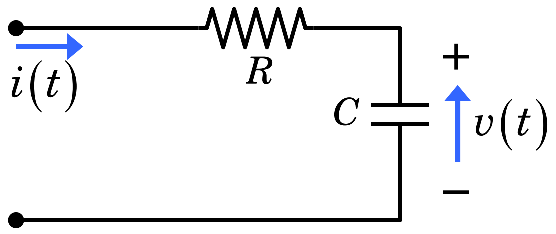
1. Signals and Classification of Signals

In this chapter, we introduce the notion of signals and examine the way in which signals are classified into various categories in accordance with their properties. We also define several important basic signals that are essential to our studies.

1.1 Signals

- Signals can manifest in many forms such as electrical voltage or current, radio wave, infrared and ultraviolet rays, lightwave, sound wave, mechanical pressure, etc.
- In signal studies, a signal is a function representing a physical quantity that conveys information about the behavior or nature of the phenomenon.
- Mathematically, a signal is represented as a function of an independent variable t.
- Usually t represents time and a signal is denoted by $x(t)$. In this case, $x(t)$ is called a *time-domain* signal.
- For example, in an RC circuit the signal $v(t)$ may represent the voltage across the capacitor and $i(t)$ the current flowing in both the resistor and capacitor.



1.1.1 Classification of Signals

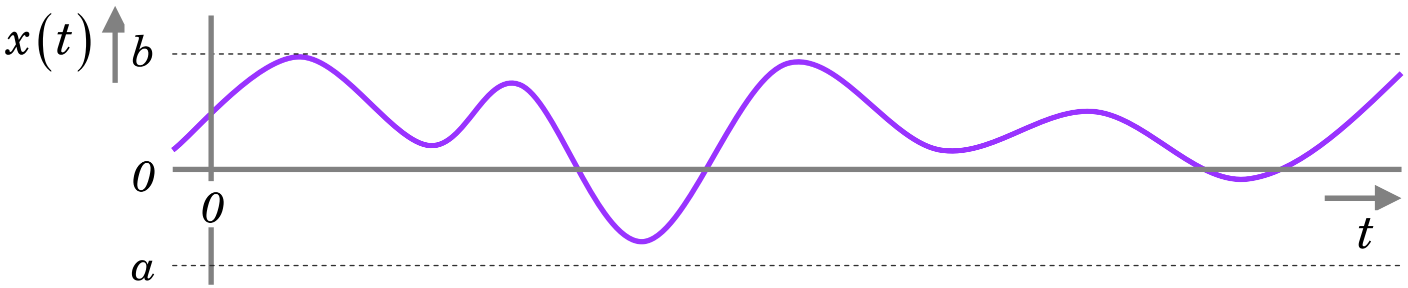
A. *Continuous-time, Analog, Discrete-time and Digital signals*

Continuous-time signal:

- A signal $x(t)$ is a **continuous-time signal** if t is a continuous variable.
- Usually depicted as a **waveform**.

ANALOG SIGNAL - A **continuous-time signal** that can take on any value in the continuous interval (a,b), where a may be $-\infty$ and b may be $+\infty$.

Graphical representation:



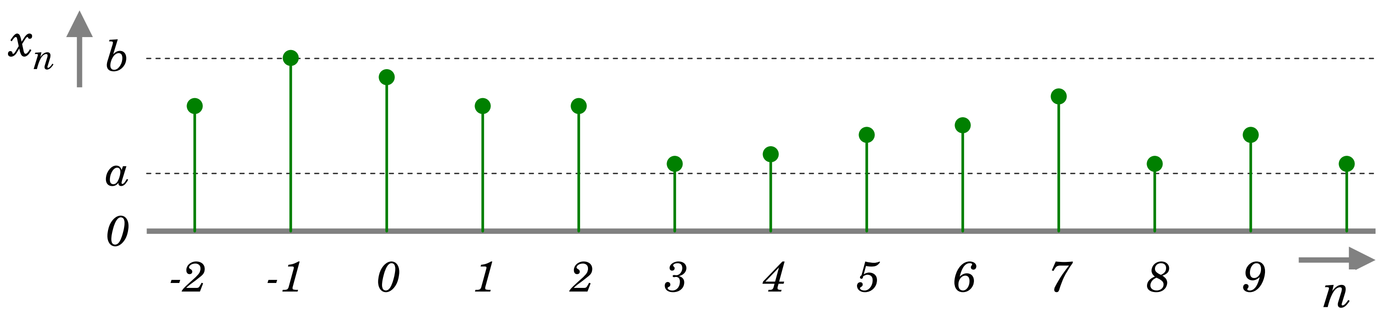
Discrete-time signal:

- A signal is discrete-time if it is defined only at discrete times.
- Usually denoted by x_n, where n is an integer, and depicted as a **sequence of numbers** such as

$$x_0, x_1, \;\cdots\cdots\; , x_n, \;\cdots\cdots$$

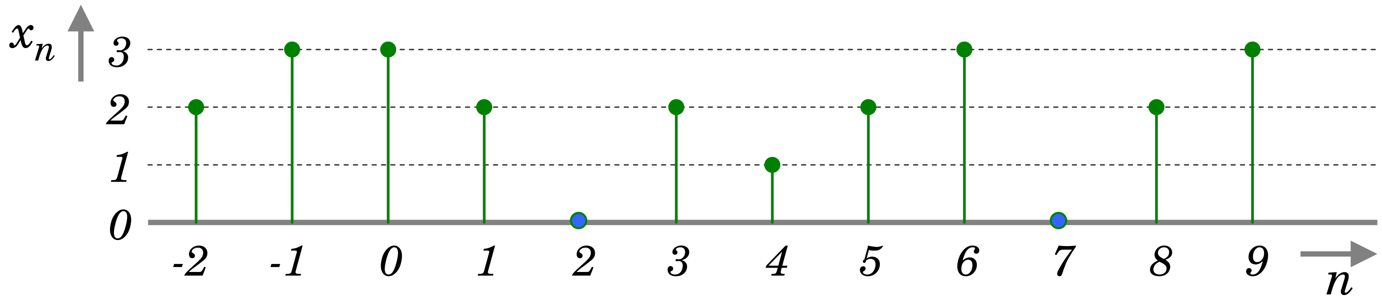
- Discrete-time signals may evolve naturally, for instance the daily closing stock market average which occurs only at the close of each day, or obtained by sampling a continuous-time signal $x(t)$ such as $x_n = x(t_n)$ where t_n are discrete time points.
- In general, a discrete-time signal can take on any value in the continuous interval (a,b), where a may be $-\infty$ and b may be $+\infty$.

Graphical representation:



DIGITAL SIGNAL - A **discrete-time signal** that can take on only a finite number of distinct values.

Graphical representation (*example* – **quaternary** [or 4-level] signal used in some digital communication systems):



In this course, we focus only on continuous-time signals and systems.

Hereon, all signals and systems are assumed continuous-time unless otherwise specified.

B. *Periodic and Nonperiodic signals*

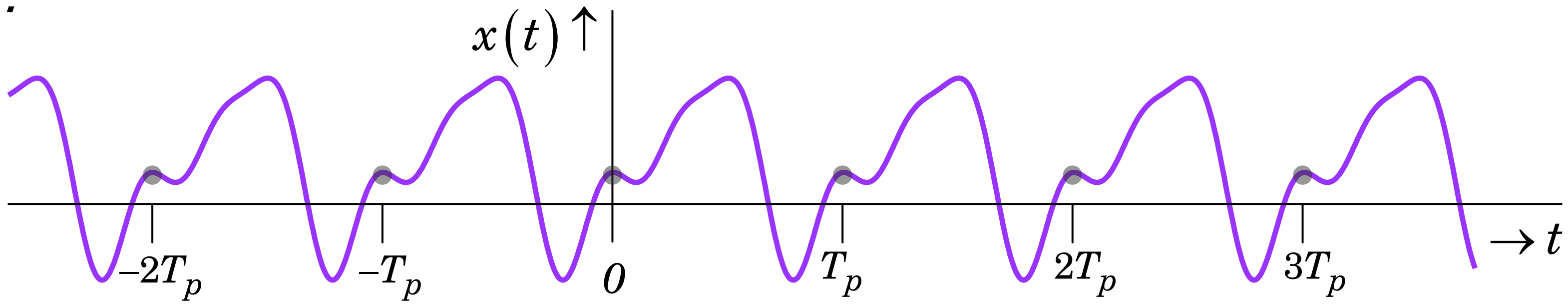
- A signal $x(t)$ is said to be **periodic with period T** if there is a positive non-zero value of T for which

$$x(t) = x(t+T); \quad \forall t. \tag{1.1}$$

- The smallest value of T which satisfes (1.1) is called the "fundamental period", or simply "period" of $x(t)$.

- The reciprocal of the "fundamental period" is called the "fundamental frequency" of $x(t)$.

Example 1-1:



This signal satisfies $x(t) = x(t+T)$ *for* $T = T_p, 2T_p, \cdots\cdots$

Period: $\boldsymbol{T_p}$ *(shortest repetition interval)*

Fundamental frequency: $\boldsymbol{f_p = 1/T_p}$

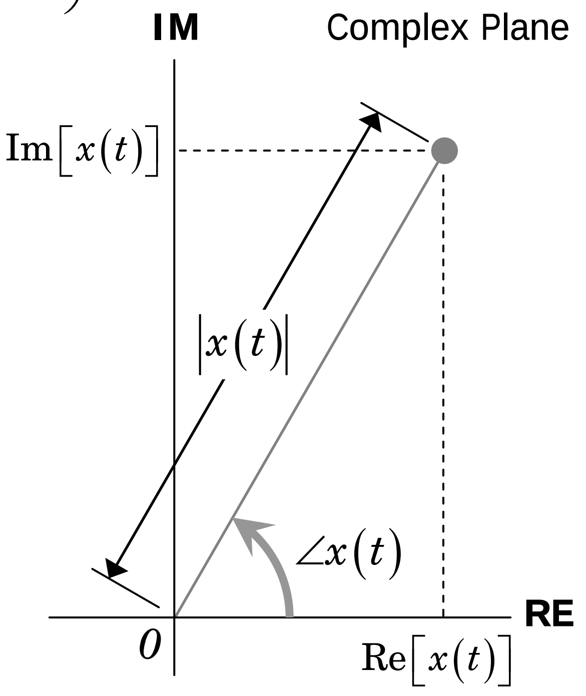
- Any signal which is not periodic is called **nonperiodic** or **aperiodic**.

C. *Real and Complex signals*

- A signal $x(t)$ is a **complex signal** if its value is a complex number.

- A general complex signal $x(t)$ may be expressed in:

$$\left(\begin{array}{r}\textbf{Cartesian form}:\; x(t)=\mathrm{Re}\left[x(t)\right]+j\,\mathrm{Im}\left[x(t)\right]\\ \textbf{Polar form}:\; x(t)=\left|x(t)\right|\exp\left(j\angle x(t)\right)\end{array}\right) \tag{1.2}$$



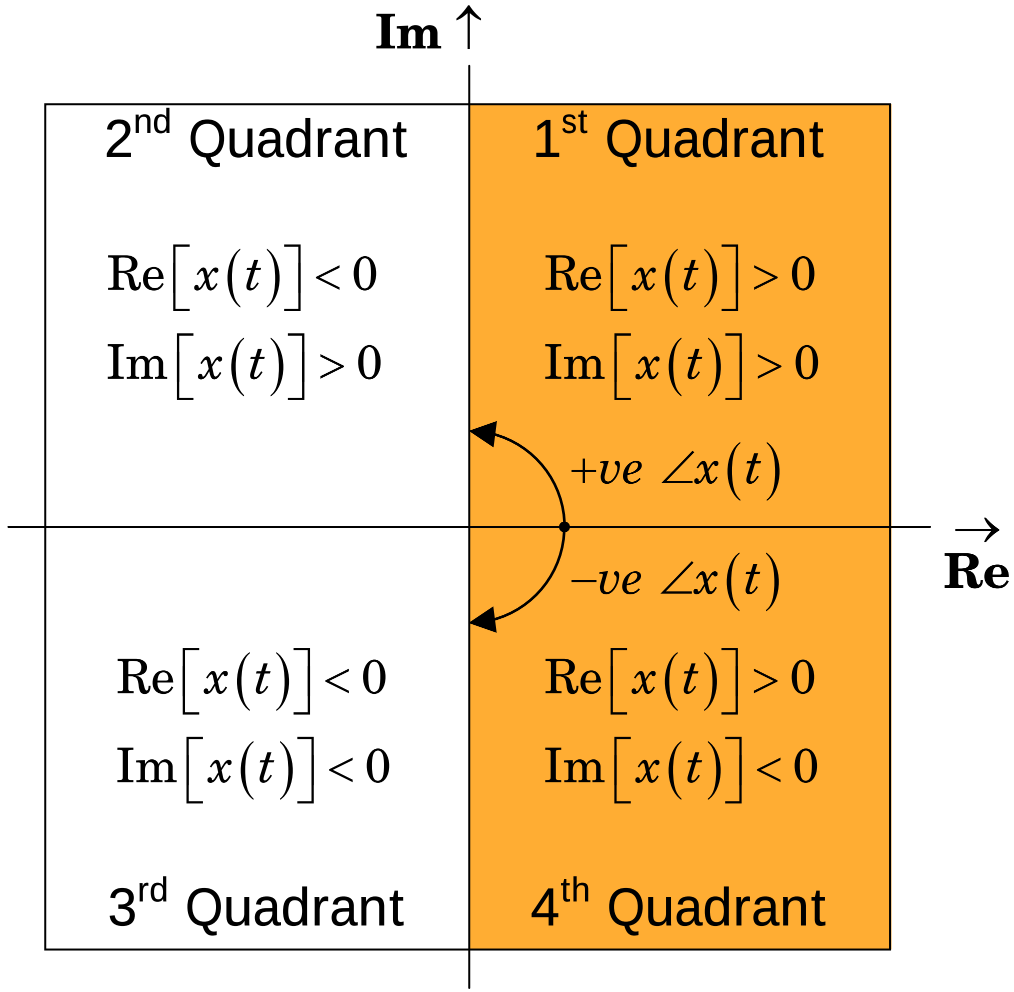
where $j=(-1)^{0.5}$ and $\left[\begin{array}{ll}\mathrm{Re}\left[x(t)\right] & : \textit{Real part of } x(t)\\ \mathrm{Im}\left[x(t)\right] & : \textit{Imaginary part of } x(t)\\ \left|x(t)\right| & : \textit{Magnitude (or Amplitude) of } x(t)\\ \angle x(t) & : \textit{Phase of } x(t)\end{array}\right]$.

By applying **Euler's formula**, $\exp(j\theta)=\cos(\theta)+j\sin(\theta)$, it can be shown that:

$$\mathrm{Re}\left[x(t)\right]=\left|x(t)\right|\cos\left(\angle x(t)\right) \qquad \mathrm{Im}\left[x(t)\right]=\left|x(t)\right|\sin\left(\angle x(t)\right)$$

$$\left|x(t)\right|=\left[\mathrm{Re}^2\left[x(t)\right]+\mathrm{Im}^2\left[x(t)\right]\right]^{0.5} \qquad \angle x(t)=\arctan\left(\frac{\mathrm{Im}\left[x(t)\right]}{\mathrm{Re}\left[x(t)\right]}\right)$$

Computing $\angle x(t)$ ➔



- A signal $x(t)$ is a **real signal** if its value is a real number. This is a special case of the complex signal when $\left[\mathrm{Im}[x(t)] = 0\right]$ or $\left[\angle x(t) = \pm n\pi\right]$.

Example 1-2:

Write $\boldsymbol{z(t)=(1-j)\exp(j2\pi t)}$ ***in Cartesian and polar forms. What is*** $\boldsymbol{\angle z(0.5)}$***?***

In Cartesian form:

$$\begin{cases} z(t)=(1-j)\big(\cos(2\pi t)+j\sin(2\pi t)\big) \\ \qquad\;\; =\cos(2\pi t)+j\sin(2\pi t)-j\cos(2\pi t)+\sin(2\pi t) \\ \qquad\;\; =\sin(2\pi t)+\cos(2\pi t)+j\left[\sin(2\pi t)-\cos(2\pi t)\right] \\ \mathrm{Re}\left[z(t)\right]=\sin(2\pi t)+\cos(2\pi t) \\ \mathrm{Im}\left[z(t)\right]=\sin(2\pi t)-\cos(2\pi t) \\ \angle z(0.5)=\tan^{-1}\left(\dfrac{\mathrm{Im}\left[z(0.5)\right]}{\mathrm{Re}\left[z(0.5)\right]}\right)=\tan^{-1}(-1)=\left(\dfrac{3\pi}{4}\right)\ \textit{or}\ \left(-\dfrac{\pi}{4}\right)\ ? \end{cases}$$

In polar form:

$$\begin{cases} z(t)=\underbrace{2^{0.5}\exp\left(-j\dfrac{\pi}{4}\right)}_{1-j}\exp(j2\pi t)=2^{0.5}\exp\left(j\left(2\pi t-\dfrac{\pi}{4}\right)\right) \\ \left|z(t)\right|=2^{0.5}\quad \textit{and}\quad \angle z(t)=2\pi t-\dfrac{\pi}{4} \\ \angle z(0.5)=\dfrac{3\pi}{4} \end{cases}$$

D. *Deterministic and Random signals*

- A **deterministic** signal is a signal in which each value of the signal is fixed and can be determined by a mathematical expression, rule, or table. Because of this the future values of the signal can be predicted with complete confidence.

Example 1-3:

$x(t) = \cos(2\pi t)$

At any time t_o, $x(t_o)$ is exactly determined as $\cos(2\pi t_o)$.

- A **random** signal has a lot of uncertainty about its behavior. The future values of the signal cannot be accurately predicted and can only be guessed (or estimated) based on the signal statistics and observation of past outcomes. The more uncertain a signal is, the more information it carries.

Example 1-4:

$x(t) = \cos(2\pi t + \varphi)$ *where* φ *can take on values in the set* $\{0, 0.5\pi, \pi, 1.5\pi\}$ *with equal probability.*

At any time t_o, $x(t_o)$ cannot be exactly determined since it can assume one of four values, $\cos(2\pi t_o), -\sin(2\pi t_o), -\cos(2\pi t_o)$ *and* $\sin(2\pi t_o)$*, with equal probabilities. In this case,* $x(t_o)$ *can only be determined in terms of statistics.*

E. *Energy and Power signals*

- $x(t)$ is said to be an **energy** signal if and only if its total energy E satisfies

$$0 < \left[E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt \right] < \infty . \tag{1.3}$$

- $x(t)$ is said to be a **power** signal if and only if its average power P satisfies

$$0 < \left[P = \lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} |x(t)|^2 \, dt \right] < \infty . \tag{1.4}$$

- Combining (1.3) and (1.4) we have $\left[P = \lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} |x(t)|^2 \, dt \to \frac{1}{2 \cdot \infty} \int_{-\infty}^{\infty} |x(t)|^2 \, dt \to \frac{E}{2 \cdot \infty} \right]$, which implies that:

$$\begin{cases} \textbf{Energy} \text{ signals have } \textit{zero average power}, \text{ because } E = \textit{finite} \text{ implies } P = 0 \\ \textbf{Power} \text{ signals have } \textit{infinite total energy}, \text{ because } P = \textit{finite} \text{ implies } E = \infty \end{cases}.$$

- Signals that satisfy neither (1.3) nor (1.4) are referred to as neither energy nor power signals.

Example 1-5:

I. $$\boldsymbol{x(t)=\begin{cases}\exp(-\alpha t); & t\ge 0\\ 0; & t<0\end{cases};\quad \alpha>0}$$

Total energy: $\int_{-\infty}^{\infty}x^2(t)\,dt=\int_0^{\infty}\exp(-2\alpha t)\,dt=\left[\dfrac{\exp(-2\alpha t)}{-2\alpha}\right]_0^{\infty}=\dfrac{1}{2\alpha}$

Finite total energy ➔ Zero average power ➔ $x(t)$ *is an* ***energy signal***

II. $$\boldsymbol{x(t)=\begin{cases}\alpha t; & t\ge 0\\ 0; & t<0\end{cases};\quad \alpha\ne 0}$$

Total energy : $\int_{-\infty}^{\infty}x^2(t)\,dt=\int_{-\infty}^{\infty}\alpha^2t^2\;dt=\left[\dfrac{\alpha^2t^3}{3}\right]_0^{\infty}=\infty$

Average power : $$\begin{cases}\lim\limits_{T\to\infty}\dfrac{1}{T}\int_{-T/2}^{T/2}\left|x(t)\right|^2dt=\lim\limits_{T\to\infty}\dfrac{1}{T}\int_0^{T/2}\alpha^2t^2dt=\lim\limits_{T\to\infty}\dfrac{1}{T}\left[\dfrac{\alpha^2t^3}{3}\right]_0^{T/2}\\ =\lim\limits_{T\to\infty}\dfrac{\alpha^2T^2}{24}=\infty\end{cases}$$

Infinite total energy and average power ➔ $x(t)$ *is neither an energy nor a power signal.*

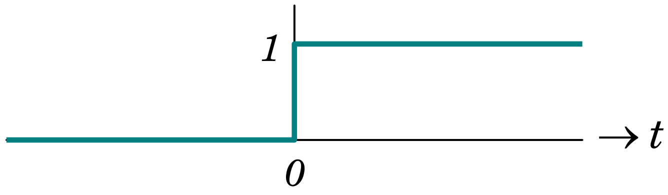
III. $$\boldsymbol{x(t)=\alpha\cos(2\pi t+\beta)}$$

$x(t)$ *is a sinusoid ➔* $x(t)$ *is a* ***power signal*** *with average power* $\alpha^2/2$.

1.2 Basic Signals

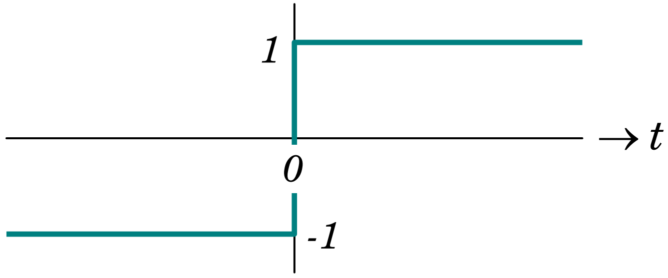
A. *The Unit Step function*

$$u(t)=\begin{cases}1; & t\geq 0\\ 0; & t<0\end{cases} \tag{1.5}$$



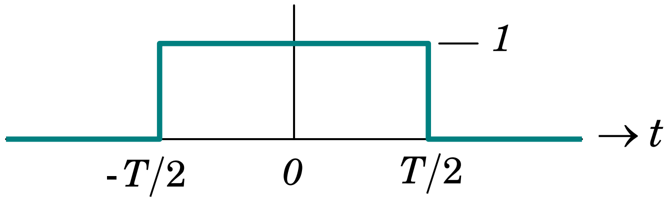
B. *The Sign (or Signum) function*

$$\begin{aligned}\operatorname{sgn}(t)&=\begin{cases}+1; & t\geq 0\\ -1; & t<0\end{cases}\\ &=2u(t)-1\end{aligned} \tag{1.6}$$



C. *The Rectangle function*

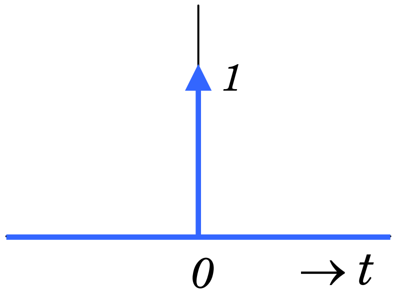
$$\begin{aligned}\operatorname{rect}\left(\frac{t}{T}\right)&=\begin{cases}1; & -T/2\leq t<T/2\\ 0; & elsewhere\end{cases}\\ &=u\left(t+\frac{T}{2}\right)-u\left(t-\frac{T}{2}\right)\end{aligned} \tag{1.7}$$



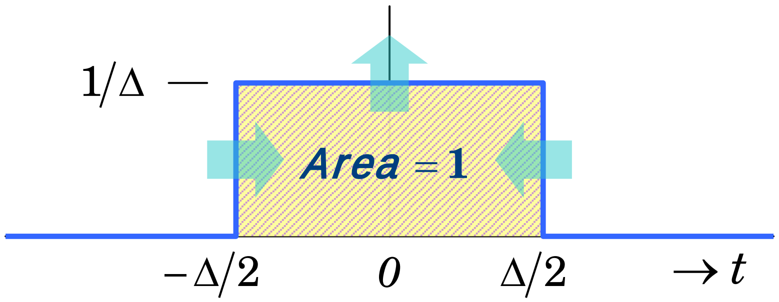
D. *The Unit Impulse (or Dirac-δ) function*

$$\delta(t)=\begin{cases}\infty; & t=0\\ 0; & t\neq 0\end{cases} \quad \text{and} \quad \int_{0^-}^{0^+}\delta(t)\,dt=1 \tag{1.8}$$

We may view the unit impulse as a limiting case of a rectangle pulse which has a unit area that is independent of its pulse width:

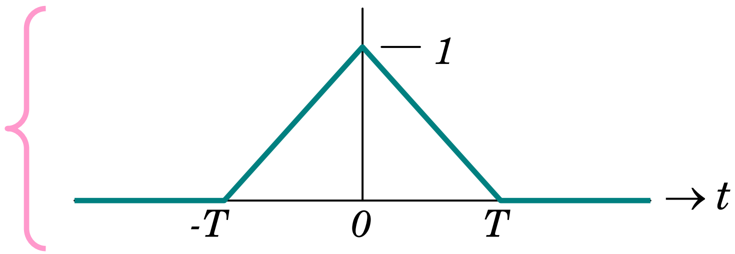


$$\delta(t)=\lim_{\Delta\to 0}\left[\frac{1}{\Delta}\operatorname{rect}\left(\frac{t}{\Delta}\right)\right]$$



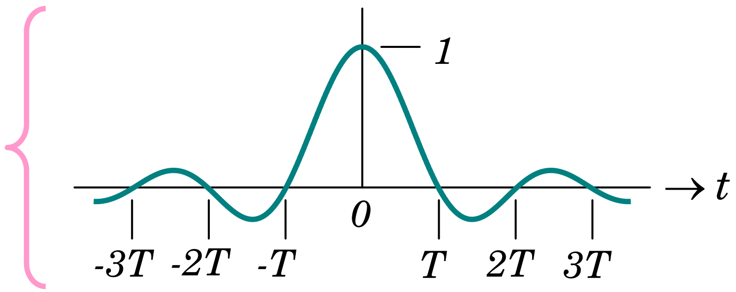
E. *The Triangle function*

$$\operatorname{tri}\left(\frac{t}{T}\right)=\begin{cases}1-|t|/T; & |t|\le T\\ 0; & |t|>T\end{cases} \tag{1.9}$$



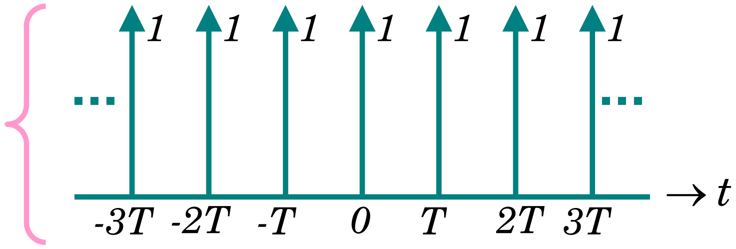
F. *The Sinc (Sine Cardinal) function*

$$\operatorname{sinc}\left(\frac{t}{T}\right)=\begin{cases}\dfrac{\sin\left(\pi\dfrac{t}{T}\right)}{\pi\dfrac{t}{T}}; & t\neq 0\\ 1; & t=0\end{cases} \tag{1.10}$$



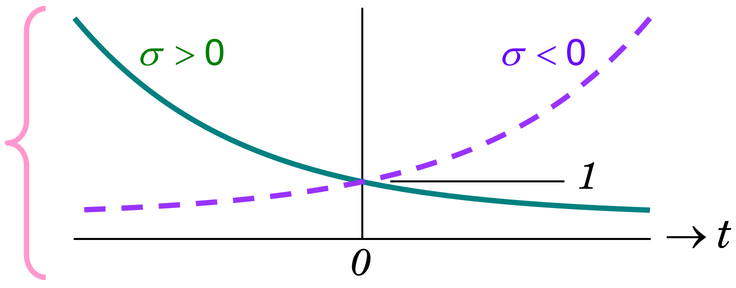
G. *The Dirac Comb function*

$$\xi_T(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT) \tag{1.11}$$



H. *Real Exponential signal*

$$x(t)=\exp(-\sigma t) \tag{1.12}$$



I. *Sinusoidal Signals*

Sinusoidal signals (or sinusoids) is a collective term for a general class of periodic signals of the form:

- $x(t) = \mu\cos(\omega_o t + \phi) = \frac{\mu}{2}\left[\exp\left[j(\omega_o t + \phi)\right] + \exp\left[-j(\omega_o t + \phi)\right]\right]$ $\begin{cases}\text{REAL SINUSOID} \\ \text{a.k.a. COSINE}\end{cases}$ (1.13)

- $x(t) = \mu\sin(\omega_o t + \phi) = \frac{\mu}{j2}\left[\exp\left[j(\omega_o t + \phi)\right] - \exp\left[-j(\omega_o t + \phi)\right]\right]$ $\begin{cases}\text{REAL SINUSOID} \\ \text{a.k.a. SINE}\end{cases}$ (1.14)

- $x(t) = \mu\exp\left[j(\omega_o t + \phi)\right] = \mu\left[\cos(\omega_o t + \phi) + j\sin(\omega_o t + \phi)\right]$ $\begin{cases}\text{COMPLEX SINUSOID} \\ \text{a.k.a.} \begin{cases}\text{COMPLEX} \\ \text{EXPONENTIAL}\end{cases}\end{cases}$ (1.15)

where

$\mu(>0)$: magnitude

ω_o : angular frequency (rad/s)

ϕ : phase (radians)

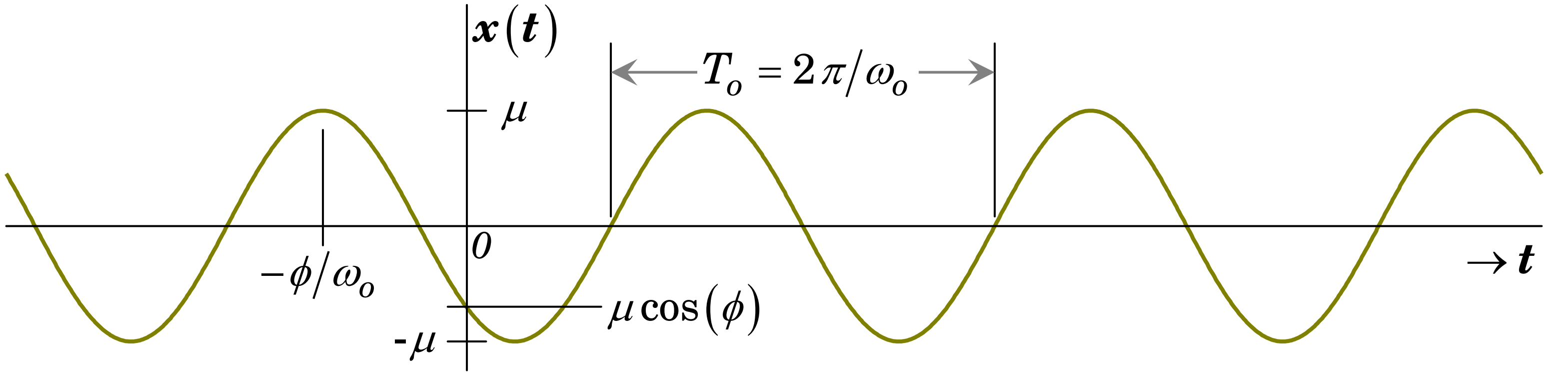
$\omega_o t + \phi$: instantaneous phase (radians)

It is also common to replace ω_o by $2\pi f_o$, where f_o is the ***cyclic frequency*** (in Hz).

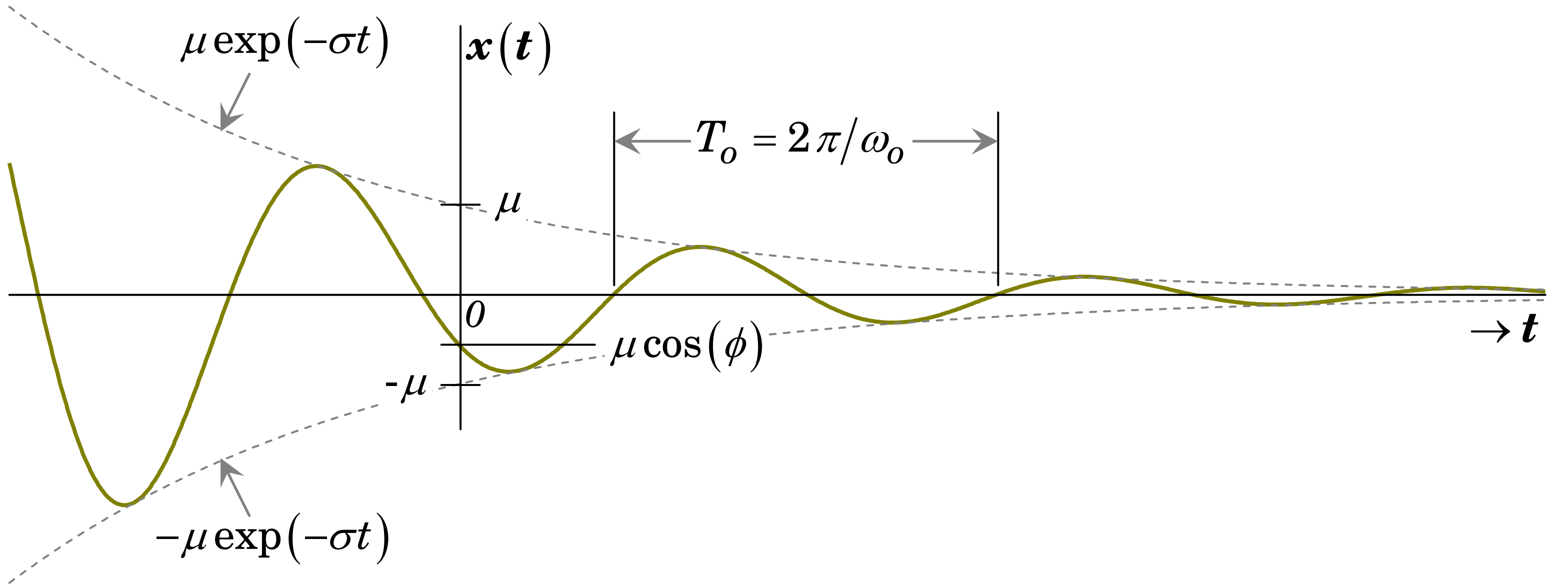
The fundamental period, T_o (in seconds), is given by $T_o = \frac{2\pi}{\omega_o} = \frac{1}{f_o}$.

Example 1-6:

Plot of a real sinusoid: $x(t) = \mu\cos(\omega_o t + \phi) = \mu\cos\left(\omega_o\left(t + \frac{\phi}{\omega_o}\right)\right)$



Sketch of an exponentially decaying real sinusoid: $x(t) = \mu\cos(\omega_o t + \phi)\exp(-\sigma t); \quad \sigma > 0$



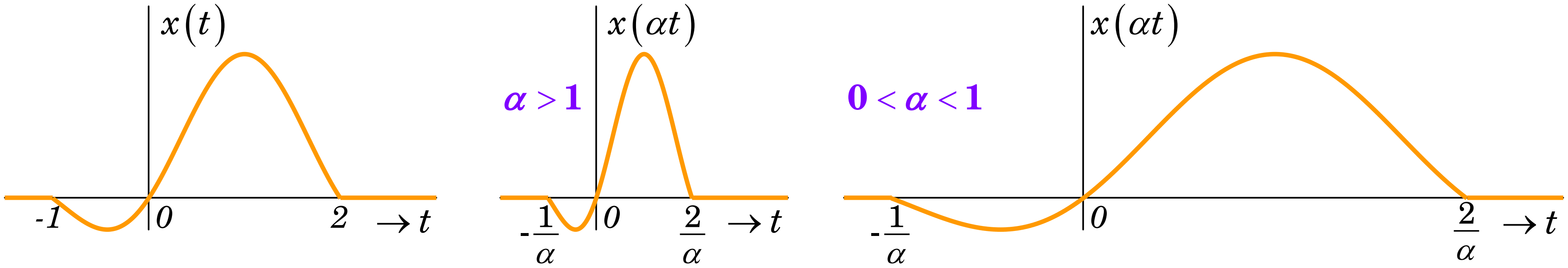
1.3 Time-Scaling, -Reversal and -Shifting of Signals

A. *Time-scaling*

Time-scaling of a signal $x(t)$ is effected by replacing the time variable t by αt, where α is a positive real number.

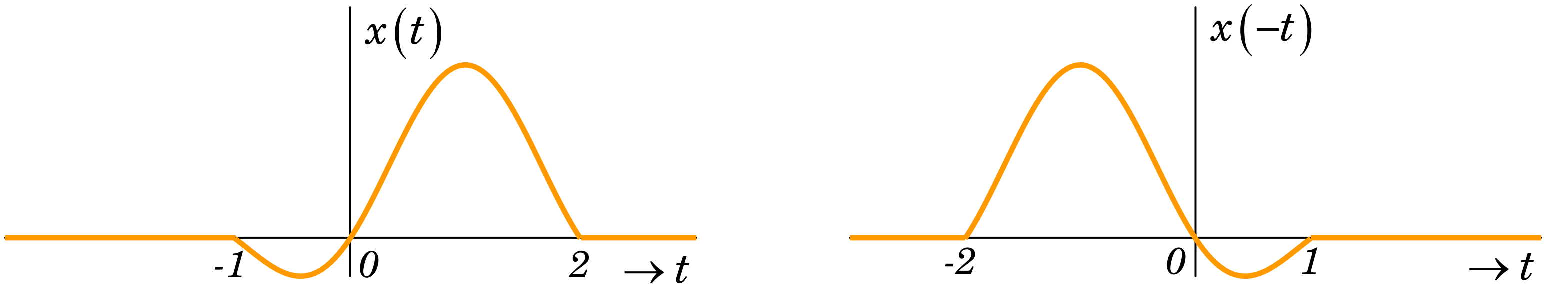
$0<\alpha<1$: uniform **expansion** of $x(t)$ along the time axis

$\alpha>1$: uniform **contraction** of $x(t)$ along the time axis



B. *Time-reversal*

Time-reversal of a signal $x(t)$ is effected by replacing the time variable t by $-t$.

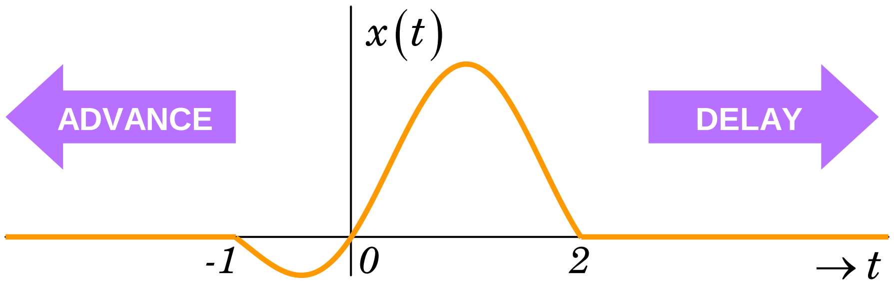


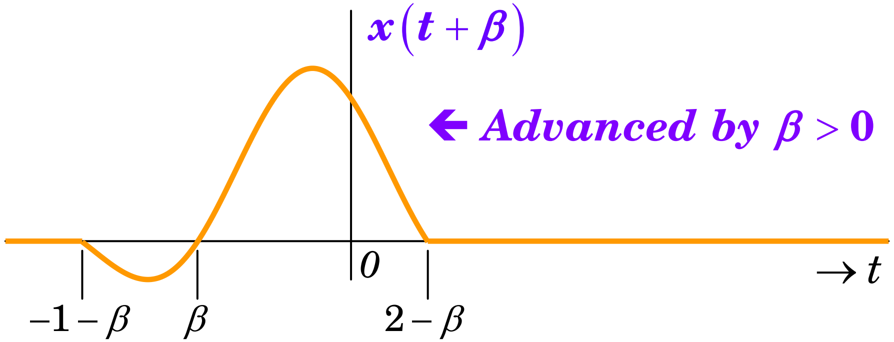
C. *Time-shifting*

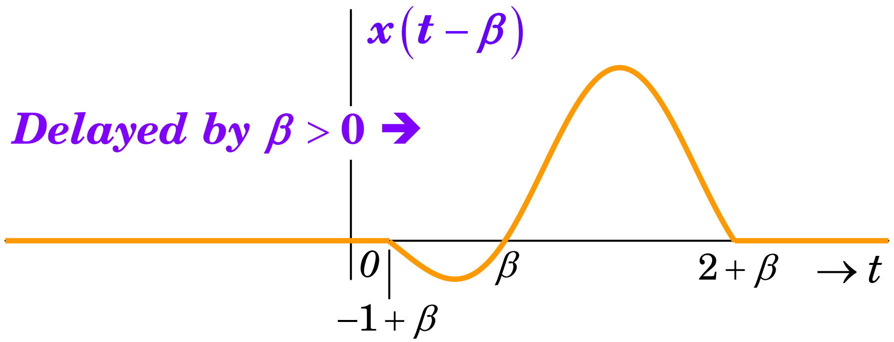
Time-shifting of a signal $x(t)$ is effected by replacing the time variable t by $(t-\beta)$, where β is a real number.

$\beta > 0$: **Delaying** $x(t)$ by β unit of time

$\beta < 0$: **Advancing** $x(t)$ by β unit of time



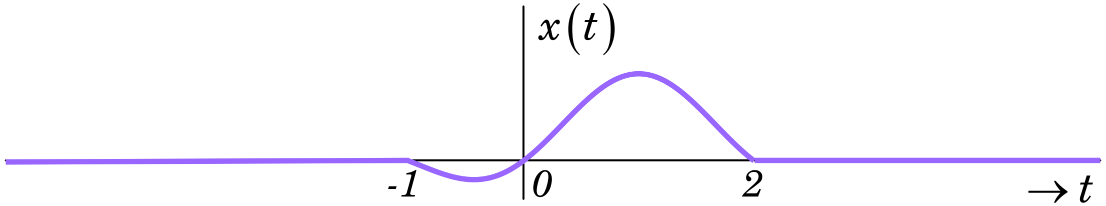




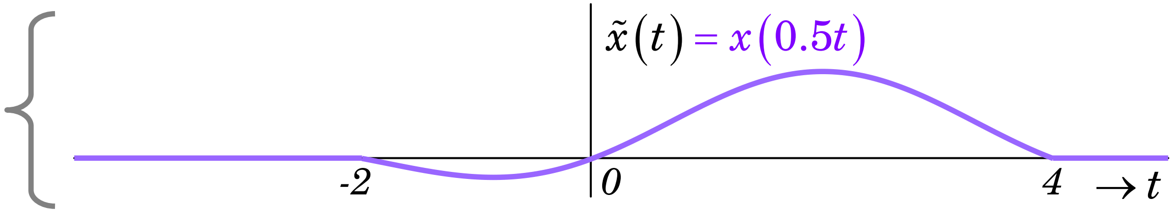
Example 1-7:

Given $x(t)$, how would $y(t) = x(1-0.5t)$ look like?

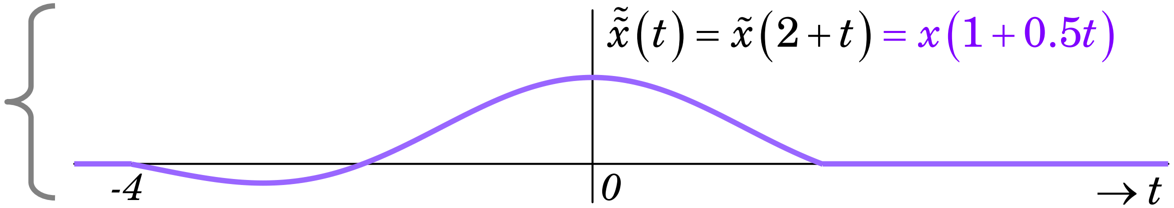
We may perform these operations on $x(t)$ sequentially, in any order, to reach $y(t)$. For example:



- *Time-scale $x(t)$ to form*
  $\tilde{x}(t) = x(0.5t)$



- *Time-shift $\tilde{x}(t)$ to form*
  $\tilde{\tilde{x}}(t) = \tilde{x}(2+t)$
  $= x(1+0.5t)$



- *Time-reverse $\tilde{\tilde{x}}(t)$ to form*
  $y(t) = \tilde{\tilde{x}}(-t)$
  $= x(1-0.5t)$