8. Some Examples of Real Applications

8.1 LTI Filters

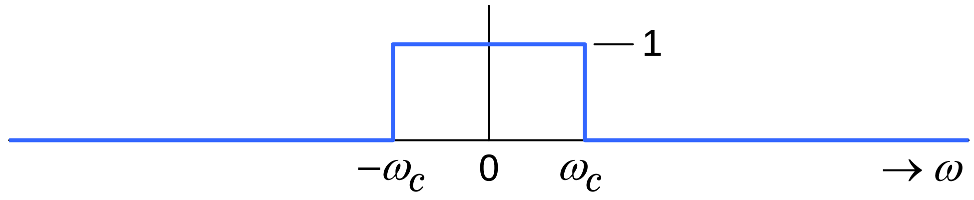
One of the most basic operations in any signal processing system is filtering. Filtering is the process by which the relative amplitudes of the frequency components in the signal are changed or perhaps some frequency components are suppressed. For continuous-time LTI systems, as discussed in the preceding chapter, we saw that the spectrum of the output is that of the input multipled by the frequency response of the system. Therefore, an LTI system acts as a filter on the input signal. Here, the word filter is used to denote a system that exhibits some sort of frequency-selective behavior.

The band of frequencies passed by the filter is referred to as the the **pass band**, and the band of frequencies rejected by the filter is called the **stop band**.

An ideal low-pass filter completely eliminates all frequencies above the cutoff frequency while passing those below unchanged: its frequency response is a rectangular function, and is a brick-wall filter. The transition region present in practical filters does not exist in an ideal filter. An ideal low-pass filter can be realized mathematically (theoretically) by multiplying a signal by the rectangular function in the frequency domain or, equivalently, convolving it with a sinc function in the time domain. Some common ideal filter types are specified below.

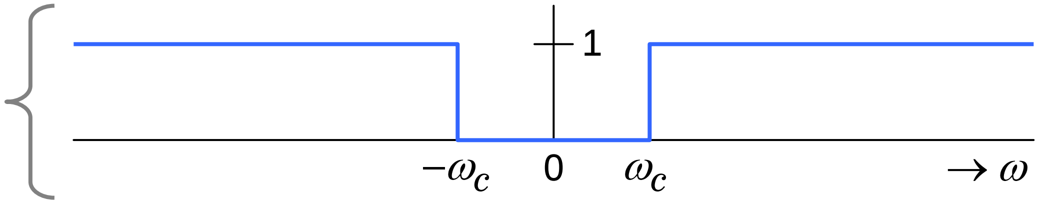
- **Ideal Low-Pass Filter (LPF)**

$$\left|H(\omega)\right| = \begin{cases} 1; & |\omega| < \omega_c \\ 0; & |\omega| > \omega_c \end{cases}$$



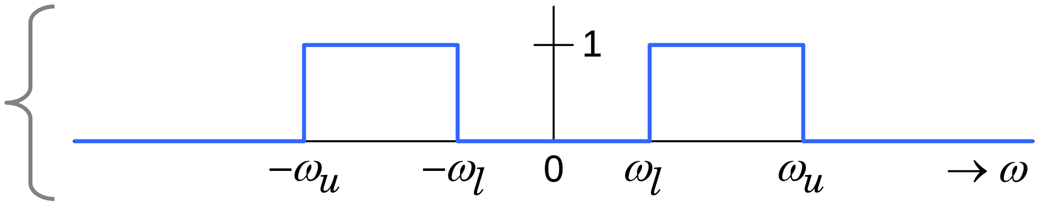
- **Ideal High-Pass Filter (HPF)**

$$|H(\omega)| = \begin{cases} 0; & |\omega| < \omega_c \\ 1; & |\omega| > \omega_c \end{cases}$$



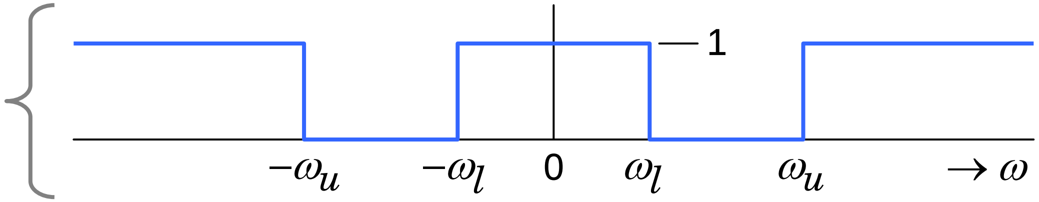
- **Ideal Band-Pass Filter (BPF)**

$$|H(\omega)| = \begin{cases} 1; & \omega_l < |\omega| < \omega_u \\ 0; & \text{otherwise} \end{cases}$$



- **Ideal Band-Stop Filter (BSF)**

$$|H(\omega)| = \begin{cases} 0; & \omega_l < |\omega| < \omega_u \\ 1; & \text{otherwise} \end{cases}$$



To avoid phase distortion in the filtering process, an ideal filter should have a linear phase characteristic over its pass band , i.e.

$$\angle H(\omega) = -\omega t_d$$

where t_d is a constant.

However, the ideal filter is impossible to realize, and so generally needs to be approximated. Some important real filters that are used to approximate the ideal filter for real-time applications include: Chebyshev filter, Butterworth filter, Bessel filter and Elliptic filter. In the following section, we shall take a closer look at the Butterworth filter.

8.1.1 Butterworth Filter Approximation for Ideal Filter

The Butterworth filter is a type of signal processing filter designed to have as flat a frequency response as possible in the passband so that it is also termed a **maximally flat magnitude filter**. It was first described in 1930 by the British engineer ***Stephen Butterworth*** in his paper entitled "On the Theory of Filter Amplifiers”.

The usual practice in filter design is to start off with a prototype low-pass filter which has a normalized corner frequency $\omega_c = 1 \ rad/s$. The final design is then obtained by frequency scaling the prototype to the desired corner frequency and impedence scaling it to suit implementation with commercially available components. The low-pass prototype can also be transformed into a high-pass, band-pass or band-stop filter.

- ***Design of an n^{th}-order Low-Pass Butterworth Filter***

  Filter order: n

  Corner frequency: 1 rad/s

  Poles: $s_i = \exp\left(j\beta_i\right)$ where $\left(\beta_i = \dfrac{\pi\left(2i-1\right)}{2n} + \dfrac{\pi}{2}\right)$

  Transfer function: $H_n\left(s\right) = \dfrac{1}{\prod_{i=1}^{n}\left(s-s_i\right)} = \sum_{i=1}^{n}\dfrac{\alpha_i}{s-s_i}$ where $\left(\alpha_i = \left[\prod_{\substack{k=1\\k\neq i}}^{n} s_i - s_k\right]^{-1} = \alpha_{n-i+1}^{*}\right)$

Illustrations of pole locations for $n = 1,2,3,4$ and 5 are shown in Fig.8-1.

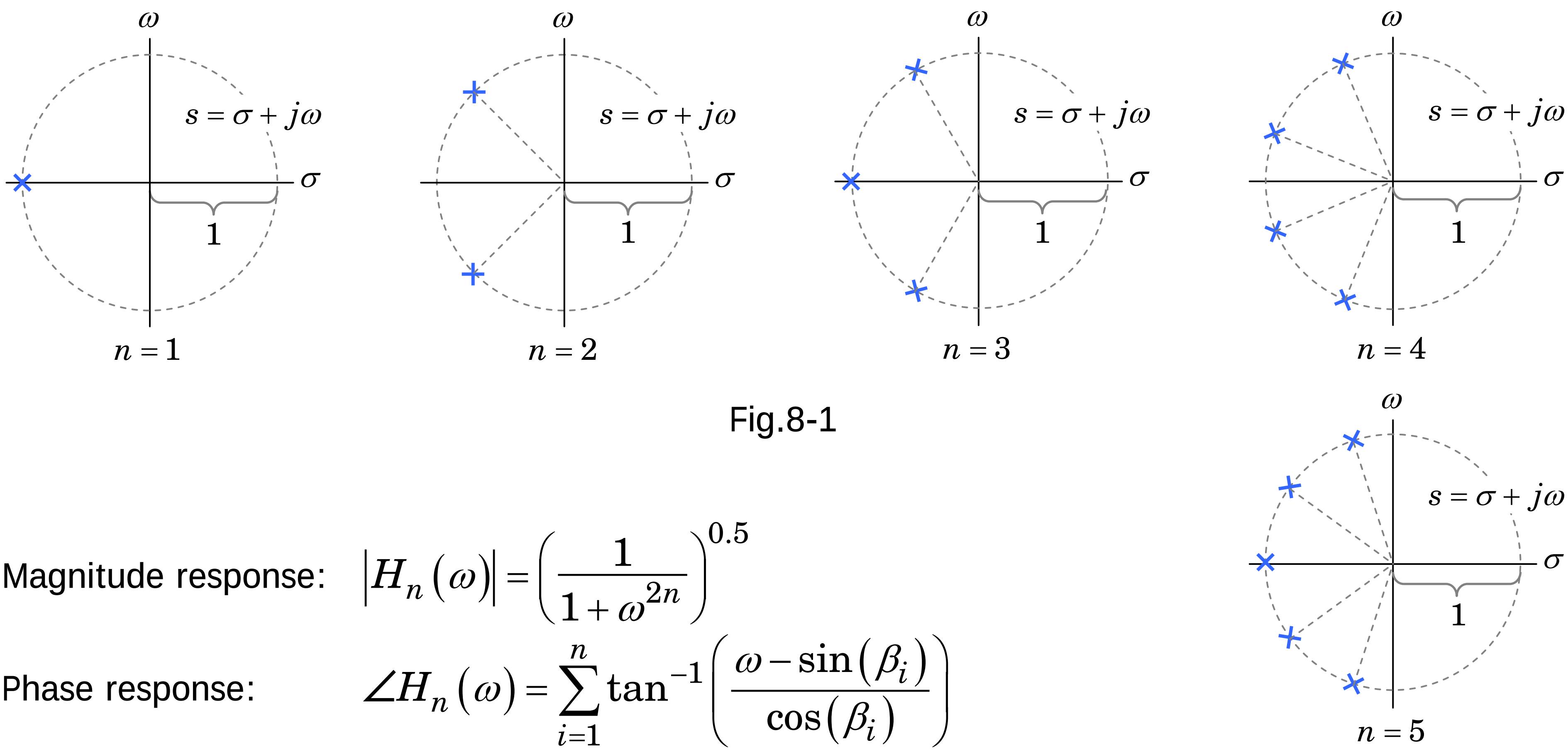


Fig.8-1

Magnitude response: $$\left|H_n(\omega)\right| = \left(\frac{1}{1+\omega^{2n}}\right)^{0.5}$$

Phase response: $$\angle H_n(\omega) = \sum_{i=1}^{n} \tan^{-1}\left(\frac{\omega - \sin(\beta_i)}{\cos(\beta_i)}\right)$$

Impulse response: $$h_n(t) = \sum_{i=1}^{n} \alpha_i \exp(s_i t)\cdot u(t)$$

Plots of $\left|H_n(\omega)\right|$, $\angle H_n(\omega)$ and $h_n(t)$ for $n = 1,2,3$ are shown in Fig.8-2.

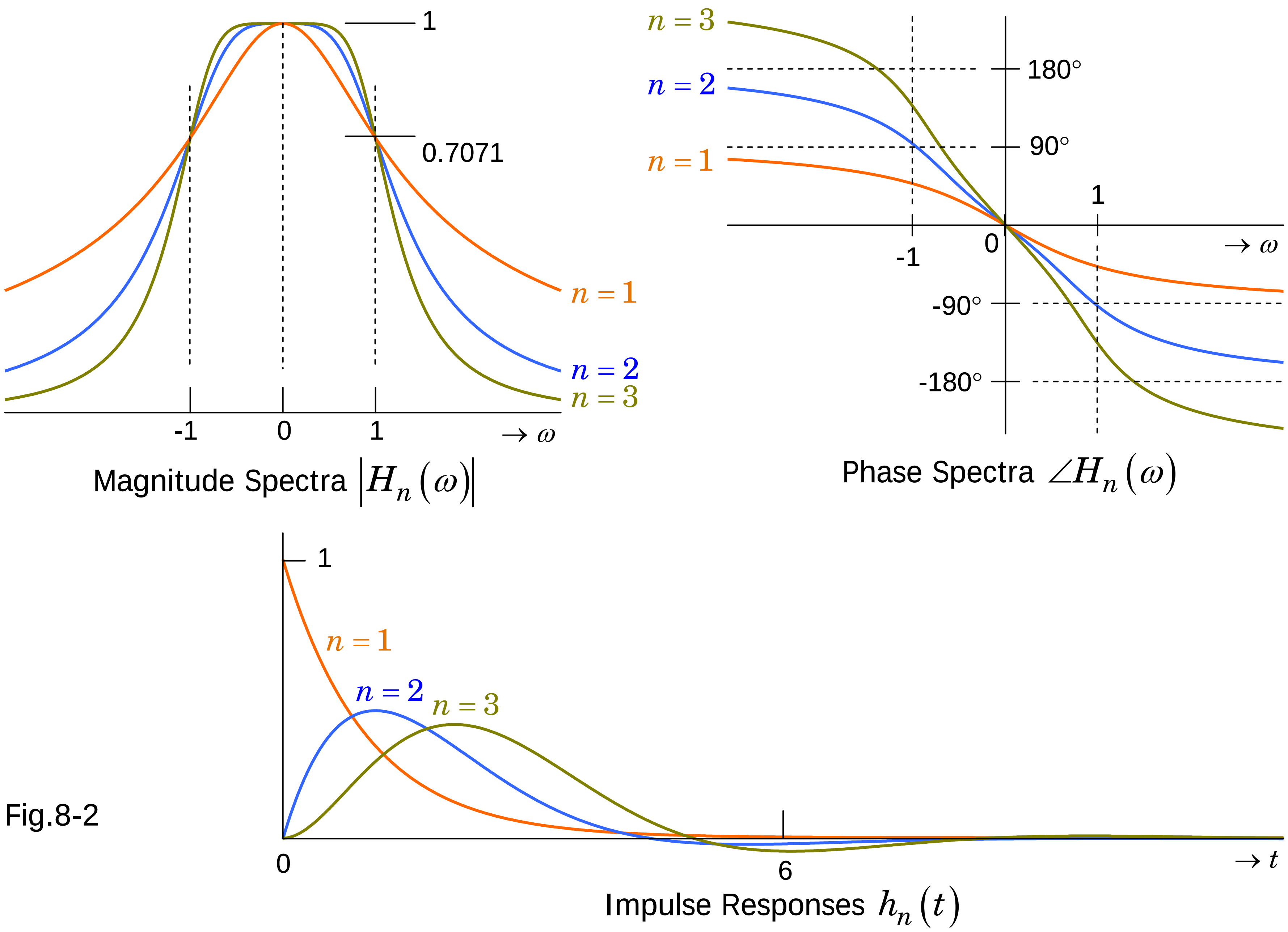

1
0.7071
$n = 1$
$n = 2$
$n = 3$
-1
0
1
$\rightarrow \omega$
Magnitude Spectra $\left|H_n(\omega)\right|$
$n = 3$
$n = 2$
$n = 1$
180°
90°
1
-1
0
$\rightarrow \omega$
-90°
-180°
Phase Spectra $\angle H_n(\omega)$
1
$n = 1$
$n = 2$
$n = 3$
0
6
$\rightarrow t$
Impulse Responses $h_n(t)$

Fig.8-2

- ***Implementation of an n^{th} -order Low-Pass Butterworth Filter***

There are a number of different filter topologies available for implementing linear analog filters.

The most often used topology for a ***passive realization*** is **Cauer topology** which uses shunt capacitors and series inductors. Fig.8-3 shows the structure of an nth-order LPF filter based on Cauer topology.

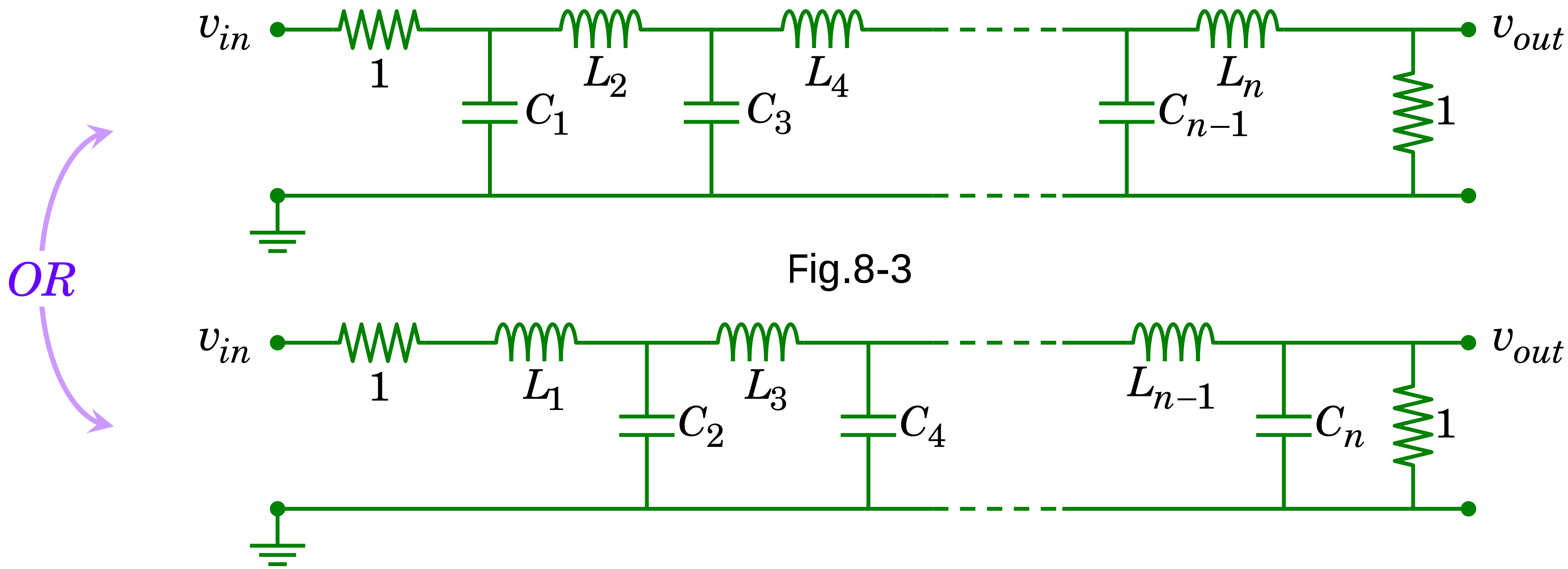


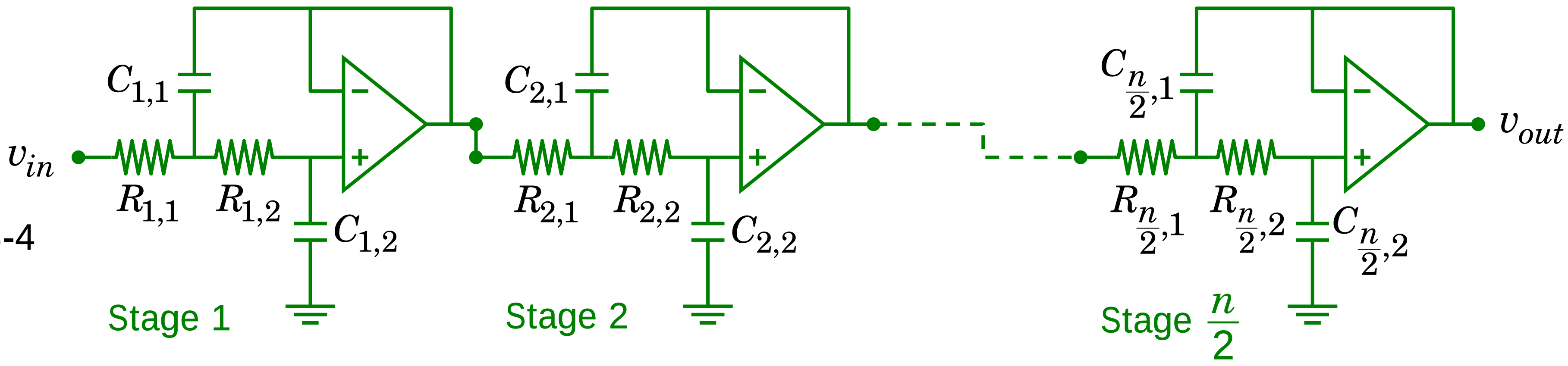
Fig.8-3

The prototype nth-order Butterworth filter transfer function $H_n(s)$ can be realized by choosing the L's and C's values according to

$$C_k, L_k = 2\sin\left(\frac{2k-1}{2n}\pi\right).$$

The most often used topology for an ***active realization*** is **Sallen–Key topology** which uses noninverting buffers (usually op amps), resistors, and capacitors. Each Sallen–Key stage implements a conjugate pair of poles; the overall filter is implemented by cascading all stages in series. If there is a real pole (in the case where n is odd), this must be implemented separately, usually as an RC circuit, and cascaded with the active stages. Fig.8-4 shows the structure of an nth-order LPF filter, where $\boldsymbol{n}$ **is even**, based on Sallen-Key topology.

Fig.8-4



The transfer function of the kth-stage is given by

$$G_k(s) = \frac{1}{1 + C_{k,2}\left(R_{k,1} + R_{k,2}\right)s + C_{k,1}C_{k,2}R_{k,1}R_{k,2}s^2}$$

The prototype nth-order Butterworth filter transfer function $H_n(s) = \prod_{k=1}^{n/2} G_k(s)$, for even n, can be realized by choosing the $R's$ and $C's$ values according to

$$\left.\begin{aligned} &C_{k,1}C_{k,2}R_{k,1}R_{k,2}=1 \\ &C_{k,2}\left(R_{k,1}+R_{k,2}\right)=-2\cos\left(\frac{2k+n-1}{2n}\pi\right);\quad k=1,2,\cdots,\frac{n}{2} \end{aligned}\right\}$$

{*This leaves two undefined component values for each stage that may be chosen at will.*

- ***Frequency Scaling and Transformations***

Let $H(s)$ be the transfer function of a prototype LPF which has a corner frequency of 1 rad/s (**normalized**). $H(s)$ can be transformed into the desired filter by replacing the s variable as follow:

Low-Pass Filter $\left(\text{Corner frequency}: \ \omega_c\right)$: $H_{LPF}(s)=H\left(\dfrac{s}{\omega_c}\right)$

High-Pass Filter $\left(\text{Corner frequency}: \ \omega_c\right)$: $H_{HPF}(s)=H\left(\dfrac{\omega_c}{s}\right)$

Band-Pass Filter $\begin{pmatrix}\text{Lower corner frequency}: & \omega_l \\ \text{Upper corner frequency}: & \omega_u\end{pmatrix}$: $H_{BPF}(s)=H\left(\dfrac{s^2+\omega_o^2}{s\left(\omega_u-\omega_l\right)}\right);\quad \omega_o^2=\omega_l\omega_u$

Band-Stop Filter $\begin{pmatrix}\text{Lower corner frequency}: & \omega_l \\ \text{Upper corner frequency}: & \omega_u\end{pmatrix}$: $H_{BSF}(s)=H\left(\dfrac{s\left(\omega_u-\omega_l\right)}{s^2+\omega_o^2}\right);\quad \omega_o^2=\omega_l\omega_u$

The corresponding transformations of circuit components are:

Low-pass PROTYPE Normalized	DESIRED FILTERS			
	Low-pass	High-pass	Band-pass	Band-stop
L	$\frac{L}{\omega_c}$	$\frac{1}{\omega_c L}$	$\frac{L}{\omega_u-\omega_l}$, $\frac{\omega_u-\omega_l}{\omega_o^2 L}$	$\frac{1}{(\omega_u-\omega_l)L}$, $\frac{(\omega_u-\omega_l)L}{\omega_o^2}$
C	$\frac{C}{\omega_c}$	$\frac{1}{\omega_c C}$	$\frac{C}{\omega_u-\omega_l}$, $\frac{\omega_u-\omega_l}{\omega_o^2 C}$	$\frac{1}{(\omega_u-\omega_l)C}$, $\frac{(\omega_u-\omega_l)C}{\omega_o^2}$

8.1.2 *Application:* Equalizers and Crossover Circuits

A common need for filter circuits is in high-performance audio systems, where certain ranges of audio frequencies need to be amplified or suppressed for best sound quality and power efficiency. You may be familiar with **equalizers**, which allow the amplitudes of several frequency ranges to be adjusted to suit the listener's taste and acoustic properties of the listening area. You may also be familiar with **crossover** networks, which block certain ranges of frequencies from reaching loudspeakers. For example, a tweeter (high-frequency speaker) is inefficient at reproducing low-frequency signals such as drum beats, so a crossover circuit is connected between the tweeter and the audio amplifier output terminals to block low-frequency signals, only passing high-frequency signals to the tweeter's connection terminals. This gives better audio system efficiency and thus better performance. Both equalizers and crossover networks are examples of filters designed to accomplish filtering of certain frequencies.

- *GRAPHIC EQUALIZER*

Example: A 4-band Graphic Equalizer:

Fig.8-5(a) shows the block diagram of a 4-band equalizer.

The low-pass filter (LPF), band-pass filters (BPF0 and BPF1) and high-pass filter (HPF) may be implemented using Butterworth, Chebyshev or other designs.

Fig.8-5(b) shows the Bode plots of the individual filter magnitude responses when 8th-order Butterworth filters are used.

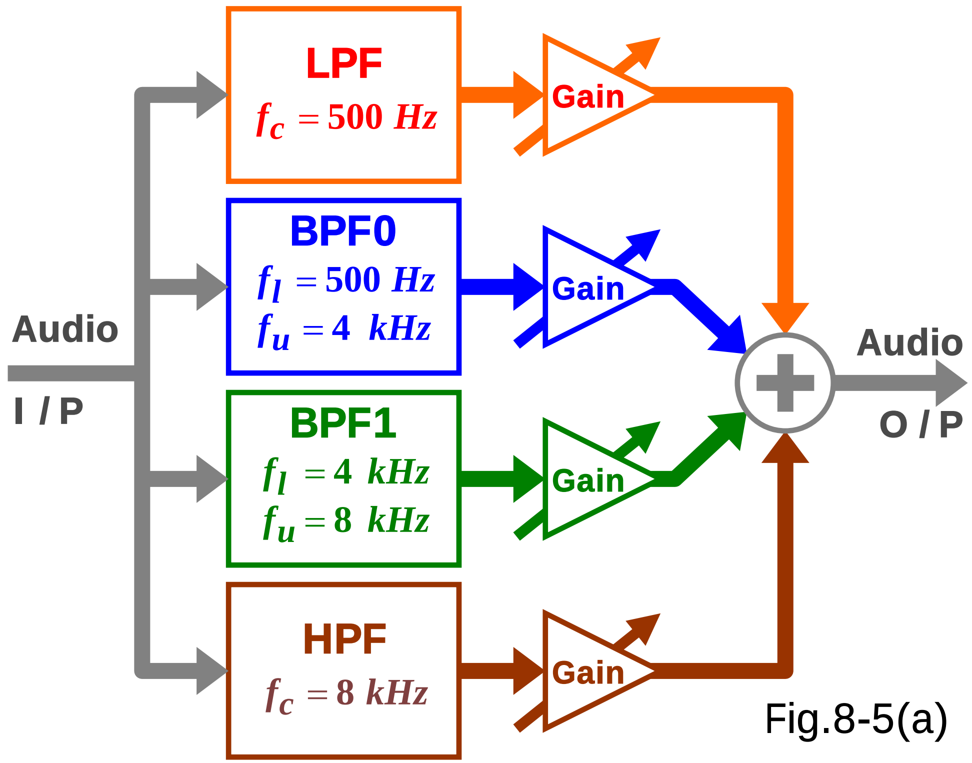


Fig.8-5(a)

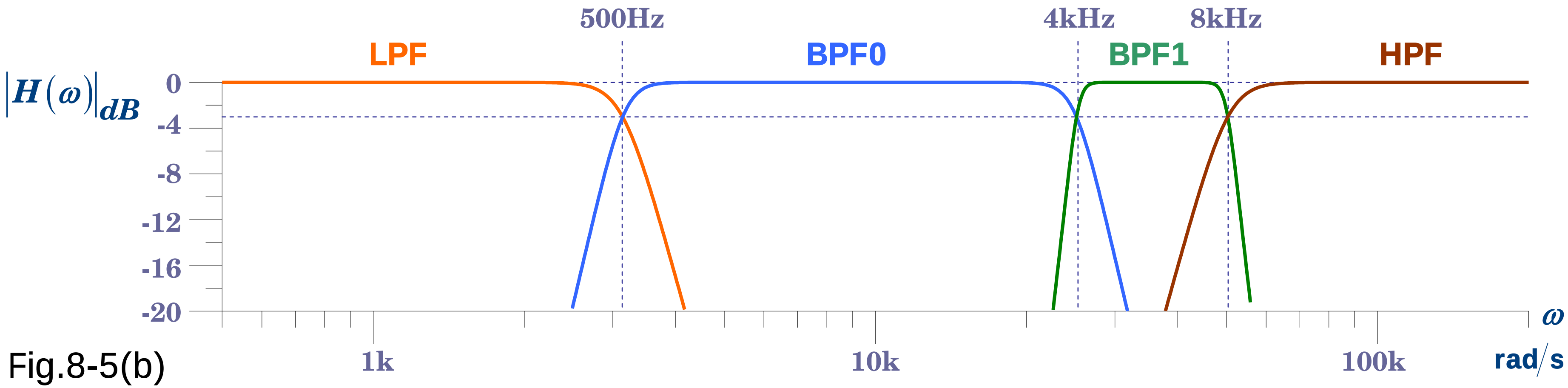


Fig.8-5(b)

Example: A Crossover Network:

Most audio crossovers use first to fourth order electrical filters. Higher orders are not generally implemented in passive crossovers for loudspeakers, but are sometimes found in electronic equipment under circumstances for which their considerable cost and complexity can be justified.

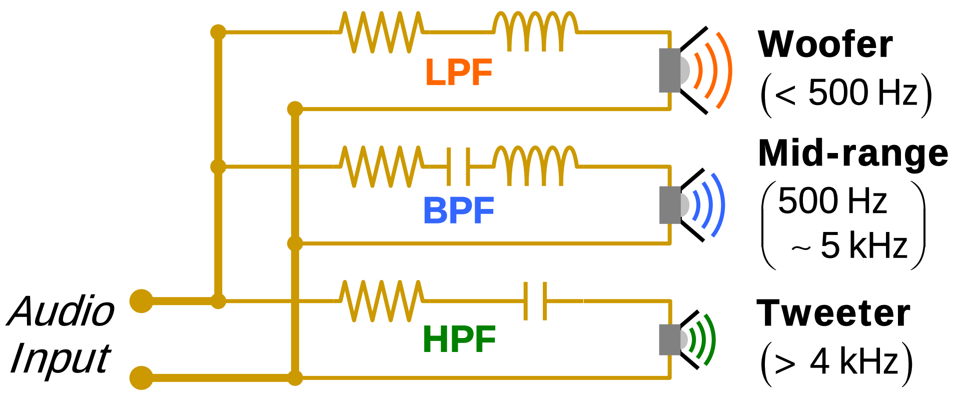


Fig.8-6(a)

Fig.8-6(a) shows the arrangement of a low order passive crossover network. Fig.8-6(b) shows the Bode plots of the magnitude responses of the LPF and HPF (1st-order Butterworth filters), and that of the BPF (2nd-order Butterworth filter).

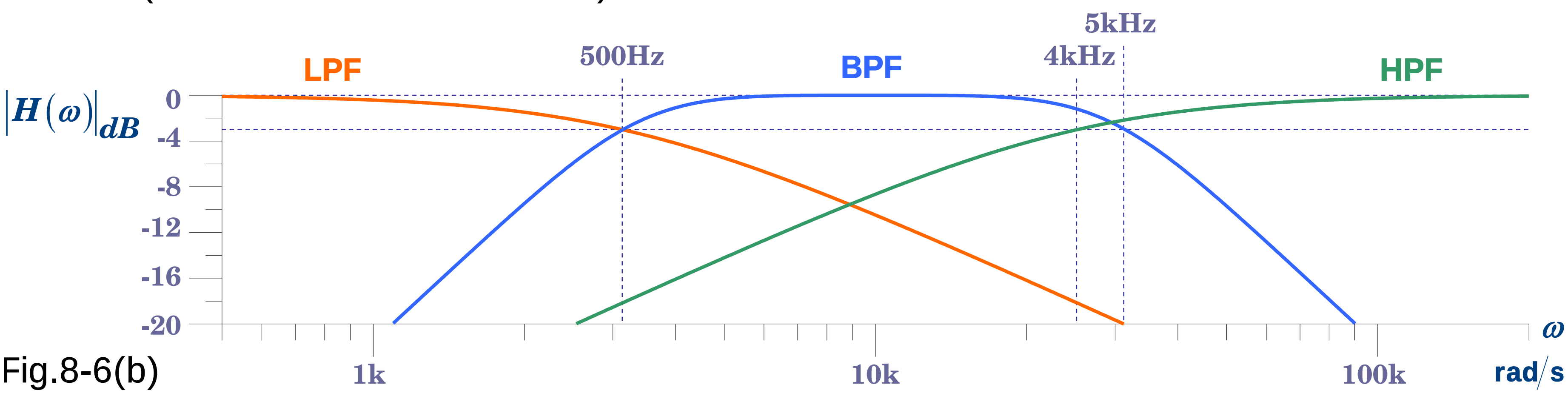


Fig.8-6(b)

The best crossover points for one environment might not be the best for another. It all depends on the loudspeakers being used and the acoustic properties of the environment. Most electronic crossovers allow you to choose from several crossover points.

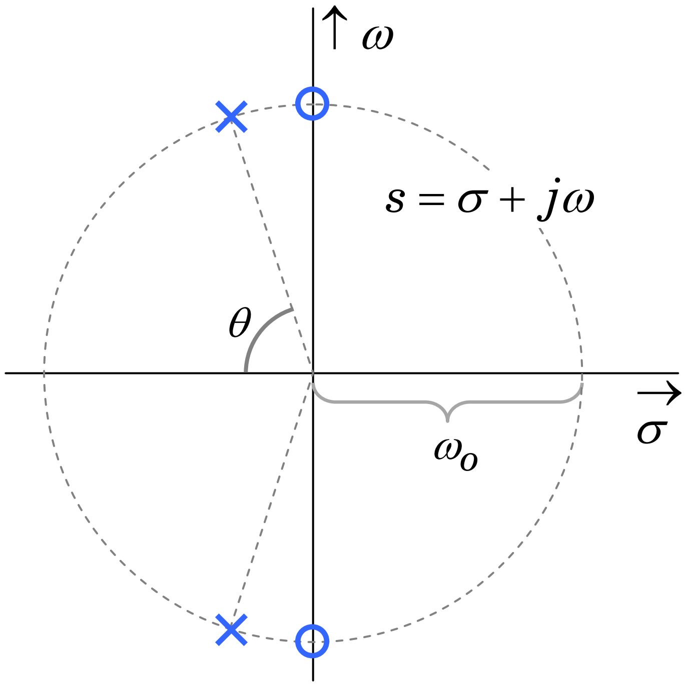
8.1.3 *Application:* Harmonics Suppression using Notch Filters

A **notch filter** is a band-stop filter with a narrow stopband (high Q factor). Notch filters are used in live sound reproduction (Public Address systems, also known as PA systems) and in instrument amplifier (especially amplifiers or preamplifiers for acoustic instruments such as acoustic guitar, mandolin, bass instrument amplifier, etc.) to reduce or prevent feedback, while having little noticeable effect on the rest of the frequency spectrum. Notch filters are also commonly used to remove the DC component of speech and acoustic signals. Other names for notch filters include *'band limit filter'*, *'T-notch filter'*, *'band-elimination filter'*, and *'band-reject filter'*.

- ***Design of a Passive Notch Filter***

  A notch filter could in theory be realised with two zeros placed at $\pm j\omega_o$. However, such a filter would not have unity gain at zero frequency, and the notch will not be sharp.

  To obtain a good notch filter, two poles are placed close the two zeros on the semicircle as shown. Since the both pole/zero pair are equal-distance to the origin, the gain is exactly one at zero frequency as well as at $\omega = \pm\infty$.



Notch Frequency : $\omega_o\,(rad/s)$

$$\begin{cases} \textbf{Poles:}\ s_{p_1}, s_{p_2} = -\omega_o\cos\theta \pm j\omega_o\sin\theta \\ \textbf{Zeros:}\ s_{z_1}, s_{z_2} = \pm j\omega_o \end{cases}$$

Transfer function:

$$H(s)=\frac{(s-s_{z_1})(s+s_{z_2})}{(s-s_{p_1})(s-s_{p_2})}$$

$$=\frac{s^2+\omega_o^2}{s^2+2\omega_o\cos(\theta)s+\omega_o^2}$$

Frequency response:

$$H(\omega)=H(s)\big|_{s=j\omega}$$

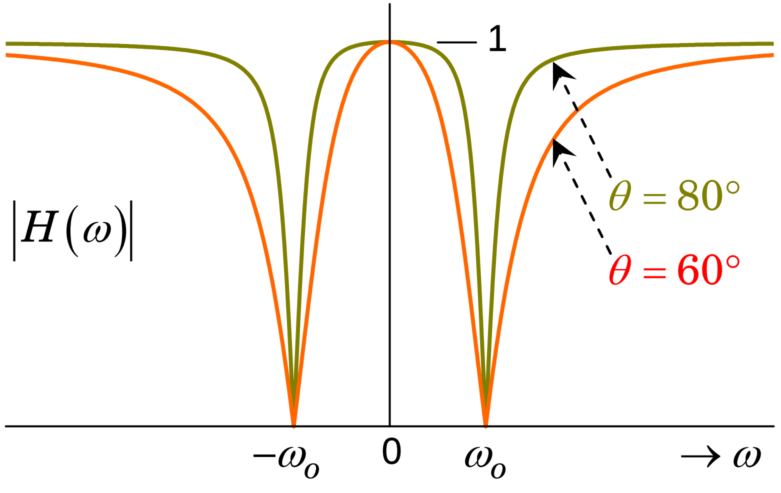
$$=\frac{\omega_o^2-\omega^2}{\omega_o^2-\omega^2+j2\omega_o\omega\cos(\theta)}$$

Magnitude response:

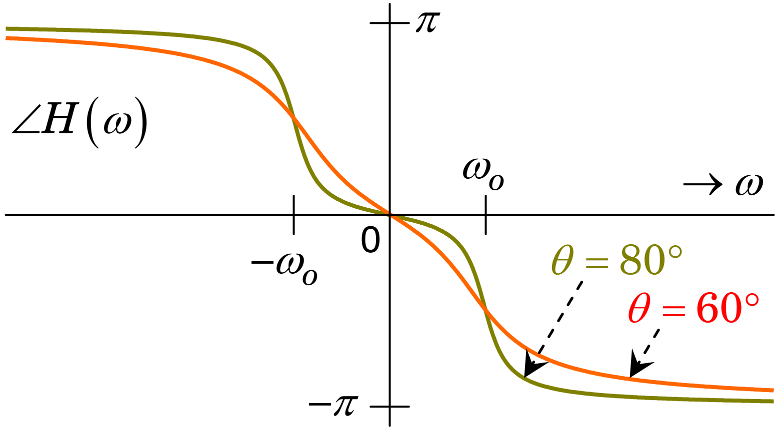
$$|H(\omega)|=\frac{|\omega_o^2-\omega^2|}{\left[(\omega_o^2-\omega^2)^2+4\omega_o^2\omega^2\cos^2(\theta)\right]^{0.5}}$$

Phase response:

$$\angle H(\omega)=-\tan^{-1}\left(\frac{2\omega_o\omega\cos(\theta)}{\omega_o^2-\omega^2}\right)$$



Magnitude Response $|H(\omega)|$



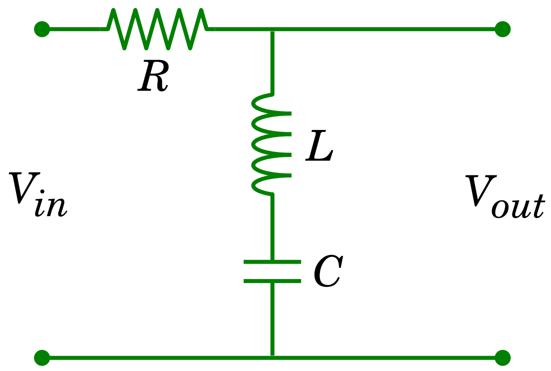
Phase Response $\angle H(\omega)$

- ***Implementation of a Passive Notch Filter***

$$H(s)=\frac{V_{out}(s)}{V_{in}(s)}=\frac{sL+\dfrac{1}{sC}}{R+sL+\dfrac{1}{sC}}$$

$$=\frac{s^2+\dfrac{1}{LC}}{s^2+\dfrac{R}{L}s+\dfrac{1}{LC}}$$

$$=\frac{s^2+\omega_o^2}{s^2+2\omega_o\cos(\theta)s+\omega_o^2}$$



Notch Filter Circuit

Equating coefficients: $\left\{\begin{array}{l}\dfrac{1}{LC}=\omega_o^2\\ \dfrac{R}{L}=2\omega_o\cos(\theta)\end{array}\right\}\rightarrow\left\{\begin{array}{l}\text{Under-determined (Infinitely many solutions):}\\ e.g.\quad R=2\quad L=\dfrac{1}{\omega_o\cos(\theta)}\quad C=\dfrac{\cos(\theta)}{\omega_o}\end{array}\right.$

8.2 *Application:* AM Radio

In radio communications, message signals are transported between a transmitter and a receiver across a wireless channel (or free space). However, the original message signals are seldom in a form that is suitable for transmission and a sinusoidal carrier is almost always employed to carry the message across.

Impressing a message onto the ***amplitude*** of a carrier for transmission is called ***Amplitude Modulation* (AM)**. Radios that employ this technology are called ***AM radios***. Fig.8-7 shows the block diagram of an AM radio system.

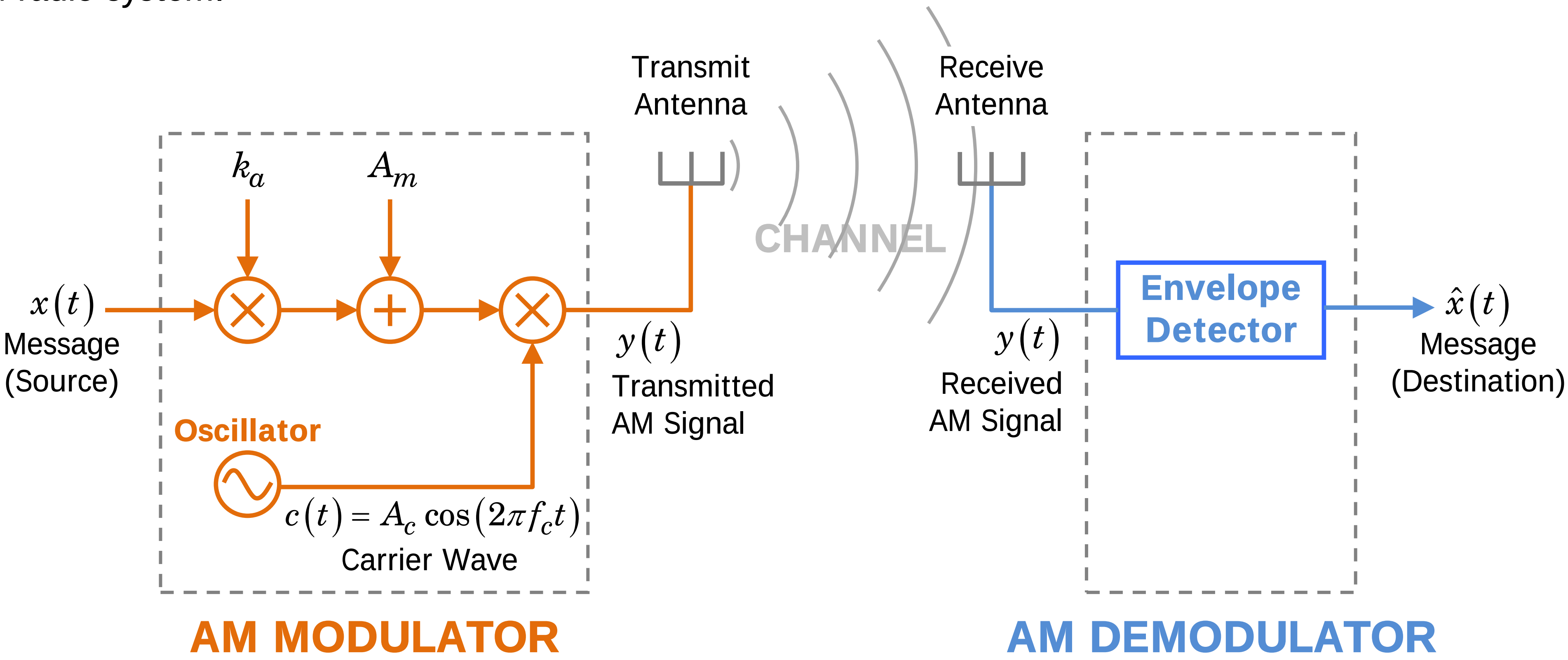


Fig.8-7

- ***Amplitude Modulation***

Message: $$x(t)\begin{cases}\text{Bounded:} & |x(t)|\le A_m \\ \left.\begin{array}{l}\text{Low-pass \&}\\ \text{Bandlimited}\end{array}\right\}: & X(f)=0;\ |f|>f_m\end{cases}$$

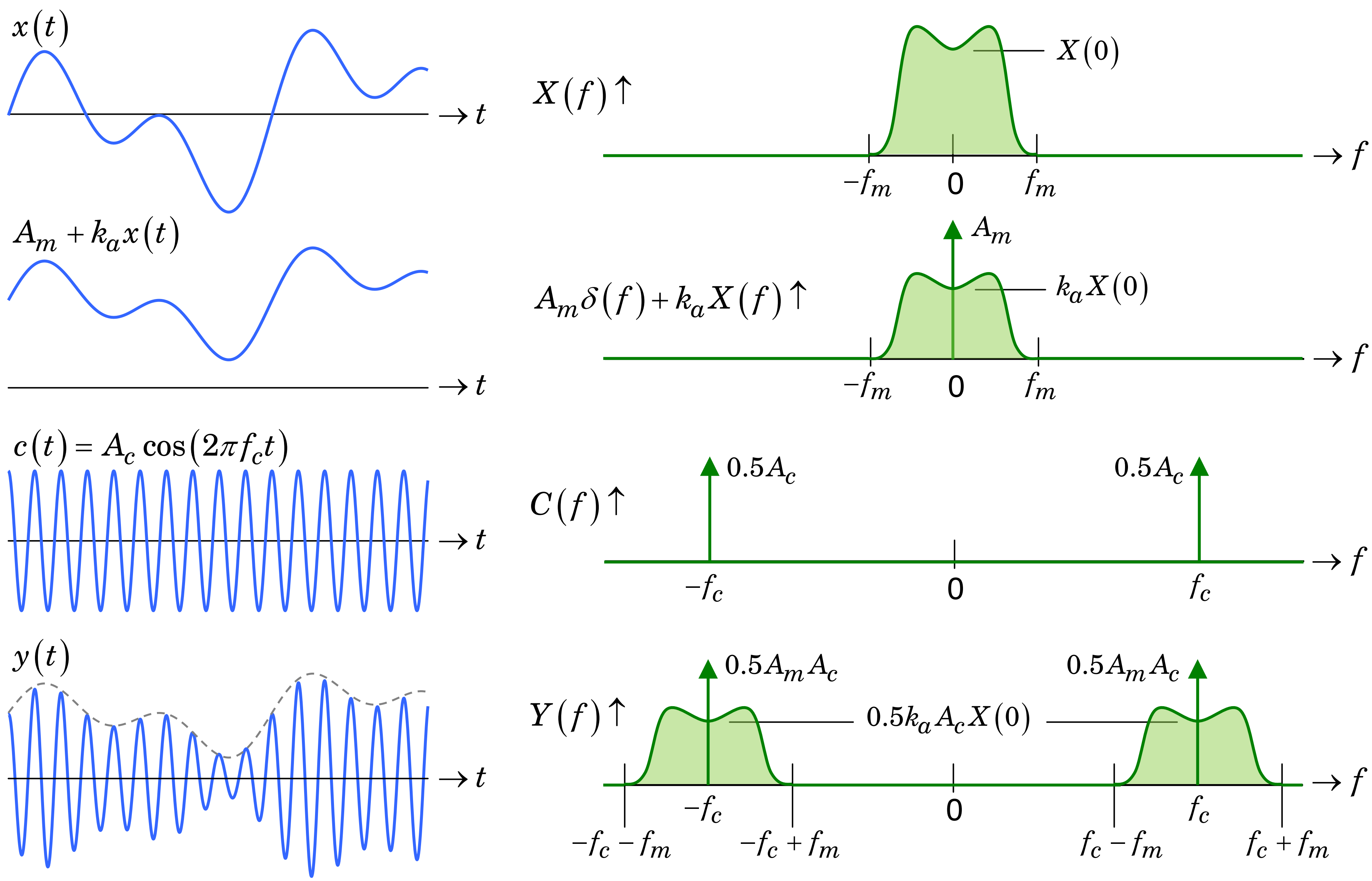
Carrier wave: $$c(t)=A_c\cos(2\pi f_c t)\ \begin{cases}\text{Carrier amplitude:} & A_c\\ \text{Carrier frequency:} & f_c >> f_m\end{cases}$$

Transmitted AM Signal: $$\begin{cases} y(t)=A_c\left[A_m+k_a x(t)\right]\cos(2\pi f_c t) \\ \begin{pmatrix}\text{Modulation index:} & 0<k_a\le 1\\ \text{Envelope:}\left[A_m+k_a x(t)\right]\ge 0 & \end{pmatrix}\end{cases}\quad \cdots\cdots\cdots\ (\clubsuit)$$

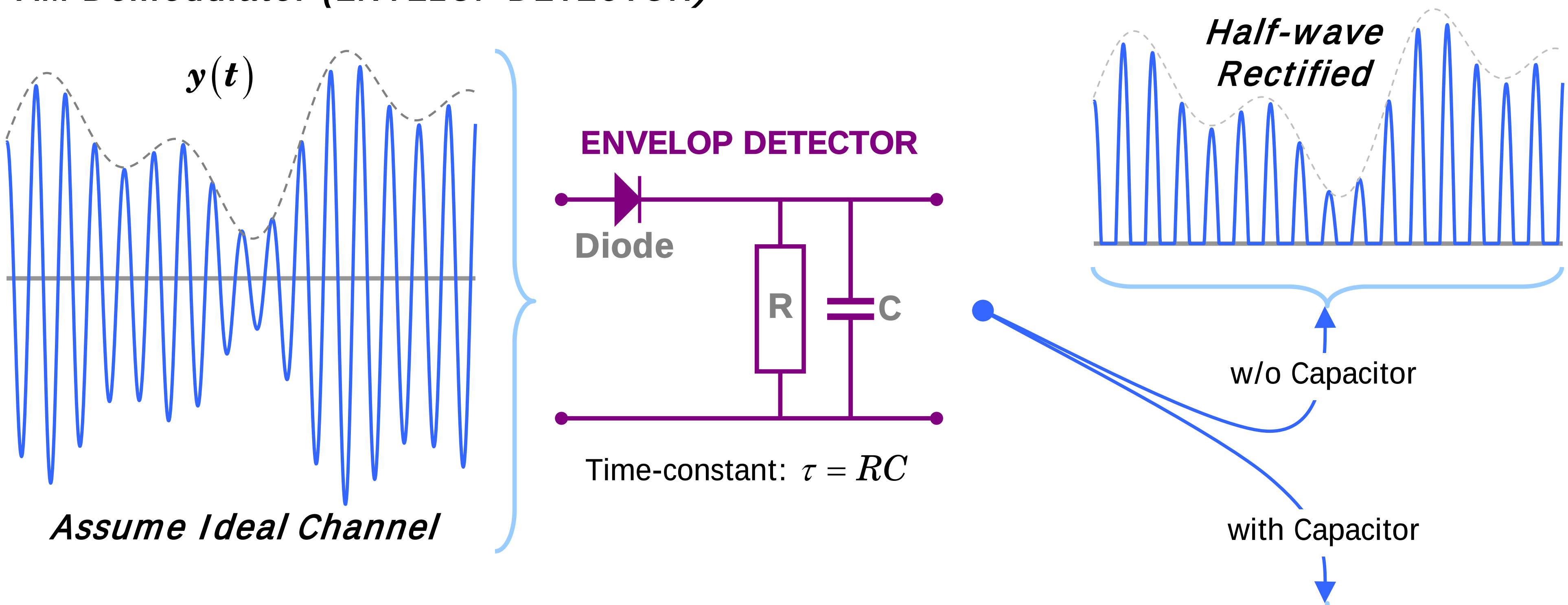
Channel: Assumed $\underbrace{\textbf{ideal band-pass}}_{\textbf{DISTORTIONLESS}}$ and **NOISELESS**

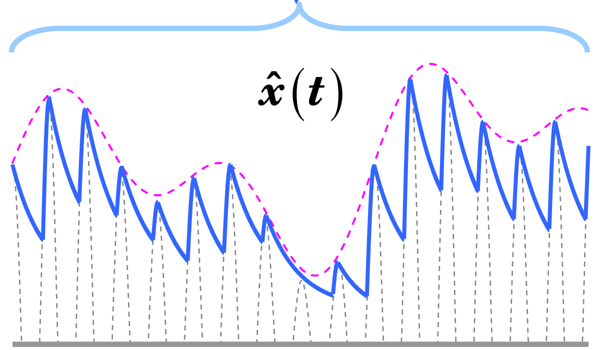
From $(\clubsuit)$, we observe that by impressing $x(t)$ on the amplitude of a sinusoidal carrier $c(t)$, the modulator transforms the message, $x(t)$, from a low-pass into a band-pass signal $y_{Tx}(t)$.

- ***Signal-flow through AM modulator***



- ***AM Demodulator (ENVELOP DETECTOR)***





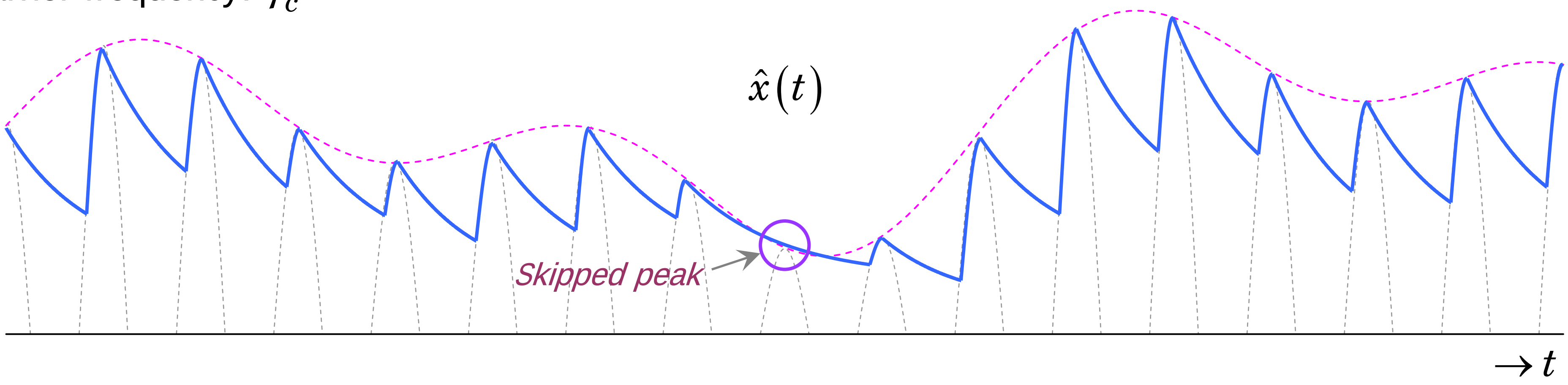
To avoid ***peak-skipping***, τ should be as small as possible. To minimize ***ripple***, τ should be as large as possible. In practice, we should therefore choose a value

$$\frac{1}{f_m} >> \tau >> \frac{1}{f_c}$$

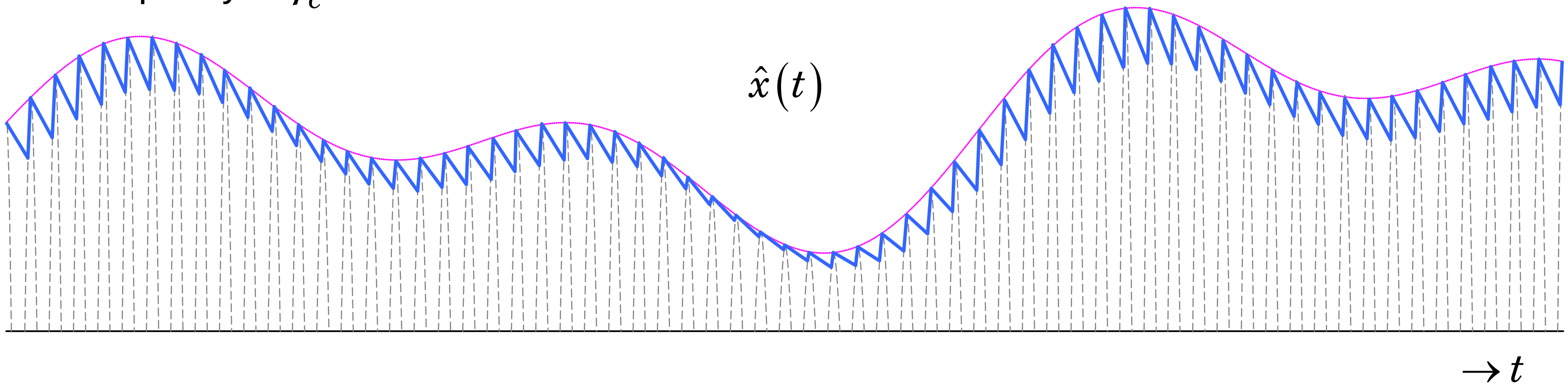
This is clearly only possible if $f_c >> f_m$. Envelope detectors only work satisfactorily when we ensure that this inequality is true.

The below figures show that, with f_m and τ fixed, distortion can be reduced by increasing the frequency, f_c, of the carrier.

Carrier frequency: f_c



Carrier frequency: $4f_c$



8.3 *Application:* Multiplexed Stereo in FM Radios

Frequency modulation (FM) is a form of modulation which conveys information over a sinusoidal carrier by varying its frequency while its amplitude remains constant (contrast this with AM, in which the amplitude of the carrier is varied while its frequency remains constant). This form of modulation is commonly used in the FM broadcast band.

Multiplexing is sending multiple signals on a carrier at the same time in the form of a single composite signal and then recovering the separate signals at the receiving end.

Stereophonic sounds consist of two channels, the left (L) channel and the right (R) channel. It is important that stereo broadcasts should be compatible with mono receivers. For this reason, the left (L) and right (R) channels are algebraically encoded into sum (L+R) and difference (L−R) signals. A mono receiver will use just the (L+R) signal so the listener will hear both channels through a single loudspeaker. A stereo receiver will add the (L−R) signal to the (L+R) signal to recover the L channel, and subtract the (L−R) signal from the (L+R) to recover the R channel.

The (L+R) main-channel signal is transmitted as baseband audio in the range of 30 Hz to 15 kHz. The (L−R) sub-channel signal is frequency-shifted to 38 kHz occupying the baseband range of 23 to 53 kHz.

A 19 kHz pilot tone, at exactly half the 38 kHz sub-carrier frequency and with a precise phase relationship to it is also generated and transmitted for the recovery of the (L−R) signal at the receiver.

The final multiplex signal from the stereo generator contains the main-channel (L+R), the pilot tone, and the sub-channel (L−R). This composite signal frequency-modulates the carrier at the FM transmitter.

Fig.8-8 shows the block diagram of a distortionless and noiseless FM multiplexed stereo radio system.

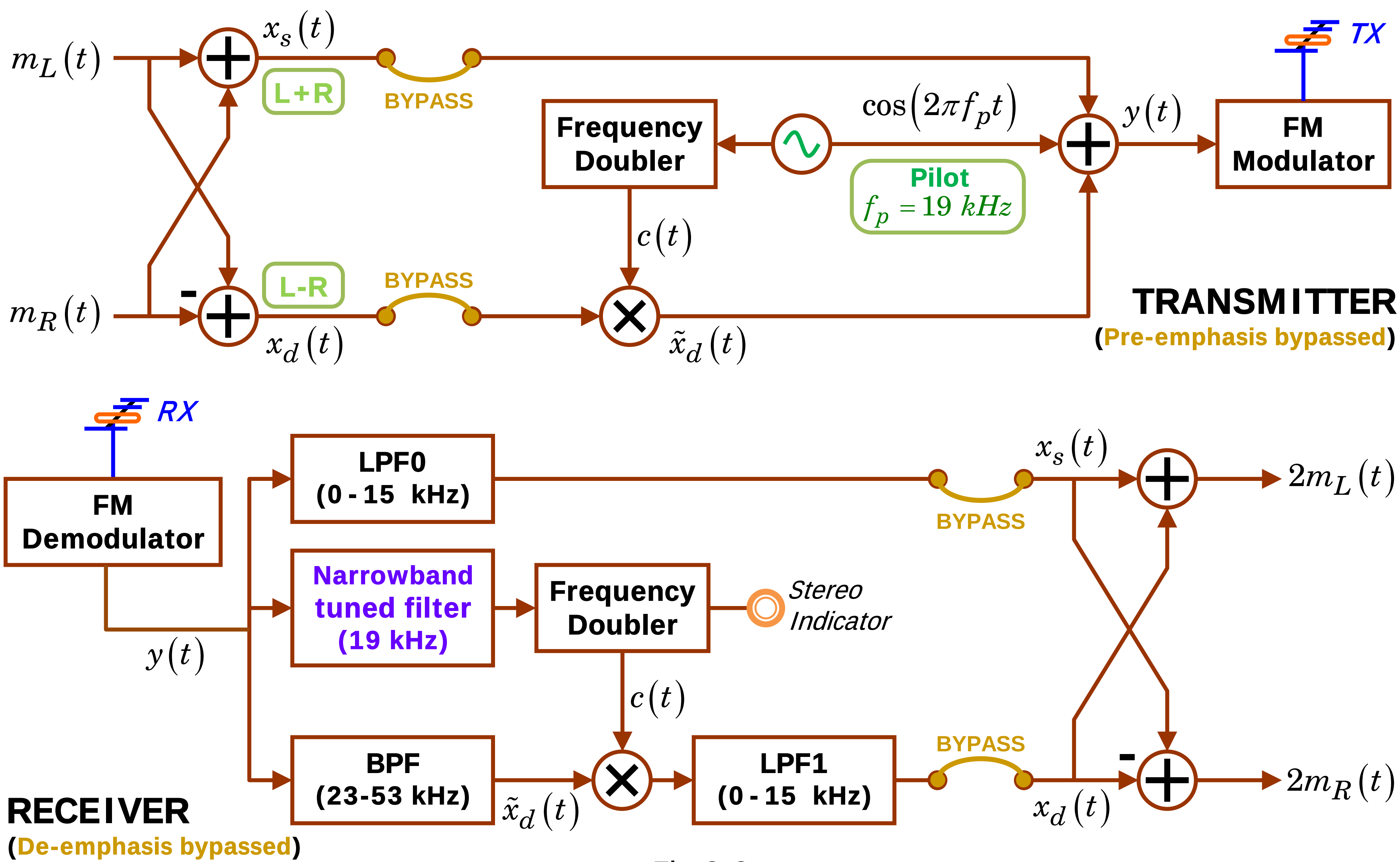
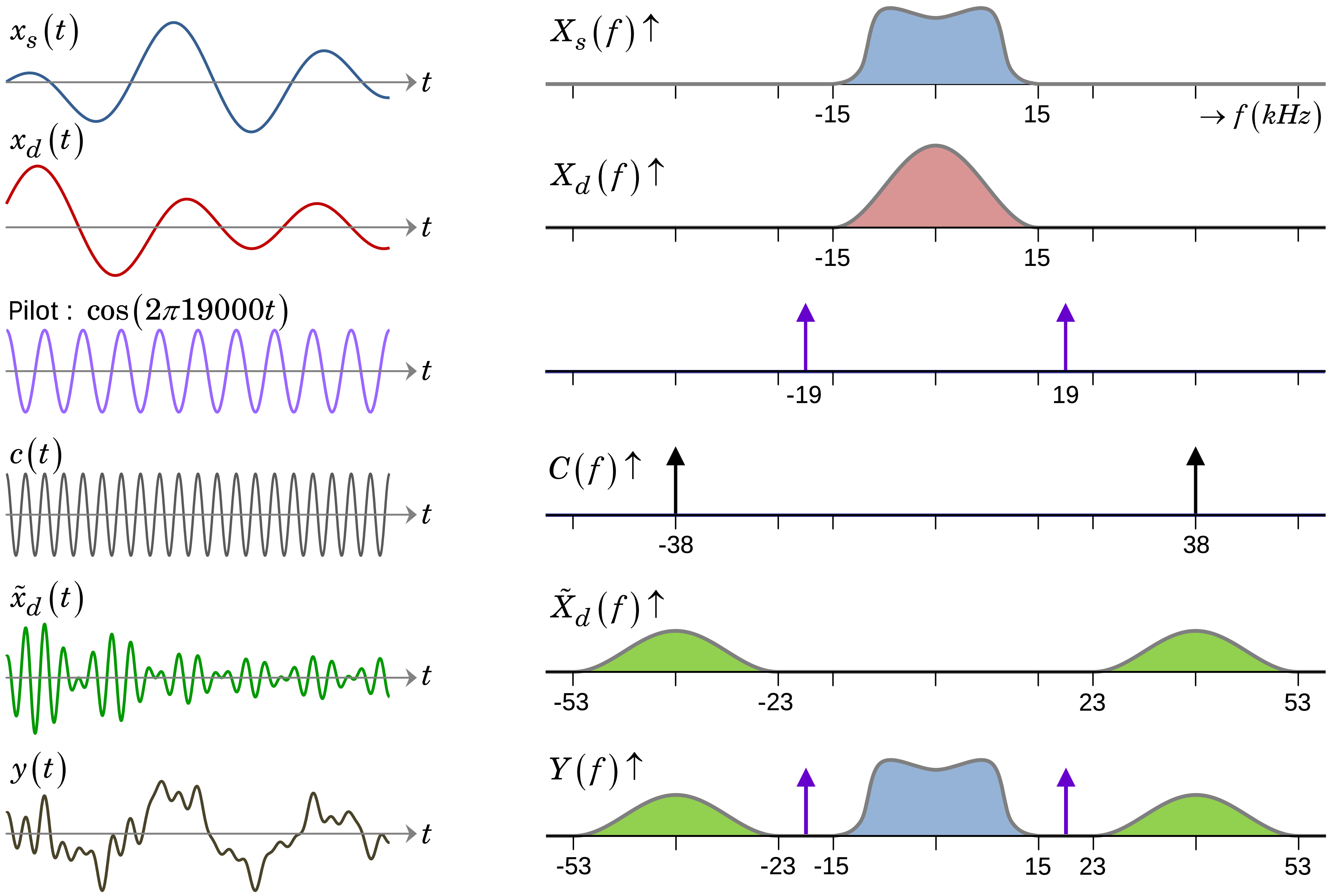


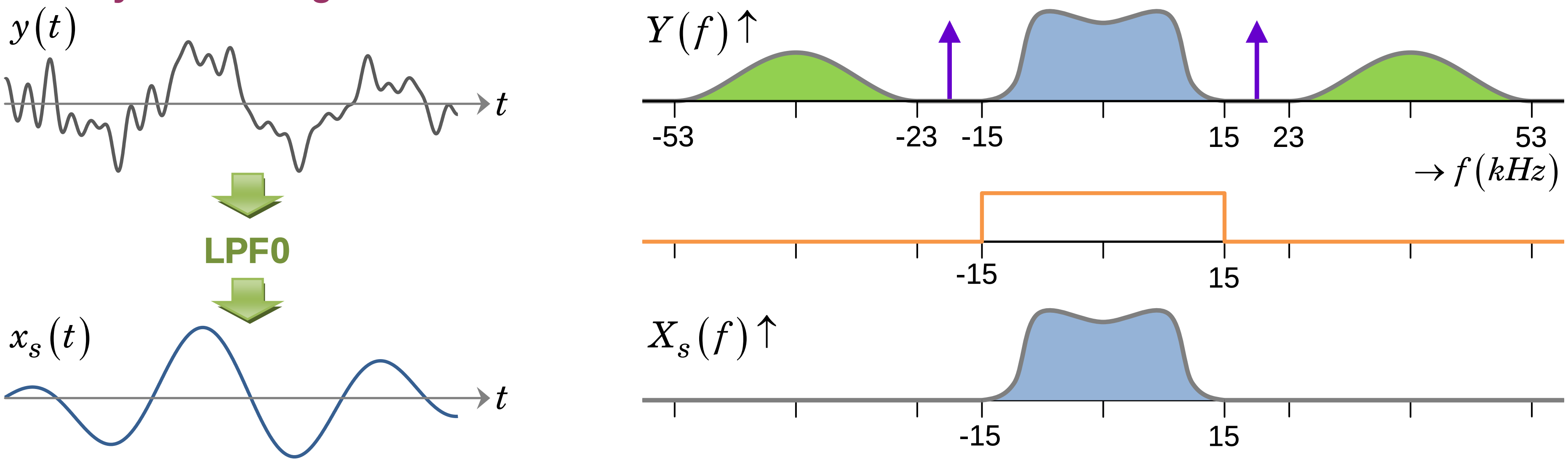
Fig.8-8

- ***Signal-flow through Stereo Multiplexer at the Transmitter***

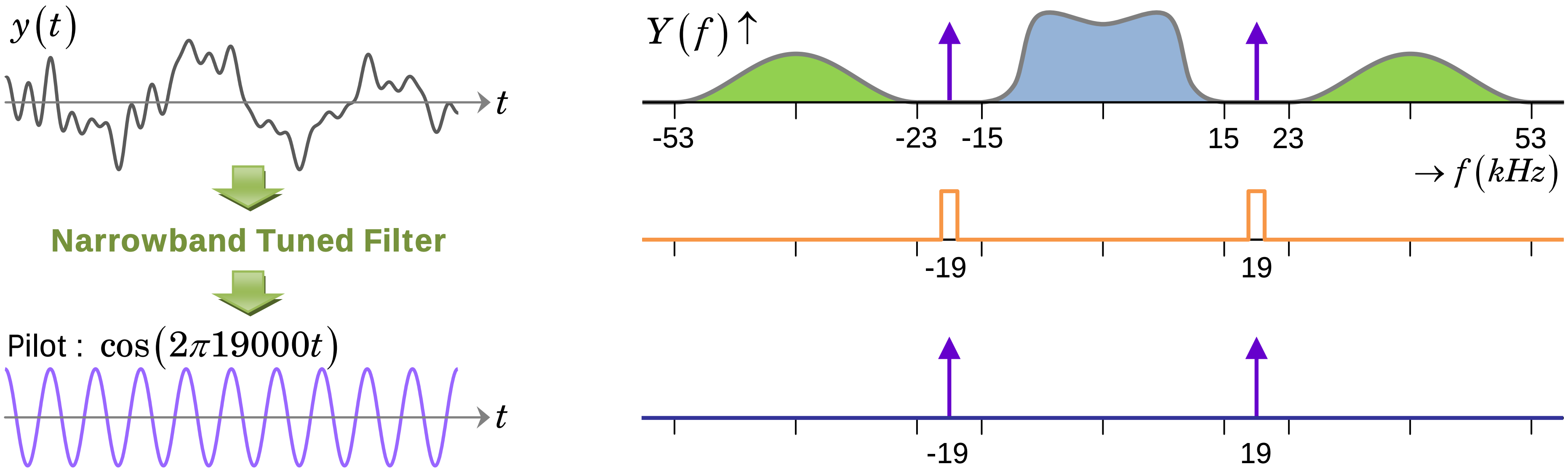


- ***Signal-flow through Stereo De-multiplexer at the Receiver***

Recovery of L+R Signal:



Recovery of Pilot Signal:



Recovery of L-R Signal: