Integrals with Vector Functions

- We will often need to use calculus (specifically vector calculus) when relating the concepts of charges, currents, electric fields, and magnetic fields.

- We will stress throughout the course, however, that this calculus is simply a means of performing superposition – e.g. treating continuous charge distributions as a collection of discrete charges and calculating the electric field due to the distribution as the sum of the contributions from each discrete charge.

- The integrals we'll typically see are of the forms

$$\left.\begin{array}{l} \int_V f\, dv \\ \int_S f\, ds \\ \int_C f\, dl \end{array}\right\} \quad \text{Scalar Volume, Surface, and Line Integrals}$$

$$\int_C \vec{E}\cdot d\vec{l} \quad \text{Vector Line Integrals}$$

$$\left.\begin{array}{l} \oint_S \vec{D}\cdot d\vec{s} \\ \int_S \vec{B}\cdot d\vec{s} \end{array}\right\} \quad \text{Vector Surface Integrals}$$

- Here are some examples (and their significance in E&M!)

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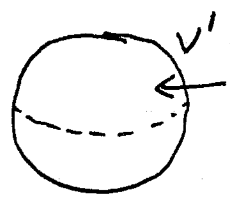
Examples

(1) Scalar Volume Integral

- Given the scalar function $\rho_v = \frac{A}{R}$ that exists in the volume defined by a sphere of radius $= b$ centered at the origin, what is

$$Q = \int_{v'} \rho_v \, dv' \quad ?$$



ρ_v exists throughout the sphere

- You could set this integral up in any of the 3 coordinate systems we've seen, however, the resulting triple integral (over the volume of the sphere) will be simplest in spherical coordinates. In spherical coordinates,

$$Q = \int \rho_v \, dv = \int_0^{2\pi}\int_0^{\pi}\int_0^{b} \underset{\uparrow \atop \rho_v}{\left(\frac{A}{R}\right)} \underbrace{R^2 \sin\theta \, dR \, d\theta \, d\phi}_{dv \text{ in spherical coordinates}}$$

- there is nothing in this integral that varies with ϕ (except $d\phi$). $\int_0^{2\pi} d\phi = 2\pi$. So Q becomes

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Examples:

(1) Scalar Volume Integral

$$Q = 2\pi A \int_0^{\pi}\int_0^{b} R\sin\theta \, dR\, d\theta$$

- integrating over R

$$Q = \pi A b^2 \int_0^{\pi} \sin\theta \, d\theta$$

- integrating over θ

$$Q = \pi A b^2(-\cos\pi + \cos 0)$$

$$\boxed{Q = 2\pi A b^2}$$

- where would this integral show up in electromagnetics?

  ⇒ If the function ρ_v represented a volume distribution of charge (C/m^3), this integral would be the total charge (C) contained in the sphere.

  ⇒ This is simply an example of an integral as a sum.

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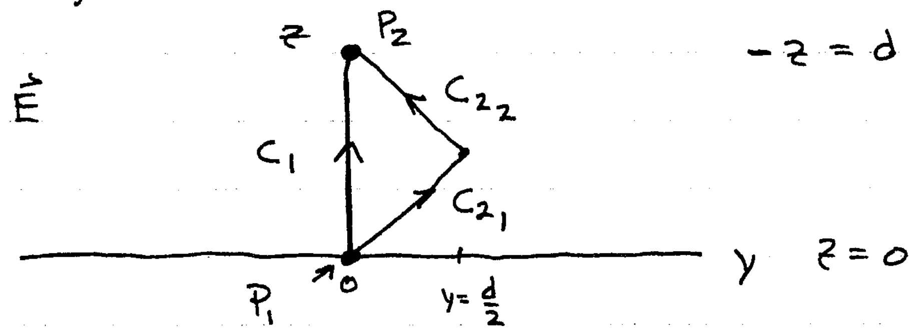
Examples

(2) Line Integrals of Vector Functions

- Assume we are given a vector

$$\vec{E} = -\frac{V_0}{d}\hat{a}_z$$

that exists everywhere (i.e. for all x, y) in the region $0 \le z \le d$



- What is $-\int_{P_1}^{P_2} \vec{E} \cdot d\vec{\ell}$?

- Again, this is simply a continuous sum, but this time it is a sum of the <u>component of $\vec{E}$ along the path from $P_1 \rightarrow P_2$</u>. This means we must take the path into account!

- Since both paths above consist of straight line segments it makes the most sense to work in Cartesian coordinates.

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Integrals with Vector Functions

Examples

(2) Line Integrals of Vector Functions

- In Cartesian coordinates, the differential length is

$$\underline{d\vec{l} = \hat{a}_x\,dx + \hat{a}_y\,dy + \hat{a}_z\,dz}$$

- For both paths from $P_1 \rightarrow P_2$, the integrand $\vec{E}\cdot d\vec{l}$ will be the same

$$\underline{\vec{E}\cdot d\vec{l}} = -\frac{V_0}{d}\hat{a}_z\cdot\left(\hat{a}_x\,dx + \hat{a}_y\,dy + \hat{a}_z\,dz\right)$$

$$\underline{= -\frac{V_0}{d}\,dz}$$

- At this point, we need to take how we are getting from $P_1 \rightarrow P_2$ into account.

- <u>For path C_1</u> this is straight forward - the integral is over a direct path, from $P_1 \rightarrow P_2$, along the z axis. So,

$$-\int_{P_1}^{P_2}\vec{E}\cdot d\vec{l} = -\int_0^d\left(-\frac{V_0}{d}\right)dz$$

$$\underline{\underline{-\int_{P_1}^{P_2}\vec{E}\cdot d\vec{l} = V_0}} \qquad \text{(over path } C_1\text{)}$$

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(2) Line Integrals of Vector Functions

- For path C_2 the process is not quite as simple (though it's not hard either!). To find the integral over this path, we simply need to account for the fact that y varies along with z over this path – this means we must express z in terms of y; simply integrating from $z=0 \rightarrow z=d$ is not correct!

- So, we can break the integral from $P_1 \rightarrow P_2$ into two parts as shown in the picture:

C_{2_1} = line segment from $P_1 = (z=0, y=0) \rightarrow (z=\frac{d}{2}, y=\frac{d}{2})$
and
C_{22} = line segment from $(z=\frac{d}{2}, y=\frac{d}{2})$ to $P_2 = (z=d, y=0)$

- For these segments of path C_2 we can express z in terms of y as follows:

On $C_{2_1} \Rightarrow$ $z = y$, so $dz = dy$

On $C_{22} \Rightarrow$ $z = d - y$, so $dz = -dy$

$\downarrow$

Integrals with Vector Functions

Examples:

(2) Line Integrals of Vector Functions

- Now

$$-\int_{P_1}^{P_2} \vec{E}\cdot d\vec{\ell} = -\int_{0}^{d/2}\left(-\frac{V_0}{d}\right)dy - \int_{d/2}^{0}\left(-\frac{V_0}{d}\right)(-dy)$$

$$= \frac{V_0}{2} - \int_{d/2}^{0} \frac{V_0}{d}\,dy$$

$$\underline{\underline{-\int_{P_1}^{P_2} \vec{E}\cdot d\vec{\ell} = V_0}} \qquad \text{(over path } C_2\text{)}$$

- So, where will we see this type of integral in electromagnetics?

  - If $\vec{E}$ represents a force and $d\vec{\ell}$ a distance, this integral is the work expended in moving an object (i.e. applying a force) over the paths C_1 and C_2. In E & M, we will define the potential (voltage) as the work done in moving a charge some distance in the presence of an electric field. The functional form of $\vec{E} = -V_0/d\ \hat{a}_z$ is actually that of the electric field in an ideal parallel-plate capacitor. [Note: $\vec{E}$ has units V/m, so $\frac{V}{m}\cdot m \cdot C$ = Joules (work)]

Integrals with Vector Functions

Examples

(2) Line Integrals of Vector Functions

- Also, we'll see that the fact the integral $-\int_{P_1}^{P_2} \vec{E} \cdot d\vec{\ell}$ does not depend on the choice of path from $P_1 \rightarrow P_2$ is a property of real electrostatic fields (this is related to conservation of energy).

(3) Surface Integrals of Vector Functions

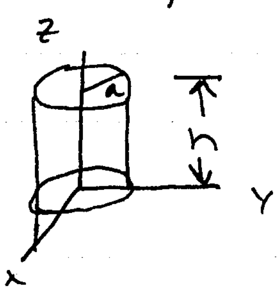
- Assume we have the following vector given in cylindrical coordinates

$$\vec{D} = \frac{\rho_L}{2\pi r} \hat{a}_r$$

- We want to find

$$Q = \oint \vec{D} \cdot d\vec{s}$$

over the closed surface of a cylinder of radius $= a$ and height $= h$ centered along the z axis.



Integrals with Vector Functions

Examples

(3) Surface Integrals of Vector Functions

- There are several things to note here:

  (a) The closed surface integral can be represented as the sum of three open surface integrals $\Rightarrow$ one over the top, one over the bottom, and one over the sidewall.

  (b) In this example, we define the vector $d\vec{s}$ to be the outward normal from each of the three surfaces above:

$$\text{top} \quad d\vec{s} = d\vec{s}_z = r\,dr\,d\phi\,\hat{a}_z$$

$$\text{bottom} \quad d\vec{s} = -d\hat{s}_z = r\,dr\,d\phi\,(-\hat{a}_z)$$

$$\text{sidewall} \quad d\vec{s} = d\vec{s}_r = r\,d\phi\,dz\,\hat{a}_r$$

So,

$$\oint \vec{D}\cdot d\vec{s} = \int_{top} \vec{D}\cdot d\hat{s}_z - \int_{bottom} \vec{D}\cdot d\vec{s}_z + \int_{sidewall} \vec{D}\cdot d\vec{s}_r$$

Since $\vec{D} = \frac{\rho_L}{2\pi r}\hat{a}_r$ and $\hat{a}_r \cdot \hat{a}_z = 0$ these integrals do not contribute to $Q = \oint \vec{D}\cdot d\vec{s}$!

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Examples

(3) Surface Integrals of Vector Functions

(b) Given this,

$$\oint \vec{D}\cdot d\vec{s} = \int_{\text{sidewall}} \vec{D}\cdot d\vec{s}_r$$

(c) Finally, we are interested in $\vec{D}\cdot d\vec{s}_r$ on the sidewall of the cylinder, i.e. at $\underline{\underline{r=a}}$.

- Taking all of this into account

$$Q = \oint \vec{D}\cdot d\vec{s} = \int \vec{D}\cdot d\vec{s}_r = \int_0^h \int_0^{2\pi} \left(\frac{\rho_L}{2\pi a}\hat{a}_r\right)\cdot\left(a\,d\phi\,dz\,\hat{a}_r\right)$$

$$= \int_0^h \int_0^{2\pi} \frac{\rho_L}{2\pi}\,d\phi\,dz$$

$$\underline{\underline{Q = \rho_L h}}$$

Integrals with Vector Functions

Examples:

(3) Surface Integrals of Vector Functions

- Where will we see this in E & M?

In E & M, integrals of this form relate to flux - that is, the total amount of a vector passing through a surface. For example, if you consider $\vec{D}$ to represent the flow of water, $\oint \vec{D} \cdot d\vec{s}$ would be the total amount of water flowing in or out of our cylinder (if this quantity is non-zero it means we have a tap or drain in our cylinder!).

- In this example, $\vec{D}$ was chosen to be the displacement field (units C/m^2) due to a linear charge density ρ_L (units C/m) along the z axis. The integral $Q = \oint \vec{D} \cdot d\vec{s} = \rho_L h$ then is the total charge enclosed by the cylindrical surface. (We will see this relationship often - it is Gauss' Law and is quite useful for determining displacement or electric fields in cases where symmetry exists!)