EE2023 TUTORIAL 1 (SOLUTIONS)

Solution to Q.1

Write z in polar form:

$$z = x + jy = |z|\exp(j\angle z).$$

Since adding integer multiples of 2π to $\angle z$ does not affect the value of z, we may also express z as

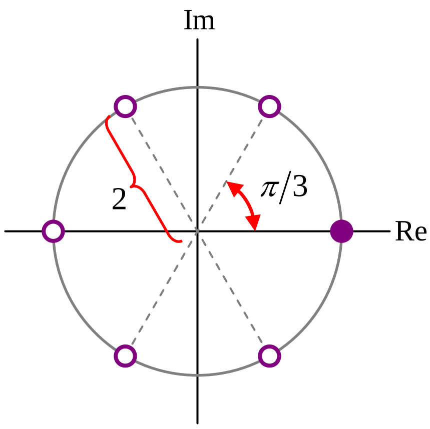
$$z = |z|\exp\big(j(\angle z + 2k\pi)\big)$$

where k is an integer. This leads to

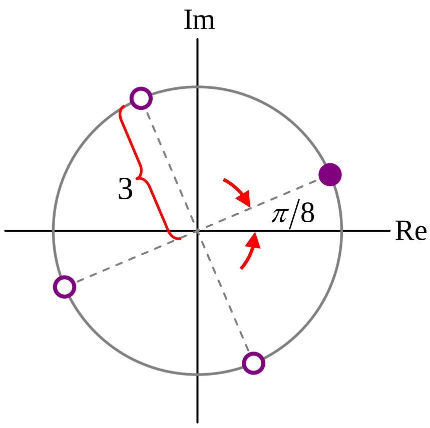
$$z^{1/N} = |z|^{1/N}\exp\left(j\left(\frac{\angle z}{N} + \frac{2k\pi}{N}\right)\right); \quad k = 0,1,\cdots,N-1,$$

which yields the N distinct values of $z^{1/N}$.

$$64^{1/6}\textbf{:}\quad \begin{cases} z = 64 \;\rightarrow\; \begin{cases} |z| = 64 \\ \angle z = 0 \end{cases} \\ 64^{1/6} = |z|^{1/N}\exp\left(j\left(\dfrac{\angle z}{N} + \dfrac{2k\pi}{N}\right)\right)\Bigg|_{z=64,\ N=6} \\ \quad = 2\exp\left(j\left(\dfrac{k\pi}{3}\right)\right); \quad k = 0,1,\cdots,5 \\ \quad = \begin{cases} 2;\ 2\exp\left(j\left(\dfrac{\pi}{3}\right)\right);\ 2\exp\left(j\left(\dfrac{2\pi}{3}\right)\right); \\ -2;\ 2\exp\left(j\left(\dfrac{4\pi}{3}\right)\right);\ 2\exp\left(j\left(\dfrac{5\pi}{3}\right)\right) \end{cases} \end{cases}$$



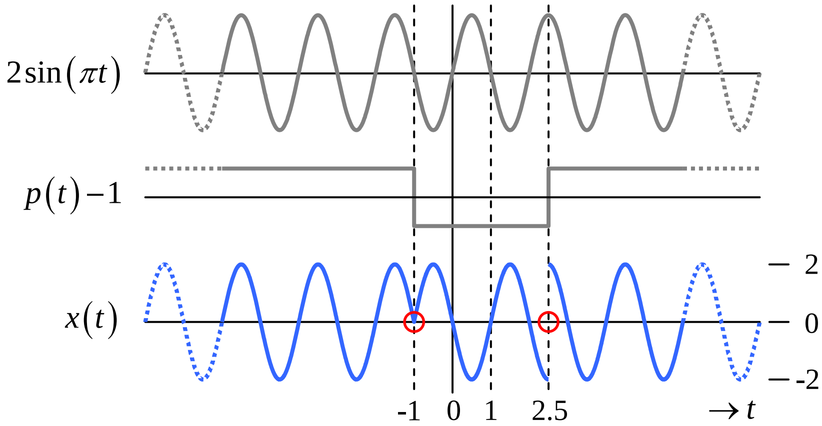
$$(j81)^{1/4}\textbf{:}\quad \begin{cases} z = j81 \;\rightarrow\; \begin{cases} |z| = 81 \\ \angle z = \dfrac{\pi}{2} \end{cases} \\ (j81)^{1/4} = |z|^{1/N}\exp\left(j\left(\dfrac{\angle z}{N} + \dfrac{2k\pi}{N}\right)\right)\Bigg|_{z=81,\ N=4} \\ \quad = 3\exp\left(j\left(\dfrac{\pi}{8} + \dfrac{k\pi}{2}\right)\right); \quad k = 0,1,\cdots,3 \\ \quad = \begin{cases} 3\exp\left(j\left(\dfrac{\pi}{8}\right)\right),\ 3\exp\left(j\left(\dfrac{5\pi}{8}\right)\right), \\ 3\exp\left(j\left(\dfrac{9\pi}{8}\right)\right),\ 3\exp\left(j\left(\dfrac{13\pi}{8}\right)\right) \end{cases} \end{cases}$$



Solution to Q.2

(a) $p(t) = 2 - 2\text{rect}\left(\dfrac{t-0.75}{3.5}\right)$

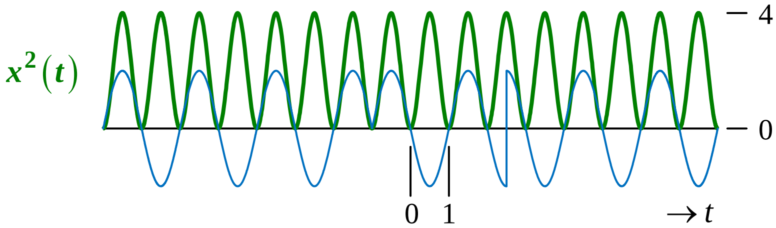
(b) By inspection, $x(t)$ is not periodic.



Notice the π rad $\left(\text{or } 180^\circ\right)$ phase jumps in $x(t)$ occurring at the zero crossings of $p(t)-1$.

(c)

$$
\begin{aligned}
x^2(t) &= 4\sin^2(\pi t)\underbrace{\left(p(t)-1\right)^2}_{1} \\
&= 4\sin^2(\pi t) \\
&= 2\left(1-\cos(2\pi t)\right)
\end{aligned}
$$



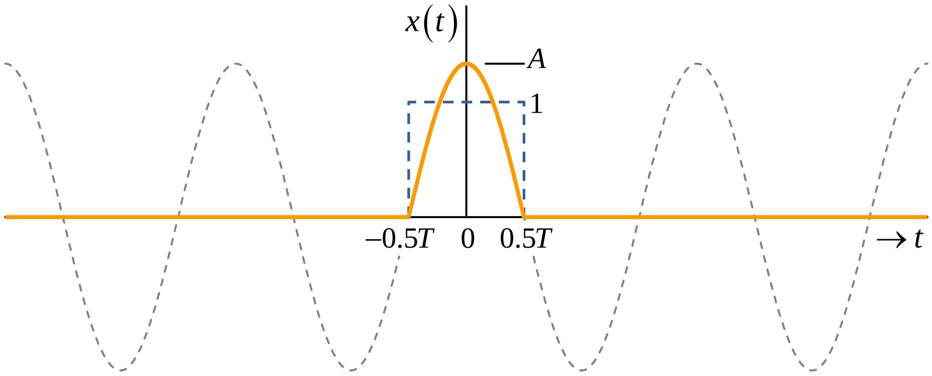
Note that $x^2(t)$ is periodic with a period of $T = 1$.

Average Power: $P = \underbrace{\dfrac{1}{T}\displaystyle\int_{-T/2}^{T/2} x^2(t)\,dt}_{\substack{x^2(t) \text{ is periodic. } \therefore \\ P \text{ can be obtained} \\ \text{by averaging over} \\ \text{one period.}}} = \displaystyle\int_{-0.5}^{0.5} 2\left(1-\cos(2\pi t)\right)dt = 2$

(d) Since the average power of $x(t)$ is finite, its total energy must be infinite. $x(t)$ is an aperiodic power signal.

Solution to Q.3

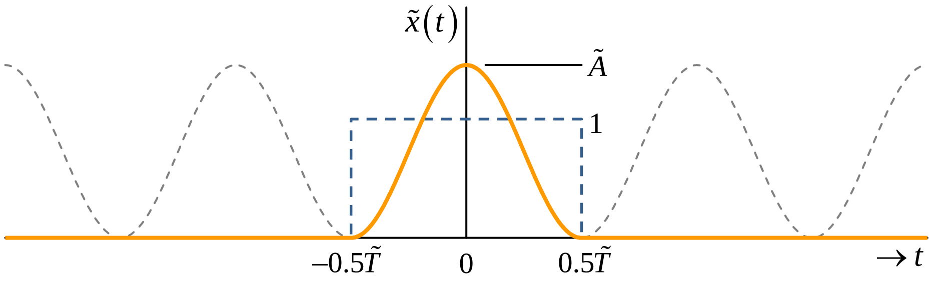
Half-cosine pulse: $x(t) = A\cos\left(\frac{\pi t}{T}\right)\text{rect}\left(\frac{t}{T}\right)$



$$x^2(t) = \frac{A^2}{2}\left[1+\cos\left(\frac{2\pi t}{T}\right)\right]\text{rect}\left(\frac{t}{T}\right)$$

$$\textbf{Energy:}\ E = \frac{A^2}{2}\int_{-0.5T}^{0.5T} 1+\underbrace{\cos\left(\frac{2\pi t}{T}\right)}_{\substack{\int\text{over one}\\ \text{period }=0}}dt = \frac{1}{2}A^2T$$

Raised-cosine pulse: $\tilde{x}(t) = \frac{\tilde{A}}{2}\left(1+\cos\left(\frac{2\pi t}{\tilde{T}}\right)\right)\text{rect}\left(\frac{t}{\tilde{T}}\right)$



$$\tilde{x}^2(t) = \frac{\tilde{A}^2}{4}\left[\frac{3}{2}+2\cos\left(\frac{2\pi t}{\tilde{T}}\right)+\frac{1}{2}\cos\left(\frac{4\pi t}{\tilde{T}}\right)\right]\text{rect}\left(\frac{t}{\tilde{T}}\right)$$

$$\textbf{Energy:}\ \tilde{E} = \frac{\tilde{A}^2}{4}\int_{-0.5\tilde{T}}^{0.5\tilde{T}} \frac{3}{2}+\underbrace{2\cos\left(\frac{2\pi t}{\tilde{T}}\right)}_{\substack{\int\text{over one}\\ \text{period }=0}}+\underbrace{\frac{1}{2}\cos\left(\frac{4\pi t}{\tilde{T}}\right)}_{\substack{\int\text{over two}\\ \text{periods }=0}}dt = \frac{3}{8}\tilde{A}^2\tilde{T}$$

Both $x(t)$ and $\tilde{x}(t)$ will have the same energy if $A^2T = \frac{3}{4}\tilde{A}^2\tilde{T}$.

Solution to Q.4

(a) i. $x(t)=\cos(3.2t)+\sin(1.6t)+\exp(j2.8t)\ \cdots \begin{cases}\cos(3.2t) & \text{has a frequency of } 3.2\ rad/s\\ \sin(1.6t) & \text{has a frequency of } 1.6\ rad/s\\ \exp(j2.8t) & \text{has a frequency of } 2.8\ rad/s\end{cases}$

Highest common factor (HCF) of $\{3.2,\ 1.6,\ 2.8\}$ exists and is equal to 0.4. Thus, $x(t)$ is periodic. Since there is no harmonic cancellation among the sinusoidal components of $x(t)$, the fundamental frequency and period of $x(t)$ are $0.4\ rad/s$ $(\text{or } 0.2/\pi\ Hz)$ and $5\pi\ s$, respectively.

REMARKS: *Although $x(t)$ is periodic with a fundamental frequency of $0.4\ rad/s$, it does not contain the fundamental frequency component itself.*

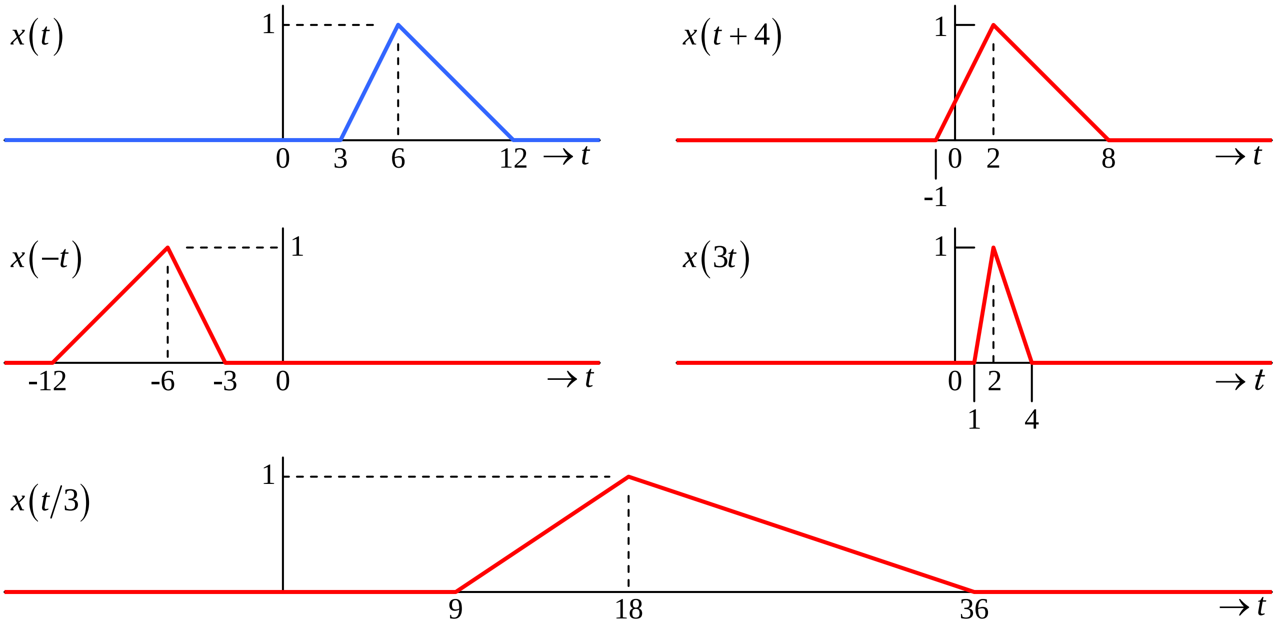
(b) ii. $x(t)=\cos(4t)+\sin(\pi t)\ \cdots \begin{cases}\cos(4t) & \text{has a frequency of } 4\ rad/s\\ \sin(\pi t) & \text{has a frequency of } \pi\ rad/s\end{cases}$

Highest common factor (HCF) of $\{4,\ \pi\}$ does not exist. Thus, $x(t)$ is not periodic.

REMARKS: *Summing sinusoids does not necessarily lead to a periodic signal unless the frequencies of the sinusoids are harmonics of a common fundamental frequency.*

Solution to Q.5

(a)



(b) We observe that $y(t)$ is a time-scaled, -reversed and –shifted version of $x(t)$.

For problems of this nature, we should start with time-scaling first since it involves linear warping of the time axis. If we were to start with time-shifting and/or time-reversal, we may have to redo them after time-scaling. However, this sequence of operation need not be followed if we are sketching the signal from the mathematical expression.

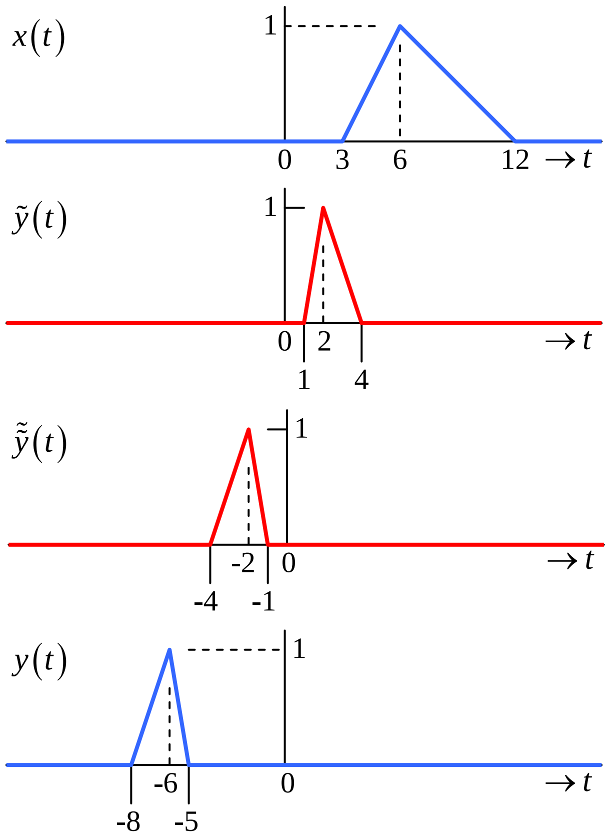
Comparing $x(t)$ and $y(t)$, we note that $y(t)$ involves time-scaling (or contraction) of $x(t)$ by a factor of 3.

Time-scaling of $x(t)$: $\tilde{y}(t) = x(3t)$

Time-reversal of $\tilde{y}(t)$: $\tilde{\tilde{y}}(t) = \tilde{y}(-t) = x(-3t)$

Time shifting of $\tilde{\tilde{y}}(t)$: $\begin{cases} y(t) = \tilde{\tilde{y}}(t+4) \\ \quad\;\; = x(-3(t+4)) \end{cases}$

$\therefore\ y(t) = x(-3(t+4))$



Solution to Q.6

Due to symmetry, $\delta(\beta t) = \delta(|\beta| t)$

We note that $\delta(|\beta| t)$ is essentially the unit impulse $\delta(t)$ with its time-width compressed by a factor of $|\beta|$. Since the area under $\delta(t)$ is 1, the area under $\delta(|\beta| t)$ must then be $\frac{1}{|\beta|}$. This leads directly to the relationship

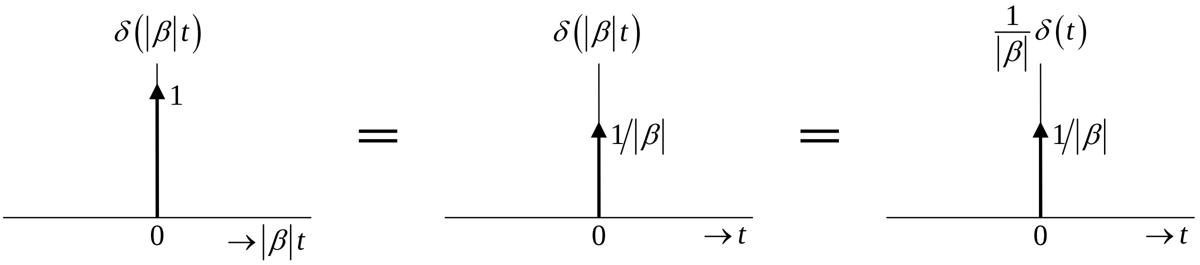
$$\delta(\beta t) = \delta(|\beta| t) = \frac{1}{|\beta|}\delta(t).$$

Mathematically:

Integrating $\delta(|\beta| t)$ over the $|\beta| t$ axis: $\int_{-\infty}^{\infty} \delta(|\beta| t)\, d(|\beta| t) = 1$

Integrating $\delta(|\beta| t)$ over the t axis: $\int_{-\infty}^{\infty} \delta(|\beta| t)\, dt = \underbrace{\int_{-\infty}^{\infty} \delta(|\beta| t)\, dt = \int_{-\infty}^{\infty} \frac{1}{|\beta|}\delta(\tau)\, d\tau}_{\text{applying a change of variable: } \tau = |\beta| t} = \frac{1}{|\beta|}$

$\delta(|\beta| t)$ is an impulse with area 1 on the $|\beta| t$ axis or an impulse with area $\frac{1}{|\beta|}$ on the t axis.



Note:

Domain-scaling of $\delta(\cdot)$ is often encountered in transforming a spectrum containing $\delta(\cdot)$ between cyclic-frequency (f) and radian frequency (ω) domain:

$$\left[\delta(\omega) = \delta(2\pi f) = \frac{1}{2\pi}\delta(f)\right] \text{ or } \left[\delta(f) = \delta\left(\frac{\omega}{2\pi}\right) = 2\pi\delta(\omega)\right].$$