Chapter 9. Line Integrals

9.1 Introduction

9.1.1 Work Done I

(i) Let $\mathbf{F}$ be a constant force acting on a particle in the displacement direction as shown in figure (i) above. Suppose the distance moved by the particle is s. The work done is given by

$$W = \|\mathbf{F}\| \times s.$$

(ii) Let $\mathbf{F}$ be a constant force acting on a particle in the direction which form an angle θ against the displacement direction (see figure (ii) above).

Suppose the distance moved by the particle is s.

The work done is given by

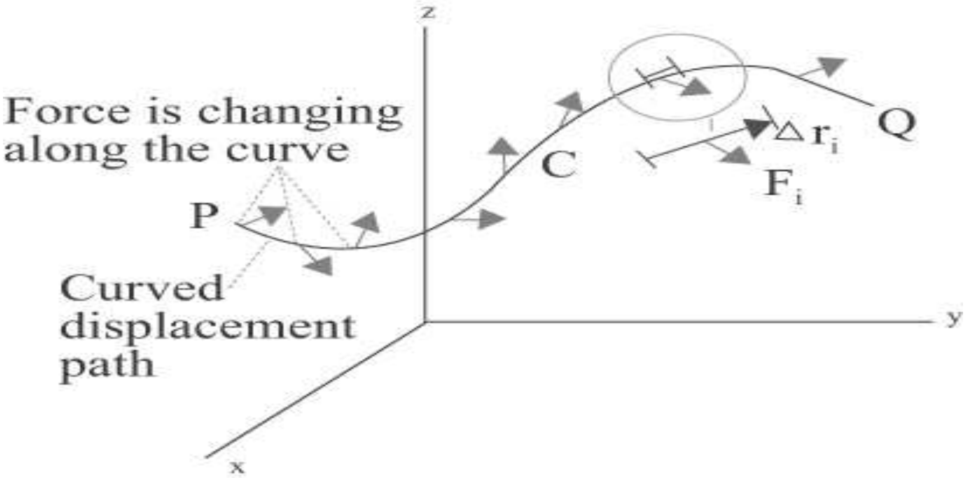
$$W = \|\mathbf{F}\| \cos\theta \times s = (\mathbf{F} \cdot \mathbf{T}) \times s = \mathbf{F} \cdot s\mathbf{T}$$

where $\mathbf{T}$ is the unit vector in the displacement direction.

9.1.2 Work Done II

Let $\mathbf{F}(x, y, z)$ be a variable force acting on a particle which moves along the curve C with vector equation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ as shown in the figure below. Suppose the particle moves from point P to point Q. What is the work done?

Solution: To solve this problem, we divide the



curve C into n segments. If a segment i is small enough, it can be treated as a straight line segment and the force within which can be assumed to be constant $\mathbf{F}_i$. Then the work done for such a segment is approximately given by

$$W_i \approx \mathbf{F}_i \cdot \Delta\mathbf{r}_i$$

where $\mathbf{r}_i = s\mathbf{T}_i$ and $\mathbf{T}_i$ is the unit tangent vector along this segment.

So the total work done is approximately

$$W_{\text{total}} \approx \sum_{1}^{n} \mathbf{F}_i \cdot \Delta \mathbf{r}_i.$$

As $n \to \infty$, we write this as

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

which gives the actual total work done.

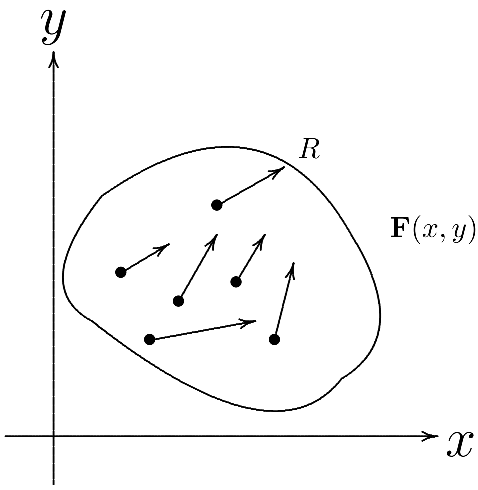
9.1.3 Vector Fields

The vector function $\mathbf{F}$ is called in general a vector field and the above integral is called the line integral of $\mathbf{F}$ along the curve C. We shall see in section 9.3.7 how to evaluate this type of integral.

9.2 Vector Fields

9.2.1 Vector field (two variables)

Let R be a region in xy-plane. A **vector field** on R is a vector function $\mathbf{F}$ that assigns to each point (x, y) in R a two-dimensional vector $\mathbf{F}(x, y)$.



We may write $\mathbf{F}(x, y)$ in terms of its component functions. That is

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$$

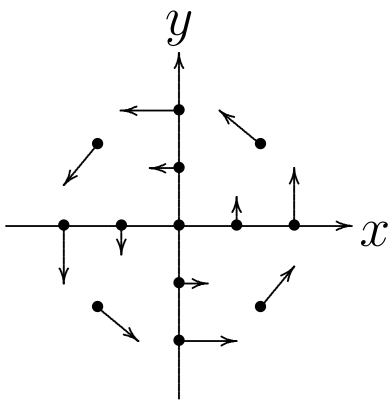
or simply $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$.

9.2.2 Vector field (three variables)

Let D be a solid region in xyz-space. A **vector field** on D is a vector function $\mathbf{F}$ that assigns to each point (x, y, z) in D a three-dimensional vector $\mathbf{F}(x, y, z)$. That is, $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$.

9.2.3 Example

A vector field in xy-plane is defined by $\mathbf{F}(x, y) = (-y)\mathbf{i} + x\mathbf{j}$. Show that $\mathbf{F}(x, y)$ is always perpendicular to the position vector of the point (x, y).



$F(x, y) = (-y)\mathbf{i} + x\mathbf{j}$

The diagram above shows the vector field $\mathbf{F}$.

9.2.4 Gradient fields

If $f(x, y)$ is a function of two variables, then

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$$

is a vector field in the xy-plane and it is called the **gradient (field)** of f.

Similarly, if $f(x, y, z)$ is a function of three variables, then

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}$$

is a vector field in the xyz-space and it is called the *gradient (field)* of f.

9.2.5 Example

The gradient field of $f(x, y) = xy^2 + x^3$ is

$$\nabla f(x, y) = (y^2 + 3x^2)\mathbf{i} + (2xy)\mathbf{j}.$$

9.2.6 Conservative fields

A vector field $\mathbf{F}$ is called a **conservative** vector field if it is the gradient of some (scalar) function. In other words, there is a function f such that $\mathbf{F} = \nabla f$. In this case, f is called a **potential** function for $\mathbf{F}$.

9.2.7 Example

By Example 9.2.5, $\mathbf{F}(x, y) = (y^2 + 3x^2)\mathbf{i} + (2xy)\mathbf{j}$ is conservative since it has a potential function $f(x, y) = xy^2 + x^3$.

9.2.8 Example

Let $\mathbf{F}(x, y) = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$. Find a potential function f for $\mathbf{F}$.

Solution: As $\nabla f = \mathbf{F}$, we have $f_x(x, y) = 3+2xy$.

Integrating with respect to x, we get $f(x, y) = 3x + x^2y+g(y)$, where $g(y)$ is an integration constant, but it could be a function of y.

Thus $f_y(x, y) = x^2 + g'(y)$ so that $x^2 + g'(y) = x^2 - 3y^2$. That is, $g'(y) = -3y^2$.

Integrating $g'(y)$ with respect to y, we obtain $g(y) = -y^3 + K$, where K is a constant.

Consequently, $f(x, y) = 3x + x^2y - y^3 + K$.

9.2.9 Example

The gravitational field given by

$$\mathbf{G} = \left(\frac{-m_1 m_2 K x}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right)\mathbf{i} + \left(\frac{-m_1 m_2 K y}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right)\mathbf{j} + \left(\frac{-m_1 m_2 K z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right)\mathbf{k}$$

is conservative because it is the gradient of the gravitational potential function

$$g(x,y,z) = \frac{m_1 m_2 K}{\sqrt{x^2+y^2+z^2}},$$

where K is the gravitational constant, m_1 and m_2 are the masses of two objects. Think of the mass m_1 at the origin that creates the field and g is the potential energy attained by the mass m_2 situated at (x, y, z).

9.2.10 Criteria of conservative fields

Throughout this chapter, we will assume the component functions of any vector field to have continuous partial derivatives, unless otherwise stated.

(a) Let $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ be a vector field on the xy-plane.

$$\text{If } \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \text{then } \mathbf{F} \text{ is conservative.}$$

(b) Let $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ be a vector field on the xyz-space.

$$\text{If } \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y},$$

then $\mathbf{F}$ is conservative.

The converse of (a) and (b) also hold.

9.2.11 Example

Consider the vector field

$$\mathbf{F}(x, y) = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}.$$

As $\dfrac{\partial(x^2 - 3y^2)}{\partial x} = 2x = \dfrac{\partial(3 + 2xy)}{\partial y}$,

$\mathbf{F}$ is conservative.

9.2.12 Example

Show that $\mathbf{F}(x, y, z) = xz\mathbf{i} + xyz\mathbf{j} - y^2\mathbf{k}$ is not conservative.

Solution: For example, $\dfrac{\partial(xyz)}{\partial x} = yz$ which is not equal to $\dfrac{\partial(xz)}{\partial y} = 0$.

So $\mathbf{F}$ is not conservative.

9.2.13 Exercise

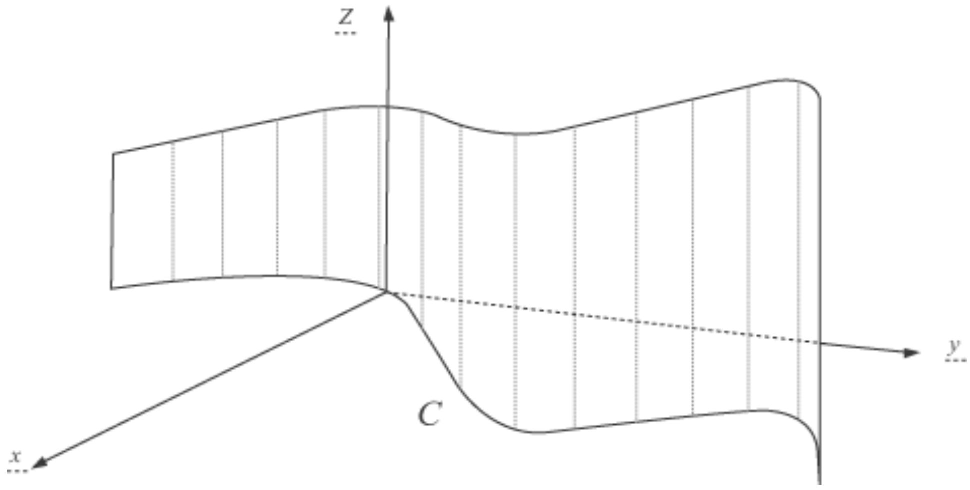
Show that the vector field $\mathbf{F}(x, y, z) = x^2\mathbf{i}+y^2\mathbf{j}+z^2\mathbf{k}$ is conservative. Find a function f such that $\nabla f = \mathbf{F}$.

9.3 Line Integrals

We have mentioned in section 9.1 that a line integral refers to an integration along a curve C. There are two types of line integrals. One is for vector fields and the other is for scalar functions.

9.3.1 Line integrals of scalar functions (Two variables)

Suppose we want to find the area of the following surface with the base, a plane curve C on the xy-plane and the top is described by a function $f(x, y)$.



Let C be described by the vector function $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ for $a \le t \le b$. We assume C is a smooth curve (meaning that $\mathbf{r}'(t) \neq 0$) and $\mathbf{r}'(t)$ is continuous for all t.

To find the surface area along C, we subdivide the curve from $\mathbf{r}(a)$ to $\mathbf{r}(b)$ into n small arcs of length $\triangle s_i$, $i = 1, \cdots n$. Pick an arbitrary point (x_i^*, y_j^*) inside the ith small arc and form the sum $\sum_{i=1}^{n} f(x_i^*, y_j^*)\triangle s_i$.

The surface area is given by

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_j^*)\triangle s_i.$$

The above limit of sum is called the **line integral of the scalar function** f along the plane curve C, and is denoted by

$$\int_C f(x, y)\, ds$$

Here, s denotes the arc length of C.

Recall from Chapter 6 that, if $\mathbf{r}(t)$ is the vector function of a curve C with $a \leq t \leq b$, the arc length of C is given by

$$s = \int_a^b \|\mathbf{r}'(t)\| dt.$$

If we replace the endpoint b by a variable t, then we obtain the arc length $s(t)$ as a function of t:

$$s(t) = \int_a^t \|\mathbf{r}'(u)\| du.$$

Then, by Fundamental Theorem of Calculus, $\dfrac{ds}{dt} = \|\mathbf{r}'(t)\|$.

Therefore, we can rewrite the line integral in terms of t:

$$\begin{aligned}\int_C f(x,y)\,ds &= \int_a^b f(x(t),y(t))\|\mathbf{r}'(t)\|dt \\ &= \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt\end{aligned}$$

where $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ is the vector equation of the plane curve C.

In other words, the formula above allows us to compute the line integral in terms of ordinary integration of single variable function (in t).

9.3.2 Example

Evaluate $\int_C (2y + x^2y)ds$, where C is the upper half of the unit circle centered at the origin.

Solution: The vector function of C is given by $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$ with $0 \le t \le \pi$.

Thus $\|\mathbf{r}'(t)\| = \sqrt{\sin^2 t + \cos^2 t}$ and

$$\begin{aligned}
\int_C (2y + x^2y)ds &= \int_0^\pi (2\sin t + \cos^2 t \sin t)\sqrt{\sin^2 t + \cos^2 t} dt \\
&= \int_0^\pi (2\sin t + \cos^2 t \sin t)\, dt \\
&= \left[-2\cos t - \frac{1}{3}\cos^3 t\right]_0^\pi \\
&= \frac{14}{3}
\end{aligned}$$

9.3.3 Line integrals of scalar functions (Three variables)

For line integral of a function $f(x, y, z)$ along a space curve C, we have the similar definitions:

$$\int_C f(x, y, z)\, ds =$$

$$\int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

9.3.4 Example

Evaluate $\displaystyle\int_C xy \sin z\, ds$, where C is the circular helix $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$, $t \in [0, \pi/2]$.

Solution:

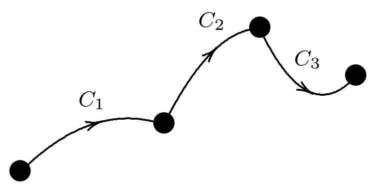
$$\begin{aligned}
&\int_C xy\sin z\,ds \\
=&\int_0^{\pi/2} (\cos t)(\sin t)(\sin t)\sqrt{\sin^2 t+\cos^2 t+1}\,dt \\
=&\sqrt{2}\int_0^{\pi/2} \cos t\sin^2 t dt \\
=&\frac{\sqrt{2}}{3}\left[\sin^3 t\right]_0^{\pi/2}=\frac{\sqrt{2}}{3}
\end{aligned}$$

9.3.5 Piecewise smooth curves

We denote the union of a finite number of (smooth) curves $C_1, C_2, \cdots, C_n$ by

$$C = C_1 + C_2 + \cdots + C_n.$$

We say C is a *piecewise-smooth* curve.



Then the line integral f along C is defined to be

$$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \cdots + \int_{C_n} f(x,y)\,ds.$$

9.3.6 Example

Evaluate $\displaystyle\int_C 9y\,ds$, where C consists of the arc C_1 of the cubic $y = x^3$ from $(0,0)$ to $(1,1)$ followed by the vertical line segment C_2 from $(1,1)$ to $(1,5)$.

Solution: We first obtain the vector function for C_1 and C_2:

For C_1, the Cartesian equation is $y = x^3$. So we may let $x = t$ and get $y = t^3$ with $0 \le t \le 1$. Hence $\mathbf{r}_1(t) = t\mathbf{i} + t^3\mathbf{j}$ and $\|\mathbf{r}'{}_1(t)\| = \sqrt{1 + (3t^2)^2}$

and

$$\int_{C_1} 9y ds = \int_0^1 9t^3\sqrt{1+9t^4}\,dt$$
$$= \frac{1}{6}\left[(1+9t^4)^{3/2}\right]_0^1 = \frac{1}{6}(10\sqrt{10}-1).$$

For C_2, since it is a vertical line that passes through $x = 1$, we have $x = 1$ and $y = t$ with $1 \le t \le 5$. Hence $\mathbf{r}_2(t) = \mathbf{i} + t\mathbf{j}$ and $\|\mathbf{r}'{}_2(t)\| = \sqrt{0+1}$.

$$\int_{C_2} 9y ds = \int_1^5 9t\,dt = 108.$$

So

$$\int_C 9y\,ds = \int_{C_1} 9y\,ds + \int_{C_2} 9y\,ds$$
$$= \frac{1}{6}(10\sqrt{10}-1) + 108 = \frac{1}{6}(10\sqrt{10}+647).$$

9.3.7 Line integrals of vector fields

Let $\mathbf{F}$ be a continuous (2 or 3 variable) vector field defined on a domain containing a smooth curve C given by a vector function $\mathbf{r}(t)$, $t \in [a, b]$. The **line integral of the vector field F** along the curve C is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt.$$

If $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, then

$$\mathbf{F}(\mathbf{r}(t)) = \mathbf{F}(x(t), y(t), z(t))$$

refers to the vector field along the curve.

Geometrically, the line integral of $\mathbf{F}$ over C is summing up the tangential component of $\mathbf{F}$ with respect to the arc length of C.

9.3.8 Example

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F}(x, y, z) = x\mathbf{i} + xy\mathbf{j} + xyz\mathbf{k}$$

and C is the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, $t \in [0, 2]$.

Solution: First $\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}$. Thus

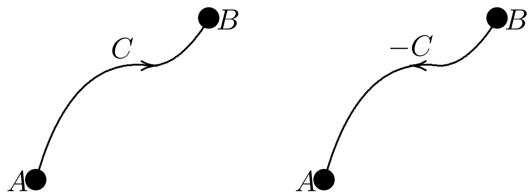
$$\begin{aligned} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) &= (t\mathbf{i} + t \cdot t^2\mathbf{j} + t \cdot t^2 \cdot t^3\mathbf{k}) \cdot (\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}) \\ &= t + 2t^4 + 3t^8. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^2 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\ &= \int_0^2 (t + 2t^4 + 3t^8)\, dt = 2782/15. \end{aligned}$$

9.3.9 Orientation of curves

The vector equation of a curve C determines an **orientation** (direction) of C. The same curve with the opposite orientation of C is denoted by $-C$.



We have

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_{C} \mathbf{F} \cdot d\mathbf{r}$$

as $\mathbf{r}'(t)$ changes sign in $-C$.

On the other hand, for line integral of scalar functions,

$$\int_{-C} f(x, y, z)\, ds = \int_{C} f(x, y, z)\, ds$$

since the arc length is always positive.

9.3.10 Line integrals in component form

Suppose

$$\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$$

and $C : \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, \quad t \in [a,b].$

Then we may write the line integral as

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P dx + Q dy.$$

Indeed

$$\begin{aligned}
&\int_C \mathbf{F} \cdot d\mathbf{r} \\
&= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\
&= \int_a^b [P(\mathbf{r}(t))\mathbf{i} + Q(\mathbf{r}(t))\mathbf{j}] \cdot \left[\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}\right] dt \\
&= \int_a^b \left[P(\mathbf{r}(t))\frac{dx}{dt} + Q(\mathbf{r}(t))\frac{dy}{dt}\right] dt \\
&= \int_C P dx + Q dy.
\end{aligned}$$

Similarly, for three variable vector field

$\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$,

we can write the line integral as

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C Pdx + Qdy + Rdz.$$

9.3.11 Example

Evaluate the line integral $\int_C y^2 dx + x dy$, where

(a) $C = C_1$ is the line segment from $(-5, -3)$ to $(0, 2)$,

(b) $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(0, 2)$.

Solution:

(a) C_1 is a line passing through the point $(-5, -3)$ and parallel to the vector $(2\mathbf{j}) - (-5\mathbf{i} - 3\mathbf{j}) = 5\mathbf{i} + 5\mathbf{j}$.

So the vector function of C_1 is given by

$\mathbf{r}(t) = (-5\mathbf{i} - 3\mathbf{j}) + t(5\mathbf{i} + 5\mathbf{j}) = (5t-5)\mathbf{i} + (5t-3)\mathbf{j}$

with $0 \le t \le 1$. Thus,

$$\begin{aligned}
\int_{C_1} y^2 dx + x dy &= \int_0^1 (5t-3)^2 \frac{dx}{dt} dt + \int_0^1 (5t-5) \frac{dy}{dt} dt \\
&= \int_0^1 (5t-3)^2 5 dt + \int_0^1 (5t-5) 5 dt \\
&= -5/6.
\end{aligned}$$

(b) By setting $y = t$, we have the vector function of

C_2 given by

$\mathbf{r}(t) = (4 - t^2)\mathbf{i} + t\mathbf{j}$ with $-3 \le t \le 2$. Thus

$$\begin{aligned}
\int_{C_2} y^2 dx + x dy &= \int_{-3}^2 t^2 \frac{dx}{dt} dt + \int_{-3}^2 (4-t^2) \frac{dy}{dt} dt \\
&= \int_{-3}^2 t^2(-2t) dt + \int_{-3}^2 (4-t^2) dt \\
&= 245/6.
\end{aligned}$$

9.3.12 The fundamental theorem for line integrals

Recall the fundamental theorem for Calculus:

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

It has the following generalization in terms of line integrals:

Let C be a smooth curve with vector function $\mathbf{r}(t)$, $t \in [a, b]$.

If f is a function of 2 or 3 variables whose gradient ∇f is continuous. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

9.3.13 Example

Find the work done by the (earth) gravitational field (see Example 9.2.9) in moving a particle of mass m from the point $(3, 4, 12)$ to the point $(1, 0, 0)$ along a curve C.

Solution:

$$W \equiv \int_C \mathbf{G} \cdot d\mathbf{r} = \int_C \nabla g \cdot d\mathbf{r} = g(1, 0, 0) - g(3, 4, 12).$$

Since the potential function $g(x, y, z) = \dfrac{mMK}{\sqrt{x^2 + y^2 + z^2}}$ where M is the mass of the earth and K the gravitational constant, we have $W = 12mMK/13$.

9.3.14 Consequences of conservative fields

(I) If $\mathbf{F}$ is a conservative vector field, then $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$ is *independent of path*,

i.e. $\displaystyle\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$ for any 2 paths C_1 and C_2 that have the same initial and terminal points.

(II) If $\mathbf{F}$ is a conservative vector field, then $\displaystyle\oint_\ell \mathbf{F}\cdot d\mathbf{r} = 0$ for any *closed* curve ℓ (i.e. a curve with terminal point coincides with its initial point).

Notation: If a curve ℓ is closed, we write the line integral as

$$\oint_\ell \mathbf{F}\cdot d\mathbf{r}.$$

9.3.15 Example

Let $\mathbf{F}(x,y) = (y^2+3x^2)\mathbf{i}+(2xy)\mathbf{j}$. Show that the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path and evaluate this integral over the curve C where C is

(i) given by $\mathbf{r}(t) = \cos t\mathbf{i} + e^t \sin t\mathbf{j}$, $t \in [0,\pi]$;

(ii) the unit circle.

Solution: We have seen in Example 9.2.5 that $\nabla f = \mathbf{F}$ where $f(x,y) = xy^2 + x^3$ is the potential function of $\mathbf{F}$. So $\mathbf{F}$ is conservative. By section 9.3.14 (I), the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path.

(i) The initial point of C is given $\mathbf{r}(0) = \mathbf{i}$ which corresponds to the coordinates $(1,0)$; and the terminal point is given $\mathbf{r}(\pi) = -\mathbf{i}$ which corresponds to the coordinates $(-1,0)$. Since $\mathbf{F} = \nabla f$, by Fundamental Theorem, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(-1,0) - f(1,0) = -2.$$

(ii) Since the unit circle is a closed path and $\mathbf{F}$ is conservative, by section 9.3.14 (II),

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

9.4 Green's Theorem

Let D be a bounded region in the xy-plane and ∂D the boundary of D. Suppose $P(x,y)$ and $Q(x,y)$ has continuous partial derivatives on D. Then

$$\oint_{\partial D} P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

The orientation of ∂D is such that, as one traverses along the boundary in this direction, the region D is always on the left hand side. We call this the **positive orientation** of the boundary.

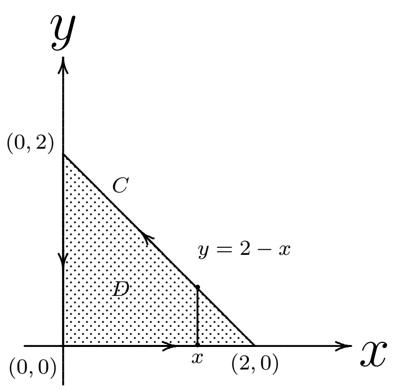
9.4.1 Example

Evaluate $\displaystyle\oint_C 2xy\, dx + xy^2 dy$, where C is the triangular curve consisting of the line segments from $(0,0)$ to $(2,0)$, from $(2,0)$ to $(0,2)$ and from $(0,2)$ to $(0,0)$.

Solution: The functions

$P(x, y) = 2xy$ and $Q(x, y) = xy^2$

have continuous partial derivatives on the xy-plane.



The region D is given by: $0 \le y \le 2-x,\ 0 \le x \le 2$.

By Green's Theorem,

$$\begin{aligned}\oint_C 2xy\ dx + xy^2 dy &= \iint_D \left[\frac{\partial(xy^2)}{\partial x} - \frac{\partial 2xy}{\partial y}\right] dA \\ &= \iint_D (y^2 - 2x)\, dydx \\ &= \int_0^2 \int_0^{2-x} (y^2 - 2x)\, dydx \\ &= -\frac{4}{3}.\end{aligned}$$

9.4.2 Example

Evaluate $\oint_C (4y - e^{x^2})dx + (9x + \sin(y^2 - 1))dy$, where C is the circle $x^2 + y^2 = 4$.

Solution: C bounds the circular disk D of radius 2 and is given the positive orientation.

By Green's Theorem,

$$\oint_C (4y - e^{x^2})dx + (9x + \sin y^2 - 1)dy$$
$$= \iint_D \left[\frac{\partial(9x + \sin y^2 - 1)}{\partial x} - \frac{\partial(4y - e^{x^2})}{\partial y} \right] dA$$
$$= \iint_D 5\, dA = 5 \iint_D dA$$
$$= 5 \times (\text{area of } D) = 5(\pi 2^2) = 20\pi.$$

9.4.3 Exercise

Evaluate by Green's Theorem

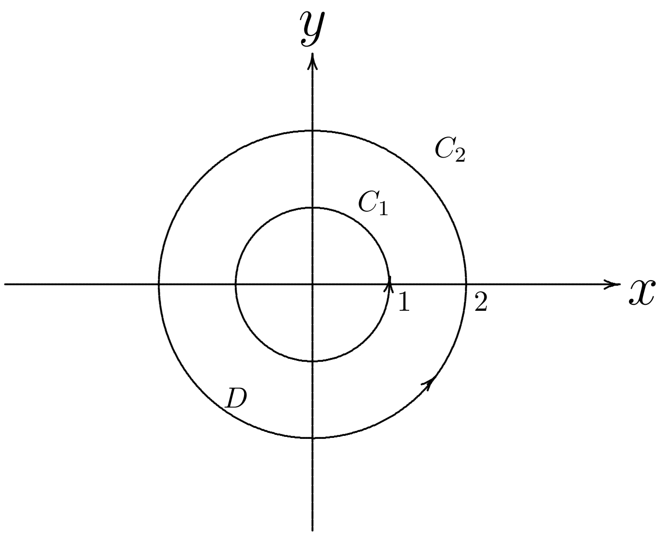
$$\oint_C e^{-x} \sin y\, dx + e^{-x} \cos y\, dy$$

where C is the rectangle with vertices at $(0,0)$, $(\pi, 0)$, $(\pi, \pi/2)$, $(0, \pi/2)$.

[Answer: $2(e^{-\pi} - 1)$]

9.4.4 Example

Let $\mathbf{F}(x, y) = y\mathbf{i} + y\mathbf{j}$ and D a region in xy-plane bounded by the two circles centered at the origin with radius 1 and 2.



Verify Green's Theorem.

Solution:

(i) Compute $\displaystyle\int_{\partial D} \mathbf{F} \cdot d\mathbf{r}$ directly:

The boundary of D is made up of two disjoint curves C_1 and C_2.

Now $C_1: \quad \mathbf{r}_1 = \cos t\mathbf{i} + \sin t\mathbf{j}$ and $C_2: \quad \mathbf{r}_2 = 2\cos t\mathbf{i} + 2\sin t\mathbf{j}$ with $t \in [0, 2\pi]$. Note that the equations give counterclockwise orientation to both curves.

However, to get positive orientation for the boundary of D, the outer boundary should traverse counterclockwise while the inner boundary should traverse clockwise.

Hence $\partial D = C_2 - C_1$.

$$
\begin{aligned}
\int_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (\sin t\mathbf{i} + \sin t\mathbf{j}) \cdot (-\sin t\mathbf{i} + \cos t\mathbf{j})\, dt \\
&= \int_0^{2\pi} (-\sin^2 t + \sin t \cos t)\, dt \\
&= \int_0^{2\pi} \frac{1}{2}(\cos 2t - 1 + \sin 2t)\, dt \\
&= \frac{1}{2}\left[\frac{\sin 2t}{2} - t - \frac{\cos 2t}{2}\right]_0^{2\pi} = -\pi
\end{aligned}
$$

Similarly, $\displaystyle\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = -4\pi.$

So $\displaystyle\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r} - \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = -3\pi.$

(ii) Using Green's Theorem, we have

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial y}{\partial x} - \frac{\partial y}{\partial y} \right) dA = \iint_D (-1)\, dA.$$

In polar coordinates, D is given by

$$1 \leq r \leq 2, \quad 0 \leq \theta \leq 2\pi.$$

So we have

$$\iint_D (-1)\, dA = \int_0^{2\pi} \int_1^2 -r \; dr \; d\theta = -3\pi.$$