NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 EXAMINATION 2005-2006

MA2214 Combinatorial Analysis

April/May 2006 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains a total of **FIVE (5)** questions and comprises **FOUR (4)** printed pages.

2. Answer **ALL** questions. The marks for each question are indicated at the beginning of the question.

3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Answer **ALL** *questions. Each question carries 20 marks.*

Question 1 [20 marks]

(a) Let n be a natural number.

(i) Evaluate $\displaystyle\sum_{k=0}^{n} \binom{n-1+k}{k}$.

(ii) Hence show that

$$\sum_{k=0}^{n} \frac{\binom{n}{k}}{\binom{2n-1}{k}} = 2.$$

(b) In the annual carnival of a community club, 8 boys take part in a game together with both of their parents. The game requires these 24 people to be divided into 8 groups of 3, such that each group consists of a boy, a male parent and a female parent, and each boy must have EXACTLY one of his parents in the same group. How many ways can we group these 24 people?

Question 2 [20 marks]

(a) Find the number of ways of distributing 30 distinct objects into 6 identical boxes with empty boxes allowed.

(b) Let $S = \{1, 2, 3, ..., n+1\}$ where $n \geq 5$, and let

$$T = \{(a, b, c, d, e, f) \epsilon S^6 \mid a, b, c, d, e \text{ are all less than } f\}.$$

By counting $|T|$ in two different ways, evaluate $\displaystyle\sum_{k=1}^{n} k^5$.

Question 3 [20 marks]

(a) Let A be the set of 17-letter permutations using ALL of the 17 letters of the word TELECOMMUNICATION. Let $B = \{CAT, TEL, MUM, CNN\}$. Find the number of elements in A which

(i) do not contain any of the four blocks in B;

(ii) contain exactly one of the four blocks in B;

(iii) contain exactly two of the four blocks in B;

(iv) contain exactly three of the four blocks in B;

(v) contain all the four blocks in B.

(b) Let $S = \{1, 2, 3, ..., n\}$ where $n \geq 10$, and let

$$T = \{(x_1, x_2, x_3, x_4, x_5, x_6) \epsilon S^6 | x_2 \geq x_1,\ x_3 \geq x_2 + 2,\ x_4 \geq x_3 + 3,\ x_6 \geq x_5 \geq x_4 + 4\}.$$

Find $|T|$ in terms of n.

Question 4 [20 marks]

(a) For each integer $n \geq 3$, let a_n denote the number of n-digit integers formed by the 9 given digits, namely, 1, 2, 3, 4, 5, 6, 7, 8 and 9, such that the total number of occurrence of the four digits 1, 2, 3 and 4 is at least 3, and the total number of occurrence of the four digits 5, 6, 7 and 8 altogether is an even number.

(i) Find a suitable generating function for a_n.

(ii) Express a_n in terms of n.

(iii) Evaluate a_8.

(b) For each positive integer n, let b_n denote the number of ways of distributing n identical objects into 9 distinct boxes labelled 1, 2, 3, 4, 5, 6, 7, 8 and 9 such that boxes 1, 2, 3 and 4 each contain an even number of objects, and the total number of objects in boxes 5, 6, 7 and 8 altogether is an odd integer.

(i) Find a suitable generating function for b_n.

(ii) Express b_n in terms of n.

(iii) Evaluate b_6.

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Question 5 [20 marks]

(a) For every natural number n, let a_n denote the number of n-digit integers formed by the six given digits, namely, 0, 1, 2, 3, 4 and 5, such that these integers contain neither a block of 12 nor a block of 23.

(i) Find a recurrence relation for a_n with the necessary initial conditions.

(ii) Hence evaluate a_6.

(b) For every integer $n > 1$, let b_n denote the number of n-digit ODD integers satisfying all of the following conditions.

(i) The first digit must be greater than 3.

(ii) The first and the last digits are distinct.

(iii) Any two adjacent digits are distinct.

Find a recurrence relation for b_n with the necessary initial conditions, and hence evaluate b_6.

END OF PAPER