ANSWERS TO MA1506 TUTORIAL 5

Question 1

The cost of making the can is a function of the radius r given by

$$
\begin{aligned}
&C(r) = J(2\pi r^2 + \frac{2V}{r}) + K\ 2\pi r \\
&\Rightarrow C'(r) = J(4\pi r - \frac{2V}{r^2}) + 2\pi K \\
&= J(4\pi r - 2\pi h) + 2\pi K = 0 \\
&\Rightarrow \frac{h}{r} = 2 + \frac{K/J}{r}
\end{aligned},
$$

where J is the cost of aluminium per square cm, V is the volume of the can [a fixed value equal to the area of the base times the height, h], and K is the cost per cm of welding on the top of the can. We have differentiated and set the derivative equal to zero so as to find the minimum cost. [It clearly is a minimum.]
Measurement shows $h \approx 12.5$ cm, $r \approx 2.5$ cm, so $K/J = 7.5$cm, so each cm of welding is about 7.5 times as expensive as 1 cm^2 of aluminium. So that is why the can has this unusual shape: welding on the top is an expensive process. [If welding were cheap, h/r would be about 2, not 5.]

Question 2

Following the standard equations for the Malthus Model [Chapter 3]:

$$
\begin{aligned}
&N = \hat{N}e^{kt}; N(0) = 10000 = \hat{N} \\
&N(2.5) = 10000e^{2.5k} = 11000 \\
&\Rightarrow e^{2.5k} = 1.1 \Rightarrow k = \frac{1}{2.5}\ell n(1.1) \\
&\qquad\qquad\qquad\quad = 0.0381 \\
&N(10) = 10000e^{10k} = 10000e^{10(0.0381)} \approx 14600 \\
&20000 = 10000e^{kt} \rightarrow t = \frac{1}{k}\ell n(2) \\
&\qquad\qquad\qquad\qquad\quad = 18.18 \text{ hours}
\end{aligned}
$$

Question 3

Let c be the number of emigrants per year. $\frac{dN}{dt} = kN - c$. You can solve this in the usual way, as a linear ODE, or use a trick: let $M = N - \frac{c}{k}$ so

$$
\begin{aligned}
\frac{dM}{dt} &= \frac{dN}{dt} = k(N - \frac{c}{k}) \\
&= kM
\end{aligned}
$$

So $M = Ae^{kt}$ i.e. $N - \frac{c}{k} = Ae^{kt}$. Let $N(0) = \hat{N}$, so

$$A = \hat{N} - \frac{c}{k} \text{ i.e. } N = \frac{c}{k} + (\hat{N} - \frac{c}{k})e^{kt}$$

Three cases: $\frac{c}{k} < \hat{N}$, so $\hat{N} - \frac{c}{k} > 0$
exponential growth $\rightarrow$ there is no point in sending out the emigrants, since the Earth's population continues to grow exponentially.
Next case: $\frac{c}{k} = \hat{N}$, $N = \frac{c}{k} = \hat{N}$. This is the desired situation, with a constant population on Earth.
Last case, $\frac{c}{k} > \hat{N}$, Earth's population decreases to zero! Presumably not what you really want.

Next, we assume an emigration rate proportional to t, so $\frac{dN}{dt} = kN - ct$. This is a linear ODE with an integrating factor e^{-kt}, [or you can use undetermined coefficients] so the solution is

$$\begin{aligned} & e^{kt} \int \frac{-ct}{e^{kt}} \\ & = -ce^{kt}\left[-\frac{1}{k}te^{-kt} - \left(\frac{1}{k^2}e^{-kt}\right) + constant\right] \begin{pmatrix} \text{integrate} \\ \text{by} \\ \text{parts} \end{pmatrix}. \\ & = \frac{c}{k}\left[t + \frac{1}{k}\right] + Ae^{kt} \end{aligned}$$

Now $\hat{N} \equiv N(0) = A + \frac{c}{k^2} \Rightarrow N(t) = \left(\hat{N} - \frac{c}{k^2}\right)e^{kt} + \frac{c}{k}\left[t + \frac{1}{k}\right]$

3 cases:
$\hat{N} - \frac{c}{k^2} > 0 \Rightarrow$ population explosion on Earth;

$\hat{N} = \frac{c}{k^2} \Rightarrow$ Earth's population grows linearly; not quite an "explosion" but still not so great;

$\hat{N} - \frac{c}{k^2} < 0$: population will grow for a while but eventually reach a maximum, followed by a decline to zero [because the exponential function will eventually defeat a linear one; note that we know that the population will grow initially because
$\frac{dN}{dt}(t = 0) = \hat{N}k > 0$.] So none of the outcomes is really satisfactory in this case.

Question 4

$N = Ae^{(B-D)t}$ (constant B and D, so we have a Malthusian situation.)
Population doubles in 20 years, so

$$2 = e^{(B-D)20} \Rightarrow B - D = \frac{\ell n 2}{20}$$

After the departure of the women, B is zero so $N = ce^{-Dt} \Rightarrow \frac{1}{2} = e^{-D\times 10} \Rightarrow D = \frac{\ell n2}{10} \Rightarrow$ $B = \frac{\ell n2}{20} + \frac{\ell n2}{10} \Rightarrow B \approx 0.10397$ i.e. about 10.397% per year.

Assumptions:

1. men and old women have same death rate as young women, which is not true in reality because men smoke, get into fights etc while on the other hand old women are indestructible;
2. the death rate of the remaining population is not changed by the departure of the girls, a very questionable assumption since morale will be affected, etc.

Question 5

$\frac{dN}{dt} = (B-D)N$. We set $B = sN$, $D=$ constant, **opposite** of the logistic model.

$$\begin{aligned} &\frac{dN}{(SN-D)N} = dt = \frac{s/D}{SN-D} - \frac{1/D}{N} \\ &\Rightarrow t + \text{constant} = \frac{1}{D}\ell n|SN-D| - \frac{1}{D}\ell n N \\ &\Rightarrow \frac{|sN-D|}{N} = ke^{Dt} \quad \otimes \end{aligned}$$

Suppose first that $N < D/s$ so that $|SN - D| = D - sN$ [remember that $|x| = -x$ for negative x.] So equation $\otimes$ becomes

$$\frac{D-sN}{N} = ke^{Dt}$$

If $N(0) = \hat{N}$ then $\frac{D-s\hat{N}}{\hat{N}} = K > 0$ since we are assuming $D - sN > 0$ always. Then $\frac{D-sN}{N} = \frac{D-s\hat{N}}{\hat{N}}e^{Dt}$.

Solving for N we get

$$N(t) = \frac{D}{s + \frac{D-s\hat{N}}{\hat{N}}e^{Dt}}$$

Suppose instead that $N > D/s,\ |sN-D| = sN - D$

$\frac{sN-D}{N} = ke^{Dt} \qquad N = \frac{D}{s-ke^{Dt}}$

[Note that setting t = 0 we have $s - \frac{D}{\hat{N}} = k > 0$.]

Clearly $N=$ const $= \frac{D}{s}$ is a solution.

So if $\hat{N} < D/s$ the population of neutrons will die out. If $\hat{N} > D/s$ it will explode (literally !), tending to infinity in a finite time given by

$0 = s - ke^{Dt}$, or $t = \frac{1}{D}\ell n\frac{s}{S-\frac{D}{\hat{N}}}$

Question 6.
The logistic equation has 3 kinds of solution, one increasing, one constant, and one decreasing. Since the number of bugs in this problem clearly increases, the relevant solution of the logistic equation is

$$N = \frac{B}{s + \left(\frac{B}{\hat{N}} - s\right) e^{-Bt}} = \frac{N_\infty}{1 + \left(\frac{N_\infty}{\hat{N}} - 1\right) e^{-Bt}}$$

Here $\hat{N} = 200$, $B = 1.5$, so at $t = 2$ we have

$$\begin{aligned}
360 &= \frac{N_\infty}{1 + \left(\frac{N_\infty}{200} - 1\right) e^{-1.5\times 2}} \\
&\Rightarrow 360 + \frac{360}{200} e^{-3} N_\infty - 360 e^{-3} = N_\infty \\
N_\infty &= \frac{360(1 - e^{-3})}{1 - \frac{360}{200} e^{-3}} \approx 376 \\
N(3) &= \frac{N_\infty}{1 + \left(\frac{N_\infty}{200} - 1\right) e^{-4.5}} \approx 372
\end{aligned}$$