Outline

Recall Example 1.3

A group of friends went kayaking at East Coast Park. 5 wore sunglasses, while 7 wore hats. How many friends went on this trip?

- Define the following sets
  T: friends who went kayaking
  S: friends who wore sunglasses
  H: friends who wore hats
- Can't use addition principle since $S \cap H \overset{?}{=} \emptyset$ and $T \overset{?}{=} S \cup H$.
- If we are further informed that everyone who went kayaking wore either sunglasses or hats as protection and there were exactly two persons who wore both sunglasses and hats.
- $|S \cap H| = 2$ and $T = S \cup H \implies T = 5 + 7 - 2 = 10.$

Inclusion-Exclusion Principle

Theorem 10.1

Let A and B be finite sets. Then

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Proof:

- LHS counts the # of elements in $A \cup B$
- $|A| + |B|$ also counts the elements in $A \cup B$ but elements belonging to both A and B are counted twice, once in $|A|$ and once in $|B|$
- Subtracting the # of elements in both A and B i.e. $|A \cap B|$, we get the exact size of $A \cup B$. $\square$

Inclusion-Exclusion Principle

Example 10.2

There are exactly positive 200 integers that are
a) less than or equal to 300 and are
b) divisible by either 2 or 3.

- Let A be set of eligible integers that are divisible by 2. Then $|A| = \frac{300}{2} = 150$.
- Let B be set of eligible integers that are divisible by 3. Then $|B| = \frac{300}{3} = 100$.
- $A \cap B$ contains integers divisible by both 2 and 3, i.e. divisible by 6.
- $|A \cap B| = \frac{300}{6} = 50$
- By IE $|A \cup B| = 150 + 100 - 50 = 200$.

Inclusion-Exclusion Principle

Example 10.3

We have invited 30 guests to a wedding and we need to allocate them into 3 tables of 10 guests each. But Alex and Jose cannot be seated at the same table, neither can Kim and Lee. In how many ways can we then allocate the guests?

- Let A be the allocation where A and J are at the same table.
- Map A to partitions of [28] into blocks of 10, 10 and 8. (Bijection)
- $|A| = \frac{28!}{10!10!8!2}$
- Let B be the allocation where K and L are at the same table. $|B| = \frac{28!}{10!10!8!2}$
- Two cases for $A \cap B$.

Inclusion-Exclusion Principle

Example 10.3 cont'd

- Case 1: All four at the same table. Map (bijectively) to $\frac{26!}{10!10!6!2}$
- Case 2: Each pair at a different table. Map to $\frac{26!}{10!8!8!2}$
- But this is 2-to-1, because there are two possible tables of 8 to insert A and J
- Hence $|A \cap B| = \frac{26!}{10!10!6!2} + 2 \times \frac{26!}{10!8!8!2}$
- By IE $|A \cup B| = 2 \times \frac{28!}{10!10!8!2} - \frac{26!}{10!10!6!2} - \frac{26!}{10!8!8!}$
- By subtraction principle, # possible allocation

$$\frac{30!}{10!10!10!3!} - \frac{28!}{10!10!8!} + \frac{26!}{10!10!6!2} + \frac{26!}{10!8!8!}$$

Inclusion-Exclusion Principle

Theorem 10.4

Let A, B and C be finite sets. Then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|.$$

Example 10.5

A group of 10 tourists is visiting Japan. In order to minimize waiting times, they split up. However, three of them, X, Y and Z do not speak Japanese and do not want to go visiting alone. In how many ways can the tourists split themselves?

- Want # partitions of [10] where 1,2 and 3 are not singletons.
- Define A_i as partitions of [10] where i is a singleton.
- Note $A_1 \cap A_2 \cap A_3 \neq \emptyset$
- Answer $= B(10) - |A_1 \cup A_2 \cup A_3|$
- $|A_i| = B(9)$
- $|A_i \cap A_j| = B(8)$ and $|A_i \cap A_j \cap A_k| = B(7)$ for distinct, i, j, k.
- By theorem 10.4, $|A_1 \cup A_2 \cup A_3| = 3B(9) - 3B(8) + B(7)$
- Answer $= B(10) - 3B(9) + 3B(8) - B(7) = 64077$.

Example 10.6

Ten people arrived at a dentist's office at different times, and they are called in random order. A person is considered unlucky if he was the ith to arrive but not among the first i people to be called. Three persons A, B and C arrived third, fifth and eighth respectively. In how many of the 10! possible orders will at least one of them be unlucky?

- Suppose A, B and C were all lucky!
- If A is lucky he must be among the first 3 persons to be called.
- If B is also lucky $\implies$ 4 possible positions to be called
- Finally if C is also lucky there are now 6 positions left.
- The remaining 7 people can be ordered in 7! ways
- Total $= 3 \times 4 \times 6 \times 7! = 9!$
- By subtraction principle $\# = 10!$ - $9!$.

Example 10.6 by IE

- S_X be case where X is unlucky
- $|S_A| = 7 \times 9!$, $|S_B| = 5 \times 9!$, $|S_C| = 2 \times 9!$
- $|S_B \cap S_A| = 5 \times 6 \times 8!$,
- $|S_C \cap S_B| = 2 \times 4 \times 8!, |S_C \cap S_A| = 2 \times 6 \times 8!$
- $|S_C \cap S_B \cap S_A| = 2 \times 4 \times 5 \times 7!$
- Answer $= 14 \cdot 9! - 50 \cdot 8! + 40 \cdot 7! = 3265920.$

Inclusion-Exclusion Principle

Theorem 10.7 (Inclusion-Exclusion Principle)

Let A_i, $i = 1, \ldots n$ be finite sets, then

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{j=1}^{n} (-1)^{j-1} \sum_{i_1,\ldots,i_j} \left|A_{i_1} \cap A_{i_2} \cdots \cap A_{i_j}\right|,$$

where $(i_1, \ldots, i_j)$ ranges over all j-element subsets of $[n]$.

Proof: WTS if $x \in$ LHS, x is counted exactly once in RHS

- Assume $x \in$ exactly k of the sets A_i, with $k > 0$.
- So $x \notin A_{i_1} \cap A_{i_2} \ldots \cap A_{i_j}$ for $j > k$.
- If $j \leq k$, there are $\binom{k}{j}$ intersections that contain x.

Inclusion-Exclusion Principle

Proof of Theorem 10.7 cont'd.

- Hence x is counted exactly $\sum_{j=1}^{k}(-1)^{j-1}\binom{k}{j}$ times.
- And $\sum_{j=1}^{k}(-1)^{j-1}\binom{k}{j} = 1 - \sum_{j=0}^{k}(-1)^{j}\binom{k}{j} = 1.$
- Now if $x \notin$ any A_i, then none of the intersection in RHS counts x

Example 10.5 cont'd

In how many ways can the tourists split themselves, if each subgroup must contain at least one person who speaks Japanese?

- Previous example allows for blocks of $\{X, Y\}$, $\{X, Y, Z\}$ etc.
- Define A_{ij} be partitions of $[10]$ where i and j forms a block.
- Then $|A_{ij}| = B(8)$ and $|A_{ijk}| = B(7)$
- Hence bad set $S = A_1 \cup A_2 \cup A_3 \cup A_{12} \cup A_{23} \cup A_{13} \cup A_{123}$

$$\begin{aligned} |S| = \sum |A_i| + \sum |A_{ij}| + |A_{123}| \\ - \sum |A_i \cap A_j| - \sum |A_i \cap A_{jk}| + \sum |A_1 \cap A_2 \cap A_3| \\ = 3B(9) + 3B(8) + B(7) - 3B(8) - 3B(7) + B(7) \end{aligned}$$

- Answer $= B(10) - 3B(9) + B(7) = 53411.$

Example 10.5 cont'd (alternative solution)

In how many ways can the tourists split themselves, if each subgroup must contain at least one person who speaks Japanese?

- Partition first those who speaks Japanese in $\left\{ {7 \atop k} \right\}$ ways.
- The groups now becomes distinct because of their members.
- Each of X, Y, Z has k distinct groups to join.
- Hence Answer $= \displaystyle\sum_{k=0}^{7} \left\{ {7 \atop k} \right\} k^3.$
- As a corollary

$$B(n+3) - 3B(n+2) + B(n) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} k^3.$$

Stirling Numbers Again

Example 10.8

Use the Inclusion-Exclusion Principle to count the # ways to distribute n distinct balls to k distinct boxes such that every box gets at least one ball.

- k^n ways to distribute distinct balls to distinct boxes
- Let A_i = set of distributions where i-th box is empty.
- Answer $= k^n - \left|\bigcup_{i=1}^{k} A_i\right|$
- $|A_i| = (k-1)^n$
- $|A_i \cap A_j| = (k-2)^n$
- $\left|A_{i_1} \cap A_{i_2} \cdots \cap A_{i_j}\right| = (k-j)^n$

Stirling Numbers Again

Example 10.8 cont'd

$$\left|\bigcup_{i=1}^{k} A_i\right| = \sum_{i=1}^{k} |A_i| - \sum_{1\le i<j\le k} |A_i \cap A_j| + \dots$$
$$+ (-1)^{j-1} \sum_{1\le i_1<\dots<i_j\le k} \left|A_{i_1} \cap A_{i_2} \dots \cap A_{i_j}\right| + \dots$$
$$= \binom{k}{1}(k-1)^n - \binom{k}{2}(k-2)^n + \dots$$
$$+ (-1)^{j-1}\binom{k}{j}(k-j)^n + \dots$$
$$\implies \# = \sum_{j=0}^{k}(-1)^j \binom{k}{j}(k-j)^n.$$

Stirling Numbers Again

Theorem 10.9

The # of surjections from $[n]$ to $[k]$ is

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n.$$

Theorem 10.10

For all positive integers $k \leq n$,

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n.$$

Stirling Numbers Again

Example 10.11

We want to divide 12 children into four playgroups. However, there are two pairs of siblings among the children, and we do not want to put siblings in different groups. How many possibilities do we have?

Treat each pair of siblings as one entity. Then answer =

$$\begin{aligned}\left\{ {10 \atop 4} \right\} &= \frac{1}{4!}\sum_{j=0}^{4}(-1)^j\binom{4}{j}(4-j)^{10}\\ &= \frac{1}{24}\left(4^{10}-\binom{4}{1}3^{10}+\binom{4}{2}2^{10}-\binom{4}{4}1^{10}\right)\\ &= 34105.\end{aligned}$$

Example 10.12

n married couples attend a conference where each person speaks exactly once. In how many ways can we schedule the speakers so that no married participants speak consecutively?

- Let $A_i = \{$ schedules where i-th couple speak consecutively $\}$
- $|A_i| = 2 \times (2n-1)!$
- $\left|A_{i_1} \cap A_{i_2} \cdots \cap A_{i_j}\right| = 2^j \times (2n-j)!$
- $\# = (2n)! - \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} 2^j (2n-j)!$
- $= \sum_{j=0}^{n} (-1)^{j} \binom{n}{j} 2^j (2n-j)!$

Derangement

Example 10.13

A country has n universities. An exchange program takes one student and send him/her to another university, such that each university receives exactly one exchange students. Find # of ways to do this.

- Let $A_i = \{$ arrangement where student i is sent to his own uni $\}$
- $|A_i| = (n-1)!$
- $\left|A_{i_1} \cap A_{i_2} \cdots \cap A_{i_j}\right| = (n-j)!$
- Hence $\# = n! - \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} (n-j)!$
- $= \sum_{j=0}^{n} (-1)^j \frac{n!}{j!}$.

Euler's ϕ

Definition 10.14

For positive integers n, $\phi(n)$ counts the total number of positive integers that are co-prime to n and not greater than n.

- k, n co-prime means their greatest common divisor is 1. Denoted as $\gcd(k, n) = 1$
- $\phi(1) = 1$
- $\phi(2) = 1$, $\phi(3) = 2$,
- $\phi(4) = 2$, $\phi(5) = 4$, $\phi(6) = 2$
- $\phi(p) = p - 1$ for primes p.

Euler's ϕ

Theorem 10.15

Let $n = pq$ where p and q are distinct primes. Then $\phi(n) = (p-1)(q-1)$.

Proof:

- Consider $[n]$.
- There are p integers divisible by q, i.e. $q, 2q, 3q, \ldots, pq$
- There are q integers divisible by p, i.e. $p, 2p, 3p, \ldots, qp$
- Only $n = pq$ is divisible by both p and q.
- By inclusion-exclusion, there are $n - p - q + 1 = (p-1)(q-1)$ integers not divisible by either p or q.

Euler's ϕ

Theorem 10.16

Let $n = p_1 p_2 \cdots p_t$, where p_i are distinct primes, then

$$\phi(n) = \prod_{i=1}^{t} (p_i - 1).$$

- Proof: Let $A_i = \{k \leq n |\ k \text{ is divisible by } p_i\ \}$
- $|A_i| = \prod_{j \neq i} p_j$ and $|A_i \cap A_j| = \prod_{k \neq i, \neq j} p_k$
- $\left|A_{i_1} \cap A_{i_2} \cdots \cap A_{i_j}\right| = \prod_{k \neq i_1, \ldots i_j} p_k$
- Hence $\phi(n) = \sum_{j=0}^{t} (-1)^j \sum_{K \subseteq [t], |K| = j} \prod_{k \in K} p_k$

Euler's ϕ

Theorem 10.16

Let $n = p_1 p_2 \cdots p_t$, where p_i are distinct primes, then

$$\phi(n) = \prod_{i=1}^{t} (p_i - 1).$$

- Suppose want by to prove this by induction.
- Induction on n requires a relationship between $\phi(n)$ and $\phi(n-1)$
- Induction on t requires a relationship between $\phi(n)$ and $\phi(n/p_t)$
- We then need the next result.

Euler's ϕ

Theorem 10.17

Let m and n be two positive integers whose greatest common divisor is 1, then $\phi(mn) = \phi(m)\phi(n)$.

Example 10.18

$\phi(12) = \phi(3)\phi(4) = 4.$

Check that:

1	2	3	4
$4+1$	$4+2$	$4+3$	8
$8+1$	$8+2$	$8+3$	12

Euler's ϕ

Proof of Theorem 10.17

$$\begin{array}{cccccc}
1 & 2 & \cdots & r & \cdots & m \\
m+1 & m+2 & \cdots & m+r & \cdots & 2m \\
2m+1 & 2m+2 & \cdots & 2m+r & \cdots & 3m \\
\vdots & & & & & \\
(n-1)m+1 & (n-1)m+2 & \cdots & (n-1)m+r & \cdots & nm
\end{array}$$

- In each row, exactly $\phi(m)$ numbers co-prime to m.
- They occupy the same column since $\gcd(jm+r, m) = \gcd(r, m)$.
- Entries in the same column have distinct remainders mod n.
- Otherwise $im + r - (jm + r) = (i - j)m$ is divisible by n.
- In these columns exactly $\phi(n)$ numbers co-prime to n
- Total $= \phi(m)\phi(n)$.

Euler's ϕ

Theorem 10.19

Let $n = p_1^{d_1} p_2^{d_2} \cdots p_t^{d_t}$, where p_i are distinct primes, then

$$\phi(n) = \prod_{i=1}^{t} p_i^{d_i - 1}(p_i - 1).$$

Proof:

- $p, 2p, \ldots p^{d-1}p$ are not co-prime to p.
- $\phi(p^d) = p^d - p^{d-1} = p^{d-1}(p-1)$.
- The general result follows from induction and Theorem 10.17.

Generalized Inclusion-Exclusion

Definition 10.20

- *Let* $P_1, P_2, \ldots, P_q$ *be some properties possessed n distinct elements*
- *Let* $E(m) = \#$ *elements satisfying exactly m properties,* $0 \leq m \leq q$*.*
- *Let* $w(P_{i_1} \ldots P_{i_j})$ *denote the* $\#$ *of elements satisfying* $P_{i_1} \ldots P_{i_j}$
- *Let* $w(m) = \sum w(P_{i_1} \ldots P_{i_m})$ *over all possible m-sized subsets*
- *Define* $w(0) = n$*.*

Generalized Inclusion-Exclusion

Theorem 10.21

$$
\begin{aligned}
E(m) &= w(m) - \binom{m+1}{m} w(m+1) + \binom{m+2}{m} w(m+2) \\
&\quad - \cdots + (-1)^{q-m} \binom{q}{m} w(q) \\
&= \sum_{j=0}^{q-m} (-1)^j \binom{m+j}{m} w(m+j)
\end{aligned}
$$

Generalized Inclusion-Exclusion

Proof of Theorem 10.21

$$E(m) = \sum_{k=m}^{q} (-1)^{k-m} \binom{k}{m} w(k)$$

- Suppose an element possess t properties $t < m$, then it contributes 0 to both LHS and RHS.
- If $t = m$, contributes 1 to LHS. RHS: contributes 1 to $w(m)$ and 0 to $w(m+k)$, $k \geq 1$.
- If $t = m + r$, contributes 0 to LHS. RHS contributes 1 to $w(m)$, $\binom{t}{m}$ times, 1 to $w(m+1)$, $\binom{t}{m+1}$ times etc.

$$\text{RHS} = \sum_{j=0}^{q-m} (-1)^j \binom{m+j}{m} \binom{m+r}{m+j} = \sum_{j=0}^{r} (-1)^j \binom{m+r}{m} \binom{r}{r-j}$$

Example 10.22

Find the number of nonnegative integer solution to $x_1 + x_2 + x_3 = 15$, *where* $x_1 \leq 5, x_2 \leq 6, x_3 \leq 7$.

- Set $P_1 = x_1 \geq 6,\ P_2 = x_2 \geq 7,\ P_3 = x_3 \geq 8$
- $E(0) = w(0) - w(1) + w(2) - w(3)$
- $w(0) = n = \binom{15+3-1}{3-1} = \binom{17}{2}$
- $w(P_1)$ counts $= x_1 + x_2 + x_3 = 15$ with $x_1 \geq 6 \implies x_1 = 6 + y_1$
- Equivalently $|w(P_1)| = \binom{9+3-1}{3-1}$, i.e. counts $y_1 + x_2 + x_3 = 9$
- Thus $w(1) = \binom{9+2}{2} + \binom{8+2}{2} + \binom{7+2}{2}$
- $w(P_1, P_2) = \binom{2+2}{2}$, i.e. counts $y_1 + y_2 + x_3 = 15 - 6 - 7$
- Thus $w(2) = \binom{4}{2} + \binom{3}{2} + \binom{2}{2}$
- $w(3) = 0$.
- Hence $E(0) = \binom{17}{2} - \binom{11}{2} - \binom{10}{2} - \binom{9}{2} + \binom{4}{2} + \binom{3}{2} + \binom{2}{2} = 10.$

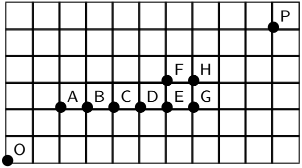
Example 10.22 Alternative solution

Find the number of nonnegative integer solution to $x_1 + x_2 + x_3 = 15$, where $x_1 \leq 5, x_2 \leq 6, x_3 \leq 7$.

- Set $t_1 = 5 - x_1,\ t_2 = 6 - x_2,\ t_3 = 7 - x_3$
- Hence $3 = t_1 + t_2 + t_3$ with $t_1 \leq 5, t_2 \leq 6, t_3 \leq 7$.
- Note constraints on t_i are always satisfied.
- Hence $\# = \binom{3+2}{2} = \binom{5}{2} = 10$

Example 10.23

Find the number of shortest NE lattice paths from O to P if
a) the sections AB, CD, EF, GH are all closed.
b) if each path must past through exactly two of the four roads above.



- p_1 to p_4 : Path passes AB, CD, EF, GH respectively.

Solution

- $w(0) = \binom{15}{5}$
- $w(1) = w(p_1) + w(p_2) + w(p_3) + w(p_4)$
- $\quad = \binom{4}{2}\binom{10}{3} + \binom{6}{2}\binom{8}{3} + \binom{8}{2}\binom{6}{2} + \binom{9}{2}\binom{5}{2}$
- $w(2) =$ $w(p_1p_2) + w(p_1p_3) + w(p_1p_4) + w(p_2p_3) + w(p_2p_4) + w(p_3p_4)$
- $\quad = \binom{4}{2}\binom{8}{3} + \binom{4}{2}\binom{6}{2} + \binom{4}{2}\binom{5}{2} + \binom{6}{2}\binom{6}{2} + \binom{6}{2}\binom{5}{2} + 0$
- $w(3) = w(p_1p_2p_3) + w(p_1p_2p_4) + w(p_1p_3p_4) + w(p_2p_3p_4)$
- $\quad = \binom{4}{2}\binom{6}{2} + \binom{4}{2}\binom{5}{2} + 0 + 0$
- $w(4) = w(p_1p_2p_3p_4) = 0$
- a) $E(0) = w(0) - w(1) + w(2) - w(3) + w(4).$
- b) $E(2) = w(2) - \binom{3}{2}w(3) + \binom{4}{2}w(4).$