MA1506
Mathematics II

Chapter 4
Laplace Transforms

4.1 Definition

Let *f(t)* be a function defined for $t \geq 0$

$$F(s) = L(f) = \int_0^{\infty} e^{-st} f(t)dt$$

Laplace transform

if integral exists

Inverse Laplace transform

$$f(t) = L^{-1}(F(s))$$

Notation: Reserve Caps for Transformed

$$f(t) \qquad F(s) = L(f)$$

$$y(t) \qquad Y(s) = L(y)$$

4.1 Convergence

By definition $$\int_0^\infty h(t)\,dt = \lim_{b\to\infty} \int_0^b h(t)dt$$

if limit exists, i.e. finite

Examples

$$\int_0^\infty 1\,dt = \lim_{b\to\infty} \int_0^b 1dt = \lim_{b\to\infty} b = \infty$$

$$\int_0^\infty t\,dt = \lim_{b\to\infty} \left.\frac{t^2}{2}\right|_0^b = \infty$$

$$\int_0^\infty \frac{1}{t}\,dt = \lim_{b\to\infty} \left.\ln t\right|_0^b = \infty$$

4.1 Convergence

$$L(f) = \int_0^\infty e^{-st} f(t)\, dt = \lim_{b\to\infty} \int_0^b e^{-st} f(t) dt$$

decrease rapidly

$$\int_0^\infty te^{-st}\, dt = -\frac{1}{s} te^{-st}\Big|_0^\infty + \int_0^\infty \frac{1}{s} e^{-st} dt$$

$$= 0 - \lim_{b\to\infty} \frac{1}{s^2} e^{-st}\Big|_0^b$$

$$= 0 + \frac{1}{s^2}$$

Example 6

4.1 Example 1 $f(t) = e^{at}, t \geq 0$

$$F(s) = L(e^{at}) = \int_0^{\infty} e^{-st} e^{at} dt$$

$$= \lim_{b \to \infty} \int_0^{b} e^{-st} e^{at} dt$$

$$\int_0^{b} e^{(a-s)t} dt = \begin{cases} b, & s = a \\ \frac{e^{b(a-s)}}{a-s} - \frac{1}{a-s}, & s \neq a. \end{cases}$$

$$F(s) = \boxed{L(e^{at}) = \frac{1}{s-a}, \quad s > a}$$

4.1 Example 2 $f(t) = 1, t \geq 0$

$$F(s) = L(e^{at}) = \frac{1}{s-a}, \quad s > a$$

Set *a=0* in Example 1

$$\boxed{L(1) = \frac{1}{s}, \quad s > 0}$$

4.1 Theorem (Linearity)

$$L(af(t) + bg(t)) = aL(f) + bL(g)$$

a, *b* are constants

$$L^{-1}(aF(s) + bG(s)) = aL^{-1}(F) + bL^{-1}(G)$$

4.1 Example 3 $\quad f(t) = \cosh at$

$$
\begin{aligned}
L(\cosh at) &= L\left(\frac{1}{2}(e^{at} + e^{-at})\right) \\
&= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right)
\end{aligned}
$$

$$
\boxed{L(\cosh at) = \frac{s}{s^2 - a^2}, \quad s > a \geq 0.}
$$

4.1 Example 4

$$F(s) = \frac{3}{s} + \frac{5}{s-7}$$

$$L(e^{at}) = \frac{1}{s-a}, \quad s > a$$

$$\begin{aligned} L^{-1}(F) &= L^{-1}\left(\frac{3}{s}\right) + 5L^{-1}\left(\frac{1}{s-7}\right) \\ &= 3 \cdot 1 + 5 \cdot e^{7t} \\ &= 3 + 5e^{7t} \end{aligned}$$

4.1 Example 5

$$L(e^{at}) = \frac{1}{s-a}, \quad s > a$$

Set $a = iw$

$$L(e^{iwt}) = \frac{1}{s-iw} = \frac{s+iw}{s^2+w^2}$$

$$\begin{aligned} L(e^{iwt}) &= L(\cos wt + i\sin wt) \\ &= L(\cos wt) + iL(\sin wt) \end{aligned}$$

$$\boxed{L(\cos wt) = \frac{s}{s^2+w^2};}$$

$$\boxed{L(\sin wt) = \frac{w}{s^2+w^2}}$$

4.1 Example 6 $f(t) = t^n, t \geq 0$

$$\begin{aligned} L(t^n) &= \int_0^\infty e^{-st} t^n dt \\ &= -\frac{1}{s} e^{-st} t^n \Big|_0^\infty + \frac{n}{s} \int_0^\infty e^{-st} t^{n-1} dt \end{aligned}$$

$$\begin{aligned} L(t^n) = \frac{n}{s} L(t^{n-1}) = \dots \\ = \frac{n(n-1)\dots 1}{s^n} L(1) = \frac{n!}{s^{n+1}} \end{aligned}$$

$$\boxed{L(t^n) = \frac{n!}{s^{n+1}}}$$

4.1 Example 7

$$F(s) = \frac{2s+5}{s^2+9}$$

$$\begin{aligned}
L^{-1}\left(\frac{2s+5}{s^2+9}\right) &= L^{-1}\left(\frac{2s}{s^2+9} + \frac{5}{s^2+9}\right) \\
&= 2L^{-1}\left(\frac{s}{s^2+9}\right) + \frac{5}{3}L^{-1}\left(\frac{3}{s^2+9}\right) \\
&= 2\cos 3t + \frac{5}{3}\sin 3t
\end{aligned}$$

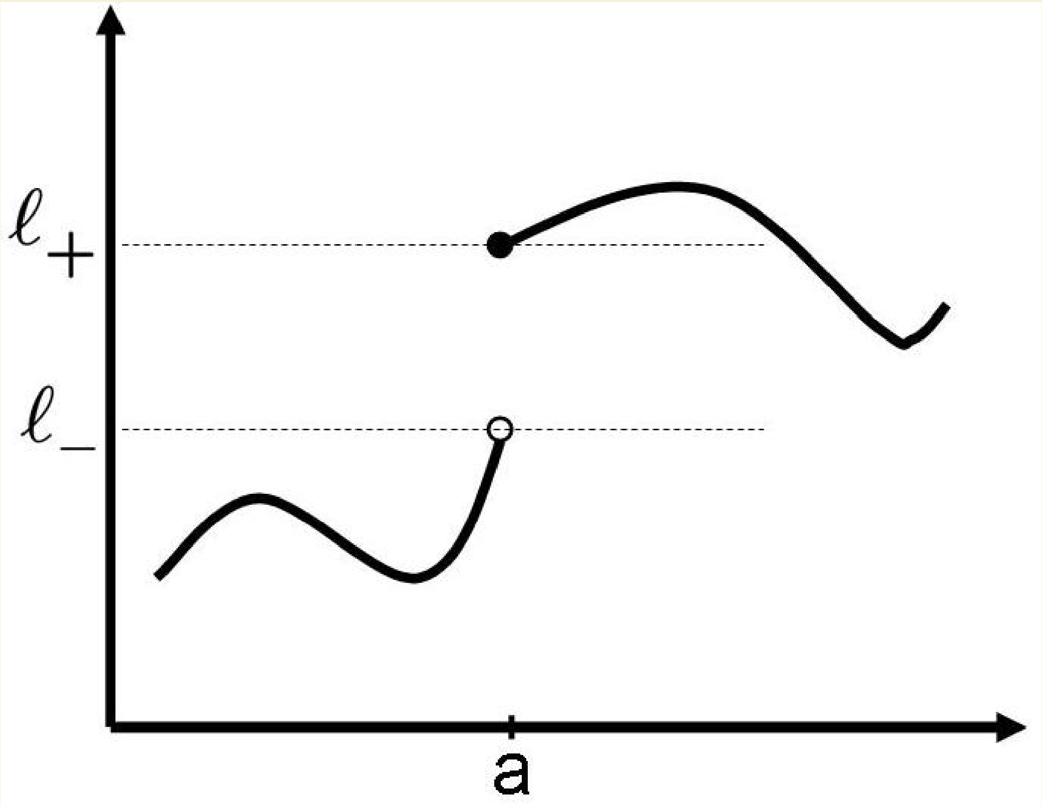
Piecewise Continuous Functions

Jump discontinuity at *a*

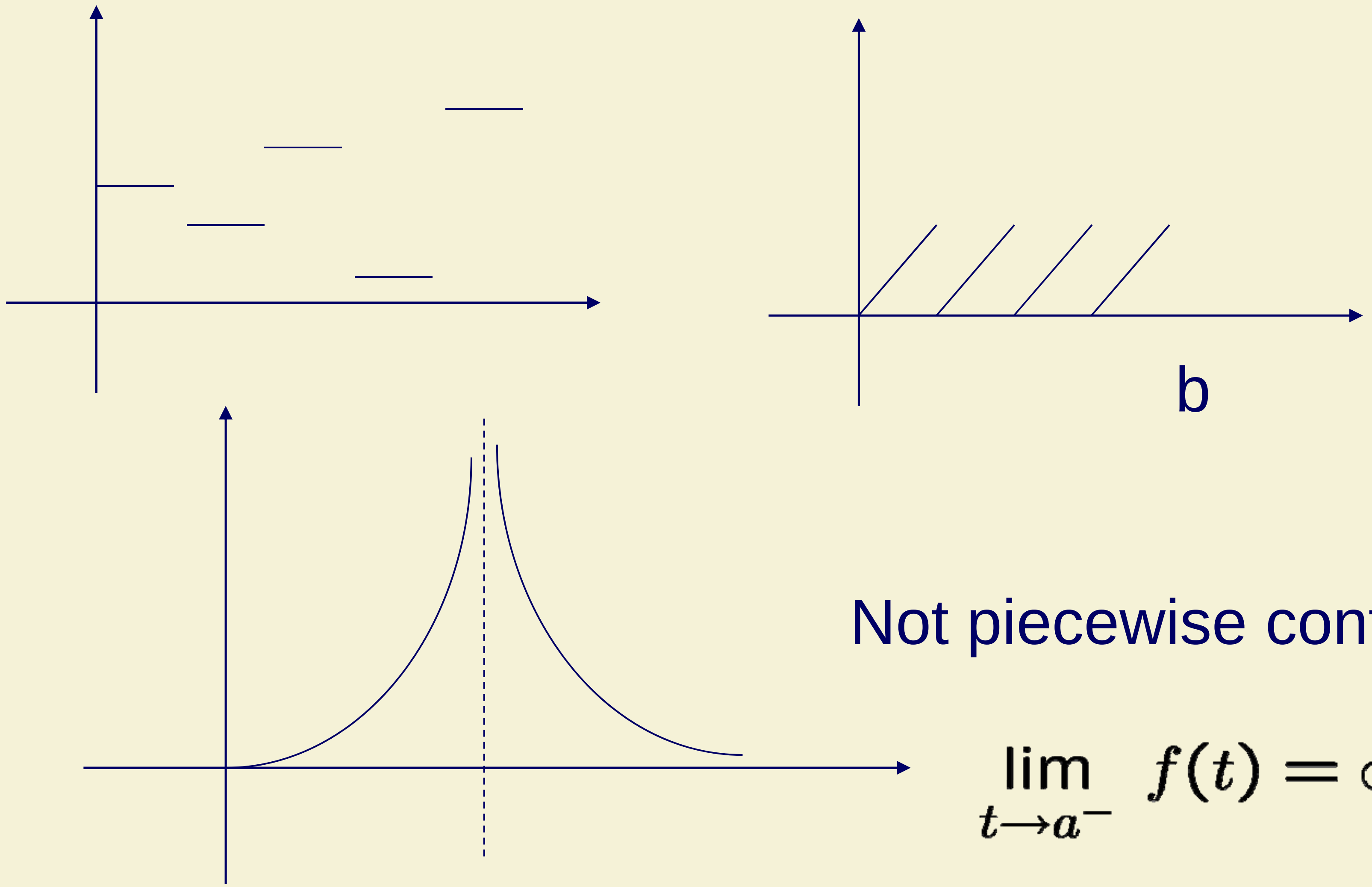
$$\lim_{t \to a^-} f(t) = \ell_-$$

$$\lim_{t \to a^+} f(t) = \ell_+$$

But *f(t)* is not continuous at *a*



Piecewise Continuous Functions



Not piecewise cont

$$\lim_{t \to a^-} f(t) = \infty$$

4.2 Transform of Derivatives and Integrals

Theorem: Suppose *f(t)* is continuous and has a well-defined Laplace transform and *f'(t)* is piecewise continuous, then

$$\boxed{L(f') = sL(f) - f(0), \quad s > a}$$

Pf

$$\begin{aligned} L(f') &= \int_0^\infty e^{-st} f'(t)dt \\ &= e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)dt \\ &= \lim_{b\to\infty} e^{-sb} f(b) - f(0) + sL(f). \end{aligned}$$

4.2 Transform of Derivatives and Integrals

if *f'(t)* is piecewise continuous

$$\int_0^\infty e^{-st}\, f'(t)dt = \int_0^{k_1} + \int_{k_1}^{k_2} + \ldots + \int^\infty$$

$$\boxed{L(f') = sL(f) - f(0), \quad s > a}$$

Initial condition

4.2 Transform of Derivatives and Integrals

$$L(f') = sL(f) - f(0), \quad s > a$$

$$\begin{aligned} L(f'') &= sL(f') - f'(0) \\ &= s(sL(f) - f(0)) - f'(0) \\ &= s^2 L(f) - s f(0) - f'(0) \end{aligned}$$

Initial conditions

4.2 Transform of Derivatives and Integrals

$$L(f'') = s^2 L(f) - sf(0) - f'(0)$$

Suppose $f(t),\ f'(t),\ f''(t),\ \ldots,\ f^{(n-1)}(t)$

are continuous and

$f^{(n)}(t)$ is piecewise continuous, then

$$\boxed{\begin{aligned} L(f^{(n)}) &= s^n L(f) - s^{n-1} f(0) - s^{n-2} f'(0) \\ &\quad - \cdots - f^{(n-1)}(0) \end{aligned}}$$

Example 8 Find *L* (*sin*^{2} *t*)

$$f(t) = \sin^2 t \;\Rightarrow\; f'(t) = 2\sin t \cos t = \sin 2t$$

$$f(0) = 0$$

Since $L(f') = sL(f) - f(0) = sL(f)$

$$L(f) = \frac{1}{s}L(f') = \frac{1}{s}L(\sin 2t)$$

$$= \frac{2}{s(s^2+4)}$$

Example 9 **Find $L(t \sin \alpha t)$**

$$f(t) = t \sin \alpha t \quad \Rightarrow \quad f(0) = 0$$

$$f'(t) = \sin \alpha t + \alpha t \cos \alpha t \Rightarrow f'(0) = 0$$

$$\begin{aligned} f''(t) &= 2\alpha \cos \alpha t - \alpha^2 t \sin \alpha t \\ &= 2\alpha \cos \alpha t - \alpha^2 f(t) \end{aligned}$$

$$\boxed{L(f'') = s^2 L(f) - s f(0) - f'(0)}$$

$$s^2 L(f) = L(2\alpha \cos \alpha t - \alpha^2 f)$$

Example 9

Find $L\ (t \sin \alpha t)$

$$L(2\alpha \cos \alpha t - \alpha^2 f) = s^2 L(f)$$

||

$$2\alpha L(\cos \alpha t) - \alpha^2 L(f)$$

Hence

$$(s^2 + \alpha^2)L(f) = 2\alpha L(\cos \alpha t) = \frac{2\alpha s}{s^2 + \alpha^2}$$

$$L(t \sin \alpha t) = \frac{2\alpha s}{(s^2 + \alpha^2)^2}$$

4.3 Solutions of IVP

$$y'' + ay' + by = r(t)$$

$$y(0) = k_0$$
$$y'(0) = k_1$$

1) Take Laplace transform

$$s^2 L(y) - sy(0) - y'(0)$$
$$+a(sL(y) - y(0)) + bL(y) = L(r)$$

2) Sub initial conditions

3) Solve for L(y)

4.3 Solutions of IVP

$$y'' + ay' + by = r(t)$$

$$y(0) = k_0$$
$$y'(0) = k_1$$

1) Take Laplace transform

2) Sub initial conditions

3) Solve for L(y)

$$L(y) = \frac{(s+a)k_0 + k_1 + R(s)}{s^2 + as + b}$$

4) Simplify and take inverse Laplace

4.3 Example 10

$$y'' + y = e^{2t} \qquad \begin{aligned} y(0) &= 0 \\ y'(0) &= 1 \end{aligned}$$

Take Laplace Transform

$$s^2 L(y) - sy(0) - y'(0) + L(y) = \frac{1}{s-2}$$

$$\begin{aligned} L(y) &= \frac{1}{s^2+1}\left(1 + \frac{1}{s-2}\right) \\ &= \frac{s-1}{(s-2)(s^2+1)} \\ &= \frac{A}{s-2} + \frac{Bs+C}{s^2+1} \end{aligned}$$

4.3 Example 10 $y'' + y = e^{2t}$ $y(0) = 0$, $y'(0) = 1$

$$L(y) = \frac{s-1}{(s-2)(s^2+1)} = \frac{A}{s-2} + \frac{Bs+C}{s^2+1}$$

$$s - 1 = A(s^2+1) + (Bs+C)(s-2)$$

Compare Coeff

1 :	$-1 = A - 2C$		$A = 1/5$
s :	$1 = -2B + C$	$\Rightarrow$	$B = -1/5$
s^2 :	$0 = A + B$		$C = 3/5$

4.3 Example 10

$$y'' + y = e^{2t}$$

$$y(0) = 0$$
$$y'(0) = 1$$

$$L(y) = \frac{1}{5} \cdot \frac{1}{s-2} - \frac{s-3}{5(s^2+1)}$$

$$= \frac{1}{5(s-2)} - \frac{s}{5(s^2+1)} + \frac{3}{5(s^2+1)}$$

Inverse transform

$$y(t) = \frac{1}{5}e^{2t} - \frac{1}{5}\cos t + \frac{3}{5}\sin t$$

Laplace -- Avoiding integration

IVP

$$\begin{tikzcd}
\dfrac{d^n y}{dt^n} + \ldots = 0 \arrow[r, "L"] \arrow[d, "\text{integration}"'] & G(L(y), s) = 0 \arrow[d] \\
y = f(t) & L(y) = F(s) \arrow[l, "L^{-1}"']
\end{tikzcd}$$

4.4 Transform of Integrals

Theorem: If *f(t)* is piecewise continuous and has a well-defined Laplace transform, then

$$L\left(\int_0^t f(\tau)d\tau\right) = \frac{1}{s}L(f) \qquad (s > 0, s > a)$$

Condition for *L(f)*

4.4 Example 11

$$L(f) = \frac{1}{s^2(s^2+w^2)}$$

$$L\left(\frac{1}{w}\sin wt\right) = \frac{1}{s^2+w^2}$$

$$\frac{1}{s(s^2+w^2)} = L\left(\frac{1}{w}\int_0^t \sin w\tau d\tau\right) = L\left(\frac{1-\cos wt}{w^2}\right)$$

$$\begin{aligned}\frac{1}{s^2(s^2+w^2)} &= L\left(\frac{1}{w^2}\int_0^t (1-\cos w\tau)d\tau\right)\\ &= L\left(\frac{1}{w^2}\left(t-\frac{\sin wt}{w}\right)\right)\end{aligned}$$

$$f(t) = \frac{1}{w^2}\left(t-\frac{\sin wt}{w}\right)$$

4.4 Example 11 $$L(f) = \frac{1}{s^2(s^2+w^2)}$$

Alternative mtd

$$\frac{1}{s^2(s^2+w^2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+w^2}$$

$$\frac{1}{s^2(s^2+w^2)} = \frac{1}{w^2 s^2} - \frac{1}{w^2(s^2+w^2)}$$

$$f(t) = \frac{1}{w^2}\left(t - \frac{\sin wt}{w}\right)$$

4.4 *s*-Shifting

If $f(t)$ has transform $F(s)$, $s > a$

$$L(e^{ct}f(t)) = F(s-c), \quad s-c > a$$

$$\begin{aligned} L(e^{ct}t^n) &= \frac{n!}{(s-c)^{n+1}} \\ L(e^{ct}\cos wt) &= \frac{s-c}{(s-c)^2+w^2} \\ L(e^{ct}\sin wt) &= \frac{w}{(s-c)^2+w^2} \end{aligned}$$

4.4 Example 12

$$y'' + 2y' + 5y = 0 \qquad \begin{aligned} y(0) &= 2 \\ y'(0) &= -4 \end{aligned}$$

$$L(y) = \frac{(s+a)k_0 + k_1 + R(s)}{s^2 + as + b}$$

$$= \frac{2(s+2) - 4}{s^2 + 2s + 5} = \frac{2s}{(s+1)^2 + 2^2}$$

$$= \frac{2(s+1)}{(s+1)^2 + 2^2} - \frac{2}{(s+1)^2 + 2^2}$$

$$y(t) = e^{-t}(2\cos 2t - \sin 2t)$$

Example 13 $$y'' - 2y' + y = e^t + t \qquad \begin{aligned} y(0) &= 1 \\ y'(0) &= 0 \end{aligned}$$

$$s^2 L(y) - sy(0) - y'(0)$$
$$-2(sL(y) - y(0)) + L(y) = \frac{1}{s-1} + \frac{1}{s^2}$$

$$(s^2 - 2s + 1)L(y) = s - 2 + \frac{1}{s-1} + \frac{1}{s^2}$$

$$L(y) = \frac{s-2}{(s-1)^2} + \frac{1}{(s-1)^3} + \frac{1}{s^2(s-1)^2}$$

Example 13

$$y'' - 2y' + y = e^t + t \qquad \begin{matrix} y(0) = 1 \\ y'(0) = 0 \end{matrix}$$

$$L(y) = \frac{s-2}{(s-1)^2} + \frac{1}{(s-1)^3} + \frac{1}{s^2(s-1)^2}$$

$$\frac{A}{s-1} + \frac{B}{(s-1)^2}$$

$$\frac{C}{s-1} + \frac{D}{(s-1)^2} + \frac{E}{s} + \frac{F}{s^2}$$

$$\begin{aligned} L(y) &= \frac{1}{s-1} - \frac{1}{(s-1)^2} + \frac{1}{(s-1)^3} \\ &\quad + \frac{1}{(s-1)^2} - \frac{2}{s-1} + \frac{1}{s^2} + \frac{2}{s} \end{aligned}$$

Example 13

$$y'' - 2y' + y = e^t + t \qquad \begin{matrix} y(0) = 1 \\ y'(0) = 0 \end{matrix}$$

$$L(y) = \frac{1}{(s-1)^3} - \frac{1}{s-1} + \frac{1}{s^2} + \frac{2}{s}$$

$$y(t) = \frac{t^2}{2}e^t - e^t + t + 2 = (\frac{t^2}{2} - 1)e^t + t + 2$$

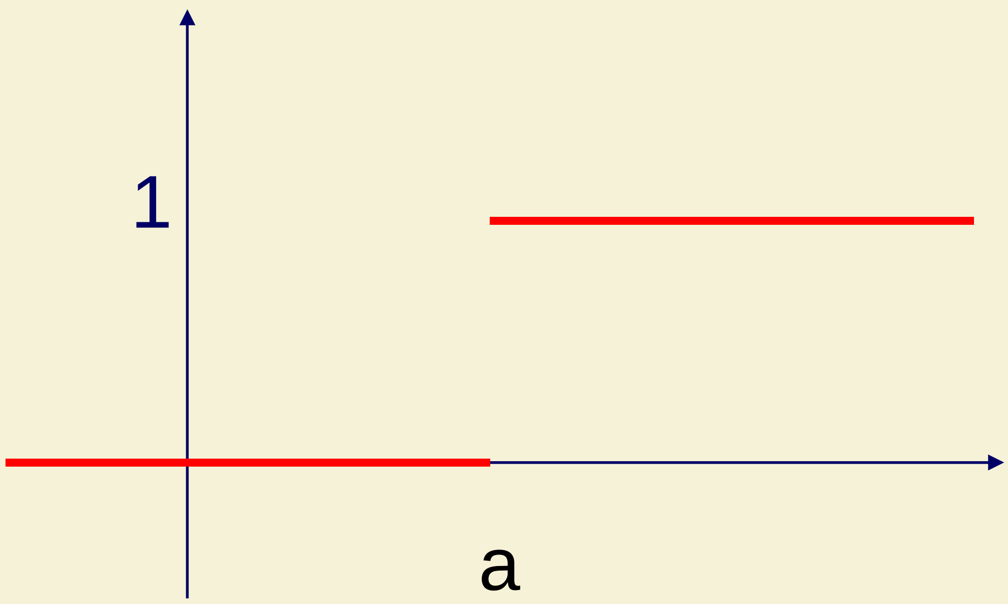
Example 13 $y'' - 2y' + y = e^t + t$ $y(0) = 1$, $y'(0) = 0$

$$(s^2 - 2s + 1)L(y) = s - 2 + \frac{1}{s-1} + \frac{1}{s^2}$$

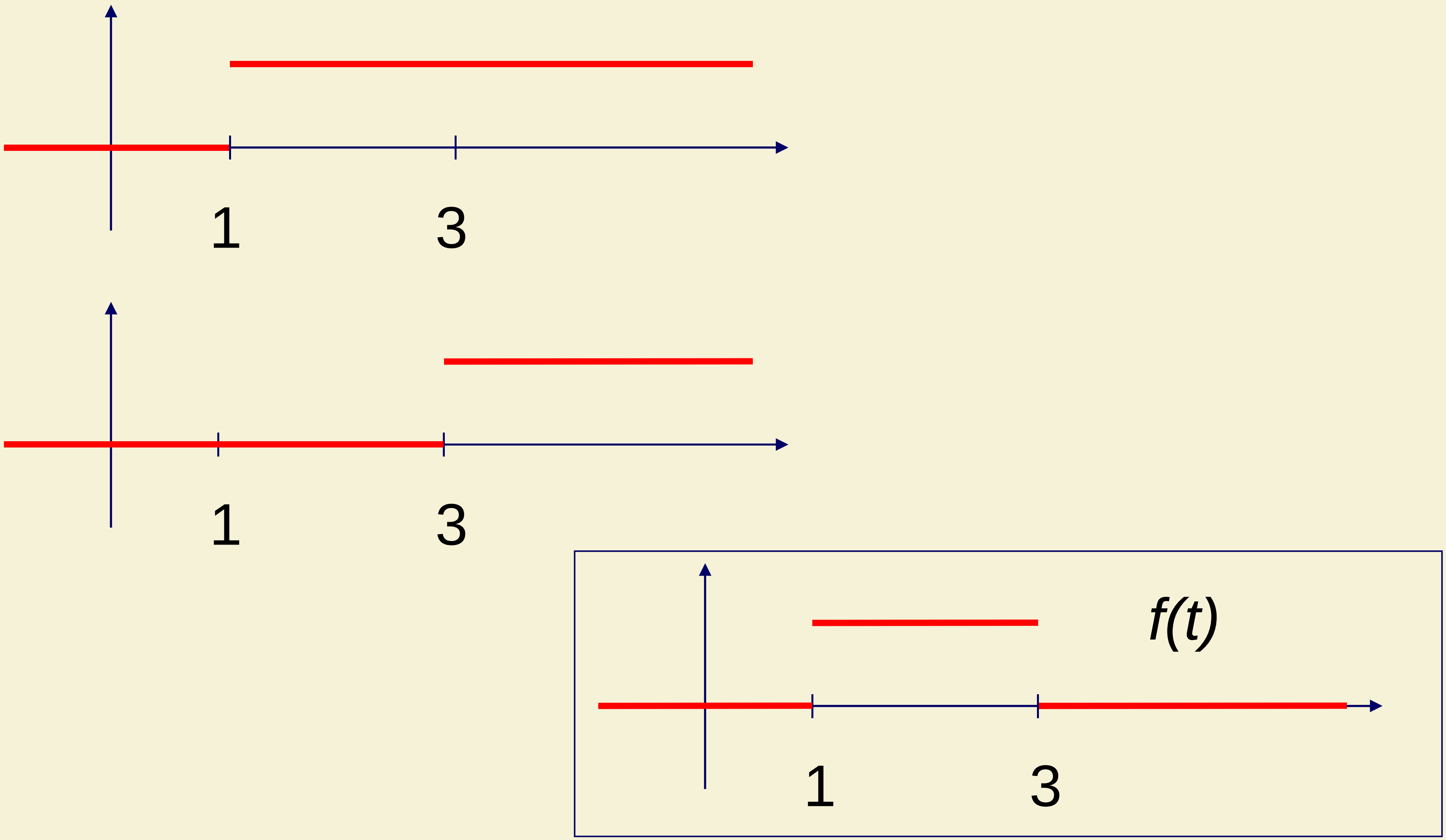
http://wims.unice.fr/wims/en_home.html

4.5 Unit Step (Heaviside) Function

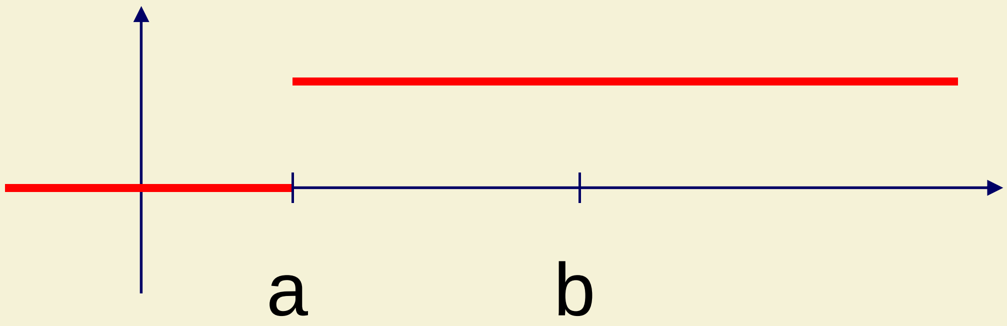
$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a. \end{cases}$$



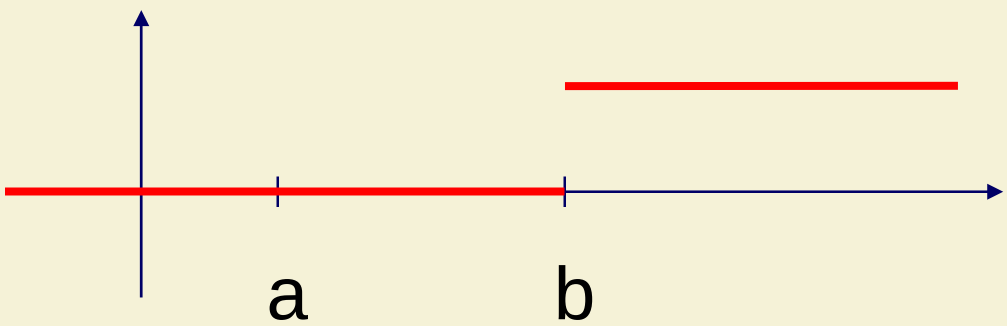
Example 14 $$f(t) = u(t-1) - u(t-3)$$



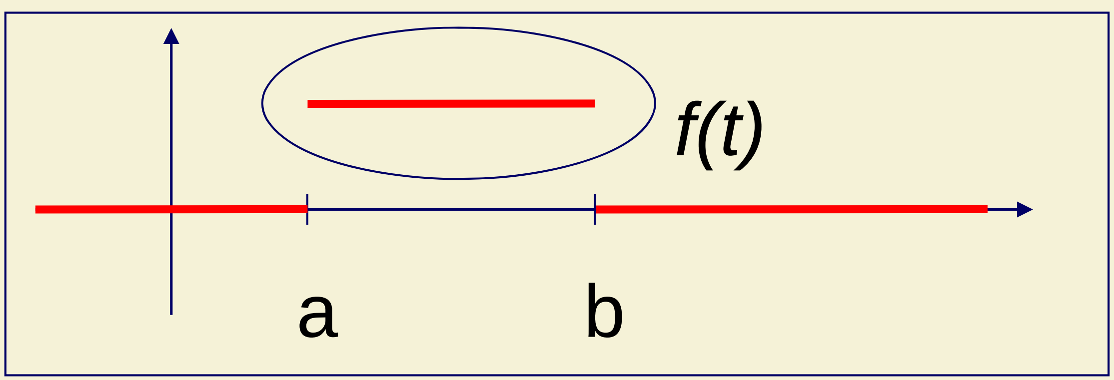
Example 14 $$f(t) = u(t-a) - u(t-b)$$



$$= \begin{cases} 0, & t < a \\ 1, & a \le t < b \\ 0, & t \ge b. \end{cases}$$



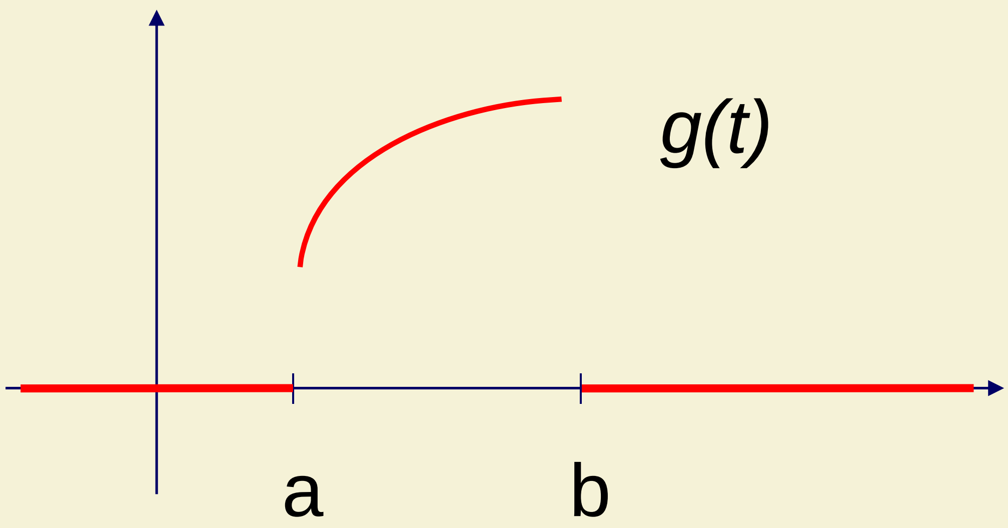
Replace with another function



Let *g(t)* be a function of *t*, if *0* < *a* < *b*

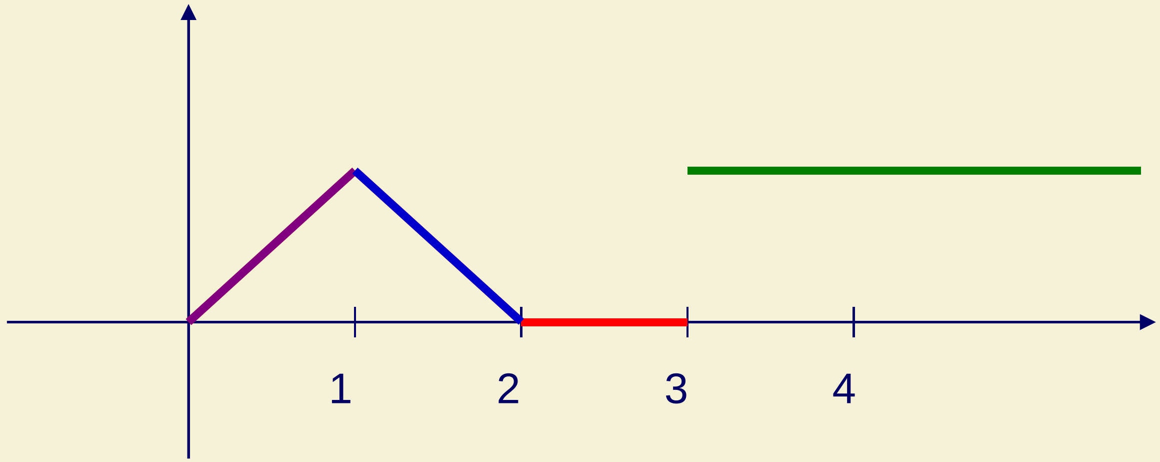
$$g(t)(u(t-a)-u(t-b)) = \begin{cases} 0, & t < a \\ g(t), & a \le t < b \\ 0, & t \ge b. \end{cases}$$

- On/Off function
- Circuit theory
- Discontinuous



Example 15 Express in terms of *u(t)*

$$f(t) = \begin{cases} t, & 0 \le t < 1 \\ 2-t, & 1 \le t < 2 \\ 0, & 2 \le t < 3 \\ 1, & t \ge 3. \end{cases}$$



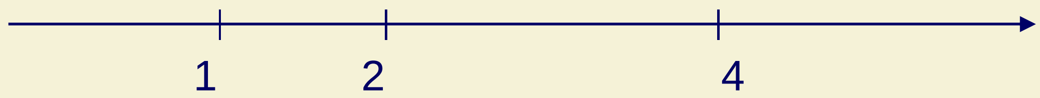
$$t(u(t) - u(t-1))$$

$$+(2-t)(u(t-1) - u(t-2))$$

$$+u(t-3)$$

Example 16

$$g(t) = 2u(t) + tu(t-1) + (3-t)u(t-2) - 3u(t-4), \ t > 0$$



$$0 < t < 1, \quad g(t) = 2\cdot 1 + t\cdot 0 + (3-t)\cdot 0 - 3\cdot 0 = 2$$

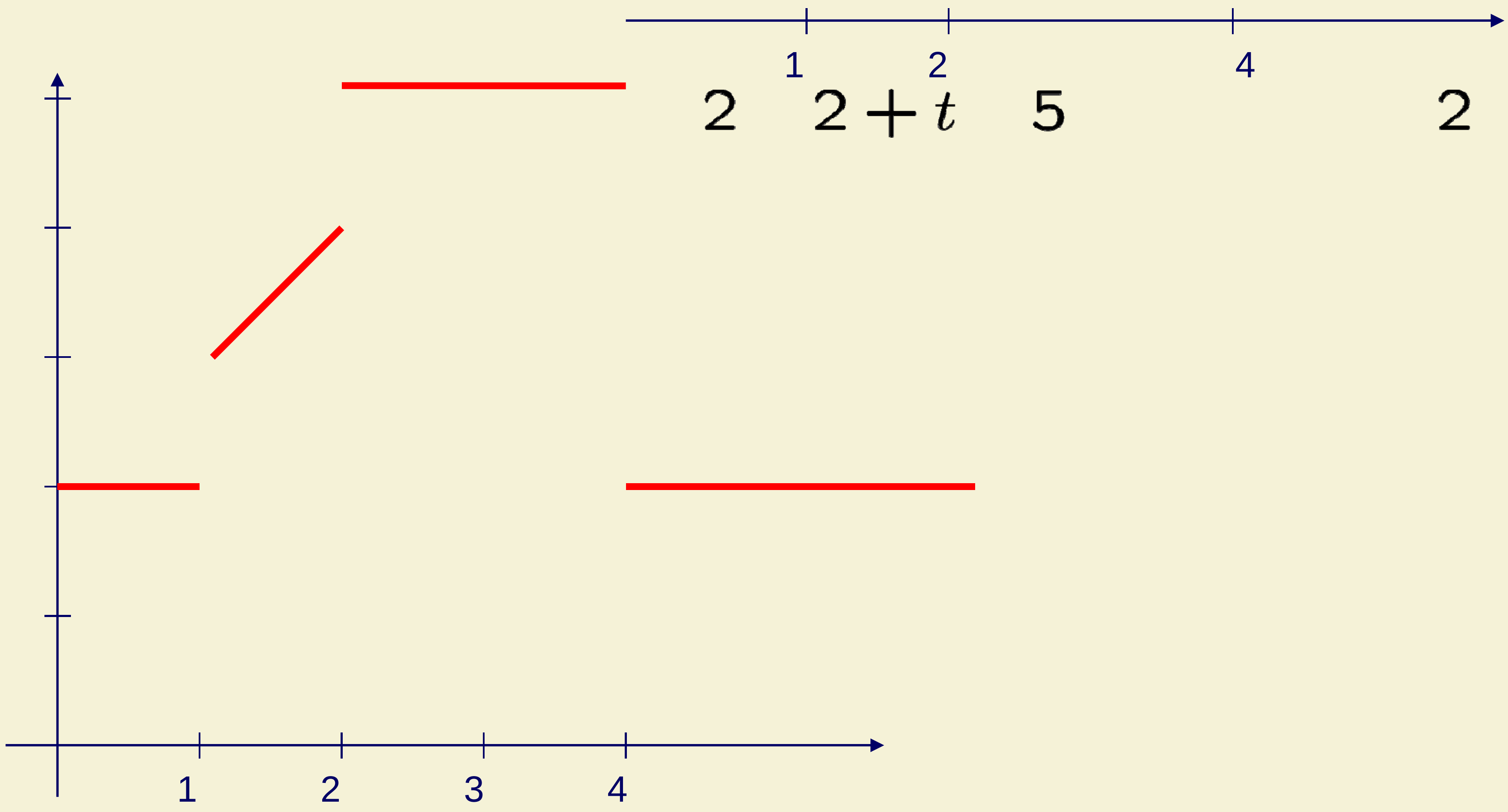
$$1 < t < 2, \quad g(t) = 2\cdot 1 + t\cdot 1 + (3-t)\cdot 0 - 3\cdot 0 = 2 + t$$

$$2 < t < 4, \quad g(t) = 2\cdot 1 + t\cdot 1 + (3-t)\cdot 1 - 3\cdot 0 = 5$$

$$t > 4, \quad g(t) = 2\cdot 1 + t\cdot 1 + (3-t)\cdot 1 - 3\cdot 1 = 2$$

Example 16

$$g(t) = 2u(t) + tu(t-1) + (3-t)u(t-2) - 3u(t-4), \; t > 0$$

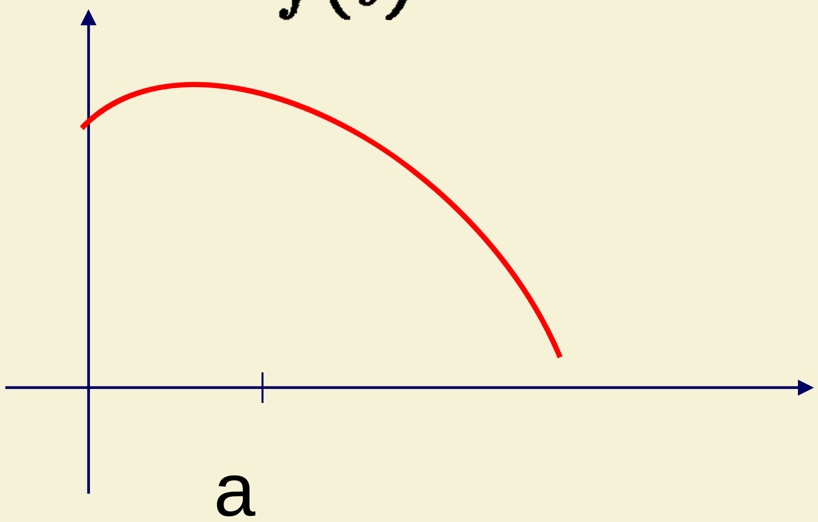


4.5 *t*-Shifting

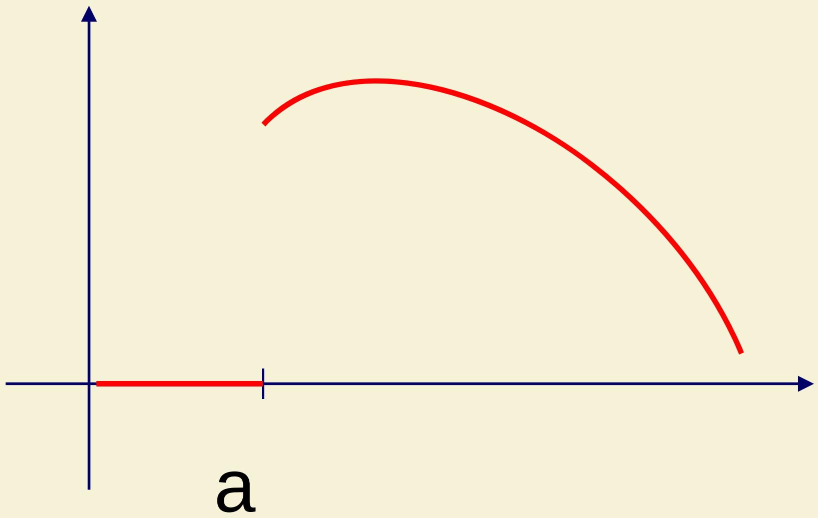
If $L(f(t)) = F(s)$ then

$$L(f(t-a)u(t-a)) = e^{-as}F(s)$$

$f(t)$



$f(t-a)u(t-a)$



4.5 *t*-Shifting

If $L(f(t)) = F(s)$ then

$$L(f(t-a)u(t-a)) = e^{-as}F(s)$$

$$L(f(t-a)u(t-a)) = \int_0^\infty e^{-st} f(t-a)u(t-a)\,dt$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt$$

$$\boxed{\tau = t - a}$$

$$= \int_0^\infty e^{-s(\tau+a)} f(\tau)\,d\tau$$

Example 17 *t*-Shifting

If $L(f(t)) = F(s)$ then

$$L(f(t-a)u(t-a)) = e^{-as}F(s)$$

$$\boxed{L(u(t-a)) = \frac{e^{-as}}{s}}$$

Example 18 $\qquad L(f(t-a)u(t-a)) = e^{-as}F(s)$

$$L(t^2u(t-1)) = L((t-1+1)^2u(t-1))$$

$$= L\Big(((t-1)^2 + 2(t-1) + 1)u(t-1)\Big)$$

$$= L\Big((t-1)^2u(t-1)\Big) + 2L\Big((t-1)u(t-1)\Big)$$

$$+L\Big(u(t-1)\Big)$$

$$\boxed{f(t) = t^2 \Rightarrow F(s) = \frac{2}{s^3}}$$

$$\boxed{f(t) = t \Rightarrow F(s) = \frac{1}{s^2}}$$

$$= e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right).$$

Example 19 $$L(f(t-a)u(t-a)) = e^{-as}F(s)$$

$$L((e^t + 1)u(t-2))$$

constant

$$= L((e^{t-2}e^2 + 1)u(t-2))$$

$$= e^2 L(e^{t-2}u(t-2)) + L(u(t-2))$$

$$\boxed{f(t) = e^t \Rightarrow F(s) = \frac{1}{s-1}}$$

$$= e^{-2s}\left(\frac{e^2}{s-1} + \frac{1}{s}\right)$$

Alternative: Example 18

$$L(f(t-a)u(t-a)) = e^{-as}F(s)$$
$$L(g(t)u(t-a)) = e^{-as}L(g(t+a))$$

$$\begin{aligned} L(t^2u(t-1)) &= e^{-s}L((t+1)^2) \\ &= e^{-s}L(t^2+2t+1) \\ &= e^{-s}\left(\frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{s}\right). \end{aligned}$$

Example 20

$$y'' + 3y' + 2y = g(t) \qquad \begin{matrix} y(0) = 0 \\ y'(0) = 1 \end{matrix}$$

where $g(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & t \ge 1. \end{cases}$

$$g(t) = u(t) - u(t-1)$$

$$L(g(t)) = \frac{1}{s} - \frac{e^{-s}}{s}$$

LHS:

$$s^2L(y) - sy(0) - y'(0) + 3(sL(y) - y(0)) + 2L(y)$$

$$= (s^2 + 3s + 2)L(y) - 1$$

Example 20

$$y'' + 3y' + 2y = g(t) \qquad \begin{aligned} y(0) &= 0 \\ y'(0) &= 1 \end{aligned}$$

$$L(g(t)) = \frac{1}{s} - \frac{e^{-s}}{s} = (s^2 + 3s + 2)L(y) - 1$$

$$L(y) = \underbrace{\frac{s+1}{s(s^2+3s+2)}} - e^{-s}\left(\frac{1}{s(s^2+3s+2)}\right)$$

$$\frac{s+1}{s(s+1)(s+2)} = \frac{1}{s(s+2)} = \frac{1}{2}\left(\frac{1}{s} - \frac{1}{s+2}\right)$$

$$\Rightarrow \quad L^{-1}\left(\frac{1}{s(s+2)}\right) = \frac{1}{2}(1 - e^{-2t})$$

Example 20

t-shifting $L(f(t-a)u(t-a)) = e^{-as}F(s)$

$$\frac{1}{s(s+1)(s+2)} = \frac{1}{2}\left(\frac{1}{s} + \frac{1}{s+2}\right) - \frac{1}{s+1}$$

$$\Rightarrow L^{-1}\left(\frac{1}{s(s+1)(s+2)}\right) = \frac{1}{2}(1+e^{-2t}) - e^{-t}$$

$$L^{-1}\left(\frac{e^{-s}}{s(s+1)(s+2)}\right)$$

$$= \left(\frac{1}{2}(1+e^{-2(t-1)}) - e^{-(t-1)}\right)u(t-1)$$

Example 20

$$y'' + 3y' + 2y = g(t) \qquad \begin{aligned} y(0) &= 0 \\ y'(0) &= 1 \end{aligned}$$

$$L(y) = \frac{s+1}{s(s^2+3s+2)} - e^{-s}\left(\frac{1}{s(s^2+3s+2)}\right)$$

$$\begin{aligned} y(t) \;=\; & \frac{1}{2}(1-e^{-2t}) - \\ & \left(\frac{1}{2}(1+e^{-2(t-1)}) - e^{-(t-1)}\right)u(t-1) \end{aligned}$$

4.6 Dirac Delta Function

Let $f_h(t)$ be defined by

$$f_h(t) = \begin{cases} 0, & t < 0 \\ 1/h, & 0 \le t \le h \\ 0, & t > h, \end{cases}$$

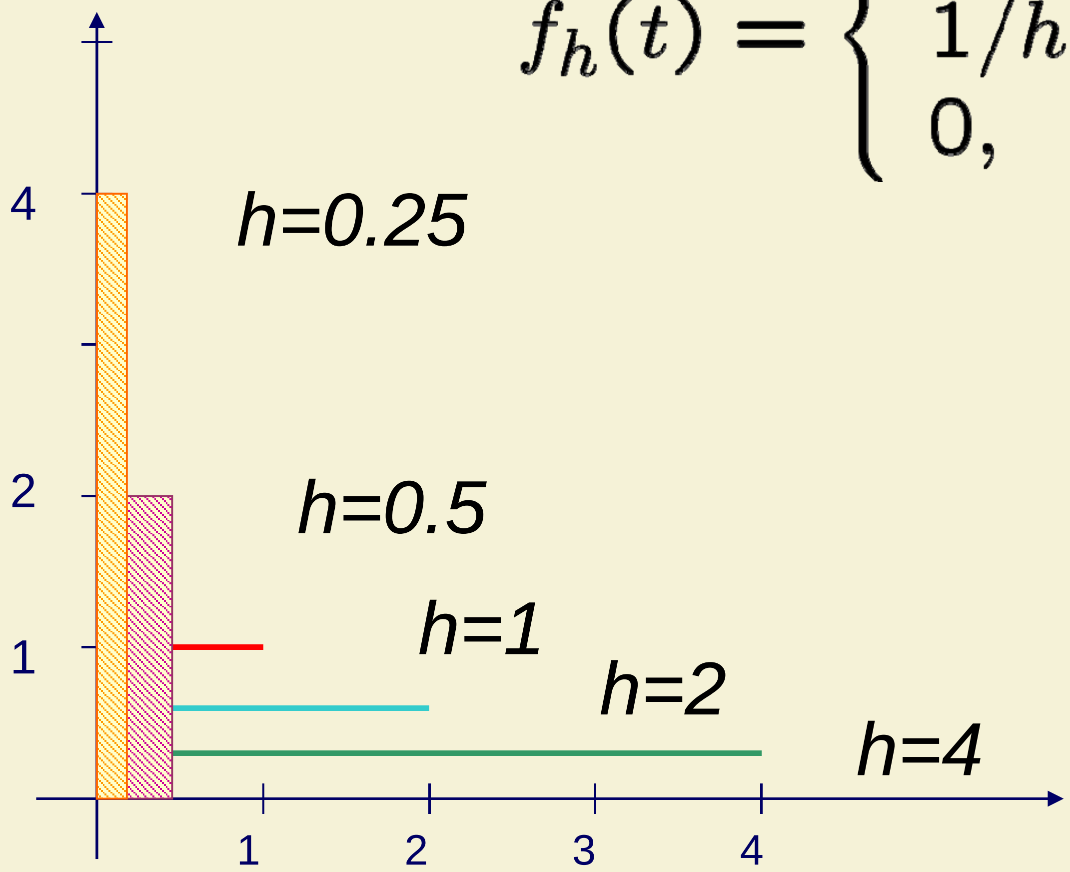
where $h>0$

$$\int_0^\infty f_h(t)\,dt = \int_0^h \frac{1}{h}\,dt = 1$$

E.g. $f_{10^{-100}}(t)$ has max value 10^{100} but area is still 1.

Examples of $f_h(t)$

$$f_h(t) = \begin{cases} 0, & t < 0 \\ 1/h, & 0 \le t \le h \\ 0, & t > h, \end{cases}$$



4.6 Dirac Delta Function

$$f_h(t) = \begin{cases} 0, & t < 0 \\ 1/h, & 0 \le t \le h \\ 0, & t > h, \end{cases}$$

$$\text{“}\delta(t) \equiv \lim_{h\to 0} f_h(t)\text{”}$$

Infinitely tall and narrow!

Practical problems: Scale

Let *h* be smaller than the smallest length in your problem

4.6 Dirac Delta Function

$$\int_0^\infty \delta(t)dt = 1$$

$$\delta(t) = 0$$

Everywhere except *t=0*

Let *g(t)* be any function

$$\int_0^\infty f_h(t)g(t)dt = \frac{1}{h}\int_0^h g(t)dt \approx g(0)$$

$$\int_0^h g(t)dt \approx g(0)h$$

MVT for integrals

4.6 Dirac Delta Function

$$\int_0^{\infty} \delta(t)dt = 1$$

$$\delta(t) = 0$$

Everywhere except *t=0*

$$\int_0^{\infty} \delta(t)g(t)dt = g(0)$$

$$\int_0^{\infty} \delta(t-a)g(t)dt = \int_{-a}^{\infty} \delta(\tau)g(\tau+a)d\tau$$

$$= \int_0^{\infty} \delta(\tau)g(\tau+a)d\tau = g(a)$$

$$\boxed{L\left(\delta(t-a)\right) = \int_0^{\infty} e^{-st}\delta(t-a)dt = e^{-as}.}$$

4.6 Dirac Delta Function

$$L\left(\delta(t-a)\right) = \int_0^\infty e^{-st}\delta(t-a)dt = e^{-as}.$$

$$L^{-1}\left(e^{-as}\right) = \delta(t-a)$$

$$L^{-1}\left(1\right) = \delta(t)$$

Example 21 : Injections

t=0 , 100 mg of morphine into a patient.
t=1 , 100 mg
half-life of morphine = 18 hours = 0.75 days

Without injections
$$y(t) = y(0)e^{-kt}$$

$$\frac{dy}{dt} = -ky$$

$$0.5 = e^{-0.75k} \Rightarrow k = \frac{\ln 2}{0.75} = 0.924$$

Example 21 : Injections

t=0 , 100 mg of morphine into a patient.
t=1 , 100 mg
half-life of morphine = 18 hours = 0.75 days

Injection Rate:

100 mg per day (but not evenly distributed)

$$\frac{dy}{dt} = -ky + 100\delta(t) + 100\delta(t-1)$$

units = 1/ time

$$sL(y) - y(0) = -0.924L(y) + 100 \times 1 + 100e^{-s}$$

Example 21 : Injections

$$sL(y) - \overset{=0}{y(0)} = -0.924L(y) + 100 \times 1 + 100e^{-s}$$

$$(s + 0.924)L(y) = 100(1 + e^{-s})$$

$$L(y) = \frac{100}{s + 0.924} + \frac{100e^{-s}}{s + 0.924}$$

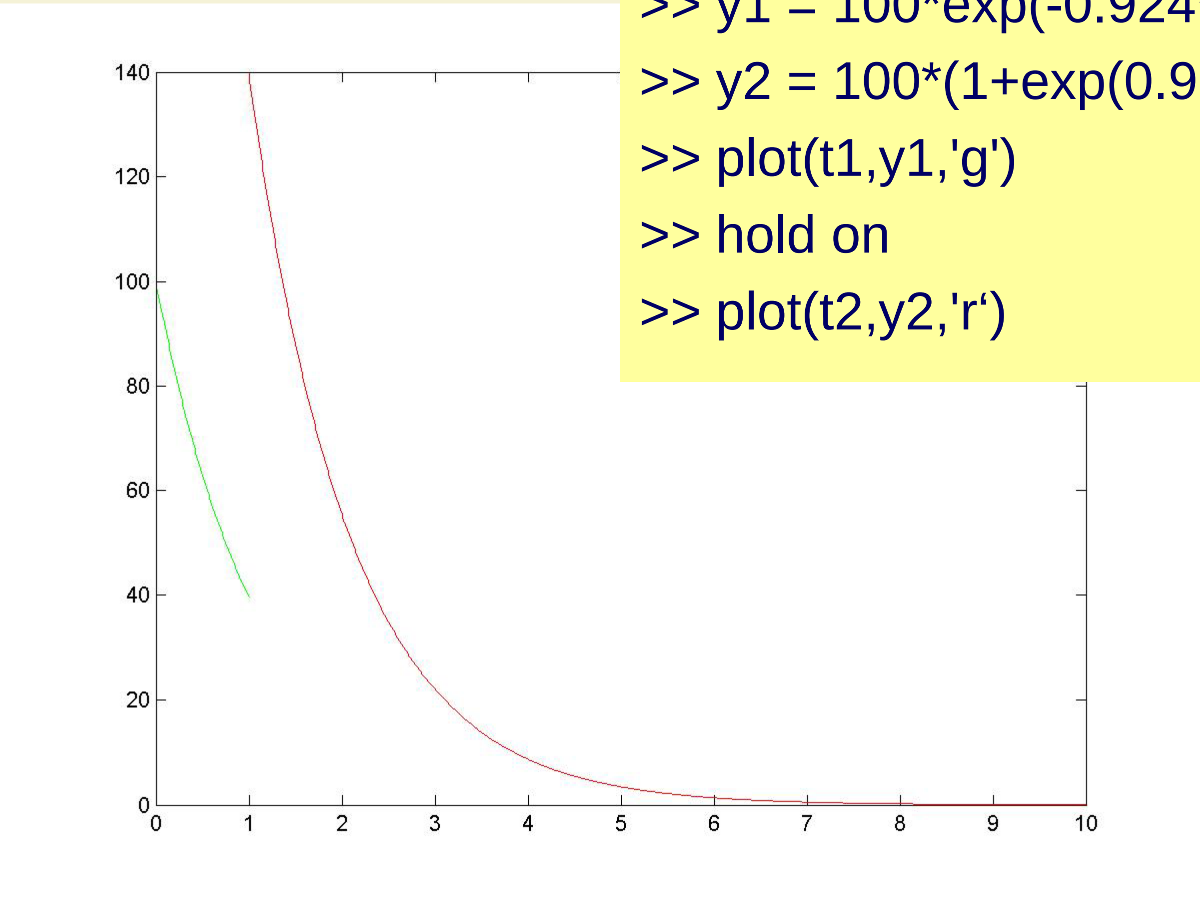
$$y = 100e^{-0.924t} + 100e^{-0.924(t-1)}u(t-1)$$

$$= \begin{cases} 100e^{-0.924t}, & 0 < t < 1 \\ 100(1 + e^{0.924})e^{-0.924t}, & t > 1. \end{cases}$$

$$y = \begin{cases} 100e^{-0.924t}, & 0 < t < 1 \\ 100(1+e^{0.924})e^{-0.924t}, & t > 1. \end{cases}$$

MATLAB plot

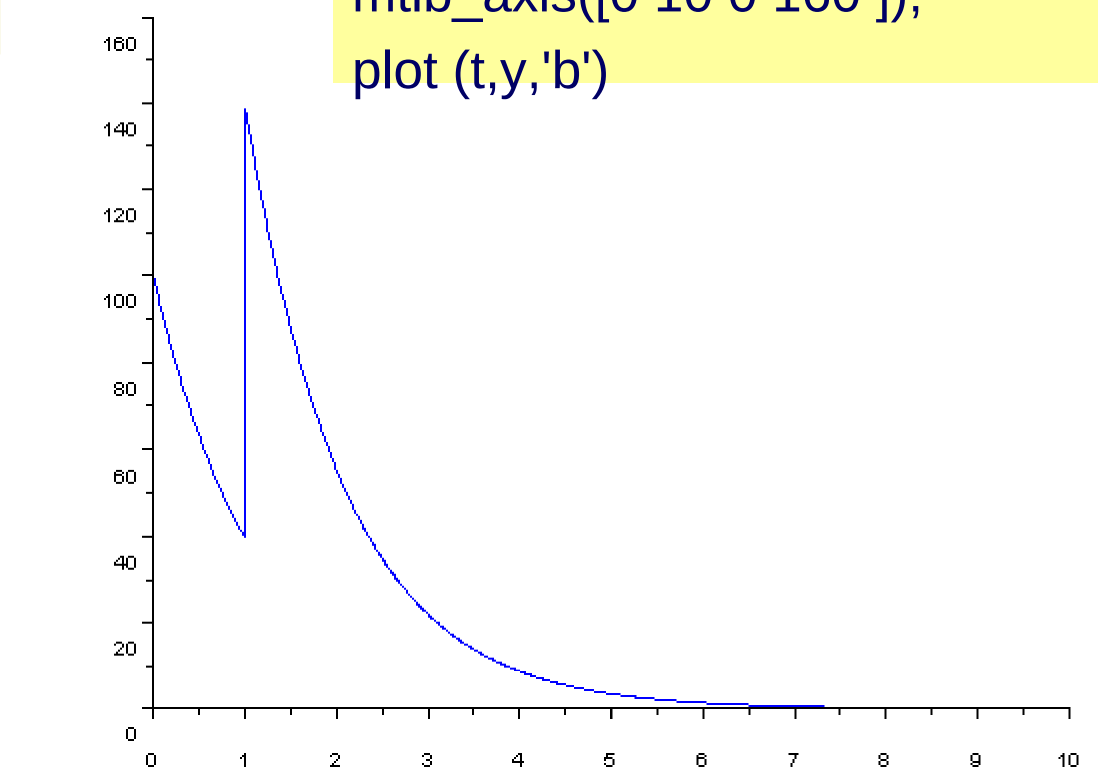
```
>> t1 = 0:0.01:1;
>> t2 = 1:0.01:10;
>> y1 = 100*exp(-0.924*t1);
>> y2 = 100*(1+exp(0.924))*exp(-0.924*t2);
>> plot(t1,y1,'g')
>> hold on
>> plot(t2,y2,'r‘)
```



$$y = 100e^{-0.924t} + 100e^{-0.924(t-1)}u(t-1)$$

scilab plot

```
t=0:0.01:10;
y = 100*exp(-0.924*t) +100*exp(-0.924*(t-1))*0.5.*(sign(t-1)+1);
//Use 0.5.*(sign(t-1)+1) to represent step function u(t-1)
mtlb_axis([0 10 0 160 ]);
plot (t,y,'b')
```



Remark

- Units of the delta function may be different in different problems

$$m\ddot{x} + b\dot{x} + kx = F$$

$$F = F_0, F_0 \cos(wt), F_0 \delta(t - a)$$

Example : Parameter Reconstruction

- Understand the system, e.g. mass spring oscillators
- Parameters unknown
- “Poke” the system with a sharp sudden force

$$\delta(t)$$

$$m\ddot{x} + b\dot{x} + kx = 0$$

$$x(0) = 0, \dot{x}(0) = 0$$

Example : Parameter Reconstruction

- Apply unit impulse at *t=1* i.e. F = $\delta(t-1)$
- Observe that solution

$$\boxed{x(t) = u(t-1)e^{-(t-1)}\sin(t-1)}$$

$$M\ddot{x} = -kx - b\dot{x} + \delta(t-1)$$

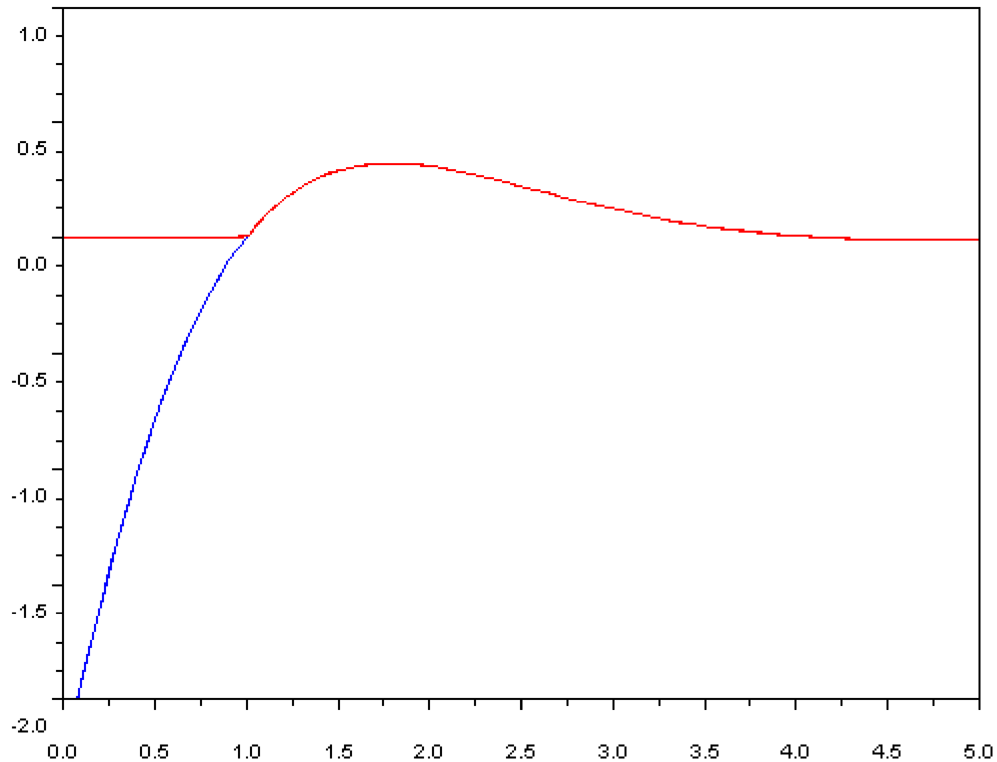
$$Ms^2X(s) = -kX(s) - bsX(s) + e^{-s}$$

$$X(s) = \frac{e^{-s}}{Ms^2 + bs + k}$$

$$\boxed{X(s) = \frac{e^{-s}}{(s+1)^2 + 1} = \frac{e^{-s}}{s^2 + 2s + 2}}$$

M=1, b=2, k=2

$$x(t) = u(t-1)e^{-(t-1)}\sin(t-1)$$



Scilab plot

```
t=linspace(0,5,100);
g = exp(-(t-1)).*sin(t-1);
plot (t,g)
x = exp(-(t-1)).*sin(t-1)*0.5.*(sign(t-1)+1);
// Use 0.5.*(sign(t-1)+1) to represent step function u(t-1)
mtlb_axis([0 5 -2 1]);
plot (t,x,'r')
```

Table of Laplace transform, $\boxed{s>0}$ $\quad F(s) = L(f) = \int_0^\infty e^{-st} f(t)dt$

1) Linearity 2) How to solve IVP

$L(e^{at}) = \frac{1}{s-a}, \quad s > a$	$L(t^n) = \frac{n!}{s^{n+1}}$
$L(\cos wt) = \frac{s}{s^2+w^2},$	$L(\sin wt) = \frac{w}{s^2+w^2}$
$L(f^{(n)}) = s^n L(f) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$, $s > a$	
$L\left(\int_0^t f(\tau)d\tau\right) = \frac{1}{s} L(f), s > a$	$L(tf(t)) = -\frac{d}{ds} F(s), s > a$
$L(e^{ct} f(t)) = F(s-c), \quad s - c > a$ **s**-shifting	
$L(f(t-a)u(t-a)) = e^{-as} F(s)$ ***t***-shifting	
$L(\delta(t-a)) = e^{-as}$	