Chapter 2. Differentiation

2.1 Derivative

2.1.1 Instantaneous Speed

Motion can be very complicated to describe; yet in "microscopic scale" it can be comprehended by the very simple and elegant Newton second law: $F = ma$.

Here a is the acceleration, which is related to change in speed. Let the distance traveled by a car be given by a function $f(t)$, where t denotes time. Then we all know that the *average* speed of the car in the time period from $t = a$ to $t = b$ is

$$\text{average speed} \quad = \frac{\text{distance}}{\text{time}} = \frac{f(b) - f(a)}{b - a}.$$

If we let the length of the time interval $[a, b]$ shrink, i.e., let b get closer and closer to a, then we obtain the average speed in shorter and shorter time intervals. As b tends to a, in the limit, we will get the *instantaneous* speed. This is the reading on the speedometer. Mathematically, this *instantaneous* rate of change is called the *derivative* of the function $f(t)$ and is denoted by $f'(t)$. Using the concept of limit, one can give the following definition.

2.1.2 Derivative

Let $f(x)$ be a given function. The derivative of f at the point a, denoted by $f'(a)$, is defined to be

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \qquad (*)$$

provided the limit exists.

An equivalent formulation of $(*)$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

If we use y as the dependent variable, i.e., $y = f(x)$, then we also use the notation

$$\left.\frac{dy}{dx}\right|_{x=a} = \frac{dy}{dx}(a) = f'(a).$$

2.1.3 Differentiable functions

If the derivative $f'(a)$ exists, we say that the function f is *differentiable* at the point a. If a function is differentiable at every point in its domain, we say that the function is differentiable.

2.1.4 Example

Let $f(x) = x^2$.

(i) Find $f'(1)$.

(ii) Show that $f(x)$ is a differentiable function and $f'(x) = 2x$.

Solution. (i) Using the definition,

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h\to 0} \frac{(1+h)^2 - 1^2}{h} \\ &= \lim_{h\to 0} \frac{1 + 2h + h^2 - 1}{h} \\ &= \lim_{h\to 0} \frac{h(2+h)}{h} = \lim_{h\to 0}(2+h) = 2. \end{aligned}$$

(ii) By doing the same calculation as above for a general x (instead of 1), we see that $f'(x) = 2x$ for every x. Hence the function $f(x) = x^2$ is differentiable.

2.1.5 Example

Let $f(x) = |x|$. f is differentiable for $x \neq 0$ but has no derivative at $x = 0$.

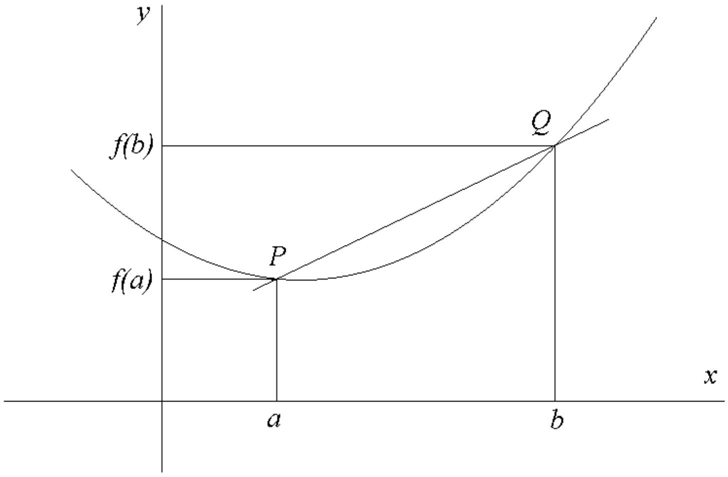
Solution. We show f has no derivative at $x = 0$.

$$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1,$$
$$\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1.$$

Hence the limit $\lim_{h \to 0} \dfrac{f(h) - f(0)}{h}$ does not exist.

2.1.6 Geometrical Meaning

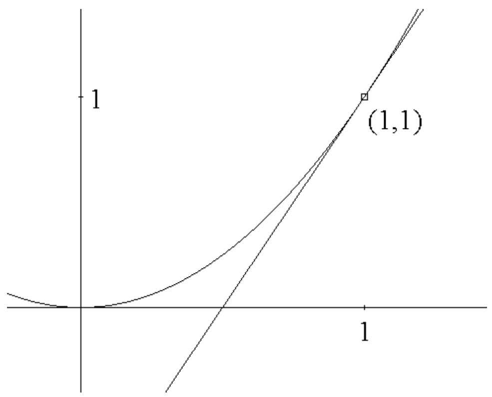
Let us start with the graph of a function f which is differentiable at a (see figure below). Then $\dfrac{f(b) - f(a)}{b - a}$ is the slope of the straight line joining the two points $P = (a, f(a))$ and $Q = (b, f(b))$ (such a line is called a secant to the graph). As b tends to a (so Q approaches P), the secant becomes the tangent, and thus, geometrically, the derivative is just the slope of the tangent to the graph.



Therefore, a function has a derivative at a point a if the slopes of the secant lines through the point $P = (a, f(a))$ and a nearby point Q on the graph approach a limit as Q approaches P. Whenever the secants fail to take up a limiting position or become vertical as Q approaches P, the derivative does not exist.

2.1.7 Example

Find equations of the lines which are tangent and normal to the curve $y = x^2$ at $x = 1$ respectively.



Solution. The slope of the tangent is given by $f'(1) = 2$. The point of contact between the slope and the curve is $(1, 1)$. Thus an equation of the slope, by the point-slope form, is $y - 1 = 2(x - 1)$.

As for the normal, the slope of the normal is $-1/2$

and it contains the same point $(1, 1)$. So an equation of the normal is $y - 1 = (-1/2)(x - 1)$.

2.1.8 Rules of Differentiation

Let k be a constant and let f and g be differentiable.

Linearity

(i) $(kf)'(x) = kf'(x)$, and

(ii) $(f \pm g)'(x) = f'(x) \pm g'(x)$.

Product Rule

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

Quotient Rule

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$

Chain Rule

Assume that the compositions $f \circ g$ and $f' \circ g$ are defined. Then

$$(f \circ g)'(x) = f'(g(x))g'(x) \equiv (f' \circ g)(x)g'(x)$$

2.1.9 Remark

The Chain Rule is often phrased in the following way: We start with a function $y = f(u)$. Then we make a change of variable $u = u(x)$ (i.e., we write the *old* variable u in terms of a *new* variable x). Substituting the change of variable $u(x)$ into the function, we get a function $y = \tilde{f}(x) = f(u(x))$ of the new variable x. Now we take the derivative of the function $\tilde{f}$ in terms of the new variable x. The Chain Rule is:

$$\frac{dy}{dx} = \left.\frac{dy}{du}\right|_{u=u(x)} \frac{du}{dx}, \qquad (**)$$

where

$$\frac{dy}{dx} = \tilde{f}'(x) = \frac{d}{dx} f(u(x)),$$

$$\frac{dy}{du} = f'(u) = \frac{d}{du} f(u), \qquad \left.\frac{dy}{du}\right|_{u=u(x)} = \left. f'(u)\right|_{u=u(x)}.$$

2.1.10 Example

Let $f(x) = x^2$ and $g(x) = x + 1$. Compute

(i) $(fg)'$, (ii) $(f/g)'$, (iii) $(f \circ g)'$, (iv) $(g \circ f)'$.

Solution.

(i) We compute

$$(fg)'(x) = 2x \cdot (x+1) + x^2 \cdot 1 = 3x^2 + 2x$$

(note that $(fg)(x) = x^3 + x^2$).

(ii) We have

$$\left(\frac{f}{g}\right)'(x) = \frac{2x \cdot (x+1) - x^2 \cdot 1}{(x+1)^2} = \frac{x^2 + 2x}{(x+1)^2}.$$

(iii) To compute $(f \circ g)'(x)$, we first note that $f'(x) = 2x$ and hence $f'(g(x)) = 2g(x) = 2(x+1)$; also, we have $g'(x) = 1$. Thus, by the Chain Rule,

$$(f \circ g)'(x) = f'(g(x))g'(x) = 2(x+1) \cdot 1 = 2x + 2$$

(note that $(f \circ g)(x) = (x+1)^2 = x^2 + 2x + 1$).

(iv) To compute $(g \circ f)'(x)$, note that $g'(x) = 1$ and hence $g'(f(x)) = 1$, and that $f'(x) = 2x$. Thus,

$$(g \circ f)'(x) = 1 \cdot 2x = 2x$$

(note that $(g \circ f)(x) = x^2 + 1$).

2.2 Other Types of Differentiation

2.2.1 Parametric Differentiation

Suppose x and y are functionally dependent but are both expressed in terms of a parameter t. Then we can differentiate y with respect to x (provided it exists) as follows:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

In other words, suppose the function $y = f(x)$ is determined by the following equations

$$\begin{cases} y = u(t), \\ x = v(t). \end{cases}$$

Then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{u'(t)}{v'(t)}.$$

2.2.2 Example

Let $x = a(t - \sin t)$ and $y = a(1 - \cos t)$. Then

$$\frac{dy}{dx} = \frac{a \sin t}{a(1 - \cos t)} = \cot \frac{t}{2}.$$

2.2.3 Implicit Differentiation

This is an application of the chain rule. This method is used when x and y are functionally dependent but this dependence is given implicitly by means of the equation

$$F(x, y) = 0.$$

In other words, the function $y = y(x)$ is determined by the above equation. To compute $\dfrac{dy}{dx}$ we may differentiate both sides of the above equation with respect to x, and solve $\dfrac{dy}{dx}$.

2.2.4 Example

Consider the function $y = y(x)$ which is determined by the equation

$$x^2 + y^2 - a^2 = 0.$$

To compute $\dfrac{dy}{dx}$, we differentiate the equation with respect to x:

$$2x + 2y\frac{dy}{dx} = 0,$$

from which we get $\dfrac{dy}{dx} = -\dfrac{x}{y}$.

Note that we used the Chain rule to get

$$\frac{d}{dx}y^2 = \frac{d}{dy}y^2 \cdot \frac{dy}{dx} = 2y\frac{dy}{dx}.$$

2.2.5 Example

Find $\dfrac{dy}{dx}$ if $2y = x^2 + \sin y$.

Solution. Differentiate both sides with respect to x,

$$2\frac{dy}{dx} = 2x + \cos\ y\frac{dy}{dx}.$$

So

$$(2 - \cos\ y)\frac{dy}{dx} = 2x \Rightarrow \frac{dy}{dx} = \frac{2x}{2 - \cos y}.$$

2.2.6 Example

Let $y = x^x$, $x > 0$. Find $\dfrac{dy}{dx}$.

Solution. $y = x^x$. Then $\ln y = x \ln x$. Differentiate both sides with respect to x,

$$\frac{1}{y}\frac{dy}{dx} = 1 + \ln x.$$

So

$$\frac{dy}{dx} = y(1 + \ln x) = x^x(1 + \ln x).$$

2.2.7 Higher Order Derivatives

Higher order derivatives are obtained when we differentiate repeatedly. Let $y = f(x)$, then the following notation is used:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x), \qquad \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = f'''(x).$$

In general, the nth derivative is denoted by

$$\frac{d^ny}{dx^n} \quad \text{or} \quad f^{(n)}(x).$$

2.2.8 Example

Let $f(x) = \sqrt{x}$. Compute $f'''(x)$.

Solution

$$f'(x) = \frac{1}{2}x^{-1/2}, \quad f''(x) = -\frac{1}{4}x^{-3/2}, \quad f'''(x) = \frac{3}{8}x^{-5/2}.$$

2.3 Maxima and Minima

2.3.1 Local and absolute extremes

A function f has a *local (relative) maximum* value at a point c of its domain if $f(x) \leq f(c)$ for all x in a neighborhood of c. The function has an *absolute maximum* value at c if $f(x) \leq f(c)$ for all x in the domain.

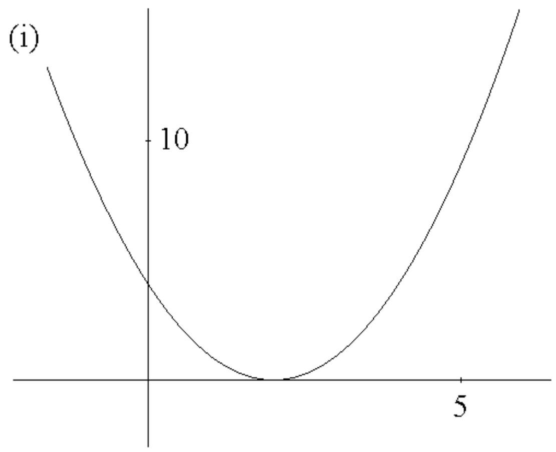
Similarly a function f has a *local (relative) minimum* value at a point c of its domain if $f(x) \geq f(c)$ for all x in a neighborhood of c. The function has an *absolute minimum* value at c if $f(x) \geq f(c)$ for all

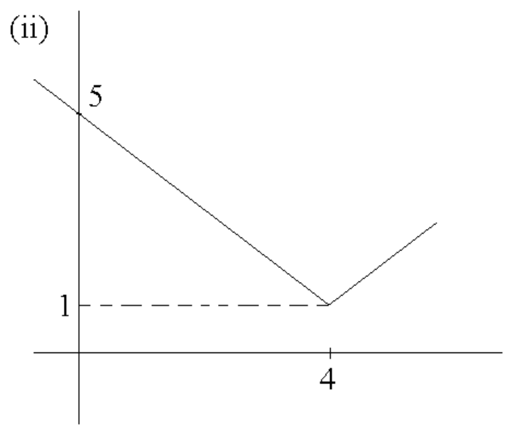
x in the domain.

Local (respectively, absolute) minimum and maximum values are called local (respectively, absolute) *extremes*.

2.3.2 Example

(i) The function $f(x) = (x-2)^2$ clearly has an absolute minimum value 0 at $x = 2$. Note that f is differentiable at $x = 2$ and $f'(2) = 0$.





(ii) Consider the graph of

$$f(x) = |x - 4| + 1 \quad \text{on} \quad [0, 6].$$

f has local maxima at $x = 0$ and $x = 6$, and a local minimum at $x = 4$. The absolute maximum is attained at $x = 0$ with $f(0) = 5$ and the local minimum is also an absolute minimum and the value is $f(4) = 1$. Note that f is not differentiable at $x = 4$.

2.3.3 Finding extreme values

Points where f can have an extreme value are

(1) Interior points where $f'(x) = 0$.

(2) Interior points where $f'(x)$ does not exist.

(3) End points of the domain of f.

2.3.4 Critical points

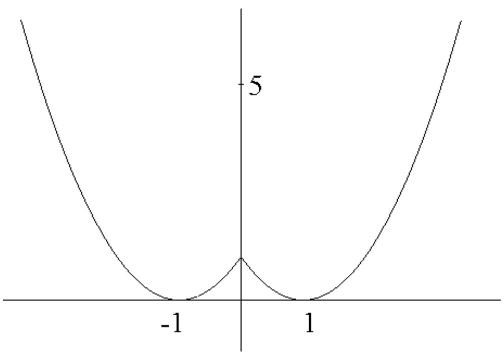
An interior point of the domain of a function f where f' is zero or fails to exist is a *critical point* of f.

2.3.5 Example

Let

$$f(x) = \begin{cases} (x-1)^2 & \text{if } x \geq 0, \\ (x+1)^2 & \text{if } x < 0. \end{cases}$$

The critical points of f are at $x = -1,\ 0,$ and 1 (we leave the verification as an exercise). Thus local or absolute extrema of f may be attained at these points.

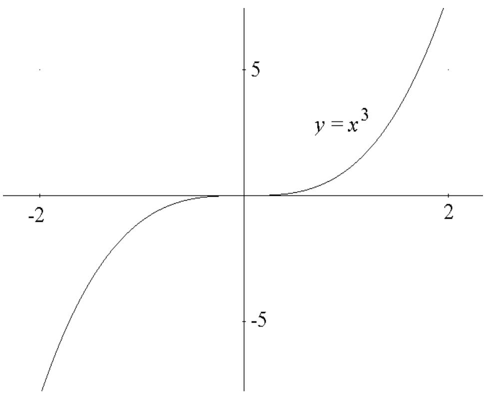


2.3.6 VERY IMPORTANT REMARK

At a critical point, a function may not have a local maximum or local minimum.

For example, for $f(x) = x^3$, $x = 0$ is a critical point.

But f does not have a local extreme value at $x = 0$.



In other words, a curve may have a horizontal tangent without having a local maximum or minimum. On the other hand a curve may have a local maximum or minimum without having a tangent line.

2.4 Increasing and Decreasing Functions

2.4.1 Definition

Let f be a function defined on an interval I. For any two points x_1 and x_2 in I,

if $x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$,

we say f is *increasing* on I;

if $x_2 > x_1 \Rightarrow f(x_2) < f(x_1)$,

we say f is *decreasing* on I.

2.4.2 Test for Increasing/Decreasing Functions

f increases on an interval I when $f'(x) > 0$ for all x on I.

f decreases on I when $f'(x) < 0$ for all x on I.

2.4.3 Example

(i) $f(x) = x^2$.

$f'(x) = 2x$ so $f'(x) > 0$ if and only if $x > 0$.

Therefore $f(x)$ is increasing on $x > 0$ and decreasing on $x < 0$.

(ii) $f(x) = \frac{2}{3}x^3 + x^2 + 2x + 1$ is increasing on any interval, since

$$f'(x) = 2x^2+2x+2 = 2\left((x + \frac{1}{2})^2 + \frac{3}{4}\right) > 0 \ \text{ for all } x.$$

(iii) Show that

$$\ln(1 + x) < x \ \text{ for all } x > 0.$$

Solution Let $f(x) = \ln(1 + x) - x$.

Then $f'(x) = \frac{1}{1+x} - 1 < 0$ for all $x > 0$.

Hence $f(x)$ is decreasing on $[0, +\infty)$ and $f(x) < f(0) = 0$ for all $x > 0$.

Thus $\ln(1 + x) - x < 0$ for all $x > 0$.

2.4.4 First Derivative Test for Local Extremes

Suppose that $c \in (a, b)$ is a critical point of f. If

(i)$f'(x) > 0$ for $x \in (a, c)$, and $f'(x) < 0$ for $x \in (c, b)$, then $f(c)$ is a local maximum.

(ii)$f'(x) < 0$ for $x \in (a, c)$, and $f'(x) > 0$ for $x \in (c, b)$, then $f(c)$ is a local minimum.

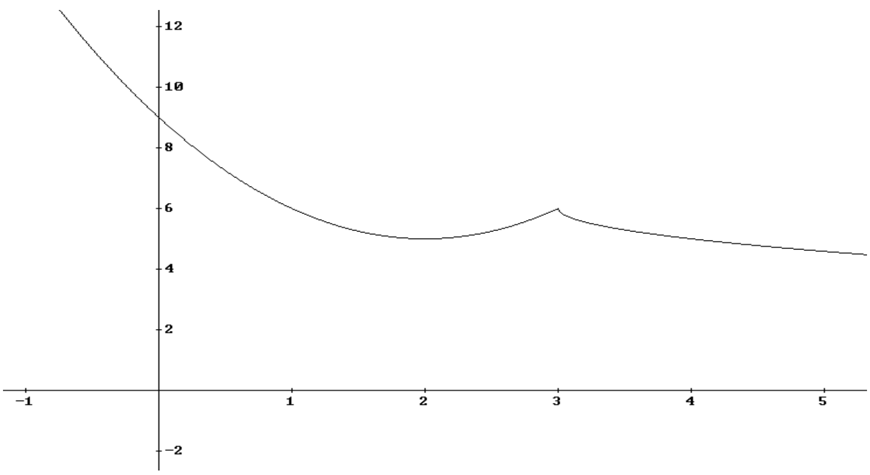
Note that the above test is applicable whether $f'(c)$ exist or not. (Cf. Second derivative test below)

2.4.5 Example

Let

$$f(x) = \begin{cases} x^2 - 4x + 9, & x \leq 3 \\ 6 - \sqrt{x-3}, & x > 3 \end{cases}$$

The function is continuous but not differentiable at $x = 3$ (give reason), so $x = 3$ is a critical point. We check that $f'(x) > 0$ for $x \in (2,3)$, and $f'(x) < 0$ for $x \in (3,4)$. Thus by First Derivative Test, f attains a local maximum at $x = 3$.



2.5 Concavity

Consider the function $y = x^3$. Since $y' = 3x^2 \geq 0$, the curve rises as x increases. But the portions on $(-\infty, 0)$ and $(0, \infty)$ turn in different ways. The slopes of the tangents decrease as the curve approaches the origin from the left, and increase as the curve moves from the origin into the first quadrant.

2.5.1 Definition

The graph of a differentiable function is *concave down* on an interval if the slope y' decreases on that interval. It is *concave up* on an interval if the slope y' increases on that interval.

2.5.2 Concavity Test

The graph of $y = f(x)$ is concave down on any interval where $y'' < 0$, and concave up on any interval where $y'' > 0$.

Notice that if $y'' < 0$, then y' decreases as x increases, and the tangent turns clockwise, so the graph is concave down. If $y'' > 0$, then y' increases as x increases, and the tangent turns anticlockwise, so the graph is concave up.

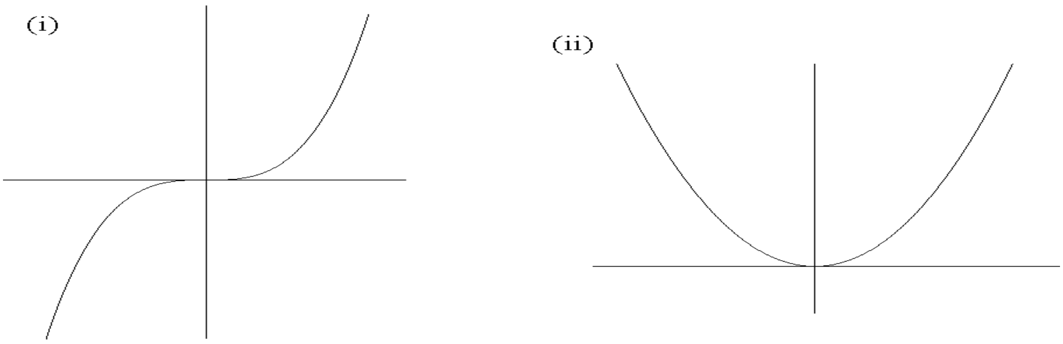
2.5.3 Example

(i) $y = x^3$. Then $y' = 3x^2$, $y'' = 6x$.

When $x < 0$, $y'' < 0$, the curve $y = x^3$ is concave down.

When $x > 0$, $y'' > 0$, the curve $y = x^3$ is concave up.

(ii) $y = x^2$. Then $y' = 2x$, and $y'' = 2$ is always positive. So the curve $y = x^2$ is concave up on $(-\infty, \infty)$.



2.5.4 Points of Inflection

A point c is a *point of inflection* of the function f if f is continuous at c and there is an open interval containing c such that the graph of f changes from concave up (or down) before c to concave down (or up) after c.

Note that the definition does not require that the function be differentiable at a point of inflection.

2.5.5 Examples.

(i) $y = x^3$ has a point of inflection at $x = 0$.

(ii) $y = \sin x$, on $(-1, 4)$. $y' = \cos x$, $y'' = -\sin x$.

In $(0, \pi)$, $y'' < 0$, curve is concave down.

When $x < 0$ or $x > \pi$, $y'' > 0$, curve is concave up.

Points of inflection are thus at $x = 0,\ \pi$.

(iii) $y = x^{\frac{1}{3}}$.

We have $y' = \frac{1}{3}x^{-\frac{2}{3}}$, and $y'' = -\frac{2}{9}x^{-\frac{5}{3}}, x \neq 0$.

The curve has an inflection point at $x = 0$ since $y'' < 0, \text{for}\, x > 0, \text{and}\, y'' > 0, \text{for}\, x < 0$.

Note that y' does not exist at $x = 0$.

2.5.6 Second Derivative Test for Local Extreme Values

If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.

If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.

2.5.7 Example

Find all local maxima and minima of the function $y = x^3 - 3x + 2$ on the interval $(-\infty, \infty)$.

The domain has no endpoints and f is differentiable everywhere. Therefore local extrema can occur only where $y' = 3x^2 - 3 = 0$, which means at $x = 1$ and $x = -1$.

We have $y'' = 6x$, so it is positive at $x = 1$ and negative at $x = -1$.

Hence $y(1) = 0$ is a local minimum value and $y(-1) = 4$ is a local maximum value.

2.6 Optimization Problems

To optimize something means to maximize or minimize some aspect of it. In the mathematical models in which functions are used to describe the things (variables) involved, we are usually required to find the absolute maximum or minimum value of a continuous function over a closed interval.

2.6.1 Finding Absolute Extreme Values

Step 1: Find all the critical points of the function in the interior.

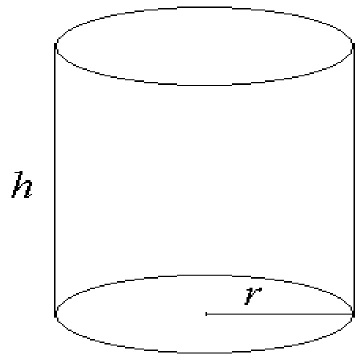
Step 2: Evaluate the functions at its critical points and at the end points of its domain.

Step 3: The largest and smallest of these values will be the absolute maximum and minimum values respectively.

2.6.2 Example.

We are asked to design a 1000cm^3 can shaped like a right circular cylinder. What dimensions will use the least material? Ignore the thickness of the material and waste in manufacturing.

Solution Let r be the radius of the circular base and h the height of the can.



We have volume

$$V = \pi r^2 h = 1000,$$

and so $h = \frac{1000}{\pi r^2}$.

The surface area

$$A = 2\pi r^2 + 2\pi r h = 2\pi r^2 + \frac{2000}{r}, \qquad r > 0.$$

Our aim is to find minimum value of A on $r > 0$.

Now $A' = 4\pi r - \frac{2000}{r^2}$. Setting $A' = 0$, we get $r =$

$(\frac{500}{\pi})^{\frac{1}{3}}$.

$$A'' = 4\pi + \frac{4000}{r^3} > 0, \qquad \text{for } r > 0.$$

Thus $r = (\frac{500}{\pi})^{\frac{1}{3}}$ leads to minimum of A. This value of r gives $h = 2r$.

Thus the dimensions of the can are $r = 5.42$cm and $h = 10.84$cm.

2.7 Indeterminate Forms

If the functions f and g are continuous at $x = a$, but $f(a) = g(a) = 0$, then the limit

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

cannot be evaluated by substituting $x = a$. To describe such a situation, we shall symbolically use the expression $\frac{0}{0}$, known as an *indeterminate form.*

2.7.1 L'Hospital's Rule

Suppose that

(1) f and g are differentiable in a neighborhood of x_0;

(2) $f(x_0) = g(x_0) = 0$;

(3) $g'(x) \neq 0$ except possibly at x_0.

Then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

In particular,

Suppose $f(a) = g(a) = 0$, $f'(a)$ and $g'(a)$ exist, and

$g'(a) \neq 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

2.7.2 Example.

(i) $$\lim_{x \to 0} \frac{3x - \sin x}{x} = \left. \frac{3 - \cos x}{1} \right|_{x=0} = 2.$$

(ii) $$\lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x} = \left. \frac{\frac{1}{2}(1+x)^{-\frac{1}{2}}}{1} \right|_{x=0} = \frac{1}{2}.$$

(iii) $$\lim_{x \to 0} \frac{x - \sin x}{x^3} = \left. \frac{1 - \cos x}{3x^2} \right|_{x=0} = \frac{1}{6}.$$

(iv) $$\lim_{x \to 0} \frac{1 - \cos x}{x + x^2} = \lim_{x \to 0} \frac{\sin x}{1 + 2x} = 0.$$

(v) $$\lim_{x \to 0} \frac{\sin x}{x^2} = \lim_{x \to 0} \frac{\cos x}{2x} = \infty\ .$$

2.7.3 Other Indeterminate Forms

If $f(x)$ and $g(x)$ both approach ∞ as $x \to a$, and $f(x)$ and $g(x)$ are differentiable, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided that the limit on the right exists. Here a may be finite or infinite.

2.7.4 Remark.

For all the other indeterminate forms (for example $\infty \cdot 0$, $\infty - \infty$), one needs to change them to either $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form and then apply L'Hopital's rule.

2.7.5 Example.

(i) (of form $\frac{\infty}{\infty}$)

$$\lim_{x \to \frac{\pi}{2}^-} \frac{\tan x}{1 + \tan x} = \lim_{x \to \frac{\pi}{2}^-} \frac{\sec^2 x}{\sec^2 x} = 1.$$

(ii) (of form $\frac{\infty}{\infty}$)

$$\lim_{x \to \infty} \frac{x - 2x^2}{3x^2 + 5} = \lim_{x \to \infty} \frac{1 - 4x}{6x} = \lim_{x \to \infty} \frac{-4}{6} = -\frac{2}{3}.$$

(iii) (of form $0 \cdot \infty$)

$$\lim_{x \to 0^+} x \cot x = \lim_{x \to 0^+} \frac{x}{\tan x} = \lim_{x \to 0^+} \frac{1}{\sec^2 x} = 1.$$

Note that we have changed it to $\frac{0}{0}$ form before we apply L'Hopital's rule.

(iv) (of form $\infty - \infty$)

$$\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x \to 0} \frac{x - \sin x}{x \sin x} = 0.$$

Note that we have changed it to $\frac{0}{0}$ form before we apply L'Hopital's rule.