EE2011 Engineering Electromagnetics

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Coulomb's Law

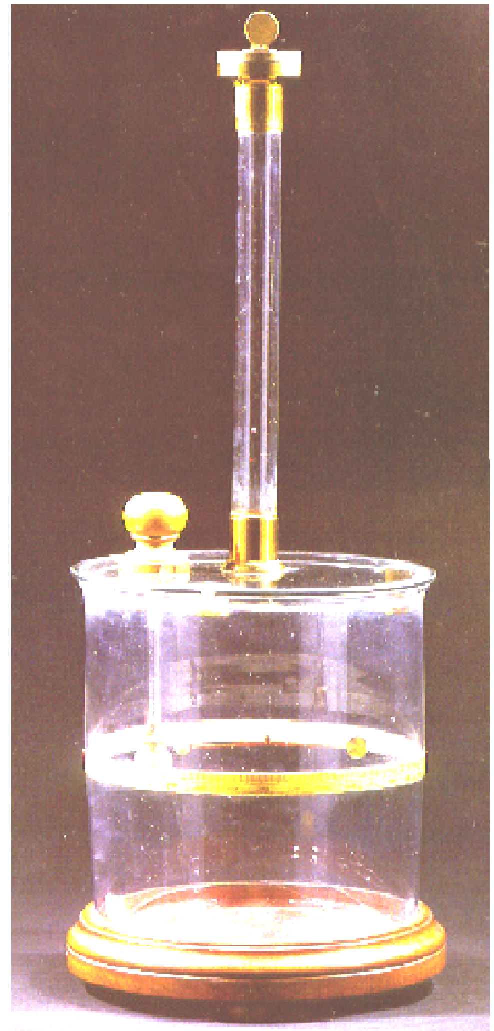
quantitative observations (in free space)

$$\vec{F} = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}\,\hat{\mathrm{u}}_r$$

- originally an empirical result
  later corroborated by further theory

- valid only for point charges
  (*i.e.* when r >> dimensions of q_1 and q_2)

- formula for pair of charges
  extend via superposition for linear systems



Coulomb's Law

vectorial addition for (linear) assembly of point charges

(attract for opposite polarity but repel for same polarity)

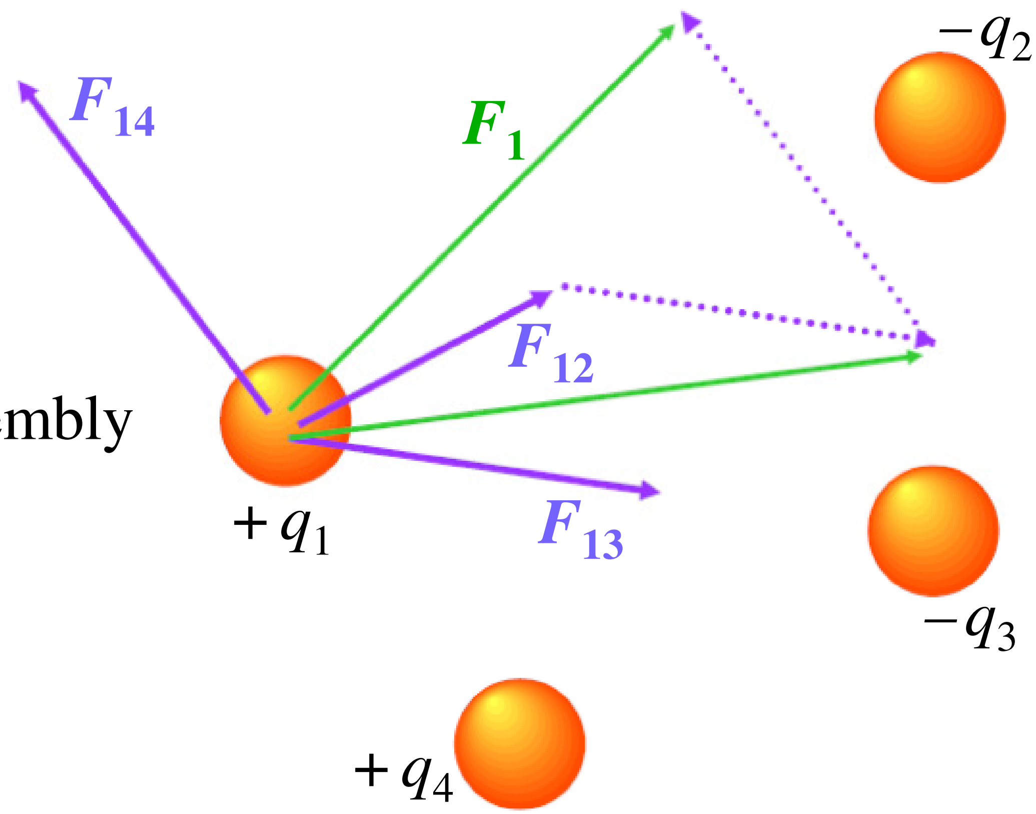
total force on q_1

$$\vec{F}_1 = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14}$$

for general N-charge assembly

total force on q_m

$$\vec{F}_\mathrm{m} = \sum_{\mathrm{n}=1}^{\mathrm{N}} \vec{F}_\mathrm{mn}\bigg|_{\mathrm{n}\neq\mathrm{m}}$$



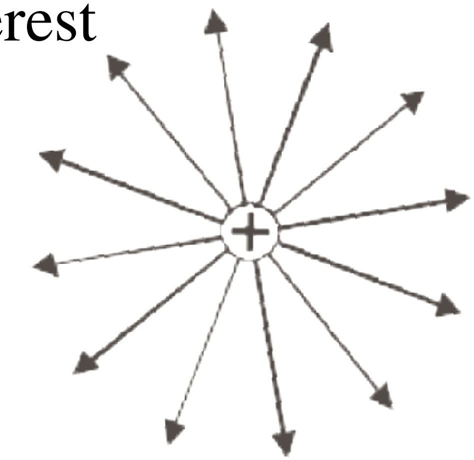
Electric Field

not convenient to employ $\vec{F}$ for analytical formulation

proceed by defining electric field $\vec{E}$ due to charge q

(a) place positive test charge q_{test} at point of interest

(b) divide Coulomb force by q_{test} magnitude



$$\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2}\,\hat{\mathrm{u}}_r$$

<u>notes</u>:

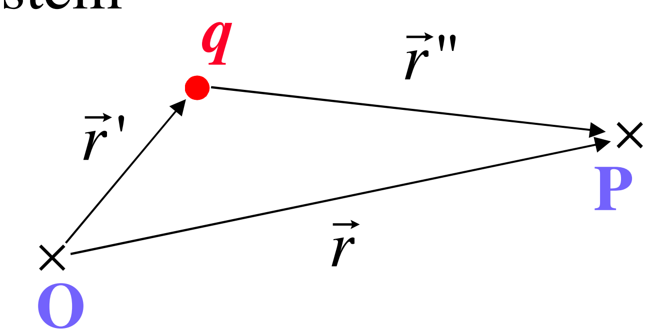
- $q_{\text{test}} << q$ so as not to perturb original electric-field pattern of q
- unit of $\mathrm{N\,C^{-1}}$ from definition but common to use $\mathrm{V\,m^{-1}}$ instead
- valid for point charge q but can extend for linear systems

Electric Field

employ superposition for linear multi-charge systems

need to account for origin of coordinate system

include magnitude of direction vector?



$$\vec{E} = \frac{q}{4\pi\varepsilon_0 (r'')^2}\,\hat{u}_{r''} \quad \text{or} \quad \vec{E} = \frac{q}{4\pi\varepsilon_0 (r'')^3}\,\vec{r}\,''$$

direction vector generally regarded as a variable

(a) assembly of discrete charges $\quad \vec{E} = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\sum_{n=1}^{N}\frac{q_n}{(r'')_n^2}(\hat{u}_{r''})_n$

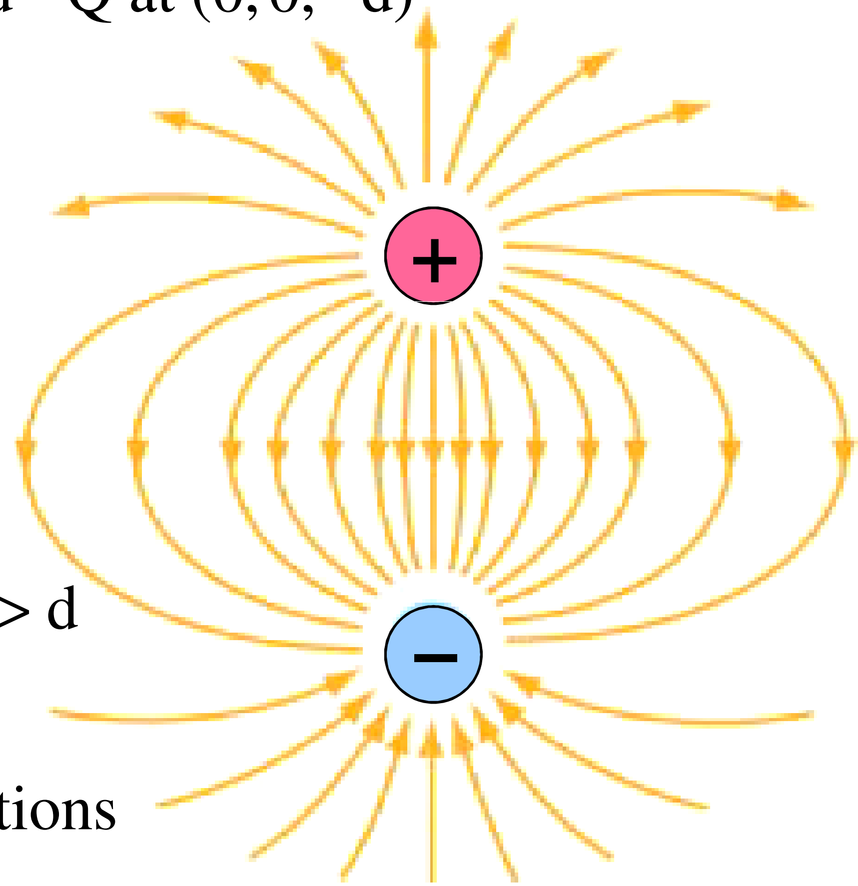
(b) volume of distributed charges $\quad \vec{E} = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\iiint_V \frac{\sigma}{(r'')^2}\,\hat{u}_{r''}\,dV$

Electric Field

Example #1: dipole (pair of point charges with opposite polarity)

$+Q$ at $(0,0,+d)$ and $-Q$ at $(0,0,-d)$

$$\vec{E} = \frac{Q\left\{x\hat{u}_x + y\hat{u}_y + (z-d)\hat{u}_z\right\}}{4\pi\varepsilon_0\left\{x^2+y^2+(z-d)^2\right\}^{\frac{3}{2}}} - \frac{Q\left\{x\hat{u}_x + y\hat{u}_y + (z+d)\hat{u}_z\right\}}{4\pi\varepsilon_0\left\{x^2+y^2+(z+d)^2\right\}^{\frac{3}{2}}}$$



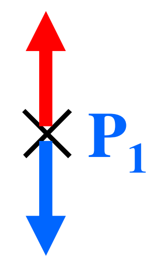
cancellation when $\sqrt{x^2+y^2+z^2} \gg d$

still need to derive residual fields

useful result for practical applications

Electric Field

residual fields of dipole when $\sqrt{x^2+y^2+z^2} >> \mathrm{d}$



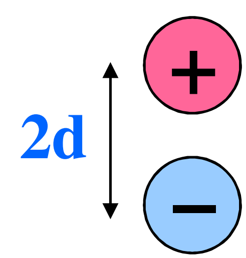
(a) at P_1 $(0, 0, z)$ where $z >> \mathrm{d}$

$$\vec{E} = \frac{\mathrm{Q}}{4\pi\varepsilon_0(z-\mathrm{d})^3}(z-\mathrm{d})\hat{\mathrm{u}}_z - \frac{\mathrm{Q}}{4\pi\varepsilon_0(z+\mathrm{d})^3}(z+\mathrm{d})\hat{\mathrm{u}}_z$$

$$= \frac{\mathrm{Q}}{4\pi\varepsilon_0 z^2}\left\{\left(1-\tfrac{\mathrm{d}}{z}\right)^{-2} - \left(1+\tfrac{\mathrm{d}}{z}\right)^{-2}\right\}\hat{\mathrm{u}}_z$$

$$= \frac{\mathrm{Q}}{4\pi\varepsilon_0 z^2}\left\{\left(1+\tfrac{2\mathrm{d}}{z}\ldots\right) - \left(1-\tfrac{2\mathrm{d}}{z}\ldots\right)\right\}\hat{\mathrm{u}}_z$$

$$\rightarrow \frac{\mathrm{Q\,d}}{\pi\varepsilon_0 z^3}\,\hat{\mathrm{u}}_z \quad \text{where } 2\,\mathrm{Q\,d} = \text{dipole moment}$$



Electric Field

residual fields of dipole when $\sqrt{x^2+y^2+z^2} >> \mathrm{d}$

(b) at P_2 $(x, 0, 0)$ where $x >> \mathrm{d}$

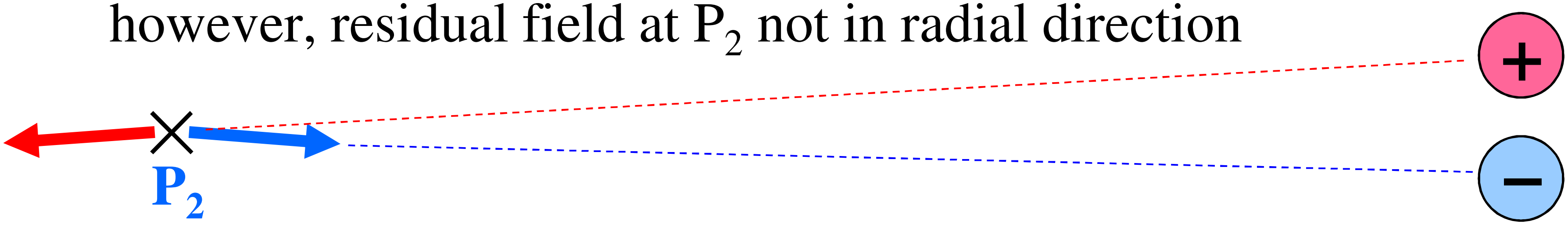
cancellation of their horizontal components

doubling of (residual) vertical components

$$\left|E_z\right| = 2\frac{\mathrm{Q}}{4\pi\varepsilon_0(x^2+\mathrm{d}^2)}\frac{\mathrm{d}}{\sqrt{x^2+\mathrm{d}^2}} \rightarrow \frac{\mathrm{Q\,d}}{2\pi\varepsilon_0 x^3}$$

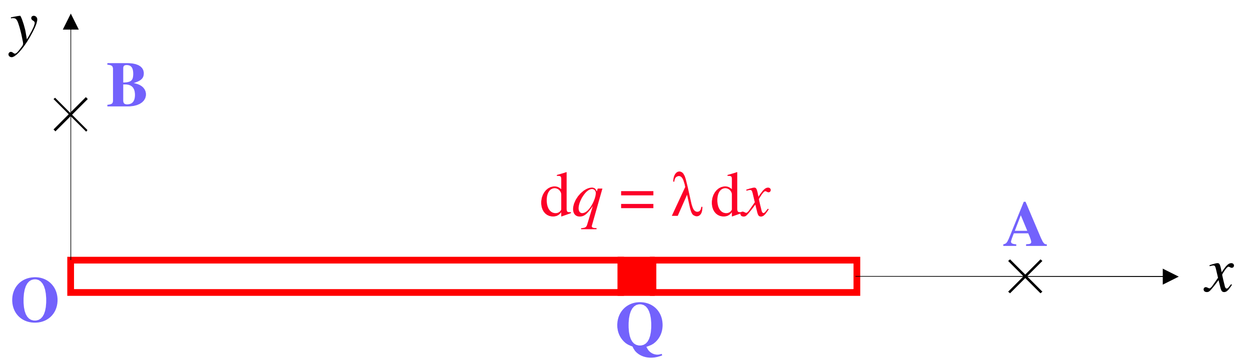
inversely proportional to r^3 (instead of r^2) for both P_1 and P_2

however, residual field at P_2 not in radial direction



Electric Field

Example #2: rod (of length L) with uniform charge density λ C m^{-1}



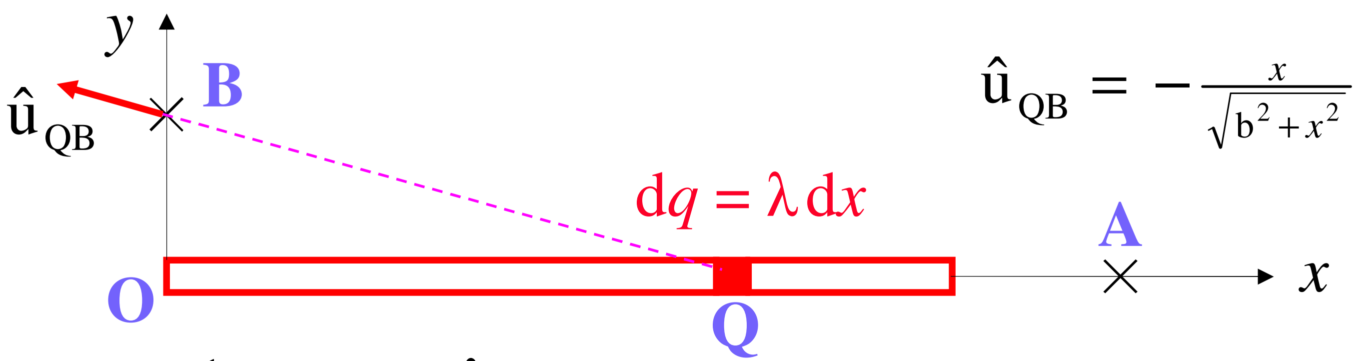
simpler case: obtain electric field at A (a, 0, 0)

$$\vec{E}_A = \frac{1}{4\pi\varepsilon_0}\int_{x=0}^{L}\frac{\lambda}{(a-x)^2}\,\hat{u}_x\, dx \quad \text{with same direction for all } d\vec{E} \text{ terms}$$

$$= \hat{u}_x \frac{\lambda}{4\pi\varepsilon_0}\int_{x=0}^{L}\frac{1}{(a-x)^2}\, dx = \hat{u}_x \frac{\lambda}{4\pi\varepsilon_0}\frac{L}{a(a-L)} \boxed{\rightarrow \hat{u}_x \frac{(\lambda L)}{4\pi\varepsilon_0 a^2} \text{ when } a \gg L}$$

Electric Field

more difficult to derive electric field at $B\,(0, b, 0)$ due to variable $\hat{u}_{QB}$



$$\hat{u}_{QB} = -\frac{x}{\sqrt{b^2 + x^2}}\,\hat{u}_x + \frac{b}{\sqrt{b^2 + x^2}}\,\hat{u}_y$$

$$\vec{E}_B = \frac{1}{4\pi\varepsilon_0}\int_{x=0}^{L}\frac{\lambda}{b^2 + x^2}\,\hat{u}_{QB}\,dx$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\left\{-\hat{u}_x\int_{x=0}^{L}\frac{x}{(b^2 + x^2)^{\frac{3}{2}}}\,dx + b\,\hat{u}_y\int_{x=0}^{L}\frac{1}{(b^2 + x^2)^{\frac{3}{2}}}\,dx\right\}$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\left\{-\hat{u}_x\left(\frac{1}{b} - \frac{1}{\sqrt{b^2 + L^2}}\right) + \hat{u}_y\,\frac{L}{b\sqrt{b^2 + L^2}}\right\}$$

$$\boxed{\rightarrow \hat{u}_y\,\frac{(\lambda L)}{4\pi\varepsilon_0 b^2} \quad \text{when } b \gg L}$$

Electric Field

need to define Φ for electric field traversing plane surface

$$\Phi \propto E_n A$$

(a) add proportionality constant *

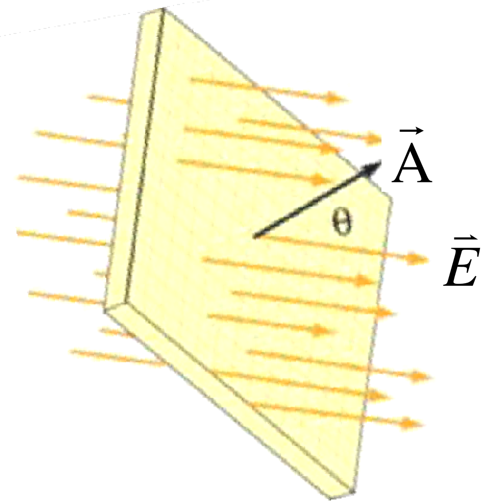
$$\Phi = \varepsilon E_n A \; *$$

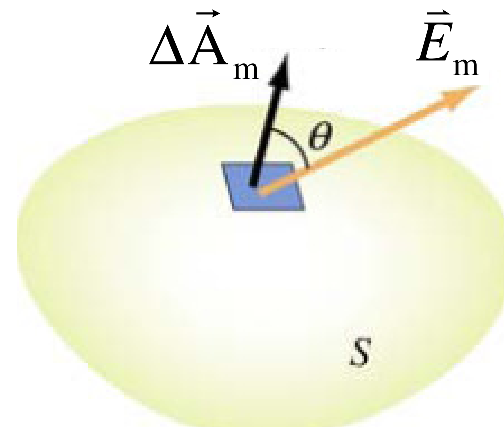
(b) account for orientation via dot product

$$\Phi = \varepsilon \vec{E} \bullet \vec{A}$$

(c) consider elemental areas for curved surface

$$\Phi = \iint \varepsilon \vec{E} \bullet d\vec{A}$$





* will later introduce another vector ($D = \varepsilon E$)

Electric Field

special property for closed surface

$$\oiint_S \varepsilon \vec{E} \bullet \mathrm{d}\vec{\mathrm{A}} = Q_{\substack{\text{total in} \\ \text{enclosure}}} = \begin{cases} \sum_{m=1}^{M} q_m & \text{for point charges} \\ \iiint_V \sigma \, \mathrm{dV} & \text{for distributed charges} \end{cases}$$

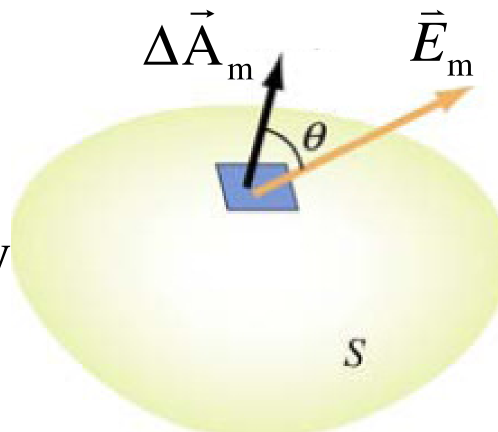
possible simplifications:

(a) if using homogeneous material (uniform ε)

$$\oiint_S \vec{E} \bullet \mathrm{d}\vec{\mathrm{A}} = \tfrac{1}{\varepsilon} Q_{\text{total}}$$

(b) if additionally adopting structural symmetry

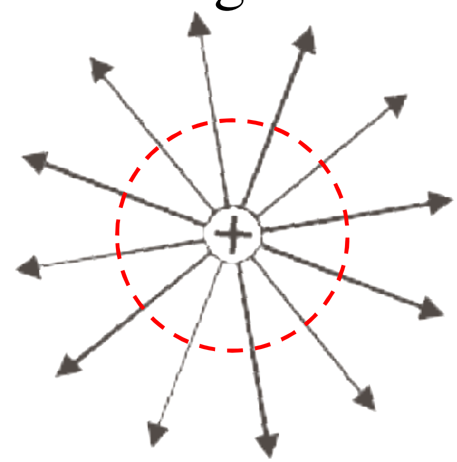
$$E_n \mathrm{A} = \tfrac{1}{\varepsilon} Q_{\text{total}}$$



Gauss's Law

macroscopic approach for deriving electric field from charges

$$\oint\!\!\oint_S \vec{E} \bullet d\vec{A} = \tfrac{1}{\varepsilon} Q_{\text{total}}$$



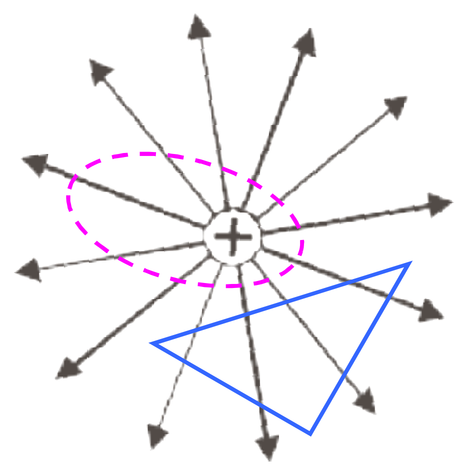
always valid but useful only with symmetry

<u>explanatory example</u>**:** point charge in free space

(a) for red Gaussian surface (with symmetry)

$$\oint\!\!\oint_{S_{\text{red}}} \vec{E} \bullet d\vec{A} = E_r \oint\!\!\oint_{S_{\text{red}}} dA_r = E_r 4\pi r^2 = \frac{q}{\varepsilon_o}$$

$$\Rightarrow \quad E_r = \frac{q}{4\pi\varepsilon_0 r^2} \qquad \textit{i.e.}\ \text{Coulomb's Law}$$



(b) for pink Gaussian surface $\oint\!\!\oint_{S_{\text{pink}}} \vec{E} \bullet d\vec{A} = \frac{q}{\varepsilon_o}$

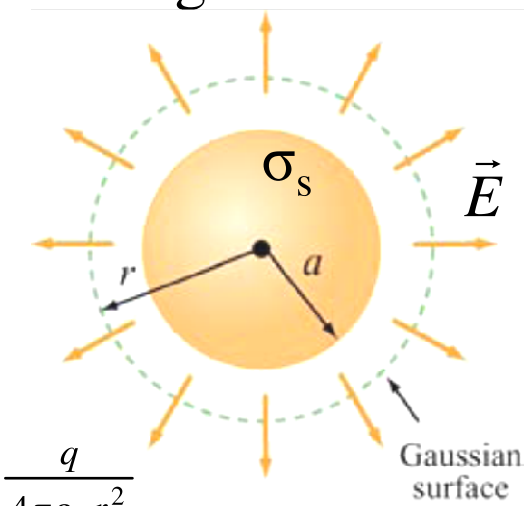
(c) for blue Gaussian surface $\oint\!\!\oint_{S_{\text{blue}}} \vec{E} \bullet d\vec{A} = 0$

Gauss's Law

Example #1: conducting sphere with distributed charges

infer from symmetry:

- no non-radial component (*i.e.* $\vec{E} = E_r \hat{u}_r$)
- E_r constant of θ and ϕ coordinates



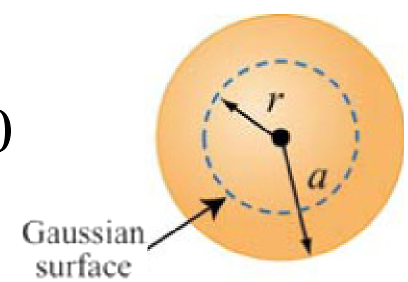
(a) apply Gauss's Law to green surface

$$E_r 4\pi r^2 = \tfrac{1}{\varepsilon_o}(\sigma_s 4\pi a^2) \Rightarrow E_r = \frac{(\sigma_s 4\pi a^2)}{4\pi\varepsilon_0 r^2} = \frac{q}{4\pi\varepsilon_0 r^2}$$

→ introduce equivalent point charge to replace charge density σ_s

(b) apply Gauss's Law to blue surface

$$\oiint_{S_{blue}} \vec{0} \bullet d\vec{A} = \tfrac{1}{\varepsilon_o}\iiint_V \sigma \, dV \Rightarrow Q_{enclosed} = 0$$



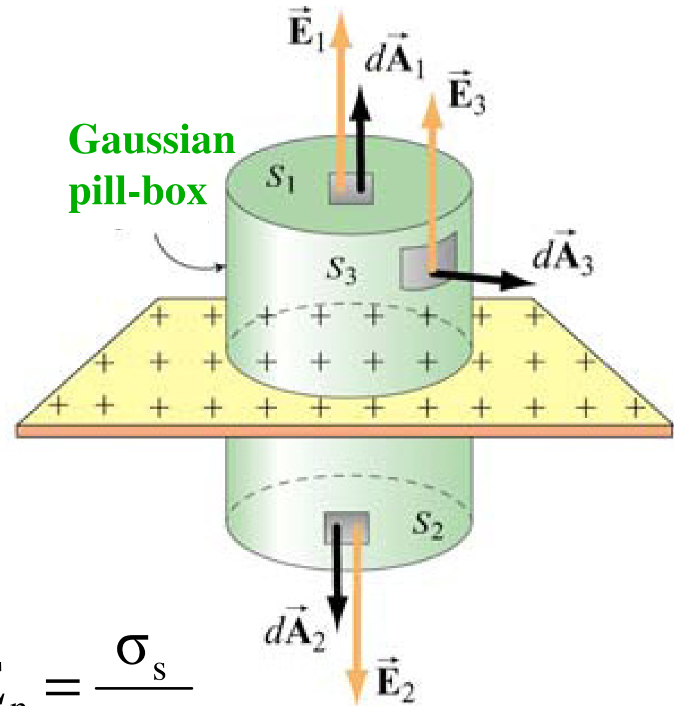
→ all charges residing at outermost surface

Gauss's Law

Example #2: large thin sheet with uniform charge density $\sigma_s \, \mathrm{C\,m^{-2}}$

infer from symmetry:

- no non-normal component ($\vec{E} = E_n \hat{u}_n$)
- E_n constant of both planar coordinates



apply Gauss's Law to green pill-box:

(a) S_1 and S_2 equi-distant from sheet

(b) no contribution at S_3 as $\vec{E}_3 \bullet d\vec{A}_3 = 0$

$$\left.\begin{aligned} \mathrm{LHS} &= (E_1 + E_2)\pi r^2 + 0 = 2E_n \pi r^2 \\ \mathrm{RHS} &= \sigma_s \pi r^2 \end{aligned}\right\} \; E_n = \frac{\sigma_s}{2\varepsilon_o}$$

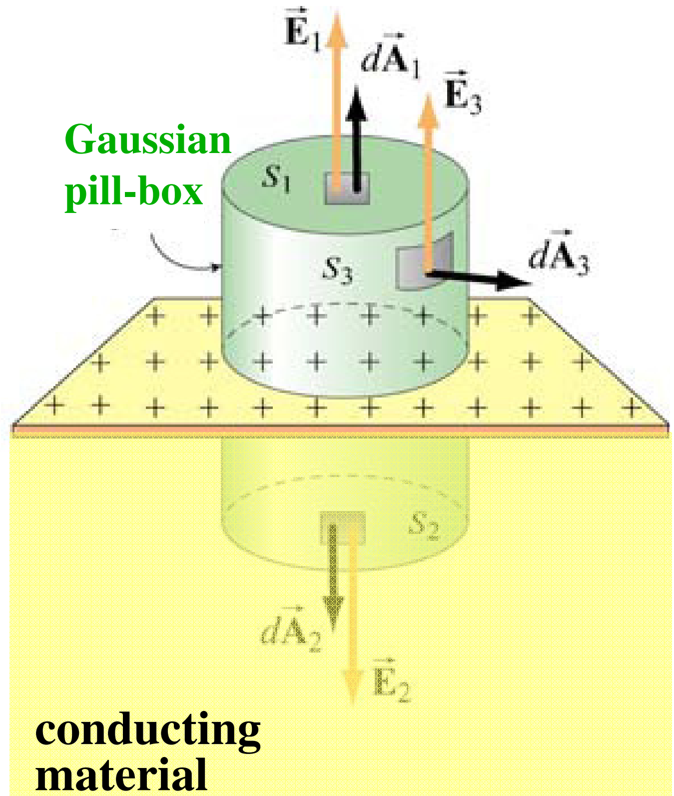
$\rightarrow$ E_n constant of all 3 coordinates (but cannot extrapolate result)

Gauss's Law

Example #3: large flat interface between metal and air

infer from symmetry:

- no non-normal component ($\vec{E} = E_n \hat{u}_n$)
- E_n constant of both planar coordinates



apply Gauss's Law to green pill-box:

(a) E = 0 in perfect electric conductor

(b) no contribution at S_3 as $\vec{E}_3 \bullet d\vec{A}_3 = 0$

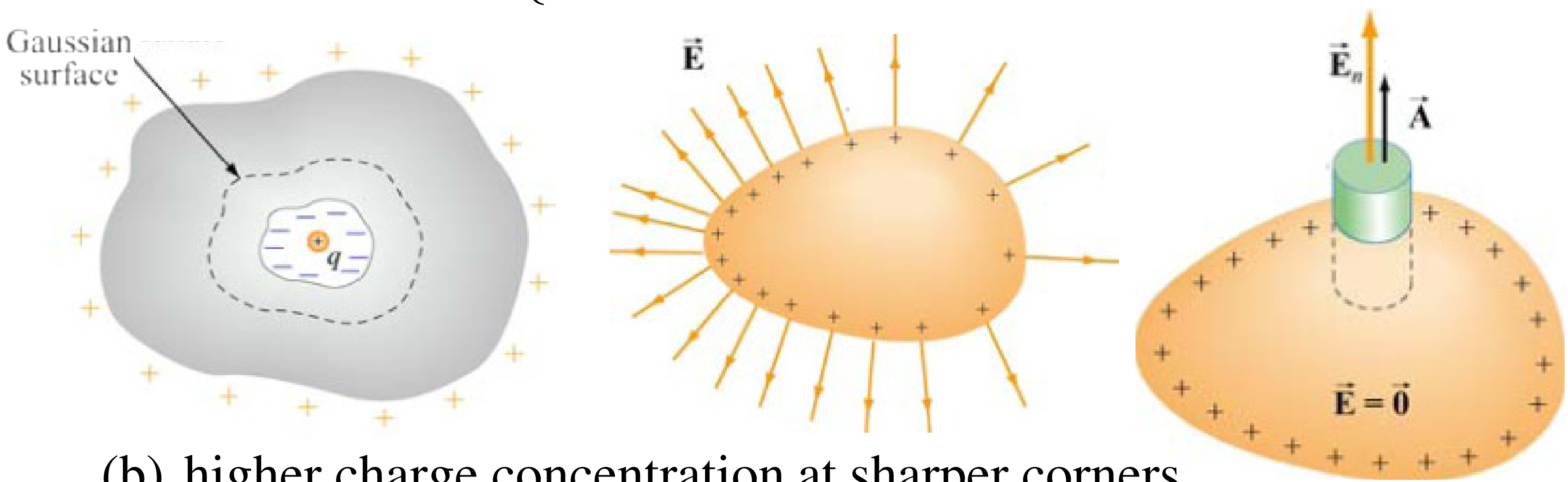
$$\left.\begin{aligned} \text{LHS} &= E_n \pi r^2 \\ \text{RHS} &= \sigma_s \pi r^2 \end{aligned}\right\} E_n = \frac{\sigma_s}{\varepsilon_o}$$

$\rightarrow$ E_n constant of all 3 coordinates (but cannot extrapolate result)

Gauss's Law

inside perfectly conducting material

(a) $Q_{\text{enclosed}} = 0 \Rightarrow \begin{cases} \textit{either} \text{ no charges within Gaussian surface} \\ \textit{or} \text{ zero sum of positive and negative charges} \end{cases}$



(b) higher charge concentration at sharper corners

(c) $\vec{E} = \frac{\sigma_s}{\varepsilon_o}\hat{u}_n$ in immediate vicinity of surface (where σ_s and $\hat{u}_n$ generally vary)

Electric Potential

inconvenient to use $\vec{E}$ when studying system of electric charges

- requires vector calculus for analysis
- cumbersome for visual representation

conservative nature of electro<u>static</u> fields

$$\oint_{\substack{\text{any}\\\text{loop}}} \vec{E} \bullet d\vec{s} = 0 \quad \rightarrow \quad \iint_{\substack{\text{any}\\\text{area}}} \nabla \times \vec{E} \bullet d\vec{A} = 0 \quad \rightarrow \quad \nabla \times \vec{E} = \vec{0}$$

define scalar potential (via null identity #1)

$\vec{E} = -\nabla V$ (or $\vec{E} = -\nabla \phi$ especially in physics textbooks)

- negative sign by convention (so as to obtain positive V values)
- also extended to time-varying $\vec{E}$ (although non-conservative)

Electric Potential

integral definition for potential

$$V_{\mathrm{P}} = -\int_{\infty}^{\mathrm{P}} \vec{E} \bullet d\vec{s} \qquad\qquad \Delta V = V_{\mathrm{P}} - V_{\mathrm{Q}} = -\int_{\mathrm{Q}}^{\mathrm{P}} \vec{E} \bullet d\vec{s}$$

(a) dimensions of definition equation

Volt x Coulomb = Newton x metre = Joule

$\Rightarrow$ Volt = Joule / Coulomb

interpret V as work done in moving unit positive charge

(b) require reference (due to need for constant of integration)

common to set $V = 0$ at infinity

but not tenable for infinitely-large systems

(c) same V (or ΔV) for any path in conservative field

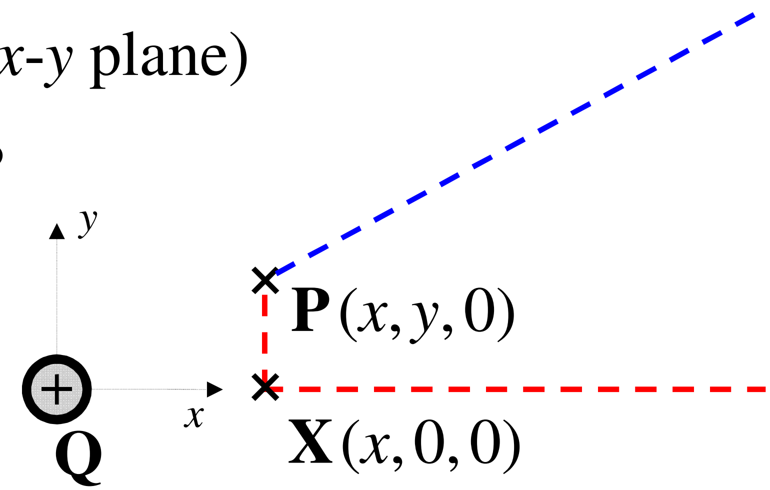
Electric Potential

Example #1: point charge Q (in x-y plane)

(a) blue (radial) path from ∞ to P

$$V_P = -\int_{\infty}^{P} \frac{Q}{4\pi\varepsilon_o r^2}\hat{u}_r \bullet dr\,\hat{u}_r$$

$$= -\frac{Q}{4\pi\varepsilon_o}\left[-\frac{1}{r}\right]_{\infty}^{r} = \frac{Q}{4\pi\varepsilon_o r}$$



(b) red path from ∞ to P via X

$$V_P = -\int_{\infty}^{x} \frac{Q}{4\pi\varepsilon_o x^2}\hat{u}_x \bullet dx\,\hat{u}_x - \int_{0}^{y} \frac{Q}{4\pi\varepsilon_o r^2}\hat{u}_r \bullet dy\,\hat{u}_y$$

$$= -\int_{\infty}^{x} \frac{Q}{4\pi\varepsilon_o x^2}dx - \int_{0}^{\sqrt{r^2-x^2}} \frac{Q\,y}{4\pi\varepsilon_o (x^2+y^2)^{\frac{3}{2}}}dy = \frac{Q}{4\pi\varepsilon_o r}$$

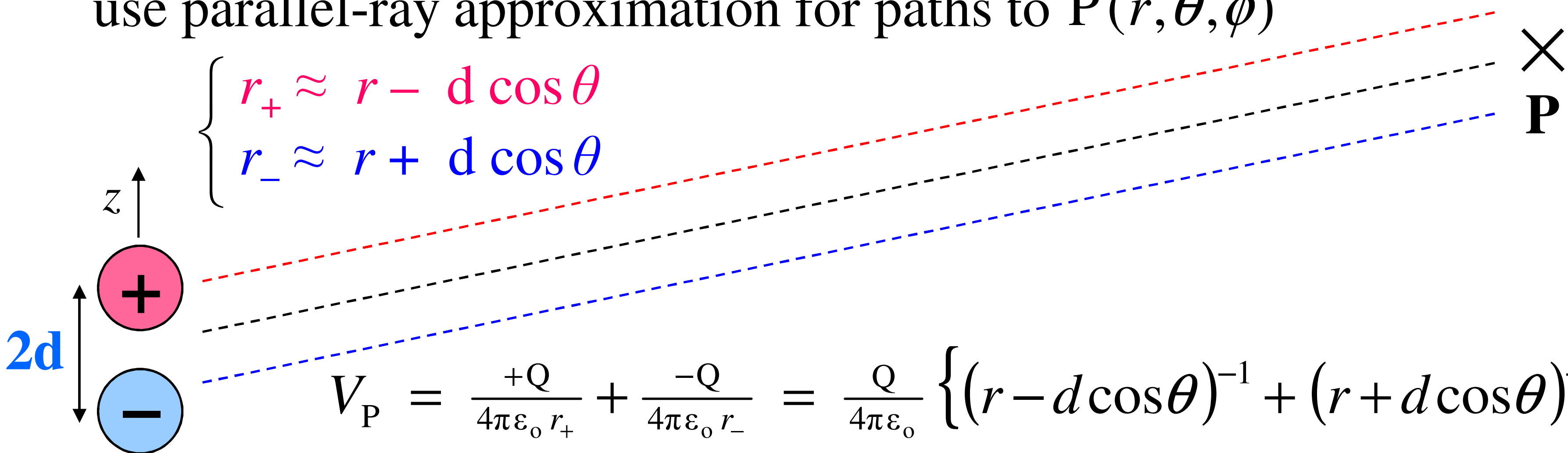
useful for deriving V of linear charge system via superposition

Electric Potential

Example #2: dipole with $r >> \mathrm{d}$

use parallel-ray approximation for paths to $\mathrm{P}(r,\theta,\phi)$

$$\begin{cases} r_+ \approx r - \mathrm{d}\cos\theta \\ r_- \approx r + \mathrm{d}\cos\theta \end{cases}$$



$$V_\mathrm{P} = \frac{+\mathrm{Q}}{4\pi\varepsilon_\mathrm{o} r_+} + \frac{-\mathrm{Q}}{4\pi\varepsilon_\mathrm{o} r_-} = \frac{\mathrm{Q}}{4\pi\varepsilon_\mathrm{o}}\left\{(r - d\cos\theta)^{-1} + (r + d\cos\theta)^{-1}\right\}$$

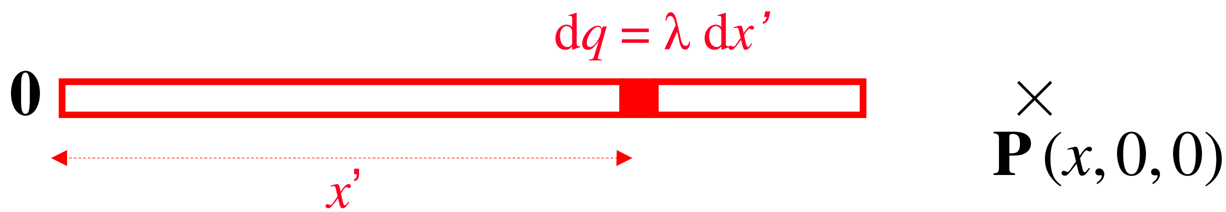
$$= \frac{\mathrm{Q}d\cos\theta}{2\pi\varepsilon_\mathrm{o} r^2}$$

easier to derive field

$$\vec{E} = -\nabla V = -\begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial \theta} \\ \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \end{bmatrix} \frac{\mathrm{Q}d\cos\theta}{2\pi\varepsilon_\mathrm{o} r^2} = \frac{\mathrm{Q}d}{4\pi\varepsilon_\mathrm{o} r^3}\begin{bmatrix} 2\cos\theta \\ \sin\theta \\ 0 \end{bmatrix}$$

Electric Potential

Example #3: rod (of length L) with linear charge density $\lambda\ \mathrm{C\,m^{-1}}$



$$dV_P = \frac{\lambda\, dx'}{4\pi\varepsilon_o\,(x - x')}$$ for each elemental length along rod

$$V_P = \frac{\lambda}{4\pi\varepsilon_o} \int_{x'=0}^{x'=x} \frac{dx'}{(x - x')} = \frac{\lambda}{4\pi\varepsilon_o} \ln \frac{x}{x - L}$$ (symmetry → depends only on x)

$$\vec{E}_P = -\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} V_P(x) = -\frac{d}{dx} V_P(x)\,\hat{u}_x = -\frac{\lambda}{4\pi\varepsilon_o} \frac{d}{dx}\left(\ln \frac{x}{x - L}\right)\hat{u}_x$$

$$= \frac{\lambda}{4\pi\varepsilon_o} \frac{L}{x(x - L)}\,\hat{u}_x$$

Electric Potential

Gauss's Law for any volume of distributed charges

$$\oiint_S \varepsilon \vec{E} \bullet d\vec{A} = Q_{\substack{\text{total in}\\\text{enclosure}}} = \iiint_V \sigma \, dV$$

$$\Rightarrow \quad \iiint_V \nabla \bullet \vec{E} \, dV = \tfrac{1}{\varepsilon} \iiint_V \sigma \, dV$$

need non-integral version for application to any particular point

$$\nabla \bullet \vec{E} = \tfrac{1}{\varepsilon} \sigma$$

more convenient to use V instead of $\vec{E}$ for analysis

$$\nabla \bullet (-\nabla V) = \tfrac{1}{\varepsilon} \sigma \;\rightarrow\; \begin{cases} \nabla^2 V = -\tfrac{1}{\varepsilon} \sigma & \text{Poisson equation} \\ \nabla^2 V = 0 & \text{Laplace equation} \end{cases}$$

partial differential equation requiring boundary conditions

Electric Potential

Laplacian operator:

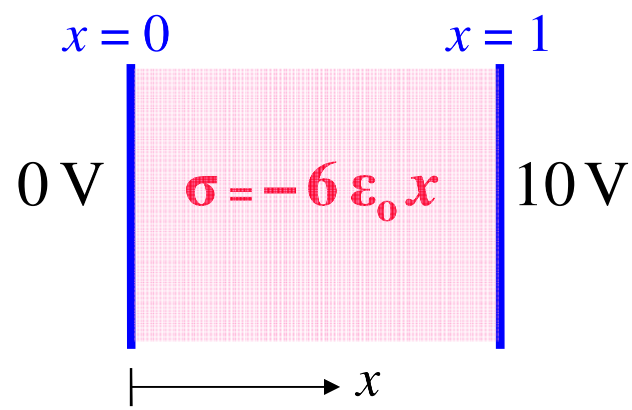
$$\nabla^2 = \nabla \bullet \nabla = \begin{cases} \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2} \\ \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \phi^2} + \dfrac{\partial^2}{\partial z^2} \\ \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial}{\partial r}\right) + \dfrac{1}{r^2 \sin\theta}\dfrac{\partial}{\partial \theta}\left(\sin\theta\dfrac{\partial}{\partial \theta}\right) + \dfrac{1}{r^2 \sin^2\theta}\dfrac{\partial^2}{\partial \phi^2} \end{cases}$$

uniqueness theorem:

only one possible solution that will satisfy partial differential equation (PDE) and all boundary conditions (BCs) regardless of approach (even by intelligent guess-work)

Electric Potential

Example #1: (one-dimensional) electron cloud between electrodes



no variations in $\hat{u}_y$ and $\hat{u}_z$ directions

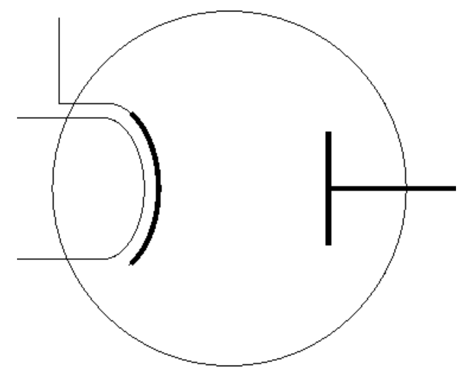
i.e. $$\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0$$

reduced to ordinary differential equation

$$\frac{d^2V}{dx^2} = -\frac{(-6\varepsilon_o x)}{\varepsilon_o} = 6x$$

$$\Rightarrow \quad V = x^3 + c_1 x + c_2$$

BCs: $$\left.\begin{aligned} V(x=0) &= 0 \\ V(x=1) &= 10 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} c_1 &= 9 \\ c_2 &= 0 \end{aligned}\right.$$

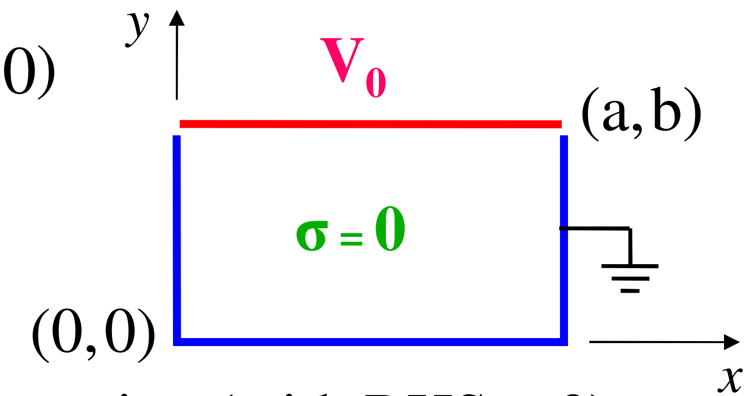


$$\therefore \quad V = x^3 + 9x \quad \text{and} \quad E_x = -3x^2 + 9$$

Electric Potential

Example #2**:** (two-dimensional) earthed trough with charged lid

no variations in $\hat{\mathrm{u}}_z$ direction (*i.e.* $\frac{\partial}{\partial z} = 0$)



reduced to PDE in two coordinates

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

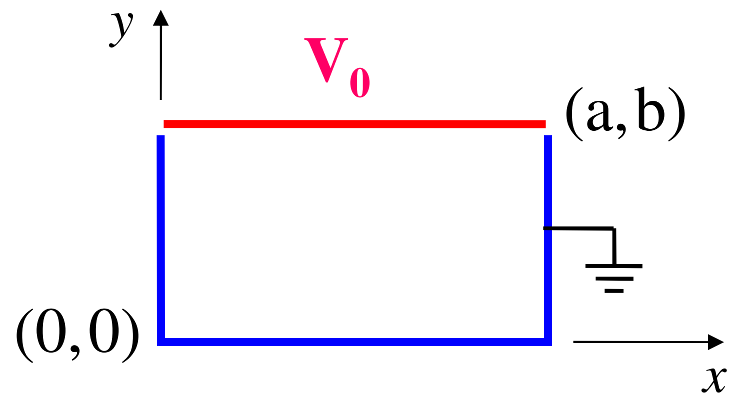
no free charges in trough → Laplace equation (with RHS = 0)

$$\text{try}\quad V(x,y) = X(x)\,Y(y) \;\rightarrow\; \begin{cases} \frac{\partial^2 V}{\partial x^2}(x,y) = \frac{\mathrm{d}^2 X}{\mathrm{d}x^2}(x)\,Y(y) \\ \frac{\partial^2 V}{\partial y^2}(x,y) = X(x)\,\frac{\mathrm{d}^2 Y}{\mathrm{d}x^2}(y) \end{cases}$$

$$\therefore\; \frac{1}{X}\frac{\mathrm{d}^2 X}{\mathrm{d}x^2}(x) + \frac{1}{Y}\frac{\mathrm{d}^2 Y}{\mathrm{d}y^2}(y) = 0 \;\rightarrow\; \begin{cases} \frac{1}{X}\frac{\mathrm{d}^2 X}{\mathrm{d}x^2}(x) = -\mathrm{c}^2 \\ \frac{1}{Y}\frac{\mathrm{d}^2 Y}{\mathrm{d}y^2}(y) = +\mathrm{c}^2 \end{cases} \qquad (\mathrm{c} = \text{constant})$$

Electric Potential

choice of sign for constant determined by boundary conditions



BC1: $V(0,y)=0 \;\rightarrow\; X = \mathrm{A}\sin(\mathrm{c}\,x)$

BC2: $V(x,0)=0 \;\rightarrow\; Y = \mathrm{B}\sinh(\mathrm{c}\,y)$

BC3: $V(\mathrm{a},y)=0 \;\rightarrow\; \sin(\mathrm{c\,a})=0$

BC4: $V(x,\mathrm{b}) = \mathrm{V}_0$

eigenvalues from BC3 $\Rightarrow$ family of solutions with $\mathrm{c}_m = \frac{m\pi}{\mathrm{a}}$

not possible to satisfy BC4 with only one eigen-solution

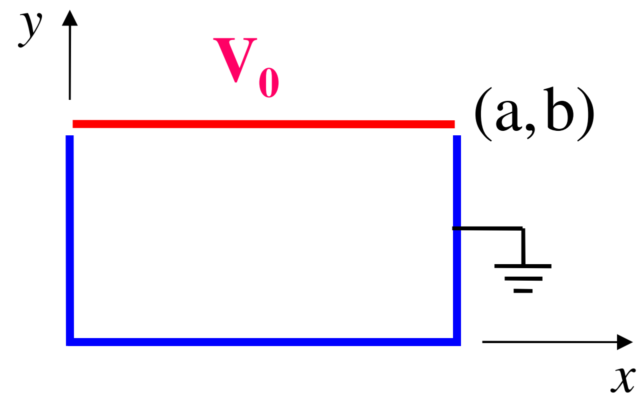
try superposition of all harmonics

$$\therefore \quad V(x,y) = X(x)\,Y(y) = \sum_m \mathrm{d}_m \sin\left(m\pi \tfrac{x}{\mathrm{a}}\right)\sinh\left(m\pi \tfrac{y}{\mathrm{a}}\right)$$

Electric Potential

obtain modal coefficient d_m from BC4 (where $y = b$ for charged lid)

$$\sum_m d_m \sin\left(m\pi \tfrac{x}{a}\right) \sinh\left(m\pi \tfrac{b}{a}\right) = V_0$$



capitalize on orthogonality of $\sin\left(m\pi \tfrac{x}{a}\right)$

(a) multiply both sides by $\sin\left(n\pi \tfrac{x}{a}\right)$

(b) integrate over $0 \le x \le a$ (using process for Fourier series)

$$\int_{x=0}^{a} \sum_m d_m \left\{\sin\left(m\pi \tfrac{x}{a}\right) \sin\left(n\pi \tfrac{x}{a}\right)\right\} \sinh\left(m\pi \tfrac{b}{a}\right) dx = \int_{x=0}^{a} V_0 \sin\left(n\pi \tfrac{x}{a}\right) dx$$

$\Rightarrow \quad d_n = \dfrac{4 V_0}{n\pi \sinh\left(n\pi \frac{b}{a}\right)}$ when n is odd n but $d_n = 0$ when n is even

$$\therefore \quad V(x, y) = \frac{4 V_0}{\pi} \sum_m \frac{\sin\left(m\pi \frac{x}{a}\right) \sinh\left(m\pi \frac{y}{a}\right)}{m \sinh\left(m\pi \frac{b}{a}\right)} \quad \text{for } m = 1, 3, 5, \ldots$$

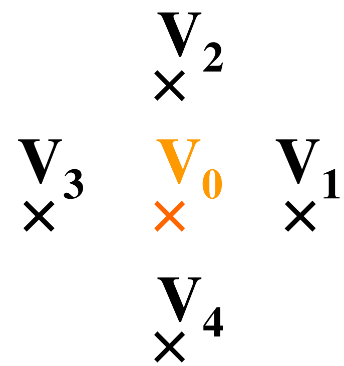
Electric Potential

approximate solutions via numerical techniques

(a) popular after availability of computational resources

(b) must check for possibility of spurious solutions

two-dimensional finite differences



$$\frac{\partial^2 V}{\partial x^2} = \frac{\Delta\left(\frac{\partial V}{\partial x}\right)}{\Delta x} = \frac{\frac{V_1 - V_0}{\delta} - \frac{V_0 - V_3}{\delta}}{\delta} = \frac{V_1 + V_3 - 2V_0}{\delta^2}$$

$$\frac{\partial^2 V}{\partial y^2} = \frac{\Delta\left(\frac{\partial V}{\partial y}\right)}{\Delta y} = \frac{\frac{V_2 - V_0}{\delta} - \frac{V_0 - V_4}{\delta}}{\delta} = \frac{V_2 + V_4 - 2V_0}{\delta^2}$$

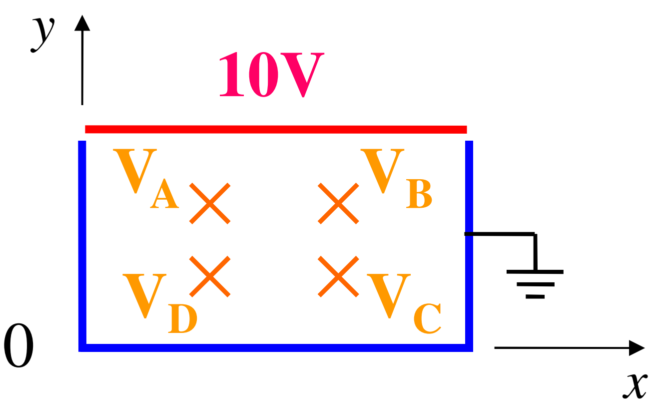
$$\rightarrow \quad \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{V_1 + V_2 + V_3 + V_4 - 4V_0}{\delta^2}$$

Electric Potential

Poisson equation in two-dimensional finite-difference form

$$\frac{V_1 + V_2 + V_3 + V_4 - 4V_0}{\delta^2} = -\frac{\sigma}{\varepsilon_0}$$

$$\Rightarrow \quad V_0 = \tfrac{1}{4}\left(\sum_{k=1}^{4} V_k + \frac{\sigma}{\varepsilon_0}\delta^2\right)$$



Laplace equation for Example #2 with $\sigma = 0$

choose (appropriate) initial values: $V_A = V_B = 5V$ and $V_C = V_D = 0$

first iteration: $V_A = \tfrac{1}{4}(10 + V_B + V_D + 0) = \frac{10+5+0+0}{4} = 3.75\ \text{V}$

$V_B = \tfrac{1}{4}(10 + 0 + V_C + V_A) = \frac{10+0+0+3.75}{4} = 3.44\ \text{V}$

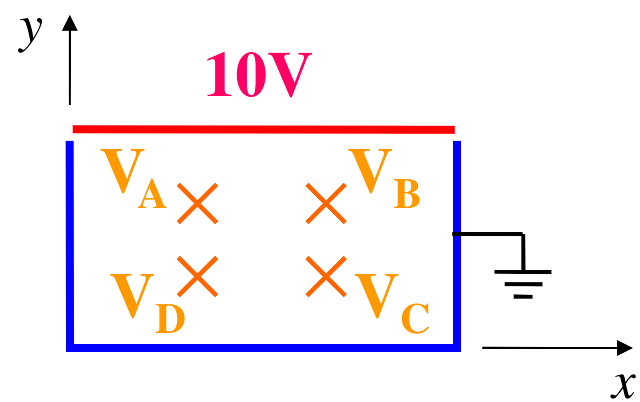
$V_C = \tfrac{1}{4}(V_B + 0 + 0 + V_D) = \frac{3.44+0+0+0}{4} = 0.86\ \text{V}$

$V_D = \tfrac{1}{4}(V_A + V_C + 0 + 0) = \frac{3.75+0.86+0+0}{4} = 1.15\ \text{V}$

next iteration: $V_A = \tfrac{1}{4}(10 + V_B + V_D + 0) = \frac{10+3.44+1.15+0}{4} = 3.65\ \text{V}$

Electric Potential

iteration #	V_A	V_B	V_C	V_D	
0	5.00	5.00	0	0	
1	3.75	3.44	0.36	1.15	
2	3.65	3.63	1.20	1.21	
3	3.71	3.73	1.24	1.24	
4	3.74	3.75	1.25	1.25	
5	3.75	3.75	1.25	1.25	→ onset of convergence



check numerical value of V_A against analytical result

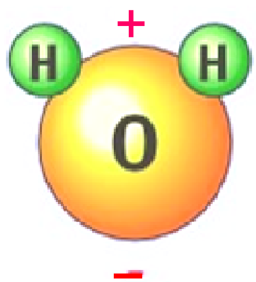
$$V\left(\frac{a}{3},\frac{2b}{3}\right) = \frac{40}{\pi}\sum_{m:\text{odd}}\frac{\sin\left(\frac{m\pi}{3}\right)\sinh\left(\frac{2m\pi}{3}\right)}{m\,\sinh(m\pi)} = 3.81\text{ V} \quad \text{for } a=b=1$$

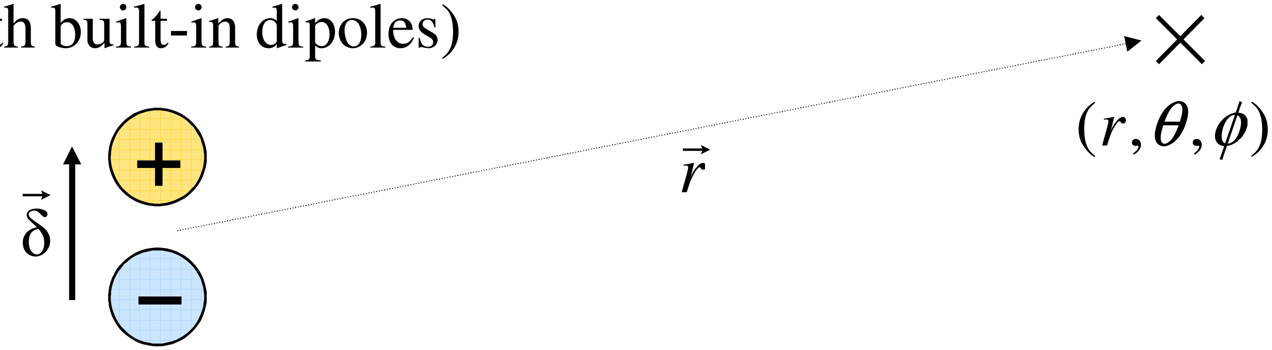
can improve accuracy by reducing mesh size (*i.e.* $\delta \ll$ a and b)

quite robust (try using $V_A = V_B = V_C = V_D = 0$ as initial values)

Dielectrics

polar material (with built-in dipoles)





define $\vec{p} = q\vec{\delta}$ for dipole $\pm$ q separated by δ

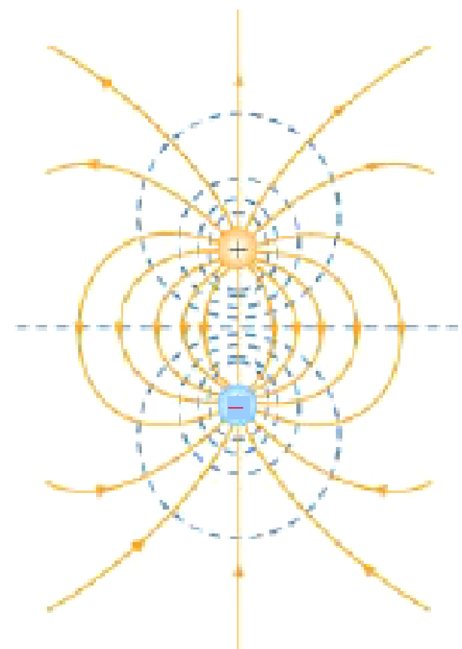
derive $\vec{E}$ from $V(r,\theta,\phi) = \frac{q\delta\cos\theta}{4\pi\varepsilon_o r^2} = \frac{\vec{p}\bullet\vec{r}}{4\pi\varepsilon_o r^3}$

not possible to consider individual dipoles

have to consider average density instead

define polarization vector $\vec{P} = \frac{1}{\text{volume}}\sum_k \vec{p}_k$

$\therefore\ dV(r,\theta,\phi) = \frac{\vec{P}\bullet\vec{r}}{4\pi\varepsilon_o r^3}d\mathrm{V}$ for dipoles in dV



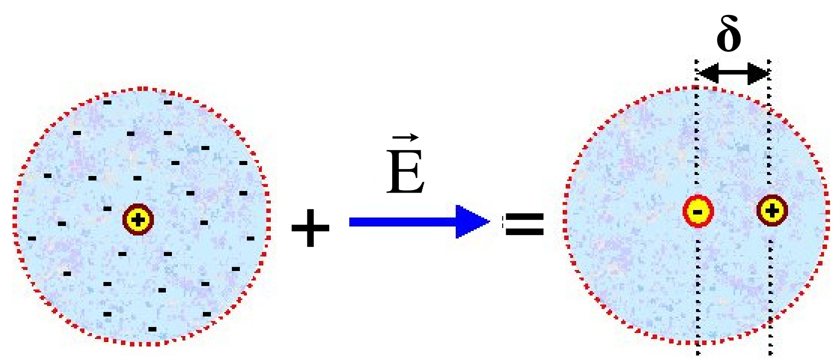
Dielectrics

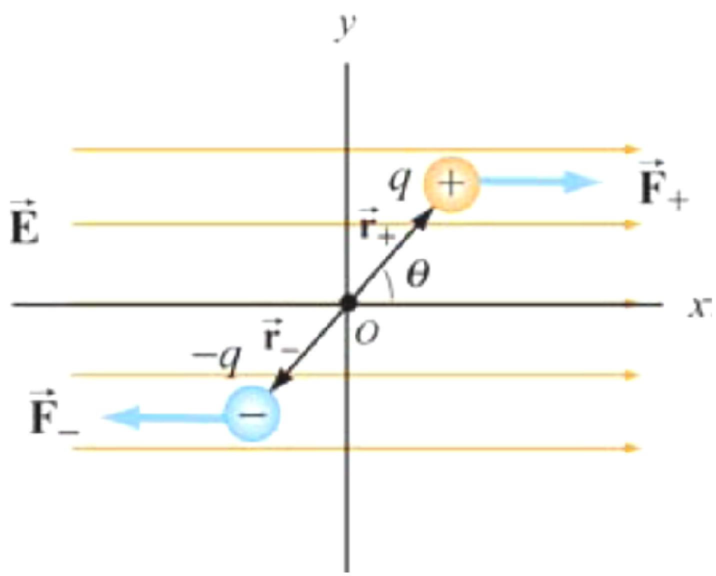
non-polar materials

common ± charge centers

separation induced by $\vec{E}$

dipole behavior in presence of $\vec{E}$





electric forces on dipole charges

no net force → no $\pm x$ shift for dipole

dipole moment due to $\pm q$ separation

rotation until alignment of forces

linear increase of $|\vec{P}|$ until saturation

Dielectrics

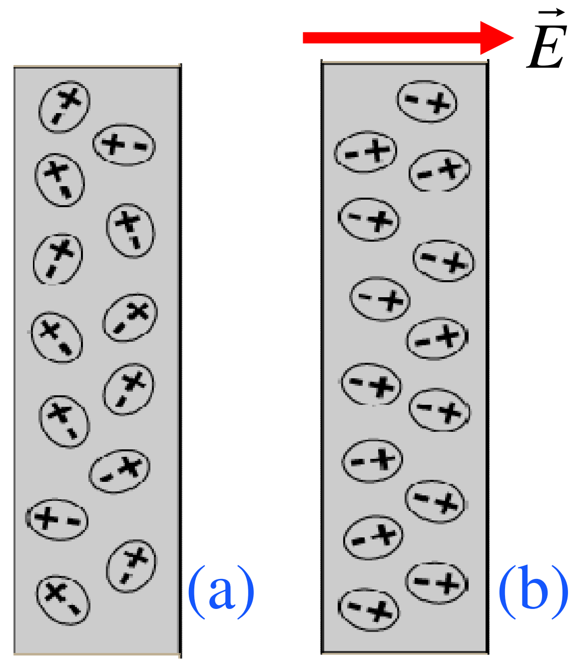
dielectric slab

(a) originally random dipole orientations

$\rightarrow$ no net electric fields from dipoles

(b) partial dipole alignment under $\vec{E}$

$\rightarrow$ need to include polarization $\vec{P}$



Gauss's Law in differential form

$$\nabla \bullet \vec{E} = \tfrac{1}{\varepsilon_0}\sigma \qquad \text{for free space}$$

$$\nabla \bullet \left(\varepsilon_0 \vec{E} + \vec{P}\right) = \sigma \qquad \text{for dielectric}$$

recast as $\nabla \bullet \vec{D} = \sigma$ by introducing $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$

electric flux density vector ($\mathrm{C\,m^{-2}}$)

Dielectrics

field-matter interactions

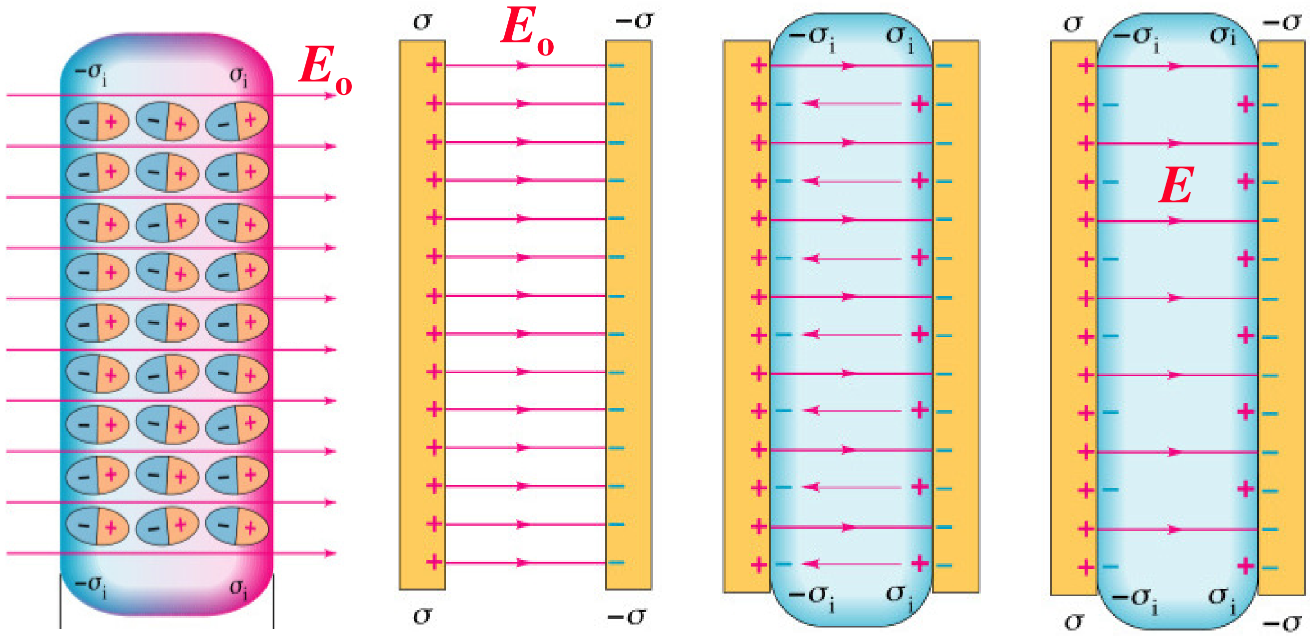
$$\vec{P} = \varepsilon_0 \chi_e \vec{E}$$

$$\vec{D} = (1+\chi_e)\,\varepsilon_0 \vec{E}$$

$$= \varepsilon_r \varepsilon_0 \vec{E} = \varepsilon \vec{E}$$

χ_e susceptibility

ε permittivity



simply replace ε_o by ε to account for presence of dielectric

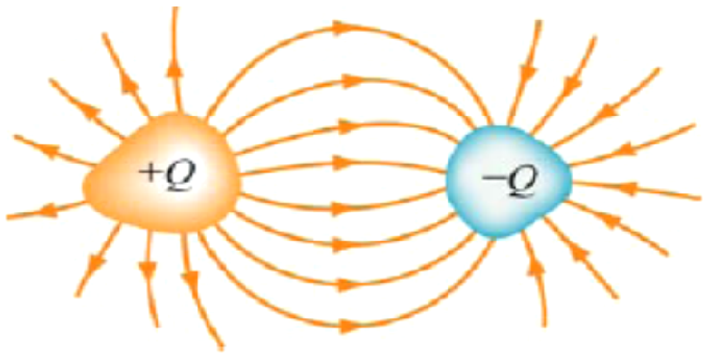
point-charge example: $\vec{E}_0 = \dfrac{q}{4\pi\varepsilon_0 r^2}\,\hat{u}_r$ becomes $\vec{E} = \dfrac{q}{4\pi\varepsilon\, r^2}\,\hat{u}_r$

$\left|\vec{E}\right| < \left|\vec{E}_o\right|$ due to net electric fields of dipoles (after alignment)

Capacitors

two conductors separated by air/dielectric with

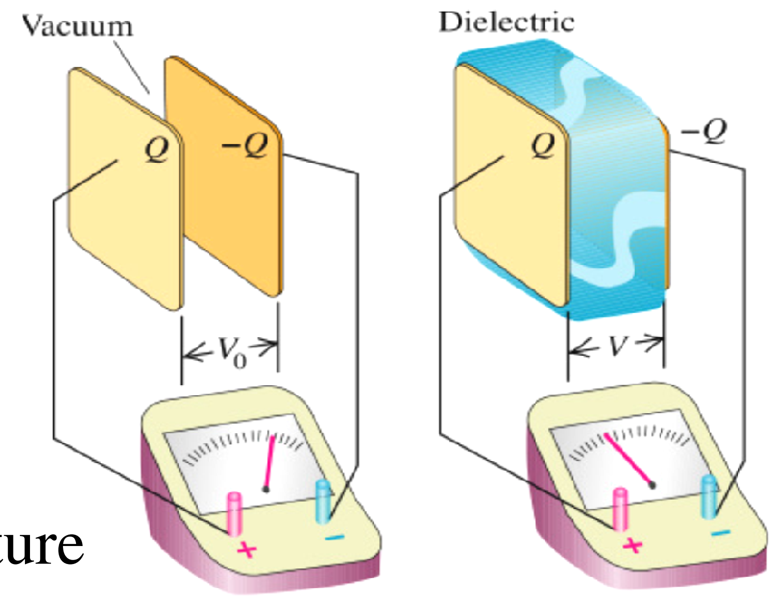
- equal and opposite charges $\pm Q$
- potential difference $\Delta V = V_1 - V_2$



experiments $\rightarrow$ $Q \propto \Delta V$

define storage parameter $C = \dfrac{Q}{\Delta V}$

$(\mathrm{F} = \mathrm{C\ V^{-1}})$



can be increased via**:**

(a) increasing dielectric permittivity

(b) reducing conductor separation

(c) proper design of conductor structure

Capacitors

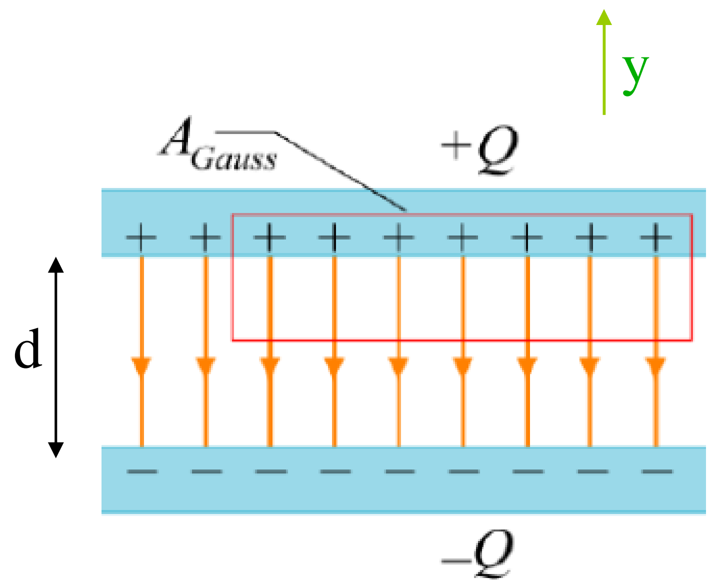
usual procedure to evaluate capacitance

(a) assume $\pm Q$ on conductors

(b) derive $\vec{E}$ from charge distribution

(c) derive ΔV via potential definition

(d) compute ratio



Example #1: large parallel plates

apply Gauss's Law to pill-box

$$E_y\, A_{\text{pill-box}} = -\frac{1}{\varepsilon_o}\sigma_s\, A_{\text{pill-box}} \quad \Rightarrow \quad E_y = -\frac{1}{\varepsilon_o}\sigma_s$$

$$\Delta V = -\frac{-Q}{\varepsilon_o A}\int_0^d dy = \frac{Q\,d}{\varepsilon_o A} \quad \Rightarrow \quad C = \frac{A\varepsilon_o}{d}$$

larger C by inserting dielectric, increasing A or decreasing d

Capacitors

Example #2: coaxial cable with uniform charge density λ C m^{-1}

apply Gauss's Law to green cylinder

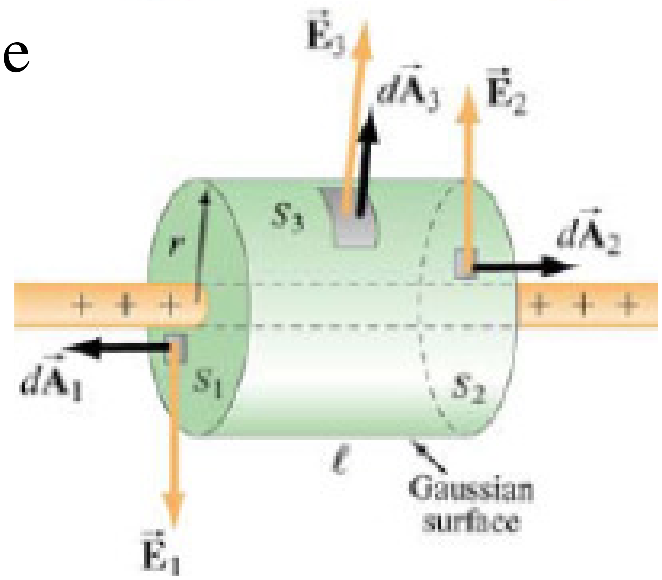
infer from symmetry (for short l):

- no non-radial component (*i.e.* $\vec{E} = E_r \hat{u}_r$)
- E_r constant of z coordinate
- zero contribution from either end surface

$$2\pi r l E_3 = \frac{1}{\varepsilon_o}\lambda l \quad \Rightarrow \quad E_3 = \frac{\lambda}{2\pi\varepsilon_o r}$$

$$V_a - V_b = -\frac{\lambda}{2\pi\varepsilon_o}\int_b^a \frac{dr}{r} = \frac{\lambda}{2\pi\varepsilon_o}\ln\left(\frac{b}{a}\right)$$

$$\frac{C}{l} = \frac{\lambda}{\Delta V} = \frac{2\pi\varepsilon_o}{\ln\left(\frac{b}{a}\right)}$$



used in Z_0 formula (only for TEM mode)

Boundary Conditions

boundary between two materials (for electrostatics)

- field behavior affected by polarization in materials
- boundary conditions for PDEs $\nabla \times \vec{E} = \vec{0}$ and $\nabla \bullet \vec{E} = \frac{1}{\varepsilon}\sigma$

need to establish field inter-relationships at boundary

(a) tangential $\vec{\mathrm{E}}$ component

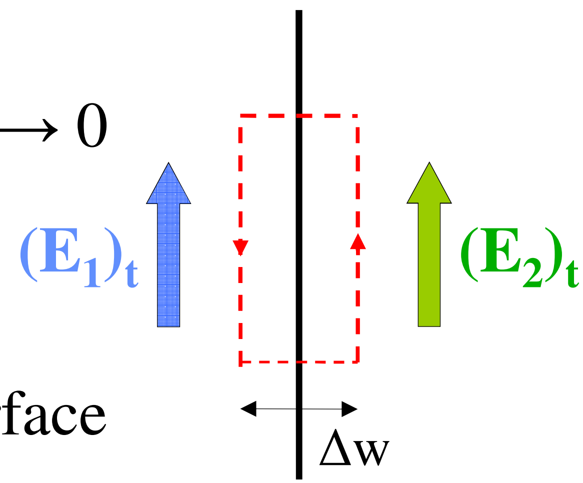
apply $\oint \vec{\mathrm{E}} \bullet d\vec{s} = 0$ to red loop where $\Delta w \to 0$

negligible contributions from widths

$\Rightarrow$ $(E_1)_t = (E_2)_t$

i.e. E_t component continuous across interface

generally extended to time-varying cases as well



Boundary Conditions

boundary between two materials (for electrostatics)

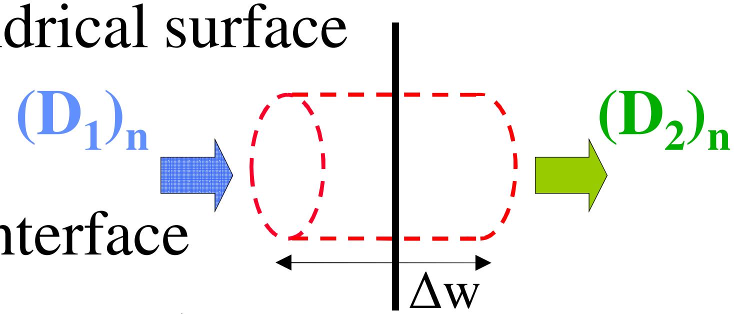
(b) normal $\vec{E}$ component but consider $\vec{D}$ instead

replace $\oiint_S \vec{E} \bullet d\vec{A} = \frac{1}{\varepsilon} Q_{\substack{\text{total in}\\ \text{enclosure}}}$ by $\oiint_S \vec{D} \bullet d\vec{A} = Q_{\substack{\text{total in}\\ \text{enclosure}}}$

apply Gauss's Law (based on $\vec{D}$) to red pill-box where $\Delta w \to 0$

negligible contributions from cylindrical surface

$\Rightarrow$ $(D_1)_n = (D_2)_n$



D_n component continuous across interface

generally extended to time-varying cases too

special case for metal where $|E| = |D| = 0$ and $Q_{enclosed} = \sigma_s A \neq 0$

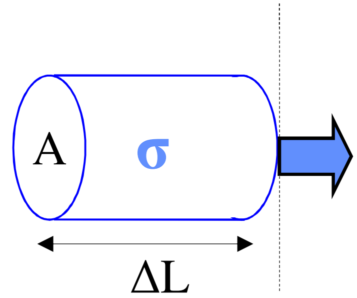
$$\vec{E}_{air} = 0\,\hat{u}_t + \frac{\sigma_s}{\varepsilon_o}\hat{u}_n$$

Currents

current density vector

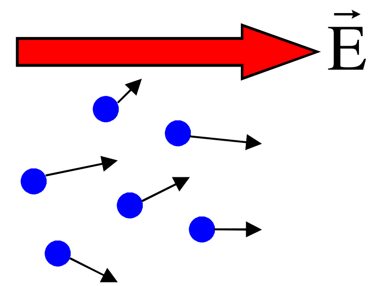
- flow of charges ΔQ contained in $A\,\Delta L$

$$I = \frac{\Delta Q}{\Delta t} = \frac{\sigma(A\,\Delta L)}{\Delta t} = A\sigma v$$



- average velocity of charges

$$\vec{v} = \frac{1}{M}\sum_{m=1}^{M} \vec{v}_m$$

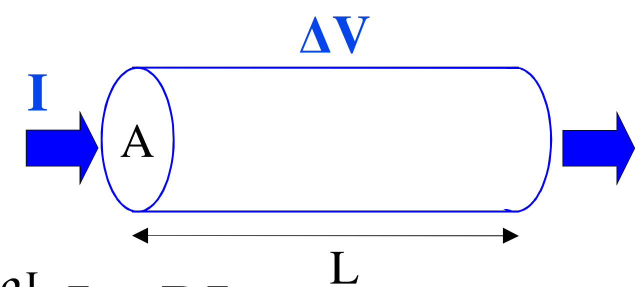


- mobility under influence of $\vec{E}$

$$\vec{v} \propto \vec{E} \quad \rightarrow \quad \vec{v} = \xi \vec{E}$$

- material property: resistivity $\rho = \frac{1}{\sigma\xi}$

$$\vec{J} = \sigma\vec{v} = \frac{1}{\rho}\vec{E}$$

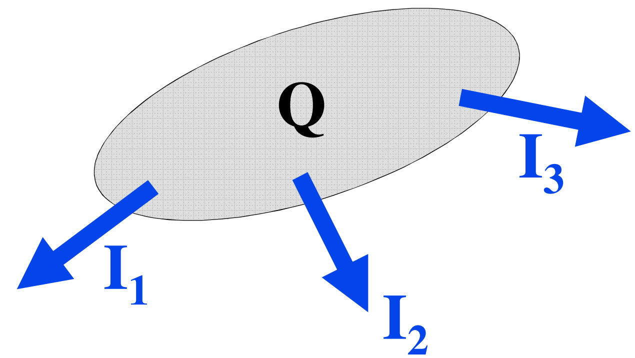


- Ohm's Law for wire $\frac{I}{A} = \frac{1}{\rho}\frac{\Delta V}{L} \rightarrow \Delta V = \frac{\rho L}{A} I = RI$

Currents

continuity equation

$$\sum_m I_m = -\frac{dQ}{dt}$$



or $$\oiint \vec{J} \bullet d\vec{A} = -\frac{d}{dt}\iiint \sigma \, dV$$

$\Rightarrow \iiint \nabla \bullet \vec{J} \, dV = -\iiint \frac{d}{dt} \sigma \, dV$ for application to any volume

$\Rightarrow \nabla \bullet \vec{J} = -\frac{d}{dt}\sigma$ for application to any point

for steady state (*i.e.* $\frac{d}{dt} = 0$), no change of total charge in enclosure

$\nabla \bullet \vec{J} = 0$ or $\sum_m I_m = 0$ Kirchhoff's Current Law

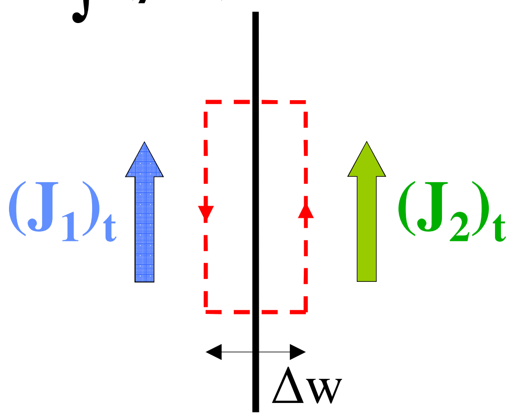
$\nabla \times \vec{E} = \nabla \times (\rho \vec{J}) = \vec{0}$ conservative property

Currents

boundary conditions (for static case)

similar to equations for $\vec{E}$ and $\vec{D} \rightarrow$ use same process with $\Delta w \approx 0$

$$\oint (\rho \vec{J}) \bullet d\vec{s} = 0$$

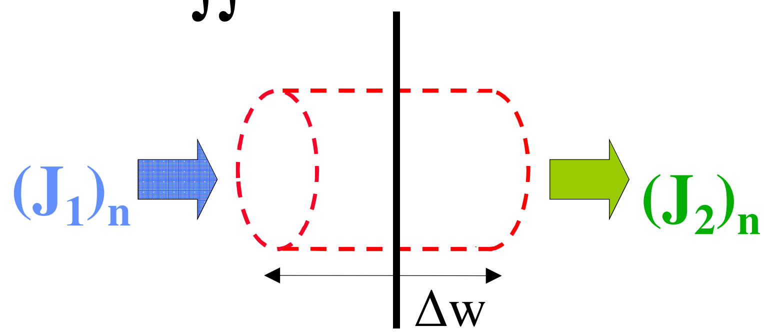


$$\rho_1 (J_1)_t = \rho_2 (J_2)_t$$

J_t discontinuous across interface

$\rightarrow$ change of incident angle for $\vec{J}$ when crossing interface

$$\oiint \vec{J} \bullet d\vec{A} = 0$$



$$(J_1)_n = (J_2)_n$$

J_n continuous across interface

Resistance

resistance between conductors (not resistance along conductor)

$$\mathrm{R} = \frac{\Delta V}{I} = \frac{-\oint \vec{E} \bullet d\vec{s}}{\oiint \vec{J} \bullet d\vec{A}} = -\rho \frac{\oint \vec{E} \bullet d\vec{s}}{\oiint \vec{E} \bullet d\vec{A}}$$

$$\mathrm{C} = \frac{Q}{\Delta V} = \frac{\oiint \vec{D} \bullet d\vec{A}}{-\oint \vec{E} \bullet d\vec{s}} = -\varepsilon \frac{\oiint \vec{E} \bullet d\vec{A}}{\oint \vec{E} \bullet d\vec{s}}$$

time constant $\mathrm{RC} = \rho\,\varepsilon$

(a) independent of design

(b) possible economy of effort

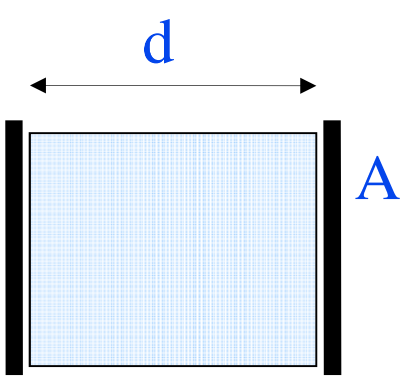


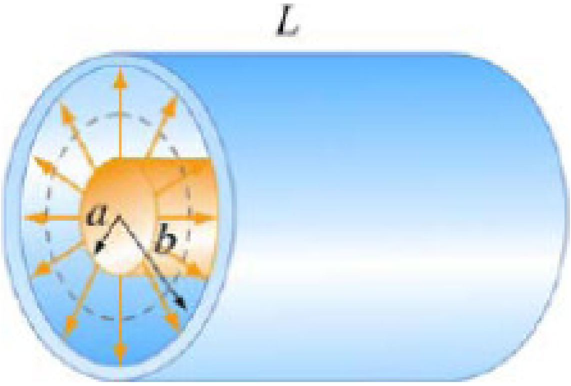
illustration: resistance between parallel plates

$$\mathrm{R} = \frac{\rho \mathrm{d}}{\mathrm{A}}$$

$$c.f.\ \mathrm{C} = \frac{\varepsilon \mathrm{A}}{\mathrm{d}}$$

Resistance

leakage conductance between conductors



re-visit coaxial cable and add $\vec{J} = \rho \vec{E}$

$$E_r = \rho J_r = \frac{\Delta V}{r \ln\left(\frac{b}{a}\right)}$$

$$I = \iint J_r \, dA = \frac{\Delta V}{\rho \ln\left(\frac{b}{a}\right)} \iint \frac{1}{r} r \, d\phi \, dz = \frac{2\pi L \Delta V}{\rho \ln\left(\frac{b}{a}\right)} \quad \Rightarrow \quad R = \frac{\rho \ln\left(\frac{b}{a}\right)}{2\pi L}$$

compare with capacitance formula (for TEM mode) $\quad C = \dfrac{2\pi \varepsilon L}{\ln\left(\frac{b}{a}\right)}$

shunt components in transmission-line model