# MA1506

Mathematics II

Section 6.6

Linear transformations usually changes direction of vectors

$$\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

---

$$\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

parallel

# Eigenvector

$$\boxed{T\vec{u} = \lambda\vec{u}}$$

eigenvalue    eigenvector

***T*** does not change the direction!

## 6.7 Eigenvalues and Eigenvectors

$$\boxed{T\vec{u} = \lambda\vec{u}}$$

eigenvalue    eigenvector

$$T\vec{0} = \lambda\vec{0}$$

trivial case,
not interesting

---

$$T\vec{u} = \lambda\vec{u} = \lambda I\vec{u} \Rightarrow (T - \lambda I)\vec{u} = \vec{0}$$

mapped to **0**

$$\boxed{\det(T - \lambda I) = 0}$$

## Example

$$\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

---

Find eigenvalues of $\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$ $\boxed{\lambda = 2, -3}$

$$\det\left( \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 0$$

$$\Rightarrow \quad \det\left( \begin{bmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix} \right) = 0$$

$$\Rightarrow \quad -(1-\lambda)(2+\lambda) - 4 = 0$$

Remark: Eigenvectors are never unique!

$$T\vec{u} = \lambda\vec{u}$$

$$\Longrightarrow \quad T(2\vec{u}) = 2T\vec{u} = 2\lambda\vec{u} = \lambda(2\vec{u}).$$

Any multiple of an eigenvector is also an eigenvector

## Finding Eigenvectors

$$\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \qquad \lambda = 2, -3$$

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$ Eigenvector associated to 2

$$(T - 2I)\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1-2 & 2 \\ 2 & -2-2 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \vec{0}$$

$$\Rightarrow \left.\begin{array}{rcl} -\alpha + 2\beta & = & 0 \\ 2\alpha - 4\beta & = & 0 \end{array}\right\}$$ Multiples of same equation

## Finding Eigenvectors

$$\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \qquad \lambda = 2, -3$$

$$\boxed{-\alpha + 2\beta = 0}$$

1 equation 2 unknowns

Choose $\alpha = 1 \Rightarrow \beta = \frac{1}{2}$

$$\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$$

Eigenvector associated to eigenvalue 2

## Finding Eigenvectors

$$\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \qquad \lambda = 2, -3$$

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$ Eigenvector associated to -3

$$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \vec{0} \Rightarrow \left.\begin{aligned} 4\alpha + 2\beta &= 0 \\ 2\alpha + \beta &= 0 \end{aligned}\right\} \text{Multiples}$$

Choose $\alpha = 1$ $\Rightarrow$ $\beta = -2$

So $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

## Example: Find eigenvalues and corr eigenvectors

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + 1 = 0$$

$$\Rightarrow \lambda = \pm i$$

$$\lambda = i$$

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} 1 \\ \beta \end{bmatrix} = \vec{0}$$

$$\Rightarrow -i - \beta = 0 \text{ and } 1 - i\beta = 0$$

$\beta = -i$ eigenvector $\begin{bmatrix} 1 \\ -i \end{bmatrix}$

## Example: Find eigenvalues and corr eigenvectors

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + 1 = 0$$

$$\Rightarrow \lambda = \pm i$$

$$\lambda = -i$$

$$\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \begin{bmatrix} 1 \\ \beta \end{bmatrix} = \vec{0}$$

$$\Rightarrow i - \beta = 0 \text{ and } 1 + i\beta = 0$$

$\beta = i$ eigenvector $\begin{bmatrix} 1 \\ i \end{bmatrix}$

## Remark

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \lambda = i \quad \lambda = -i$$

$$\begin{bmatrix} 1 \\ -i \end{bmatrix} \quad \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Rotation through 90 degrees,

Every real vector should change direction

## 6.8 Diagonal Form of A Linear Transformation

***T*** with respect to $\widehat{i}, \widehat{j}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$T\widehat{i} = a\widehat{i} + c\widehat{j} = \begin{bmatrix} a \\ c \end{bmatrix}$$

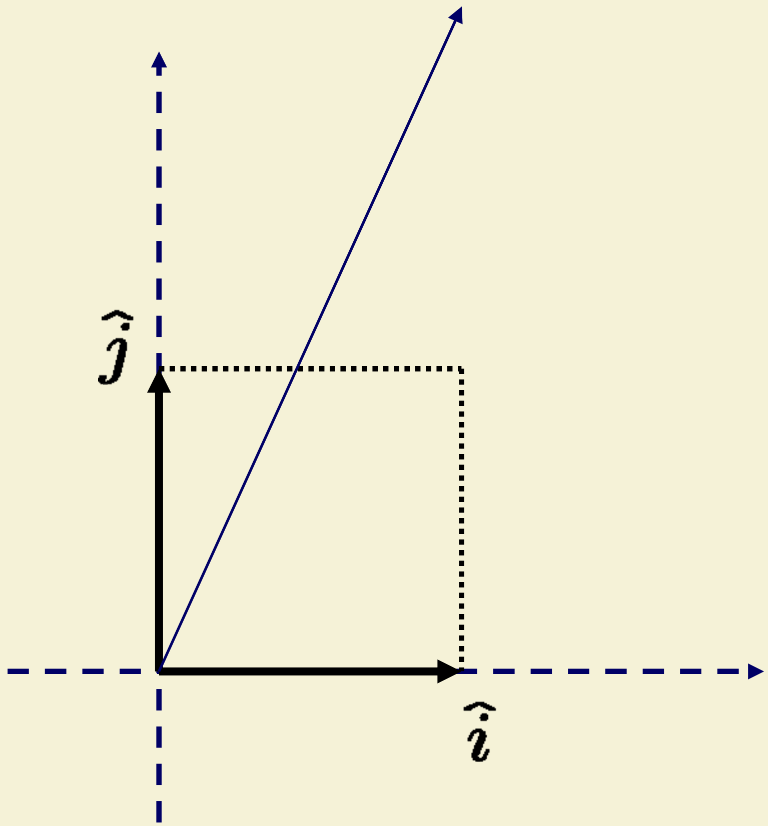
$$T\widehat{j} = b\widehat{i} + d\widehat{j} = \begin{bmatrix} b \\ d \end{bmatrix}$$

In 2D, we can replace $\widehat{i}, \widehat{j}$

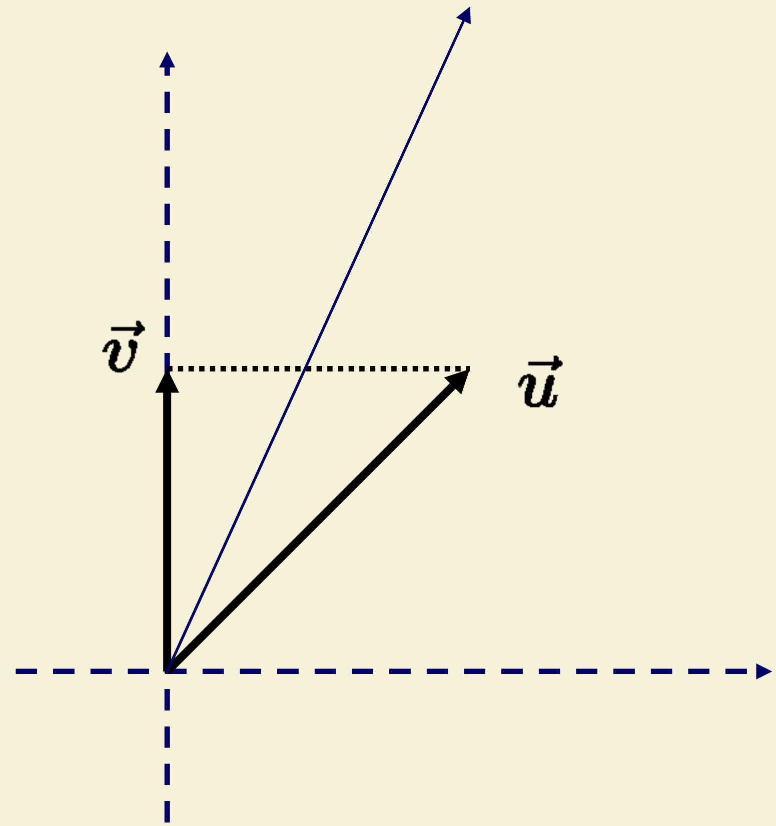
with any pair of non parallel $\vec{u}, \vec{v}$

$$\vec{w} = \alpha\vec{u} + \beta\vec{v}$$

$$\vec{w} = 1\hat{i} + 2\hat{j}$$




$$\vec{w} = 1\vec{u} + 1\vec{v}$$



## 6.8 Diagonal Form of A Linear Transformation

$(\vec{u}, \vec{v})$ called a basis

$$\begin{aligned}\vec{u} &= P_{11}\hat{i} + P_{21}\hat{j} = \begin{bmatrix} P_{11} \\ P_{21} \end{bmatrix}, \\ \vec{v} &= P_{12}\hat{i} + P_{22}\hat{j} = \begin{bmatrix} P_{12} \\ P_{22} \end{bmatrix}.\end{aligned}$$

$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$ maps $(\hat{i}, \hat{j})$ to $(\vec{u}, \vec{v})$

$\vec{u}, \vec{v}$ not parallel ➡ det $\boldsymbol{P} \neq 0$

Example $(\vec{u}, \vec{v})$ forms a basis

$$\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \det\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) = 1$$

Components of a vector changes wrt new basis

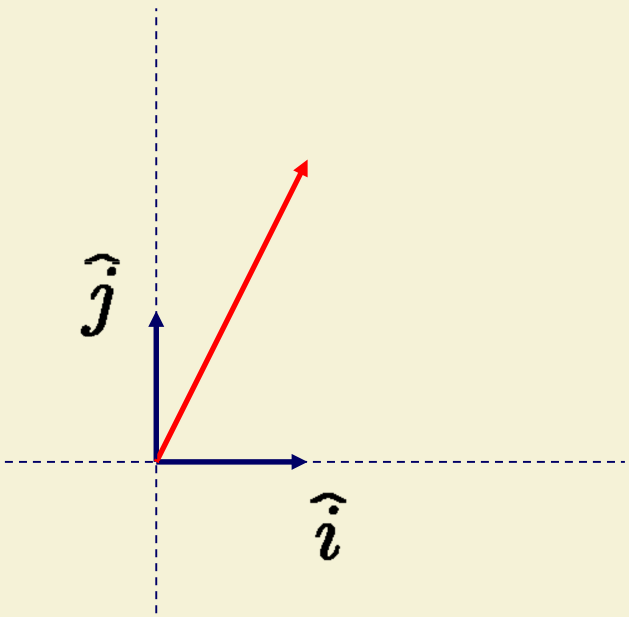
$$1\hat{i} + 2\hat{j} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

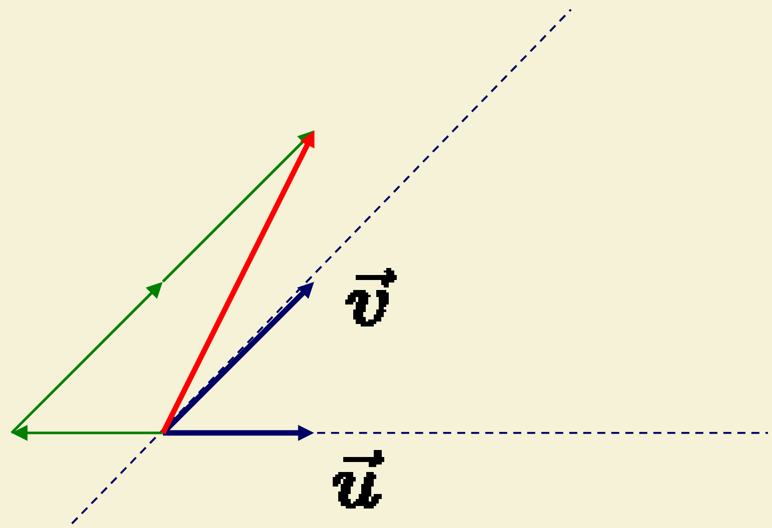
$$= -\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = -\vec{u} + 2\vec{v}$$

New component: $\begin{bmatrix} -1 \\ 2 \end{bmatrix}_{(\vec{u},\vec{v})}$

$$1\hat{i} + 2\hat{j} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix}_{(\vec{u},\vec{v})}$$



## Computing Components Systematically

$\vec{u} = P\hat{i}$ $\qquad \vec{v} = P\hat{j}$ $\qquad \det P \neq 0$

$\hat{i} = P^{-1}\vec{u}$ $\qquad \hat{j} = P^{-1}\vec{v}$

To compute new components

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \alpha\vec{u} + \beta\vec{v}$$

$$P^{-1}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \alpha P^{-1}\vec{u} + \beta P^{-1}\vec{v}$$

$$= \alpha\hat{i} + \beta\hat{j}$$

## Computing Components

$$P^{-1}\begin{bmatrix}1\\2\end{bmatrix} = \alpha\hat{i} + \beta\hat{j}$$

$$P^{-1} = \begin{bmatrix}1 & -1\\0 & 1\end{bmatrix} \Rightarrow P^{-1}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}-1\\2\end{bmatrix}$$

$$\boxed{\begin{bmatrix}-1\\2\end{bmatrix}_{(\vec{u},\vec{v})} = P^{-1}\begin{bmatrix}1\\2\end{bmatrix}_{(\hat{i},\hat{j})}.}$$

# Computing Components

$$T = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}_{(\hat{i},\hat{j})}$$

different matrix relative to $(\vec{u}, \vec{v})$

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}_{(\hat{i},\hat{j})} \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{(\hat{i},\hat{j})} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}_{(\hat{i},\hat{j})}$$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{(\vec{u},\vec{v})}$ map $P^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{(\hat{i},\hat{j})} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ to $P^{-1} \begin{bmatrix} 5 \\ -2 \end{bmatrix}_{(\hat{i},\hat{j})} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{(\vec{u},\vec{v})} P^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{(\hat{i},\hat{j})} = P^{-1} \begin{bmatrix} 5 \\ -2 \end{bmatrix}_{(\hat{i},\hat{j})}$$

## Computing Components

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{(\vec{u},\vec{v})} P^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{(\hat{i},\hat{j})} = P^{-1} \begin{bmatrix} 5 \\ -2 \end{bmatrix}_{(\hat{i},\hat{j})}$$

$$\Rightarrow \quad P \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{(\vec{u},\vec{v})} P^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{(\hat{i},\hat{j})} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}_{(\hat{i},\hat{j})}$$

but $\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}_{(\hat{i},\hat{j})} \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{(\hat{i},\hat{j})} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}_{(\hat{i},\hat{j})}$

$$\Rightarrow \quad P \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{(\vec{u},\vec{v})} P^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}_{(\hat{i},\hat{j})}$$

# Change of Basis

$$P\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{(\vec{u},\vec{v})} P^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}_{(\hat{i},\hat{j})}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{(\vec{u},\vec{v})} = P^{-1}\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}_{(\hat{i},\hat{j})} P = \begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix}$$

Matrix of ***T*** relative to $(\vec{u},\vec{v})$

Multiply $\boldsymbol{P^{-1}}$ to left and ***P*** to right

## Diagonalization

***T*** has eigenvectors $\vec{e}_1, \vec{e}_2$

if $(\vec{e}_1, \vec{e}_2)$ form a basis

$$\begin{aligned} T\vec{e}_1 &= \lambda_1\vec{e}_1 + 0\vec{e}_2, \\ T\vec{e}_2 &= 0\vec{e}_1 + \lambda_2\vec{e}_2, \end{aligned}$$

Matrix of ***T*** relative to $(\vec{e}_1, \vec{e}_2)$

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}_{(\vec{e}_1, \vec{e}_2)}$$

diagonal

## Example

$\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$ has eigenvectors $\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$$\Rightarrow P = \begin{bmatrix} 1 & 1 \\ \frac{1}{2} & -2 \end{bmatrix}$$

Matrix of ***T*** relative to $(\vec{e}_1, \vec{e}_2)$

$$P^{-1} \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} P = -\frac{2}{5} \begin{bmatrix} -2 & -1 \\ -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \frac{1}{2} & -2 \end{bmatrix}$$

$$= -\frac{2}{5} \begin{bmatrix} -2 & -1 \\ -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & 6 \end{bmatrix}$$

$$= -\frac{2}{5} \begin{bmatrix} -5 & 0 \\ 0 & -\frac{15}{2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$$

## Summary

We say a matrix A is diagonalizable if

$$\boxed{A = PDP^{-1}}$$

Matrix of eigenvectors

Diagonal matrix of eigenvalues

Fact: If A is 2x2, just need to find two non parallel eigenvectors

## Summary

Remark: Diagonalizable has nothing to do with invertible.

But if a matrix A is diagonalizable

$$A = PDP^{-1}$$

$$\det A = \det P \times \det D \times \det P^{-1}$$

= product of eigenvalues

## Example

$$\begin{bmatrix} 1 & \tan\theta \\ 0 & 1 \end{bmatrix}$$ has eigenvalues

$$\det\left(\begin{bmatrix} 1-\lambda & \tan\theta \\ 0 & 1-\lambda \end{bmatrix}\right) = (1-\lambda)^2 = 0$$

only one eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Not possible to diagonalize

# 6.9 Application: Markov Chains

$$M = \begin{bmatrix} R \to R & S \to R \\ R \to S & S \to S \end{bmatrix} = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}.$$

How to compute $\boldsymbol{M^k}$ for large $\boldsymbol{k}$ ?

## 6.19 Application: Markov Chains

$$P^{-1}MP = D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\begin{aligned} M &= PDP^{-1} \\ M^2 &= PDP^{-1}PDP^{-1} = PD^2P^{-1} \\ M^3 &= MM^2 = PDP^{-1}PD^2P^{-1} = PD^3P^{-1} \end{aligned}$$

$$\Rightarrow \quad M^{30} = PD^{30}P^{-1}$$

$$\lambda_1 = 0.3 \qquad \lambda_2 = 1$$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ \frac{4}{3} \end{bmatrix}$$

$$D^{30} = \begin{bmatrix} \lambda_1^{30} & 0 \\ 0 & \lambda_2^{30} \end{bmatrix}$$

# 6.9 Markov Chains

$$\lambda_1 = 0.3 \qquad \lambda_2 = 1$$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ \frac{4}{3} \end{bmatrix}$$

$$M^{30} = PD^{30}P^{-1}$$

$$P = \begin{bmatrix} 1 & 1 \\ -1 & \frac{4}{3} \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} \frac{4}{7} & -\frac{3}{7} \\ \frac{3}{7} & \frac{3}{7} \end{bmatrix}$$

$$D^{30} = \begin{bmatrix} 0.3^{30} & 0 \\ 0 & 1 \end{bmatrix} \approx \begin{bmatrix} 2 \times 10^{-16} & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} M^{30} &= \begin{bmatrix} 1 & 1 \\ -1 & \frac{4}{3} \end{bmatrix} \begin{bmatrix} 2 \times 10^{-16} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{4}{7} & -\frac{3}{7} \\ \frac{3}{7} & \frac{3}{7} \end{bmatrix} \\ &\approx \begin{bmatrix} \frac{3}{7} & \frac{3}{7} \\ \frac{4}{7} & \frac{4}{7} \end{bmatrix}. \end{aligned}$$

## 6.10 Application: Trace of a Matrix

Let ***M*** be a square matrix.

The trace of ***M*** , denoted ***Tr (M)*** is the sum of the diagonal entries

$$Tr\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2, \quad Tr\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 15, \quad Tr\begin{bmatrix} 1 & 5 & 16 \\ 7 & 2 & 15 \\ 11 & 9 & 8 \end{bmatrix} = 11$$

# 6.10 Trace of a Matrix

$$TrMN = \sum_i \sum_j M_{ij} N_{ji} = \sum_j \sum_i N_{ji} M_{ij} = TrNM$$

$$Tr(P^{-1}AP) = Tr(APP^{-1}) = TrA$$

Trace is independent of basis

For a diagonalizable matrix ***A***,

***Tr(A)*** = sum of its eigenvalues.

Use this to check your calculations