Chapter 7
Functions of Several Variables

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Overview

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Overview

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Introduction

Introduction

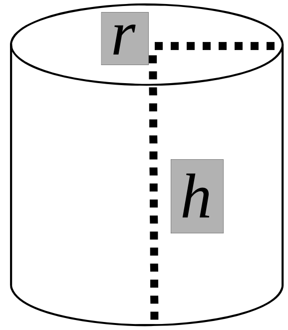
Objective:

To extend some methods of single-variable differential calculus to functions of several variables.

In many practical situations, the value of a quantity may depend on more than one variable.

Introduction - Example

$$\boxed{V = \pi r^2 h}$$



Output of a factory

------ amount of capital invested and the size of manpower.

Current in electrical circuit

------ capacitance, electromotive force,

impedance and resistance in the circuit.

Functions of 2 Variables

$f(x, y)$ ------ a rule that assigns to each (x, y) a real number $f(x, y)$, where x and y are real.

$z = f(x, y)$ ------ z is a function of x and y

x and y ------ independent variables

z ------ dependent variables

In general, $z = f(x_1, x_2, \cdots, x_n)$ ------- function of n variables

Domain of 2 Variables

$$\text{Domain of } f = D_f$$
$$= \{\, (x, y) \mid f(x, y) \text{ is defined}\}$$

Domain of 2 Variables - Example

Let $f(x, y) = 3 + x \sin y - x^2 y^5$.

$$D_f = \{(x, y) \mid x, y \text{ are real}\}$$

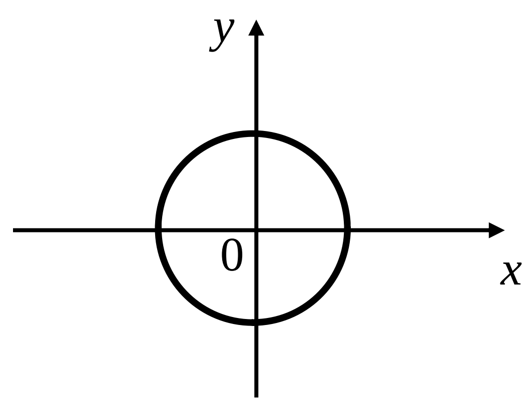
Domain of 2 Variables - Example

Let $f(x, y) = \sqrt{x^2 + y^2 - 1}.$

$\sqrt{\text{Positive}}$

$x^2 + y^2 - 1 \geq 0$

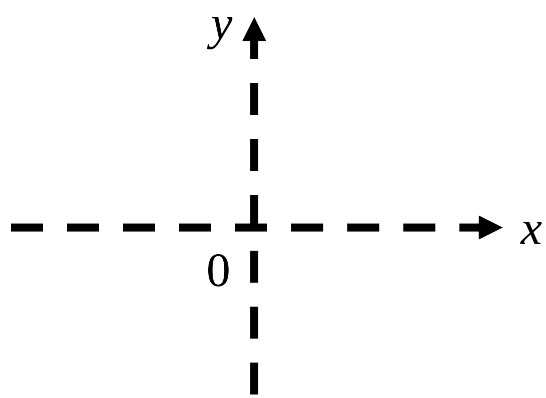
$$D_f = \{(x, y) \mid x^2 + y^2 \geq 1\}$$



Domain of 2 Variables - Example

Let $f(x, y) = \dfrac{1}{xy}$.

$$D_f = \{(x, y) \mid x \neq 0 \text{ and } y \neq 0\}$$



Geometric Representation

Geometric Representation

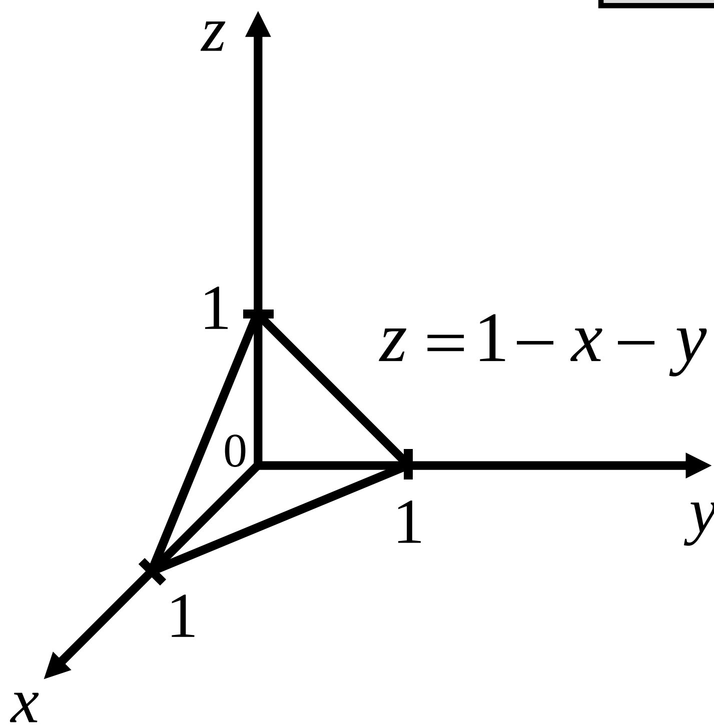
$$\boxed{y = f(x) \text{ ------ a curve in } xy\text{-plane}}$$

$$\boxed{z = f(x, y) \text{ ------ a surface in 3-D space}}$$

Geometric Representation - Example

$$\boxed{z = 1 - x - y}$$

$$\boxed{x + y + z = 1}$$



Cartesian Equation of plane:

$$ax + by + cz = d, \text{ where } d = ax_0 + by_0 + cz_0.$$

Geometric Representation - Example

$\boxed{z = f(x, y) \text{ ------ a surface in 3-D space}}$

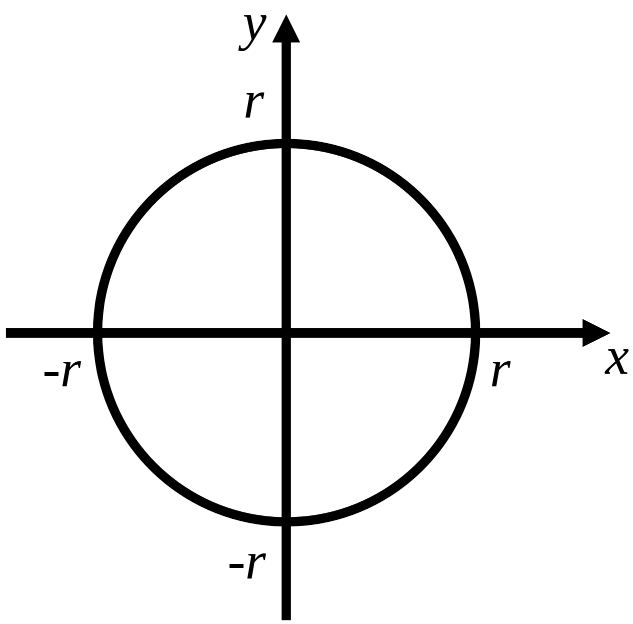
Pause and Think !!!

Question: How to "plot" the surface

$$\boxed{z = x^2 + y^2}$$

$$x^2 + y^2 = r^2$$

circle center (0,0) with radius r.



Geometric Representation - Example

$z = f(x, y)$ ------ a surface in 3-D space

Pause and Think !!!

Question: How to "plot" the surface

$$z = x^2 + y^2$$

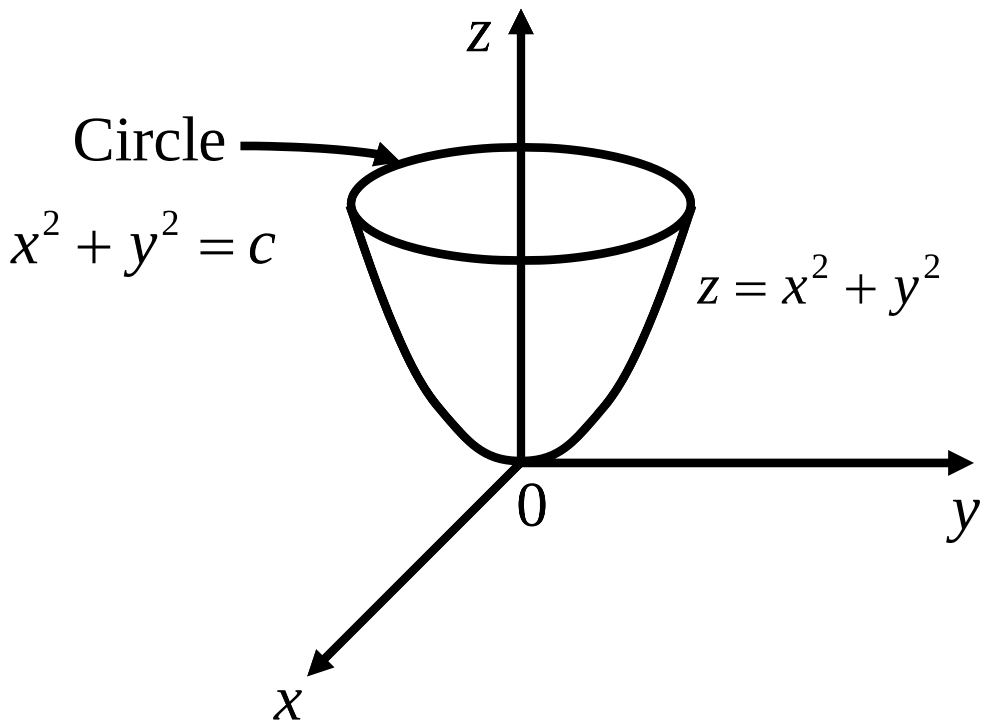
Fix $z = 1$ $\quad$ $1 = x^2 + y^2$

Fix $z = 2$ $\quad$ $2 = x^2 + y^2$

Circles

Geometric Representation - Example

$$\boxed{z = x^2 + y^2}$$

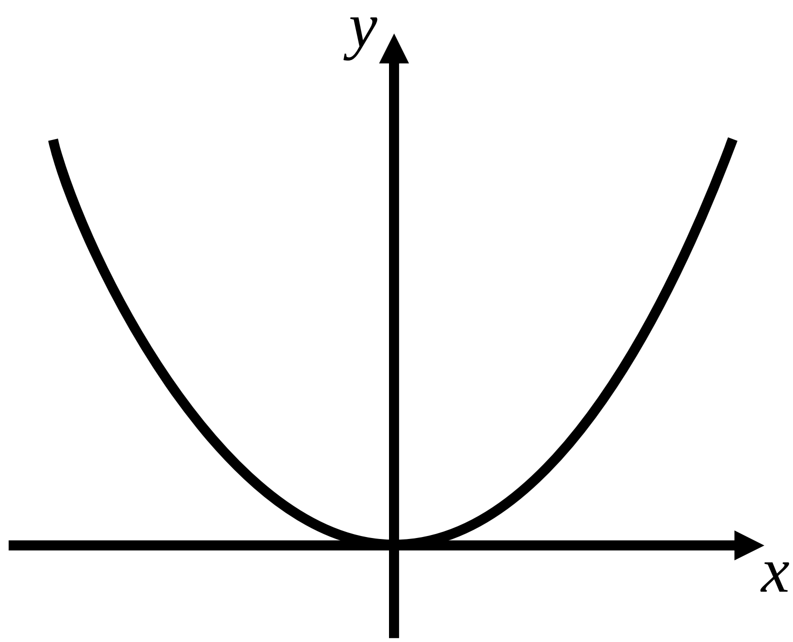


For a fix value of z, we get a circle

Any plane parallel to the $xy-$plane intersects the surface ------ a circle

$y = x^2$

parabola



Geometric Representation - Example

$z = f(x, y)$ ------ a surface in 3-D space

Pause and Think !!!

Question: How to "plot" the surface

$$\boxed{z = x^2 + y^2}$$

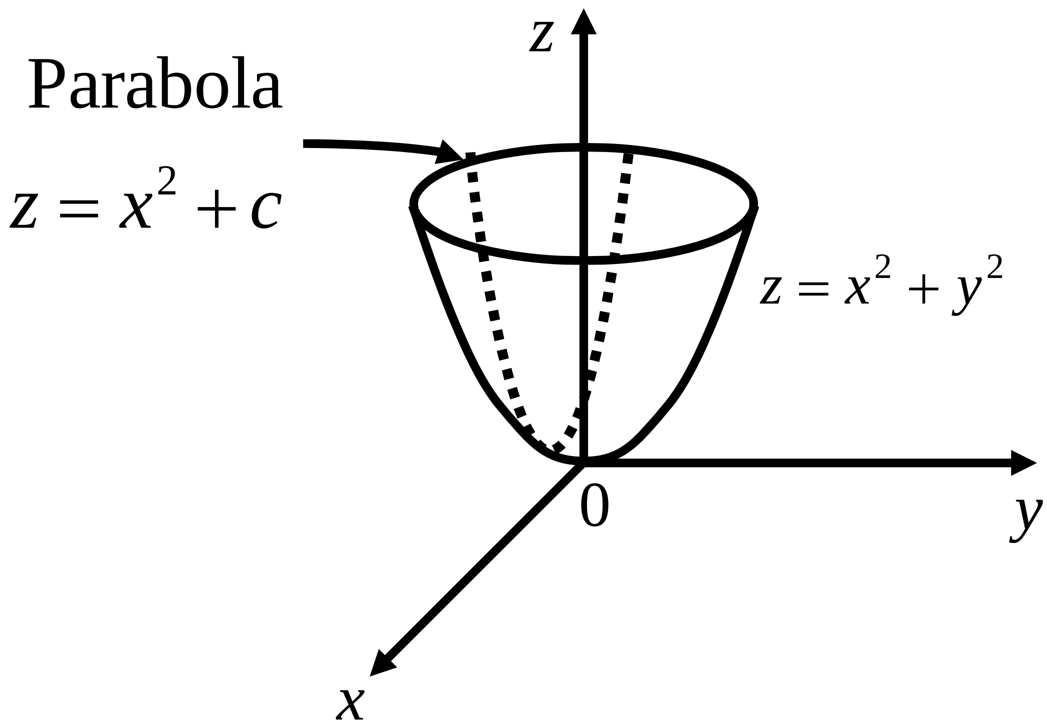
Fix $y = 0$ $z = x^2$

Fix $y = 1$ $z = x^2 + 1$

Parabola

Geometric Representation - Example

$$\boxed{z = x^2 + y^2}$$



For a fix value of y, we get a parabola

Any plane parallel to $xz-$plane intersects the surface ------ a parabola

Geometric Representation - Example

$$z = 9 - x^2 - y^2$$

Fix $z = 0$ $\quad x^2 + y^2 = 9$

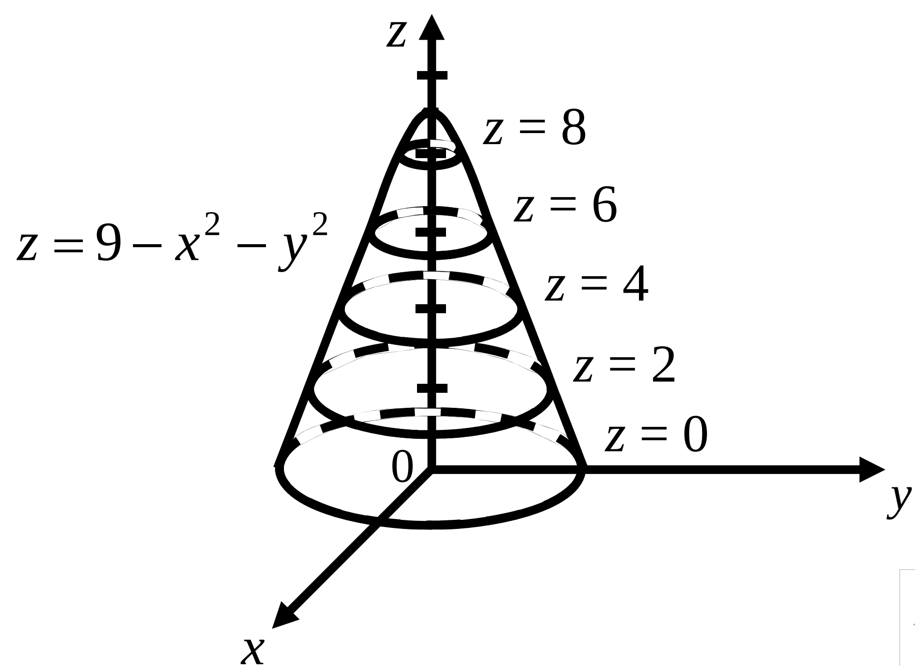
Fix $z = 2$ $\quad x^2 + y^2 = 7$

Circles

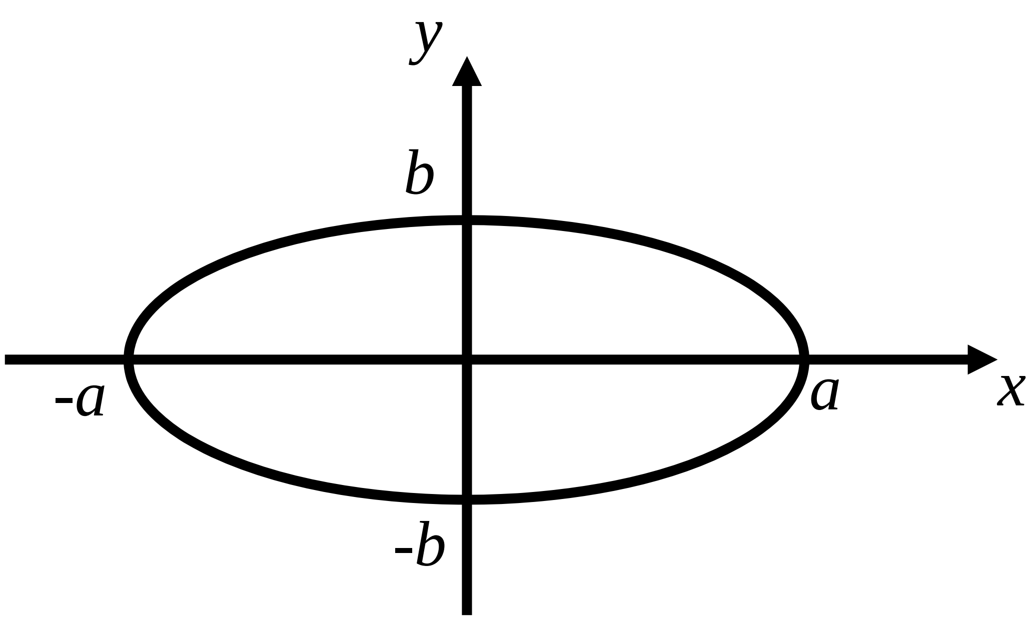
Fix $y = 0$ $\quad z = 9 - x^2$

Parabola

Inverted

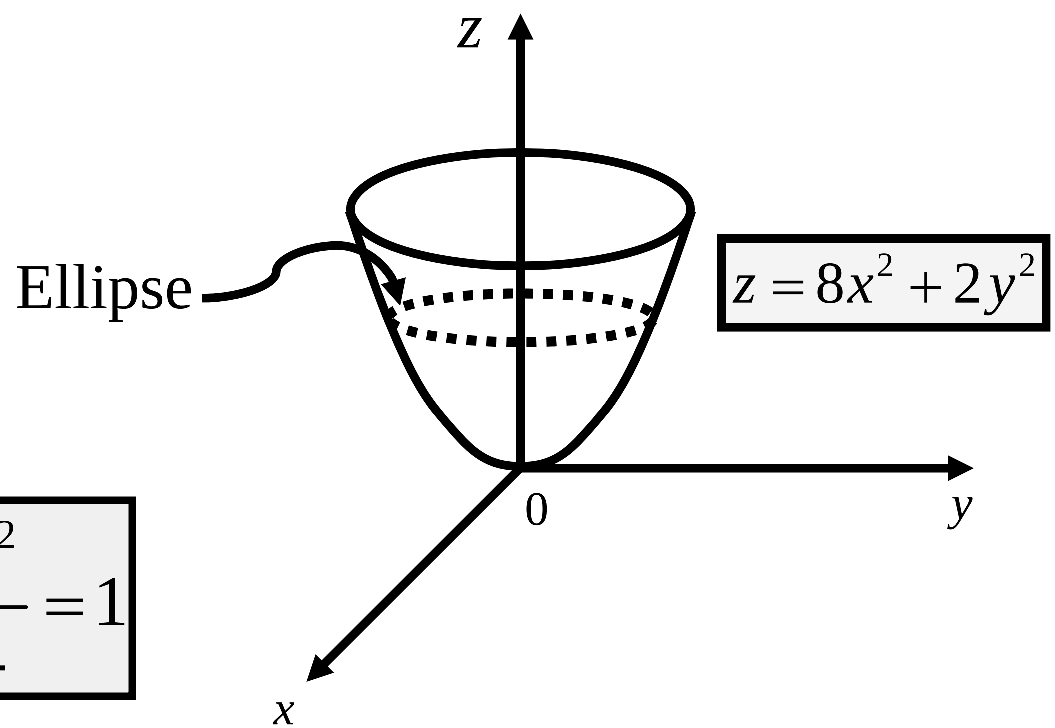


$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{ellipse}$$



Geometric Representation - Example

$$\boxed{z = 8x^2 + 2y^2}$$



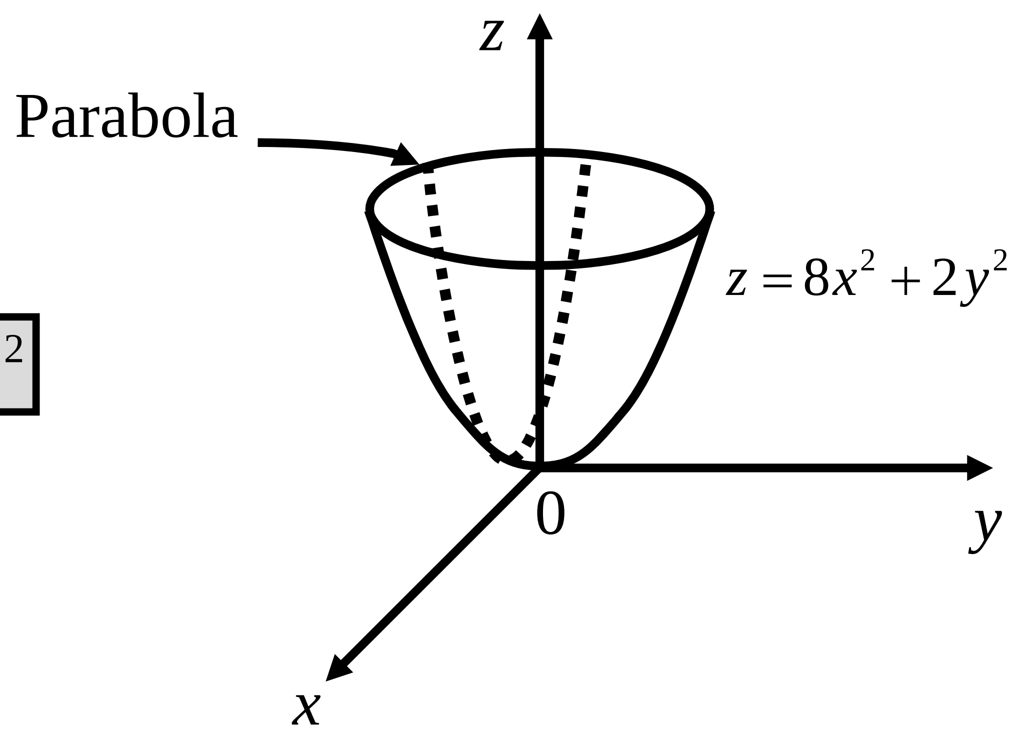
$\boxed{\text{Fix } z = 8}$ $\boxed{x^2 + \frac{y^2}{4} = 1}$

For a fix value of z, we get an ellipse

Any plane parallel to $xy -$ plane intersects the surface ------ an ellipse

Geometric Representation - Example

$$\boxed{z = 8x^2 + 2y^2}$$



Fix $y = 0$ $\quad z = 8x^2$

Parabola

For a fix value of y, we get a parabola

Any plane parallel to $xz-$plane intersects the surface ------ a parabola

Partial Derivatives

Let $f(x, y) = x^2 - 2xy + 3y^3$.

If we fix $y = 2$, we get :

$f(x, 2) = x^2 - 4x + 24$ a function in x

We may think of $f(x, 2)$ as $g(x) = x^2 - 4x + 24$

We may find $g'(x) = 2x - 4$

Let $f(x, y) = x^2 - 2xy + 3y^3$.

If we fix $x = -1$, we get :

$$f(-1, y) = 1 + 2y + 3y^3$$

a function in y

We may think of $f(-1, y)$ as $h(y) = 1 + 2y + 3y^3$

We may find $h'(y) = 2 + 9y^2$

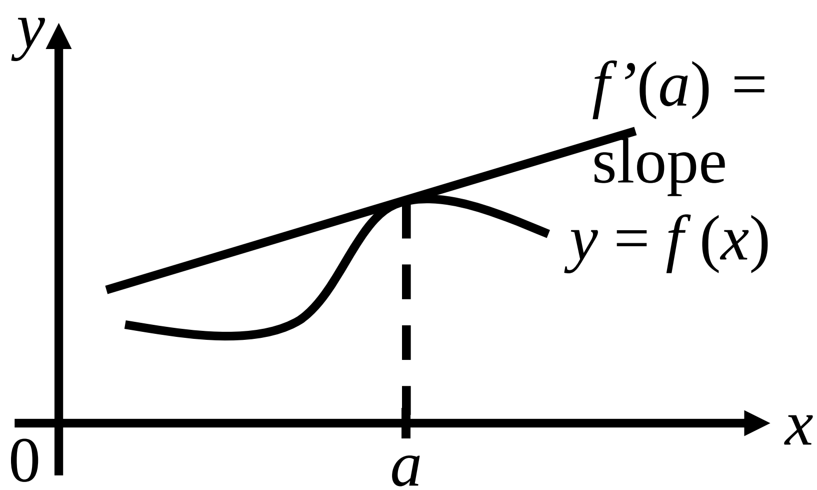
Partial Derivatives

Given $f(x, y)$, to find its derivative $w.r.t$ one of the 2 variables when the other is held constant.

Partial Derivatives

Recall that for a single variable function $f(x)$,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$



Partial Derivatives

The ***partial derivative*** of f $w.r.t$ x at (a,b) is defined as

$$\lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h}$$

With respect to x, we fix $y=b$

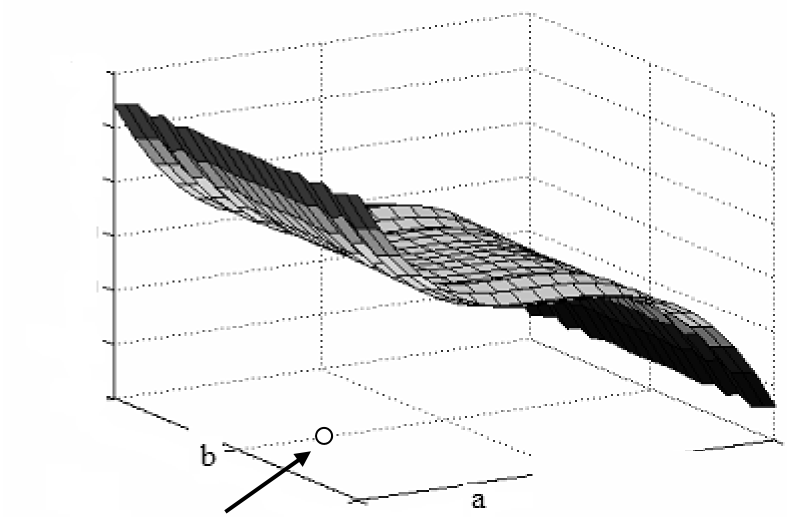
We write

$$\left.\frac{\partial f}{\partial x}\right|_{(a,b)} = f_x(a,b) = \lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h}$$

if the limit exists.

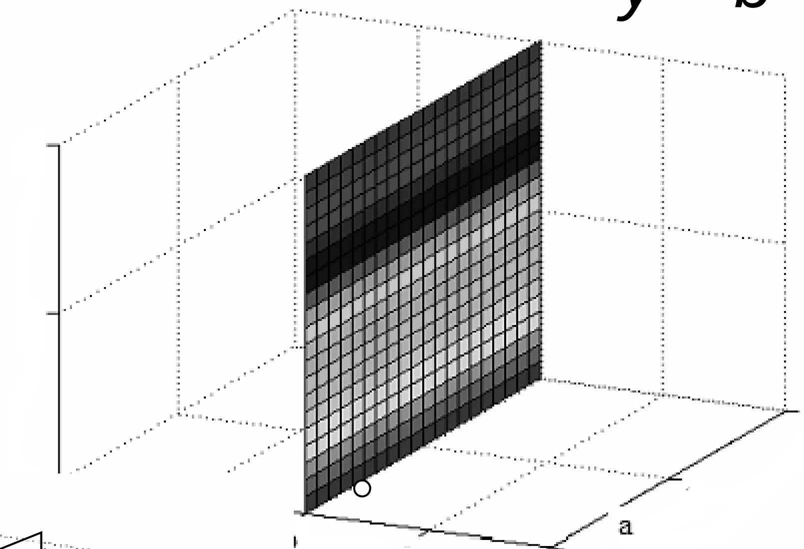
$\boxed{\boldsymbol{f_x(a, b)}}$

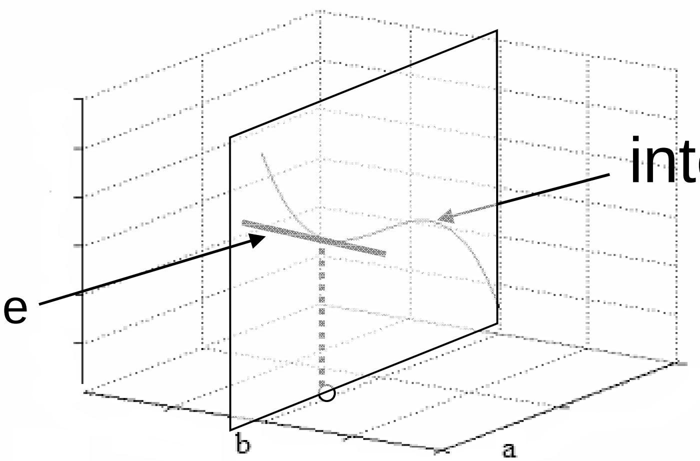
$z = f(x, y)$



$(x,y) = (a,b)$

$y = b$





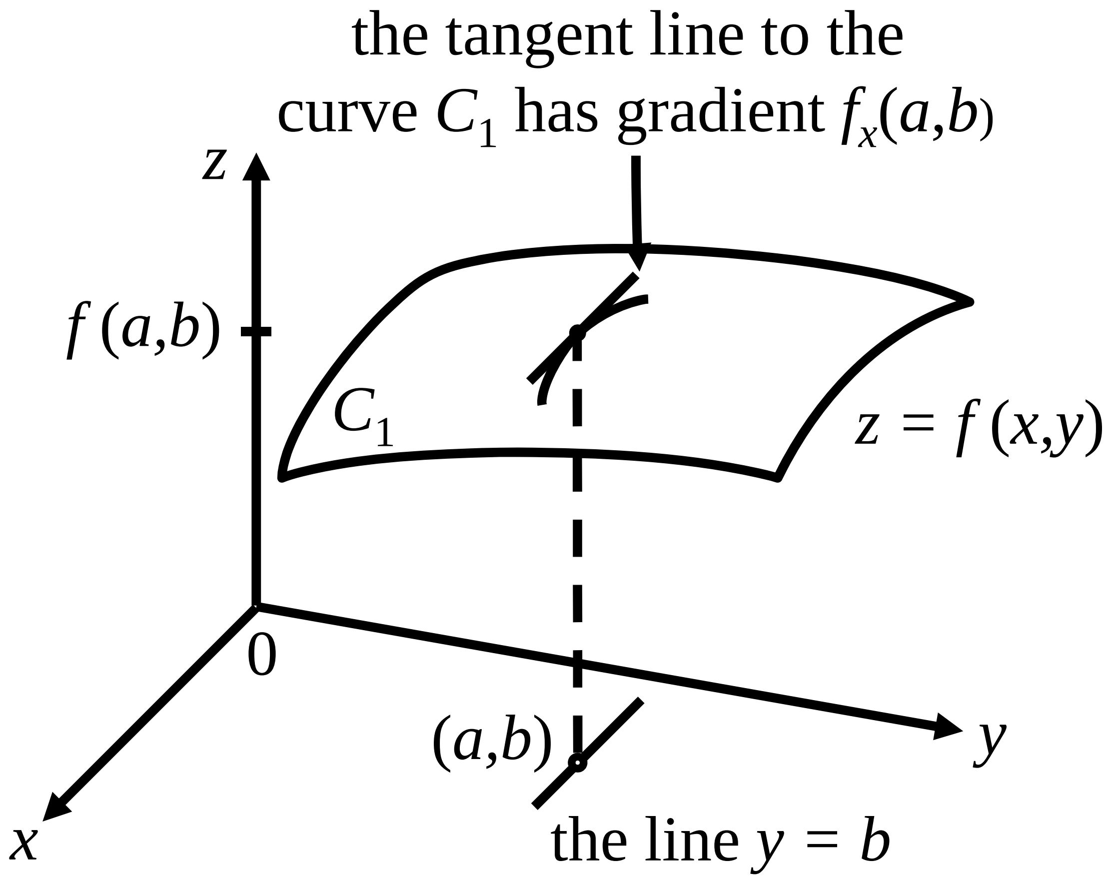
intersection: z = f(x,b)

gradient of the tangent line

$\Downarrow$

$f_x(a, b)$

Geometric Interpretation



Partial Derivatives

Likewise, the ***partial derivative*** of f $w.r.t$ y at (a,b) is defined as

$$\lim_{h\to 0}\frac{f(a,\ b+h\ )-f(a,\ b\ )}{h}$$

With respect to y, we fix $x=a$

We write

$$\left.\frac{\partial f}{\partial y}\right|_{(a,b)}=f_y(a,b)=\lim_{h\to 0}\frac{f(a,b+h)-f(a,b)}{h}$$

if the limit exists.

Partial Derivatives

If $z = f(x, y)$, we also write

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} \quad \& \quad f_y = \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y}$$

Given $z = f(x, y)$, how to compute f_x and f_y ?

Partial Derivatives – Example

Given $z = f(x, y)$, how to compute f_x and f_y ?

Let $f(x,y) = (x^3 + y)\cos(y^2)$.

Find $f_x(2,0)$ and $f_y(2,0)$.

Given $z = f(x, y)$, to compute f_x , we treat y terms as constants.

$$f_x(x,y) = \frac{d}{dx}\Big(\ (x^3 + y)\cos(y^2)\ \Big)$$

$$= \cos(y^2)\ \frac{d}{dx}\Big(\ (x^3 + y)\ \Big)$$

$$\frac{d}{dx}\Big(\ (k\ f(x)\ \Big) = k\frac{d}{dx}\Big(f(x)\Big)$$

$$= \cos(y^2)\ (3x^2 + 0)$$

$$= 3x^2\cos(y^2)$$

Thus, $f_x(2,0) = 3(2)^2\cos(0^2) = 12$

Partial Derivatives – Example

Given $z = f(x, y)$, how to compute f_x and f_y ?

Let $f(x, y) = (x^3 + y)\cos(y^2)$.

Find $f_x(2,0)$ and $f_y(2,0)$.

Given $z = f(x, y)$, to compute f_y , we treat x terms as constants.

$$f_y(x, y) = \frac{d}{dy}\Big(\ (x^3 + y)\cos(y^2) \ \Big)$$

$$= (x^3 + y)\frac{d}{dy}\Big(\cos(y^2)\Big) + \cos(y^2)\frac{d}{dy}\Big(\ (x^3 + y) \ \Big)$$

$$= (x^3 + y)\Big(-\sin(y^2)\Big)2y + \cos(y^2) \ (0 + 1)$$

$$= -2y(x^3 + y)\sin(y^2) + \cos(y^2)$$

Product Rule

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Thus, $\quad f_y(2,0) = 0 + \cos(0^2) = 1$

Note that : $f_x(2,0) \neq f_y(2,0)$

Partial Derivatives – Example

Let $z = x^3 \sin(y^2 + x)$.

Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

$$\frac{\partial z}{\partial x} = \frac{d}{dx}\left(x^3 \sin(y^2 + x) \right)$$

$$= x^3 \cos(y^2 + x) + 3x^2 \sin(y^2 + x)$$

treat y terms as constants

$$\frac{\partial z}{\partial y} = \frac{d}{dy}\left(x^3 \sin(y^2 + x) \right)$$

$$= x^3 \cos(y^2 + x) \cdot (2y)$$

$$= 2x^3 y \cos(y^2 + x)$$

treat x terms as constants

Partial Derivatives – Example

Let $z = e^{xy} \ln y$.

Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

$$\frac{\partial z}{\partial x} = (\ln y)e^{xy} \cdot y$$

treat y terms as constants

$$\frac{\partial z}{\partial y} = e^{xy}\left(\frac{1}{y}\right) + xe^{xy} \ln y$$

treat x terms as constants

Higher Order Partial Derivatives

The ***2nd order partial derivatives*** of f are:

$$f_{xx} = (f_x)_x = \frac{\partial^2 f}{\partial x^2}$$

$$f_{yy} = (f_y)_y = \frac{\partial^2 f}{\partial y^2}$$

$$f_{xy} = (f_x)_y = \frac{\partial^2 f}{\partial y \partial x}$$

$$f_{yx} = (f_y)_x = \frac{\partial^2 f}{\partial x \partial y}$$

Higher Order Partial Derivatives

If $z = f(x, y)$, we also have the following notation:

$$f_{xx} = \frac{\partial^2 z}{\partial x^2}$$

$$f_{yy} = \frac{\partial^2 z}{\partial y^2}$$

$$f_{xy} = \frac{\partial^2 z}{\partial y \partial x}$$

$$f_{yx} = \frac{\partial^2 z}{\partial x \partial y}$$

Higher Order Partial Derivatives

Notation:

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

f_{xy} --- do from left to right

$\dfrac{\partial^2 f}{\partial y \partial x}$ --- do from right to left

Example $\boxed{\text{Let } f(x,y) = 4x^3 + x^2y^3 - 6y^2.}$

---

constants

$$f_x = \frac{\partial}{\partial x}(4x^3 + x^2\overbrace{y^3} - \overbrace{6y^2})$$
$$= 4(3x^2) + 2xy^3 - 0$$
$$= 12x^2 + 2xy^3$$

constants

$$f_{xy} = \frac{\partial}{\partial y}(\overbrace{12x^2} + \overbrace{2x}y^3)$$
$$= 0 + 2x(3y^2)$$
$$= 6xy^2$$

---

constants

$$f_y = \frac{\partial}{\partial y}(\overbrace{4x^3} + \overbrace{x^2}y^3 - 6y^2)$$
$$= 0 + x^2 3y^2 - 12y$$
$$= 3x^2y^2 - 12y$$

constants

$$f_{yx} = \frac{\partial}{\partial y}(3x^2\overbrace{y^2} - \overbrace{12y})$$
$$= 6xy^2 - 0$$
$$= 6xy^2$$

Note that : In this example, we have $f_{xy} = f_{yx}$.

Example

Let $z = f(x, y) = x^3 \sin(y^2 + x)$.

$$f_x = x^3 \cos(y^2 + x) + 3x^2 \sin(y^2 + x)$$

$$f_y = 2x^3 y \cos(y^2 + x)$$

$$f_{xy} = -2x^3 y \sin(y^2 + x) + 6x^2 y \cos(y^2 + x)$$

$$f_{yx} = -2x^3 y \sin(y^2 + x) + 6x^2 y \cos(y^2 + x)$$

Note that : In this example, again we have $f_{xy} = f_{yx}$.

Question :

Is it true that

$$f_{xy}(a,b) = f_{yx}(a,b) \qquad ???$$

Note

Let $f(x, y)$ be a function defined on a region D containing (a,b). If f_x, f_y, f_{xy}, f_{yx} are all continuous in D, then

$$\boxed{f_{xy}(a,b) = f_{yx}(a,b).}$$

Example

Let $f(x,y) = xy + \dfrac{e^y}{y^2+1}$.

Find f_{yx}.

---

needs Quotient Rule to differentiate

$$f_y = \frac{\partial}{\partial y}\left(xy + \left(\frac{e^y}{y^2+1}\right)\right)$$

Difficult !!!

Example

Let $f(x,y) = xy + \dfrac{e^y}{y^2+1}$.

Find f_{yx}.

$$f_{xy} = f_{yx}$$

$$f_x = \frac{\partial}{\partial x}\left(xy + \frac{e^y}{y^2+1}\right)$$

$$= y$$

treated as constant since differentiating with respect to x

Easy !!!

$$f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}(y) = 1$$

$$f_{yx} = f_{xy} = 1$$

Example

Let $f(x, y) = xy^3 + \dfrac{\ln y}{\sin y}$.

Find $f_{yx}(1,\ 3)$.

$$\boxed{f_{xy} = f_{yx}}$$

$$\boxed{f_x = y^3}$$

$$\boxed{f_{xy} = 3y^2}$$

$$\boxed{f_{yx}(1,\ 3) = f_{xy}(1,\ 3) = 27}$$

Remark

For function in three variables $f(x, y, z)$,
we can similarly define :

$$f_x = \frac{\partial f}{\partial x}, \quad f_y = \frac{\partial f}{\partial y} \quad \text{and} \quad f_z = \frac{\partial f}{\partial z}$$

The Chain Rule

Example

Ideal Gas Law: $pV = kT,$ where

p is the pressure of the gas

V is the volume of the gas

k is a constant

T is the temperature of the gas

The Chain Rule - Example

$$\boxed{\text{Ideal Gas Law: } pV = kT}$$

$$p = \frac{kT}{V} \quad \Rightarrow \quad \frac{\partial p}{\partial V} = -\frac{kT}{V^2}$$

$$V = \frac{kT}{p} \quad \Rightarrow \quad \frac{\partial V}{\partial T} = \frac{k}{p}$$

$$T = \frac{pV}{k} \quad \Rightarrow \quad \frac{\partial T}{\partial p} = \frac{V}{k}$$

Pause and Think !!!

What is the value of

$$\frac{dz}{dy}\frac{dy}{dx}\frac{dx}{dz} = \quad ???$$

Answer :

The Chain Rule - Example

$$\boxed{\text{Ideal Gas Law: } pV = kT}$$

$$p = \frac{kT}{V} \qquad \Rightarrow \qquad \frac{\partial p}{\partial V} = -\frac{kT}{V^2}$$

$$V = \frac{kT}{p} \qquad \Rightarrow \qquad \frac{\partial V}{\partial T} = \frac{k}{p}$$

$$T = \frac{pV}{k} \qquad \Rightarrow \qquad \frac{\partial T}{\partial p} = \frac{V}{k}$$

$$\text{Note : } \quad \frac{\partial p}{\partial V} \cdot \frac{\partial V}{\partial T} \cdot \frac{\partial T}{\partial p} = -\frac{kT}{VP} = -1 \quad \text{but} \quad \frac{dz}{dy}\frac{dy}{dx}\frac{dx}{dz} = 1.$$

The Chain Rule

Chain rule for functions of 1 variable

If $w = f(x)$, then w is a function in x.

Suppose $x = g(t)$, then

$$w = f(g(t))$$

is a function in t.

$$w \xrightarrow{\frac{dw}{dx}} x \xrightarrow{\frac{dx}{dt}} t$$

$$\boxed{\frac{dw}{dt} = \frac{dw}{dx} \cdot \frac{dx}{dt}}$$

The Chain Rule

Chain rule for functions of 2 variables

If $z = f(x, y)$, then z is a function in 2 variables x and y.

Suppose $x = x(t)$ and $y = y(t)$, then

$$z = f(x(t), y(t))$$

is a function in one variable t.

From ***two*** variables x and y becomes ***one*** variable t.

We may now find $\dfrac{dz}{dt}$.

Example

Given that $z = 3xy^2 + x^4 y$, where $x = \sin 2t$ and $y = \cos t$.

Find $\dfrac{dz}{dt}$.

$$z = 3xy^2 + x^4 y$$

$$= 3(\sin 2t)(\cos t)^2 + (\sin 2t)^4(\cos t)$$

$$= 3\sin 2t\cos^2 t + \sin^4 2t\cos t$$

Find $\dfrac{dz}{dt}$ by differentiating with respect to t

Example

Given that $z = 3xy^2 + x^4 y$, where $x = \sin 2t$ and $y = \cos t$.

Find $\dfrac{dz}{dt}$.

$$z = 3xy^2 + x^4 y$$

$$= 3(\sin 2t)(\cos t)^2 + (\sin 2t)^4(\cos t)$$

$$= 3\sin 2t \cos^2 t + \sin^4 2t \cos t$$

Instead of writing z in terms of t and find

$\dfrac{dz}{dt}$ by differentiating with respect to t,

we want to have a chain rule instead.

The Chain Rule

Chain rule for functions of more than 1 variable.

$$\boxed{z = f(x,y) \text{ and } x = x(t),\ y = y(t), \text{ so } z \text{ is a function of } t.}$$

$$\begin{tikzcd} & x \arrow[dr, "\frac{dx}{dt}"] & \\ z \arrow[ur, "\frac{\partial z}{\partial x}"] \arrow[dr, "\frac{\partial z}{\partial y}"'] & & t \\ & y \arrow[ur, "\frac{dy}{dt}"'] & \end{tikzcd}$$

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}}$$

Example

Given that $z = 3xy^2 + x^4 y$, where $x = \sin 2t$ and $y = \cos t$.

Find $\dfrac{dz}{dt}$.

$$z = 3xy^2 + x^4 y$$

$$\frac{\partial z}{\partial x} = 3y^2 + 4x^3 y$$

$$\frac{\partial z}{\partial y} = 6xy + x^4$$

$$x = \sin 2t$$

$$\frac{dx}{dt} = 2\cos 2t$$

$$y = \cos t$$

$$\frac{dy}{dt} = -\sin t$$

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}}$$

$$\boxed{\frac{dz}{dt} = (3y^2 + 4x^3 y)(2\cos 2t) + (6xy + x^4)(-\sin t)}$$

The Chain Rule - Example

Let $z = x^2 + xy + y^2$, where $x = \cos t$ and $y = \sin t$.

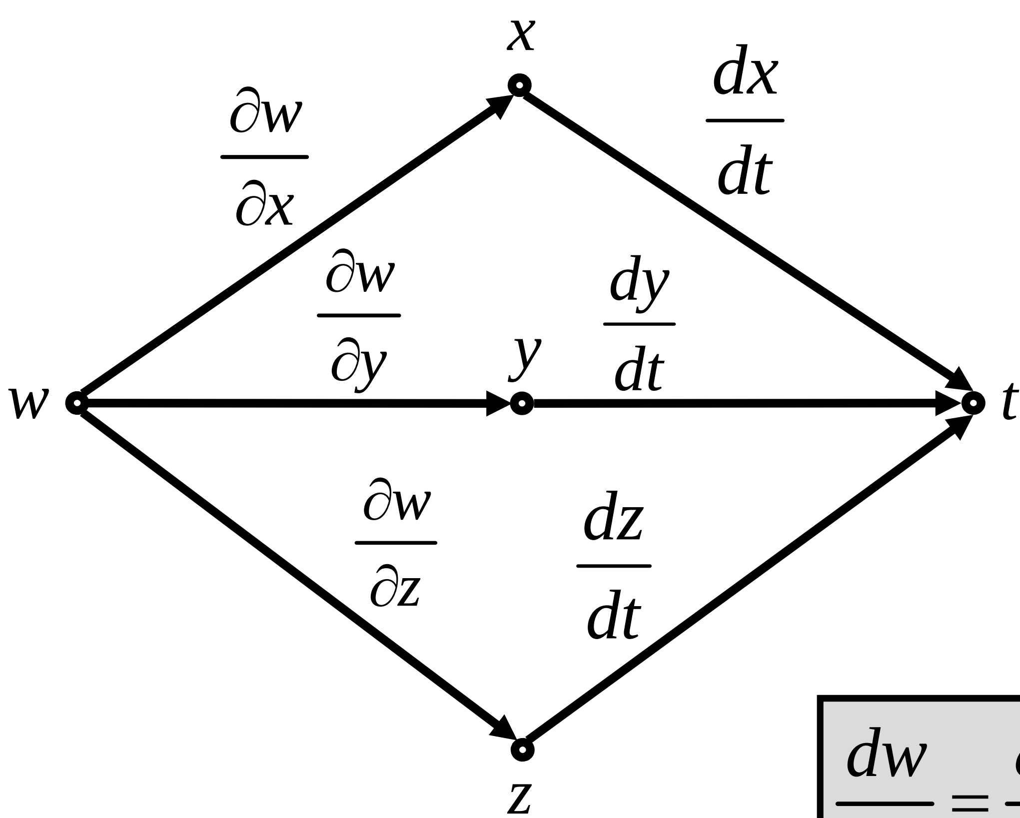
$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= (2x + y)(-\sin t) + (x + 2y)\cos t$$

The Chain Rule

$w = f(x, y, z)$ and $x = x(t),\ y = y(t), z = z(t),$

so w is a function of t.



$$\frac{dw}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}$$

The Chain Rule - Example

Let $w = z - \sin xy$, where $x = t,\ y = \ln t$ and $z = e^{t-1}$.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$

$$= (-\cos xy)\,y\cdot 1 + (-\cos xy)\,x\cdot\frac{1}{t} + 1\cdot e^{t-1}$$

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt}$$

The Chain Rule

If $z = f(x, y)$, then z is a function in 2 variables x and y.

Suppose $x = x(s,t)$ and $y = y(s,t)$, then

$$z = f(x(s,t), y(s,t))$$

is a function in two variables s and t.

$$z(s,t)$$

$$\frac{\partial z}{\partial s} \qquad \frac{\partial z}{\partial t}$$

Example

Let $z = e^{2x}\cos 3y$, where $x = st^2$ and $y = s^2 t$.

Find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$.

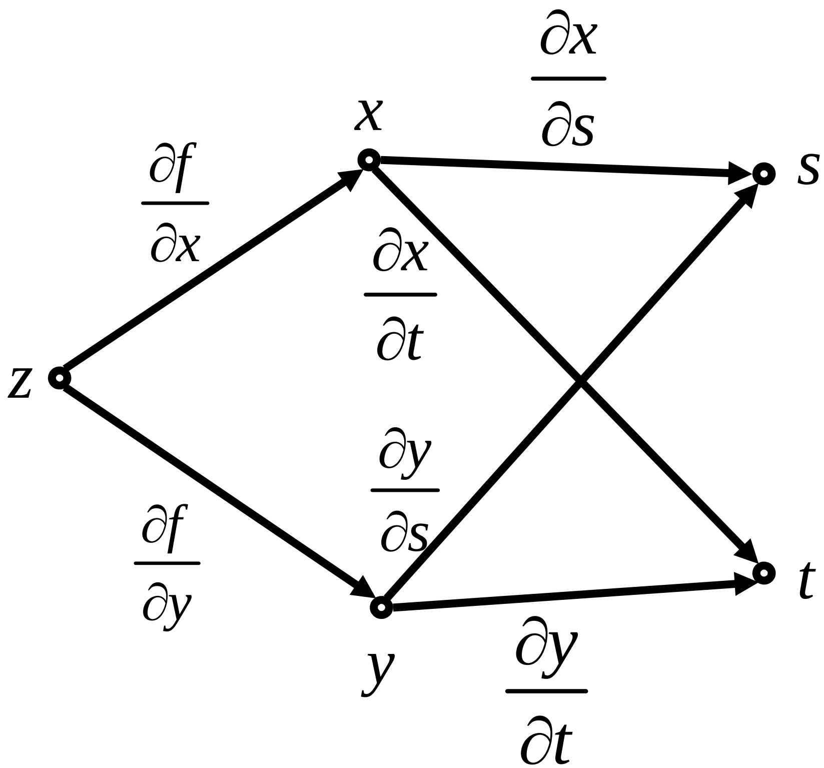
$$
\begin{aligned}
z &= e^{2x}\cos 3y \\
&= e^{2(st^2)}\cos 3(s^2 t) \\
&= e^{2st^2}\cos(3s^2 t)
\end{aligned}
$$

$$\frac{\partial z}{\partial s} \qquad\qquad \frac{\partial z}{\partial t}$$

The Chain Rule

Chain rule for functions of 2 variables

$$\boxed{\begin{array}{l} z = f(x,y) \text{ and } x = x(s,t),\ y = y(s,t), \\ \text{so } z \text{ is a function of } s \text{ and } t. \end{array}}$$



$$\boxed{\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}}$$

and

$$\boxed{\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}}$$

Let $z = e^{2x}\cos 3y$, where $x = st^2$ and $y = s^2 t$.

Find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$
$$= (2e^{2x}\cos 3y)t^2 + (-3e^{2x}\sin 3y)(2st).$$

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}$$

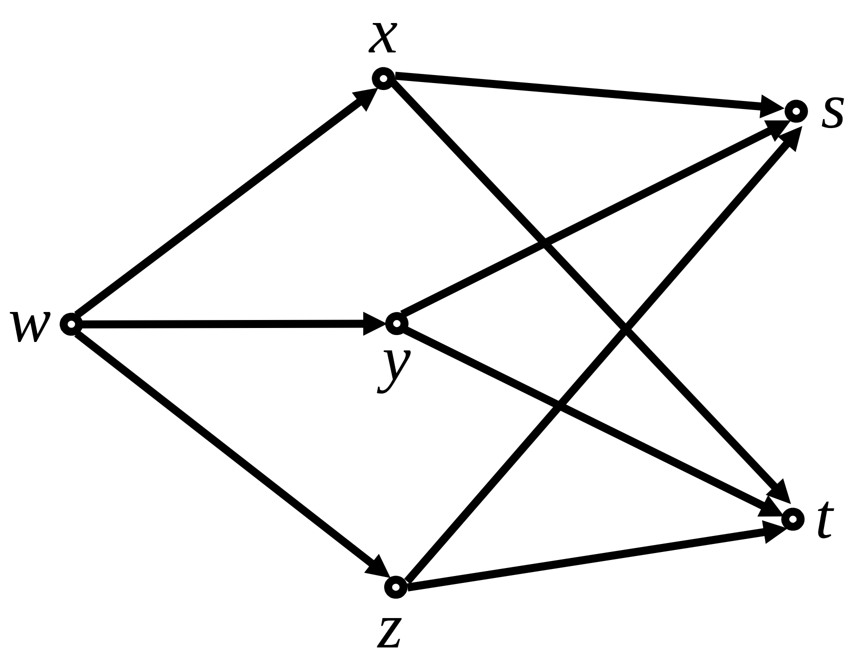
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$
$$= (2e^{2x}\cos 3y)(2st) + (-3e^{2x}\sin 3y)(s^2).$$

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}$$

The Chain Rule

$w = f(x, y, z)$ and $x = x(s,t)$, $y = y(s,t)$, $z = z(s,t)$,

so w is a function of s and t.



$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial s}$$

and

$$\frac{\partial w}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial t}$$

Directional Derivatives

Directional Derivatives

Let $z = f(x, y)$.

$$\left.\frac{\partial f}{\partial x}\right|_{(a,b)} = f_x(a,b) \text{ is the } \boldsymbol{rate\ of\ change} \text{ of } f \text{ w.r.t } x$$

(along direction of x-axis) at (a,b).

$$\left.\frac{\partial f}{\partial y}\right|_{(a,b)} = f_y(a,b) \text{ is the } \boldsymbol{rate\ of\ change} \text{ of } f \text{ w.r.t } y$$

(along direction of y-axis) at (a,b).

Question:

How about the rate of change of f along an arbitrary direction ???

Directional Derivatives

The directional derivative of f at (a,b) in the direction of ***unit vector*** $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is

$$D_{\mathbf{u}} f(a,b) = \lim_{h\to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$$

if the limit exists.

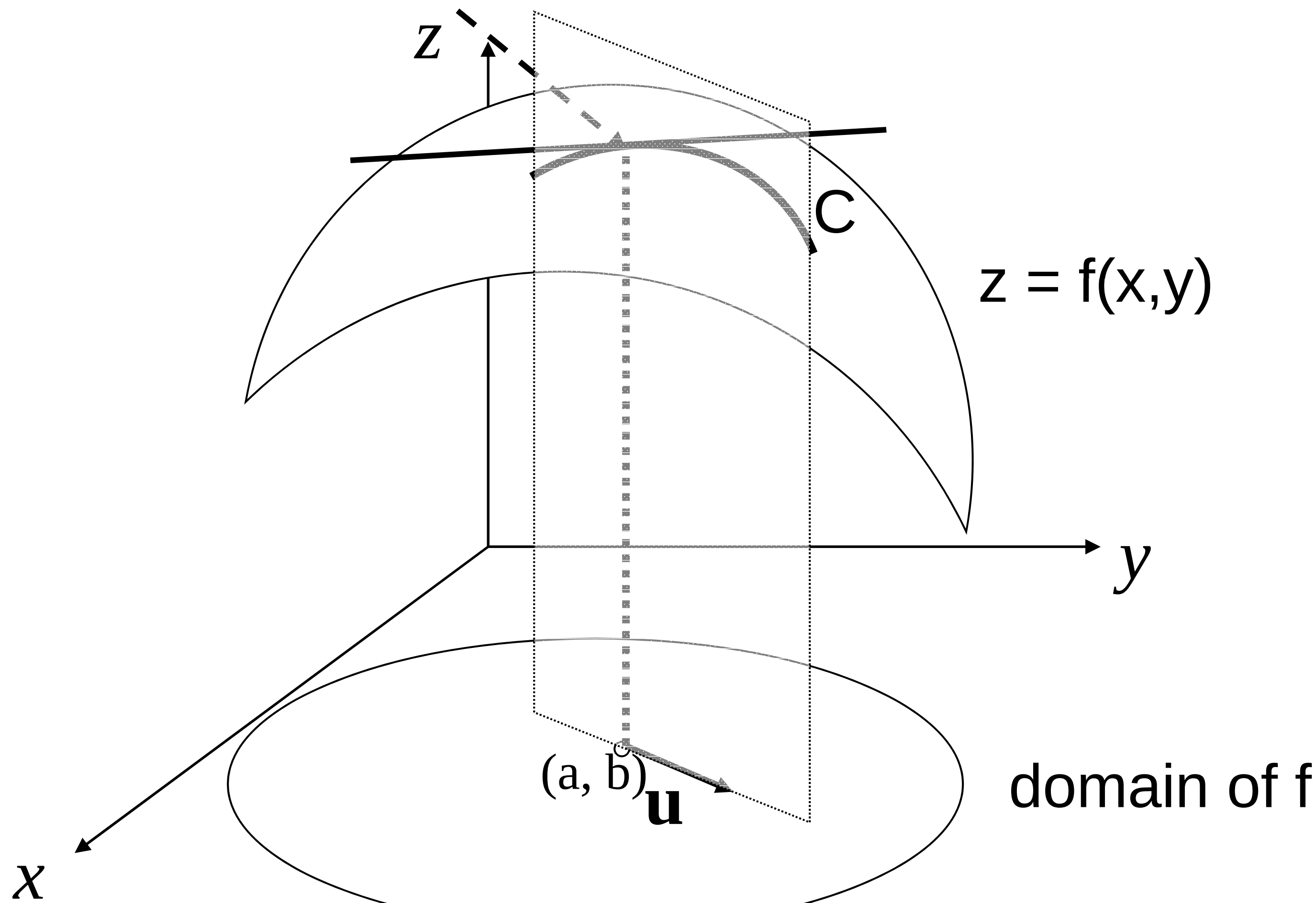
For directional derivative, we need to specify

(1) the point we are interested in (a,b),

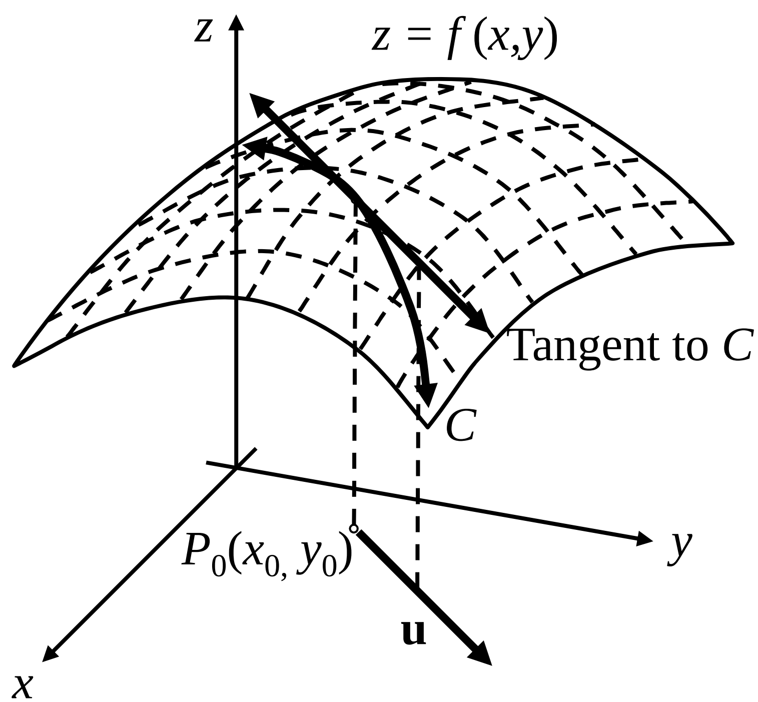
(2) the direction we are looking at $\mathbf{u}$.

Directional Derivatives - Geometrical

$D_{\mathbf{u}} f(a,b)$ = gradient of the tangent line of curve C at the point (a, b, f(a, b))



Geometrical Meaning



$D_u f(a,b)$ gives the gradient of the tangent line to the curve C at the point (a,b).

Directional Derivatives

$D_{\mathbf{u}} f(a,b)$: Directional derivative of f at point (a,b) in the direction $\mathbf{u}$.

Note : $\mathbf{u}$ is a unit vector.

At the same point (a,b), we can have directional derivative in different directions **u**.

Directional Derivatives

The directional derivative of f at (a,b) in the direction of ***unit vector*** $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is

$$D_{\mathbf{u}} f(a,b) = \lim_{h\to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$$

if the limit exists.

Note that :

$$D_{\mathbf{i}} f(a,b) = f_x(a,b)$$

$$D_{\mathbf{j}} f(a,b) = f_y(a,b)$$

Directional Derivatives

Question:

How to compute $D_{\mathbf{u}} f(a,b)$???

Formula :

$$D_{\mathbf{u}} f(a,b) = f_x(a,b)u_1 + f_y(a,b)u_2$$

where $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is a ***unit*** vector.

Directional Derivatives - Example

Let $f(x,y) = x^2 - 3xy^2 + 2y^3$.

Find $D_{\mathbf{u}} f(2,1)$, where $\mathbf{u} = \dfrac{\sqrt{3}}{2}\mathbf{i} + \dfrac{1}{2}\mathbf{j}$.

$$f_x = 2x - 3y^2 \text{ and } f_y = -6xy + 6y^2$$

$$f_x(2,1) = 1 \text{ and } f_y(2,1) = -6$$

Thus, $D_{\mathbf{u}} f(2,1) = (1)\left(\dfrac{\sqrt{3}}{2}\right) + (-6)\left(\dfrac{1}{2}\right) = \dfrac{\sqrt{3}-6}{2}$.

$$D_{\mathbf{u}} f(a,b) = f_x(a,b)u_1 + f_y(a,b)u_2$$

where $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is a ***unit*** vector.

Physical Meaning

The directional derivative $D_{\mathbf{u}} f(a,b)$ measures the change in the value df of a function f when we move a small distance dt from the point (a,b) in the direction of the vector $\mathbf{u}$:

$$df = D_{\mathbf{u}} f(a,b) \cdot dt$$

Directional Derivatives - Example

Let $f(x, y) = x^2 y^3 + 1$. Estimate how much the value of f will change if a point Q moves 0.1 unit from (2,1) towards (3,0).

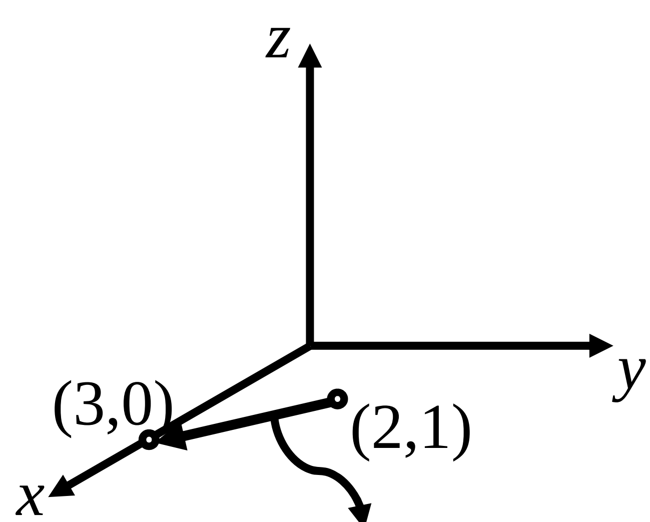
First need to find the unit vector parallel to the vector that starts at (2,1) and ends at (3,0).

Q moves in the direction $(3\mathbf{i} + 0\mathbf{j}) - (2\mathbf{i} + \mathbf{j}) = \mathbf{i} - \mathbf{j}$.

$\mathbf{u}$ is parallel to $\mathbf{i} - \mathbf{j}$.

$$\|\mathbf{i} - \mathbf{j}\| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\text{Unit vector } \mathbf{u} = \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j})$$



$(3\mathbf{i} + 0\mathbf{j}) - (2\mathbf{i} + \mathbf{j})$
$= \mathbf{i} - \mathbf{j}$

$$D_{\mathbf{u}} f(a,b) = f_x(a,b)u_1 + f_y(a,b)u_2$$

where $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is a ***unit*** vector.

Directional Derivatives - Example

Let $f(x, y) = x^2 y^3 + 1$. Estimate how much the value of f will change if a point Q moves 0.1 unit from (2,1) towards (3,0).

$\mathbf{u}$ is parallel to $\mathbf{i} - \mathbf{j}$. Unit vector $\mathbf{u} = \dfrac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j})$

$$f_x = 2xy^3 \quad \text{and} \quad f_y = 3x^2y^2$$

$$f_x(2,1) = 4 \quad \text{and} \quad f_y(2,1) = 12$$

Thus, $D_{\mathbf{u}} f(2,1) = (4)\left(\dfrac{1}{\sqrt{2}}\right) + (12)\left(-\dfrac{1}{\sqrt{2}}\right) = -\dfrac{8}{\sqrt{2}}$

$$D_{\mathbf{u}} f(a,b) = f_x(a,b)u_1 + f_y(a,b)u_2$$

where $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is a ***unit*** vector.

Directional Derivatives - Example

Let $f(x,y) = x^2 y^3 + 1$. Estimate how much the value of f will change if a point Q moves 0.1 unit from (2,1) towards (3,0).

$\mathbf{u}$ is parallel to $\mathbf{i} - \mathbf{j}$. Unit vector $\mathbf{u} = \dfrac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j})$

$$f_x = 2xy^3 \quad \text{and} \quad f_y = 3x^2y^2$$

$$f_x(2,1) = 4 \quad \text{and} \quad f_y(2,1) = 12$$

Thus, $D_{\mathbf{u}} f(2,1) = (4)\left(\dfrac{1}{\sqrt{2}}\right) + (12)\left(-\dfrac{1}{\sqrt{2}}\right) = -\dfrac{8}{\sqrt{2}}$

$$df = D_{\mathbf{u}} f(2,1) \cdot dt = \left(-\frac{8}{\sqrt{2}}\right)(0.1) \approx 0.57$$

$$\boxed{df = D_{\mathbf{u}} f(a,b) \cdot dt}$$

Past Exam Question

Let $f(x,y)$ be a differentiable function of two variables such that $f(2,1)=1506$ and $\dfrac{\partial f}{\partial x}(2,1)=4$. It was found that if the point moved from (2,1) a distance 0.1 unit towards (3,0), the value of f became 1505. Estimate the value of $\dfrac{\partial f}{\partial y}(2,1)$.

Functions of Three Variables

We can also define directional derivatives for functions of three variables. Let f be a function of x, y and z. The directional derivative of f at (a,b,c) in the direction of a unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ in the xyz space is

$$D_{\mathbf{u}} f(a,b,c) = \lim_{h\to 0} \frac{f(a+hu_1, b+hu_2, c+hu_3) - f(a,b,c)}{h}$$

if this limit exists.

Functions of Three Variables

Similarly, we have the formula

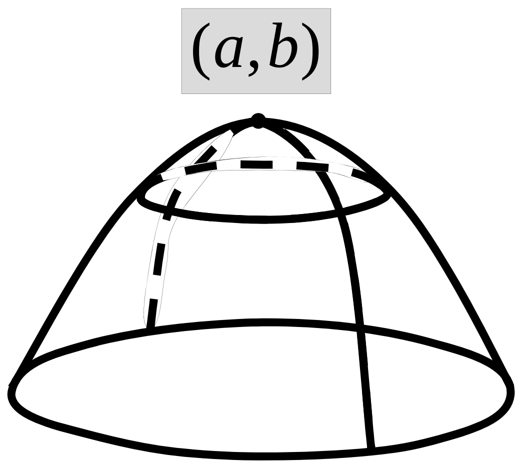
$$D_{\mathbf{u}} f(a,b,c) = f_x(a,b,c)u_1 + f_y(a,b,c)u_2 + f_z(a,b,c)u_3$$

since $df = D_{\mathbf{u}} f(a,b,c) \cdot dt$.

Maximum and Minimum Values

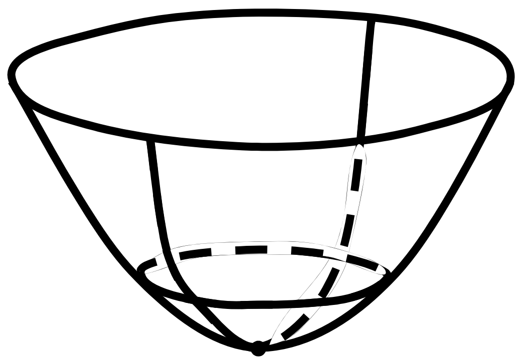
Local Maximum and Minimum

$f(x,y)$ has a ***local maximum*** at (a,b) if $f(x,y) \le f(a,b)$ for all points (x,y) near (a,b). The number $f(a,b)$ is called a ***local maximum value***.



Local Maximum and Minimum

$f(x, y)$ has a ***local minimum*** at (a,b) if $f(x, y) \geq f(a,b)$ for all points (x, y) near (a,b). The number $f(a,b)$ is called a ***local minimum value***.



(a,b)

Critical Points

A point (a,b) is called a critical point of f if

(i) $f_x(a,b)=0$ and $f_y(a,b)=0$; or

(ii) $f_x(a,b)$ or $f_y(a,b)$ does not exist.

Pause and Think !!!

Suppose $f_x(a,b)=0$ and $f_y(a,b)=0.$

Question:

What can you say about $D_{\mathbf{u}}f(a,b)$???

Pause and Think !!!

Suppose $f_x(a,b) = 0$ and $f_y(a,b) = 0$.

Question:

What can you say about $D_{\mathbf{u}} f(a,b)$???

Answer :

Pause and Think !!!

Suppose $f_x(a,b)=0$ and $f_y(a,b)=0.$

Question:

Must the point (a,b) be

a local maximum / minimum of f ???

Pause and Think !!!

Suppose $f_x(a,b)=0$ and $f_y(a,b)=0.$

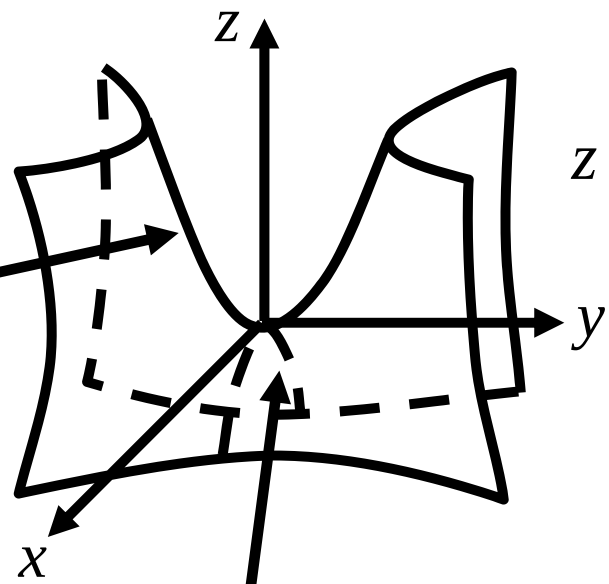
Question:

Must the point (a,b) be

a local maximum / minimum of f ???

Answer :

Saddle Point



$z = 2y^2 - 3x^2$

$\boxed{\text{Fix } x = 0}$ $\boxed{z = 2y^2}$

local minimum

$\boxed{\text{Fix } y = 0}$ $\boxed{z = -3x^2}$

local maximum

Critical Points

Let (a,b) be a point of f with $f_x(a,b)=0$ and $f_y(a,b)=0$.

We say (a,b) is a saddle point of f

if there are some directions

along which f has a local maximum at (a,b)

and some directions

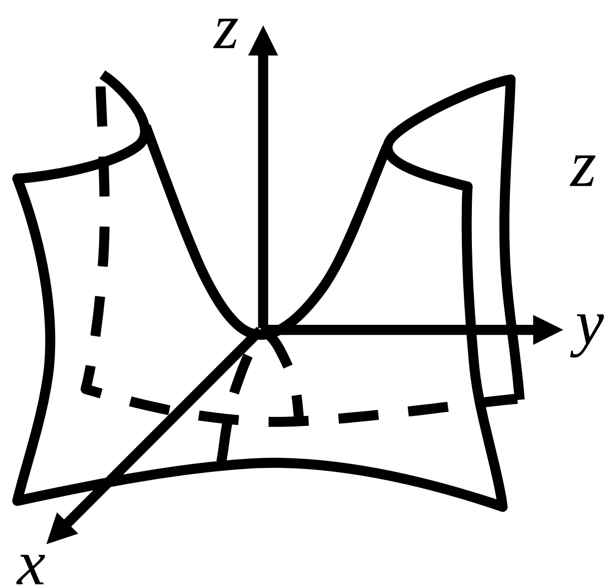
along which f has a local minimum at (a,b).

Fix $x=0$ $\quad z=2y^2$

local minimum

Fix $y=0$ $\quad z=-3x^2$

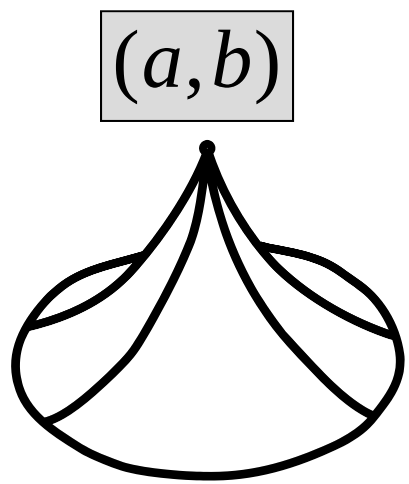
local maximum



$z=2y^2-3x^2$

Critical Points

f may have a local maximum or minimum at (a,b), where $f_x(a,b)$ or $f_y(a,b)$ does not exist.



Critical Points - Example

$$\text{Let } z = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

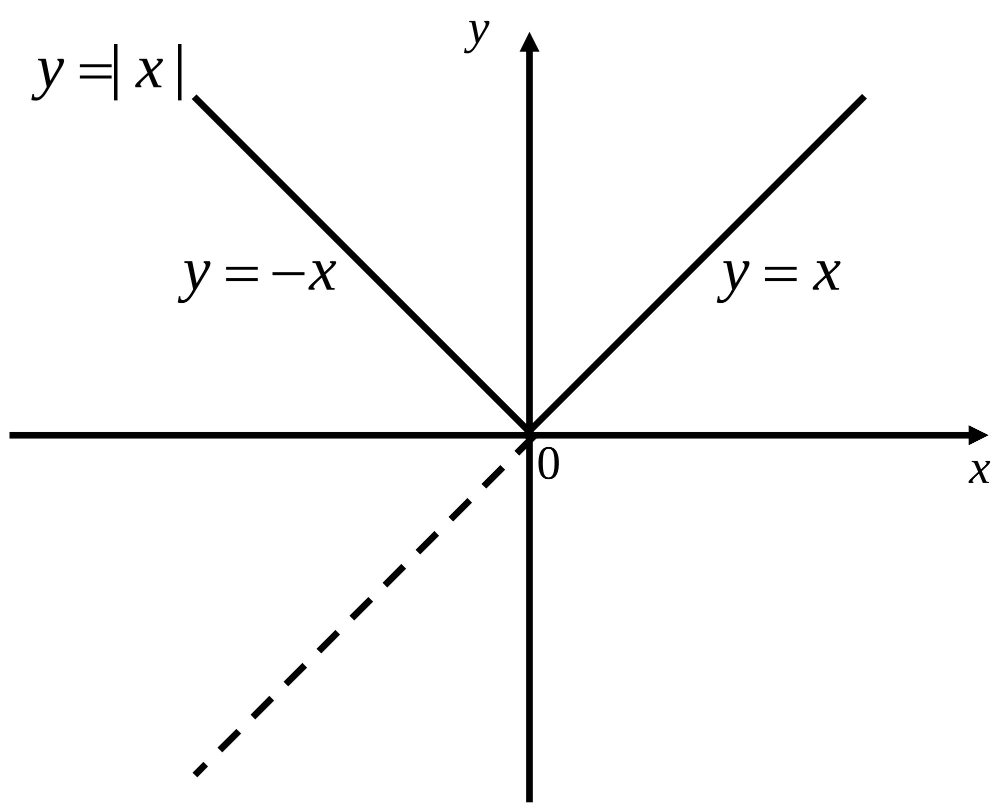
z has a local minimum at $(0,0)$

but $\left.\dfrac{\partial z}{\partial x}\right|_{(0,0)}$ does not exist.

Let $f(x) = |x|$.

Show that f is differentiable for $x \neq 0$ and has no derivative at $x = 0$.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

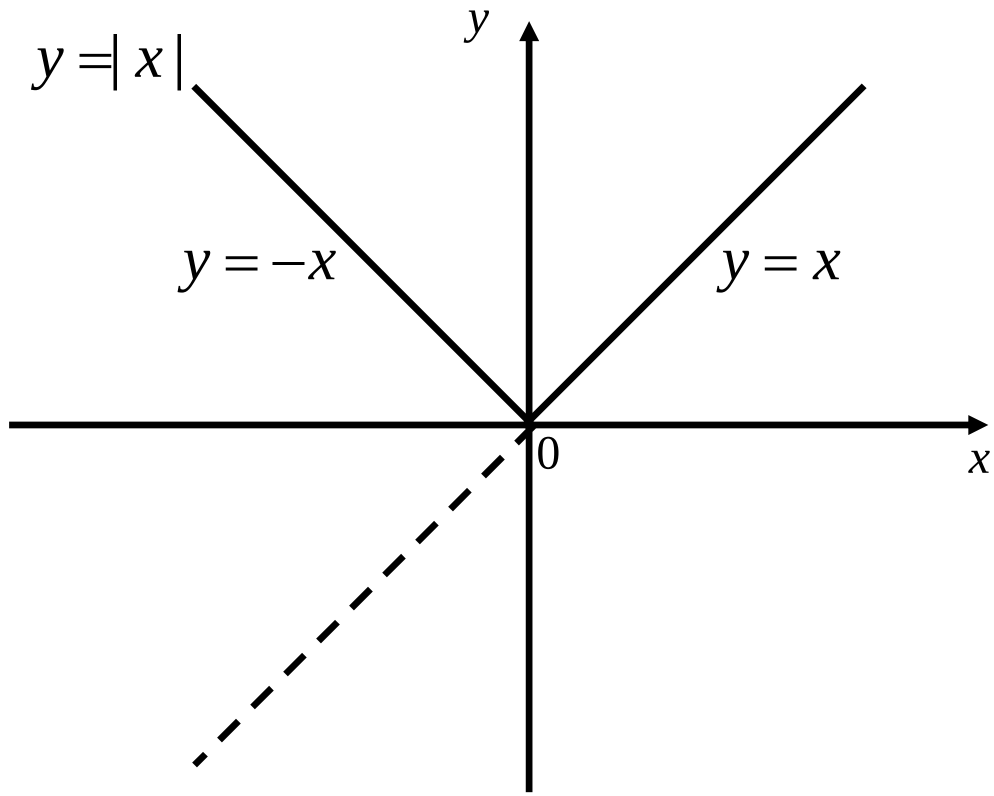


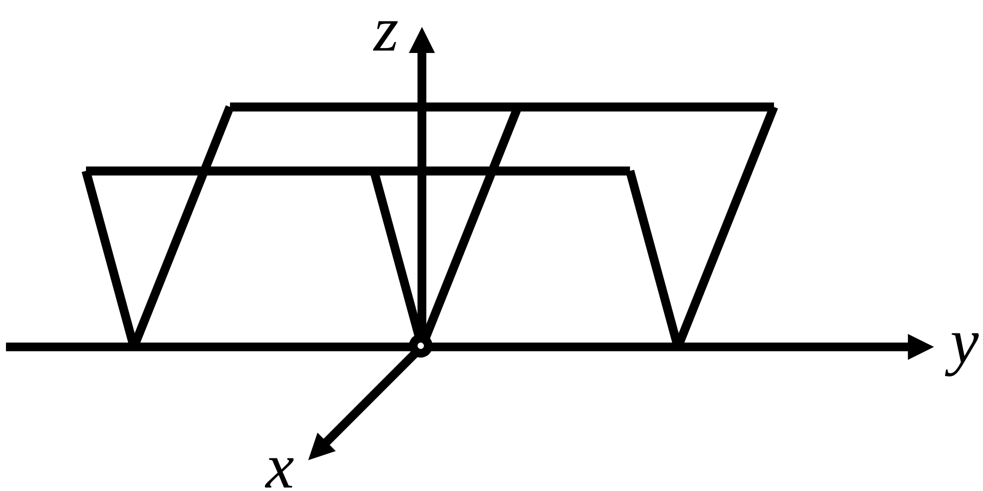
Critical Points - Example

$$\text{Let } z = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

z has a local minimum at $(0,0)$

but $\left.\dfrac{\partial z}{\partial x}\right|_{(0,0)}$ does not exist.





Critical Points

Second Derivative Test:

Assume that f and its first and second partial derivatives are continuous in a region containing (a,b) such that $f_x(a,b)=0$ and $f_y(a,b)=0$.

$$\text{Let} \quad D = f_{xx}(a,b)\,f_{yy}(a,b) - f_{xy}(a,b)^2$$

We will use the value of D to test for local maximum / minimum and saddle points

Critical Points

$$\boxed{D = f_{xx}(a,b)\, f_{yy}(a,b) - f_{xy}(a,b)^2}$$

(a) If $D > 0$ and $f_{xx}(a,b) > 0$,

then f has a ***local minimum*** at (a,b).

(b) If $D > 0$ and $f_{xx}(a,b) < 0$,

then f has a ***local maximum*** at (a,b).

(c) If $D < 0$,

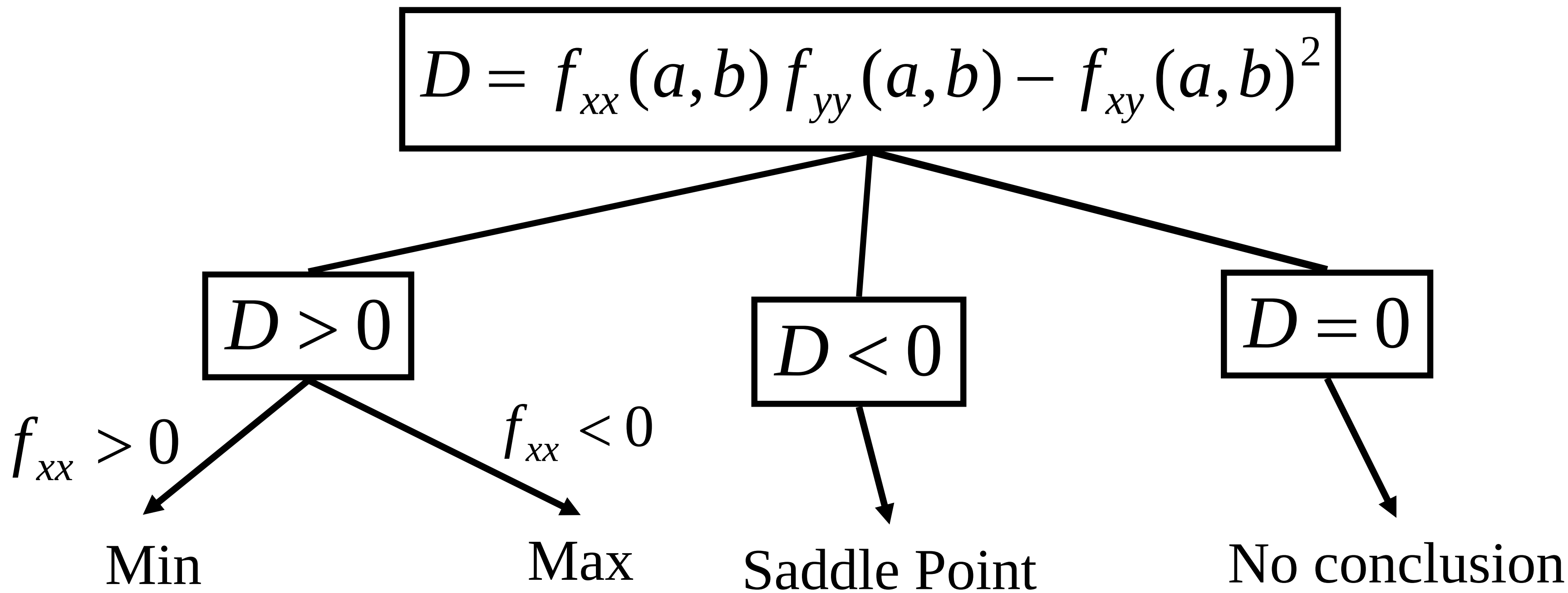
then f has a ***saddle point*** at (a,b).

(d) If $D = 0$,

then ***no conclusion*** can be drawn.

Critical Points

Given a function $f(x, y)$ and point (a,b) such that $f_x(a,b) = 0 = f_y(a,b)$.



Critical Points - Example

Find and classify all the critical points of

$$f(x,y) = x^3 + y^3 + 3x^2 - 3y^2 - 8.$$

$$f_x = 3x^2 + 6x = 0$$
$$x^2 + 2x = 0$$
$$x(x+2) = 0$$
$$x = 0, -2$$

$$f_y = 3y^2 - 6y = 0$$
$$y^2 - 2y = 0$$
$$y(y-2) = 0$$
$$y = 0, 2$$

4 critical points: $(0,0)$, $(0,2)$, $(-2,0)$, $(-2,2)$.

Critical Points - Example

Find and classify all the critical points of

$$f(x,y) = x^3 + y^3 + 3x^2 - 3y^2 - 8.$$

$f_x = 3x^2 + 6x$

$f_y = 3y^2 - 6y = 0$

$f_{xy} = 0$

At $(0,0)$, $D = -36 < 0$

saddle point at (0,0)

$$D = f_{xx}(a,b) f_{yy}(a,b) - f_{xy}(a,b)^2$$

At $(0,2)$, $D = 36 > 0$ and $f_{xx}(0,2) = 6 > 0$

local minimum at (0,2)

At $(-2,0)$, $D = 36 > 0$ and $f_{xx}(-2,0) = -6 < 0$

local maximum at $(-2,0)$

At $(-2,2)$, $D = -36 < 0$

saddle point at $(-2,-2)$

Critical Points - Example

Find and classify all the critical points of

$$f(x, y) = y^3 + 3x^2y - 3x^2 - 3y^2 + 2.$$

$$f_x = 6xy - 6x$$

$$f_y = 3y^2 + 3x^2 - 6y$$

$$6xy - 6x = 0 \quad \text{------} \quad (1)$$

$$3y^2 + 3x^2 - 6y = 0 \quad \text{------} \quad (2)$$

Solve (1) and (2) : simultaneous equatons

Solving $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$ yields 4 critical points:

$$(0,0), \ (0,2), \ (1,1), \ (-1,1).$$

Find and classify all the critical points of

$$f(x,y) = y^3 + 3x^2y - 3x^2 - 3y^2 + 2.$$

$$f_{xx} = 6y - 6, \quad f_{yy} = 6y - 6 \quad \text{and} \quad f_{xy} = 6x$$

$$D = f_{xx} f_{yy} - f_{xy}^{\;2}$$

	(0,0)	(0,2)	(1,1)	(-1,1)
f_{xx}	-6	6	0	0
f_{yy}	-6	6	0	0
f_{xy}	0	0	6	-6
D	36	36	-36	-36
	Max	Min	Saddle Points	

Critical Points

$$\boxed{D = f_{xx}(a,b)\, f_{yy}(a,b) - f_{xy}(a,b)^2}$$

(a) If $D > 0$ and $f_{xx}(a,b) > 0$,

then f has a ***local minimum*** at (a,b).

(b) If $D > 0$ and $f_{xx}(a,b) < 0$,

then f has a ***local maximum*** at (a,b).

(c) If $D < 0$,

then f has a ***saddle point*** at (a,b).

(d) If $D = 0$,

then ***no conclusion*** can be drawn.

$$D = f_{xx}(a,b)\,f_{yy}(a,b) - f_{xy}(a,b)^2$$

(a) If $D > 0$ and $f_{xx}(a,b) > 0$,

then f has a ***local minimum*** at (a,b).

(b) If $D > 0$ and $f_{xx}(a,b) < 0$,

then f has a ***local maximum*** at (a,b).

Pause and Think !!!

Note that :

the value of D depends on f_{yy}

but we don't "ask" f_{yy} when checking for

local maximum / minimum of f.

Why ???

End