NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2000–2001

MA2214 Combinatorial Analysis

October/November 2000 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **EIGHT (8)** questions and comprises **FIVE (5)** printed pages.

2. Answer **ALL** questions in **Section A**. Each question in Section A carries 12 marks.

3. Answer not more than **TWO (2)** questions from **Section B**. Each question in Section B carries 20 marks.

4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A

Answer **all** *the questions in this section. Section A carries a total of 60 marks.*

Question 1 [12 marks]

Let $r \geq 2$ be a positive integer. Find the number of ways to distribute $2r$ identical objects into 4 distinct boxes numbered 1 to 4 in each of the following cases:

(i) There is no restriction.

(ii) There are no empty boxes.

(iii) Box 1 and Box 2 are both empty.

(iv) Box 1 and Box 2 contain the same number of objects.

(v) Box 1 and Box 2 contain different number of objects.

(vi) The number of objects in Box 1 is strictly less than the number of objects in Box 2.

Question 2 [12 marks]

Using the Pigeon-hole Principle, or otherwise, prove that

(a) for any set S of 5 points on the circumference of a unit circle, there exist two points in S whose distance is at most $\sqrt{2}$ apart;

(b) in a class of 105 students, there exist at least 5 students whose surnames start with the same letter;

(c) in a set T of 17 positive integers, there exist 5 integers in T whose sum is divisible by 5.

Question 3 [12 marks]

(a) In how many ways can you arrange all the seven letters of the word "SINGING" in a row such that no two adjacent letters are the same?

(b) Let S be the set of all sequences (a_1, a_2, a_3, a_4) with 4 terms, where $a_i \in \{0, 1, 2, ..., 9\}$ for each $i = 1, 2, 3, 4$. Define three properties P_1, P_2, P_3 of the elements of S as follows:

For each $x = (a_1, a_2, a_3, a_4) \in S$ and each $i = 1, 2, 3$, we say that

$$\text{“}x \text{ satisfies } P_i \text{ if } a_i = a_{i+1}\text{”}.$$

(i) Find $|S|$;

(ii) Find the number of elements of S satisfying P_i, for each $i = 1, 2, 3$;

(iii) Find the number of elements of S satisfying P_i and P_j with $i \neq j$, for each $i, j = 1, 2, 3$;

(iv) Find the number of elements of S satisfying P_1, P_2 and P_3;

(v) Find the number of elements of S that do not contain two equal consecutive terms.

Question 4 [12 marks]

Find the generating function for each of the following sequences (a_r). Express your answers as quotients of polynomials.

(a) $a_r = r$, for each non-negative integer r.

(b) a_r is the number non-negative integer solutions of the equation:

$$x_1 + x_2 + x_3 + x_4 + x_5 = r,$$

for each non-negative integer r.

(c) a_r is the number of ways to distribute r identical objects into 3 distinct boxes, where the first box contains an even number of objects, the second box contains an odd number of objects and the third box contains at least 2 objects, for each non-negative integer r.

(d) a_r is the number of ways to partition r into exactly 4 parts, all of which are even, for each non-negative integer r.

Question 5 [12 marks]

(a) Solve the recurrence relation

$$a_n - a_{n-1} - a_{n-2} + a_{n-3} = 0,$$

given that $a_0 = 1$, $a_1 = 2$, and $a_2 = -1$.

(b) For each positive integer n, let a_n be the number of binary sequences of length n with an even number of 1's and b_n be the number of binary sequences of length n with an odd number of 1's. Express a_n in terms of a_{n-1} and b_{n-1}. Hence, or otherwise, find a_n.

SECTION B

Answer not more than **two** *questions from this section. Each question in this section carries 20 marks.*

Question 6 [20 marks]

(a) Let S be the set of all permutations of the numbers 1, 2, 3, 4 and 5. Define five properties P_1, P_2, P_3, P_4 and P_5 of the elements $x = a_1a_2a_3a_4a_5$ of S by saying that

"x satisfies P_i if $a_i = i$, for each $i = 1, 2, 3, 4, 5$".

Find the number of elements of S satisfying exactly one of these five properties.

(b) A sequence of 8 numbers is chosen at random (with repetitions allowed) from the set

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$

What is the probability that the sequence contains five consecutive 0's?

Question 7 [20 marks]

(a) Find the generating function for the sequence (a_r), where a_r is the number of integer solutions of the equation:

$$x_1 + x_2 + x_3 = r$$

with $1 \le x_1 \le 3$, $2 \le x_2 \le 4$, and $x_3 \ge 0$. Hence, or otherwise, find a_{50}.

(b) Find the generating function for the sequence (a_r), where a_r is the number of partitions of r into parts of sizes 1, 2 or 4. Hence, or otherwise, find a_{40}.

(c) Find the exponential generating function for the sequence (a_r), where a_r is the number of ways to distribute r distinct objects into 3 distinct boxes such that the total number of objects in the first two boxes is even and the third box is not empty. Hence, or otherwise, determine a_r.

Question 8 [20 marks]

(a) Solve the system of recurrence relations:

$$\begin{cases} 3a_n - 2a_{n-1} - b_{n-1} = 0 \\ 3b_n - a_{n-1} - 2b_{n-1} = 0, \end{cases}$$

given that $a_0 = 2$ and $b_0 = -1$.

(b) A sequence $(a_1, a_2, ..., a_k)$ of positive integers is said to be an *ordered-partition* of a positive integer n if $a_1 + a_2 + \cdots + a_k = n$. Let a_n be the number of ordered-partitions of n. Find a recurrence relation for (a_n). Hence, or otherwise, determine a_n.

(c) For each non-negative integer n, let a_n denote the number of ordered-partitions of n into parts, the sizes of which are 1 or 2. Find a recurrence relation for (a_n). Hence, or otherwise, determine a_n.

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