NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATIONS 2004-2005

MA1505 Mathematics 1

May 2005 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write your matriculation number neatly in the space below.**
2. Do not insert loose papers into this booklet. This booklet will be collected at the end of the examination.
3. This examination paper contains a total of **TEN (10)** questions and comprises **TWENTY-THREE (23)** printed pages.
4. Answer **ALL** 10 questions. The marks for each question are indicated at the beginning of the question.
5. **Write your solution in the space below each question.**
6. Calculators may be used. However, you should lay out systematically the various steps in your calculations.

Matriculation Number:

For official use only. Do not write in the boxes below.

1	2	3	4	5

6	7	8	9	10	Total

Answer **all** *the questions.*

Question 1 [10 marks]

Find the following limits, if they exist:

(a) $\lim_{x \to 0} \dfrac{\ln(\cos^6 mx)}{\ln(\cos^2 nx)}$, where m and n are positive constants;

(b) $\lim_{x \to +\infty} \left(\dfrac{x-2}{x+2} \right)^{2x}$.

(More space for the solution to Question 1.)

Question 2 [10 marks]

Let $f : \mathbb{R} \longrightarrow \mathbb{R}, \quad f(x) = 2x^5 - 5x^4 + 3.$

(i) Find the intervals on which f is increasing or decreasing.

(ii) Find the intervals on which the graph of f is concave up or concave down.

(iii) Find the relative extrema, if any, of f.

(iv) Sketch the graph of f, indicating clearly any relative extrema.

(*More space for the solution to Question 2.*)

(*More space for the solution to Question 2.*)

Question 3 [10 marks]

Find the derivative $\dfrac{dy}{dx}$ of the following functions:

(a) $y = (\ln x)^x$ for $x > 1$;

(b) $y = \displaystyle\int_{-x}^{x^2} \frac{5}{3+t^4}\, dt$ for all $x \in \mathbb{R}$.

(*More space for the solution to Question 3.*)

Question 4 [10 marks]

The region R in the first quadrant is bounded by the graph of

$$f : \{x \in \mathbb{R} \mid x > 5\} \longrightarrow \mathbb{R}, \qquad f(x) = \frac{4}{(x-1)\sqrt{x-5}},$$

the $x-$axis and the lines $x = 7$ and $x = 9$. Find volume of the solid generated when R is revolved about the $x-$axis.

(*The graph need not be sketched.*)

(More space for the solution to Question 4.)

Question 5 [10 marks]

Determine whether the following series converge or diverge, showing clearly all the calculations leading to your conclusion:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(3n)!}{3^{4n}\,(n!)^3}$;

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{n \ln n}{n^3+3}$.

(*More space for the solution to Question 5.*)

Question 6 [10 marks]

Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}(3x-2)^n.$$

(More space for the solution to Question 6.)

Question 7 [10 marks]

Find the Fourier series of the function

$$f(x) = |x| \quad \text{for} \quad -\pi < x < \pi, \qquad \text{and} \quad f(x+2\pi) = f(x).$$

Hence, find the value of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

(More space for the solution to Question 7.)

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Question 8 [10 marks]

Solve the differential equation

$$y'' - 4y = (8x-2)e^{2x}.$$

(*More space for the solution to Question 8.*)

Question 9 [10 marks]

Solve the differential equation

$$e^{x}y' \;=\; \sin\left(e^{-x}\right) \;+\; e^{x}\cos\left(e^{-x}\right) \;-\; e^{-x} \;-\; e^{x}y.$$

(*More space for the solution to Question 9.*)

Question 10 [10 marks]

Use Laplace transforms to solve the initial value problem

$$y'' + 9y = 27(t-2)u(t-2), \quad \text{with } y(0)=0,\ y'(0)=0,$$

where $u(t-2) = \begin{cases} 0 & \text{if } t < 2 \\ 1 & \text{if } t > 2. \end{cases}$

(*More space for the solution to Question 10.*)

Some Formulas

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y \qquad \cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\sin 2x = 2 \sin x \cos x \qquad \cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1 \qquad \int \frac{1}{x} \, dx = \ln|x| + C$$

$$\int \sin kx \, dx = -\frac{\cos kx}{k} + C, \quad k \neq 0 \qquad \int \cos kx \, dx = \frac{\sin kx}{k} + C, \quad k \neq 0$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C \qquad \int \sec^2 x \, dx = \tan x + C$$

$$\int \sec x \tan x \, dx = \sec x + C \qquad \int u \, dv = uv - \int v \, du$$

$$\int \frac{a}{a^2 + x^2} \, dx = \tan^{-1} \frac{x}{a} + C, \quad a \text{ is a positive constant}$$

Variation of parameters: $y_p = uy_1 + vy_2$

$$u = -\int \frac{y_2 r}{y_1 y_2' - y_1' y_2} \, dx, \qquad v = \int \frac{y_1 r}{y_1 y_2' - y_1' y_2} \, dx$$

Fourier series of a function f of period $2L$:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \, dx,$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx, \quad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx, \quad n = 1, 2, 3, \ldots$$

Laplace transforms: $F(s) = \mathcal{L}(f(t)) = \int_0^{\infty} e^{-st} f(t) dt$

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}, \ s > a \qquad \mathcal{L}(t^n) = \frac{n!}{s^{n+1}}, \ n = 0, 1, 2, \ldots$$

$$\mathcal{L}(\sin wt) = \frac{w}{s^2 + w^2} \qquad \mathcal{L}(\cos wt) = \frac{s}{s^2 + w^2}$$

$$\mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0) \qquad \mathcal{L}(f''(t)) = s^2 \mathcal{L}(f(t)) - sf(0) - f'(0)$$

$$\mathcal{L}(e^{ct} f(t)) = F(s - c) \qquad \mathcal{L}(f(t-a)u(t-a)) = e^{-as} F(s), \ a > 0$$

END OF PAPER