Calculating Integrals in ECE311

In this class we will frequently encounter integrals that contain scalar or vector functions, such as:

$$\int_V f\,dv \quad , \quad \int_C V\,d\vec{l} \quad , \quad \int_S \vec{A}\cdot d\vec{s}$$

It is really important to understand what these integrals mean and how to calculate them.

Good news : There is an easy and systematic way to calculate them!

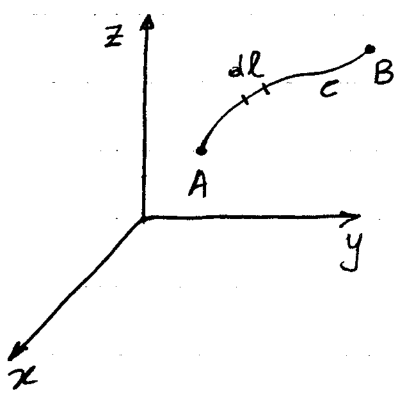
Bad news : You need to read this handout to find out this way, if you don't already know it.

In the following, we are showing you how to calculate these integrals based on their kind. It is assumed that you are already familiar with all three coordinate systems (cartesian, cylindrical and spherical) as well as with their differential quantities (lengths, surfaces and volumes).

A. Line Integrals

In this type, you have to integrate a scalar or vector quantity over a line that connects two points: A and B.

A1. Line Integral with Scalar Functions



The integral has the form:

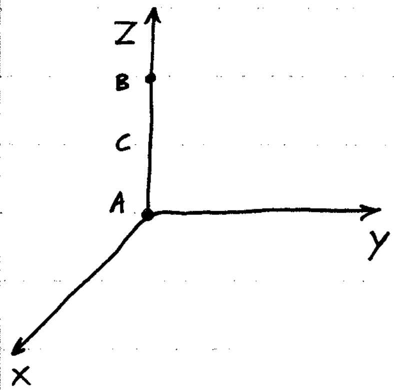
$$I = \int_{AB} f \, dl \quad \text{or} \quad I = \int_{c} f \, dl$$

both scalars

Although in general the line c may have a complicated expression, in 311 it will almost always have a very simple shape in one of the coordinate systems. The trick is to pick the coordinate system that "fits" the line. Then you can easily find dl from the geometry.

Note also that as you integrate from A to B, you only evaluate the function f along the given line connecting A and B.

Example 1



Evaluate the integral $I = \int_{AB} f \, d\ell$

where $f(x, y, z) = 2z$ on the line c shown in the figure. The points A and B have the following coordinates:

A (0,0,0)

B (0, 0, 0.1)

Solution

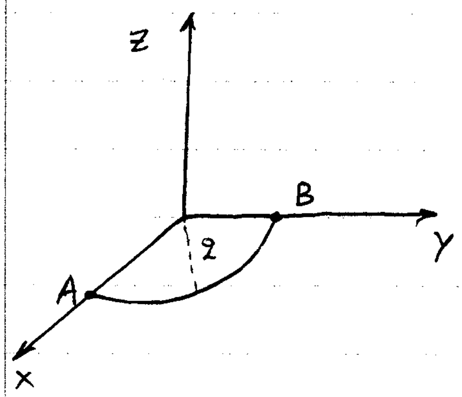
The given line c is obviously "very well suited" for the cartesian coordinate system. In this coordinate system:

$$d\ell = dz \quad , \quad x = 0, \; y = 0 \; \text{ and } \; 0 < z < 0.1$$

$$\text{Thus} \quad I = \int_{AB} f \, d\ell = \int_{A}^{B} f \, d\ell = \int_{0}^{0.1} \underbrace{2z}_{f} \, \underbrace{dz}_{d\ell} = z^2 \Big|_{0}^{0.1} = 0.01$$

Note: If the function f were given as $f(x, y, z) = 2z + yx$, then on the line c we would have $x = 0$ and $y = 0$. Thus the function f on the line c would still be given as $f(0, 0, z) = 2z$ since $x = y = 0$ on c

Example 2

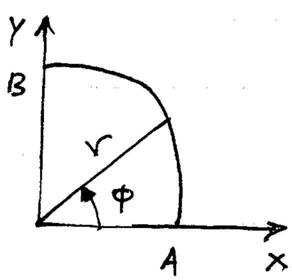


Evaluate the integral $I = \int_A^B f\, d\ell$

where $f(x,y,z) = x$ on the line c shown in the figure. The line c is an arc of a circle with a radius 2 and center at the origin.

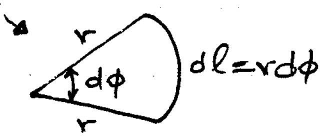
Solution

The given line c is obviously "very well suited" for the cylindrical coordinate system. In this system:



$$z = 0\,, \quad r = 2\,, \quad 0 < \phi < \pi/2\,, \quad d\ell = r\, d\phi$$

reminder



Also (from golden rules) $x = r\cos\phi = 2\cos\phi.$

Hence $$I = \int_A^B f\, d\ell = \int_0^{\pi/2} \underbrace{2\cos\phi}_{f}\, \underbrace{(2\, d\phi)}_{d\ell} = 4\int_0^{\pi/2} \cos\phi\, d\phi =$$

$$= 4\sin\phi \Big|_0^{\pi/2} = 4$$

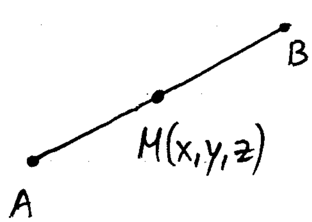
General case for curve c $\boxed{\text{not required}}$

If the curve c is not so simple, the following procedure needs to be followed:

a) Find a parametric representation of the given curve. In other words, choose a parameter (e.g. t) and express any point $P(x,y,z)$ on the curve as $P(x(t), y(t), z(t))$. Every value of t has to represent only one point on the curve. Therefore, two values of t (namely t_A and t_B) correspond to the two end points A and B. Of course you can find a number of different parametric representations, but you obviously choose the simplest one.

Here are a few examples of how you do this:

- Line segment with end points $A(x_A, y_A, z_A)$ and $B(x_B, y_B, z_B)$



Consider a point $M(x,y,z)$ on the line segment AB. Since the vectors $\vec{AM}$ and $\vec{AB}$ lie along the same direction, there is a parameter $t \in \mathbb{R}$ such that:

$$\vec{AM} = t\,\vec{AB}$$

We now have:

$$\vec{AM} = (x - x_A)\,\vec{a_x} + (y - y_A)\,\vec{a_y} + (z - z_A)\,\vec{a_z}$$

$$\vec{AB} = (x_B - x_A)\,\vec{a_x} + (y_B - y_A)\,\vec{a_y} + (z_B - z_A)\,\vec{a_z}$$

From the previous two formulas:

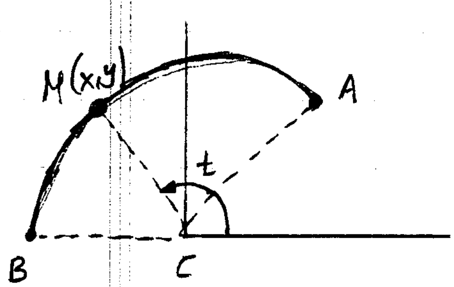
$$\begin{cases} x - x_A = t\,(x_B - x_A) \Rightarrow x = x_A + t\,(x_B - x_A) \\ y - y_A = t\,(y_B - y_A) \Rightarrow y = y_A + t\,(y_B - y_A) \\ z - z_A = t\,(z_B - z_A) \Rightarrow z = z_A + t\,(z_B - z_A) \end{cases}$$

These equations constitute the parametric representation of the line segment. The two end points (A and B) correspond to $t_A = 0$ and $t_B = 1$ respectively

- <u>Arc of a circle lying on the xy plane</u>

Consider that $C(a,b)$ and R are the circle's center and radius respectively. The coordinates of a point $M(x,y)$ on the circle can be expressed as:

$$x = a + R\cos t$$
$$y = b + R\sin t$$



The parameter t now represents the angle between the CM and the positive axis Ox (we called this parameter ϕ in cylindrical coordinates). In this case, t_A and t_B depend on the exact locations of A and B on the arc.

- General curve given by the equations $\left\{ \begin{array}{l} F(x,y,z)=0 \\ G(x,y,z)=0 \end{array} \right\}$

Every curve can be expressed by such equations. In this case, we typically set one of the x, y or z equal to t and then find the other two as a function of t. For example, if the given curve is defined by

$$\begin{aligned} x + e^{y} - z &= 0 \\ x\cos y + z &= 0 \end{aligned}$$

we can set $y = t$. The curve's parametric expression is then given by:

$$\left\{ \begin{aligned} x &= -\frac{e^{t}}{1+\cos t} \\ y &= t \\ z &= \frac{e^{t}\cos t}{1+\cos t} \end{aligned} \right\}$$

b) Calculate the derivatives:

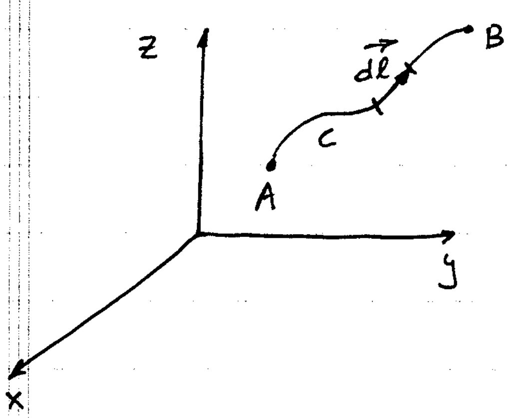
$$\dot{x} = \frac{dx}{dt}, \quad \dot{y} = \frac{dy}{dt}, \quad \dot{z} = \frac{dz}{dt}$$

c) The line integral is given by:

$$I = \int_c f(x,y,z)\,d\ell = \int_{t_A}^{t_B} f(x(t), y(t), z(t))\,\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\,dt$$

Note: This general procedure is a really powerful technique (try it on examples 1 and 2!) In class, however, we restrict ourselves to integrals that can be very easily calculated in cartesian, cylindrical and spherical coordinate systems. This is the reason we often do not need this technique.

A2. Line Integral with Vector Functions



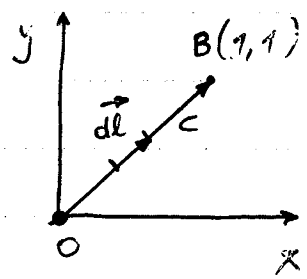
This integral has the form

$$I = \int_A^B \vec{F} \cdot \vec{d\ell}$$

The idea is again to evaluate the integral in the most "convienient" coordinate system, i.e. in the coordinate system that the curve c has the simplest description.

Note: Since $\vec{F}\cdot\vec{d\ell}$ is a dot product of two vectors, the integral I will be a scalar.

Example 3



Evaluate the integral

$$I = \int_{O}^{B} \vec{F}\cdot\vec{d\ell}$$

on the line segment OB. The vector function $\vec{F}$ is given by:

$$\vec{F} = y\,\vec{a_x} + z\,\vec{a_y} + x\,\vec{a_z}$$

Solution

Since the curve C is a line segment, the cartesian system is the most appropriate one. In this system:

$$\vec{d\ell} = \vec{a_x}\,dx + \vec{a_y}\,dy + \vec{a_z}\,dz$$ (differential length in cartesian coordinates)

Therefore:

$$\vec{F}\cdot\vec{d\ell} = \left(y\,\vec{a_x} + z\,\vec{a_y} + x\,\vec{a_z}\right)\cdot\left(dx\,\vec{a_x} + dy\,\vec{a_y} + dz\,\vec{a_z}\right)$$

$$= y\,dx + z\,dy + x\,dz$$

However the line c lies on the xy plane. Hence on the line c:

$$z = 0, \; dz = 0$$

Thus on the line c: $\vec{F} \cdot \vec{dl} = y\,dx + \cancel{z}^{\,0} dy + x\,\cancel{dz}^{\,0} =$

$$= y\,dx$$

Also when we integrate from $O \to B$ we move along a line with an equation $y = x$. This means:

$$y = x \Rightarrow dy = dx$$

and y (as well as x) changes from 0 (at the origin) to 1 (at B).

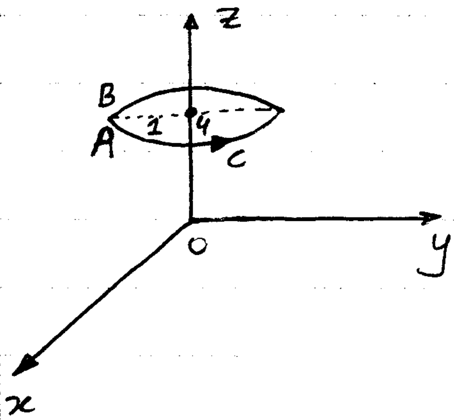
So the integral becomes:

$$I = \int_0^B \vec{F} \cdot \vec{dl} = \int_0^B y\,dx = \int_0^1 y\,dy = \frac{1}{2}$$

because $\vec{F} \cdot \vec{dl} = y\,dx$ on the line OB

because $x = y \Rightarrow dx = dy$ and $y_{min} = 0$, $y_{max} = 1$ on the line c

Example 4



Evaluate the integral

$$I = \oint_c \vec{F} \cdot \vec{dl} \quad \text{on the}$$

circle c with center at (0,0,4) and radius of 1. The function $\vec{F}$ is given by:

$$\vec{F} = (x-z)\,\vec{a_x} + z\,\vec{a_y} + (x+y)\,\vec{a_z}$$

<u>Note</u>: The symbol $\oint_c$ denotes a line integral on a <u>closed</u> curve c. You can assume the start and end points (A and B) anywhere you like on c, but you have to keep the given direction of c (as shown in the figure).

Solution

Since the curve c is a circle, the cylindrical coordinate system is the most appropriate. ~~th~~ this system:

$$\vec{dl} = dr\,\vec{a_r} + r\,d\phi\,\vec{a_\phi} + dz\,\vec{a_z}$$ (differential length in cylindrical system)

As in the previous example, we could simply calculate

$\vec{F} \cdot \vec{dl}$ and then simplify be evaluating it on the given line c. However, we can do these simplifications now.

In other words, on the line c we have:

$$\left.\begin{array}{l} r = 1 \Rightarrow dr = 0 \\ 0 \leqslant \phi \leqslant 2\pi \\ z = 4 \Rightarrow dz = 0 \end{array}\right\} \text{ Thus } \vec{dl} = \cancelto{0}{dr}\,\vec{a_r} + \cancelto{1}{r}\,d\phi\,\vec{a_\phi} + \cancelto{0}{dz}\,\vec{a_z}$$

$$= d\phi\,\vec{a_\phi}$$

Now in order to evaluate $\vec{F} \cdot \vec{dl}$ we need to calculate $\vec{F}$ on the line c in cylindrical coordinates.

From the golden rules: $x = \cancelto{1 \text{ on } c}{r}\cos\phi = \cos\phi$

$$y = \cancelto{1 \text{ on } c}{r}\sin\phi = \sin\phi$$

$$z = 4 \text{ on } c$$

Hence: $\vec{F} = (x - z)\,\vec{a_x} + z\,\vec{a_y} + (x+y)\,\vec{a_z}$

$$= (\cos\phi - 4)\,\vec{a_x} + 4\,\vec{a_y} + (\cos\phi + \sin\phi)\,\vec{a_z}$$

Now: $\vec{F} \cdot \vec{dl} = \left[(\cos\phi - 4)\,\vec{a_x} + 4\vec{a_y} + (\cos\phi + \sin\phi)\vec{a_z}\right] \cdot d\phi\,\vec{a_\phi}$

or $\vec{F} \cdot \vec{dl} = (\cos\phi - 4)\,d\phi\left[\vec{a_x} \cdot \vec{a_\phi}\right] + 4\,d\phi\left[\vec{a_y} \cdot \vec{a_\phi}\right] + (\cos\phi + \sin\phi)\,d\phi\left[\vec{a_z} \cdot \vec{a_\phi}\right]$

From the golden rules:

$$\vec{a_x} \cdot \vec{a_\phi} = -\sin\phi$$
$$\vec{a_y} \cdot \vec{a_\phi} = \cos\phi$$
$$\vec{a_z} \cdot \vec{a_\phi} = 0$$

Therefore: $\vec{F} \cdot \vec{dl} = \left[-(\cos\phi - 4)\sin\phi + 4\cos\phi + 0 \right] d\phi$

$$= \left[4(\sin\phi + \cos\phi) - \sin\phi\cos\phi \right] d\phi$$

Finally:

$$I = \oint_C \vec{F} \cdot \vec{dl} = \int_0^{2\pi} \left[4(\sin\phi + \cos\phi) - \sin\phi\cos\phi \right] d\phi$$

$$= 4(-\cos\phi)\Big|_0^{2\pi} + 4\sin\phi\Big|_0^{2\pi} - \frac{1}{2}\sin^2\phi\Big|_0^{2\pi} = 0 + 0 + 0 = 0$$

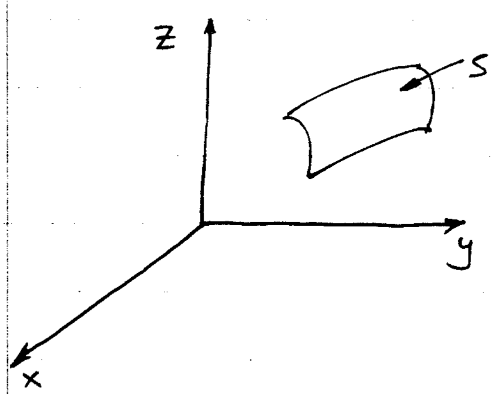
General case for curve c [not required]

As in the previous case, if c is not a simple curve in one of the three coordinate systems, a general procedure similar to the one presented before needs to be followed. If you are interested in this, please come and talk to your instructor.

B. Surface Integrals

In this type, you have to integrate a scalar or vector quantity over an open or closed surface.

B1. Surface Integral with Scalar Functions

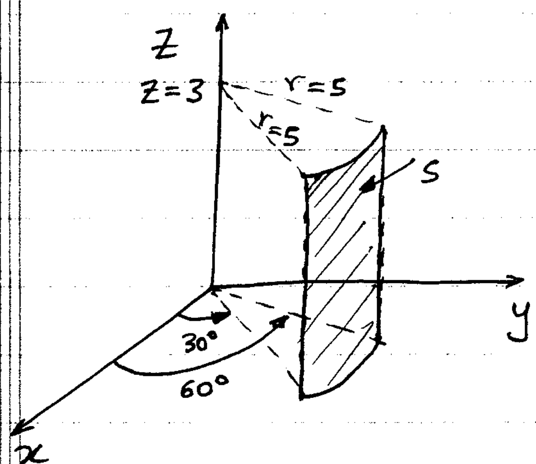


This integral has the form:

$$I = \int_S f \, dS$$

As usual, we will only discuss the case in which dS has a very simple expression in one of the three basic coordinate systems. The trick is to select the coordinate system that dS has a known form (for example $dS_x = dy\,dz$ or $dS_\phi = dr\,dz$ or $dS_\theta = R\sin\theta \, dR \, d\phi$ etc). Then you need to find the integral limits by simply observing the given geometry.

Example 5



Evaluate the integral

$$I = \int_S f \, ds$$ on the cylindrical surface shown in the figure (it is described by: $r=5$, $30° < \phi < 60°$, $0 < z < 3$).

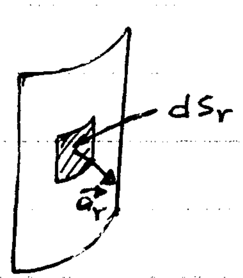
The function $f = 1$.

Solution

$$\left(\text{Since } f = 1 : \quad I = \int_S f \, ds = \int_S ds = \text{area of given surface.} \right)$$

Since the surface has cylindrical symmetry (it is part of a cylinder in this case), the cylindrical coordinate system needs to be chosen for the fastest calculation.

Since the surface is perpendicular to $\vec{a_r}$, $ds = ds_r = r \, d\phi \, dz$



A different way to see this, is that in order to cover the given surface, you need to vary ϕ (from $\frac{\pi}{6}$ to $\frac{\pi}{3}$) and z (from 0 to 3), while $r = 5 = \text{const}$

Thus, you need to choose the ds that <u>does not</u> include dr! In other words $ds = ds_r$

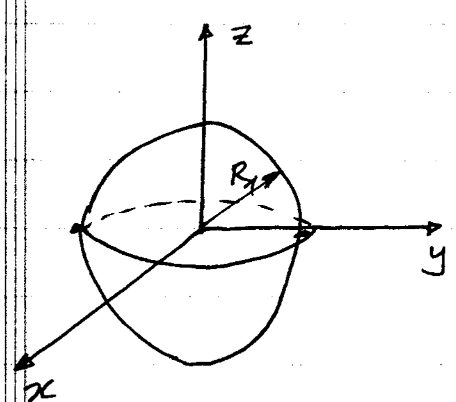
Finally: $I = \int_S ds = \int_{z=0}^{3} \int_{\phi=\pi/6}^{\pi/3} r \, dz \, d\phi$

Note that $r = 5 = \text{const}$ on the given surface, hence:

$$I = \int_0^3 \int_{\pi/6}^{\pi/3} 5 \, dz \, d\phi = 5\left(z \Big|_0^3 \right)\left(\phi \Big|_{\pi/6}^{\pi/3} \right) = \frac{5\pi}{2}$$

Note that you have to convert to radians before you calculate the integral.

Example 6



Evaluate the integral $I = \int_S f \, ds$ on the spherical surface given on the figure (its equation is $x^2 + y^2 + z^2 = R_1^2$). The function f is given by:

$$f(x,y,z) = x^2 + z$$

Solution

The given surface is a sphere, so the spherical coordinate system seems to be the most appropriate!

The surface S is described by:

$$S: \quad 0 \le \phi \le 2\pi, \quad 0 \le \theta \le \pi, \quad R = R_1 = \text{const}$$

Can you see which ds you choose? Obviously $R = R_1 = \text{const}$ so it does not have dR! Equivalently ds is normal to $\vec{a_R}$. No matter what way you think about it:

$$ds = ds_R = R^2 \sin\theta \, d\theta \, d\phi = R_1^2 \sin\theta \, d\theta \, d\phi$$

Thus:

$$I = \int_0^{2\pi}\int_0^{\pi} (x^2 + z) \, R_1^2 \sin\theta \, d\theta \, d\phi$$

Obviously you need to tranform $x^2 + z$ to spherical coordinates. From the "golden rules":

$$x = R_1 \sin\theta \cos\phi \quad , \quad z = R_1 \cos\theta$$

Thus $\quad x^2 + z = R_1^2 \sin^2\theta \cos^2\phi + R_1 \cos\theta$

Now:

$$I = \int_0^{2\pi}\int_0^{\pi} \left(R_1^2 \sin^2\theta \cos^2\phi + R_1 \cos\theta \right) R_1^2 \sin\theta \, d\theta \, d\phi =$$

$$= \int_0^{2\pi}\int_0^{\pi} R_1^4 \sin^3\theta \cos^2\phi \, d\theta d\phi + \int_0^{2\pi}\int_0^{\pi} R_1^3 \sin\theta \cos\theta \, d\theta d\phi$$

$$= R_1^4 \int_0^{\pi} \sin^3\theta \, d\theta \int_0^{2\pi} \cos^2\phi \, d\phi + R_1^3 \int_0^{\pi} \sin\theta\cos\theta \, d\theta \int_0^{2\pi} d\phi$$

- $\displaystyle\int_0^{\pi} \sin^3\theta \, d\theta = \int_0^{\pi} \sin^2\theta \sin\theta \, d\theta = -\int_0^{\pi} (1-\cos^2\theta) \, d(\cos\theta)$

$$= \left(\frac{1}{3}\cos^3\theta - \cos\theta \right) \Big|_0^{\pi} = 4/3$$

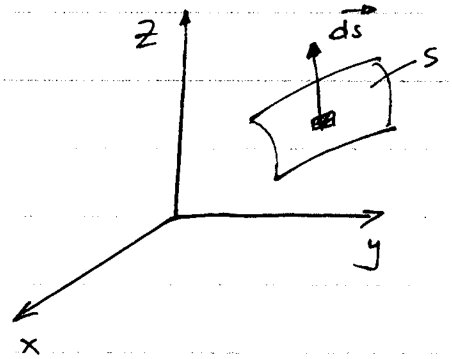
- $\displaystyle\int_0^{2\pi} \cos^2\phi \, d\phi = \int_0^{2\pi} \frac{1+\cos 2\phi}{2} \, d\phi = \frac{1}{2}\int_0^{2\pi} d\phi + \cancel{\frac{1}{2}\int_0^{2\pi} \cos 2\phi \, d\phi}^{\,0}$

$$= \frac{1}{2} 2\pi = \pi$$

- $\displaystyle\int_0^{\pi} \sin\theta\cos\theta \, d\theta = \int_0^{\pi} \sin\theta \, d(\sin\theta) = 0$

Thus $I = R_1^4 \left(\frac{4}{3}\right)\pi = \frac{4}{3}\pi R_1^3$

B2. Surface Integral with Vector Functions

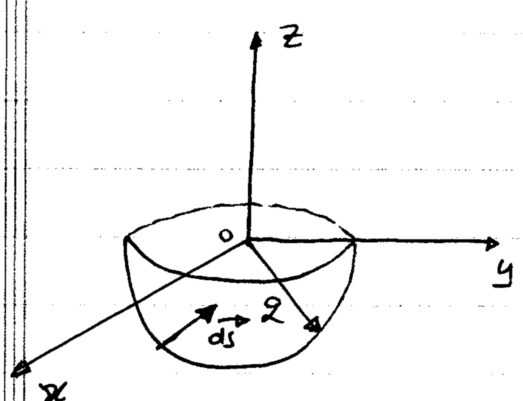


This integral has the form:

$$I = \int_S \vec{F} \cdot \vec{ds}$$

The calculation of this integral is very similar to B1 (surface integral with scalar functions). The difference is that you have vector functions, which means that $\vec{ds}$ is a vector and has a direction. In all applications of this integral in electromagnetics, the direction of $\vec{ds}$ will be always specified by a rule. For example, if S is a closed surface, it will be pointing outward from the respective surface. For, now, $\vec{ds}$ will be specified by the statement of the problem.

Example 7



Evaluate the integral

$$I = \int_S \vec{F} \cdot \vec{ds}$$ on the half-sphere

shown in the figure (bottom half of $x^2+y^2+z^2=4$).

$\vec{ds}$ points towards the origin (inwards)

The function $\vec{F}$ is given by $\vec{F} = -3\,\vec{a_z}$

Solution

Following arguments similar to the examples shown in B1, the spherical coordinate system will be used.

Obviously $\vec{ds} = -\vec{a_R}\, R^2 \sin\theta\, d\theta d\phi = -4\,\vec{a_R} \sin\theta\, d\theta d\phi$

- negative sign because $\vec{ds}$ points inwards (towards origin)
- $R = 2$ (const.) on the given surface

Now calculate $\vec{F} \cdot \vec{ds}$:

$$\vec{F} \cdot \vec{ds} = \left(-3\,\vec{a_z}\right) \cdot \left(-4\,\vec{a_R} \sin\theta\, d\theta d\phi\right) =$$

$$= 12\left(\vec{a_z} \cdot \vec{a_R}\right) \sin\theta\, d\theta d\phi = 12 \sin\theta \cos\theta\, d\theta d\phi$$

since from golden rules: $\vec{a_z} \cdot \vec{a_R} = \cos\theta$

(Alternatively, from golden rules, $\vec{a_z} = \vec{a_R}\cos\theta - \vec{a_\theta}\sin\theta$.

Thus $\vec{F} \cdot \vec{ds} = \left(-3\,\vec{a_R}\cos\theta + 3\,\vec{a_\theta}\sin\theta\right) \cdot \left(-4\,\vec{a_R}\sin\theta\, d\theta d\phi\right)$

$$= 12\,\cancelto{1}{\left(\vec{a_R} \cdot \vec{a_R}\right)} \sin\theta\cos\theta\, d\theta d\phi - 12\,\cancelto{0}{\left(\vec{a_\theta} \cdot \vec{a_R}\right)} \sin^2\theta\, d\theta d\phi$$

$$= 12 \sin\theta \cos\theta\, d\theta d\phi \;)$$

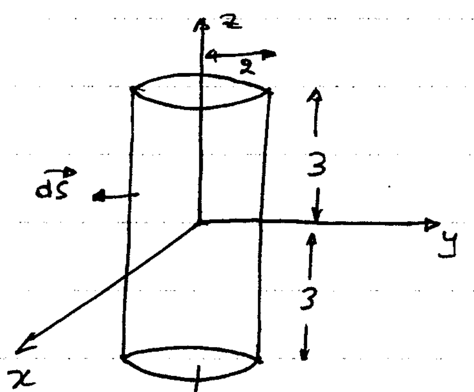
Therefore the integral can be expressed as:

$$\int_S \vec{F} \cdot \vec{ds} = \int_{\phi=0}^{2\pi} \int_{\theta=\pi/2}^{\pi} 12 \sin\theta \cos\theta \, d\theta d\phi =$$

$$= (2\pi)\, 12 \int_{\pi/2}^{\pi} \sin\theta \, d(\sin\theta) = 12\pi \sin^2\theta \Big|_{\pi/2}^{\pi} =$$

$$= -12\pi$$

Example 8 (Cheng example 2-15, p. 41)



Evaluate the integral

$I = \oint_S \vec{F} \cdot \vec{ds}$ over the surface of the closed cylinder shown in the figure ($r = 2$, $-3 \le z \le 3$, $0 \le \phi \le 2\pi$). The function $\vec{F}$ is given by:

$$\vec{F} = \vec{a_r}\, \frac{k_1}{r} + \vec{a_z}\, k_2 z$$

$\vec{ds}$ points outwards.

Solution

First note that the symbol $\oint_S$ denotes an integral over a closed surface.

Because of cylindrical symmetry, we will work with the cylindrical coordinate system. However, in this example, $\vec{ds}$ does not have the same expression everywhere on the surface S. It is easy to see that:

$$\begin{cases} \vec{ds} = \vec{a_z}\, ds_z = \vec{a_z}\, r\,dr\,d\phi & \text{on the top face} \\ \vec{ds} = -\vec{a_z}\, ds_z = -\vec{a_z}\, r\,dr\,d\phi & \text{on the bottom face} \\ \vec{ds} = \vec{a_r}\, ds_r = \vec{a_r}\, r\,d\phi\,dz & \text{on the side wall} \end{cases}$$

This means that you need to "break down" the integral into three sub-integrals:

$$\oint_S \vec{F}\cdot\vec{ds} = \underbrace{\int_{\substack{\text{top}\\\text{face}}} \vec{F}\cdot\vec{ds}}_{I_1} + \underbrace{\int_{\substack{\text{bottom}\\\text{face}}} \vec{F}\cdot\vec{ds}}_{I_2} + \underbrace{\int_{\substack{\text{side}\\\text{wall}}} \vec{F}\cdot\vec{ds}}_{I_3}$$

Top face:

$$I_1 = \int_{\phi=0}^{2\pi}\int_{r=0}^{2} \left(\vec{a_r}\,\frac{k_1}{r} + \vec{a_z}\,k_2 z\right)\cdot\vec{a_z}\, r\,dr\,d\phi =$$

$$= \int_0^{2\pi} \int_0^{2} k_2\, r\, \cancel{z}^{\,3 \text{ on the top face}}\, dr\, d\phi \qquad \left(\text{since } \vec{a_r} \cdot \vec{a_z} = 0,\ \vec{a_z} \cdot \vec{a_z} = 1 \right)$$

$$= 3k_2\,(2\pi)\,\frac{r^2}{2}\Big|_0^2 = 12\,k_2\,\pi$$

<u>Bottom face</u>. Similarly $I_2 = 12\,k_2\,\pi$. Keep in mind that now $\vec{ds} = -\vec{a_z}\, r\,dr\,d\phi$, $z = -3$

<u>Side wall</u>

$$I_3 = \int_{z=-3}^{3} \int_{\phi=0}^{2\pi} \left(\vec{a_r}\,\frac{k_1}{r} + \vec{a_z}\,k_2 z \right) \cdot \left(\vec{a_r}\, r\, d\phi\, dz \right) =$$

$$= \int_{-3}^{3} \int_0^{2\pi} k_1\, d\phi\, dz \qquad \left(\text{since } \vec{a_r} \cdot \vec{a_r} = 1,\ \vec{a_z} \cdot \vec{a_r} = 0 \right)$$

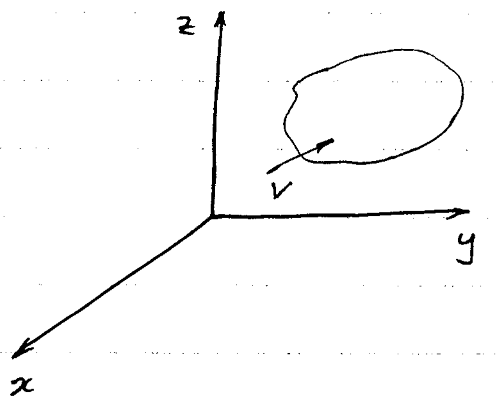
$$= 12\pi k_1$$

Finally: $I = 12\pi k_2 + 12\pi k_2 + 12\pi k_1 = 12\pi\,(k_1 + 2k_2)$

C. Volume Integrals

In this type, you have to integrate a scalar or a vector quantity in a volume.

C1. Volume Integral with Scalar Functions

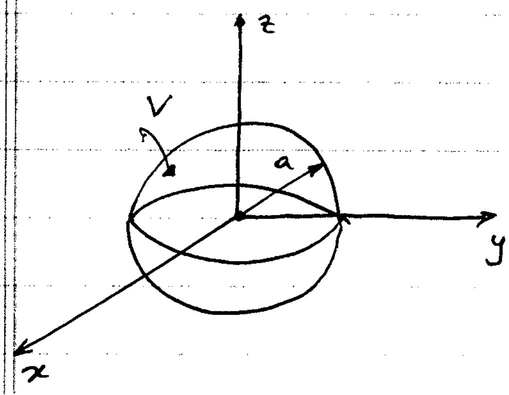


This integral has the form:

$$I = \int_V f \, dv$$

As in the previous cases, this integral is very easy to compute if you choose the "right" (= most convenient) coordinate system. Then dv will be known (look at "differential elements" handout) and you only have to find the limits of the integral by observing the geometry.

Example 9



Evaluate the integral

$$I = \int_V f\, dv \quad \text{inside the volume}$$

bounded by the sphere shown in the figure (its equation is $x^2+y^2+z^2 \leq a^2$). The function f is given by:

$$f = 3\left(x^2+y^2+z^2\right)$$

Solution

Because of the spherical volume, the spherical coordinate system will be used. Hence:

$$dv = R^2 \sin\theta \, dR\, d\theta\, d\phi$$

with: $\quad 0 \leq R \leq a, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi$

From golden rules, you can transform the given function to spherical coordinates:

$$f = 3\left(x^2+y^2+z^2\right) = 3\left[R^2 \sin^2\theta \cos^2\phi + R^2 \sin^2\theta \sin^2\phi + R^2 \cos^2\theta\right]$$

$$= 3R^2\left[\sin^2\theta\left(\cos^2\phi + \sin^2\phi\right) + \cos^2\theta\right] = 3R^2$$

Now the integral becomes:

$$I = \int_V f\, dv = \int_{R=0}^{a} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} (3R^2)(R^2 \sin\theta \, dR\, d\theta\, d\phi)$$

$$= 3 \left.\frac{R^5}{5}\right|_0^a \left.\cos\theta\right|_\pi^0 (2\pi) = \frac{12\pi a^5}{5}$$

C2. Volume Integral with Vector Functions

This integral has the form $I = \int_V \vec{F}\, dv$. We won't use it much in 311, so we won't discuss it. Notice that $\vec{F}\, dv$ is <u>NOT</u> a dot product (compare with $\vec{F} \cdot \vec{dl}$, $\vec{F} \cdot \vec{ds}$). In any case the evaluation of this integral is identical to C1.