Problem 3.45 Vector field $\mathbf{E}$ is characterized by the following properties: (a) $\mathbf{E}$ points along $\hat{\mathbf{R}}$, (b) the magnitude of $\mathbf{E}$ is a function of only the distance from the origin, (c) $\mathbf{E}$ vanishes at the origin, and (d) $\nabla \cdot \mathbf{E} = 12$, everywhere. Find an expression for $\mathbf{E}$ that satisfies these properties.

Solution: According to properties (a) and (b), $\mathbf{E}$ must have the form

$$\mathbf{E} = \hat{\mathbf{R}} E_R$$

where E_R is a function of R only.

$$\begin{aligned}
\nabla \cdot \mathbf{E} = \frac{1}{R^2} \frac{\partial}{\partial R} (R^2 E_R) &= 12, \\
\frac{\partial}{\partial R} (R^2 E_R) &= 12R^2, \\
\int_0^R \frac{\partial}{\partial R} (R^2 E_R)\, dR &= \int_0^R 12R^2\, dR, \\
R^2 E_R \big|_0^R &= \frac{12R^3}{3} \bigg|_0^R, \\
R^2 E_R &= 4R^3.
\end{aligned}$$

Hence,

$$E_R = 4R,$$

and

$$\mathbf{E} = \hat{\mathbf{R}} 4R.$$

Problem 3.47 For the vector field $\mathbf{E} = \hat{\mathbf{r}}10e^{-r} - \hat{\mathbf{z}}3z$, verify the divergence theorem for the cylindrical region enclosed by $r = 2$, $z = 0$, and $z = 4$.

Solution:

$$\begin{aligned}
\oint \mathbf{E} \cdot d\mathbf{s} &= \int_{r=0}^{2} \int_{\phi=0}^{2\pi} \left((\hat{\mathbf{r}}10e^{-r} - \hat{\mathbf{z}}3z) \cdot (-\hat{\mathbf{z}}r\,dr\,d\phi) \right)\big|_{z=0} \\
&\quad + \int_{\phi=0}^{2\pi} \int_{z=0}^{4} \left((\hat{\mathbf{r}}10e^{-r} - \hat{\mathbf{z}}3z) \cdot (\hat{\mathbf{r}}r\,d\phi\,dz) \right)\big|_{r=2} \\
&\quad + \int_{r=0}^{2} \int_{\phi=0}^{2\pi} \left((\hat{\mathbf{r}}10e^{-r} - \hat{\mathbf{z}}3z) \cdot (\hat{\mathbf{z}}r\,dr\,d\phi) \right)\big|_{z=4} \\
&= 0 + \int_{\phi=0}^{2\pi} \int_{z=0}^{4} 10e^{-2}2\,d\phi\,dz + \int_{r=0}^{2} \int_{\phi=0}^{2\pi} -12r\,dr\,d\phi \\
&= 160\pi e^{-2} - 48\pi \approx -82.77, \\
\iiint \nabla \cdot \mathbf{E}\,d\mathcal{V} &= \int_{z=0}^{4} \int_{r=0}^{2} \int_{\phi=0}^{2\pi} \left(\frac{10e^{-r}(1-r)}{r} - 3 \right) r\,d\phi\,dr\,dz \\
&= 8\pi \int_{r=0}^{2} (10e^{-r}(1-r) - 3r)\,dr \\
&= 8\pi \left(-10e^{-r} + 10e^{-r}(1+r) - \frac{3r^2}{2} \right)\bigg|_{r=0}^{2} \\
&= 160\pi e^{-2} - 48\pi \approx -82.77.
\end{aligned}$$

Problem 3.52 Verify Stokes's theorem for the vector field $\mathbf{B} = (\hat{\mathbf{r}} r\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)$ by evaluating:

(a) $\oint_C \mathbf{B}\cdot d\mathbf{l}$ over the semicircular contour shown in Fig. P3.52(a), and

(b) $\int_S (\nabla\times\mathbf{B})\cdot d\mathbf{s}$ over the surface of the semicircle.

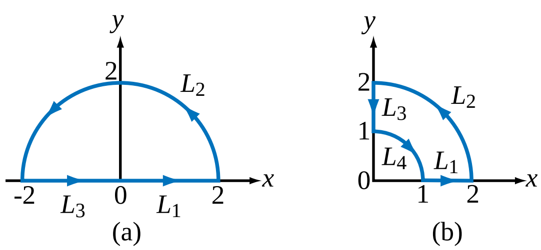

Figure P3.52: Contour paths for (a) Problem 3.52 and (b) Problem 3.53.

Solution:

(a)

$$\oint \mathbf{B}\cdot d\mathbf{l} = \int_{L_1}\mathbf{B}\cdot d\mathbf{l} + \int_{L_2}\mathbf{B}\cdot d\mathbf{l} + \int_{L_3}\mathbf{B}\cdot d\mathbf{l},$$

$$\mathbf{B}\cdot d\mathbf{l} = (\hat{\mathbf{r}} r\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)\cdot(\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}} r\,d\phi + \hat{\mathbf{z}}\,dz) = r\cos\phi\,dr + r\sin\phi\,d\phi,$$

$$\int_{L_1}\mathbf{B}\cdot d\mathbf{l} = \left(\int_{r=0}^{2} r\cos\phi\,dr\right)\Bigg|_{\phi=0,\ z=0} + \left(\int_{\phi=0}^{0} r\sin\phi\,d\phi\right)\Bigg|_{z=0}$$

$$= \left(\tfrac{1}{2}r^2\right)\Big|_{r=0}^{2} + 0 = 2,$$

$$\int_{L_2}\mathbf{B}\cdot d\mathbf{l} = \left(\int_{r=2}^{2} r\cos\phi\,dr\right)\Bigg|_{z=0} + \left(\int_{\phi=0}^{\pi} r\sin\phi\,d\phi\right)\Bigg|_{r=2,\ z=0}$$

$$= 0 + (-2\cos\phi)\big|_{\phi=0}^{\pi} = 4,$$

$$\int_{L_3}\mathbf{B}\cdot d\mathbf{l} = \left(\int_{r=2}^{0} r\cos\phi\,dr\right)\Bigg|_{\phi=\pi, z=0} + \left(\int_{\phi=\pi}^{\pi} r\sin\phi\,d\phi\right)\Bigg|_{z=0}$$

$$= \left(-\tfrac{1}{2}r^2\right)\Big|_{r=2}^{0} + 0 = 2,$$

$$\oint \mathbf{B}\cdot d\mathbf{l} = 2+4+2 = 8.$$

(b)

$$
\begin{aligned}
\nabla\times\mathbf{B} &= \nabla\times(\hat{\mathbf{r}}r\cos\phi+\hat{\boldsymbol{\phi}}\sin\phi)\\
&= \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial}{\partial\phi}0-\frac{\partial}{\partial z}(\sin\phi)\right)+\hat{\boldsymbol{\phi}}\left(\frac{\partial}{\partial z}(r\cos\phi)-\frac{\partial}{\partial r}0\right)\\
&\quad+\hat{\mathbf{z}}\frac{1}{r}\left(\frac{\partial}{\partial r}(r(\sin\phi))-\frac{\partial}{\partial\phi}(r\cos\phi)\right)\\
&= \hat{\mathbf{r}}0+\hat{\boldsymbol{\phi}}0+\hat{\mathbf{z}}\frac{1}{r}(\sin\phi+(r\sin\phi))=\hat{\mathbf{z}}\sin\phi\left(1+\frac{1}{r}\right),\\
\iint\nabla\times\mathbf{B}\cdot d\mathbf{s} &= \int_{\phi=0}^{\pi}\int_{r=0}^{2}\left(\hat{\mathbf{z}}\sin\phi\left(1+\frac{1}{r}\right)\right)\cdot(\hat{\mathbf{z}}r\,dr\,d\phi)\\
&= \int_{\phi=0}^{\pi}\int_{r=0}^{2}\sin\phi(r+1)\,dr\,d\phi=\left(\left(-\cos\phi\left(\tfrac{1}{2}r^2+r\right)\right)\Big|_{r=0}^{2}\right)\Big|_{\phi=0}^{\pi}=8.
\end{aligned}
$$

Problem 3.58 Find the Laplacian of the following scalar functions:

(a) $V_1 = 10r^3 \sin 2\phi$

(b) $V_2 = (2/R^2)\cos\theta \sin\phi$

Solution:

(a)

$$\begin{aligned}
\nabla^2 V_1 &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_1}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V_1}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} \\
&= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}(10r^3 \sin 2\phi)\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}(10r^3 \sin 2\phi) + 0 \\
&= \frac{1}{r}\frac{\partial}{\partial r}(30r^3 \sin 2\phi) - \frac{1}{r^2}(10r^3)4\sin 2\phi \\
&= 90r\sin 2\phi - 40r\sin 2\phi = 50r\sin 2\phi.
\end{aligned}$$

(b)

$$\begin{aligned}
\nabla^2 V_2 = {} & \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V_2}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V_2}{\partial \theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V_2}{\partial \phi^2} \\
= {} & \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial}{\partial R}\left(\frac{2}{R^2}\cos\theta\sin\phi\right)\right) \\
& + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\left(\frac{2}{R^2}\cos\theta\sin\phi\right)\right) \\
& + \frac{1}{R^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}\left(\frac{2}{R^2}\cos\theta\sin\phi\right) \\
= {} & \frac{4}{R^4}\cos\theta\sin\phi - \frac{4}{R^4}\cos\theta\sin\phi - \frac{2}{R^4}\frac{\cos\theta}{\sin^2\theta}\sin\phi \\
= {} & -\frac{2}{R^4}\frac{\cos\theta\sin\phi}{\sin^2\theta}.
\end{aligned}$$