

Problem 5.6 A 20-turn rectangular coil with sides $l = 30$ cm and $w = 10$ cm is placed in the y - z plane as shown in Fig. P5.6.

- (a) If the coil, which carries a current $I = 10$ A, is in the presence of a magnetic flux density

$$\mathbf{B} = 2 \times 10^{-2}(\hat{\mathbf{x}} + \hat{\mathbf{y}}2) \quad (\text{T}),$$

determine the torque acting on the coil.

- (b) At what angle ϕ is the torque zero?
(c) At what angle ϕ is the torque maximum? Determine its value.

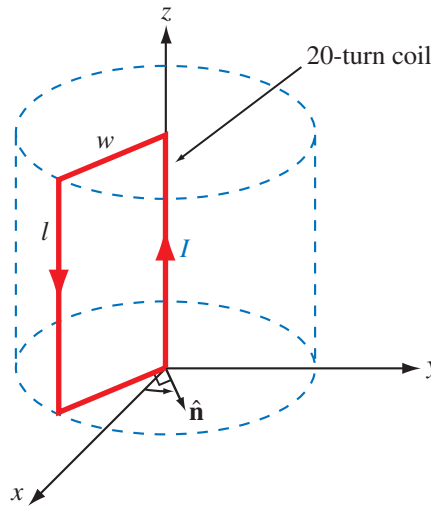


Figure P5.6: Rectangular loop of Problem 5.6.

Solution:

- (a) The magnetic field is in direction $(\hat{\mathbf{x}} + \hat{\mathbf{y}}2)$, which makes an angle $\phi_0 = \tan^{-1} \frac{2}{1} = 63.43^\circ$.

The magnetic moment of the loop is

$$\mathbf{m} = \hat{\mathbf{n}}NIA = \hat{\mathbf{n}}20 \times 10 \times (30 \times 10) \times 10^{-4} = \hat{\mathbf{n}}6 \quad (\text{A}\cdot\text{m}^2),$$

where $\hat{\mathbf{n}}$ is the surface normal in accordance with the right-hand rule. When the loop is in the negative- y of the y - z plane, $\hat{\mathbf{n}}$ is equal to $\hat{\mathbf{x}}$, but when the plane of the loop is moved to an angle ϕ , $\hat{\mathbf{n}}$ becomes

$$\hat{\mathbf{n}} = \hat{\mathbf{x}} \cos \phi + \hat{\mathbf{y}} \sin \phi,$$

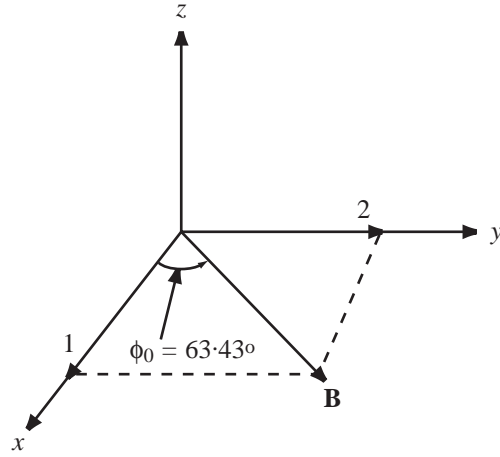


Figure P5.6: (a) Direction of \mathbf{B} .

$$\begin{aligned}\mathbf{T} &= \mathbf{m} \times \mathbf{B} = \hat{\mathbf{n}} 6 \times 2 \times 10^{-2} (\hat{\mathbf{x}} + \hat{\mathbf{y}} 2) \\ &= (\hat{\mathbf{x}} \cos \phi + \hat{\mathbf{y}} \sin \phi) 6 \times 2 \times 10^{-2} (\hat{\mathbf{x}} + \hat{\mathbf{y}} 2) \\ &= \hat{\mathbf{z}} 0.12 [2 \cos \phi - \sin \phi] \quad (\text{N} \cdot \text{m}).\end{aligned}$$

(b) The torque is zero when

$$2 \cos \phi - \sin \phi = 0,$$

or

$$\tan \phi = 2, \quad \phi = 63.43^\circ \text{ or } -116.57^\circ.$$

Thus, when $\hat{\mathbf{n}}$ is parallel to \mathbf{B} , $\mathbf{T} = 0$.

(c) The torque is a maximum when $\hat{\mathbf{n}}$ is perpendicular to \mathbf{B} , which occurs at

$$\phi = 63.43 \pm 90^\circ = -26.57^\circ \text{ or } +153.43^\circ.$$

Mathematically, we can obtain the same result by taking the derivative of \mathbf{T} and equating it to zero to find the values of ϕ at which $|\mathbf{T}|$ is a maximum. Thus,

$$\frac{\partial T}{\partial \phi} = \frac{\partial}{\partial \phi} (0.12(2 \cos \phi - \sin \phi)) = 0$$

or

$$-2 \sin \phi + \cos \phi = 0,$$

which gives $\tan \phi = -\frac{1}{2}$, or

$$\phi = -26.57^\circ \text{ or } 153.43^\circ,$$

at which $\mathbf{T} = \hat{\mathbf{z}}0.27 \text{ (N}\cdot\text{m)}$.

Problem 5.10 An infinitely long, thin conducting sheet defined over the space $0 \leq x \leq w$ and $-\infty \leq y \leq \infty$ is carrying a current with a uniform surface current density $\mathbf{J}_s = \hat{\mathbf{y}}J_s$ (A/m). Obtain an expression for the magnetic field at point $P = (0, 0, z)$ in Cartesian coordinates.

Solution:

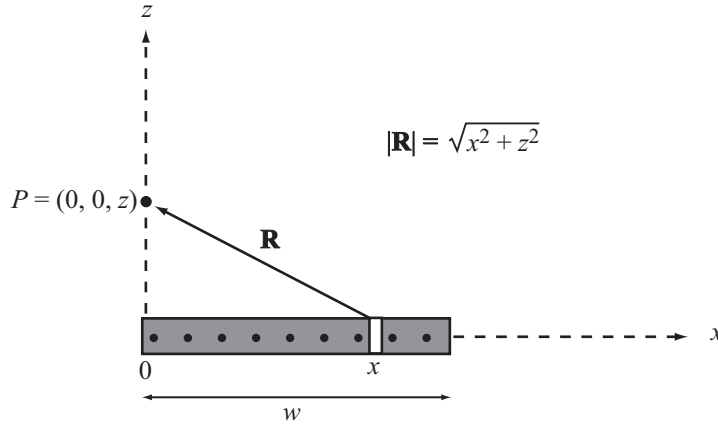


Figure P5.10: Conducting sheet of width w in x - y plane.

The sheet can be considered to be a large number of infinitely long but narrow wires each dx wide lying next to each other, with each carrying a current $I_x = J_s dx$. The wire at a distance x from the origin is at a distance vector \mathbf{R} from point P , with

$$\mathbf{R} = -\hat{\mathbf{x}}x + \hat{\mathbf{z}}z.$$

Equation (5.30) provides an expression for the magnetic field due to an infinitely long wire carrying a current I as

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} = \frac{\hat{\boldsymbol{\phi}}I}{2\pi r}.$$

We now need to adapt this expression to the present situation by replacing I with $I_x = J_s dx$, replacing r with $R = (x^2 + z^2)^{1/2}$, as shown in Fig. P5.10, and by assigning the proper direction for the magnetic field. From the Biot–Savart law, the direction of \mathbf{H} is governed by $\mathbf{l} \times \mathbf{R}$, where \mathbf{l} is the direction of current flow. In the present case, \mathbf{l} is in the $\hat{\mathbf{y}}$ direction. Hence, the direction of the field is

$$\frac{\mathbf{l} \times \mathbf{R}}{|\mathbf{l} \times \mathbf{R}|} = \frac{\hat{\mathbf{y}} \times (-\hat{\mathbf{x}}x + \hat{\mathbf{z}}z)}{|\hat{\mathbf{y}} \times (-\hat{\mathbf{x}}x + \hat{\mathbf{z}}z)|} = \frac{\hat{\mathbf{x}}z + \hat{\mathbf{z}}x}{(x^2 + z^2)^{1/2}}.$$

Therefore, the field $d\mathbf{H}$ due to the current I_x is

$$d\mathbf{H} = \frac{\hat{\mathbf{x}}z + \hat{\mathbf{z}}x}{(x^2 + z^2)^{1/2}} \frac{I_x}{2\pi R} = \frac{(\hat{\mathbf{x}}z + \hat{\mathbf{z}}x)J_s dx}{2\pi(x^2 + z^2)},$$

and the total field is

$$\begin{aligned}\mathbf{H}(0, 0, z) &= \int_{x=0}^w (\hat{\mathbf{x}}z + \hat{\mathbf{z}}x) \frac{J_s dx}{2\pi(x^2 + z^2)} \\ &= \frac{J_s}{2\pi} \int_{x=0}^w (\hat{\mathbf{x}}z + \hat{\mathbf{z}}x) \frac{dx}{x^2 + z^2} \\ &= \frac{J_s}{2\pi} \left(\hat{\mathbf{x}}z \int_{x=0}^w \frac{dx}{x^2 + z^2} + \hat{\mathbf{z}} \int_{x=0}^w \frac{x dx}{x^2 + z^2} \right) \\ &= \frac{J_s}{2\pi} \left(\hat{\mathbf{x}}z \left(\frac{1}{z} \tan^{-1} \left(\frac{x}{z} \right) \right) \Big|_{x=0}^w + \hat{\mathbf{z}} \left(\frac{1}{2} \ln(x^2 + z^2) \right) \Big|_{x=0}^w \right) \\ &= \frac{5}{2\pi} \left[\hat{\mathbf{x}} \tan^{-1} \left(\frac{w}{z} \right) + \hat{\mathbf{z}} \frac{1}{2} (\ln(w^2 + z^2) - \ln(0 + z^2)) \right] \quad \text{for } z \neq 0, \\ &= \frac{5}{2\pi} \left[\hat{\mathbf{x}} \tan^{-1} \left(\frac{w}{z} \right) + \hat{\mathbf{z}} \frac{1}{2} \ln \left(\frac{w^2 + z^2}{z^2} \right) \right] \quad (\text{A/m}) \quad \text{for } z \neq 0.\end{aligned}$$

An alternative approach is to employ Eq. (5.24a) directly.

Problem 5.14 Two parallel, circular loops carrying a current of 40 A each are arranged as shown in Fig. P5.14. The first loop is situated in the x - y plane with its center at the origin, and the second loop's center is at $z = 2$ m. If the two loops have the same radius $a = 3$ m, determine the magnetic field at:

- (a) $z = 0$
- (b) $z = 1$ m
- (c) $z = 2$ m

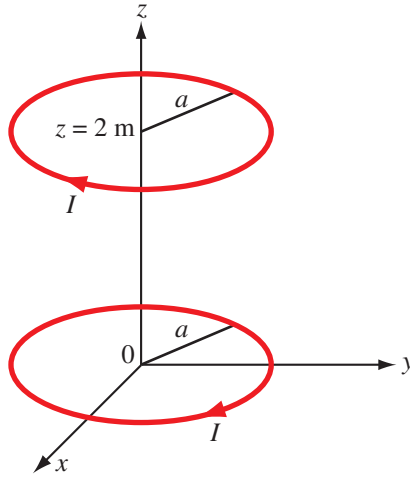


Figure P5.14: Parallel circular loops of Problem 5.14.

Solution: The magnetic field due to a circular loop is given by (5.34) for a loop in the x - y plane carrying a current I in the $+\hat{\phi}$ -direction. Considering that the bottom loop in Fig. is in the x - y plane, but the current direction is along $-\hat{\phi}$,

$$\mathbf{H}_1 = -\hat{\mathbf{z}} \frac{Ia^2}{2(a^2 + z^2)^{3/2}},$$

where z is the observation point along the z -axis. For the second loop, which is at a height of 2 m, we can use the same expression but z should be replaced with $(z - 2)$. Hence,

$$\mathbf{H}_2 = -\hat{\mathbf{z}} \frac{Ia^2}{2[a^2 + (z - 2)^2]^{3/2}}.$$

The total field is

$$\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2 = -\hat{\mathbf{z}} \frac{Ia^2}{2} \left[\frac{1}{(a^2 + z^2)^{3/2}} + \frac{1}{[a^2 + (z - 2)^2]^{3/2}} \right] \text{ A/m.}$$

(a) At $z = 0$, and with $a = 3$ m and $I = 40$ A,

$$\mathbf{H} = -\hat{\mathbf{z}} \frac{40 \times 9}{2} \left[\frac{1}{3^3} + \frac{1}{(9+4)^{3/2}} \right] = -\hat{\mathbf{z}} 10.5 \text{ A/m.}$$

(b) At $z = 1$ m (midway between the loops):

$$\mathbf{H} = -\hat{\mathbf{z}} \frac{40 \times 9}{2} \left[\frac{1}{(9+1)^{3/2}} + \frac{1}{(9+1)^{3/2}} \right] = -\hat{\mathbf{z}} 11.38 \text{ A/m.}$$

(c) At $z = 2$ m, \mathbf{H} should be the same as at $z = 0$. Thus,

$$\mathbf{H} = -\hat{\mathbf{z}} 10.5 \text{ A/m.}$$

Problem 5.17 In the arrangement shown in Fig. P5.17, each of the two long, parallel conductors carries a current I , is supported by 8-cm-long strings, and has a mass per unit length of 1.2 g/cm . Due to the repulsive force acting on the conductors, the angle θ between the supporting strings is 10° . Determine the magnitude of I and the relative directions of the currents in the two conductors.

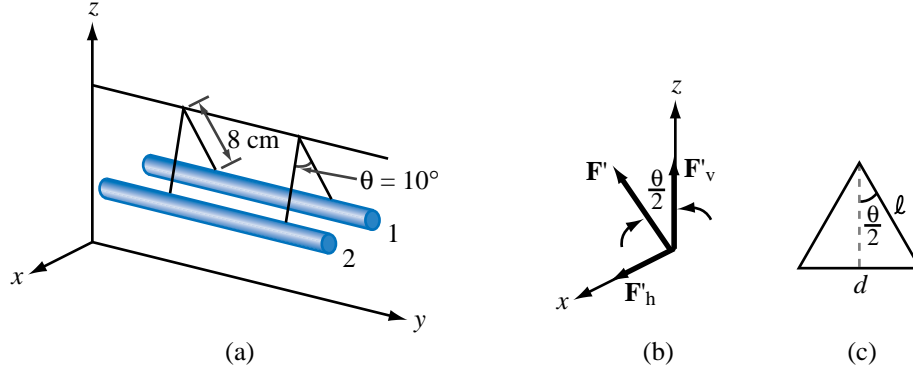


Figure P5.17: Parallel conductors supported by strings (Problem 5.17).

Solution: While the vertical component of the tension in the strings is counteracting the force of gravity on the wires, the horizontal component of the tension in the strings is counteracting the magnetic force, which is pushing the wires apart. According to Section 5-3, the magnetic force is repulsive when the currents are in opposite directions.

Figure P5.16(b) shows forces on wire 1 of part (a). The quantity \mathbf{F}' is the tension force per unit length of wire due to the mass per unit length $m' = 1.2 \text{ g/cm} = 0.12 \text{ kg/m}$. The vertical component of \mathbf{F}' balances out the gravitational force,

$$F'_v = m'g, \quad (23)$$

where $g = 9.8 \text{ (m/s}^2\text{)}$. But

$$F'_v = F' \cos(\theta/2). \quad (24)$$

Hence,

$$F' = \frac{m'g}{\cos(\theta/2)}. \quad (25)$$

The horizontal component of \mathbf{F}' must be equal to the repulsion magnitude force given by Eq. (5.42):

$$F'_h = \frac{\mu_0 I^2}{2\pi d} = \frac{\mu_0 I^2}{2\pi[2\ell \sin(\theta/2)]}, \quad (26)$$

where d is the spacing between the wires and ℓ is the length of the string, as shown in Fig. P5.16(c). From Fig. 5.16(b),

$$F'_h = F' \sin(\theta/2) = \frac{m'g}{\cos(\theta/2)} \sin(\theta/2) = m'g \tan(\theta/2). \quad (27)$$

Equating Eqs. (26) and (27) and then solving for I , we have

$$I = \sin(\theta/2) \sqrt{\frac{4\pi\ell m'g}{\mu_0 \cos(\theta/2)}} = \sin 5^\circ \sqrt{\frac{4\pi \times 0.08 \times 0.12 \times 9.8}{4\pi \times 10^{-7} \cos 5^\circ}} = 84.8 \quad (\text{A}).$$

Problem 5.22 A long cylindrical conductor whose axis is coincident with the z -axis has a radius a and carries a current characterized by a current density $\mathbf{J} = \hat{\mathbf{z}}J_0/r$, where J_0 is a constant and r is the radial distance from the cylinder's axis. Obtain an expression for the magnetic field \mathbf{H} for

(a) $0 \leq r \leq a$

(b) $r > a$

Solution: This problem is very similar to Example 5-5.

(a) For $0 \leq r_1 \leq a$, the total current flowing within the contour C_1 is

$$I_1 = \iint \mathbf{J} \cdot d\mathbf{s} = \int_{\phi=0}^{2\pi} \int_{r=0}^{r_1} \left(\frac{\hat{\mathbf{z}}J_0}{r} \right) \cdot (\hat{\mathbf{z}}r dr d\phi) = 2\pi \int_{r=0}^{r_1} J_0 dr = 2\pi r_1 J_0.$$

Therefore, since $I_1 = 2\pi r_1 H_1$, $H_1 = J_0$ within the wire and $\mathbf{H}_1 = \hat{\phi}J_0$.

(b) For $r \geq a$, the total current flowing within the contour is the total current flowing within the wire:

$$I = \iint \mathbf{J} \cdot d\mathbf{s} = \int_{\phi=0}^{2\pi} \int_{r=0}^a \left(\frac{\hat{\mathbf{z}}J_0}{r} \right) \cdot (\hat{\mathbf{z}}r dr d\phi) = 2\pi \int_{r=0}^a J_0 dr = 2\pi a J_0.$$

Therefore, since $I = 2\pi r H_2$, $H_2 = J_0 a/r$ within the wire and $\mathbf{H}_2 = \hat{\phi}J_0(a/r)$.
