IEEE Revision Lecture for MA1505

Tutor: Hu Hengnan

16 Nov 2010

Outline

In these two hour, we are going to follow this arrangement:

- Take about 20 minutes to demonstrate some basic concepts and useful formulas.
- Take about 70 minutes or so to work through all the problems in detail.
- The rest of time is up to you.

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Part I

Related Concepts and Formulas

Statistics of the All 13 Problems

- Taylor Series: 3 Qns
- Fourier Series: 1 Qns and 3 other Qns mentioned
- Series sums: 4 Qns
- line integral: 1 Qns related
- Surface integral: 4 Qns

Taylor Series

- What is Taylor Series:In brief,change function $f(x)$ to the sum of Power series. For example the Taylor series of $f(x)$ at $x = a$ is the following Power series at $x = a$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

- Classical results:

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n t^n$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

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Fourier Series

- What is Fourier Series: In brief, change a periodic function to the sum of its trigonometric series.For example, if for $f(x+2\pi) = f(x)$:

  $$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

  where

  $$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

  $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \text{for } n = 1,2,\ldots$$

  $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx, \text{for } n = 1,2,\ldots$$

- Then for function $f(x)$ of any period $p = 2L > 0$:

  $$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos \frac{n\pi}{L}x + b_n \sin \frac{n\pi}{L}x).$$

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For function $f(x)$ which defined only on the interval $[0, L]$

- Cosine:Extend the function to be an even function

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x$$

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$$a_0 = \frac{1}{L} \int_0^L f(x)dx,$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx, \text{for } n = 1,2,\ldots$$

- Sine:Extend the function to be an odd function

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Series Sums

- General definition: nth term a_n, nth partial sum$S_n = \sum_{k=1}^{n} a_k$.
- Geometric series: $\sum_{n=1}^{\infty} ar^{n-1}$,where $a \neq 0$ is called the first term,r is called the ratio. Converge to $\frac{a}{1-r}$ if $|r| < 1$.
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line integral:Three methods

- Using the fundamental theorem of Calculus:

$$\int_a^b F'(x)dx = F(b) - F(a).$$

- Using the following equality for conservative fields: If f is function of 2 or 3 variables whose gradient ∇f is continuous, then

$$\int_C \nabla f \cdot d\mathbf{r} = f(r(b)) - f(r(a)).$$

- Using the Stokes' Theorem($3 variables$) or Green's Theorem($2 variables$) change to surface integral. But here be careful of the orientation of surface should be consistent with the orientation of the closed curve.

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- Change the surface integral $\int\int_S fdS$ to Double integral $\int\int_D gdA$ where D is the a region in the uv-plane.Be careful of the orientation. Then change to iterated integral to $\int_a^b \int_{l_1(u)}^{l_2(u)} gdvdu$ or $\int_c^d \int_{l_1(v)}^{l_2(v)} gdudv$ to get the answer.
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- For closed surface,using Divergence's Theorem change to integral over the solid bounded by the surface. Here also be careful with the surface orientation should consist with the orientation of the solid.

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Part II

Solutions for All Problems

Problem

If

$$f(x) = \int_0^x te^{t^3}\,dt$$

use Taylor series to find $f^{(1505)}(0)$.
(Leave your answer in terms of factorials)

Doing some simple calculation:
$f'(x) = xe^{x^3}$ so $f'(0) = 0$
$f^{(2)}(x) = e^{x^3} + 3x^3e^{x^3}$ so $f^{(2)}(0) = e^0 = 1$
$f^{(3)}(x) = (12x^2 + 9x^5)e^{x^3}$ so $f^{(3)}(0) = 0$
$f^{(4)}(x) = (24x + 81x^4 + 27x^7)e^{x^3}$ so $f^{(4)}(0) = 0$
$f^{(5)}(x) = (24 + 4 \times 81x^3 + 24 \times 3x^3 + 7 \times 27x^6 + 81 \times 3x^6 + 27 \times 3x^9)e^{x^3}$
so $f^{(5)}(0) = 24$

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Question 3(a) of 2007-2008 Semester 2

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Question 3(a) of 2007-2008 Semester 2

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If

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The shortcut: Actually very standard

Here we have to use the Taylor expansion for the function e^x which is:

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \ldots + \frac{1}{n!}x^n + \ldots = \sum_{n=0}^{+\infty} \frac{1}{n!}x^n$$

then we get:

$$e^{x^3} = \sum_{n=0}^{+\infty} \frac{1}{n!}(x^3)^n = \sum_{n=0}^{+\infty} \frac{1}{n!}x^{3n}$$

$$xe^{x^3} = x\sum_{n=0}^{+\infty} \frac{1}{n!}x^{3n} = \sum_{n=0}^{+\infty} \frac{1}{n!}x^{3n+1}$$

With the above formulas we can get the answer but please be careful.
If $f(x) = f(0) + f'(0)x + \ldots + \frac{1}{1505!}f^{(1505)}(0)x^{1505} + \ldots$
then $f'(x) = f'(0) + \ldots + \frac{1}{1505!}f^{(1505)}(0) \times 1505x^{1504} + \ldots$
so we get that the corresponding coefficients equal then we have:

$$\frac{1}{1505!}f^{(1505)}(0) \times 1505 = \frac{1}{n!} \text{ where } 3n + 1 = 1504.$$

Question 2(a) of 2009-2010 Semester 1

Problem

Let $f(x) = \frac{23-4x}{7-2x}$ and let $\sum_{n=0}^{\infty} c_n(x-2)^n$ be the Taylor series for $f(x)$ at $x = 2$. Find the ***exact value*** *of $c_0 + c_{2000}$*

- Rewrite the function $f(x)$ as $\frac{15-4(x-2)}{3-2(x-2)}$.Denote $t = x - 2$, then $f(t) = \frac{15-4t}{3-2t} = \frac{5-\frac{4t}{3}}{1-\frac{2t}{3}}$. We need to find the Taylor series of $f(t)$ at $t = 0$.
- As we know $\frac{1}{1-\frac{2t}{3}} = \sum_{n=0}^{\infty}(\frac{2t}{3})^n$. Then we have $f(t) = \sum_{n=0}^{\infty}(\frac{5\times2^n}{3^n})t^n + \sum_{n=0}^{\infty} -(\frac{2^{n+2}}{3^{n+1}})t^{n+1} = 5 + \sum_{n=1}^{\infty}(\frac{5\times2^n-2^{n+1}}{3^n})t^n = 5 + \sum_{n=1}^{\infty}(\frac{2^n}{3^{n-1}})t^n$.
- Then $f(x) = 5 + \sum_{n=1}^{\infty}(\frac{2^n}{3^{n-1}})(x-2)^n$.So $c_0 = 5$ and $c_{2000} = \frac{2^{2000}}{3^{1999}}$.

Problem

Let

$$g(x) = \frac{x-16}{x^2-16x+68}.$$

If $f(x)$ *is a function such that* $f(8) = 8$ *and* $f'(x) = g(x)$*, then use Taylor series to find the value of* $f^{(2009)}(8)$*.*
(You may give your answer in terms of factorials.)

Answer to the Question 2(b) of 2008-2009 Semester 2

- The Taylor series of $g(x)$ at 8. Denote $t = x - 8$, then $g(t) = \frac{t-8}{t^2+4}$. We can get the Taylor series of $g(t)$ at0 in this way:
  1. Rewrite $g(t)$ as $\frac{1}{(\frac{t}{2})^2+1}\frac{t}{4} - 2\frac{1}{(\frac{t}{2})^2+1}$
  2. We know that $\frac{1}{(\frac{t}{2})^2+1} = \sum_{n=0}^{\infty}(-1)^n(\frac{t}{2})^{2n}$
  3. Then $g(t) = \sum_{n=0}^{\infty}(-1)^n(\frac{t}{2})^{2n}\frac{t}{4} - \sum_{n=0}^{\infty}(-1)^n(\frac{t}{2})^{2n} = \sum_{n=0}^{\infty}(-1)^n[-(\frac{1}{2})^{2n-1}t^{2n} + (\frac{1}{2})^{2n+2}t^{2n+1}]$

  Then we have the Taylor series of $g(x)$ at 8:

$$g(x) = \sum_{n=0}^{\infty}(-1)^n[-(\frac{1}{2})^{2n-1}(x-8)^{2n} + (\frac{1}{2})^{2n+2}(x-8)^{2n+1}]$$

- Suppose the Taylor series of $f(x)$ at 8 is $\sum_{n=0}^{\infty}\frac{f^{(n)}(8)}{n!}(x-8)^n$. Then we have $f'(x) = \sum_{n=0}^{\infty}\frac{f^{(n+1)}(8)}{n!}(x-8)^n$. Compare with the coefficient, we get that $\frac{f^{(2009)}(8)}{2008!} = (-1)^{1004} \times (-1)(\frac{1}{2})^{2007}$. Then $f^{(2009)}(8) = -\frac{2008!}{2^{2007}}$.

Problem

Let $f(x) = 0$*,if* $-3 < x < 0$*;* $f(x) = x$*,if* $0 < x < 3$*;* $f(x+6) = f(x)$*,for all* x*.Suppose that the Fourier series of* $f(x)$ *is:*

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos \frac{n\pi x}{3} + b_n \sin \frac{n\pi x}{3})$$

Find the positive value of n such that

$$2 + 27\pi^2 a_n = 0.$$

Use the formula:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx.$$

Here we have $2L = 6$,then

$$a_n = \frac{1}{3}\int_{-3}^{3} f(x)\cos\frac{n\pi x}{3}dx = \frac{1}{3}\int_{0}^{3} x\cos\frac{n\pi x}{3}dx.$$

Here we use the integral by parts theorem to calculate.

$$a_n = \frac{1}{3}\int_0^3 x\cos\frac{n\pi x}{3}dx = \frac{1}{3}\int_0^3 x\frac{3}{n\pi}d\sin\frac{n\pi x}{3} = \frac{-1}{n\pi}\int_0^3 \sin\frac{n\pi x}{3}dx = \frac{3}{n^2\pi^2}[\cos n\pi - 1].$$

So if we need to satisfy $2 + 27\pi^2 a_n = 0$, $a_n = -\frac{2}{27\pi^2}$. Then we can see that $n = 9$.

Problem

For $-\frac{\pi}{2} < x < \frac{\pi}{2}$*,the series:*

$$\sin^2 x + \sin^4 x + \sin^6 x + \ldots + \sin^{2k} x + \ldots$$

converges. Find its sum.

This is a very standard geometric series. If we denote $t = \sin x$, then we have $|t| < 1$. The geometric series becomes:

$$\sum_{n=1}^{\infty} t^2 (t^2)^{n-1} = \frac{t^2}{1-t^2}.$$

Then the result is:

$$\frac{\sin^2 x}{1-\sin^2 x} = \tan^2 x.$$

Problem

Let $g(x) = x^4$ *for* $-1 \le x \le 1$ *and* $g(x+2) = g(x)$ *for all* x*.The Fourier series of* $g(x)$ *is*

$$g(x) = \frac{1}{5} + \frac{8}{\pi^4}\sum_{n=1}^{\infty}(-1)^n \frac{n^2\pi^2 - 6}{n^4}\cos n\pi x.$$

Given the sum of the series $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$*,use this Fourier series to find the sum of the series*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}.$$

(The above series S *and the Fourier series need not be derived. Give the exact value of the sum in terms of* π*.)*

Answer to Question 3(b) of 2009-2010 Semester 2

- Take $x = 0$, then LHS$= 0$

$$\text{RHS} = \frac{1}{5} + \frac{8}{\pi^4}\sum_{n=1}^{\infty}(-1)^n \frac{n^2\pi^2 - 6}{n^4}.$$

So we have $\pi^2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} + 6\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = -\frac{1}{5} \times \frac{\pi^4}{8} = -\frac{\pi^4}{40}$.

- It's obvious that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = -\frac{\pi^2}{12}$. So by the above equality, we have
$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = \frac{1}{6}[-\frac{\pi^4}{40} - \pi^2 \times (-\frac{\pi^2}{12})] = \frac{1}{6}[\frac{\pi^4}{12} - \frac{\pi^4}{40}] = \frac{7\pi^4}{720}$.

Answer to Question 3(b) of 2009-2010 Semester 2

- Take $x = 0$, then LHS$= 0$

$$\text{RHS}= \frac{1}{5} + \frac{8}{\pi^4} \sum_{n=1}^{\infty} (-1)^n \frac{n^2\pi^2 - 6}{n^4}.$$

  So we have $\pi^2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} + 6 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = -\frac{1}{5} \times \frac{\pi^4}{8} = -\frac{\pi^4}{40}$.

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  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = \frac{1}{6}[-\frac{\pi^4}{40} - \pi^2 \times (-\frac{\pi^2}{12})] = \frac{1}{6}[\frac{\pi^4}{12} - \frac{\pi^4}{40}] = \frac{7\pi^4}{720}$.

Question 3(b) of 2008-2009 Semester 2

Problem

Let $u(t) = 0$,if $-1 < t < 0$; $u(t) = \sin \pi t$,if $0 < x < 1$; $u(t+2) = u(t)$,for all x.The Fourier series of $u(t)$ is:

$$u(t) = \frac{1}{\pi} + \frac{1}{2}\sin \pi t - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} \cos 2n\pi t$$

Use this Fourier series to find the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)(2n+1)}.$$

Hence find the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}.$$

(You need not derive the above Fourier series.Give the exact value of each sum in terms of π)

Answer to Question 3(b) of 2008-2009 Semester 2

- Take $t = \frac{1}{2}$,then LHS$= u(\frac{1}{2}) = 1$ and

$$\text{RHS}= \frac{1}{\pi} + \frac{1}{2} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)(2n+1)}.$$

  So we have $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)(2n+1)} = \frac{\pi-2}{4}$.

- In the following, we need to find the relation between the sums of series,$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$ and $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)(2n+1)}$ which we denote by A and B respectively.Using the fact that:

$$\frac{1}{(2n-1)(2n+1)} = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right).$$

  Then we have:

$$B = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)(2n+1)} = \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) = \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2n-1} + \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n+2} \frac{1}{2n+1}.$$

  We know that $A - 1 = \sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{2n-1} = \sum_{n=1}^{\infty} (-1)^{n+2} \frac{1}{2n+1}$.
  So we have $B = \frac{1}{2}A + \frac{1}{2}(A-1)$ Then $A = B + \frac{1}{2} = \frac{\pi}{4}$.

Question 3(b) of 2007-2008 Semester 2

Problem

Let $f(x) = x$,if $0 < x < 1$;$f(x) = 2 - x$,if $1 < x < 2$.The cosine half-range expansion of $f(x)$ is:

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} cos(2n-1)\pi x$$

(You need not derive this Fourier series)
Use the above cosine half-range expansion to find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.Hence find the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$.
(Give the exact values in terms of π)

Answer to Question 3(b) of 2007-2008 Semester 2

- For the equality, let's take x=0. Then the LHS$= f(0) = 0$,and the RHS$= \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$. Then we'll get that

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{1}{2} \times \frac{\pi^2}{4} = \frac{\pi^2}{2}$$

- The sums of two series,$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$ and $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$ denoted by P and Q,respectively.We know the value of Q. Here we need to find the relation between P and Q.For P, we divide the sum into two parts,shown below:

$$P = \sum_{n=2k-1}\frac{(-1)^{n+1}}{n^2} + \sum_{n=2k}\frac{(-1)^{n+1}}{n^2} = \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} + \sum_{k=1}^{\infty}\frac{-1}{4}\frac{1}{(k)^2}$$

Then we see that $P = Q + \sum_{k=1}^{\infty}\frac{-1}{4}\frac{1}{(k)^2}$. Denote $\sum_{k=1}^{\infty}\frac{1}{(k)^2}$ by R. Then we see that $P = Q - \frac{1}{4}R$.

Answer to Question 3(b) of 2007-2008 Semester 2 Continu

In the following we have to find the relation of Q and R. Similarly we divide R into to two parts:

$$R = \sum_{2k-1} \frac{1}{(2k-1)^2} + \sum_{2k} \frac{1}{(2k)^2} = Q + \frac{1}{4}R.$$

So we have $\frac{3}{4}R = Q$,i.e.$R = \frac{4}{3}Q$.In summary we have

$$P = Q - \frac{1}{4}R = Q - \frac{1}{4} \times \frac{4}{3}Q = \frac{2}{3}Q = \frac{\pi^2}{3}.$$

Problem

Let S be the cone described by

$$z = \sqrt{x^2 + y^2}, \text{ where } 0 \leq z \leq 4.$$

If $\boldsymbol{F}(x, y, z) = y\boldsymbol{i} - x\boldsymbol{j} + z^2\boldsymbol{k}$, find the surface integral $\int\int_S \boldsymbol{F} \bullet d\boldsymbol{S}$,when the orientation of S is given by the inner normal vector.

Answer to Question 8(a) of 2007-2008 Semester 2

- Parametric presentation of the surface:

$$\mathbf{r}(u,v) = u\mathbf{i} + v\mathbf{j} + \sqrt{u^2+v^2}\mathbf{k}, \text{ where } u^2+v^2 \leq 16$$

  The region for (u, v) is denoted as D.Now we check whether the orientation of the surface under this parametric presentation is the same as the orientation given. $\vec{r}_u = 1\mathbf{i} + \frac{u}{\sqrt{u^2+v^2}}\mathbf{k}$,
  $\vec{r}_v = 1\mathbf{j} + \frac{v}{\sqrt{u^2+v^2}}\mathbf{k}$, then

$$\vec{r}_u \times \vec{r}_v = \left(-\frac{u}{\sqrt{u^2+v^2}}\right)\mathbf{i} + \left(-\frac{v}{\sqrt{u^2+v^2}}\right)\mathbf{j} + 1\mathbf{k}.$$

  So we know that these orientation are the same.
- The required integral is

$$\int\int_S \mathbf{F} \bullet d\mathbf{S} = \int\int_D \mathbf{F}(\mathbf{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) dA = \int\int_D u^2 + v^2 dA.$$

  Using the polar coordinate change, we get the required integral equals to

$$\int_0^4 \int_0^{2\pi} r^2 r d\theta dr = 128\pi.$$

Problem

Let S be the closed surface that consists of

1. *the upper hemisphere*

$$x^2 + y^2 + z^2 = 1, z \geq 0,$$

   together with

2. *the base of points $(x, y, 0)$, where $0 \leq x^2 + y^2 \leq 1$.*

If $\mathbf{F}(x, y, z) = 4x\mathbf{i} - z^2\mathbf{j} + e^{xy}\mathbf{k}$, use the Divergence Theorem to find the surface integral $\int\int_S \mathbf{F} \bullet d\mathbf{S}$, where the orientation of S is given by the outer normal vector.

Answer to Question 8(b) of 2007-2008 Semester 2

- The solid bounded by the surface is just the upper ball,denoted by V. And orientation of the surface is given with the outward orientation.
- Use Divergence Theorem, we get $\int\int_S \mathbf{F} \bullet d\mathbf{S} = \int\int\int_V div(\mathbf{F})dV = 4\int\int\int_V dV = 4 \times \frac{1}{2} \times \frac{4\pi}{3} = \frac{8\pi}{3}$.

Question 6(a) of 2009-2010 Semester 1

Problem

Find the ***exact value*** *of the double integral*

$$\int\int_D \sqrt{|x-y|}dxdy,$$

where D is the rectangular region : $0 \leq x \leq 1$ *and* $0 \leq y \leq 2$

- We divide the region D into two parts,denoted as D_1 and D_2.D_1 is the region $\{(x,y) \in D : x \geq y\}$. $D_2 = \{(x,y) \in D : x \leq y\}$. Then we have:

$$\int\int_D \sqrt{|x-y|}dxdy = \int_{D_1} \sqrt{x-y}dA + \int_{D_2} \sqrt{y-x}dA.$$

- For integral over D_1,

$$\int\int_{D_1} \sqrt{x-y}\,dA = \int_0^1 \int_y^1 \sqrt{x-y}\,dxdy = \int_0^1 \frac{2}{3}(1-y)^{\frac{3}{2}}dy = \frac{4}{15}.$$

- For integral over D_2,

$$\int\int_{D_2} \sqrt{y-x}\,dA = \int_0^1 \int_x^2 \sqrt{y-x}\,dydx = \int_0^1 \frac{2}{3}(2-x)^{\frac{3}{2}}dx = \frac{4}{15}(4\sqrt{2}-1).$$

- In summary, so the required integral is $\frac{4}{15} + \frac{4}{15}(4\sqrt{2}-1) = \frac{16\sqrt{2}}{15}$.

Question 6(b) of 2009-2010 Semester 1

Problem

Find the ***exact value*** *of the iterated integral*

$$\int_0^6 [\int_x^6 \frac{2xy}{\ln(1+y^2)^{(1+x^2)}} dy] dx.$$

- It's obvious that $\ln(1+y^2)^{(1+x^2)} = (1+x^2)\ln(1+y^2)$. Then we have the original integral is $\int_0^6 [\int_x^6 \frac{2xy}{\ln(1+y^2)^{(1+x^2)}} dy] dx = \int_0^6 \frac{2x}{1+x^2} [\int_x^6 \frac{y}{\ln(1+y^2)} dy] dx$. For every $0 \le x \le 6$,it's impossible to calculate the integral $\int_x^6 \frac{y}{\ln(1+y^2)} dy$. So we need to change the order of the iterated integral.

- First we need to change the iterated integral to Double integral. The region of the Double integral is $D = \{(x, y) : 0 \leq x \leq 6, x \leq y \leq 6\}$. Then we have $\int_0^6 [\int_x^6 \frac{2xy}{\ln(1+y^2)^{(1+x^2)}} dy] dx = \int\int_D \frac{2xy}{\ln(1+y^2)^{(1+x^2)}} dA$
- In the original iterated integral D is taken as type A region. Then here we need to take D as type B region. Then we have

$$\int\int_D \frac{2xy}{\ln(1+y^2)^{(1+x^2)}} dA = \int_0^6 [\int_0^y \frac{2xy}{\ln(1+y^2)^{(1+x^2)}} dx] dy =$$
$$\int_0^6 \frac{y}{\ln(1+y^2)} [\int_0^y \frac{2x}{1+x^2} dx] dy = \int_0^6 y dy = 18.$$

Question 5(a) of 2009-2010 Semester 2

Problem

On a certain mountain, the elevation z above a point (x, y) in an xy-plane at sea level is

$$z = f(x, y) = 3205 - 0.02x^2 - 0.01y^2,$$

where x,y and z are in meters. The positive x-axis points east, and the positive y-axis points north. A mountain climber is at the point $P(200, 300, 1505)$. Find the direction,given as a unit vector $a\mathbf{i} + b\mathbf{j}$, of the steepest ascent. If it's negative, it means that the climber does not ascent

- How to understand steepest ascent? Given a unit vector $\vec{u}$ and a small distance dt, the ascent approximately equals $D_{\vec{u}}(P) \cdot dt$. If it's negative, it means the climber declines other than increases. The most important value is $D_{\vec{u}}(P)$.

Answer to Question 5(a) of 2009-2010 Semester 2

- Similar with the *Qn 3 of Tu6* , we can find the direction which gives the largest directional derivative $D_{\vec{u}}(P)$. Suppose unit vector $\vec{u} = a\mathbf{i} + b\mathbf{j}$, then $D_{\vec{u}}(P) = (f_x(P), f_y(P)) \cdot (a, b)$.By the rule of vector inner product, when vector (a, b) and $(f_x(P), f_y(P))$ are parallel and with the same direction, their inner product gets maximum. so we know that $(a, b) = (\frac{f_x(p)}{\sqrt{f_x^2(P)+f_y^2(P)}}, \frac{f_y(P)}{\sqrt{f_x^2(P)+f_y^2(P)}})$.
  Simple calculation we know that $f_x(P) = -8$,$f_y(P) = -6$,so $(a, b) = (-\frac{4}{5}, -\frac{3}{5})$. Then direction is $-\frac{4}{5}\mathbf{i} - \frac{3}{5}\mathbf{j}$.