Combinatorial Analysis MA2214

Dr Toh Pee Choon

National University of Singapore

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Introduction

This is the lecture notes for the course. We shall follow closely the textbook **Introduction to Enumerative Combinatorics** by Miklós Bóna. In the foreword, Richard Stanley says

> What could be a more basic mathematical activity than counting the number of elements of a finite set? The misleading simplicity that defines the subject of enumerative combinatorics is in fact one of its principal charms. Who would suspect the wealth of ingenuity and of sophisticated techniques that can be brought to bear on a such an apparently superficial endeavour?

Outline

Sets

We are interested in enumerating sets.

Definition 1.1

Let $|X|$ denote the number of elements of a finite set X.

When We Add

Example 1.2

A group of friends went kayaking at East Coast Park. 5 fell into the water at one point or another, while 7 finished the day without getting wet. How many friends went on this trip?

- Define the following sets.
  T: friends who went kayaking
  W: friends who fell into the water
  D: friends who did not get wet
- Clearly $W \cap D = \emptyset$ and $T = W \cup D$
- Hence
$$|T| = |W| + |D| = 5 + 7 = 12.$$

When We Add

Example 1.3

A group of friends went kayaking at East Coast Park. 5 wore sunglasses, while 7 wore beach shorts. How many friends went on this trip?

- Define the following sets.
  T: friends who went kayaking
  S: friends who wore sunglasses
  B: friends who wore beach shorts
- Insufficient information:

$$S \cap B \stackrel{?}{=} \emptyset \qquad \text{and} \qquad T \stackrel{?}{=} S \cup B.$$

When We Add

Theorem 1.4 (Addition Principle)

If A and B are disjoint finite sets, i.e. $A \cap B = \emptyset$, then

$$|A \cup B| = |A| + |B|.$$

Proof: Both sides of the identity counts the same set. The left hand side counts the set $A \cup B$ directly, while the right hand side counts the elements of A and B separately. In either case, each element is counted exactly once since A and B are disjoint. $\square$

When We Add

Theorem 1.5 (Generalized Addition Principle)

If $A_1, A_2, \ldots, A_n$ pairwise disjoint finite sets, i.e. $A_i \cap A_j = \emptyset, i \neq j$, then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots |A_n|.$$

Or

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i|.$$

Sets

Definition 1.6

Let $A - B$ denote the set of elements that are in A but not in B.

For example if $A = \{2, 3, 5, 7\}$ and $B = \{4, 5, 7\}$, then

$$A - B = \{2, 3\}.$$

When We Add

Theorem 1.7 (Subtraction Principle)

If A be a finite set and $B \subseteq A$, then

$$|A - B| = |A| - |B|.$$

Proof: Check that $(A - B) \cup B = A$ and $(A - B) \cap B = \emptyset$. By the Addition Principle,

$$|A - B| + |B| = |A|.$$

$\square$

When We Add

Example 1.8

Count the number of positive integers less than or equal to 1000 that have at least two different digits.

Let X, Y and Z denote different digits, classify integers ≤ 1000 as
1) X
2) XX, XY
3) XXX, XXY, XYX, XYY, XYZ
4) 1000

Let A be the set of positive integers less than or equal to 1000 and B be the set of integers less than or equal to 1000 that do not have different digits. Then

$$|A - B| = |A| - |B| = 1000 - 27 = 973.$$

When We Add

Example 1.8

Alternative Presentation

Count the number of positive integers less than or equal to 1000 that have at least two different digits.

$A = \{n \in \mathbb{Z}^+ \mid n \leq 1000\} \implies |A| = 1000.$

$B = \{n \in \mathbb{Z}^+ \mid n \leq 1000, n \text{ does not have two different digits}\}$

$\quad = \{n \in \mathbb{Z}^+ \mid n \text{ has exactly one, two or three repeated digits}\}$

$\implies |B| = 27.$

$$|A - B| = |A| - |B| = 1000 - 27 = 973.$$

When We Multiply

Example 1.9

Suppose there are two types of bicycles for sale, mountain bikes or racers. Each comes in 3 colours: black, silver or red. How many different choices of bicycles do you have?

- Use (T, C) to denote each possible bike, where T stands for the type: Mountain bike or Racers and C stand for the colour: Black, Silver or Red.
- The various different choices are:
  $(M, B), (M, S), (M, R)$
  $(R, B), (R, S), (R, R)$
- A total of $2 \times 3 = 6$ choices.

Sets: Ordered pairs and tuples

Definition 1.10

Let X and Y denote (not necessarily finite) sets. Then a direct (or Cartesian) product $X \times Y$ is the collection of all ordered pairs (x, y) where $x \in X$ and $y \in Y$.

Note that order is critical and in general $(x, y) \neq (y, x)$.

Definition 1.11

Let $X_1, \ldots, X_n$ be sets. Then the direct product $X_1 \times \cdots \times X_n$ is the collection of all ordered $n-$tuples $(x_1, \ldots, x_n)$ where $x_i \in X_i$.

When We Multiply

Theorem 1.12 (Product Principle)

If X and Y are finite sets, then the number of ordered pairs $(x, y) \in X \times Y$ is $|X| \times |Y|$ i.e

$$|X \times Y| = |X| \times |Y|.$$

Theorem 1.13 (Generalized Product Principle)

If $X_1, X_2, \ldots, X_n$ are finite sets, then the number of ordered tuples $(x_1, x_2, \ldots, x_n) \in X_1 \times X_2 \times \cdots \times X_n$ is $|X_1| \times |X_2| \times \cdots \times |X_n|$.

When We Multiply

Example 1.14

The number of two-digit positive integers is 90.

- View each integer as an ordered pair, i.e. $34 \leftrightarrow (3, 4)$. Then we are counting $X \times Y$ where
- $X = \{1, 2, 3, \ldots, 9\}$,
- $Y = \{0, 1, 2, 3, \ldots, 9\}$.
- So $|X \times Y| = 9 \times 10 = 90$.

When We Multiply

Example 1.15

For any positive integer k, the number of k-digit positive integers is $9 \cdot 10^{k-1}$.

- View each integer as a k-tuple in $X_1 \times X_2 \times \cdots \times X_k$.
- $X_1 = \{1, 2, 3, \ldots, 9\}$,
- $X_i = \{0, 1, 2, 3, \ldots, 9\}$ for $1 < i \leq k$.
- So $|X_1 \times X_2 \times \cdots \times X_k| = 9 \times 10^{k-1}$.

When We Multiply

Example 1.16

How many four digit positive integers both start and end in even digits?

First digit comes from $\{2, 4, 6, 8\}$,
and the last digit comes from $\{0, 2, 4, 6, 8\}$.
So the total number is

$$4 \times 10 \times 10 \times 5 = 2000.$$

When We Add And Multiply

Example 1.17

Choose a password of 4 to 7 digits from $0, 1, \ldots, 9$ *without any other restrictions. How many choices do I have?*

- Let A_i be a password of i-digits then $|A_i| = 10^i$ by Product Principle.
- Note that $A_i \cap A_j = \emptyset$ if $i \neq j$.
- Hence total number of choices, by Addition Principle, is

$$\left|\bigcup_{i=4}^{7} A_i\right| = \sum_{i=4}^{7} |A_i| = \sum_{i=4}^{7} 10^i$$
$$= 10^4(1 + 10 + 100 + 1000) = 11,110,000.$$

When We Add And Multiply

Example 1.18

A thief observed that my password has 5 digits, does not start with 0 and contains the digit 8. In the worst case, how many attempts does he require to find my password?

- It is difficult to compute the number of 5-digit passwords containing at least one 8.
- Let $B = \{5 \text{ digit integer } P \mid P \text{ does not contain } 8\}$.
  Then $|B| = 8 \times 9^4$.
- Let $A = \{5 \text{ digit integer } P \mid \text{ no restrictions } \}$.
  Then $|A| = 9 \times 10^4$.
- The maximum number of attempts is

$$|A - B| = 9 \times 10^4 - 8 \times 9^4 = 37512.$$

When We Add And Multiply

Example 1.19

A 3-course set lunch at a campus cafe offers the following choices: Pick one out of two soups, one out of five main courses and a dessert. Dessert is either one of three sundaes or a coffee with a cookie. Coffee choices are a long black or a flat white, and for each coffee, you can choose between a chocolate chip or a wholemeal cookie. How many different meal choices do you have?

- You either pick a sundae or coffee+cookie for dessert. (Addition Principle)
- Choices for (Soup, Main, Sundae) $= 2 \times 5 \times 3 = 30$.
- Choices for (Soup, Main, Coffee, Cookie) $= 2 \times 5 \times 2 \times 2 = 40$.
- So there are a total of $30+40=70$ choices.

Outline

Repetitions Not Allowed

Example 2.1

Eight runners participate in a long distance race. If there are no ties, in how many ways can the race end?

- Previously our choices were independent. Now if runner A wins the race, he cannot finish third or fourth etc.
- There are 8 choices for the winner. After picking the winner we have 7 choices for the runner-up. After which, 6 choices are left for third place etc.
- Total number of ways

$$8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40320.$$

Repetitions Not Allowed

Definition 2.2

Let n be a positive integer, then the number

$$n \times n - 1 \times \cdots \times 2 \times 1$$

is called n-factorial and denoted by $n!$.

Remark: We define $0! = 1$

Definition 2.3

We use $[n]$ to denote the set

$$\{1, 2, \ldots, n-1, n\}.$$

Repetitions Not Allowed

Definition 2.4

A permutation of a finite set S is a (ordered) list of the elements of S containing each element exactly once.

Theorem 2.5

The number of permutations of $[n]$ is $n!$.

Repetitions Not Allowed

Theorem 2.6

Let n and k be positive integers so that $n \geq k$. Then the number of ways to make a k-element list from $[n]$ without repetition is

$$n(n-1)(n-2)\cdots(n-k+1).$$

Definition 2.7

We define the k-th falling factorial as

$$(n)_{\underline{k}} = n(n-1)(n-2)\cdots(n-k+1).$$

Remark: $(n)_{\underline{n}} = n!$.

Repetitions Not Allowed

Example 2.8

You are planning a holiday. You can go north to Malaysia and visit 4 out of 6 possible destinations or go south to Indonesia and visit any two of Jakarta, Bali or Bintan. How many different itineraries can you have?

You either go north or south (Addition Principle)
Total number of itineraries:

$$(6)_{\underline{4}} + (3)_{\underline{2}} = 6 \times 5 \times 4 \times 3 + 3 \times 2 = 366.$$

When We Divide

Example 2.9

Assume several families are invited to a children's party and each family comes with two children. If there are altogether 10 children at the party, how many families are present?

- The answer is obviously $10/2 = 5$.
- In general, if we want to count a set S, it might be easier to count another set T if there is a certain relation between the two.

When We Divide

Definition 2.10

Let S and T be finite sets, and let d be a fixed positive integer. We say that the function f from T onto S is d-to-one if for each element $s \in S$ there exists exactly d elements $t \in T$ so that $f(t) = s$.

Example 2.11

Let $T = \{-4, -3, -2, -1, 1, 2, 3, 4\}$ and $S = \{1, 4, 9, 16\}$, then the function $f : T \to S$ defined by $f(t) = t^2$ is 2-to-one.

When We Divide

Theorem 2.12 (Division Principle)

Let S and T be finite sets so that there exists a d-to-one function from T onto S. Then

$$|S| = \frac{|T|}{d}.$$

When We Divide

Example 2.13

The number of different seating arrangements for n people around a circular table is $(n-1)!$.

- $\#$ of ways to arrange n people in a line is $n!$.
- Construct a function from any linear seating to a circular seating by wrapping and joining the two ends.
- It remains to show this function is n-to-one.
- Start with any circular seating. There are exactly n ways to 'break' it into a linear seating such that the immediate anticlockwise person is the first and the immediate clockwise person is the last.
- Hence $\#$ of circular seatings $= \frac{n!}{n} = (n-1)!$.

When We Divide

Example 2.14

The number of ways to form a necklace from $n \geq 3$ distinct gemstones is $\frac{(n-1)!}{2}$.

- Circular seating vs necklace: you can flip over a necklace.
- There is a 2-to-one function from # of circular seating onto # of different necklaces.
- Hence # of ways to form a necklace $= \frac{(n-1)!}{2}$.

When We Divide

Definition 2.15

Let S and T be finite sets. If the function f from T onto S is one-to-one then we call f a bijection.

Theorem 2.16 (Bijection Principle)

Let S and T be finite sets. If a bijection exists between S and T, then $|S| = |T|$.

Bijection Principle

Example 2.17

For any positive integer n, the number of divisors of n that are larger than $\sqrt{n}$ is equal to the number of divisors of n that are smaller than $\sqrt{n}$

Bijection Principle

Proof: Let $S = \{s \text{ divides } n \mid s > \sqrt{n}\}$ and $T = \{s \text{ divides } n \mid s < \sqrt{n}\}$. Define $f : S \to T$ by $f(s) = n/s$.
Check well-defined: If s is a divisor of n, then n/s also divides n. If

$$f(s) \geq \sqrt{n} \implies n = s \times f(s) > \sqrt{n}\sqrt{n} = n$$

a contradiction.
Check 1-to-1: If

$$f(s) = f(s') \implies \frac{n}{s} = \frac{n}{s'} \implies s = s'.$$

Check onto: Let t be a divisor of n, then n/t is also a divisor and

$$f(n/t) = \frac{n}{n/t} = t.$$

Bijection Principle

Example 2.18

The integer 1000 has exactly eight divisors that are larger than $\sqrt{1000}$.

It suffices to check the interval from 1 to 31 since $\sqrt{1000} \approx 31.62$. The divisors are 1, 2, 4, 5, 8, 10, 20, 25.

Double Barrelled Race option

Example 2.19

The Singapore government announced that from 2011, children born to parents of different races can adopt a double barrelled race option. For example if one parent is Chinese and the other is Indian, the child's race can be recorded as Chinese, Indian, Chinese-Indian or Indian-Chinese.
If one spouse is Malay-Chinese and the other is Indian-Thai, how many different possibilities can the child's race be recorded?

- Either double race or single race (Addition Principle) $(4)_{\underline{2}} + (4)_{\underline{1}} = 4 \times 3 + 4 = 16$.
- (Bijection) X $\mapsto$ X-X
  $\#$ of ways $= 4 \times 4 = 16$

Double Barrelled Race option

Example 2.20

In fact, there is an added restriction, that the first component race must be taken from the first component race of the parents if both parents are registered with a double-barrelled race. For example, if one spouse is Malay-Chinese and the other is Indian-Thai, the first component race for their child can only be Malay or Indian. What is the actual number of possibilities?

- (Addition Principle) $2 \times 3 + (2)_{\underline{1}} = 6 + 2 = 8$.
- (Division Principle) 1) X $\mapsto$ X-X,
  2) $\{$I,T$\}$ $\mapsto$ I and $\{$M,C$\}$ $\mapsto$ M for first component
  2 to 1 mapping, so # is 16/2=8.

Outline

Selecting A Subset

Example 3.1

The number of ways to choose a chairman, secretary and treasurer from 8 people is $8 \times 7 \times 6 = (8)_{\underline{3}}$.

Example 3.2

The number of ways to choose a three member committee from 8 people is $\frac{8\times7\times6}{3!}$.

The answer is no longer $8 \times 7 \times 6$ because choices like (A, B, C) and (B, A, C) are distinct.
Let f be the function that sends an ordered triple (A, B, C) to an unordered set $\{A, B, C\}$.
Then f is 6-to-one.

Selecting A Subset

Theorem 3.3

Let n be a positive integer and k be a non-negative integer so that $n \geq k$. Then the number of all k-element subsets of $[n]$ is

$$\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \frac{(n)_{\underline{k}}}{k!}.$$

Definition 3.4

We define binomial coefficients by

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}.$$

Set $\binom{n}{k} = 0$ if $n < k$.

Identities for Binomial Coefficients

Theorem 3.5 (Symmetry)

$$\binom{n}{k} = \binom{n}{n-k}$$

Algebraic Proof:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}.$$

Combinatorial Proof:

$\binom{n}{k}$ counts the number of ways to pick a subset of k elements from n elements. After choosing k elements, exactly $n-k$ elements are left. So there is a bijection from the set of k elements subset to the set of $n-k$ elements subset. By the bijection principle $\binom{n}{k} = \binom{n}{n-k}$.

Example 3.6

In how many ways can a committee of 5 be formed from a group of 11 people consisting of 4 tutors and 7 students if
(i) there is no restriction?
(ii) there must be exactly 2 tutors?
(iii) there must be at least 3 tutors?
(iv) tutor X and student Y cannot be both in the committee?

(i) $\binom{11}{5}$,
(ii) $\binom{4}{2}\binom{7}{3}$
(iii) $\binom{4}{3}\binom{7}{2} + \binom{4}{4}\binom{7}{1}$
(iv) $\binom{11}{5} - \binom{9}{3}$ (subtraction principle).

Example 3.7

Find # ways to place 8 rooks on an 8×8 chessboard, so that no two of them attack each other.

- Each rook must occupy a unique column
- For the leftmost rook, we have 8 rows to choose
- 7 rows left for second rook
- Answer $= (8)_{\underline{8}} = 8!$ ways.

Example 3.8

Find # ways to place some rooks on an 8×8 chessboard, so that no two of them attack each other.

- 1 way if no rooks to place
- For k rooks, $\binom{8}{k}$ ways to choose columns
- $(8)_{\underline{k}}$ ways to choose the rows
- Answer by product and addition principle (if we include the no rooks case)

$$1 + \sum_{k=1}^{8} \binom{8}{k} (8)_{\underline{k}} = \sum_{k=0}^{8} \binom{8}{k} (8)_{\underline{k}}$$

if we define $(n)_{\underline{0}} = 1$.

Identities for Binomial Coefficients

Theorem 3.9 (Pascal's relation)

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Combinatorial Proof:
LHS counts the number of ways to pick a subset of k elements from n elements.
RHS counts the same number except that among the n elements, designate one as X. Now any k-element subset either contains or does not contains X. By the addition principle, the number of ways to pick a k-element subset without X is $\binom{n-1}{k}$ and the number of ways to pick a k-element subset with X is $\binom{n-1}{k-1}$.

Pascal triangle

$\binom{n}{k}$											k
$n = 0$						1					
$n = 1$					1		1				
$n = 2$				1		2		1			
$n = 3$			1		3		3		1		
$n = 4$		1		4		6		4		1	
$n = 5$	1		5		10		10		5		1

Theorem 3.10 (Binomial Theorem)

Let n be a positive integer, then

$$(X+Y)^n = \sum_{k=0}^{n} \binom{n}{k} X^k Y^{n-k}.$$

Proof: The left hand side is

$$(X+Y)(X+Y)\cdots(X+Y),$$

with factor $(X+Y)$ appearing n times. The number of ways to choose k copies of X to form X^k is $\binom{n}{k}$. Note that each term should have exactly n factors, thus if we have picked k copies of X, there is only one way to pick $n-k$ copies of Y from the remaining factors. $\square$

Example 3.11

Find the last three digits of 9^{100}.

Solution:

$$
\begin{aligned}
(10-1)^{100} &= \sum_{k=0}^{100} \binom{100}{k} 10^k (-1)^{100-k} \\
&= 1 - \binom{100}{1} 10 + \binom{100}{2} 10^2 + \sum_{k=3}^{100} \binom{100}{k} 10^k (-1)^k
\end{aligned}
$$

Since 1000 divides every term except the first, the last three digits are 001.

Binary Sequences

Definition 3.12

A binary sequence of length n is a sequence consisting of n 0s and 1s concatenated together.

For example, these are all the 16 sequences of length 4.
$0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111,$
$1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111.$

Theorem 3.13

$\#$ of binary sequences of length $n = 2^n$.

Binary Sequences

Example 3.14

Prove the following identity using a combinatorial argument.

$$2^n = \sum_{k=0}^{n} \binom{n}{k}.$$

- count $\#$ of ways to pick a subset from n distinct objects
- bijection to set of binary sequences of length n
- line up the n objects, 1 means the object is picked
- total $\#$ of ways $= 2^n$ (bijection principle)
- RHS counts the $\#$ of ways to pick a subset of size k for all k.

Binary Sequences

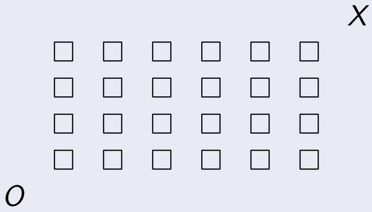
Remark: By the bijection principle again, $\binom{n}{k}$ counts the $\#$ of binary sequences of length n with exactly k 1s (or exactly k 0s.)

Thus we can interpret $2^n = \displaystyle\sum_{k=0}^{n} \binom{n}{k}$ as two different ways to count the total number of binary sequences of length n.

Lattice Paths

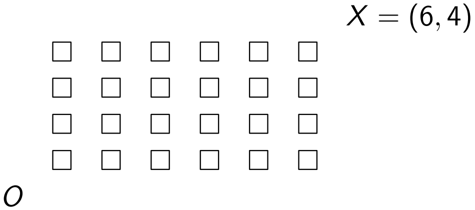
Example 3.15

In a certain town consisting of 24 buildings, the streets form a square grid like below.



How many different ways are there to get from $O = (0,0)$ *to* $X = (6,4)$ *using the least number of steps?*

Lattice Paths



No matter which path you choose, you need to travel a total of 10 blocks, 6 of which eastwards and 4 of which northwards. Two possible paths would be NNNNEEEEEE or NENENENEEE.
Thus it suffices to decide which of the ten blocks to travel northwards and the total number of ways is $\binom{10}{4}$. (Note bijection with binary sequences.)

Lattice Paths

Example 3.16

In a the same town, how many different ways are there to get from $O = (0,0)$ to point $P = (4,2)$ then to the point $X = (6,4)$ in the least number of steps.

To get from O to P, there are $\binom{6}{2}$ ways and to get from $P = (4,2)$ to $X = (6,4)$, there are $\binom{4}{2}$ ways. In total by the product principle, there are $\binom{6}{2}\binom{4}{2} = 15 \times 6 = 90$ ways.

Lattice Paths

Example 3.17

In a the same town, how many different ways are there to get from $O = (0,0)$ to point $X = (6,4)$,in the least number of steps, if you want to stop at either $A = (3,2)$ or $B = (2,3)$.

$O \to A \to X = \binom{5}{2}\binom{5}{2}$ ways
$O \to B \to X = \binom{5}{3}\binom{5}{1}$ ways
By the addition principle, answer $= 10^2 + 10 \times 5 = 150$ ways.

Remark: In future we shall refer to these as NE lattice paths and we always assume that we take the shortest path possible, i.e. without retracing, detours etc.

Lattice Paths

Example 3.18

Prove the following identity using a combinatorial argument.

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

- LHS counts the # of NE lattice paths in an $n \times n$ grid
- Write $\binom{n}{k}^2 = \binom{n}{k}\binom{n}{n-k}$
- This counts how far north one travels in the first n steps
- The value of k should range from 0 to n

Example 3.18

Alternative interpretation.

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

- LHS counts the # of ways to choose n objects from $2n$ objects
- Divide these $2n$ into two groups
- with k objects in the first group and $n-k$ in the second group
- Again k should range from 0 to n

Example 3.19

$$n\binom{2n-1}{n-1} = \sum_{k=1}^{n} k\binom{n}{k}^2.$$

- Count the # of ways to choose n students from n boys and n girls with the restriction that at least one boy be the monitor
- LHS picks the monitor first then the remaining $n-1$ students
- RHS counts the number of ways to pick k boys, select the monitor and then pick the $n-k$ girls.
- Note the use of the product and addition principle.

Recall $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}$ can be interpreted as finding a n students from n boys and n girls.

What if we want to choose n students from r boys and s girls? (For simplicity we first consider $n \le \min\{r, s\}$.)

Is the following correct?

$$\binom{r+s}{n} = \sum_{k=0}^{n} \binom{r}{k}\binom{s}{n-k}$$

Chu-Vandermonde identity

Theorem 3.20 (Chu-Vandermonde)

For arbitrary nonnegative r, s and n

$$\binom{r+s}{n} = \sum_{k=0}^{n} \binom{r}{k} \binom{s}{n-k}.$$

Proof: LHS is the coefficient of X^n in

$$(1+X)^{r+s} = (1+X)^r (1+X)^s$$

To pick the coefficient of X^n from the RHS of the above, we pick k copies of X from $(1+X)^r$ and the remaining $n-k$ copies from $(1+X)^s$, for all possible values of k. $\square$

Chu-Vandermonde identity

Theorem 3.21 (Chu-Vandermonde)

For arbitrary nonnegative r, s and n

$$\binom{r+s}{n} = \sum_{k=0}^{n} \binom{r}{k} \binom{s}{n-k}.$$

Proof: LHS is the coefficient of X^n in

$$(1+X)^{r+s} = (1+X)^r (1+X)^s$$

To pick the coefficient of X^n from the RHS of the above, we pick k copies of X from $(1+X)^r$ and the remaining $n-k$ copies from $(1+X)^s$, for all possible values of k. $\square$

Example 3.22

A coach must choose 4 players out of 20, 10 of whom are offensive players and 10 are defensive. How many possibilities are there if the coach must choose at least 1 offensive and 1 defensive player?

<u>Method 1</u>: Let A be all subsets of 4 players without restriction and B be all subsets of 4 players without at least 1 offensive and 1 defensive player.
Answer $= |A - B| = |A| - |B| = \binom{20}{4} - 2\binom{10}{4}$.
<u>Method 2</u>: Choices are must a) 1O+3D , b) 2O+2D and c) 3O+1D
Answer $= \binom{10}{1}\binom{10}{3} + \binom{10}{2}\binom{10}{2} + \binom{10}{3}\binom{10}{1}$.
Observe that $\binom{20}{4} = 2\binom{10}{4} + \binom{10}{1}\binom{10}{3} + \binom{10}{2}\binom{10}{2} + \binom{10}{3}\binom{10}{1}$.

More identities

Example 3.23

Prove the following identity for positive n

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k+1}.$$

$$\begin{aligned} 0 = (1-1)^n &= \sum_{j=0}^{n} (-1)^j \binom{n}{j} \\ &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{2k} \binom{n}{2k} + \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{2k+1} \binom{n}{2k+1}. \end{aligned}$$

Note: we can further deduce that the sum is 2^{n-1}.

Pascal's triangle

$\binom{n}{k}$	k
$n=0$	1
$n=1$	1 1
$n=2$	1 2 1
$n=3$	1 3 3 1
$n=4$	1 4 6 4 1
$n=5$	1 5 10 10 5 1

Note that

$$1+1+1+1=4 \text{ i.e. } \binom{0}{0}+\binom{1}{0}+\binom{2}{0}+\binom{3}{0}=\binom{4}{1}$$

$$1+2+3+4=10 \text{ i.e. } \binom{1}{1}+\binom{2}{1}+\binom{3}{1}+\binom{4}{1}=\binom{5}{2}$$

$$1+3+6=10 \text{ i.e. } \binom{2}{2}+\binom{3}{2}+\binom{4}{2}=\binom{5}{3}$$

Chu's Hook identity

Theorem 3.24 (Chu)

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}.$$

Proof: By induction on n. When $n = 0$, we have $\binom{1}{1} = 1 = \binom{0}{0}$.

$$\begin{aligned} \text{Consider } \sum_{j=r}^{n} \binom{j}{r} &= \sum_{j=r}^{n-1} \binom{j}{r} + \binom{n}{r} \\ &= \binom{n-1+1}{r+1} + \binom{n}{r} \quad \text{by hypothesis} \\ &= \binom{n+1}{r+1} \quad \text{by Pascal's relation} \qquad \square \end{aligned}$$

Outline

Example 4.1

How many different six letter words can you form with 3 As, 2 Bs and 1 C?

- Note the position of the letter is important
- Choose 3 elements from [6] and label these A, i.e if $\#2$ is chosen, then the 2rd letter is A
- Choose 2 from the remaining numbers to be labeled B
- Choose 1 from the remaining numbers to be labeled C
- Total $\#$ of ways $\binom{6}{3}\binom{3}{2}\binom{1}{1}$

Remark:

$$\binom{6}{3}\binom{3}{2}\binom{1}{1} = \frac{6!}{3!3!} \times \frac{3!}{2!1!} \times \frac{1!}{1!0!} = \frac{6!}{3!2!1!}.$$

Example 4.1: Alternative solution

- Consider $A_1, A_2, A_3, B_1, B_2,\ C$ (distinct letters)
- There are 6! permutations but we are overcounting
- Now map all A_i to A.
- This is a 3! to 1 map. Thus by the division principle, there are $6!/3!$ words (A indistinguishable)
- Continue to map B_i to B and C_i to C to get $\frac{6!}{3!2!1!}$

Example 4.2

In a dance competition, every dancer must choreograph a 30 step dance consisting of 15 steps of type A, 10 steps of type B and 5 steps of type C. How many different dances are there?

Note the sequences of moves are important and this is exactly like the previous problem
Answer: $\frac{30!}{15!10!5!}$.

Multinomial Coefficients

Theorem 4.3

The number of ways to arrange n objects of k different types in a line, with the requirement that there are a_i objects of type i is

$$\frac{n!}{a_1!a_2!\cdots a_k!}.$$

- $a_1 + a_2 + \cdots + a_k = n$
- These are known as multinomial coefficients
- When $k = 2$, these are exactly the binomial coefficients
- We denote this # by $\binom{n}{a_1, a_2, \ldots, a_k}$

Example 4.4

A quality controller has to visit one factory a day. In the next eight days, she will visit each of the four factories, A, B, C and D, twice. The controller is free to choose the order in which she visits these factories but the two visits to factory A cannot be on consecutive days. In how many ways can she proceed?

- Without restrictions, # of ways $= \binom{8}{2,2,2,2}$
- Itineraries that are forbidden $= \binom{7}{1,2,2,2}$
- By subtraction principle $\frac{8!}{16} - \frac{7!}{8} = 1890.$

Multinomial Theorem

Theorem 4.5 (Multinomial Theorem)

Let n be a positive integer and $x_1, \ldots, x_k$ be variables, then

$$\begin{aligned}(x_1 + x_2 + \cdots + x_k)^n \\ = \sum_{\substack{a_1+a_2+\cdots+a_k=n \\ a_i \geq 0}} \binom{n}{a_1, a_2, \ldots, a_k} x_1^{a_1} x_2^{a_2} \ldots x_k^{a_k}\end{aligned}$$

Remark: The sum is over all possible k-tuples $(a_1, \ldots, a_k)$ such that $a_i \geq 0$ and $a_1 + a_2 + \cdots a_k = n$.

Multinomial Theorem

Example 4.6

$$
\begin{aligned}
(x+y+z)^3 &= \sum \binom{3}{a_1, a_2, a_3} x^{a_1} y^{a_2} z^{a_3} \\
&= \binom{3}{3,0,0} x^3 + \binom{3}{2,1,0} x^2 y + \binom{3}{2,0,1} x^2 z \\
&+ \binom{3}{1,2,0} xy^2 + \binom{3}{1,1,1} xyz + \binom{3}{1,0,2} xz^2 \\
&+ \binom{3}{0,3,0} y^3 + \binom{3}{0,2,1} y^2 z + \binom{3}{0,1,2} yz^2 + \binom{3}{0,0,3} z^3
\end{aligned}
$$

There are 10 distinct monomials.

Multinomial Theorem

Example 4.7

Find the coefficient of x^6 in $(1+x+x^2)^9$.

By the multinomial theorem we want to sum all the coefficients of $\binom{9}{a,b,c}1^a x^b x^{2c}$ such that $b+2c=6$. Hence the required coefficient is

$$\begin{aligned}
&\binom{9}{3,6,0}+\binom{9}{4,4,1}+\binom{9}{5,2,2}+\binom{9}{6,0,3}\\
=&\binom{9}{3}+\binom{9}{4}\binom{5}{4}+\binom{9}{5}\binom{4}{2}+\binom{9}{6}\\
=&1554.
\end{aligned}$$

Example 4.7: Alternative approach

Find the coefficient of x^6 in $(1+x+x^2)^9$.

Use the binomial theorem twice:

$$(1+(x+x^2))^9 = \sum_{j=0}^{9} \binom{9}{j} (x(1+x))^j = \sum_{j=0}^{9} \binom{9}{j} x^j \left(\sum_{k=0}^{j} \binom{j}{k} x^k \right)$$

We only need terms where $j+k=6 \implies k=6-j$.

$$(1+(x+x^2))^9 = \sum_{j=0}^{6} \binom{9}{j} x^j \left(\binom{j}{6-j} x^{6-k} \right) + \dots$$

$$\implies [x^6](1+(x+x^2))^9 = \sum_{j=0}^{6} \binom{9}{j} \binom{j}{6-j} = \sum_{j=3}^{6} \binom{9}{j} \binom{j}{6-j}.$$

Tiling Problem

Example 4.8

Find the # ways to pave a 1×7 *block with tiles of size* 1×1, 1×2, 1×3. *For example,*

$$\boxed{\;2\;}\boxed{1}\boxed{\quad 3 \quad}\boxed{1} \qquad \boxed{\quad 3 \quad}\boxed{1}\boxed{1}\boxed{1}\boxed{1}$$

Let a, b and c be # of 1×1, 1×2 and 1×3 tiles used, then

$$\sum \binom{a+b+c}{a, b, c} \text{ for all } a,\ b,\ c \text{ satisfying } a+2b+3c=7$$

$$= \binom{3}{1,0,2} + \binom{3}{0,2,1} + \binom{4}{2,1,1} + \binom{5}{4,0,1} + \binom{4}{1,3,0} + \binom{5}{3,2,0} + \binom{6}{5,1,0} + \binom{7}{7,0,0} = 44$$

Tiling Problem: Digression

Example 4.8: Using recurrence relations

Find the # ways to pave a 1×7 block with tiles of size 1×1, 1×2, 1×3.

Let a_n denote the # of ways to tile a $1 \times n$ block. Then a_n satisfies

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}.$$

Since $a_1 = 1, a_2 = 1 + 1, a_3 = 1 + 2 + 1$, we have

$$\begin{aligned} a_4 &= 1 + 2 + 4 = 7 \\ a_5 &= 2 + 4 + 7 = 13 \\ a_6 &= 4 + 7 + 13 = 24 \\ a_7 &= 7 + 13 + 24 = 44. \end{aligned}$$

Example 4.9

Which is a bigger number ? $(n!)^n$ or $(n^2)!$.

n	$(n!)^n$	$(n^2)!$
0	1	1
1	1	1
2	2^2	$4!$
3	6^3	$9!$
4	$(24)^4$	$16!$

$(n^2)! \geq (n!)^n$. In fact, $\frac{(n^2)!}{(n!)^n}$ is an integer!

Proof:

Consider n types of objects with exactly n items of each type. The total number of permutations is

$$\frac{(n^2)!}{n! \cdots n!} = \frac{(n^2)!}{(n!)^n}.$$

Example 4.10

There are 128 players in the men's singles in the Australian Open 2011. How many ways are there to pair up the players in round 1?

Method 1: Let $n = 64$, then # of ways =

$$\binom{2n}{2}\binom{2n-2}{2}\cdots\binom{4}{2}\binom{2}{2} = \binom{2n}{2,2,\ldots,2} = \frac{(2n)!}{2^n}.$$

Is this right?

Example 4.10: Alternative Method 1

There are $2n$ players in the men's singles in the Australian Open 2011. How many ways are there to pair up the players in round 1?

A. Method 1: Arrange all players in a line $= (2n)!$ ways.
Match up the $2k-1$-th player with the $2k$-th player.
But the pairs are ordered and within each pair the two players are ordered.
Example: 128! would give us possibly:

$$\begin{array}{ccc} Fed - Nad, & X - Y, & Z - E, \ldots \\ Nad - Fed, & X - Y, & Z - E, \ldots \\ X - Y, & Fed - Nad, & Z - E, \ldots \end{array}$$

Hence answer $= \dfrac{(2n)!}{2^n n!}$.

Example 4.10: Alternative Method 2

A. Method 2: Player 1 has $2n-1$ choices of opponents. The next player has $2n-3$ choices of opponents. So total choices:

$$\begin{aligned}
&= (2n-1)(2n-3)\cdots(3)(1) \\
&= \frac{(2n)(2n-2)\cdots(4)(2)}{2^n n!}(2n-1)(2n-3)\cdots(3)(1) \\
&= \frac{(2n)!}{2^n n!}
\end{aligned}$$

One could also write it as

$$\begin{aligned}
2^n(n-\frac{1}{2})(n-\frac{1}{2}-1)\cdots(n-\frac{1}{2}-n+1) &= 2^n(n-\frac{1}{2})_{\underline{n}} \\
&= (-2)^n(-\frac{1}{2})_{\underline{n}}
\end{aligned}$$