NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2003-2004

MA1506 MATHEMATICS II

April 2004 Time allowed: 2 hours 30 minutes

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided below.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **TEN (10)** questions and comprises **FORTY ONE (41)** printed pages.
3. Answer **ALL** questions. Write your answers and working in the spaces provided inside the booklet following each question.
4. The marks for each question are indicated at the beginning of the question.
5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Matriculation Number:

For official use only. Do not write below this line.

Question	1	2	3	4	5	6	7	8	9	10
Marks										

Answer **all** *the questions.*

Question 1 [10 marks]

Consider the system of equations

$$\begin{array}{rcrcrcrcrcr} x_1 & + & 3x_2 & + & 2x_3 & + & 3x_4 & - & 7x_5 & = & 14 \\ 2x_1 & + & 6x_2 & + & x_3 & - & 2x_4 & + & 5x_5 & = & -2 \\ x_1 & + & 3x_2 & - & x_3 & & & + & 2x_5 & = & -1. \end{array}$$

(i) Find the reduced row echelon form of the augmented matrix of the system.

(ii) Solve the system if it is consistent. Otherwise explain why the system is inconsistent.

Show your working below and on the next three pages.

(Working spaces for Question 1 - Indicate your parts clearly)

(More working spaces for Question 1)

(More working spaces for Question 1)

Question 2 **[10 marks]**

Find the absolute maximum value and absolute minimum value of

$$g(x, y) = xy - x - 3y$$

on the triangular region in the xy-plane with vertices $(0,0)$, $(4,0)$ and $(0,4)$.

Show your working below and on the next three pages.

(Working spaces for Question 2 - Indicate your parts clearly)

(More working spaces for Question 2)

(More working spaces for Question 2)

Question 3 [10 marks]

(a) Let $\mathbf{F} = e^y\mathbf{i} + (xe^y + e^z)\mathbf{j} + ye^z\mathbf{k}$ be a force field in the xyz-space.

(i) Find the potential function f for $\mathbf{F}$ such that $f(0,0,0) = 0$.

(ii) Suppose $\mathbf{F}$ moves a particle P from the point $(0,2,0)$ along a straight line to the point $(1,2,3)$, and then from $(1,2,3)$ along a straight line to $(3,10,5)$, and finally from $(3,10,5)$ along a straight line to $(4,0,3)$. Find the total work done by $\mathbf{F}$ on P along the above path.

(b) Suppose D is a plane region with boundary curve ∂D (positively oriented). Show that

$$\frac{1}{2}\oint_{\partial D} xdy - ydx$$

is the area of D.

Show your working below and on the next three pages.

(Working spaces for Question 3 - Indicate your parts clearly)

(More working spaces for Question 3)

(More working spaces for Question 3)

Question 4 [10 marks]

Let D be a solid region bounded by the cylinder $x^2 + y^2 = 1$, the planes $z = x + 2$ (on top) and $z = 0$ (below).

(i) Compute the triple integral $\iiint_D x^3 + xy^2 \, dV$.

(ii) Use part (i) to compute the surface integral $\iint_S \mathbf{F} \bullet d\mathbf{S}$ where

$$\mathbf{F} = x^4\mathbf{i} - x^3z^2\mathbf{j} + 4xy^2z\mathbf{k}$$

and S is the surface enclosing the region D with positive orientation.

Show your working below and on the next three pages.

(Working spaces for Question 4 - Indicate your parts clearly)

(More working spaces for Question 4)

(More working spaces for Question 4)

Question 5 [10 marks]

Use Laplace transforms to find the solution $w(x,t)$ of

$$w_x - w_t - 2w \;=\; 0, \qquad w(x,0) = e^{-x},$$

which is bounded for $x > 0$, $t > 0$.

(The Laplace transform of e^{at} is $\dfrac{1}{s-a}$.)

Show your working below and on the next three pages.

(Working spaces for Question 5 - Indicate your parts clearly)

(More working spaces for Question 5)

(More working spaces for Question 5)

Question 6 **[10 marks]**

(i) Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 6 & 9 \\ 1 & 6 \end{bmatrix}.$$

(ii) Solve the linear system

$$y_1' = 6y_1 + 9y_2, \quad y_2' = y_1 + 6y_2$$

given the initial conditions

$$y_1(0) = 3, \quad y_2(0) = 0.$$

Show your working below and on the next three pages.

(Working spaces for Question 6 - Indicate your parts clearly)

(More working spaces for Question 6)

(More working spaces for Question 6)

Question 7 [10 marks]

(a) Determine the equation of the plane containing the point $P(1,3,1)$ and the space curve

$$C: \quad x = \sin t, \quad y = \sin t, \quad z = \sin t + 2.$$

(b) A particle is thrown upward from the top of a building 160 feet high with an elevation of 45^0 with the horizontal. If the initial speed was 32 feet per second, how far from the base of the building will the particle strike the ground? Give your answer *correct to one decimal place.* [Take g = 32 ft/s^2.]

Show your working below and on the next three pages.

(Working spaces for Question 7 - Indicate your parts clearly)

(More working spaces for Question 7)

(More working spaces for Question 7)

Question 8 **[10 marks]**

(a) Let $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$.

(i) Find curl $\mathbf{F}$.

(ii) Use Stoke's Theorem to evaluate the line integral $\int_C \mathbf{F} \bullet d\mathbf{r}$ where C is the triangle with vertices $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ oriented counter-clockwise when viewed from above.

(b) Is there a vector field $\mathbf{G}$ such that

$$\text{curl } \mathbf{G} = yz\mathbf{i} + xyz\mathbf{j} + xy\mathbf{k}?$$

Justify your answer.

Show your working below and on the next three pages.

(Working spaces for Question 8 - Indicate your parts clearly)

(More working spaces for Question 8)

(More working spaces for Question 8)

Question 9 [10 marks]

Evaluate the following iterated integral by changing to spherical coordinates:

$$\int_{-\sqrt{3}}^{\sqrt{3}} \int_{0}^{\sqrt{3-x^2}} \int_{\sqrt{(x^2+y^2)/3}}^{\sqrt{4-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dzdydx$$

Show your working below and on the next three pages.

(Working spaces for Question 9 - Indicate your parts clearly)

(More working spaces for Question 9)

(More working spaces for Question 9)

Question 10 [10 marks]

(a) If $u(x,t)$ satisfies the boundary value problem

$$u_x = 2u_t, \qquad u(0,t) = 3e^{-t} + 4e^{t},$$

use the method of separation of variables to find $u(x,t)$.

(b) Let c is a positive constant. Consider the heat equation

$$u_t = c^2 u_{xx}, \tag{1}$$

with boundary conditions

$$u(0,t) = 0, \qquad u(\pi,t) = 0 \qquad \text{for all } t, \tag{2}$$

and the initial condition

$$u(x,0) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 \le x \le 2 \\ 0 & \text{if } 2 < x \le \pi. \end{cases} \tag{3}$$

Suppose

$$u_n(x,t) = B_n e^{-n^2 t/2} \sin nx, \quad \text{where } B_n \text{ is a constant,}$$

is a solution of (1) and (2) for each $n = 1, 2, 3, \ldots$

(i) Find suitable values of B_n so that a solution $u(x,t)$ can be obtained which satisfies (1), (2) and (3). (*Leave your answer in terms of n.*)

(ii) Find the value of c.

Show your working below and on the next three pages.

(Working spaces for Question 10 - Indicate your parts clearly)

(More working spaces for Question 10)

(More working spaces for Question 10)

[END OF PAPER]