$

$$\therefore I = \int \frac{1}{u}\left(\frac{2u\,du}{u^2-1}\right) = \int \frac{2u}{(u+1)(u-1)} du$$

$$= \int \left(\frac{1}{u-1} - \frac{1}{u+1}\right) du$$

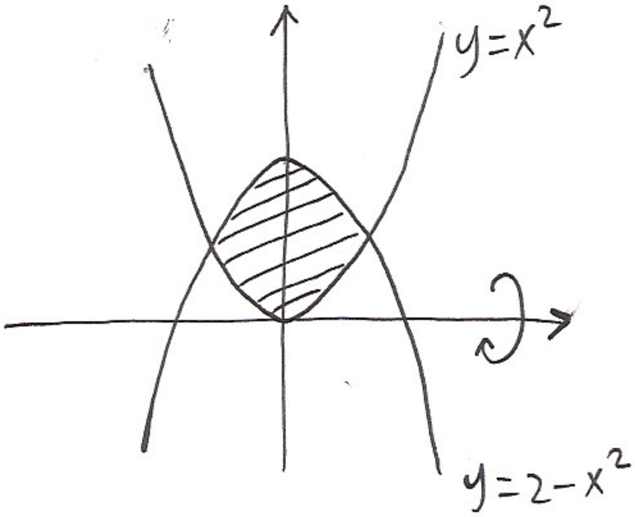
$$= \ln|u-1| - \ln|u+1| + C$$

$$= \ln(u-1) - \ln(u+1) + C \quad (\because u>1)$$

$$= \ln \frac{u-1}{u+1} + C$$

$$= \ln \frac{\sqrt{1+e^x} - 1}{\sqrt{1+e^x} + 1} + C$$

8). A



$$\begin{cases} y = x^2 \\ y = 2 - x^2 \end{cases} \Rightarrow x^2 = 2 - x^2 \Rightarrow x = \pm 1$$

$$Vol = \pi \int_{-1}^{1} \left[(2-x^2)^2 - (x^2)^2 \right] dx$$

$$= \pi \int_{-1}^{1} \left[4 - 4x^2 + x^4 - x^4 \right] dx$$

$$= \pi \left[4x - \frac{4}{3} x^3 \right]_{-1}^{1}$$

$$= \frac{16}{3} \pi$$

9). A

Observe that $1+x+x^2+x^3$ is a geometric progression and its sum is

$$\frac{1-x^4}{1-x}$$

$$\therefore f(x) = \ln(1+x+x^2+x^3)$$

$$= \ln \frac{1-x^4}{1-x}$$

$$= \ln(1-x^4) - \ln(1-x)$$

$$= \ln\{1+(-x^4)\} - \ln\{1+(-x)\}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-x^4)^n}{n} - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-x)^n}{n}$$

$$= \sum_{n=1}^{\infty} \frac{-x^{4n}}{n} + \sum_{n=1}^{\infty} \frac{x^n}{n}$$

Observe that both 2009 and 2010 are not divisible by 4,

$$\therefore C_{2009} = \frac{1}{2009}, \quad C_{2010} = \frac{1}{2010}$$

$$\therefore C_{2009} + C_{2010} = \frac{1}{2009} + \frac{1}{2010}$$

10). E

$$\sum_{n=1}^{\infty} \left(\frac{5^n + (-1)^n}{n^3}\right)(x-2)^n$$

$$\lim_{n\to\infty} \left| \frac{\frac{5^{n+1} + (-1)^{n+1}}{(n+1)^3}(x-2)^{n+1}}{\frac{5^n + (-1)^n}{n^3}(x-2)^n} \right|$$

$$= \lim_{n\to\infty} \left| \left(\frac{5 + \frac{(-1)^{n+1}}{5^n}}{1 + \frac{(-1)^n}{5^n}} \right) \cdot \left(\frac{n}{n+1}\right)^3 \cdot (x-2) \right|$$

$$= 5|x-2|$$

$$\therefore\ 5|x-2| < 1 \Rightarrow |x-2| < \underline{\underline{\frac{1}{5}}}$$