Chapter 4. Sequences and Series

4.1 Infinite sequences

An infinite sequence (or sequence) of real numbers is an infinite succession of numbers which is usually given by some rule.

We shall denote an infinite sequence by

$$a_1, a_2, a_3, \cdots, a_n, \cdots,$$

and we shall often write the sequence as $\{a_n\}$; and for each n, the number a_n is called a term of the sequence.

4.1.1 Example

(i) The sequence

$$0, 1, 2, \cdots, n-1, \cdots$$

is defined by the rule $a_n = n - 1$.

(ii) The sequence

$$1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots$$

is defined by $a_n = \frac{1}{n}$.

(iii) If $a_n = (-1)^{n+1}(\frac{1}{n})$, the sequence is

$$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \cdots .$$

(iv) If $a_n = \frac{n-1}{n}$, the sequence is

$$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots .$$

(v) If $a_n = (-1)^{n+1}$, the sequence is

$$1, -1, 1, -1, \cdots .$$

(vi) If $a_n = 3$, the sequence is

$$3, 3, 3, \cdots$$

4.1.2 Limits of sequences

A number L is called the <u>limit</u> of a sequence $\{a_n\}$, if for sufficiently large n, we can get a_n as close as we want to a number L.

We write $\lim\limits_{n\to\infty} a_n = L$, or simply, $a_n \to L$,

Note that the limit of a sequence $\{a_n\}$ is unique.

4.1.3 Convergent and divergent

Not all sequences have limits.

If $\{a_n\}$ has a limit, we say the sequence is convergent and $\{a_n\}$ converges to L.

If $\{a_n\}$ does not have a limit, we say $\{a_n\}$ is divergent.

4.1.4 **Example**

(i) $0, 1, 2, 3, \cdots$ is divergent.

(ii) $1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots$ is convergent, its limit is 0.

(iii) $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots$ converges to 1.
(since $\lim\limits_{n\to\infty} \dfrac{n-1}{n} = 1$. See example 4.1.6 (ii).)

(iv) $1, -1, 1, -1, \cdots$ is divergent.

(v) $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \cdots$. converges to 0.

(vi) If c is any real number, $c, c, c, \cdots$ clearly converges to c. Such a sequence is called a constant sequence.

4.1.5 Some Rules on Limits

Let $\lim_{n\to\infty} a_n = A$, and $\lim_{n\to\infty} b_n = B$, with A and B real numbers.

(1) Sum rule: $\lim_{n\to\infty}(a_n + b_n) = A + B$.

(2) Difference rule: $\lim_{n\to\infty}(a_n - b_n) = A - B$.

(3) Product rule: $\lim_{n\to\infty}(a_n b_n) = AB$.

(4) Quotient rule: $\lim_{n\to\infty}\dfrac{a_n}{b_n} = \dfrac{A}{B}$, if $B \neq 0$.

Using the above rules, we obtain:

4.1.6 Example

(i) $\lim_{n\to\infty}(-\frac{1}{n}) = (-1)\lim_{n\to\infty}\frac{1}{n} = -1(0) = 0.$

(ii) $\lim\limits_{n\to\infty} \dfrac{n-1}{n} = \lim\limits_{n\to\infty}(1 - \dfrac{1}{n})$

$= \lim\limits_{n\to\infty} 1 - \lim\limits_{n\to\infty} \dfrac{1}{n} = 1 - 0 = 1.$

(iii) $\lim\limits_{n\to\infty} \dfrac{5}{n^2} = 5 \lim\limits_{n\to\infty} \dfrac{1}{n} \cdot \lim\limits_{n\to\infty} \dfrac{1}{n}$

$= 5 \cdot 0 \cdot 0 = 0.$

(iv) $\lim\limits_{n\to\infty} \dfrac{4-7n^6}{n^6+3} = \lim\limits_{n\to\infty} \dfrac{\frac{4}{n^6} - 7}{1 + \frac{3}{n^6}} = \dfrac{0-7}{1+0} = -7.$

(Note that all the four rules in (4.1.5) are used in this example.)

4.1.7 Sequence and function

Let $\{a_n\}$ be a sequence. Suppose there is a function $f(x)$ such that $a_n = f(n)$.

If $\lim\limits_{x\to\infty} f(x) = L$, then

$$\lim_{n\to\infty} a_n = L.$$

4.1.8 Example

Consider the sequence $\{a_n\}$ where $a_n = \sqrt{n+1} - \sqrt{n}$.

Then $a_n = f(n)$ where

$f(x)$ is the function $\sqrt{x+1} - \sqrt{x}$. So

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty}(\sqrt{x+1}-\sqrt{x}) = \lim_{x\to\infty} \frac{1}{\sqrt{x+1}+\sqrt{x}} = 0.$$

4.1.9 Example

Show that $\displaystyle\lim_{n\to\infty} \frac{\ln n}{n} = 0$.

Consider the function $\frac{\ln x}{x}$ (defined for $x > 0$).

$$\lim_{x\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

We apply L'Hopital's rule in the second equality above.

4.2 Infinite series

4.2.1 Definition.

Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is called an infinite series. The term a_n is the nth term of the series.

For example,

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots$$

is an infinite series whose nth term is $\frac{1}{2^n}$.

The sequence $\{s_n\}$ defined by

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_n &= a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k \end{aligned}$$

is called the sequence of partial sums of the series.

The number s_n is called the nth partial sum.

If the sequence of partial sums $\{s_n\}$ converges to a limit L, then we say that the series is convergent and that its sum is L. We write

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots = L.$$

If the sequence of partial sums does not converge, we say that the series is divergent.

Note that

$$\sum_{n=1}^{\infty} a_n, \quad \sum_{k=1}^{\infty} a_k, \quad \sum_{r=1}^{\infty} a_r, \quad \sum a_n, \quad \sum a_k, \quad \sum a_r$$

represent the same series.

4.2.2 Geometric series

The series

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

is called a geometric series, where a and r are fixed numbers, a is called the first term, and r is the (common) ratio.

For this series, the nth partial sum s_n is given by

$$\begin{aligned} s_n &= a + ar + ar^2 + \cdots + ar^{n-1} \\ rs_n &= ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n. \end{aligned}$$

Thus $s_n - rs_n = a - ar^n$,

$$\Longrightarrow s_n = a\frac{1-r^n}{1-r}, \qquad r \neq 1.$$

If $r = 1$, then clearly $s_n = na \to \infty$ (or $-\infty$), if $a \neq 0$, and the series is divergent.

If $|r| < 1$, then $r^n \to 0$. Thus

$$s_n \to \frac{a}{1-r},$$

and the sum of the series is $\dfrac{a}{1-r}$.

If $|r| > 1$, then $|r|^n \to \infty$, and the series diverges.

We summarize this as follow:

4.2.3 Convergence of Geometric series

The geometric series

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$$

with $a \neq 0$ converges to the sum $\dfrac{a}{1-r}$ if $|r| < 1$ and it diverges if $|r| \geq 1$.

4.2.4 Example

(i) $\dfrac{1}{9} + \dfrac{1}{27} + \dfrac{1}{81} + \cdots$ is a geometric series whose first term is $\frac{1}{9}$ and common ratio $\frac{1}{3}$. It converges to $\frac{1}{6}$.

(ii) $4 - 2 + 1 - \dfrac{1}{2} + \dfrac{1}{4} - \cdots$ converges to $\frac{8}{3}$.

4.2.5 Some rules on series

If $\sum a_n = A$, and $\sum b_n = B$, then

(1) Sum rule. $\sum(a_n + b_n) = A + B$.

(2) Difference rule. $\sum(a_n - b_n) = A - B$.

(3) Constant multiple rule. $\sum(ka_n) = kA$.

4.2.6 Ratio test

Let $\sum a_n$ be a series, and let

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho.$$

Then

(1) the series converges if $\rho < 1$.

(2) the series diverges if $\rho > 1$.

(3) no conclusion if $\rho = 1$.

4.2.7 **Example**

(i) $a_1 = 1$, $a_{n+1} = \dfrac{n}{2n+1} a_n$ and the series is

$$\sum a_n = 1 + \frac{1}{3} + \frac{1\cdot 2}{3\cdot 5} + \frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7} + \cdots .$$

For this series $\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{n}{2n+1} \to \dfrac{1}{2}$ as $n \to \infty$. So the ratio test implies the convergence of the series.

(ii) $\sum \dfrac{(n!)^2}{(2n)!}$

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{(n+1)!(n+1)!}{(2n+2)!}\,\frac{(2n)!}{n!n!}\\ &= \frac{(n+1)(n+1)}{(2n+2)(2n+1)} = \frac{n+1}{2(2n+1)} \to \frac{1}{4}.\end{aligned}$$

So the given series is convergent by ratio test.

(iii) $\sum \dfrac{3^n}{2^n+5}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{3^{n+1}}{2^{n+1}+5}\frac{2^n+5}{3^n} = 3\cdot\frac{1+5\cdot 2^{-n}}{2+5\cdot 2^{-n}} \to \frac{3}{2}.$$

By ratio test, $\sum \dfrac{3^n}{2^n+5}$ is divergent.

(iv) $\sum \frac{1}{n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}} \to 1.$$

We cannot draw conclusion from ratio test.

(In fact, this series is divergent.)

(v) $\sum \frac{1}{n^2}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n^2}{(n+1)^2} = \frac{1}{(1+\frac{1}{n})^2} \to 1.$$

We cannot draw conclusion from ratio test.

(In fact, this series is convergent.)

4.3 Power Series

4.3.1 Power series about $x = 0$

A **power series** about $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

where $c_0, c_1, \ldots, c_n, \ldots$ are constants while x is a variable.

So a power series can be regarded as a function for x where it converges.

4.3.2 Example

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots.$$

This power series about $x = 0$ converges to $\dfrac{1}{1-x}$ when $|x| < 1$.

We state this as

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots, \quad -1 < x < 1.$$

4.3.3 Power series about $x = a$

More generally, a **power series** about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots .$$

The number a is called the centre of the power series.

4.3.4 Convergence of power series

We observe that a power series always behaves in exactly **one** of the following three ways:

Case 1. The series $\sum c_n(x-a)^n$ converges at $x = a$ and diverges elsewhere.

Case 2. There is a positive number h such that the series converges for all x in the interval $(a-h, a+h)$ but diverges for all $x > a + h$ and $x < a - h$.

The series may or may not converge at either of the end points $x = a - h$ and $x = a + h$.

Case 3. The series converges for every x.

4.3.5 Radius of Convergence

The number h in case 2 of the previous section is called the *radius of convergence* of the series.

Note that we can describe all the points in the interval in this case as $|x-a| < h$. Here a is at the center of the interval.

If the power series converges for all x, we say that the radius of convergence is infinite ∞.

If it converges only at a, we say that the radius of convergence is zero.

4.3.6 Example

(i) Find the radius of convergence of the power series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots .$$

<u>Solution.</u> We apply ratio test to the series with nth term $u_n = (-1)^{n-1} \dfrac{x^n}{n}$.

$$\left| \frac{u_{n+1}}{u_n} \right| = \frac{n}{n+1} |x| \to |x| \text{ as } n \to \infty.$$

Therefore, the series converges for $|x| < 1$. It diverges if $|x| > 1$.

The radius of convergence is therefore equal to 1.

(ii) For the series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{(n+1)!}\,\frac{n!}{x^n}\right| = \frac{|x|}{n+1} \to 0 \text{ as } n \to \infty.$$

Therefore, the series converges for all x.

The radius of convergence is therefore equal to ∞.

(iii) For the series $\displaystyle\sum_{n=0}^{\infty} n!x^n = 1+x+2!x^2+3!x^3+\cdots.$

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = (n+1)|x| \to \infty$$

as $n \to \infty$ unless $x = 0$.

Therefore, the series diverges for all x except $x = 0$.

The radius of convergence is zero.

4.3.7 Differentiation and Integration of power series

If $\sum c_n(x-a)^n$ has radius of convergence h,

it defines a function f:

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, \quad a-h < x < a+h.$$

(i) The function f has derivatives of all orders in $(a-h, a+h)$. The derivatives can be obtained by differentiating the power series term-by-term:

$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}, \\ f''(x) &= \sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2}, \ldots . \end{aligned}$$

The differentiated series converges for $a-h < x < a+h$.

(ii) The function f has anti-derivatives in $(a-h, a+h)$. The anti-derivatives can be obtained by integrating the power series term-by-term:

$$\int f(x)dx = \sum_{0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + c.$$

The integrated series converges for $a-h < x < a+h$.

4.3.8 **Example**

$$\begin{aligned} f(x) &= \frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots, \\ &\quad -1 < x < 1 \\ f'(x) &= \frac{1}{(1-x)^2} = 1 + 2x + \cdots + nx^{n-1} + \cdots, \\ &\quad -1 < x < 1 \\ f''(x) &= \frac{2}{(1-x)^3} = 2 + 6x + \cdots + n(n-1)x^{n-2} \\ &\quad + \cdots, \quad -1 < x < 1 \end{aligned}$$

4.3.9 Example

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots, \quad -1 < t < 1$$

So

$$\begin{aligned}\ln(1+x) &= \int_0^x \frac{dt}{1+t} \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + - \cdots, \\ &\quad -1 < x < 1.\end{aligned}$$

4.4 Taylor Series

4.4.1 Definition

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point.

The **Taylor series** of f at a is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots \quad (1)$$

4.4.2 Example

The Taylor series of e^x at $x = 0$ is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

This follows from the formula (1) and the fact that

$$\frac{d}{dx}e^x = e^x.$$

The radius of convergence of this series is ∞.

4.4.3 Example

The Taylor series of $\sin x$ and $\cos x$ at $x = 0$ are

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

This follows from the formula (1) and the fact that

$$\frac{d}{dx} \sin x = \cos x$$

and

$$\frac{d}{dx} \cos x = -\sin x$$

The radius of convergence of these two series is ∞.

4.4.4 Example

The Taylor series of $\ln(1+x)$ at $x=0$ is

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n}$$

This follows from integrating the following geometric series from 0 to x:

$$\frac{1}{1+t} = \sum_{n=0}^{\infty}(-1)^n t^n$$

The radius of convergence of this series is 1.

4.4.5 Example

The Taylor series of $\tan^{-1} x$ at $x=0$ is

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

This follows from integrating the following geometric series from 0 to x:

$$\frac{1}{1+t^2} = \sum_{n=0}^{\infty} (-1)^n t^{2n}$$

The radius of convergence of this series is 1.

4.4.6 Example

Find the Taylor series of $\dfrac{1}{2x+1}$ at $x = -2$.

Solution.

$$\begin{aligned}
\frac{1}{2x+1} &= \frac{1}{2(x+2)-4+1} = \frac{1}{-3+2(x+2)} \\
&= -\frac{1}{3}\left(\frac{1}{1-(\frac{2}{3}(x+2))}\right) \\
&= -\frac{1}{3}\sum_{0}^{\infty}\left(\frac{2}{3}(x+2)\right)^n \\
&= \sum_{0}^{\infty}\left(-\frac{2^n}{3^{n+1}}\right)(x+2)^n
\end{aligned}$$

The radius of convergence of this series is $3/2$.

4.4.7 Taylor polynomials

The nth order **Taylor polynomial** of f at a is

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \;=\; f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

It provides the best polynomial approximation of degree n.

4.4.8 Example

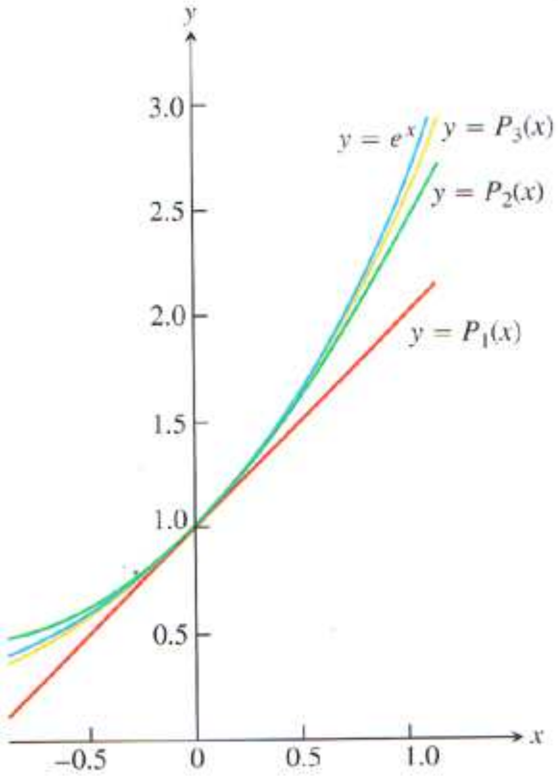
The Taylor polynomials of e^x at $x=0$ of order 1, 2 and 3:

$$P_1(x) = 1 + x$$

$$P_2(x) = 1 + x + \frac{x^2}{2!}$$

$$P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

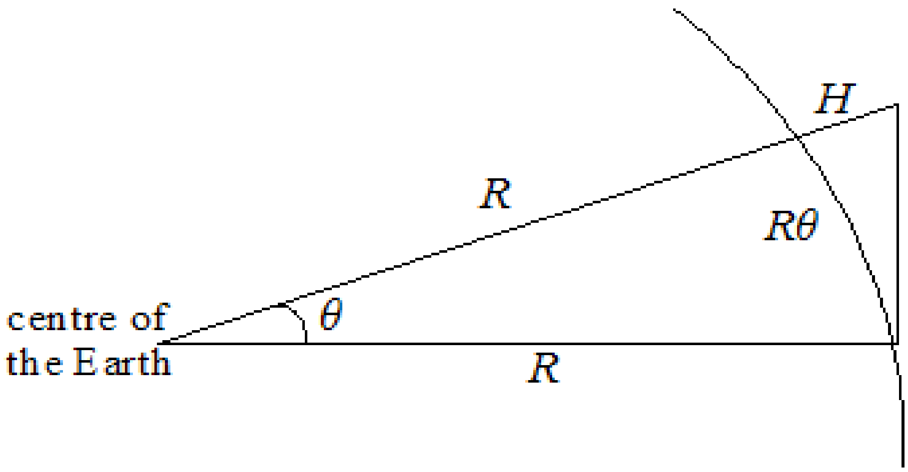
In the diagram below, notice the very close agreement between e^x and its Taylor polynomials near $x = 0$.



Note that the graph of $P_1(x)$ is in fact the tangent line of e^x at $x = 0$.

4.4.9 An application of Taylor polynomials

Suppose you are at the top of a lighthouse, height H above sea level. How far out to sea can you see? The most distant spots are called the HORIZON.



From the diagram you can see that the ray of light from the centre of the earth, define a right angled triangle. Simple trigonometry gives us

$$\frac{R}{R+H} = \cos\theta$$

and we can write this as

$$\left(1+\frac{H}{R}\right)^{-1} = \cos\theta.$$

Now of course H/R and θ are extremely small numbers. The radius of the Earth is about 6370 km, while a very tall lighthouse might be 0.1 km tall.

For the left hand side, we approximate it with the order 1 Taylor polynomial:

$$\frac{1}{1+\frac{H}{R}} \approx 1 - \frac{H}{R}$$

while for the right hand side, we approximate it with the order 2 Taylor polynomial (note that the order 1 Taylor polynomial of $\cos\theta$ is 1):

$$\cos\theta \approx 1 - \frac{\theta^2}{2}.$$

So we obtain

$$1 - \frac{H}{R} = 1 - \frac{\theta^2}{2}$$

approximately. Thus

$$R^2\theta^2 = 2RH$$

and so the distance to the horizon (measured along the curved surface of the ocean) is

$$R\theta = (2R)^{1/2}H^{1/2} = 113H^{1/2}$$

where everything is measured in kilometers.

For a lighthouse 100m (=0.1km) in height, that comes to about 35.7 km. Notice that if you double the height of your lighthouse, you don't double the distance you can see (or from which you can be seen)!