

Example 6-1:

How to use (6.2) to find $\mathcal{L}\{\sin(\omega_o t)u(t)\}$.

$$\begin{aligned}
 \mathcal{L}\{\sin(\omega_o t)u(t)\} &= \int_{0^-}^{\infty} \sin(\omega_o t) \exp(-st) dt \\
 &= \frac{1}{2j} \int_{0^-}^{\infty} [\exp(j\omega_o t) - \exp(-j\omega_o t)] \exp(-st) dt \\
 &= \frac{1}{2j} \int_{0^-}^{\infty} [\exp((j\omega_o - s)t) - \exp(-(j\omega_o + s)t)] dt \\
 &= \frac{1}{2j} \left[\frac{\exp((j\omega_o - s)t)}{(j\omega_o - s)} - \frac{\exp(-(j\omega_o + s)t)}{-(j\omega_o + s)} \right]_{0^-}^{\infty} \\
 &= \frac{1}{2j} \left[-\frac{1}{(\textcolor{red}{j}\omega_o - s)} - \frac{1}{(\textcolor{red}{j}\omega_o + s)} \right] \cdots \underbrace{\text{Re}[s] > 0}_{\text{Region of convergence}} \\
 &= \frac{\omega_o}{s^2 + \omega_o^2}
 \end{aligned}$$

The range of values of s for which the Laplace transform converges is called the region of convergence (ROC).

Can you find $\mathcal{L}\{\cos(\omega_o t)u(t)\}$?

• Multiple Pole Case

If $D(s)$ has multiple roots, i.e., it contains factors of the form $(s + p_n)^r$, we say that $-p_n$ is a **multiple pole of $F(s)$ with multiplicity r** . The expansion of $F(s)$ will consist of terms of the form

$$\frac{\gamma_1}{s + p_n} + \frac{\gamma_2}{(s + p_n)^2} + \cdots + \frac{\gamma_r}{(s + p_n)^r} \quad (6.19)$$

where

$$\gamma_{r-k} = \frac{1}{k!} \frac{d^k}{ds^k} \left[(s + p_n)^r F(s) \right] \Big|_{s=-p_n}; \quad k = 0, 1, \dots, r-1. \quad (6.20)$$

Example 6.4:

$$F(s) = \frac{s^2 + 2s + 5}{(s + 3)(s + 5)^2} = \frac{\alpha_1}{(s + 3)} + \frac{\gamma_1}{(s + 5)} + \frac{\gamma_2}{(s + 5)^2}$$

Using (6.18) : $\alpha_1 = (s + 3)F(s) \Big|_{s=-3} = 2$

Using (6.20) : $\gamma_2 = \underbrace{(s + 5)^2 F(s)}_{\text{with } k=0} \Big|_{s=-5} = -10$

Using (6.20) : $\gamma_1 = \underbrace{\frac{d}{ds} \left[(s + 5)^2 F(s) \right]}_{\text{with } k=1} \Big|_{s=-5} = \frac{d}{ds} \left[\frac{s^2 + 2s + 5}{s + 3} \right] \Big|_{s=-5} = \frac{s^2 + 6s + 1}{(s + 3)^2} \Big|_{s=-5} = -1$

$$\therefore F(s) = \frac{2}{(s + 3)} - \frac{1}{(s + 5)} - \frac{10}{(s + 5)^2} \rightarrow \left(f(t) = \mathcal{L}^{-1} \{ F(s) \} = (2e^{-3t} - e^{-5t} - 10te^{-5t})u(t) \right)$$

Errata

B. $F(s)$ is an Improper Rational Function ($M \geq N$)

If $M \geq N$, we can apply long division to express $F(s)$ in the form

$$F(s) = \frac{N(s)}{D(s)} = Q(s) + \frac{R(s)}{D(s)} \quad (6.21)$$

such that the $\begin{cases} \text{Quotient} & : Q(s) \text{ is a polynomial in } s \text{ with degree } (M - N), \\ \text{Remainder} & : R(s) \text{ is a polynomial in } s \text{ with degree strictly less than } N. \end{cases}$

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The inverse Laplace transform of $R(s)/D(s)$, which is now a **proper rational function**, can be computed by first expanding into partial fractions.

The inverse Laplace transform of $Q(s)$ can be computed by using

$$\mathcal{L}^{-1}\{s^k\} = \frac{d^k}{dt^k} \delta(t); \quad k = 0, 1, 2, \dots \quad (6.22)$$

Example 6.5:

$$F(s) = \frac{2s^2 + 10s + 10}{(s+1)(s+3)} = 2 + \underbrace{\frac{2s+4}{(s+1)(s+3)}}_{\text{by long division}}$$

$$\therefore F(s) = 2 + \frac{1}{s+1} + \frac{1}{s+3} \rightarrow \left(f(t) = \mathcal{L}^{-1}\{F(s)\} = 2\delta(t) + e^{-t}u(t) + e^{-3t}u(t) \right)$$

6.4 Relationship between the Fourier Transform and the Laplace Transform

$$\text{Fourier Transform } \left\{ \right\}: \mathfrak{T}\{f(t)\} = \int_{-\infty}^{\infty} f(t) \exp(-j\omega t) dt \quad (6.23)$$

$$\text{Bilateral Laplace Transform } \left\{ \right\}: \tilde{F}(s) = \int_{-\infty}^{\infty} f(t) \exp(-st) dt \quad (6.24)$$

$$\text{Unilateral Laplace Transform } \left\{ \right\}: F(s) = \int_{0^-}^{\infty} f(t) \exp(-st) dt \quad (6.25)$$

Errata



- From (6.23) and (6.24), we see that the Fourier transform is a special case of the **bilateral** Laplace transform in which $\mathbf{s} = \mathbf{j}\omega$, that is

$$\mathfrak{T}\{f(t)\} = \tilde{F}(s) \Big|_{s=j\omega} \quad (6.26)$$

- Setting $\mathbf{s} = \boldsymbol{\sigma} + \mathbf{j}\omega$ in (6.24), we have

$$\begin{aligned} \tilde{F}(\sigma + j\omega) &= \int_{-\infty}^{\infty} f(t) \exp(-(\sigma + j\omega)t) dt = \int_{-\infty}^{\infty} [f(t) \exp(-\sigma t)] \exp(-j\omega t) dt \\ &= \mathfrak{T}\{f(t) \exp(-\sigma t)\} \end{aligned} \quad (6.27)$$

which shows that the **bilateral** Laplace transform of $f(t)$ can be viewed as the Fourier transform of $f(t)e^{-\sigma t}$.

- Considering the Laplace transform as a generalization of the Fourier transform where the frequency variable is generalized from $j\omega$ to $s = \sigma + j\omega$, the complex variable \mathbf{s} is often referred to as the *complex frequency*.



- It should not be automatically assumed that the Fourier transform of $f(t)$ is the Laplace transform with s replaced by $j\omega$. If $f(t)$ is **absolutely integrable**, i.e. $\underbrace{\int_{-\infty}^{\infty} |f(t)| dt}_{\text{Dirichlet's 4}^{\text{th}}\text{-condition}} < \infty$, then the

Fourier transform of $f(t)$ can be obtained from its bilateral Laplace transform with s replaced by $j\omega$. This is not generally true if $f(t)$ is not absolutely integrable.

The above relationship between the Fourier Transform and the *Bilateral* Laplace Transform extends fully to *Unilateral* Laplace transform if $f(t)$ is, in addition, a right-sided function.

Example 6.6:

	$f(t)$	$F(s)$	$\Im\{f(t)\}$	Right-sided?	Absolutely Integrable?	$\Im\{f(t)\} = F(j\omega)$
<i>Unit Impulse</i>	$\delta(t)$	1	1	Yes	Yes	Yes
<i>Unit Step</i>	$u(t)$	$\frac{1}{s}$	$\pi\delta(\omega) + \frac{1}{j\omega}$	Yes	No	No
<i>Exponential</i>	$\exp(- \alpha t)u(t)$	$\frac{1}{s+ \alpha }$	$\frac{1}{j\omega+ \alpha }$	Yes	Yes	Yes

Example 6.7: Series RC Circuit

For the RC circuit shown, find the voltage $v_c(t)$ across the capacitor C .

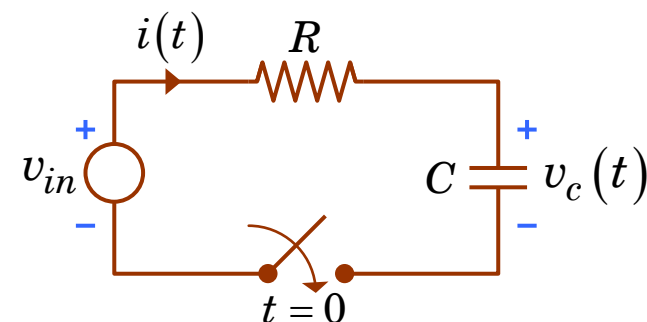
$$RC \underbrace{\frac{dv_c(t)}{dt}}_{i(t)} + v_c(t) = v_{in}$$

$$sRCV_c(s) - RCv_c(0^-) + V_c(s) = \frac{v_{in}}{s}$$

$$V_c(s) = \frac{RCv_c(0^-)}{sRC + 1} + \frac{v_{in}}{s(sRC + 1)} = \frac{RCv_c(0^-)}{sRC + 1} + \frac{v_{in}}{s} - \frac{v_{in}RC}{sRC + 1} \quad (\clubsuit)$$

$$v_c(t) = \mathcal{L}^{-1}\{V_c(s)\} = \left. \begin{aligned} &\mathcal{L}^{-1}\left\{\frac{v_c(0^-) - v_{in}}{s + \frac{1}{RC}}\right\} + \mathcal{L}^{-1}\left\{\frac{v_{in}}{s}\right\} \\ &= \left[v_c(0^-) - v_{in}\right] \exp\left(-\frac{t}{RC}\right) + v_{in} \end{aligned} \right\} \quad (\heartsuit)$$

Errata



From (\clubsuit) and (\heartsuit) , we observe that: $\underbrace{\lim_{t \rightarrow \infty} v_c(t) = \lim_{s \rightarrow 0} sV_c(s)}_{\text{Final Value Theorem}} = v_{in}$

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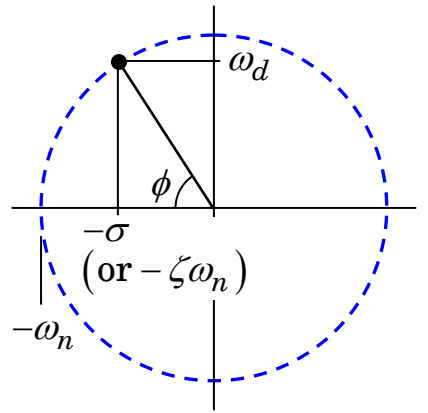
Unilateral Laplace Transform: $X(s) = \int_{0^-}^{\infty} x(t) \exp(-st) dt$

ROC added

	$x(t)$	$X(s)$	ROC
Unit Impulse	$\delta(t)$	1	All s
Unit Step	$u(t)$	$1/s$	$\text{Re}[s] > 0$
Ramp	$tu(t)$	$1/s^2$	$\text{Re}[s] > 0$
n^{th} order Ramp	$t^n u(t)$	$\frac{n!}{s^{n+1}}$	$\text{Re}[s] > 0$

	$x(t)$	$X(s)$	ROC
Exponential	$\exp(-\alpha t)u(t)$	$1/(s + \alpha)$	$\text{Re}[s] > -\text{Re}[\alpha]$
Damped Ramp	$t \exp(-\alpha t)u(t)$	$1/(s + \alpha)^2$	$\text{Re}[s] > -\text{Re}[\alpha]$
Cosine	$\cos(\omega_o t)u(t)$	$s/(s^2 + \omega_o^2)$	$\text{Re}[s] > 0$
Sine	$\sin(\omega_o t)u(t)$	$\omega_o/(s^2 + \omega_o^2)$	$\text{Re}[s] > 0$

	$x(t)$	$X(s)$	ROC
Damped Cosine	$\exp(-\alpha t) \cos(\omega_o t)u(t)$	$(s + \alpha)/[(s + \alpha)^2 + \omega_o^2]$	$\text{Re}[s] > -\text{Re}[\alpha]$
Damped Sine	$\exp(-\alpha t) \sin(\omega_o t)u(t)$	$\omega_o/[(s + \alpha)^2 + \omega_o^2]$	$\text{Re}[s] > -\text{Re}[\alpha]$

	$x(t)$	$X(s)$	
Step response of 1 st order system	$[1 - \exp(-at)]u(t)$	$\frac{a}{s(s + a)}$	
Step response of 2 nd order underdamped system	$K \left\{ 1 - \frac{\exp(-\omega_n \zeta t)}{\sqrt{1 - \zeta^2}} \sin[\omega_n \sqrt{1 - \zeta^2} t + \phi] \right\} u(t)$	$\frac{K \omega_n^2}{s(s^2 + 2\zeta \omega_n s + \omega_n^2)}$	
	$K \left\{ 1 - \frac{\sqrt{\sigma^2 + \omega_d^2} \exp(-\sigma t)}{\omega_d} \sin[\omega_d t + \phi] \right\} u(t)$	$\frac{K \omega_d^2}{s[(s + \sigma)^2 + \omega_d^2]}$	