MA1506

Mathematics II

Chapter 2

Oscillations

Overview (LT notes vs Textbook)

Chpt 1 : 1^{st} + 2^{nd} Order (quantitative)

Chpt 2 : Harmonic oscillators (qualitative)

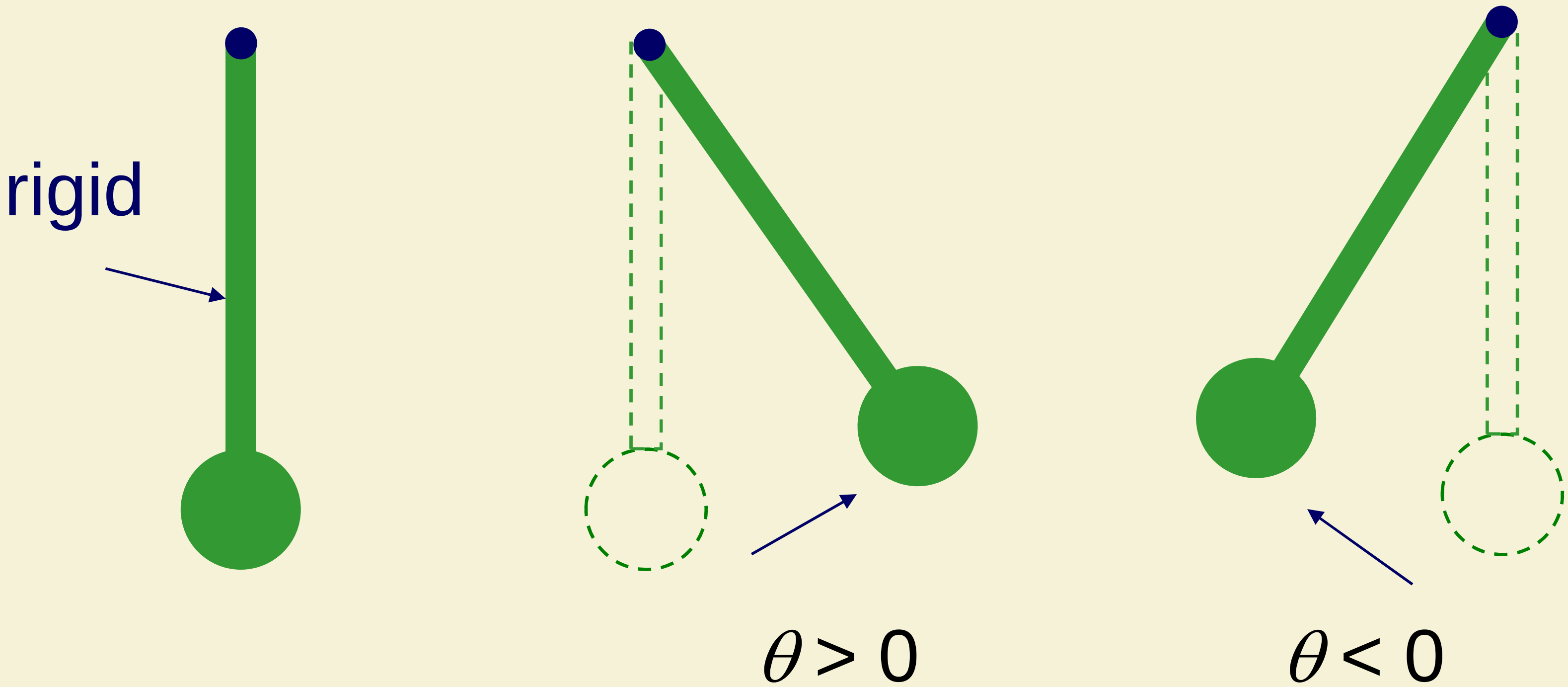
Chpt 3 : Math modelling

<u>Farlow et al.</u>

Chpt 1-2: 1^{st} order + modelling

Chpt 4: Harmonic oscillator + 2^{nd} order

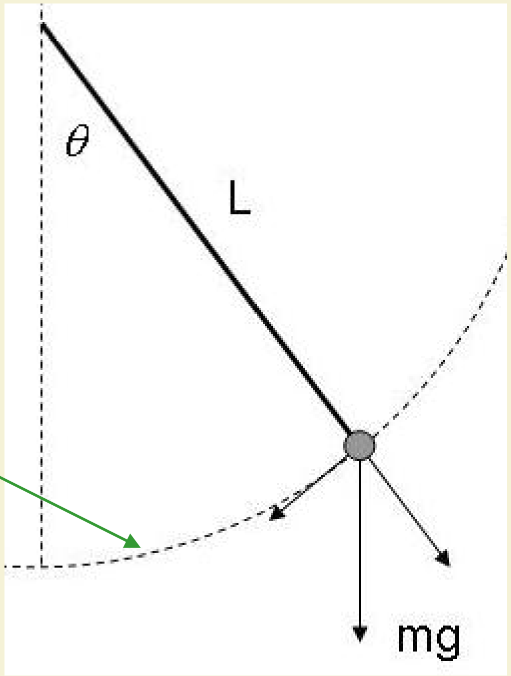
2.1 Pendulum as example of harmonic oscillator



- Consider angular displacement
- What is the range of θ?

2.1 Harmonic Oscillator

Arc length = $L\theta$



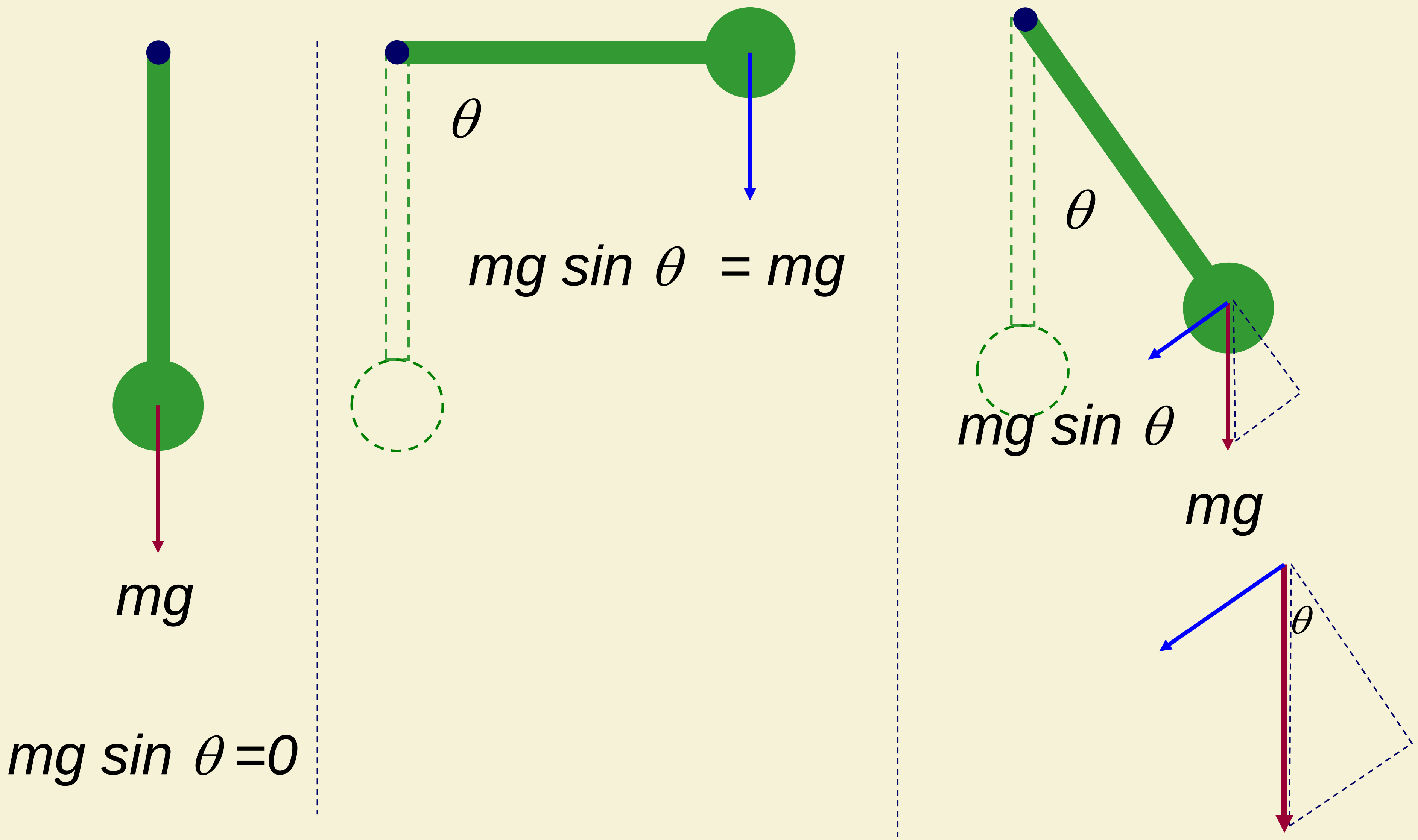
$\theta > 0$

$$\dot{\theta} = \frac{d\theta}{dt}$$

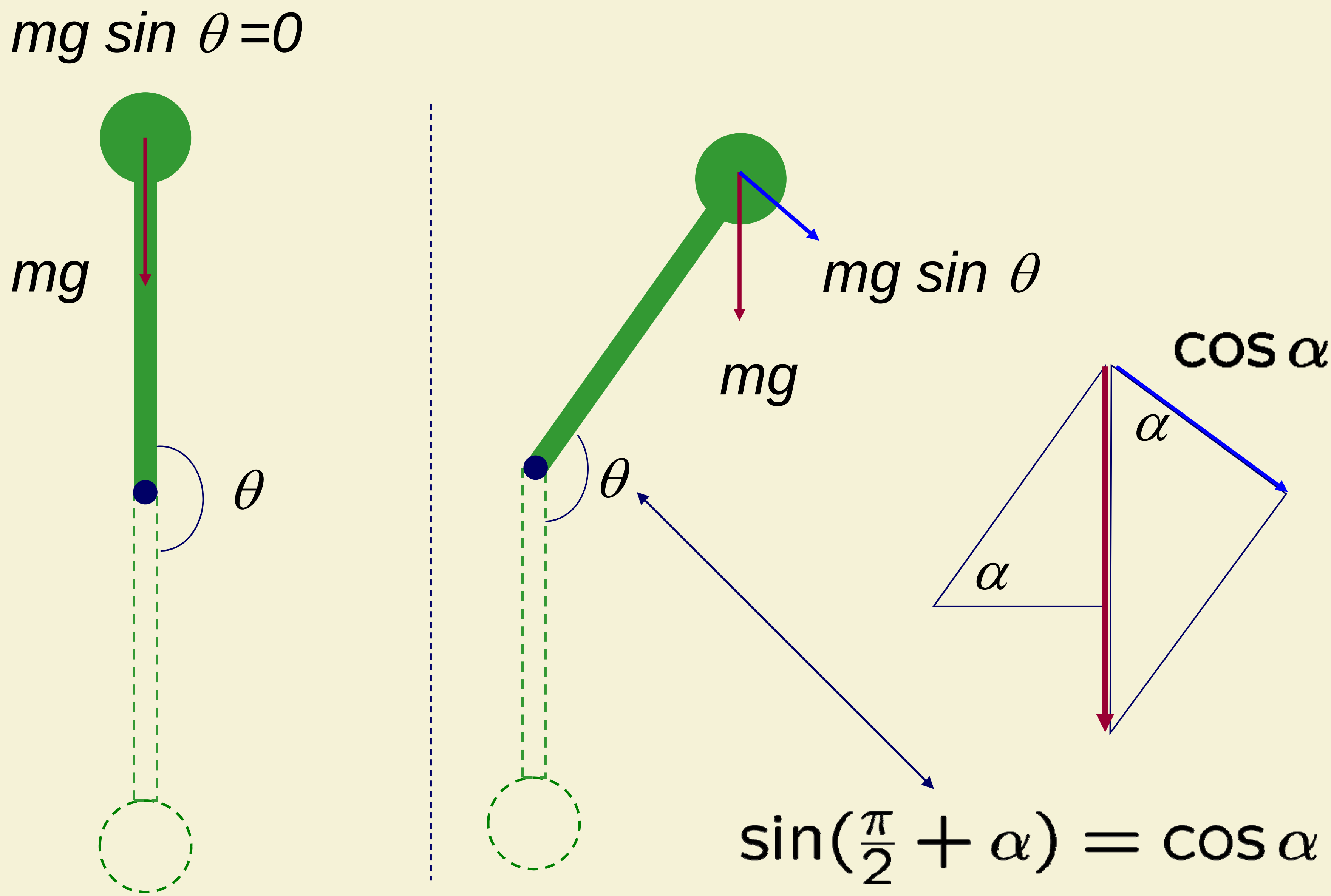
$$\ddot{\theta} = \frac{d^2\theta}{dt^2}$$

$$\boxed{mL\ddot{\theta} = -mg\sin\theta}$$

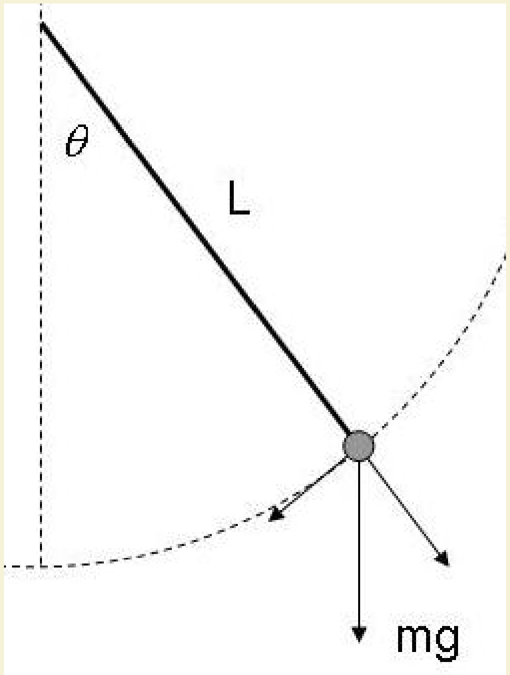
What is the acceleration in each case?



What is the acceleration in each case?



Remark



$$\dot{\theta} = \frac{d\theta}{dt} \qquad \ddot{\theta} = \frac{d^2\theta}{dt^2}$$

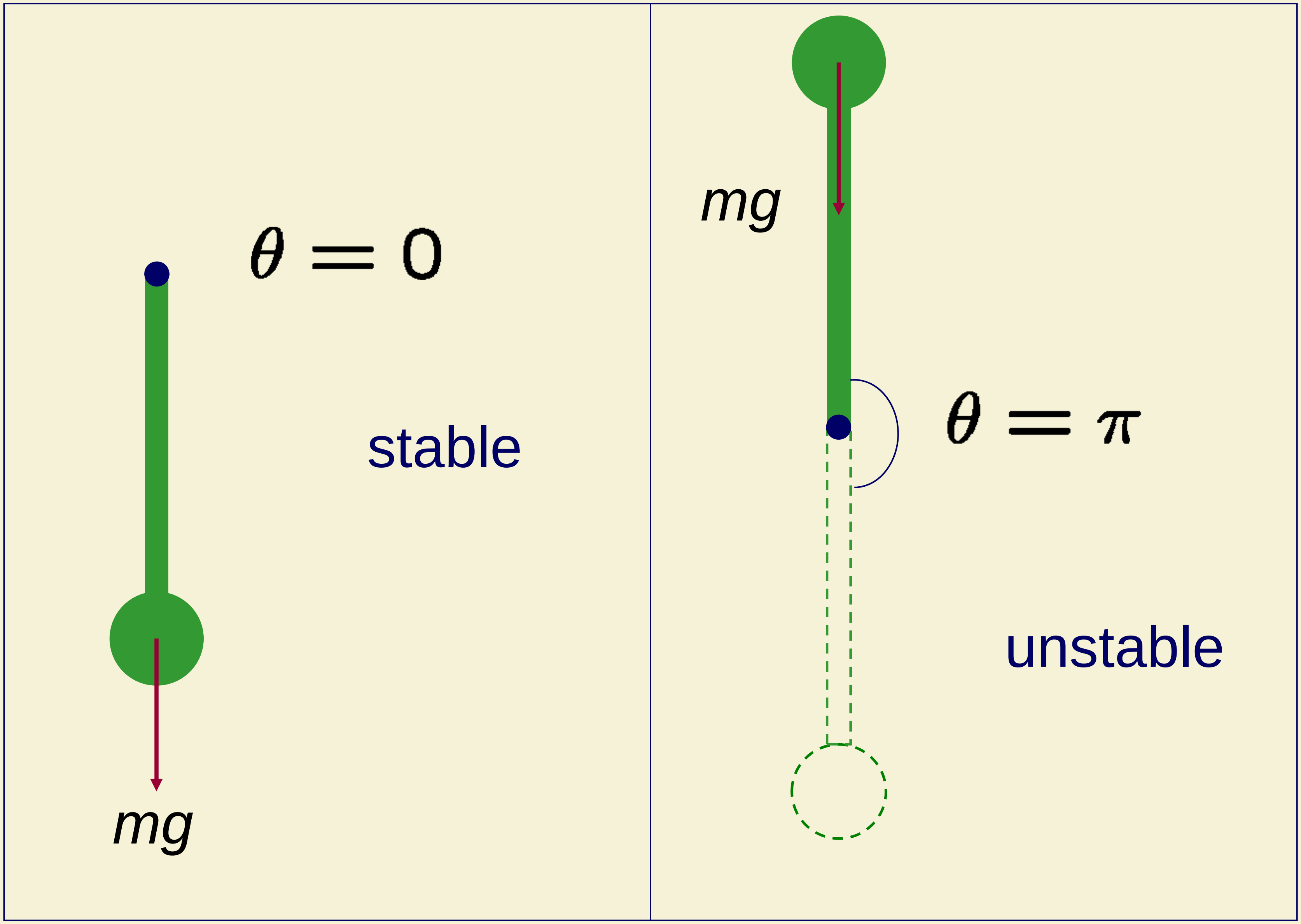
$$\boxed{mL\ddot{\theta} = -mg\sin\theta}$$

Non linear 2nd Order Hom. d.e.

$$\theta \mapsto y$$
$$t \mapsto x$$

$$mL\frac{d^2y}{dx^2} = -mg\sin y$$

Equilibrium solutions i.e. does not change over time



Non linear

2.1 Unstable Case

$$mL\ddot{\theta} = -mg\sin\theta$$

By Taylor's Theorem at $\theta = \pi$

$$f(\theta) = f(\pi) + f'(\pi)(\theta - \pi) + \frac{1}{2}f''(\pi)(\theta - \pi)^2 + \ldots$$

$$\sin(\theta) = 0 - (\theta - \pi) - 0 + \frac{1}{6}(\theta - \pi)^3 + \ldots$$

$$\sin(\theta) \approx -(\theta - \pi)$$

$$\boxed{mL\ddot{\theta} = -mg\sin\theta \approx mg(\theta - \pi)}$$

2.1 Unstable case

$$mL\ddot{\theta} = mg(\theta - \pi)$$

Let $\phi = \theta - \pi$ $\Rightarrow$ $\ddot{\phi} = \ddot{\theta}$

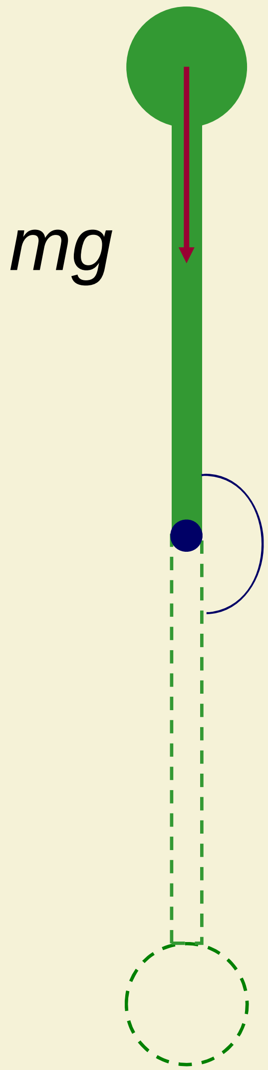
$\boxed{\text{Instability: } \ddot{\phi} = \frac{g}{L}\phi}$

$\lambda^2 = \frac{g}{L}$ Aux Eq

$$\phi = Ae^{(\sqrt{g/L})t} + Be^{-(\sqrt{g/L})t}$$

$$\theta = \phi + \pi = \underbrace{Ae^{(\sqrt{g/L})t}}_{\text{big}} + Be^{-(\sqrt{g/L})t} + \pi$$

Justification that A is usually nonzero



$$\theta = \phi + \pi = \underbrace{Ae^{(\sqrt{g/L})t}}_{\text{big}} + Be^{-(\sqrt{g/L})t} + \pi$$

Small displacement from equilibrium

$$\theta(0) = \pi + d \Rightarrow A + B = d$$

$$\theta'(0) = 0 \Rightarrow A\sqrt{g/L} - B\sqrt{g/L} = 0$$

$$\Rightarrow A = B$$

Example:

An eccentric professor likes to balance pendula near their unstable equilibrium point. In a given performance, the pendulum is initially slightly away from that point, and is initially at rest. The prof's skill is such that <u>he can stop the pendulum from falling provided that the angular deviation from the vertical angle does not double.</u> If the shortest pendulum for which he can perform this trick is 9.8 centimetres long, estimate the speed of his reflexes.

$$\phi(0) = d$$

$$\dot{\phi}(0) = 0$$

$$\Rightarrow \quad \phi(t) = \frac{d}{2}\left(e^{(\sqrt{g/L})t} + e^{(-\sqrt{g/L})t}\right)$$

$$= d\cosh((\sqrt{g/L})t) = d\cosh(10t)$$

$$\Rightarrow \quad 2d = d\cosh(10t) \Rightarrow t = \cosh^{-1}(2)/10 \approx 0.132$$

2.1 Stable Case

$$mL\ddot{\theta} = -mg\sin\theta$$

By Taylor's Theorem at $\theta = 0$

$$\sin(\theta) = 0 + \theta - 0 - \frac{1}{6}(\theta)^3 + \ldots$$

$$\boxed{mL\ddot{\theta} = -mg\theta}$$

$\ddot{\theta} = -\frac{g}{L}\theta =$ [redacted] $\omega^2\theta$ $\qquad \omega^2 = \frac{g}{L}$

big difference

2.1 Stable Case

$$\ddot{\theta} = -\omega^2\theta$$

$$\theta = C\cos(\omega t) + D\sin(\omega t)$$

Trigo identity (R cosine formula)

$$C\cos(x) + D\sin(x) = R\cos(x-\gamma)$$
$$= R\cos x\cos\gamma + R\sin x\sin\gamma$$

$$R = \sqrt{C^2 + D^2}, \quad \tan\gamma = D/C$$

$$\theta = A\cos(\omega t - \delta) \qquad -A \le \theta \le A$$

amplitude

Phase angle

2.1 SHM

$$\theta = A\cos(\omega t - \delta)$$

$$\theta = A\cos(\omega t - \delta) = A\cos\left(\omega\left(t + \frac{2\pi}{\omega}\right) - \delta\right)$$

two unknowns

angular frequency

period $\quad \dfrac{2\pi}{\omega} = 2\pi\sqrt{L/g}$

= time taken for θ to return to initial value

$\boxed{\text{stability} \quad \ddot{\theta} = -\omega^2\theta}$

Summary

Instability: $\ddot{\phi} = \frac{g}{L}\phi$

$$\phi = Ae^{(\sqrt{g/L})t} + Be^{-(\sqrt{g/L})t}$$

stability $\ddot{\theta} = -\omega^2\theta$

SHM $\theta = A\cos(\omega t - \delta)$

Remark : SHM is everywhere $\ddot{x} = f(x)$

Assume $f(0) = 0$

By Taylor's Theorem

$$f(x) = \underbrace{f(0)}_{= 0} + f'(0)x + \ldots$$

SHM for motion near equilibrium point

$$\ddot{x} = f'(0)x$$

Stability depends on sign

2.2 Oscillator Phase Plane

Let *x(t)* be a solution to a SHM problem

Define $y = \dot{x}$ $\qquad \psi = \delta - \omega t$

$$\begin{aligned} x &= A\cos(\omega t - \delta) = A\cos\psi \\ y &= -A\omega\sin(\omega t - \delta) = A\omega\sin\psi \end{aligned}$$

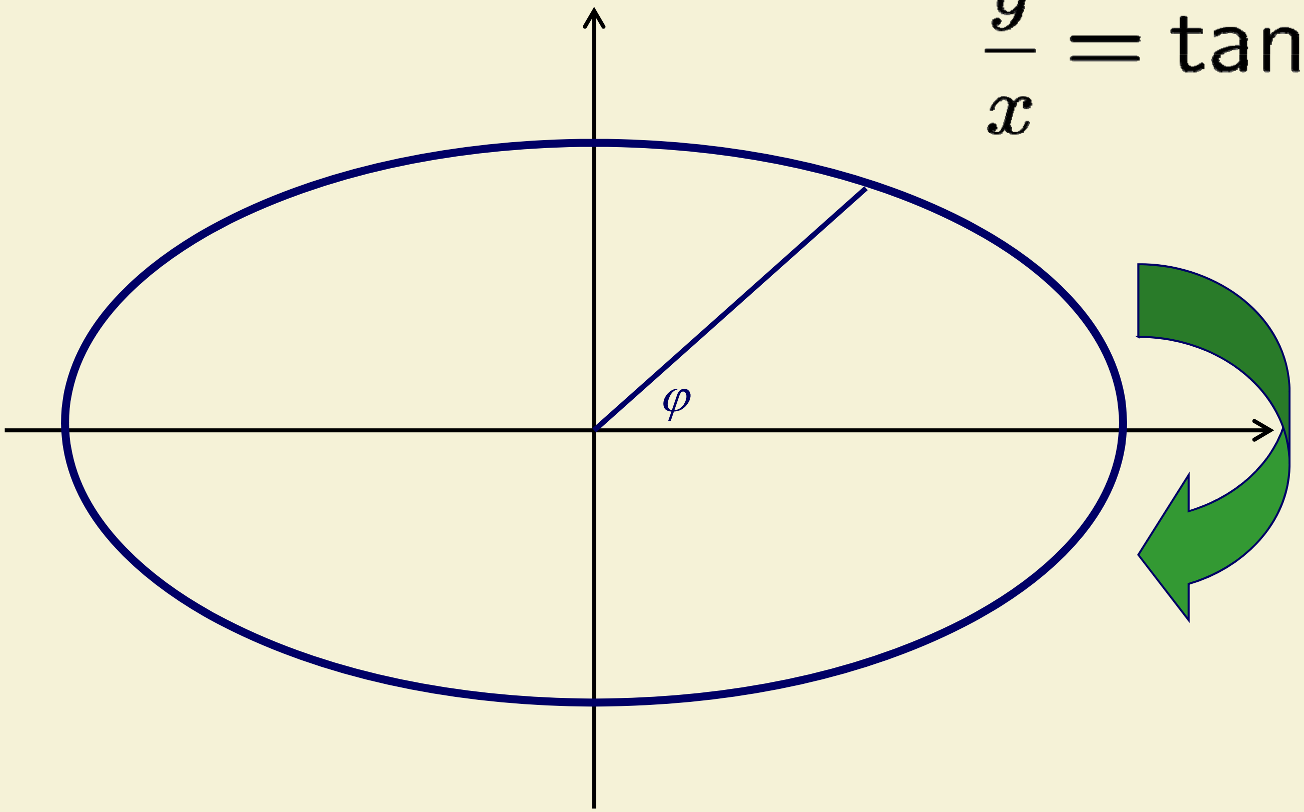
Ellipse $\dfrac{x^2}{A^2} + \dfrac{y^2}{A^2\omega^2} = \cos^2 + \sin^2 = 1$

2.2 Phase Plane

$$\frac{x^2}{A^2} + \frac{y^2}{A^2\omega^2} = 1$$

$$\begin{aligned} x &= A\cos\psi \\ y = \dot{x} &= A\omega\sin\psi \end{aligned}$$

$$\frac{y}{x} = \tan\varphi = \omega\tan\psi$$



$$\varphi \propto \psi$$

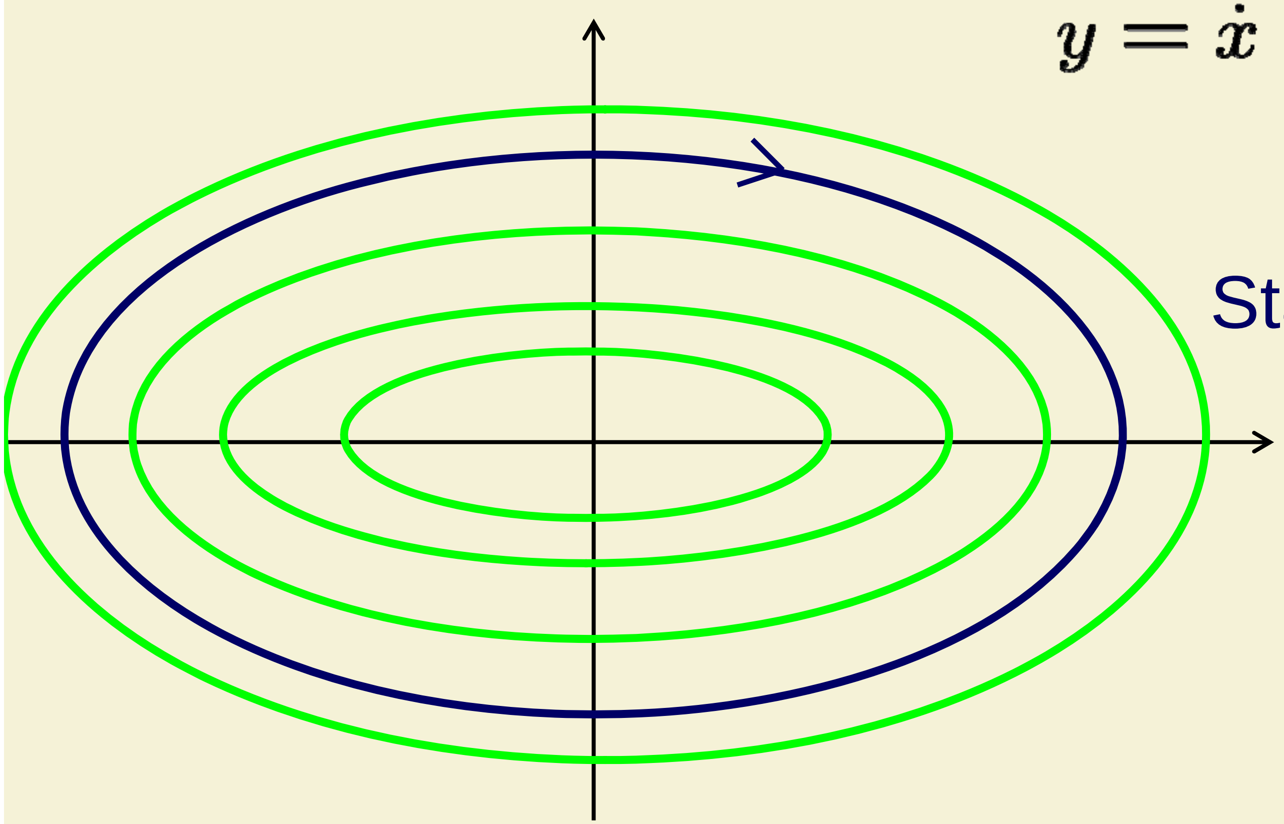
$$\psi = \delta - \omega t$$

2.2 Phase Plane of SHM

$$\ddot{x} = -\omega^2 x$$

$$x = A\cos\psi$$

$$y = \dot{x} = A\omega\sin\psi$$



Stable Equilibrium at

$$x = \dot{x} = 0$$

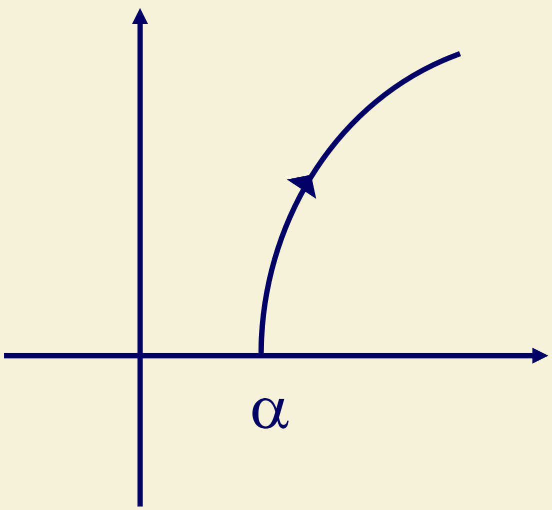
http://www.aw-bc.com/ide/idefiles/media/JavaTools/smpharos.html

2.2 Phase Plane $\ddot{x} = +\omega^2 x$

Initial displacement $x(0) = \alpha, \;\; \dot{x} = 0$

$$x(t) = \frac{1}{2}\alpha\left(e^{\omega t} + e^{-\omega t}\right) = \alpha \cosh(\omega t) \quad > 0$$

$$y(t) = \dot{x}(t) = \alpha\omega \sinh(\omega t) \quad > 0 \text{ when } t > 0$$

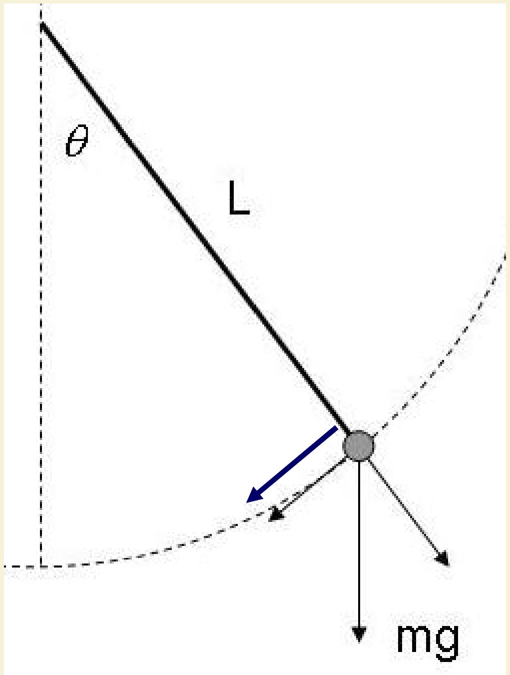


$\cosh^2 - \sinh^2 = 1$

$\Rightarrow (\frac{x}{\alpha})^2 - (\frac{y}{\alpha\omega})^2 = 1$

unstable equilibrium

2.3 Damped Harmonic motion



air resistance $\propto \dot{\theta}$

anti-motion

$$mL\ddot{\theta} = -mg\sin\theta - SL\dot{\theta} \approx -mg\theta - SL\dot{\theta}$$

$$\boxed{m\ddot{\theta} + S\dot{\theta} + \frac{mg}{L}\theta = 0}$$

2.3 Forced Damped Harmonic motion

air resistance $\propto \dot{\theta}$

attach a motor

$$mL\ddot{\theta} = -mg\sin\theta - SL\dot{\theta} + F(t)$$

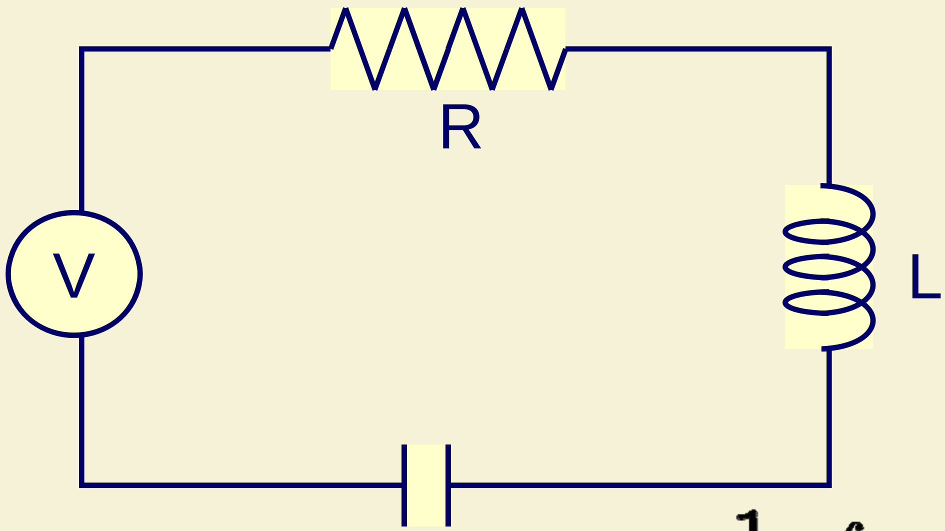
$$\boxed{m\ddot{\theta} + S\dot{\theta} + \frac{mg}{L}\theta = \frac{1}{L}F(t)}$$

$$\begin{aligned} \theta &\mapsto y \\ t &\mapsto x \end{aligned} \qquad m\frac{d^2y}{dx^2} + S\frac{dy}{dx} + \frac{mg}{L}y = \frac{1}{L}F(x)$$

2.4 Models of Electrical Circuits

Voltage Drop

$$V = IR$$



$$V(t) = L\frac{dI}{dt}$$

$$V = \frac{1}{C}\int I(t)dt$$

$$V(t) = RI + L\dot{I} + \frac{1}{c}\int I\ dt$$

2.4 Models of Electrical Circuits

Define $Q = \int I \; dt$ $\Rightarrow$ $\dot{Q} = I$

$$\boxed{L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = V(t)}$$

Forced damped harmonic oscillator

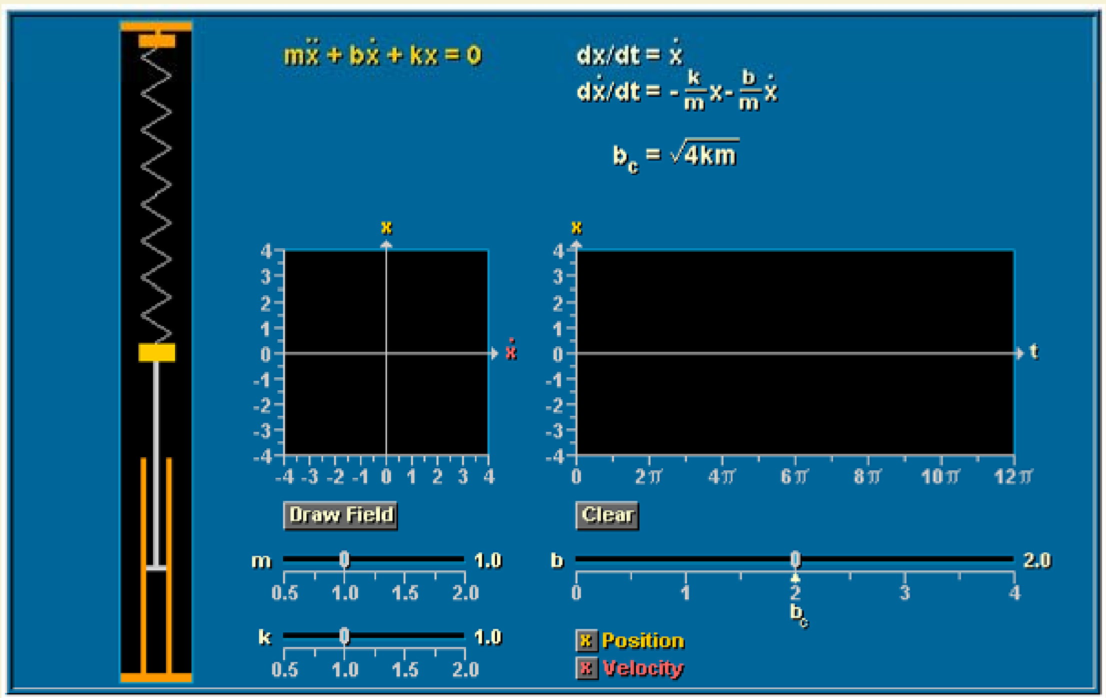
$$m\ddot{\theta} + S\dot{\theta} + \frac{mg}{L}\theta = \frac{1}{L}F(t)$$

2.5 Damped, Unforced Oscillators

$$\boxed{m\ddot{x} + b\dot{x} + kx = 0}$$

$m,\ k > 0\ ,\quad b \geq 0$

spring constant　　Damping constant



Mass Spring Oscillator

Textbook p196

http://www.aw-bc.com/ide/idefiles/media/JavaTools/massprng.html

2.5 Damped, Unforced Oscillators

$$\boxed{m\ddot{x} + b\dot{x} + kx = 0}$$

m, $k > 0$, $b \geq 0$

spring constant

Damping constant

$$\Downarrow$$

$$m\lambda^2 + b\lambda + k = 0$$

Case a: two real roots — Over damping

Case b: double root — Critical damping

Case c: complex roots — Under damping

2.5 Damped, Unforced Oscillators (2 real roots)

$$\boxed{\ddot{x} + 3\dot{x} + 2x = 0}$$

$$\lambda = -1, -2$$

Will we ever get positive roots?

$$B_1 e^{-t} + B_2 e^{-2t}$$

Goes to zero rapidly $\Rightarrow$ overdamping

2.5 Damped, Unforced Oscillators (double root)

$$\boxed{\ddot{x} + 6\dot{x} + 9x = 0}$$

$$\Downarrow$$

$$\lambda = -3$$

Critical damping

$$\Downarrow$$

$$B_1 e^{-3t} + B_2 t e^{-3t}$$

Also goes to zero rapidly

http://www.aw-bc.com/ide/idefiles/media/JavaTools/vibedamp.html

2.5 Damped, Unforced Oscillators (complex roots)

$$\boxed{\ddot{x} + 2\dot{x} + 26x = 0}$$

$$\Downarrow$$

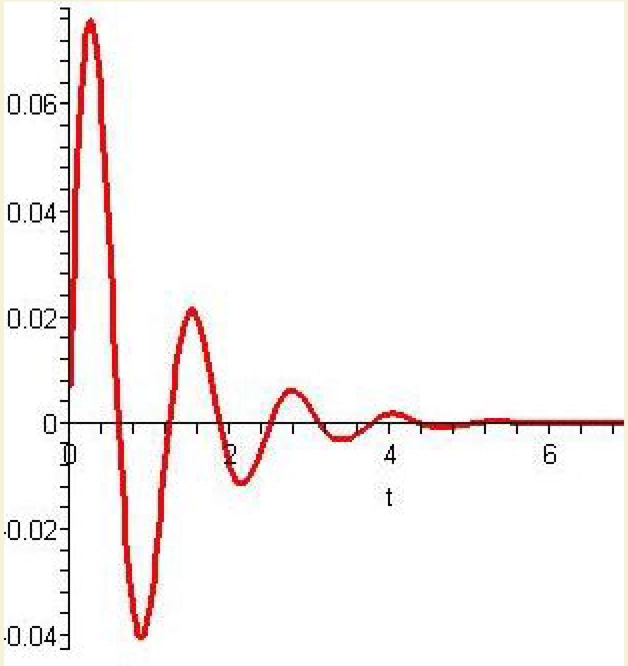
$$\lambda = -1 \pm 5i$$

$$\Downarrow$$

$$B_1 e^{-t} \cos(5t) + B_2 e^{-t} \sin(5t)$$

$$x = Ae^{-t} \cos(5t - \delta)$$

underdamped



Underdamped, Unforced Oscillators (complex roots)

$$\boxed{m\ddot{x} + b\dot{x} + kx = 0}$$

$$x(t) = Ae^{\frac{-bt}{2m}}\cos(\beta t - \delta)$$

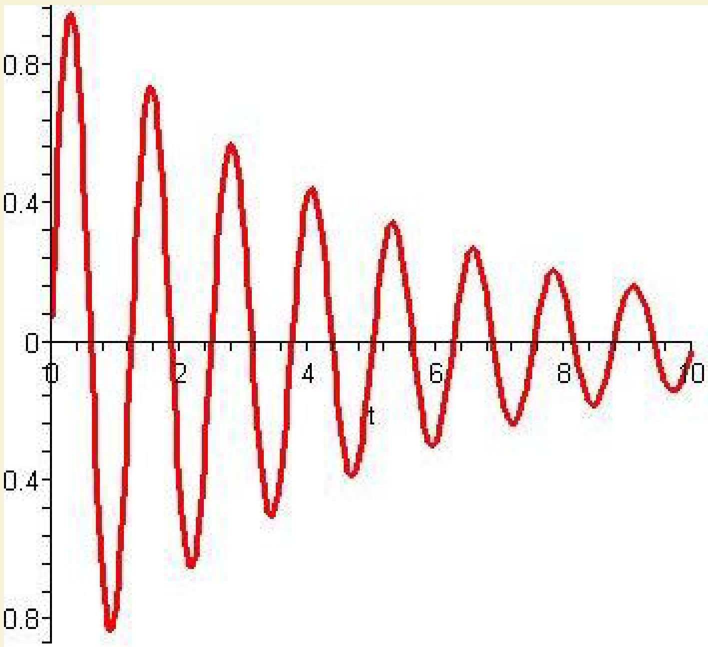
time varying amplitude

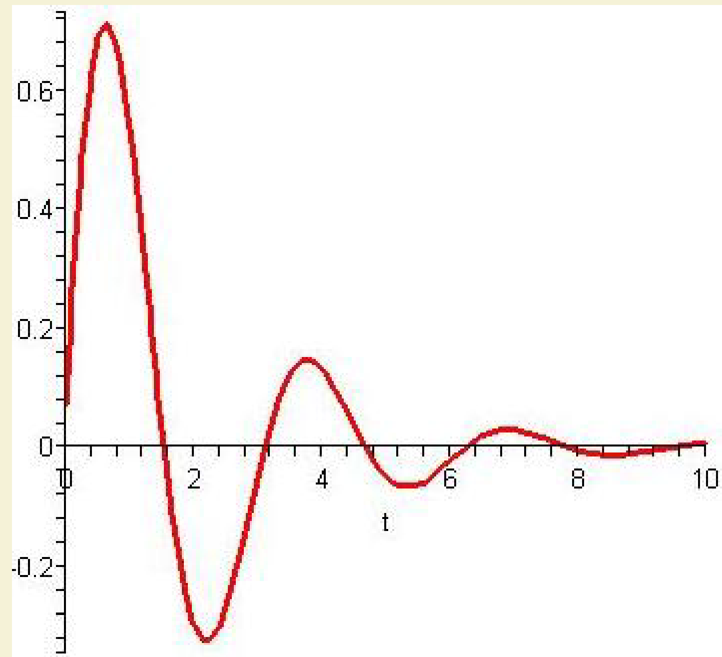
$$\beta = \frac{1}{2m}\sqrt{4mk - b^2}$$ Quasi-frequency

$$\frac{2\pi}{\beta}$$ Quasi-period

Underdamped, Unforced Oscillators (complex roots)

$$x(t) = Ae^{\frac{-bt}{2m}} \cos(\beta t - \delta)$$



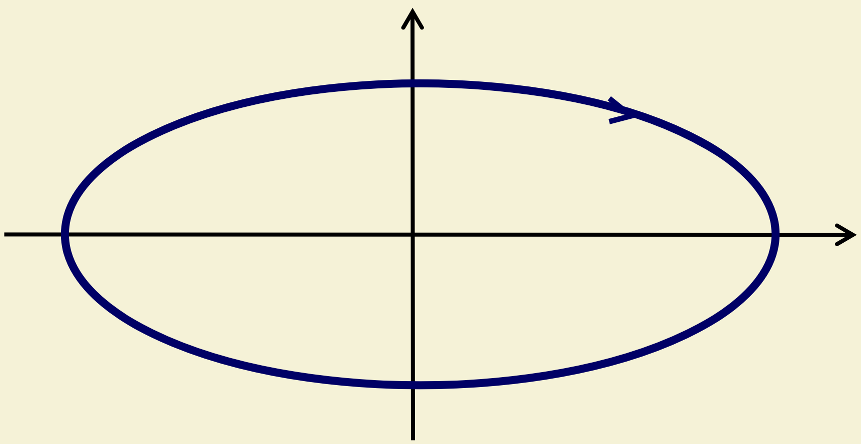


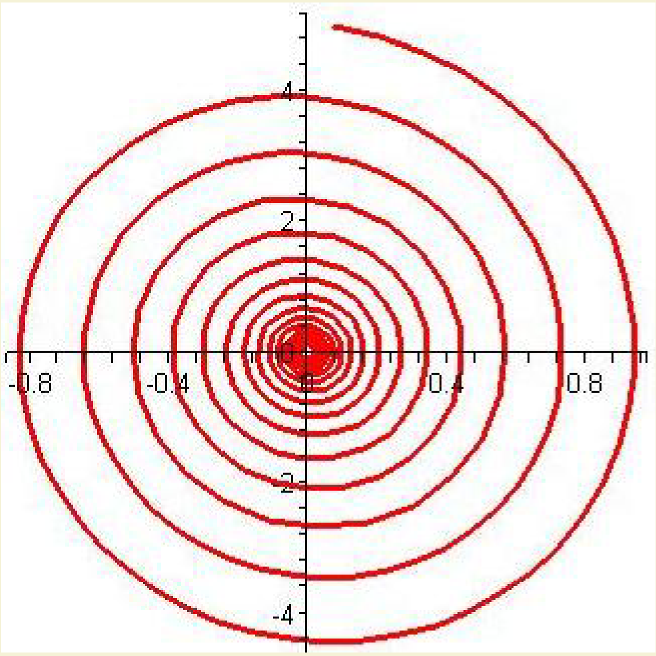
large $\dfrac{2m}{b}$ small $\dfrac{2\pi}{\beta}$

small $\dfrac{2m}{b}$ large $\dfrac{2\pi}{\beta}$

Underdamped, Unforced Oscillators (complex roots)

$$x(t) \;=\; A\cos(\omega t - \delta) \qquad x(t) = Ae^{\frac{-bt}{2m}}\cos(\beta t - \delta)$$

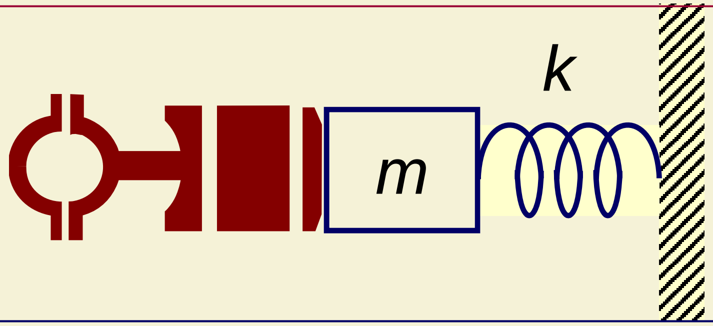




Phase Plane: Plot x $(x, \dot{x})$

2.6 Forced Oscillators

$F_0 \cos(\alpha t)$



Natural frequency

when $F_0 = 0$

$$F_{\text{spring}} = -kx$$

$$\omega = \sqrt{k/m}$$

$$m\ddot{x} = -kx \Rightarrow \ddot{x} = -\omega^2 x$$

$$\boxed{m\ddot{x} + kx = F_0 \cos \alpha t}$$ nonhomogeneous

$$m\ddot{z} + kz = F_0\ e^{i\alpha t}$$ Solve, take real part

2.6 Forced Oscillators

$$m\ddot{x} + kx = F_0 \cos \alpha t$$

$$m\ddot{z} + kz = F_0 \; e^{i\alpha t}$$

Try $z = Ce^{i\alpha t}$

$$mC(i\alpha)^2 e^{i\alpha t} + C \; k \; e^{i\alpha t} = F_0 e^{i\alpha t}$$

$$\Rightarrow \quad C = \frac{F_0}{k - m\alpha^2} = \frac{F_0/m}{\omega^2 - \alpha^2}$$

$$\boxed{\omega = \sqrt{k/m}}$$

Gen Solution

$$x = A \cos(\omega t - \delta) + \frac{F_0/m}{\omega^2 - \alpha^2} \cos(\alpha t)$$

2.6 Forced Oscillators

$$m\ddot{x} + kx = F_0 \cos \alpha t$$

$$x = A\cos(\omega t - \delta) + \frac{F_0/m}{\omega^2 - \alpha^2}\cos(\alpha t)$$

$$\dot{x} = -A\omega \sin(\omega t - \delta) - \frac{\alpha F_0/m}{\omega^2 - \alpha^2}\sin(\alpha t)$$

Assume $x(0) = \dot{x}(0) = 0$

$$0 = A\cos(\delta) + \frac{F_0/m}{\omega^2 - \alpha^2} \Rightarrow A = -\frac{F_0/m}{\omega^2 - \alpha^2}$$

$$0 = A\omega \sin(\delta) \Rightarrow \delta = 0$$

$$\boxed{x = \frac{F_0/m}{\omega^2 - \alpha^2}\Big(\cos(\alpha t) - \cos(\omega t)\Big)}$$

2.6 Forced Oscillators

$$m\ddot{x} + kx = F_0 \cos \alpha t$$

$$x = \frac{F_0/m}{\omega^2 - \alpha^2}\big(\cos(\alpha t) - \cos(\omega t)\big)$$

Use $\boxed{\cos A - \cos B = -2 \sin\left(\frac{A-B}{2}\right) \sin\left(\frac{A+B}{2}\right)}$

$$\Rightarrow \quad x = \underbrace{\frac{2F_0/m}{\alpha^2 - \omega^2} \sin\left[\left(\frac{\alpha - \omega}{2}\right)t\right]}_{A(t)} \sin\left[\left(\frac{\alpha + \omega}{2}\right)t\right]$$

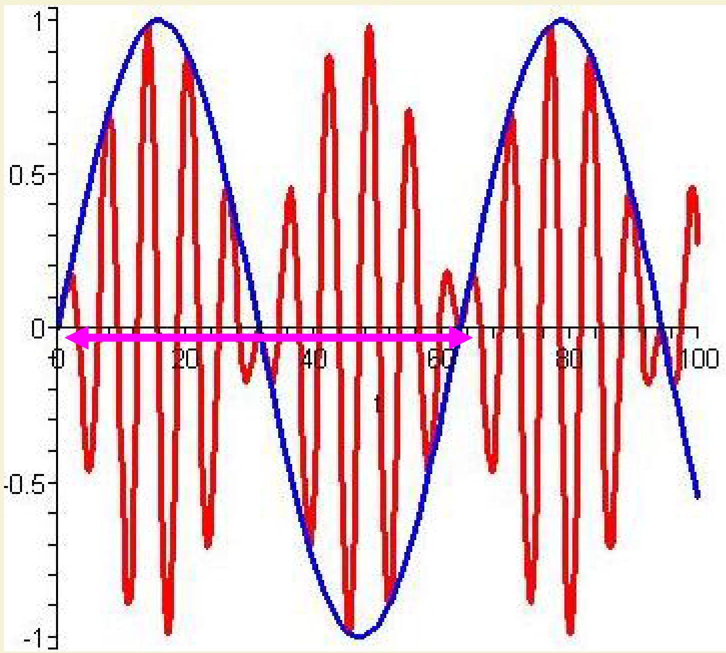
small, i.e. low frequency

2.6 Forced Oscillators

$$m\ddot{x} + kx = F_0 \cos \alpha t$$

$$x = A(t) \sin\left[\left(\frac{\alpha + \omega}{2}\right) t\right]$$

where $$A(t) = \frac{2F_0/m}{\alpha^2 - \omega^2} \sin\left[\left(\frac{\alpha - \omega}{2}\right) t\right]$$



$$\frac{\alpha - \omega}{2}$$

Beat frequency

http://www.school-for-champions.com/science/sound_beat.htm

2.6 Forced Oscillators

$$m\ddot{x} + kx = F_0 \cos \alpha t$$

$$A(t) = \frac{2F_0/m}{\alpha^2 - \omega^2} \sin\left[\left(\frac{\alpha - \omega}{2}\right) t\right]$$

Max value

$$\lim_{\alpha\to\omega} A(t) = \lim_{\alpha\to\omega} \frac{2F_0/m}{\alpha + \omega} \times \frac{\sin\left[\frac{\alpha-\omega}{2} t\right]}{\alpha - \omega}$$

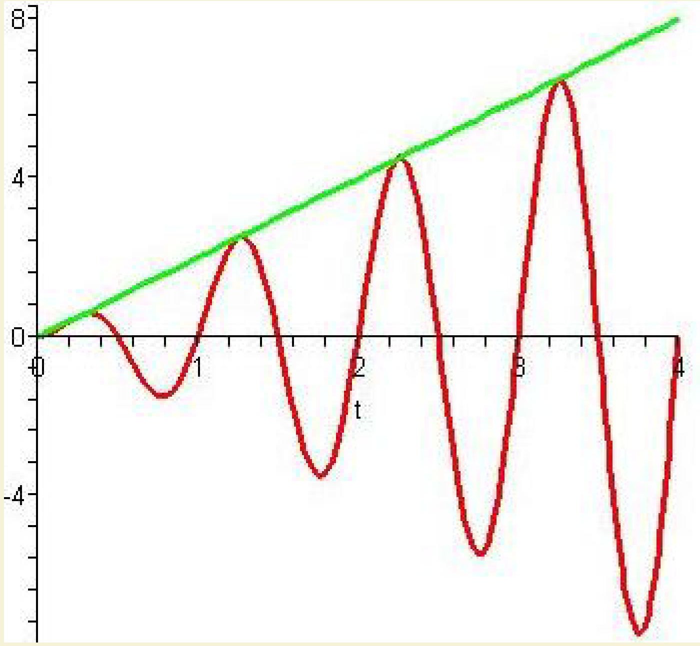
$$= \frac{F_0}{m\omega} \times \frac{t}{2} = \frac{F_0 t}{2m\omega}$$

2.6 Forced Oscillators

$$m\ddot{x} + kx = F_0 \cos \alpha t$$

$$x = A(t) \sin \left[\left(\frac{\alpha + \omega}{2} \right) t \right]$$

$$\lim x = \frac{F_0 t}{2m\omega} \sin(\omega t)$$



Oscillations go out of control! Resonance

2.6 Forced Oscillators (with friction)

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos \alpha t$$

$$m\ddot{z} + b\dot{z} + kz = F_0 e^{i\alpha t}$$

Try $z = ce^{i\alpha t}$

$$c = \frac{F_0}{k - m\alpha^2 + ib\alpha} = \frac{F_0(k - m\alpha^2 - ib\alpha)}{(k - m\alpha^2)^2 + b^2\alpha^2}$$

Take Real Part

$$\frac{F_0(k - m\alpha^2 - ib\alpha)}{(k - m\alpha^2)^2 + b^2\alpha^2} \times (\cos(\alpha t) + i\sin(\alpha t))$$

$$\boxed{x(t) = \frac{F_0(k - m\alpha^2)\cos(\alpha t) + F_0 b\alpha \sin(\alpha t)}{(k - m\alpha^2)^2 + b^2\alpha^2}}$$

2.6 Forced Oscillators (with friction)

$$x(t) = \frac{F_0(k - m\alpha^2)\cos(\alpha t) + F_0 b\alpha \sin(\alpha t)}{(k - m\alpha^2)^2 + b^2\alpha^2}$$

+ Gen Sol of $m\ddot{x} + b\dot{x} + kx = 0$

Tends to zero rapidly (Transient)

$$x(t) = \frac{\frac{1}{m}F_0\cos(\alpha t - \gamma)}{\sqrt{(\omega^2 - \alpha^2)^2 + \frac{b^2}{m^2}\alpha^2}}$$

$$\omega = \sqrt{k/m}$$

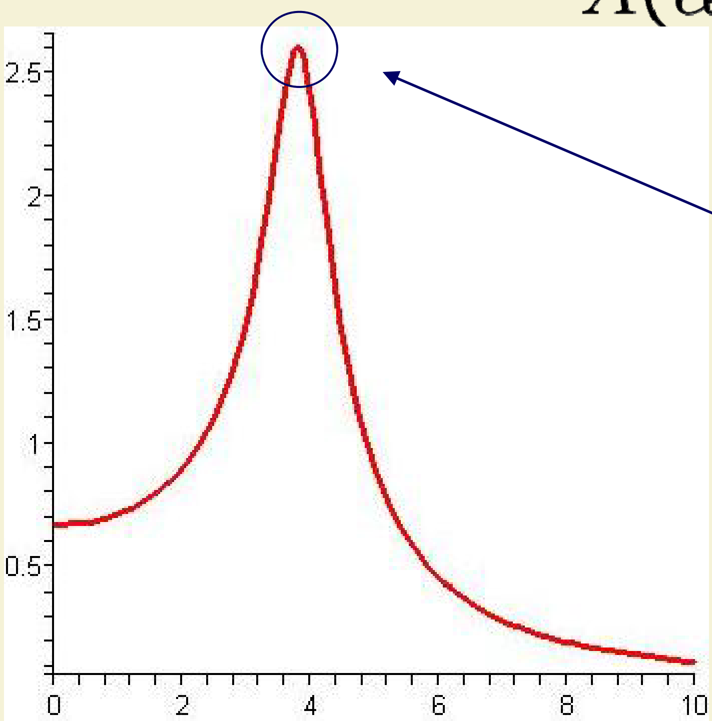
Oscillation at frequency α

http://www.aw-bc.com/ide/idefiles/media/JavaTools/vibefdmp.html

2.6 Forced Oscillators (with friction)

$$x(t) = \frac{\frac{1}{m} F_0 \cos(\alpha t - \gamma)}{\sqrt{(\omega^2 - \alpha^2)^2 + \frac{b^2}{m^2}\alpha^2}}$$

$$A(\alpha) = \frac{F_0/m}{\sqrt{(\omega^2 - \alpha^2)^2 + \frac{b^2}{m^2}\alpha^2}}$$



Max when $\alpha = \sqrt{\omega^2 - \frac{b^2}{2m^2}}$

Amplitude response curve

http://www.acoustics.salford.ac.uk/feschools/waves/shm.htm

2.7 Conservation of Energy

$$\frac{d}{dx}\left(\frac{1}{2}\dot{x}^2\right) = \dot{x}\frac{d\dot{x}}{dx} = \frac{dx}{dt}\frac{d\dot{x}}{dx} = \ddot{x}$$

SHM: $$m\ddot{x} = m\frac{d}{dx}(\frac{1}{2}\dot{x}^2) = -kx$$

Integrate $$\frac{1}{2}m\dot{x}^2 = -\frac{1}{2}kx^2 + E$$

$$\Rightarrow \quad E = \underbrace{\frac{1}{2}m\dot{x}^2}_{\text{kinetic}} + \underbrace{\frac{1}{2}kx^2}_{\text{potential}}$$

2.7 Conservation of Energy (friction)

SHM: $m\ddot{x} = -kx - b\dot{x}$

$$m\frac{d}{dx}(\frac{1}{2}\dot{x}^2) + kx = -b\dot{x}$$

$$\boxed{\frac{d}{dx}\left(\frac{1}{2}\dot{x}^2\right) = \ddot{x}}$$

Integrate $E = \int -b\dot{x}dx$

$$\boxed{E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2}$$

$$\Rightarrow \frac{dE}{dx} = -b\dot{x}$$

$$\frac{dE}{dt} = \frac{dx}{dt}\frac{dE}{dx} = \dot{x}\frac{dE}{dx} = -b\dot{x}^2 \leq 0$$

Energy decreasing, i.e. heat loss

http://www.aw-bc.com/ide/idefiles/media/JavaTools/vibedenr2.html

2.7 Conservation of Energy (General 1D Case)

Potential Energy: $V(x) = -\int^x F(y)dy$

Function of position

$$F = m\ddot{x}$$

$$-\frac{dV}{dx} = \frac{d}{dx}\left(\frac{1}{2}m\dot{x}^2\right)$$

$$\frac{d}{dx}\left(\frac{1}{2}m\dot{x}^2 + V(x)\right) = 0$$

$$\boxed{\frac{1}{2}m\dot{x}^2 + V(x) = E}$$

2.7 Conservation of Energy

SHM: $$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

$\boxed{y = \dot{x}}$

$$\frac{1}{2}kx^2 + \frac{1}{2}my^2 = E$$ Ellipse!

Unstable motion: $$\ddot{x} = +\omega^2 x$$

$$\frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2 = E$$ Hyperbola!