NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

MA1506 Laboratory 3 (scilab)
Semester II 2010/2011

Part A: Working With Matrices

Note: This worksheet is meant to complement chapters 5 and 6 of the lectures.

We can input an $m \times n$ matrix A by

$A = [$ row 1; row 2; ... ; row m $]$

where the n entries of each row are separated by one or more blank spaces. For example:

```
--> A=[3 2 -1 ; 0 1 0 ; 1 2 2]
```

The following commands perform basic operations on matrices A and B:

A+B	matrix addition
A-B	matrix subtraction
t∗A	scalar multiplication, with t scalar
A∗B	matrix multiplication
A^n	raising a square matrix A to a positive integral power n
A'	transpose of A
inv(A)	inverse of an invertible square matrix A
det(A)	compute the determinant of a square matrix A
trace(A)	compute the trace of a square matrix A

Practice

1. In Chapter 5, we constructed a matrix M to forecast weather. To predict the weather 4 days from now and 30 days from today, we can do the following:

```
--> M=[ 0.6 0.3 ; 0.4 0.7 ]
--> M^4
--> M^30
```

2. In Chapter 6, we learnt that rotation about z-axis and x-axis in 3 dimensions do not commute. Use scilab to verify that the two matrix products are really different.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

```
--> A=[ 1 0 0 ; 0 0 -1 ; 0 1 0]
--> B=[ 0 -1 0 ; 1 0 0 ; 0 0 1]
--> A*B
--> B*A
```

3. Compute the transpose of the matrix $M = \begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 9 \end{bmatrix}$ and $N = \begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 9 \\ 2 & 1 & 0 \end{bmatrix}$.

   Verify that $N^T + N$ is symmetric and $N^T - N$ is anti-symmetric.

```
--> M=[ 1 2 4 ; 6 8 9]
--> M'
--> N=[ 1 2 4 ; 6 8 9 ; 2 1 0]
--> N' + N
--> N' - N
```

4. Using M and N defined previously, predict what happens if we try to perform the matrix addition $M + N$ and the matrix multiplications MN and NM. Verify your prediction.

```
--> M + N
--> M*N
--> N*M
```

5. Determine if the following matrices, $C1 = \begin{bmatrix} 2 & 7 & 5 \\ 1 & 3 & -1 \\ 4 & 13 & 3 \end{bmatrix}$ and $C2 = \begin{bmatrix} 2 & 7 & 5 \\ 1 & 3 & -1 \\ 4 & 13 & 4 \end{bmatrix}$

   are invertible.

```
--> C1=[ 2 7 5 ; 1 3 -1 ; 4 13 3]
--> C2=[ 2 7 5 ; 1 3 -1 ; 4 13 4]
--> det(C1)
--> det(C2)
--> inv(C2)
--> inv(C2)*C2
```

6. Let $D = \begin{bmatrix} 5 & 7 & 9 \\ 8 & 8 & 1 \\ 20 & 4 & 6 \end{bmatrix}$ and $E = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Verify that $(DE)^{-1} = E^{-1}D^{-1}$ and $(DE)^T = E^T D^T$.

```
--> D=[ 5 7 9 ; 8 8 1 ; 20 4 6]
--> E=[ 1 2 4 ; 1 1 1 ; 1 1 0]
--> inv(D*E)
--> inv(E)*inv(D)
--> (D*E)'
--> E'*D'
```

7. Recall that in previous labs, we defined an array of values with

```
--> x = 0: 0.2 : 1
--> x*x
```

We should really view this as a 1×6 row vector $\vec{x}$. Hence we get an error when we multiply x to itself. To get the dot product $\vec{x} \cdot \vec{x}$, when $\vec{x}$ is a row vector, we use $\vec{x}\vec{x}^T$, i.e.

```
--> x*x'
```

What will happen if we use $\vec{x}^T\vec{x}$?

```
--> x'*x
```

8. What happens now if $\vec{y}$ is a row vector?

```
--> y = [2 ; 1 ; 5]
--> y*y'
--> y'*y
```

9. Consider the following linear system of equations.

$$\begin{aligned} x_1 - x_2 + x_3 &= 4 \\ x_1 + x_2 &= 1 \\ x_1 + 2x_2 - x_3 &= 0. \end{aligned}$$

We rewrite this system as a matrix equation $A\vec{x} = \vec{b}$ and calculate the determinant of A. Note that $\vec{b}$ is a column vector.

```
--> A= [1 -1 1; 1 1 0; 1 2 -1]
--> b= [ 4; 1; 0]
--> det(A)
```

Since $\det(A) \neq 0$, the matrix is non-singular. We can then solve the system by finding the inverse of A. The required solution is $\vec{x} = A^{-1}\vec{b}$.

```
--> x= inv(A)*b
```

10. In 1966, Leontief used his input-output model to analyze the Israeli economy by dividing it into three segments: Agriculture (A), Manufacturing (M), and Energy (E), as shown in the following technology matrix.

Output \ Input	A	M	E
A	\$0.30	\$0.00	\$0.00
M	\$0.10	\$0.20	\$0.20
E	\$0.05	\$0.01	\$0.02

The export demands on the Israeli economy are listed as follows: Agriculture: \$140 million, Manufacturing: \$20 million and Energy: \$2 million.

To find the total output for each sector required to meet both internal and external demand, we must solve the following system

$$\begin{aligned} A &= 0.30A + 0.00M + 0.00E + 140 \\ M &= 0.10A + 0.20M + 0.20E + 20 \\ E &= 0.05A + 0.01M + 0.02E + 2. \end{aligned}$$

Using the technology matrix T, we have $\vec{x} = (I_3 - T)^{-1}\vec{b}$.

```
--> T=[ .3 0 0 ; .1 .2 .2 ; .05 .01 .02]
--> b=[140; 20; 2]
--> x= inv(eye(3,3)- T)*b
```

The required output is approximately, A = \$200 m , M = \$53 m and E = \$13 m. Note that eye(3,3) is the scilab command for the 3×3 identity matrix.

Part B: Eigenvectors And Eigenvalues

In chapter 6, we saw that the matrix $\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$ has an eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ with corresponding eigenvalue 2. This can be easily computed using the following commands:

```
--> A=[ 1 2 ; 2 -2]
--> [P D]=mtlb_eig(A)
```

The second command computes the eigenvectors of the matrix A and stores them as column vectors in the matrix P. At the same time, the corresponding eigenvalues are stored as the diagonal entries of the matrix D. To work with the eigenvector corresponding to eigenvalue 2, we extract the second column of P and call it v.

```
--> v = P(:,2)
--> A*v
```

From our understanding of eigenvectors, Av should give us $2v$. Note that v is not $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ but a multiple of it. Remember that eigenvectors are never unique, and the **mtlb_eig** function will compute eigenvectors with lengths 1. To get our familiar $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, we multiply the vector v by the scalar $1/v(2)$, where $v(2)$ is the second coordinate of the vector v

```
--> y = v/v(2)
--> A*y
```

We should recognize that that the **mtlb_eig** function is actually trying to diagonalize the matrix A. Recall that $A = PDP^{-1}$, where D is the diagonal matrix with eigenvalues of A as its entries, and P is a square matrix where the columns are the corresponding eigenvectors. Verify this by

```
--> P*D*inv(P)
```

Exercise 3

1. Compute the determinant of

$$\begin{bmatrix} 1 & 1 & 13 & 6.5 & 1.5 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & -1 & 3 & 1 \\ 1 & 7 & -1 & 4 & 9 \\ 1.5 & 2 & 21 & 3 & 1 \end{bmatrix}.$$

   (i) 615.75
   (ii) -516.5
   (iii) 765.0
   (iv) 716.25

2. A car rental agency has three branches A, B and C. The company policy allows cars to be rented from and returned to any one of the three branches. A statistical study revealed that the chances of cars being returned to the same branch where they were rented are 70%, 50% and 40% respectively. There is a 20% likelihood of cars rented from branch A being returned to branch B. A 30% chance of cars rented from branch B being returned to branch C and also a 30% chance of cars rented from branch C being returned to branch A. Assuming the company started with 100 cars at each branch, in the long run, approximately how many cars will remain at branch C?

   (i) 23
   (ii) 70
   (iii) 134
   (iv) 32

3. A small country's economy is divided into three segments: Electronics (E), Manufacturing (M), and Pharmaceutical (P), as shown in the following technology matrix.

Output \ Input	E	M	P
E	\$0.15	\$0.20	\$0.00
M	\$0.10	\$0.30	\$0.00
P	\$0.12	\$0.18	\$0.40

The export demands are as follows: Electronics: \$80 million, Manufacturing: \$20 million and Pharmaceutical: \$10 million.

The electronics output is approximately (nearest million)

(i) \$ 104 million

(ii) \$ 100 million

(iii) \$ 89 million

(iv) \$ 80 million

4. Find the eigenvalues and a matrix P that diagonalizes $\begin{bmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{bmatrix}$.

5. Find the eigenvalues and a matrix P that diagonalizes $\begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix}$.

6. Find the eigenvalues and a matrix P that diagonalizes $\begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix}$.

7. In order to find the eigenvalues of a matrix A, we solve the characteristic equation

$$0 = \det(A - \lambda I) = c_n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0,$$

which is a polynomial equation in λ. The coefficients of the characteristic polynomial can be found with the command

```
--> poly(A,'s')
```

There is a remarkable theorem called the Cayley-Hamilton Theorem which states that a square matrix A satisfies its characteristic equation. Hence

$$c_nA^n + c_{n-1}A^{n-1} + \cdots + c_1A + c_0I_n = 0.$$

Verify this for the three matrices given above.

—The End—