7. Linear Time-Invariant Systems

Linear time invariant (LTI) systems are generally modeled using differential equations in the time domain, or transfer functions in the Laplace transform domain. Transfer functions are defined by the ratio of the output (response) to the input (excitation) in the *s-domain* where s is the Laplace variable. This is illustrated in Fig.7-1.

$$\mathcal{L}\{x(t)\} = X(s) \longrightarrow \boxed{\boldsymbol{H(s)}} \longrightarrow Y(s) = \mathcal{L}\{y(t)\}$$

Fig.7-1 Transfer function: $H(s) = \dfrac{Y(s)}{X(s)} = \mathcal{L}\{h(t)\}$

The input-output relationship of a system can then be expressed in the *s-domain* as

$$Y(s) = H(s)X(s) \tag{7.1}$$

or in the *time-domain* (by taking inverse Laplace transform on both sides of (7.1)) as

$h(t)$ is called the impulse response of the LTI system.
This will be discussed in Section 7.3.1 on Pg 7.15.

$$y(t) = \underbrace{\mathcal{L}^{-1}\{H(s)X(s)\} = \overbrace{h(t)*x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau}^{\text{convolution}}}_{\text{see Chapter 6, eqn.(6.10)}} \tag{7.2}$$

7.1 System Model

- Begin with the general time-domain differential equation model of a system (as defined in Chapter 5, eqn.(5.8))

$$\sum_{n=0}^{N} a_n \frac{d^n y(t)}{dt^n} = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m}. \tag{7.3}$$

- The transfer function $H(s)$ is obtained by taking the Laplace transform on both sides of (7.3), **assuming zero initial conditions**. This results in:

$$\Rightarrow \quad \sum_{n=0}^{N} a_n Y(s) s^n = \sum_{m=0}^{M} b_m X(s) s^m$$

$$\Rightarrow \quad H(s) = \frac{Y(s)}{X(s)} = \left(\sum_{m=0}^{M} b_m s^m \Big/ \sum_{n=0}^{N} a_n s^n \right) = \frac{b_M s^M + b_{M-1} s^{M-1} + \cdots + b_0}{a_N s^N + a_{N-1} s^{N-1} + \cdots + a_0}$$

$$\Rightarrow \quad H(s) = K \frac{\left(\frac{s}{z_1}+1\right)\left(\frac{s}{z_2}+1\right)\cdots\left(\frac{s}{z_M}+1\right)}{\left(\frac{s}{p_1}+1\right)\left(\frac{s}{p_2}+1\right)\cdots\left(\frac{s}{p_N}+1\right)}; \qquad K = \frac{b_0}{a_0} \tag{7.4a}$$

$$\Rightarrow \quad H(s) = K' \frac{(s+z_1)(s+z_2)\cdots(s+z_M)}{(s+p_1)(s+p_2)\cdots(s+p_N)}; \qquad K' = K \cdot \frac{p_1 p_2 \cdots p_N}{z_1 z_2 \cdots z_M} \tag{7.4b}$$

where $-p_n$ and $-z_m$ are the poles and zeros, respectively, of the system $H(s)$. Note, from (7.4), the different ways that a transfer function can be factorized.

- The general transfer function in (7.4) is said to be an N^{th}-order system with N poles and M zeros.
- The poles $(-p_n)$ and zeros $(-z_m)$ are roots obtained by, respectively, solving
$$a_N s^N + a_{N-1}s^{N-1} + \cdots + a_0 = 0 \tag{7.5}$$
and
$$b_M s^M + b_{M-1}s^{M-1} + \cdots + b_0 = 0. \tag{7.6}$$
- To express $H(s)$ in the forms of (7.4), we must have knowledge of $-p_n$ and $-z_m$. Otherwise, we will have to solve for the roots of (7.5) and (7.6).

Example 7-1:

1^{st} *- order system* : $a_1 s + a_0 = 0$ ➔ $-p_1 = -a_0/a_1$

2^{nd} *- order system* : $a_2 s^2 + a_1 s + a_0 = 0$ ➔ $-p_1, -p_2 = \dfrac{-a_1 \pm \left(a_1^2 - 4a_2 a_0\right)^{0.5}}{2a_2}$

Higher order systems : *No exact algebraic formula to evaluate the roots. Roots are often found thru iterative algorithms such as the* ***Laguerre's method****.*

- For practical systems, $N > M$. We say that the system has a ***pole-zero excess*** of $(N - M)$

7.2 System Stability (Role of Poles and Zeros)

To understand the role of poles and zeros, consider the output

$$Y(s) = H(s)X(s) = K \cdot \frac{p_1 p_2 \cdots p_N}{z_1 z_2 \cdots z_M} \frac{(s+z_1)(s+z_2)\cdots(s+z_M)}{(s+p_1)(s+p_2)\cdots(s+p_N)} X(s) \quad \cdots \quad \text{cf } (7.4\text{b})$$

where $X(s)$ is the input. $X(s)$ may contain poles and zeros too. We distinguish the poles of $X(s)$ from the poles of $H(s)$ by referring to the former as ***input poles*** while the latter as ***system poles***.

Example 7-2 (Step Response of a General LTI System)

Suppose the input is a unit step signal $x(t) = u(t)$ which has a Laplace transform of $X(s) = 1/s$. Then

$$\begin{aligned} Y_{step}(s) &= K \cdot \frac{p_1 p_2 \cdots p_N}{z_1 z_2 \cdots z_M} \frac{(s+z_1)(s+z_2)\cdots(s+z_M)}{(s+p_1)(s+p_2)\cdots(s+p_N)} \cdot \frac{1}{s} \\ &= \frac{\alpha_1}{s+p_1} + \frac{\alpha_2}{s+p_2} + \cdots + \frac{\alpha_N}{s+p_N} + \frac{K}{s} \end{aligned} \tag{7.7}$$

where α_n are the constants obtained after partial factorization. Taking the inverse Laplace transform of $Y_{step}(s)$,

$$y_{step}(t) = \underbrace{\left[\alpha_1 e^{-p_1 t} + \alpha_2 e^{-p_2 t} + \cdots + \alpha_{N} e^{-p_N t}\right] u(t)}_{y_{tr}(t)} + \underbrace{Ku(t)}_{y_{ss}(t)} = y_{tr}(t) + y_{ss}(t) \tag{7.8}$$

where

$y_{tr}(t)$ is the transient response which depends on the system poles $(-p_n;\ n = 1, 2, \cdots, N)$,

$y_{ss}(t)$ is the steady-state response which is constant.

Furthermore,

$$\lim_{t\to\infty} y_{tr}(t) = \begin{cases} 0 & \cdots \text{ if all the system poles have negative real part } \left(\forall n:\ \mathrm{Re}[-p_n] < 0\right) \\ \infty & \cdots \text{ if at least one system pole has positive real part } \left(\exists n:\ \mathrm{Re}[-p_n] > 0\right) \end{cases},$$

$$\lim_{t\to\infty} y_{ss}(t) = K.$$

From Example 7-2, we infer that, in general, the *system output* $y(t)$ *will be **bounded*** if:

- *Input* $x(t)$ *is bounded*
- *All system poles* $(-p_n;\ n = 1, 2, \cdots, N)$ *have negative real parts*

This leads us to the definition of system stability. Stability is defined in the context of ***bounded-input bounded-output*** **(BIBO)**. The different notions of stability are as follow:

- **BIBO stable:** if $\lim_{t\to\infty} y_{tr}(t) = 0$. The condition for this is that **all system poles must have negative real parts**.
- **Unstable:** if $\lim_{t\to\infty} y_{tr}(t) = \infty$. This occurs when **at least one system pole has positive real parts**.
- **Marginally: Stable** if $\lim_{t\to\infty} y_{tr}(t)$ has no fixed value or is non-zero. For example, $y_{tr}(t)$ could be sinusoidal and hence has no fixed steady-state value. This happens when the **system poles are on the imaginary axis, including the origin**. (Note: Systems with multiple pole(s) on the imaginary axis are UNSTABLE)

The above provides a convenient way to check if a system is BIBO stable without having to calculate the output response $y(t)$ at all. System transfer functions are therefore very useful from this point of view. An overall view of the transient response in relation to the pole locations on the complex *s-plane* is shown in Fig.7-2.

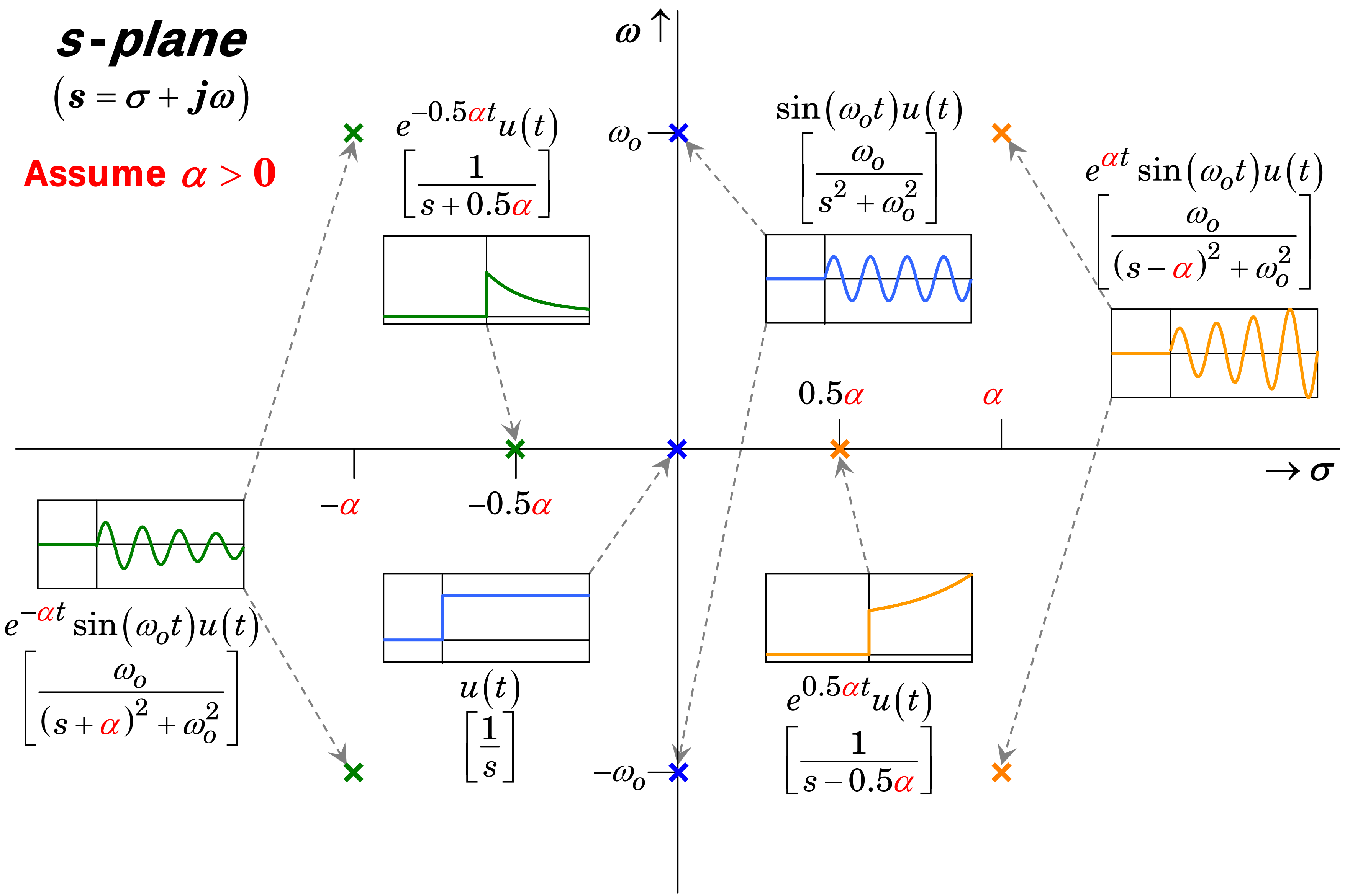


Fig.7-2: Responses of systems due to a unit impulse input

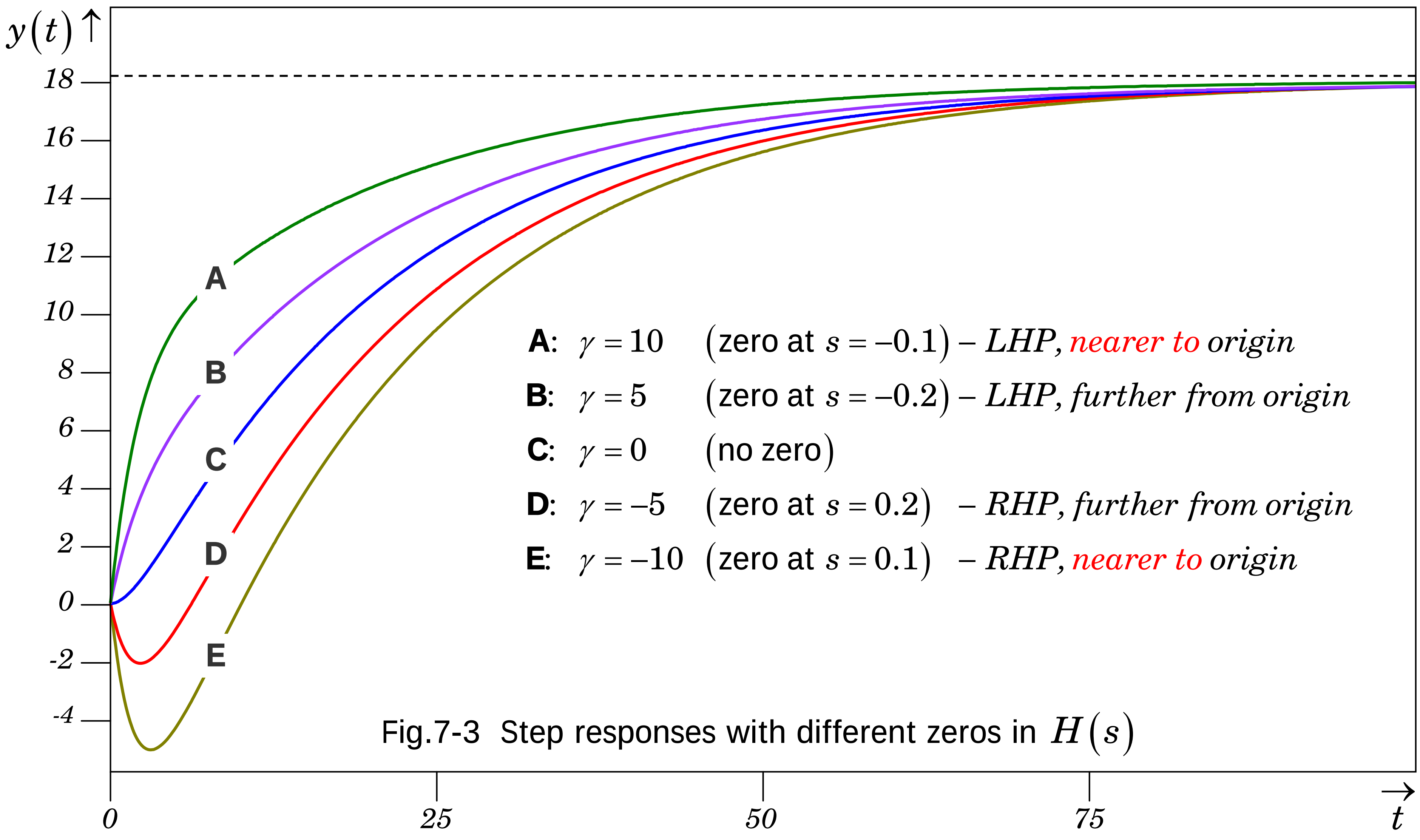


Fig.7-3 Step responses with different zeros in $H(s)$

Notice how the location of the zero affects $y(t)$.

7.2.1 DE and TF of First-order Systems

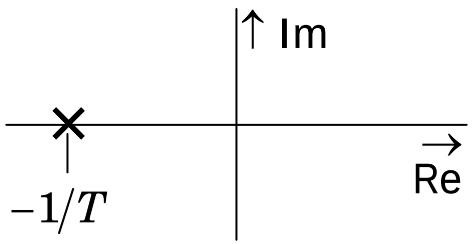
- Differential equation (DE) of a linear first-order system is generally written as

$$T\frac{dy(t)}{dt} + y(t) = Kx(t) \tag{7.9}$$

where $\begin{cases} x(t) & \text{: system input} \\ y(t) & \text{: system output} \\ K & \text{: DC (or Static) gain} \\ T & \text{: time-constant} \end{cases}$

- Transfer function (TF):

$$TsY(s) + Y(s) = KX(s) \rightarrow \begin{cases} H(s) = \dfrac{Y(s)}{X(s)} = \dfrac{K}{Ts+1} \\ \\ \text{Pole: } s_1 = -1/T \end{cases} \tag{7.10}$$



7.2.2 DE and TF of Second-order Systems

- Differential equation (DE) of a linear second-order system is generally written as

$$\frac{d^2y(t)}{dt^2}+2\zeta\omega_n\frac{dy(t)}{dt}+\omega_n^2y(t)=K\omega_n^2x(t) \tag{7.11}$$

$$\text{where}\begin{cases} x(t) & : \text{system input} \\ y(t) & : \text{system output} \\ \zeta & : \text{damping ratio} \\ \omega_n & : \text{undamped natural frequency}\left(\text{takes on particular meaning only when } 0\le\zeta<1\right) \\ K & : \text{DC (or Static) gain} \end{cases}$$

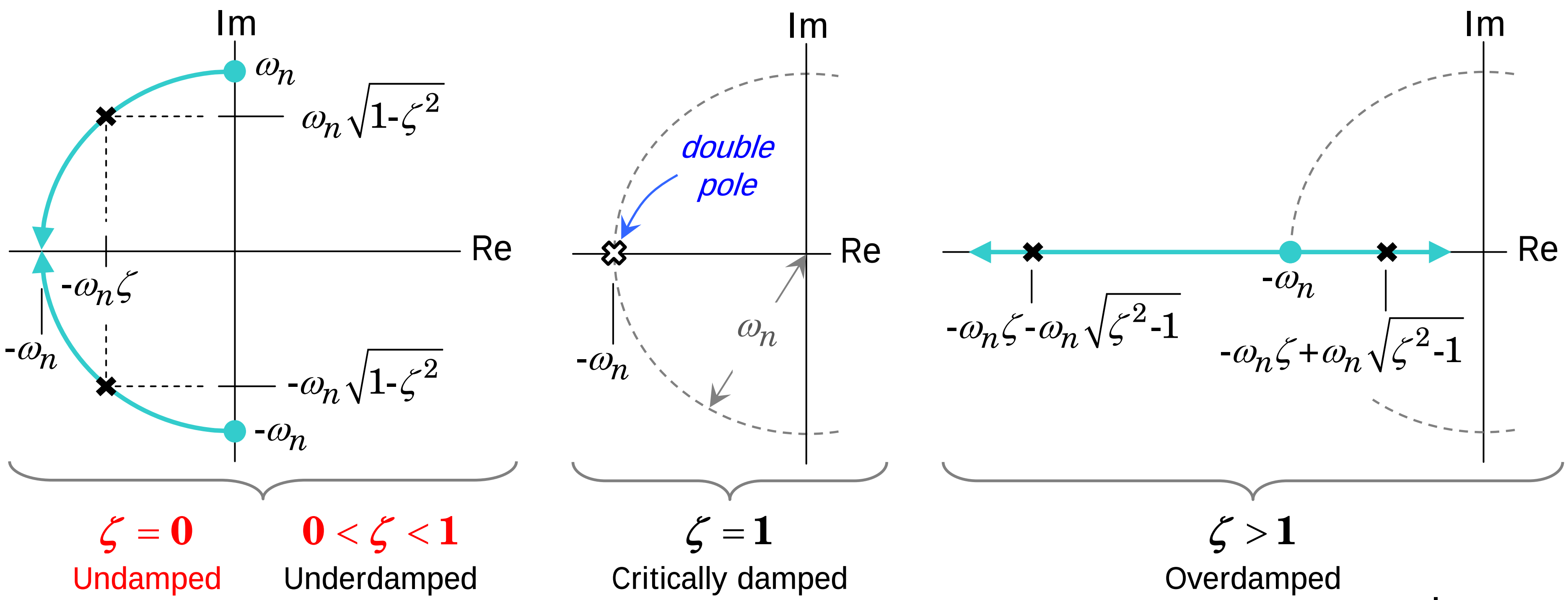
- Transfer function (TF):

$$s^2Y(s)+2\zeta\omega_n sY(s)+\omega_n^2Y(s)=K\omega_n^2X(s)$$

$$\downarrow$$

$$\overbrace{\begin{matrix} H(s)=\dfrac{Y(s)}{X(s)}=\dfrac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \\ Poles:\quad s_{1,2}=-\omega_n\zeta\pm\omega_n\sqrt{\zeta^2-1} \end{matrix}\quad\begin{cases} \zeta=0 & :imaginary\ poles\ \cdots\cdots\cdots\ (Undamped) \\ 0\le\zeta<1 & :complex\ poles\ \cdots\cdots\ (Underdamped) \\ \zeta=1 & :double\ pole\ \cdots\cdots\ (Critically\ damped) \\ \zeta>1 & :distinct\ real\ poles\ \cdots\ (Overdamped) \end{cases}}$$

(7.12)



- SPECIAL NOTATIONS for Underdamped Second-Order Systems

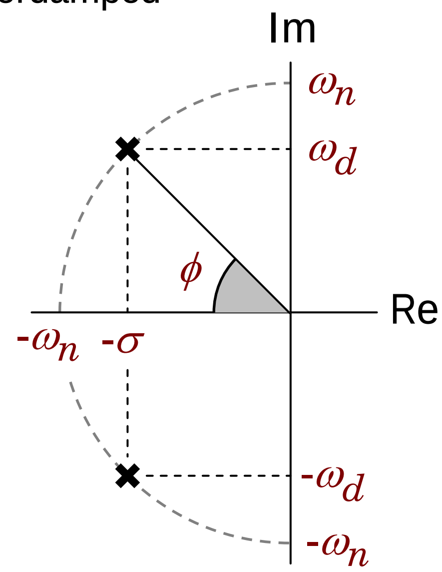
$$\sigma=\omega_n\zeta=\omega_n\cos(\phi)$$

$$\omega_d=\omega_n\sqrt{1-\zeta^2}=\omega_n\sin(\phi)\ \cdots\ \textbf{damped natural frequency}$$

$$H(s)=\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}=\frac{K\left(\sigma^2+\omega_d^2\right)}{(s+\sigma)^2+\omega_d^2}$$

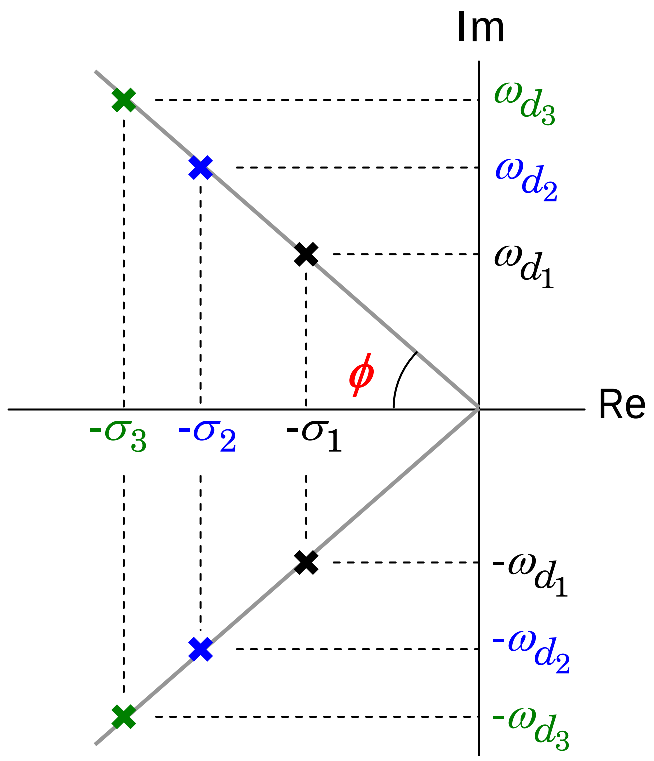
Poles located at $s=-\sigma\pm j\omega_d$

$(\diamond)$

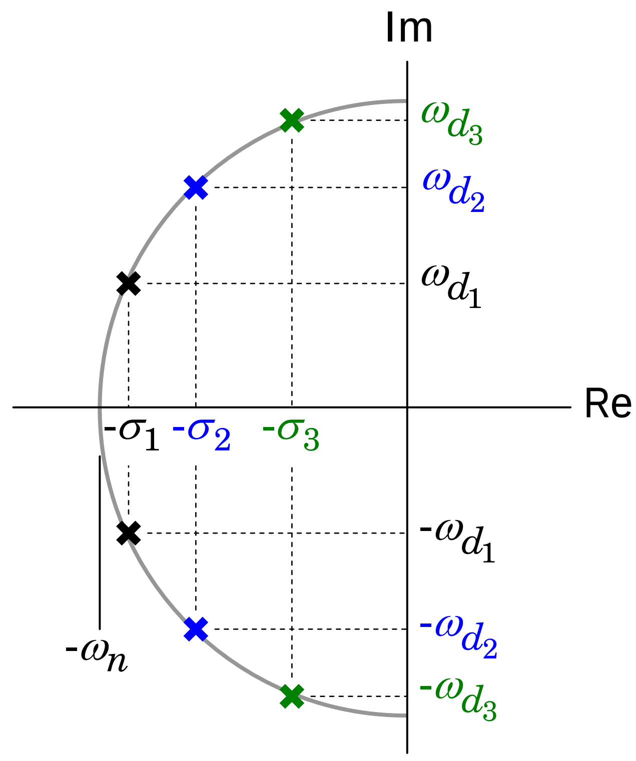


Note:

$$\left(\underbrace{\phi = \cos^{-1}(\zeta)}_{\text{Independent of } \omega_n}\right) \qquad \left(\underbrace{|s_{1,2}| = \omega_n}_{\text{Independent of } \zeta}\right)$$



Poles with same ζ are located on the same line with $\phi = \cos^{-1}(\zeta)$. Different poles have different σ and ω_d values.

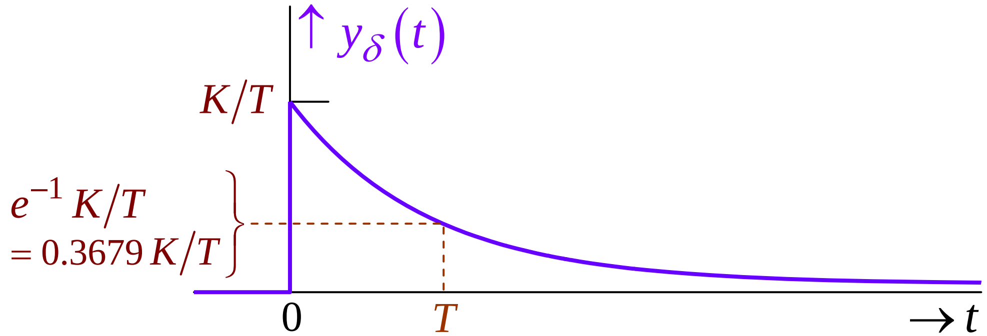


Poles with the same ω_n are located on an arc with a radius of ω_n. Different poles on the same arc have different σ and ω_d values.

- **First-order System:**

$$H(s) = \frac{K}{Ts+1}$$

$$y_\delta(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{K}{T}\exp\left(-\frac{t}{T}\right)u(t)$$



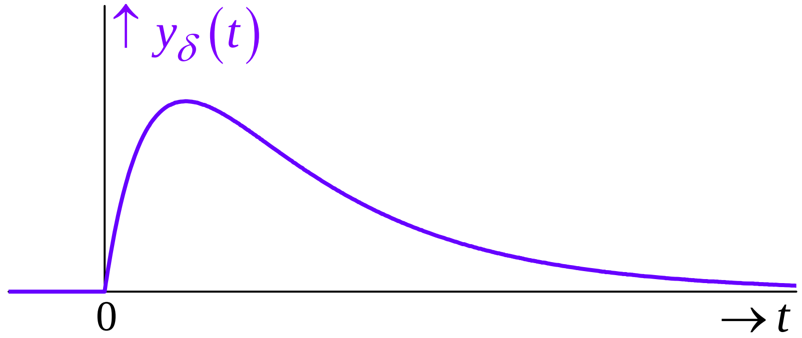
- **Second-order System:**

  Overdamped $(\zeta > 1)$: Two distinct REAL poles

$$H(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{K\omega_n^2}{(s+a)(s+b)}$$

$$\frac{K\omega_n^2(b-a)^{-1}}{(s+a)} + \frac{K\omega_n^2(a-b)^{-1}}{(s+b)}$$

$$\cdots\cdots\cdots \begin{bmatrix} a = \omega_n\zeta + \omega_n\left(\zeta^2 - 1\right)^{1/2} \\ b = \omega_n\zeta - \omega_n\left(\zeta^2 - 1\right)^{1/2} \end{bmatrix} \quad (\text{✶})$$



$$y_\delta(t) = \mathcal{L}^{-1}\{H(s)\} = \left[K_1\exp(-at) + K_2\exp(-bt)\right]u(t)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \left[K_1 = \frac{K\omega_n^2}{b-a} \quad K_2 = \frac{K\omega_n^2}{a-b}\right] \quad (\text{✶})$$

7.3.2 Step Response

Suppose the system input is a unit step, i.e. $x(t)=u(t)$, and the corresponding system output is $y(t)=y_{step}(t)$.

$$\underset{\textit{Unit Step}}{u(t)} \longrightarrow \boxed{\boldsymbol{H}(\boldsymbol{s})} \longrightarrow \underset{\textit{Step Response}}{y_{step}(t)}$$

Then (7.1) becomes

$$Y_{step}(s)=H(s)\mathcal{L}\{u(t)\}=H(s)\frac{1}{s}. \tag{7.15}$$

and in the time-domain, using Laplace transform *Property E* (see Chapter **6**, Section **6.2**),

$$y_{step}(t)=\mathcal{L}^{-1}\left\{H(s)\cdot\frac{1}{s}\right\}=\int_{0^-}^{t}h(\tau)\,d\tau\,. \tag{7.16}$$

$\boldsymbol{y_{step}(t)=\int_{0^-}^{t}h(\tau)\,d\tau}$ is called the ***step response*** of a causal system.

From (7.14) and (7.16), we have

$$\left[\boldsymbol{y_{step}(t)=\int_{0^-}^{t}y_{\delta}(\tau)\,d\tau}\right] \quad \text{and} \quad \left[\boldsymbol{y_{\delta}(\tau)=\frac{d}{dt}y_{step}(t)}\right] \tag{7.17}$$

Hence,

$$y(t)=\left(\underbrace{A\left[A_1\exp(-p_1t)+A_2\exp(-p_2t)+\cdots\cdots\right]}_{\text{Transient Response: } y_{tr}(t)}+\underbrace{A\left[C\cos(\tilde{\omega}t)+\frac{D}{\tilde{\omega}}\sin(\tilde{\omega}t)\right]}_{\text{Steady-state Response: } y_{ss}(t)}\right)u(t)$$

We shall drop the $u(t)$ with the understanding that the following results apply only to $t \geq 0$.

- ***Transient Response*** :

$$\begin{cases} y_{tr}(t)=A\left[A_1\exp(-p_1t)+A_2\exp(-p_2t)+\cdots\cdots\right] \\ \text{Note: } \lim_{t\to\infty} y_{tr}(t)=0, \text{ i.e. the transient response} \\ \qquad\qquad \text{decays to zero.} \end{cases}$$

- ***Steady-state Response*** :

$$\begin{cases} y_{ss}(t)=A\left[C\cos\tilde{\omega}t+\frac{D}{\tilde{\omega}}\sin\tilde{\omega}t\right]=AM_{\tilde{\omega}}\sin\left(\tilde{\omega}t+\phi_{\tilde{\omega}}\right) \\ \textit{where}\left(\underbrace{M_{\tilde{\omega}}=\left[C^2+(D/\tilde{\omega})^2\right]^{0.5}}_{M_{-\tilde{\omega}}=M_{\tilde{\omega}}} \quad \underbrace{\phi_{\tilde{\omega}}=\tan^{-1}\frac{\tilde{\omega}C}{D}}_{\phi_{-\tilde{\omega}}=-\phi_{\tilde{\omega}}}\right) \end{cases} \tag{7.18}$$

Clearly, the **steady-state** response is obtained by multiplying the amplitude of the input sinusoid by $M_{\tilde{\omega}}$ and adding $\phi_{\tilde{\omega}}$ to its phase. Hence, **dropping the tilde** ˜, the function

$$\boldsymbol{M_\omega \exp\left(j\phi_\omega\right)} \tag{7.19}$$

assumes the meaning of ***frequency response*** of the system, where $\boldsymbol{M_\omega}$ and $\boldsymbol{\phi_\omega}$ are, respectively, the magnitude and phase response of the system.

7.4 Frequency Response

- ***Frequency response*** is an intrinsic property of LTI systems as it characterizes how sinusoidal signals are altered in going through the system.

- We have alluded to the notion of frequency response of an LTI system in Chapters 2 when we discussed the convolution property of the Fourier transform.

- For a ***causal*** and ***stable*** system, we have shown in Chapter 6 Section 6.4 that the system ***frequency response*** is given by

$$H(j\omega) = H(s)\big|_{s=j\omega}\ , \tag{7.20}$$

which is equivalent to the Fourier transform of the system impulse response $h(t)$.

- Like poles and zeros, ***frequency response*** is another important measure that is often used to quantify the behavior of an LTI system. The measure is in terms of the magnitude and phase response as defined by

$$H(j\omega) = \underbrace{|H(j\omega)|}_{\substack{\text{Magnitude} \\ \text{Response}}} \exp\big[\, j \underbrace{\angle H(j\omega)}_{\substack{\text{Phase} \\ \text{Response}}} \,\big] \tag{7.21}$$

- Comparing with (7.19) and (7.21), we have $\left(\boldsymbol{M_\omega} = \left|\boldsymbol{H}(j\omega)\right|,\ \boldsymbol{\phi_\omega} = \angle\boldsymbol{H}(j\omega)\right)$. The steady-state sinusoidal response given by (7.18) can then be summarized as follows.

NEW FACTS

$$A\cos(\tilde{\omega}t+\psi)u(t) \longrightarrow \boxed{H(j\omega)} \longrightarrow y_{SS}(t) = A\left|H(j\tilde{\omega})\right|\cos\left(\tilde{\omega}t+\psi+\angle H(j\tilde{\omega})\right)$$

$$A\sin(\tilde{\omega}t+\psi)u(t) \longrightarrow \boxed{H(j\omega)} \longrightarrow y_{SS}(t) = A\left|H(j\tilde{\omega})\right|\sin\left(\tilde{\omega}t+\psi+\angle H(j\tilde{\omega})\right)$$

$$\underbrace{Ae^{j(\tilde{\omega}t+\psi)}u(t)}_{\text{Sinusoid turned on at } t=0} \longrightarrow \boxed{H(j\omega)} \longrightarrow \underbrace{y_{SS}(t) = A\left|H(j\tilde{\omega})\right|e^{j\left(\tilde{\omega}t+\psi+\angle H(j\tilde{\omega})\right)}}_{\text{Steady-state Sinusoidal Response}}$$

$t \to \infty$

OLD FACTS from 'Signals' Part

$$x(t) = A\cos\left(2\pi\tilde{f}t+\psi\right) \longrightarrow \boxed{H(f)} \longrightarrow y(t) = A\left|H(\tilde{f})\right|\cos\left(2\pi\tilde{f}t+\psi+\angle H(\tilde{f})\right)$$

$$x(t) = A\sin\left(2\pi\tilde{f}t+\psi\right) \longrightarrow \boxed{H(f)} \longrightarrow y(t) = A\left|H(\tilde{f})\right|\sin\left(2\pi\tilde{f}t+\psi+\angle H(\tilde{f})\right)$$

$$\underbrace{x(t) = Ae^{j\left(2\pi\tilde{f}t+\psi\right)}}_{\text{Sinusoid turned on at } t=-\infty} \longrightarrow \boxed{H(f)} \longrightarrow \underbrace{y(t) = A\left|H(\tilde{f})\right|e^{j\left(2\pi\tilde{f}t+\psi+\angle H(\tilde{f})\right)}}_{\mathfrak{I}^{-1}\{H(f)X(f)\}}$$

- In Chapter 2, we examined the frequency content of signals for frequency ranging from $-\infty$ to ∞. However, for LTI systems, it is customary to make use of Bode diagrams, which has a frequency axis extending only from 0 to ∞, to describe their responses to different frequencies.

Example 7-4 :

Differentiator: $\boldsymbol{H(s) = Ks} \rightarrow \boldsymbol{H(j\omega) = jK\omega}$

Magnitude response : $\left|H_{dB}(j\omega)\right|_{dB} = \left(20\log_{10}K + 20\log_{10}\omega\right)\text{dB}$ (7.22)

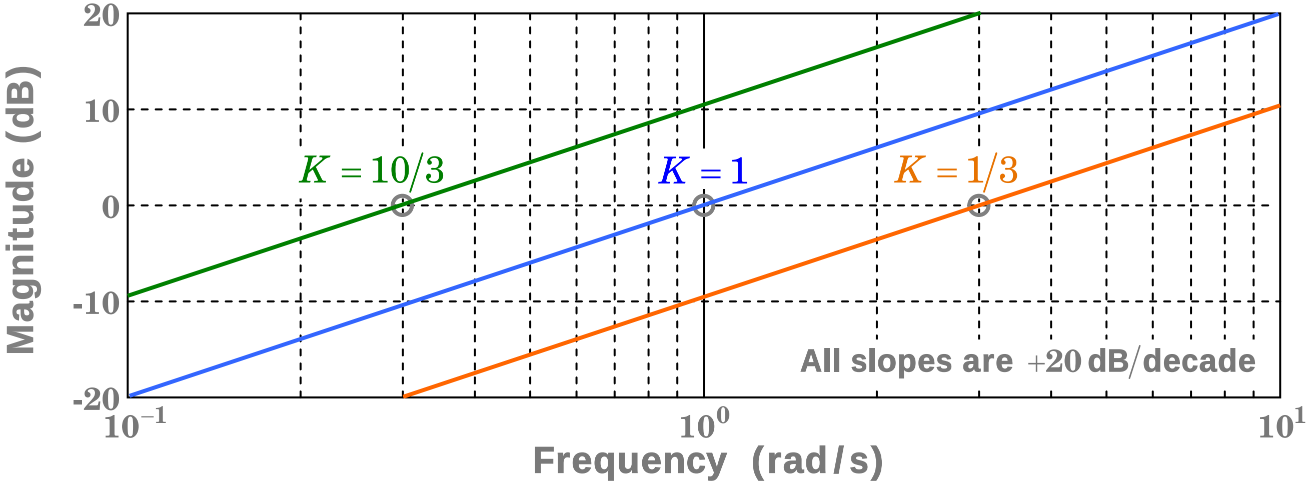
Phase response : $\angle H(j\omega) = \tan^{-1}(K\omega/0) = 90°$ (7.23)

From (7.22), we note that if $\left|H(j\omega)\right|$ *in dB is plotted against the* $\log_{10}\omega$ *, then*

- *A straight line is obtained.*
- *The slope of this straight line is* $+20\text{dB}$ *for every 10 times increase in frequency. We call this a* $+20\,\text{dB/decade}$ *slope.*

Bode plots of differentiators with different K values are shown in Fig.7-4.

Fig.7-4(a)



Example 7-5

Integrator: $\boldsymbol{H(s) = K/s} \rightarrow \boldsymbol{H(j\omega) = -jK/\omega}$

Magnitude response : $\left|H(j\omega)\right|_{dB} = (20\log_{10} K - 20\log_{10}\omega)\,\text{dB}$ (7.24)

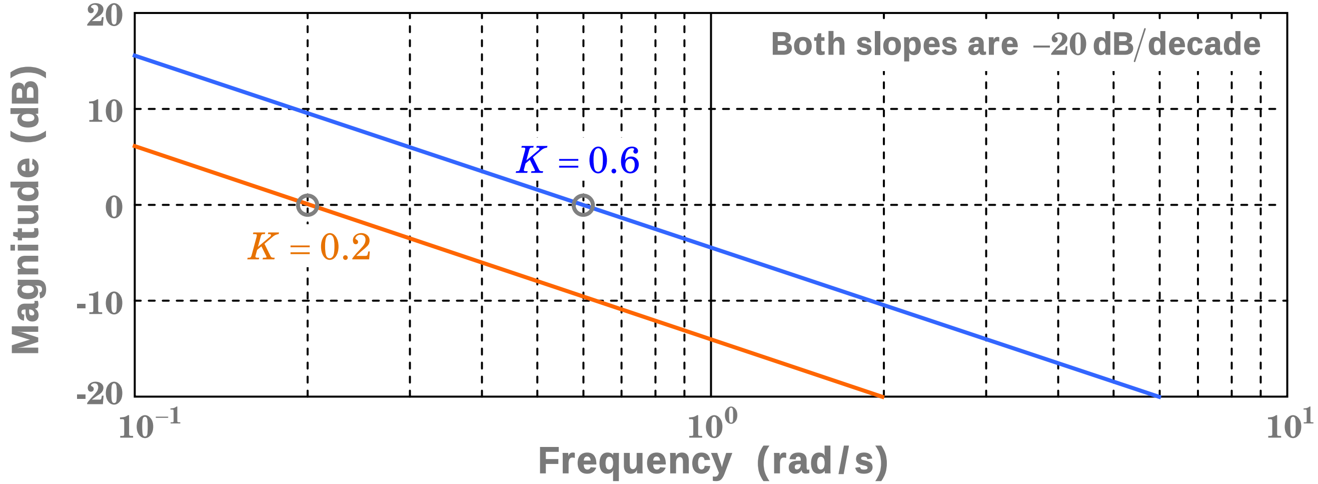
Phase response : $\angle H(j\omega) = \tan^{-1}\left(-(K/\omega)/0\right) = -90°$ (7.25)

From (7.24), we note that if $\left|H(j\omega)\right|$ *in dB is plotted against the* $\log_{10}\omega$, *then*

- *A straight line is obtained.*
- *The slope of this straight line is* -20dB *for every 10 times increase in frequency. We call this a* -20 dB/decade *slope.*

Bode plots of integrators with different K ***values are shown in Fig.7-5.***

Fig.7-5(a)



Example 7-6

First order system with one real pole: $\boldsymbol{H(s) = \frac{K}{sT+1}}$ ➔ $\boldsymbol{H(j\omega) = \frac{K}{j\omega T+1}}$

Pole: $s = -1/T$

Magnitude response: $$\left|H(j\omega)\right|_{dB} = \left(20\log_{10} K - 20\log_{10}\sqrt{\omega^2T^2+1}\right)\text{dB} \tag{7.26}$$

Phase response: $$\angle H(j\omega) = \angle\frac{1}{j\omega T+1} = -\tan^{-1}(\omega T) \tag{7.27}$$

Case I: $\omega << \boldsymbol{1/T}$

$$\left|H(j\omega)\right|_{dB} \approx 20\log_{10} K \text{ dB} \tag{7.28}$$

$$\angle H(j\omega) = -\tan^{-1}(\omega T) \to 0° \tag{7.29}$$

Case II: $\omega = \boldsymbol{1/T}$

$$\left|H(j\omega)\right|_{dB} \approx \left(20\log_{10} K - 20\log_{10}\sqrt{2}\right) \text{ dB} = 20\log_{10} K - 3.01 \text{ dB} \tag{7.30}$$

$$\angle H(j\omega) = -\tan^{-1}(1) = -45° \tag{7.31}$$

Case I: $\omega >> \boldsymbol{1/T}$

$$\left|H(j\omega)\right|_{dB} \approx \left(20\log_{10} K - 20\log_{10}\omega T\right) \text{ dB} \tag{7.32}$$

$$\angle H(j\omega) = -\tan^{-1}(\omega T) \to -90° \tag{7.33}$$

Example 7-7:

Second order system with two real poles: $\begin{cases} \boldsymbol{H(s) = \dfrac{K}{(sT_1+1)(sT_2+1)}} \quad \textbf{or} \\ \boldsymbol{H(j\omega) = \dfrac{K}{(jT_1\omega+1)(jT_2\omega+1)}} \end{cases}$

Poles: $s=-1/T_1$ and $s=-1/T_2$ (*Assume* $T_1 > T_2$)

Magnitude response: $$\left|H(j\omega)\right|_{dB} = K_{dB} + \left|\frac{1}{jT_1\omega+1}\right|_{dB} + \left|\frac{1}{jT_2\omega+1}\right|_{dB} \qquad (7.34)$$

Phase response: $$\angle H(j\omega) = \angle\frac{1}{jT_1\omega+1} + \angle\frac{1}{jT_2\omega+1} \qquad (7.35)$$

Summary of the Magnitude response: (See example in Fig.7-7(a))

- Low frequency magnitude is the same as DC gain ie $K_{dB} = 20\log_{10}K$ dB
- Since this is a second order overdamped system, there are two corner frequencies, one corresponding to each pole. Since $T_1 > T_2$, the first corer frequency is at $1/T_1$ and the second at $1/T_2$.

Example 7-8

Integrator cascaded with a first order system: $\begin{cases} \boldsymbol{H(s) = \dfrac{1}{s(sT+1)}} \quad \textbf{or} \\ \boldsymbol{H(j\omega) = \dfrac{1}{j\omega(jT\omega+1)}} \end{cases}$

The same method of constructing the magnitude and phase responses applies. The complete response is the sum of responses of $\dfrac{1}{s}$ and $\dfrac{1}{sT+1}$.

Magnitude response:
$$\begin{cases} \left|H(j\omega)\right|_{dB} = \left|\dfrac{1}{j\omega}\right|_{dB} + \left|\dfrac{1}{jT\omega+1}\right|_{dB} \\ \qquad\qquad\quad = -20\log_{10}\omega + \left|\dfrac{1}{jT\omega+1}\right|_{dB} \end{cases} \tag{7.36}$$

Phase response:
$$\begin{cases} \angle H(j\omega) \equiv \angle\dfrac{1}{j\omega} + \angle\dfrac{1}{jT\omega+1} \\ \qquad\qquad = -90^\circ + \angle\dfrac{1}{jT\omega+1} \end{cases} \tag{7.37}$$

Bode plots of $\boldsymbol{H(s) = \dfrac{1}{s(sT+1)}}$ $(\textit{with } \boldsymbol{T = 0.1})$ ***are shown in Fig.7-8.***

Example 7-9

Second order underdamped system: $\boldsymbol{H(s) = \dfrac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}}$ $\begin{cases} \textbf{Poles are complex} \\ \textbf{Damping ratio, } \zeta < 1 \end{cases}$

$H(s)$ can be rewritten in terms of the <u>normalized</u> frequency, $\tilde{\omega} = \dfrac{\omega}{\omega_n}$. Thus

$$H(s) = \frac{K}{\tilde{s}^2 + 2\zeta\tilde{s} + 1} \quad \text{or} \quad H(j\omega) = \frac{K}{\left(1 - \tilde{\omega}^2\right) + j2\tilde{\omega}\zeta} \tag{7.38}$$

where $\tilde{s} = \dfrac{s}{\omega_n} = j\tilde{\omega}$. Bode diagrams can be drawn in terms of the **normalized frequency $\tilde{\omega}$**.

Magnitude response: $$\left|H(j\omega)\right|_{dB} = 20\log_{10} K + 20\log_{10}\frac{1}{\sqrt{\left(1-\tilde{\omega}^2\right)^2 + 4\tilde{\omega}^2\zeta^2}} \text{ dB} \tag{7.39}$$

Phase response: $$\angle H(j\omega) = -\tan^{-1}\left(\frac{2\tilde{\omega}\zeta}{1-\tilde{\omega}^2}\right) \tag{7.40}$$

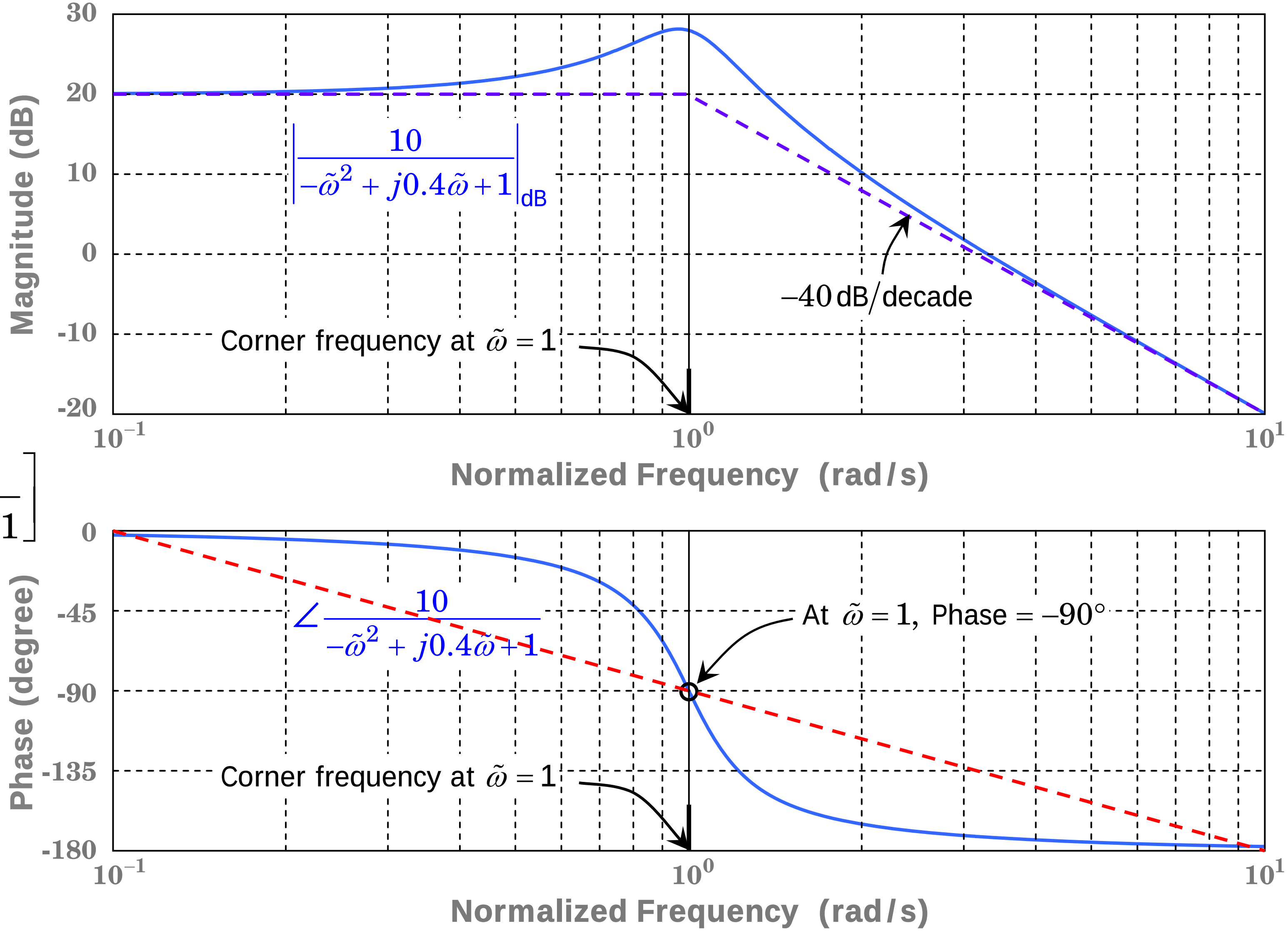
The approximate Bode straight-line magnitude plot is constructed based on the critically-damped case as follows:

- Corner frequency: $\tilde{\omega} = \mathbf{1}$ or $\omega = \omega_n$
- Slope change at the corner frequency: **-40 dB/decade**.

Bode plots of $\boldsymbol{H(s) = \dfrac{K}{\tilde{s}^2 + 2\zeta\tilde{s} + 1}}$ $\left(\textit{with } \boldsymbol{K = 10,\ \zeta = 0.2}\right)$ ***are shown in Fig.7-9***

Fig.7-9

$$\left[H(s) = \frac{10}{\tilde{s}^2 + 0.4\tilde{s} + 1} \right]$$



Note that the magnitude response has a 'hump' or peak around $\tilde{\omega}=1$ or $\omega=\omega_n$. In general, the smaller the damping ratio, ζ, the larger is this peak. The family of Bode diagrams for different values of damping ratio is shown in Fig.7-10.

Fig.7-10

$$\left[H(s) = \frac{1}{\tilde{s}^2 + 2\zeta\tilde{s} + 1} \right]$$