CHAPTER 7

SYSTEMS OF FIRST-ORDER ODEs

7.1. ROMEO AND JULIET

We all know that many (most) relationships have their ups and downs. Let's try to model this fact.

Romeo loves Juliet, but Juliet believes in a more subtle approach and finds Romeo's excessive enthusiasm rather repulsive - the more he loves her, the less she likes him. On the other hand, when he loses interest, she fears losing him and begins to see his good side. Romeo is more straightforward: his love for Juliet increases when she is warm to him, and decreases when she is cold. Let $R(t)$ represent Romeo's feelings and $J(t)$ Juliet's. We can model the lovers' feelings by:

$$\begin{cases} \frac{dR}{dt} = aJ & R(0) = \alpha \\ \frac{dJ}{dt} = -bR & J(0) = \beta \end{cases}$$

where a, b are positive constants and α and β represent their feelings when they first meet.

$$\frac{dR}{dt} = aJ \quad \text{----} \quad ①$$

$$\frac{dJ}{dt} = -bR \quad \text{----} \quad ②$$

$$\frac{d}{dt}① \Rightarrow \frac{d^2R}{dt^2} = a\frac{dJ}{dt} \quad \text{----} \quad ③$$

Substitute ② into ③

$$\Rightarrow \frac{d^2R}{dt^2} = -abR$$

$$\Rightarrow \frac{d^2R}{dt^2} + abR = 0$$

characteristic equation:

$$\lambda^2 + ab = 0$$

$$\Rightarrow \lambda = \pm\sqrt{ab}\, i \quad (\because ab > 0)$$

$$\therefore R = c_1 \cos\sqrt{ab}\,t + c_2 \sin\sqrt{ab}\,t$$

$$① \Rightarrow J = \frac{1}{a}\frac{dR}{dt} = \frac{1}{a}\left\{ -c_1\sqrt{ab}\sin\sqrt{ab}\,t + c_2\sqrt{ab}\cos\sqrt{ab}\,t \right\}$$

$$= -c_1\sqrt{\frac{b}{a}}\sin\sqrt{ab}\,t + c_2\sqrt{\frac{b}{a}}\cos\sqrt{ab}\,t$$

$$R(0) = \alpha \Rightarrow c_1 = \alpha$$

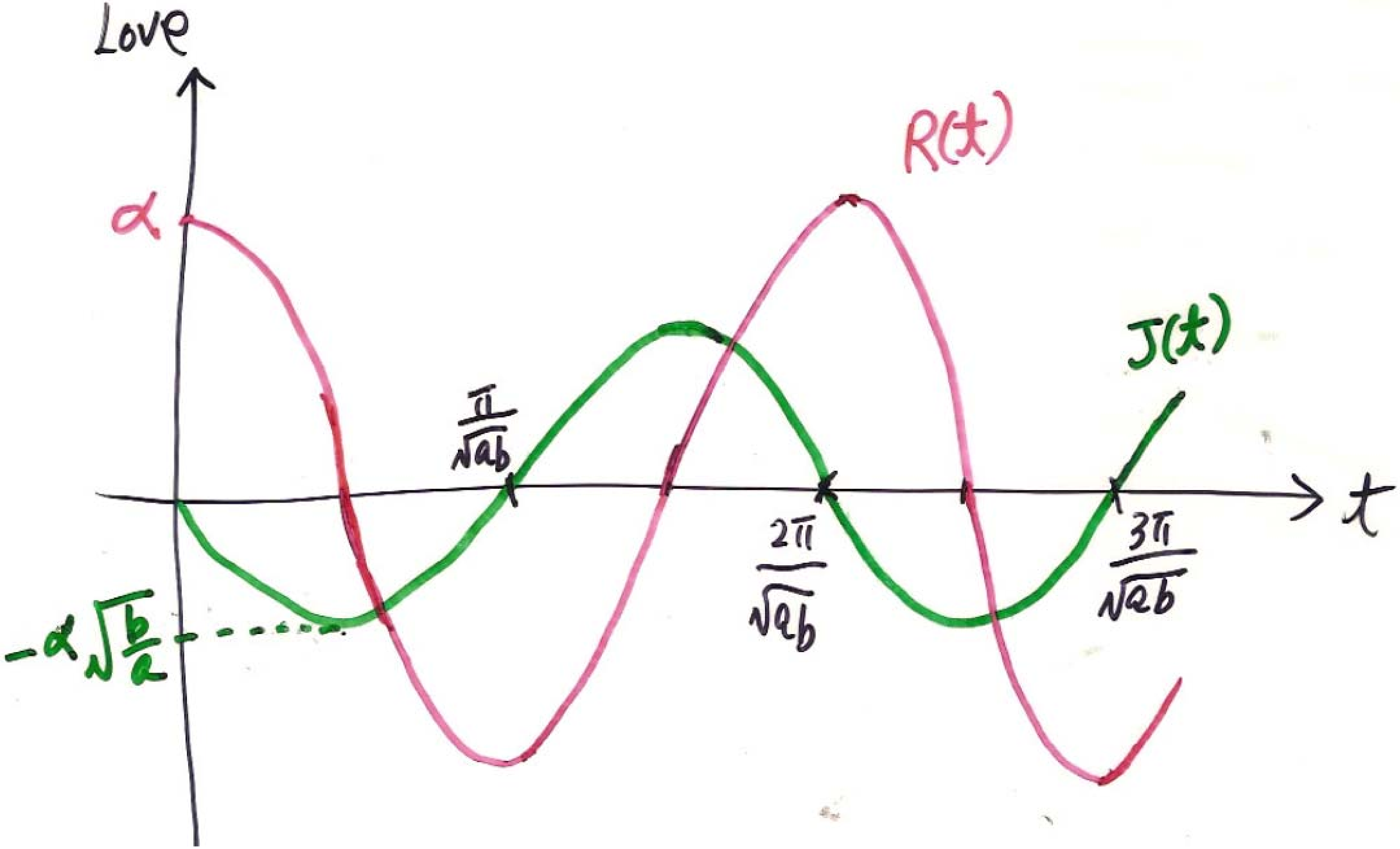
$$J(0) = \beta \Rightarrow c_2 = \beta\sqrt{\frac{a}{b}}$$

$$\therefore \begin{cases} R = \alpha \cos \sqrt{ab}\, t + \beta \sqrt{\frac{a}{b}} \sin \sqrt{ab}\, t \\ J = \beta \cos \sqrt{ab}\, t - \alpha \sqrt{\frac{b}{a}} \sin \sqrt{ab}\, t \end{cases}$$

$$\begin{cases} R = \alpha \cos\sqrt{ab}\, t + \beta\sqrt{\dfrac{a}{b}} \sin\sqrt{ab}\, t \\ J = \beta \cos\sqrt{ab}\, t - \alpha\sqrt{\dfrac{b}{a}} \sin\sqrt{ab}\, t \end{cases}$$

Suppose $\alpha > 0$ and $\beta = 0$.

$$\begin{cases} R = \alpha \cos\sqrt{ab}\, t \\ J = -\alpha\sqrt{\dfrac{b}{a}} \sin\sqrt{ab}\, t \end{cases}$$



Love
R(t)
J(t)
α
$-\alpha\sqrt{\frac{b}{a}}$
$\frac{\pi}{\sqrt{ab}}$
$\frac{2\pi}{\sqrt{ab}}$
$\frac{3\pi}{\sqrt{ab}}$
t

Note:

(i) One period $= \dfrac{2\pi}{\sqrt{ab}}$.

(ii)

R	+	−	−	+
J	−	−	+	+

$0 \qquad \dfrac{\pi}{2\sqrt{ab}} \qquad \dfrac{\pi}{\sqrt{ab}} \qquad \dfrac{3\pi}{2\sqrt{ab}} \qquad \dfrac{2\pi}{\sqrt{ab}}$

Over one period of time, they are only in love a quarter of the time in $\frac{3\pi}{2\sqrt{ab}} \leq t \leq \frac{2\pi}{\sqrt{ab}}$ (both +ve here).

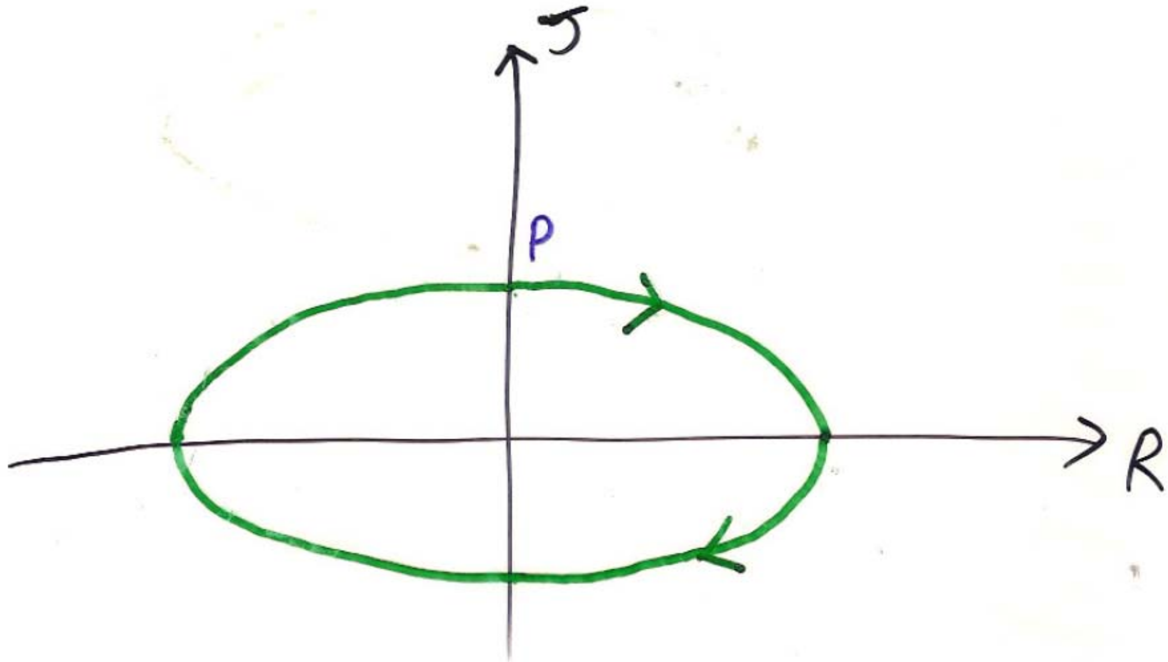
They are out of love three quarters of the time in $0 \leq t \leq \frac{3\pi}{2\sqrt{ab}}$ (both of opposite sign here).

The phase trajectory:

$$R = \alpha \cos\sqrt{ab}\,t \Rightarrow \cos\sqrt{ab}\,t = \frac{R}{\alpha}$$

$$J = -\alpha\sqrt{\frac{b}{a}}\sin\sqrt{ab}\,t \Rightarrow \sin\sqrt{ab}\,t = -\frac{J}{\alpha\sqrt{\frac{b}{a}}}$$

$$\therefore \quad \frac{R^2}{\alpha^2} + \frac{J^2}{\alpha^2 b/a} = 1 : \text{ an ellipse.}$$

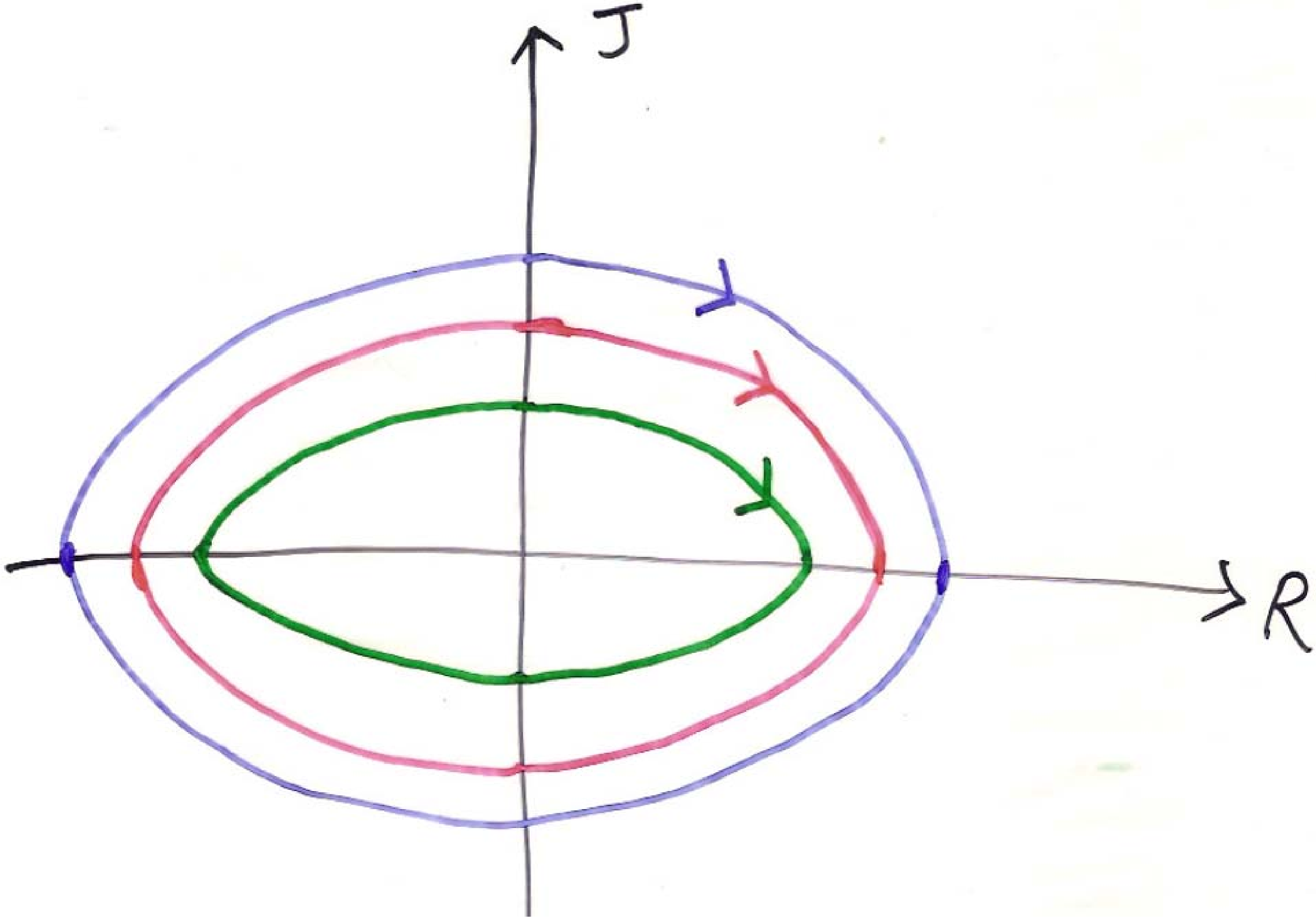


Question: How do we know the direction of the arrow?

Answer: Check one point. Any point will do so take one which is simple to check.
Let us take P. at P, $R=0$ and $J>0$
$\therefore \frac{dR}{dt} = aJ$ is +ve. i.e. $R\uparrow$. i.e. "$\rightarrow$" points to right.

When α changes, we get a whole set of ellipses.

The phase plane diagram:



Note : The NO CROSSING RULE holds

Equilibrium solutions:

$$\begin{cases} \dfrac{dR}{dt} = aJ \\ \dfrac{dJ}{dt} = -bR \end{cases}$$

a, b +ve constants

This is an autonomous system (i.e. the R.H.S. is independent of time t)

To find equilibrium solutions, we set R.H.S. $= 0$

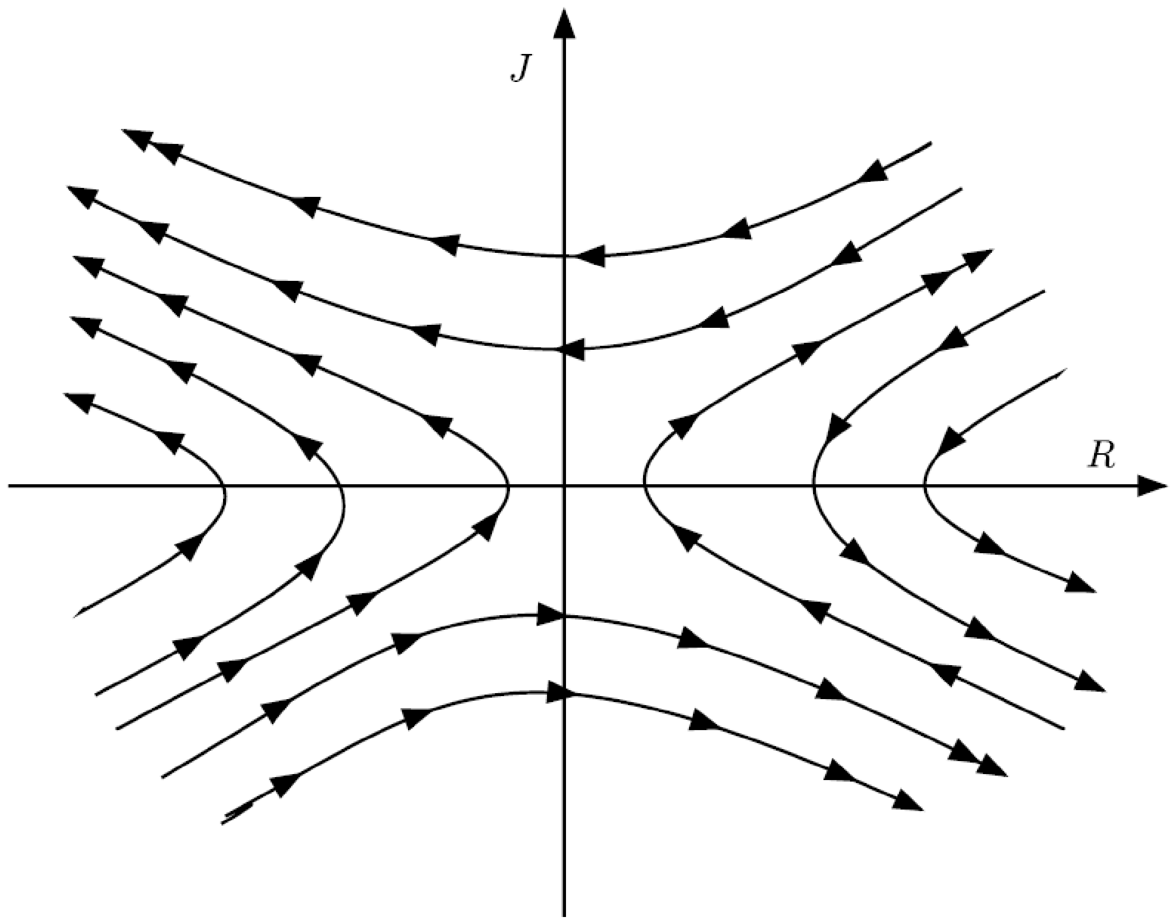
$$\therefore \begin{cases} aJ = 0 \\ -bR = 0 \end{cases}$$

$\because$ a, b are +ve constants

$\therefore$ $(R, J) = (0, 0)$ is the only equilibrium solution.

Note: $(R, J) = (0, 0)$ is a <u>stable equilibrium</u> since a small displacement will put it on an orbit of an ellipse which still stays "close" to $(0,0)$.

Contrast this with a phase diagram like the one in the next picture:



$(R, J) = (0, 0)$ is a point of equilibrium, but a slight push along the positive J axis will cause the point to be swept off to infinity in the second quadrant, with $J \to +\infty$, $R \to -\infty$. Clearly

UNSTABLE equilibrium! So stability can be determined just by a glance at the phase diagram.

VERY OFTEN, WE DON'T CARE ABOUT THE DETAILS OF THE SOLUTIONS — THE PHASE PLANE DIAGRAM ALREADY TELLS US ALL WE NEED TO KNOW FOR MANY APPLIED PROBLEMS!

7.2. HOW TO SOLVE SYSTEMS OF SIMULTANEOUS ODEs

Let's consider the general system:

$$\begin{cases} \dfrac{dx}{dt} = ax + by \\ \dfrac{dy}{dt} = cx + dy \end{cases}$$

a, b, c, d constants.

$$\therefore \quad \underbrace{\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix}}_{\frac{d\vec{u}}{dt}} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\vec{u}} \qquad (*)$$

Idea: For the one unknown function case

$$\frac{dx}{dt} = ax$$

we have $\frac{dx}{x} = a\,dt$

$\therefore \quad \ln|x| = at + \text{constant}$

$\therefore \quad x = x_0 e^{at} = e^{at} x_0$

where $x_0 = \text{constant}$

This suggest that we try

$$\vec{u} = e^{\lambda t}\vec{u_0} \quad \text{where } \vec{u_0} = \text{constant} \neq \vec{0}.$$

for (*).

$$(*) \Rightarrow \lambda e^{\lambda t}\vec{u_0} = B\vec{u} = e^{\lambda t}B\vec{u_0}$$

$$\Rightarrow \lambda\vec{u_0} = B\vec{u_0}$$

$\Rightarrow \lambda$ is an eigenvalue of B,
$\vec{u_0}$ is the corresponding eigenvector.

$$B - \lambda I = \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$$

$$\therefore \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 \Rightarrow (a-\lambda)(d-\lambda) - bc = 0$$

$$\Rightarrow \lambda^2 - (a+d)\lambda + (ad-bc) = 0$$

$$\Rightarrow \lambda^2 - (\mathrm{Tr}B)\lambda + (\det B) = 0$$

$$\therefore \lambda = \frac{1}{2}\left\{(\mathrm{Tr}B) \pm \sqrt{(\mathrm{Tr}B)^2 - 4(\det B)}\right\}$$

Case 1: $(\mathrm{Tr}\, B)^2 > 4 \det B$

There are two real eigenvalues

$\lambda_1 \neq \lambda_2$.

Let $\vec{u}_1$, $\vec{u}_2$ be the corresponding eigenvectors.

Then the general solution of $(*)$ is

$$\vec{u} = c_1 e^{\lambda_1 t} \vec{u}_1 + c_2 e^{\lambda_2 t} \vec{u}_2 .$$

Case 2: $(\mathrm{Tr}B)^2 = 4 \det B$

We have a double root $\lambda_1 = \lambda_2 = \lambda$.

This case is more complicated.

In practice, the measurements are never exact and so it is rare to have $(\mathrm{Tr}\,B)^2 = 4\det B$.

We will not go into too much on the details here.

Case 3: $(\mathrm{Tr}B)^2 < 4 \det B$.

This is similar to Case 1 but now $\lambda_1, \lambda_2, \vec{u}_1, \vec{u}_2$ are complex.

The general solution in complex form is

$$\vec{u} = c_1 e^{\lambda_1 t} \vec{u}_1 + c_2 e^{\lambda_2 t} \vec{u}_2 .$$

This can usually be written in real form as a combination of $e^{\alpha t}\cos\beta t$ and $e^{\alpha t}\sin\beta t$ where $\lambda_1, \lambda_2 = \alpha \pm i\beta$. The general solution then becomes this form

$$\begin{cases} x = c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t \\ y = c_3 e^{\alpha t}\cos\beta t + c_4 e^{\alpha t}\sin\beta t. \end{cases}$$

Example Solve

$$\left\{ \begin{array}{l} \frac{dx}{dt} = -4x + 3y \\ \frac{dy}{dt} = -2x + y \end{array} \right\}$$

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad (*)$$

$$\therefore B = \begin{pmatrix} -4 & 3 \\ -2 & 1 \end{pmatrix}$$

$$\mathrm{Tr}B = -4 + 1 = -3$$

$$\det B = -4 + 6 = 2$$

$$\lambda = \frac{1}{2}\left\{ (\mathrm{Tr}B) \pm \sqrt{(\mathrm{Tr}B)^2 - 4(\det B)} \right\}$$

$$= \frac{1}{2}\left\{ -3 \pm \sqrt{9-8} \right\} = -1 \text{ or } -2 .$$

Case 1: $\lambda = -1$.

$$B - \lambda I = B + I = \begin{pmatrix} -3 & 3 \\ -2 & 2 \end{pmatrix}$$

$$\therefore -3x + 3y = 0$$

$$\therefore x = t \Rightarrow y = t$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ t \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Case 2 : $\lambda = -2$

$$B - \lambda I = B + 2I = \begin{pmatrix} -2 & 3 \\ -2 & 3 \end{pmatrix}$$

$$\therefore -2x + 3y = 0$$

$$x = 3t \Rightarrow y = 2t$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3t \\ 2t \end{pmatrix} = t \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

The general solution of (*) is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

i.e.

$$\begin{cases} x = c_1 e^{-t} + 3c_2 e^{-2t} \\ y = c_1 e^{-t} + 2c_2 e^{-2t} \end{cases}$$

Example Solve

$$\left\{ \begin{array}{l} \frac{dx}{dt} = 4x - 5y \\ \frac{dy}{dt} = 2x - 2y \end{array} \right\}$$

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 4 & -5 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad (*)$$

$$\therefore B = \begin{pmatrix} 4 & -5 \\ 2 & -2 \end{pmatrix}$$

$$\mathrm{Tr}B = 4 - 2 = 2$$

$$\det B = -8 + 10 = 2$$

$$\lambda = \frac{1}{2}\left\{ (\mathrm{Tr}B) \pm \sqrt{(\mathrm{Tr}B)^2 - 4(\det B)} \right\}$$

$$= \frac{1}{2}\left\{ 2 \pm \sqrt{4-8} \right\}$$

$$= 1 \pm i$$

Case 1: $\lambda = 1+i$

$$B - \lambda I = \begin{pmatrix} 3-i & -5 \\ 2 & -3-i \end{pmatrix}$$

$$\therefore \ (3-i)x - 5y = 0$$

$$x = 5t \Rightarrow y = (3-i)t$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5t \\ (3-i)t \end{pmatrix} = t\begin{pmatrix} 5 \\ 3-i \end{pmatrix}$$

Case 2 : $\lambda = 1 - i$

$$B - \lambda I = \begin{pmatrix} 3+i & -5 \\ 2 & -3+i \end{pmatrix}$$

$$\therefore (3+i)x - 5y = 0$$

$$x = 5t \Rightarrow y = (3+i)t$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5t \\ (3+i)t \end{pmatrix} = t \begin{pmatrix} 5 \\ 3+i \end{pmatrix}$$

$$\therefore \vec{u}_1 = e^{(1+i)t}\begin{pmatrix} 5 \\ 3-i \end{pmatrix}$$

and $$\vec{u}_2 = e^{(1-i)t}\begin{pmatrix} 5 \\ 3+i \end{pmatrix}$$

are two linearly independent solutions of $(*)$.

Now $\vec{u}_1 = e^t(\cos t + i\sin t)\begin{pmatrix} 5 \\ 3-i \end{pmatrix}$

$$= \begin{pmatrix} 5e^t(\cos t + i\sin t) \\ e^t\{(3\cos t + \sin t) \\ \quad + i(-\cos t + 3\sin t)\} \end{pmatrix}$$

$$\vec{u}_2 = e^t(\cos t - i\sin t)\begin{pmatrix} 5 \\ 3+i \end{pmatrix}$$

$$= \begin{pmatrix} 5e^t(\cos t - i\sin t) \\ e^t\{(3\cos t + \sin t) \\ \quad + i(\cos t - 3\sin t)\} \end{pmatrix}$$

$$\therefore \vec{u}_1 + \vec{u}_2 = \begin{pmatrix} 10e^t \cos t \\ \\ 2e^t(3\cos t + \sin t) \end{pmatrix} = \vec{v}_1$$

is also a solution.

$$\vec{u}_1 - \vec{u}_2 = \begin{pmatrix} 10i e^t \sin t \\ 2i e^t(-\cos t + 3\sin t) \end{pmatrix}$$

$$= i \underbrace{\begin{pmatrix} 10 e^t \sin t \\ 2e^t(-\cos t + 3\sin t) \end{pmatrix}}_{\vec{V}_2}$$

is also a solution

$\therefore \vec{V}_2$ is also a solution

Observe that $\vec{V_1}, \vec{V_2}$ are linearly independent.

$\therefore$ The general solution of $(*)$ is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \vec{V_1} + c_2 \vec{V_2}$$

$$= \begin{pmatrix} 10c_1 e^t \cos t + 10 c_2 e^t \sin t \\ \\ 2c_1 e^t(3\cos t + \sin t) \\ + 2c_2 e^t(-\cos t + 3\sin t) \end{pmatrix}$$

i.e. $$\begin{cases} x = 10e^{t}(c_1 \cos t + c_2 \sin t) \\ y = 2e^{t}(3c_1 - c_2)\cos t \\ \qquad + 2e^{t}(c_1 + 3c_2)\sin t \end{cases}$$

Note

By doing a similar calculation, we can see that in the complex case: $\lambda = \alpha \pm \beta i$

the general solution is of the form

$$\begin{cases} x = c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t \\ y = c_3 e^{\alpha t}\cos\beta t + c_4 e^{\alpha t}\sin\beta t \end{cases}$$

with c_3, c_4 related to c_1, c_2 in a certain way that can be found by substituting this into (*).

Recall that we can write

$$A\cos\beta t + B\sin\beta t = C\cos(\beta t + \gamma)$$

$$\therefore \begin{cases} x = Ce^{\alpha t}\cos(\beta t + \gamma) \\ y = De^{\alpha t}\cos(\beta t + \delta) \end{cases}$$

$\therefore$ The phase trajectory is a "spiral".

Case 1: $\alpha > 0$

UNSTABLE

Case 2: $\alpha < 0$

STABLE

Case 3 : $\alpha = 0$

CENTRE

The form of the general solution:

For the system

$$\underbrace{\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix}}_{\frac{d\vec{u}}{dt}} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\vec{u}} \qquad (*)$$

Let λ_1, λ_2 be the eigenvalues of B.

Case 1: λ_1, λ_2 real; $\lambda_1 \neq \lambda_2$

General solution is of the form

$$\begin{cases} x = \alpha e^{\lambda_1 t} + \beta e^{\lambda_2 t} \\ y = \gamma e^{\lambda_1 t} + \delta e^{\lambda_2 t} \end{cases}$$

Case 2: $\lambda_1 = \lambda_2$

General solution is of the form

$$\begin{cases} x = \alpha e^{\lambda_1 t} + \beta t e^{\lambda_1 t} \\ y = \gamma e^{\lambda_1 t} + \delta t e^{\lambda_1 t} \end{cases}$$

Case 3: $\lambda_1, \lambda_2 = p \pm iq$; p, q are real.

General solution is of the form

$$\begin{cases} x = \alpha e^{pt} \cos qt + \beta e^{pt} \sin qt \\ y = \gamma e^{pt} \cos qt + \delta e^{pt} \sin qt \end{cases}$$

In all 3 cases; $\alpha, \beta, \gamma, \delta$ can be determined in terms of 2 parameters say c_1 and c_2 by substituting into (*)

Observe that

$$\begin{cases} x = 0 \\ y = 0 \end{cases}$$

is an <u>equilibrium solution</u> of (*)

$\therefore$ $e^{\lambda_1 t}$, $e^{\lambda_2 t}$ will dominate in cases 1 and 2,

e^{pt} will dominate in case 3

$\therefore$ The stability of $\binom{0}{0}$ depends on the sign of the real part of the eigenvalues.

Recall:

$$\det B = \lambda_1 \lambda_2 \qquad \left(\because B = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P^{-1} \right)$$

$$\operatorname{Tr} B = \lambda_1 + \lambda_2$$

(i) Both λ_1, λ_2 are real.

$\det B < 0 \Rightarrow \lambda_1, \lambda_2$ are of opposite signs

$\Rightarrow$ unstable

$\det B > 0 \Rightarrow \lambda_1, \lambda_2$ are of the same signs

$\therefore \det B > 0, \ \mathrm{Tr} B > 0 \Rightarrow \lambda_1, \lambda_2$ +ve

$\Rightarrow$ unstable

and $\det B > 0, \ \mathrm{Tr} B < 0 \Rightarrow \lambda_1, \lambda_2$ −ve

$\Rightarrow$ stable

(ii) Both λ_1, λ_2 are complex

$\therefore \lambda_1, \lambda_2 = p \pm iq, \quad q \neq 0$

$\det B = \lambda_1 \lambda_2 = p^2 + q^2 > 0$ always

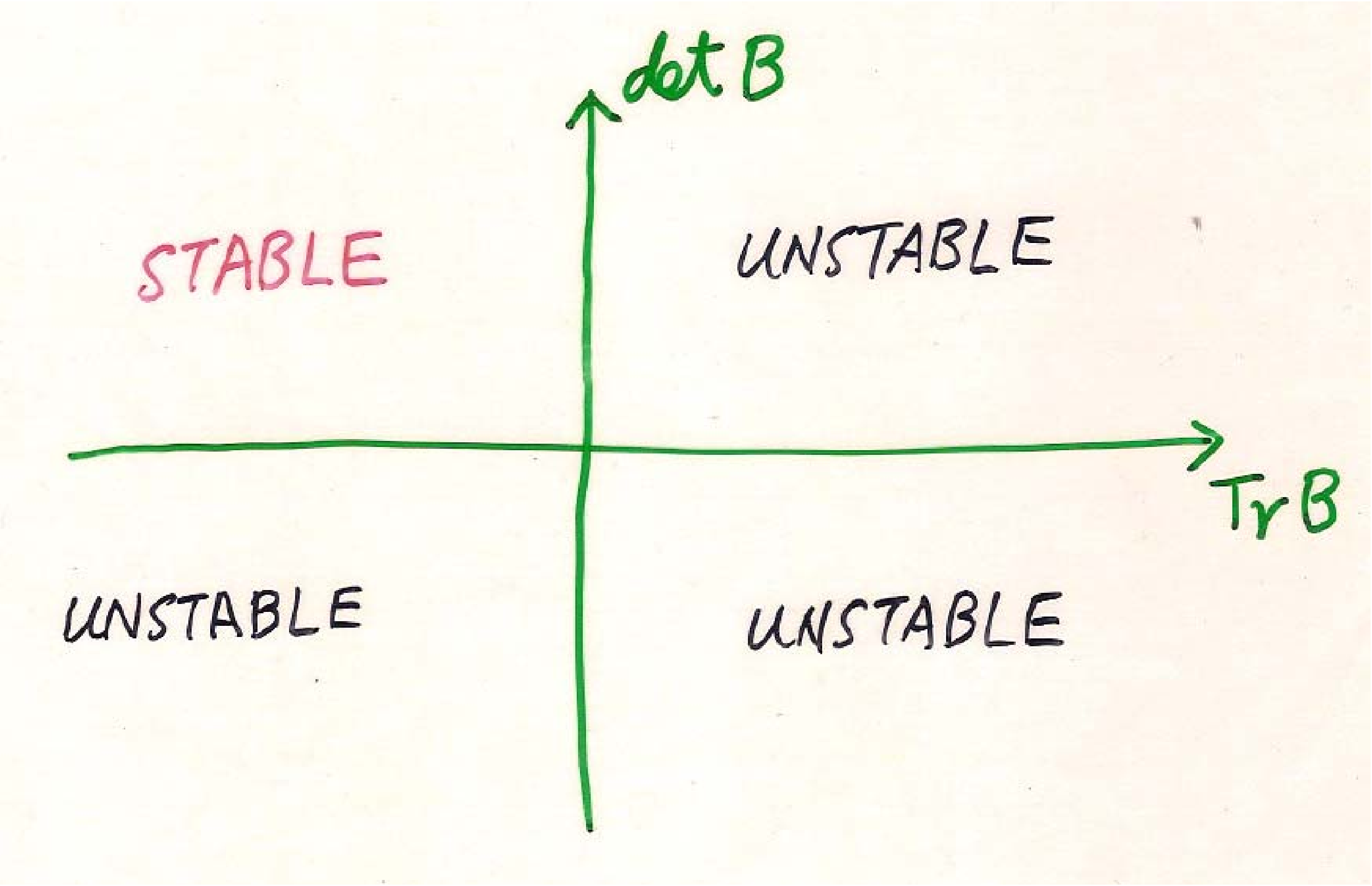
$\mathrm{Tr}B = 2p$

$\therefore \mathrm{Tr}B < 0 \Rightarrow$ stable

$\mathrm{Tr}B > 0 \Rightarrow$ unstable

$\mathrm{Tr}B = 0 \Rightarrow$ centre

SUMMARY



Example

(i) $$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$\det = 4 > 0$, $\mathrm{Tr} = 2 + 2 = 4 > 0$

$\therefore$ unstable

(ii) $$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\det = 1 - 4 < 0$$

$\therefore$ unstable

(iii) $$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$\det = -2+3 > 0$, $\mathrm{Tr} = -2+1 < 0$

$\therefore$ stable

7.3. PHASE PLANE

For the system

$$\underbrace{\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix}}_{\frac{d\vec{u}}{dt}} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\vec{u}} \qquad (*)$$

Case 1: B has two real distinct eigenvalues $\lambda_1 < \lambda_2$.

i.e. $(\mathrm{Tr} B)^2 > 4(\det B)$.

(i) Both are +ve.

i.e. $0 < \lambda_1 < \lambda_2$

$\Leftrightarrow \det B > 0, \ \mathrm{Tr} B > 0$

Let $\vec{u}_1 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ be an eigenvector

corresponding to λ_1.

Let $\vec{u}_2 = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}$ be an eigenvector

corresponding to λ_2.

We know that

$$\begin{pmatrix} x \\ y \end{pmatrix} = \pm e^{\lambda_1 t} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

is a solution of $(*)$

$$\therefore \quad \begin{cases} x = \alpha e^{\lambda_1 t} \\ y = \beta e^{\lambda_1 t} \end{cases} \quad \text{or} \quad \begin{cases} x = -\alpha e^{\lambda_1 t} \\ y = -\beta e^{\lambda_1 t} \end{cases}$$

$$\therefore \quad \frac{y}{x} = \frac{\beta}{\alpha}$$

i.e. $y = \frac{\beta}{\alpha} x$ a straight line with slope $= \frac{\beta}{\alpha}$ passing through $\binom{0}{0}$.

Assume first that α, β both > 0.
There are two parts of this line:

For $\begin{cases} x = \alpha e^{\lambda_1 t} \\ y = \beta e^{\lambda_1 t} \end{cases}$ ..... (L_1)

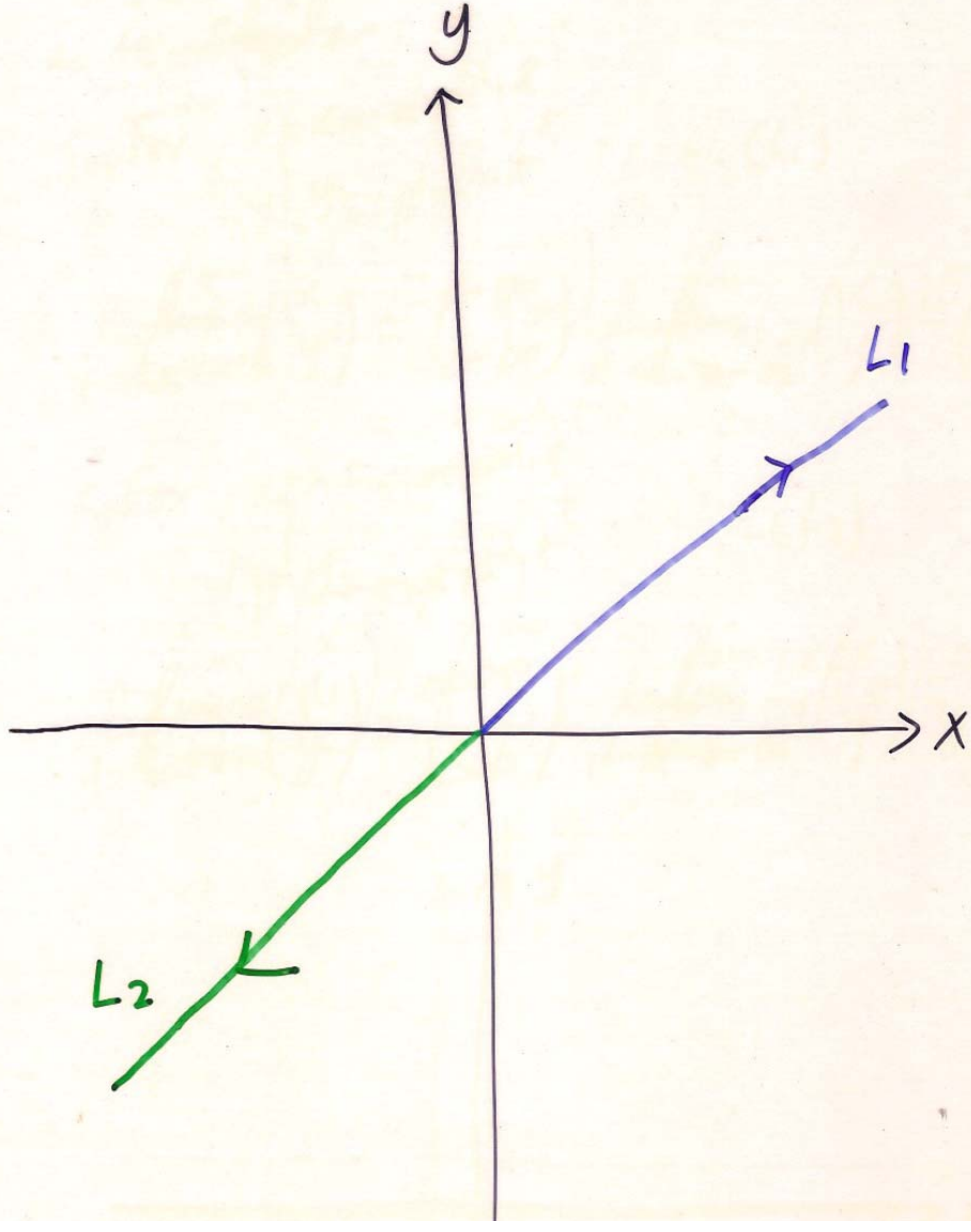
$$\lim_{t \to \infty} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \infty \\ \infty \end{pmatrix} \qquad (\because \lambda_1 > 0)$$

and $\displaystyle \lim_{t \to -\infty} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (\because \lambda_1 > 0)$

and for $\begin{cases} x = -\alpha e^{\lambda_1 t} \\ y = -\beta e^{\lambda_1 t} \end{cases}$ ---- (L_2)

$$\lim_{t \to \infty} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\infty \\ -\infty \end{pmatrix}$$

$$\lim_{t \to -\infty} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$


y
x
L1
L2

The case when α, β both < 0

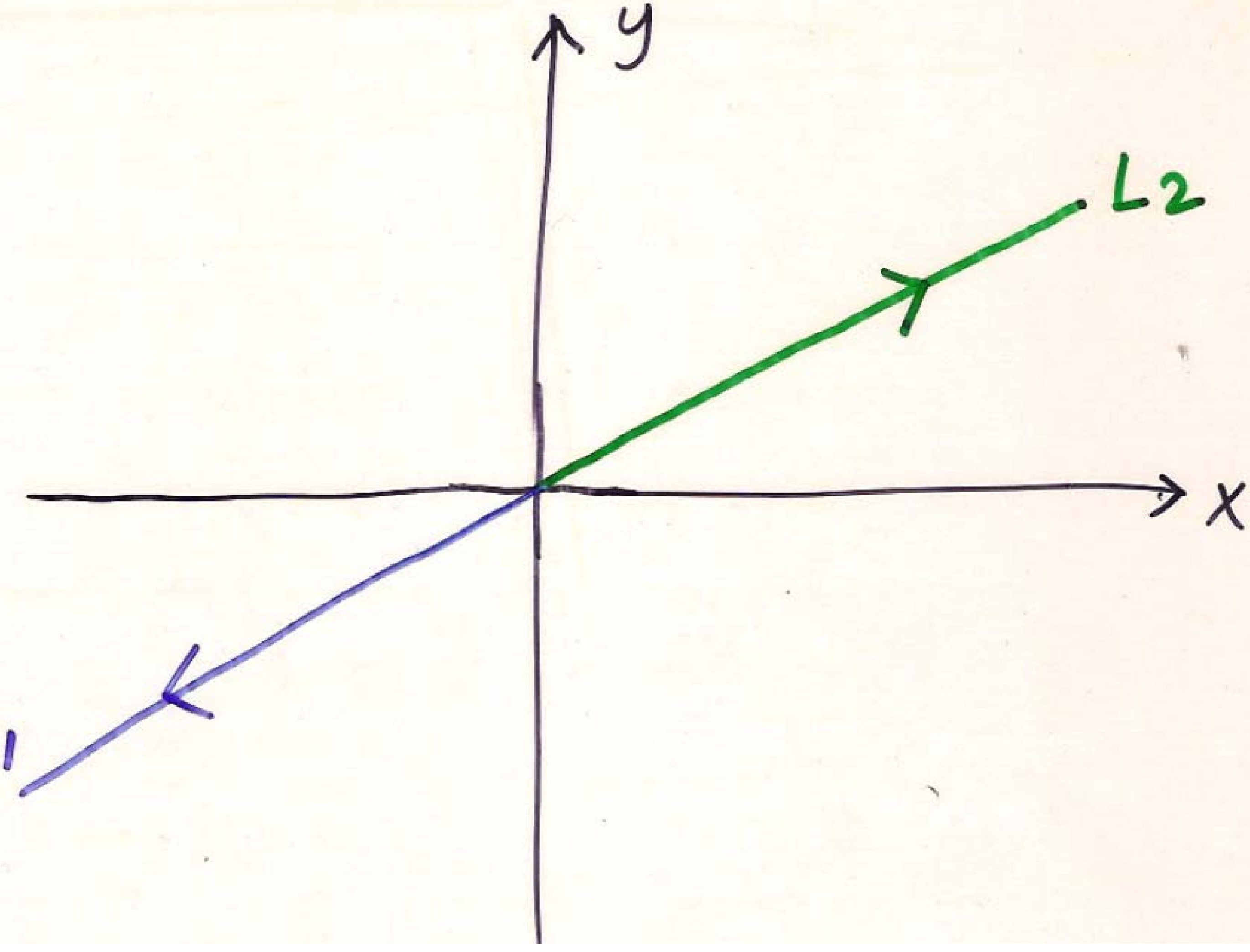
is similar:

For $\begin{cases} x = \alpha e^{\lambda_1 t} \\ y = \beta e^{\lambda_1 t} \end{cases}$ ---- (L_1)

$$\lim_{t \to \infty} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\infty \\ -\infty \end{pmatrix}, \quad \lim_{t \to -\infty} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For $\begin{cases} x = -\alpha e^{\lambda_1 t} \\ y = -\beta e^{\lambda_1 t} \end{cases}$ ---- (L_2)

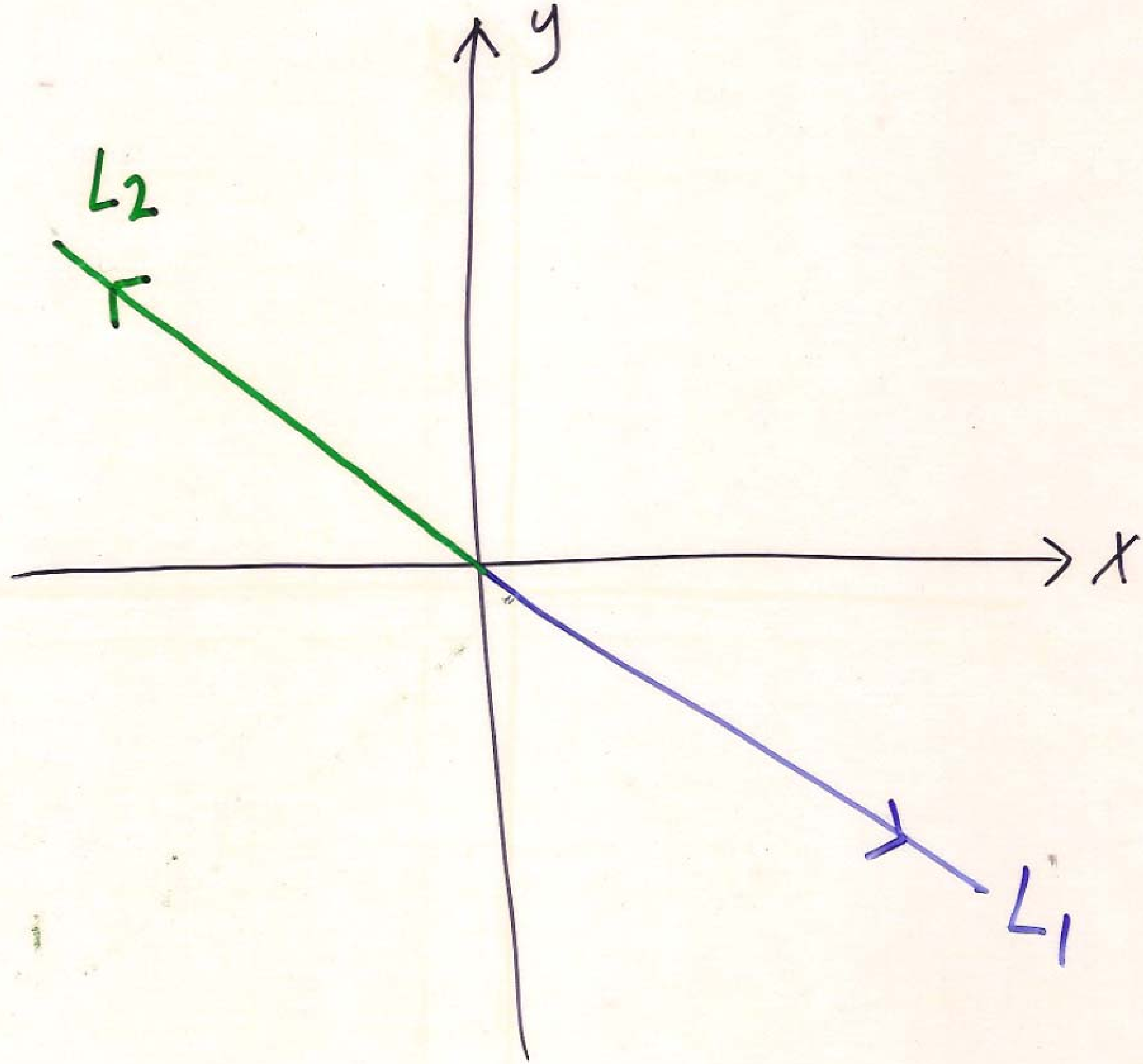
$$\lim_{t\to\infty} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \infty \\ \infty \end{pmatrix}, \quad \lim_{t\to-\infty} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$


y
x
L2
L1

For $\alpha > 0$, $\beta < 0$:

For (L_1) $\lim_{t \to \infty} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \infty \\ -\infty \end{pmatrix}$

For (L_2) $\lim_{t \to \infty} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\infty \\ \infty \end{pmatrix}$

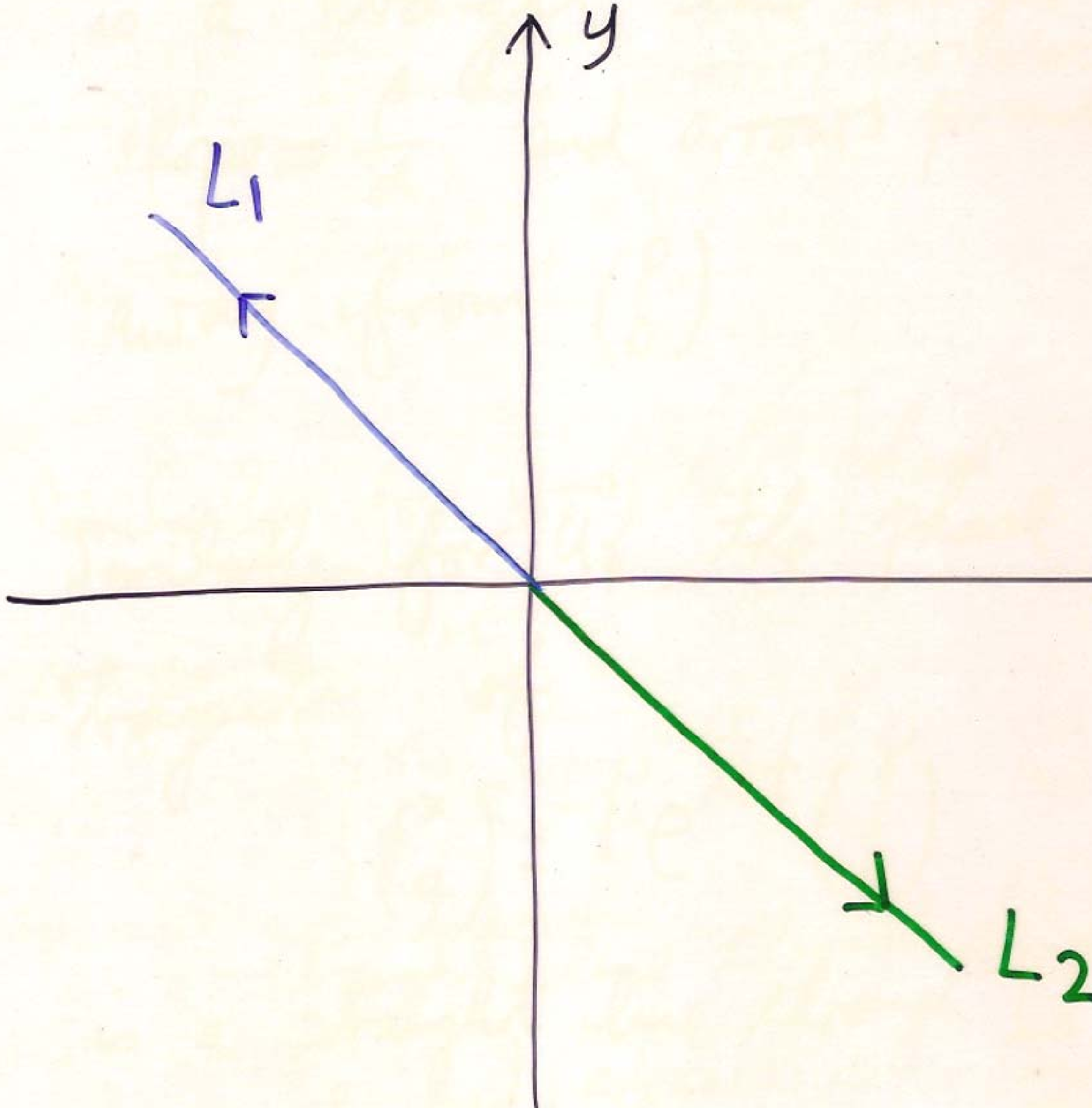


y
x
L2
L1

For the last possibility $\alpha < 0$, $\beta > 0$

For (L_1) $\lim_{t \to \infty} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\infty \\ \infty \end{pmatrix}$

For (L_2) $\lim_{t \to \infty} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \infty \\ -\infty \end{pmatrix}$


y
x
L1
L2

So we see that in all cases for $\vec{u}_1$,

the phase trajectories of

$$\begin{pmatrix} x \\ y \end{pmatrix} = \pm e^{\lambda_1 t} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

is a straight line through $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ with slope $= \frac{\beta}{\alpha}$ and arrows pointing away from $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Similarly for $\vec{u}_2$, the phase trajectories of

$$\begin{pmatrix} x \\ y \end{pmatrix} = \pm e^{\lambda_2 t} \begin{pmatrix} \gamma \\ \delta \end{pmatrix}$$

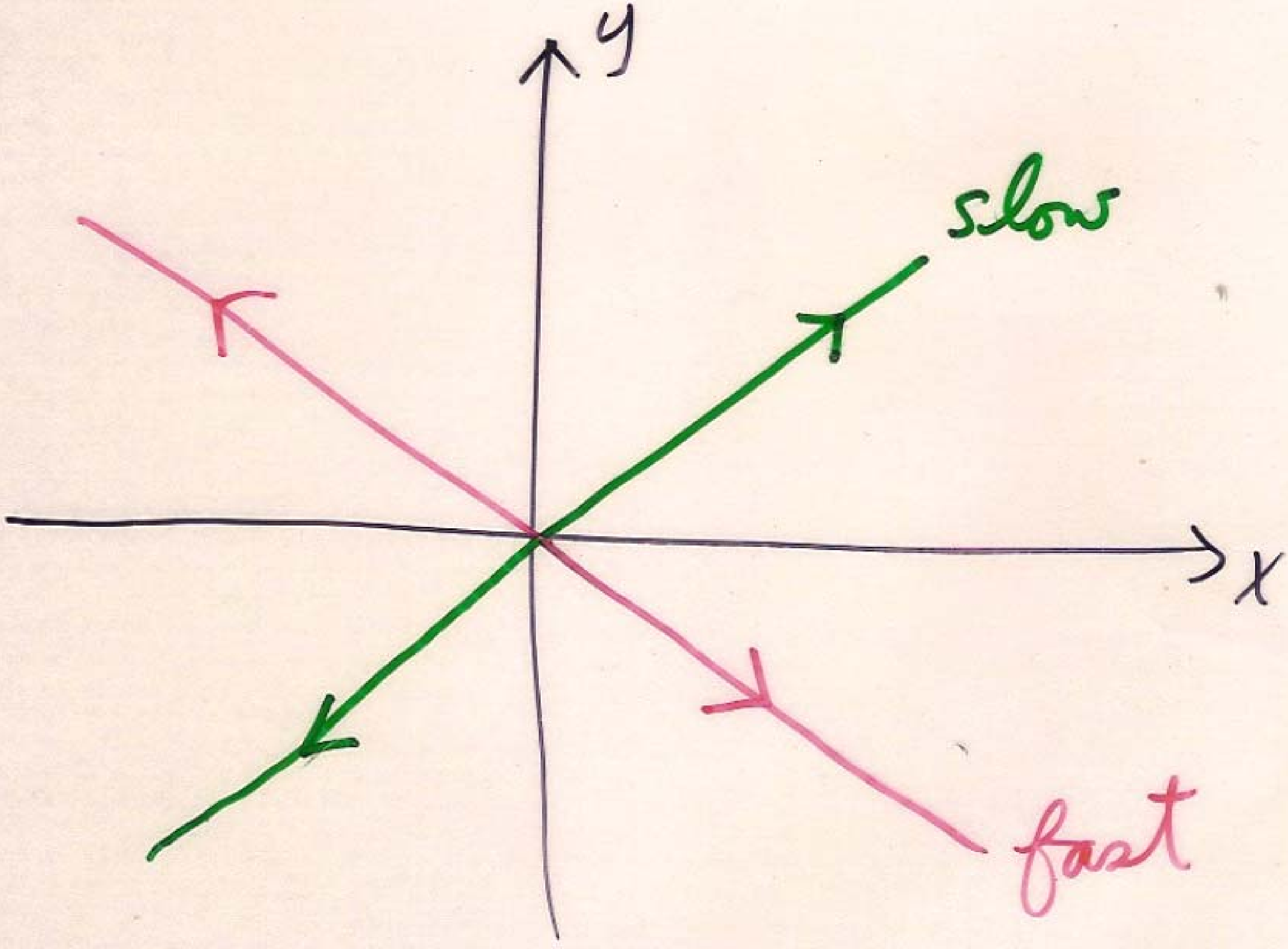
is a straight line through $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ with slope $= \frac{\delta}{\gamma}$ and arrows pointing away from $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

This gives us four trajectories on the phase plane diagram.

Since $\lambda_2 > \lambda_1 > 0$, the term $e^{\lambda_2 t}$ will dominate over the term $e^{\lambda_1 t}$ when $t \to \infty$.

We call $\vec{u}_2$ the <u>fast direction</u> and $\vec{u}_1$ the <u>slow direction</u>.

A typical picture looks like



<u>Question</u>: Does this diagram contradict the "NO CROSSING RULE"?

<u>Answer</u>: No. There are four trajectories (the four rays from $\binom{0}{0}$) and they only "meet" at $\binom{0}{0}$ at time $t = -\infty$. So they do not really cross each other in finite time.

Now we look at the phase trajectory of other solutions of (*):

Any such solution looks like

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{\lambda_1 t} \vec{u}_1 + c_2 e^{\lambda_2 t} \vec{u}_2$$

$$= c_1 e^{\lambda_1 t} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + c_2 e^{\lambda_2 t} \begin{pmatrix} \gamma \\ \delta \end{pmatrix}$$

i.e.
$$\begin{cases} x = c_1 \alpha e^{\lambda_1 t} + c_2 \gamma e^{\lambda_2 t} \\ y = c_1 \beta e^{\lambda_1 t} + c_2 \delta e^{\lambda_2 t} \end{cases}$$

This is the parametric equation of a curve which we denote by (C).

(C) can be rewritten as

$$\begin{cases} x = e^{\lambda_2 t}\left\{c_1 \alpha e^{(\lambda_1-\lambda_2)t} + c_2 \gamma\right\} \\ y = e^{\lambda_2 t}\left\{c_1 \beta e^{(\lambda_1-\lambda_2)t} + c_2 \delta\right\} \end{cases}$$

$\because \lambda_1 < \lambda_2$

$\therefore \lambda_1 - \lambda_2$ is $-ve$

$$\therefore \lim_{t \to \infty} \left\{ c_1 \alpha e^{(\lambda_1 - \lambda_2)t} + c_2 \gamma \right\} = c_2 \gamma$$

$$\lim_{t \to \infty} \left\{ c_1 \beta e^{(\lambda_1 - \lambda_2)t} + c_2 \delta \right\} = c_2 \delta$$

$\therefore$ When t is large, we have

$$\begin{cases} x \simeq c_2 \gamma e^{\lambda_2 t} \\ y \simeq c_2 \delta e^{\lambda_2 t} \end{cases}$$

$$\therefore \quad \begin{pmatrix} x \\ y \end{pmatrix} \simeq c_2 e^{\lambda_2 t} \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = c_2 e^{\lambda_2 t} \vec{u_2}$$

i.e. (C) is approximately // to $\vec{u_2}$ when t is large.

Next (C) can also be rewritten as

$$x = e^{\lambda_1 t}\left\{c_1\alpha + c_2\gamma e^{(\lambda_2-\lambda_1)t}\right\}$$

$$y = e^{\lambda_1 t}\left\{c_1\beta + c_2\delta e^{(\lambda_2-\lambda_1)t}\right\}$$

$\because \lambda_2 > \lambda_1$

$\therefore \lambda_2 - \lambda_1$ is +ve

$$\therefore \lim_{t \to -\infty} \left\{ c_1 \alpha + c_2 \gamma e^{(\lambda_2 - \lambda_1)t} \right\} = c_1 \alpha$$

$$\lim_{t \to -\infty} \left\{ c_1 \beta + c_2 \delta e^{(\lambda_2 - \lambda_1)t} \right\} = c_1 \beta$$

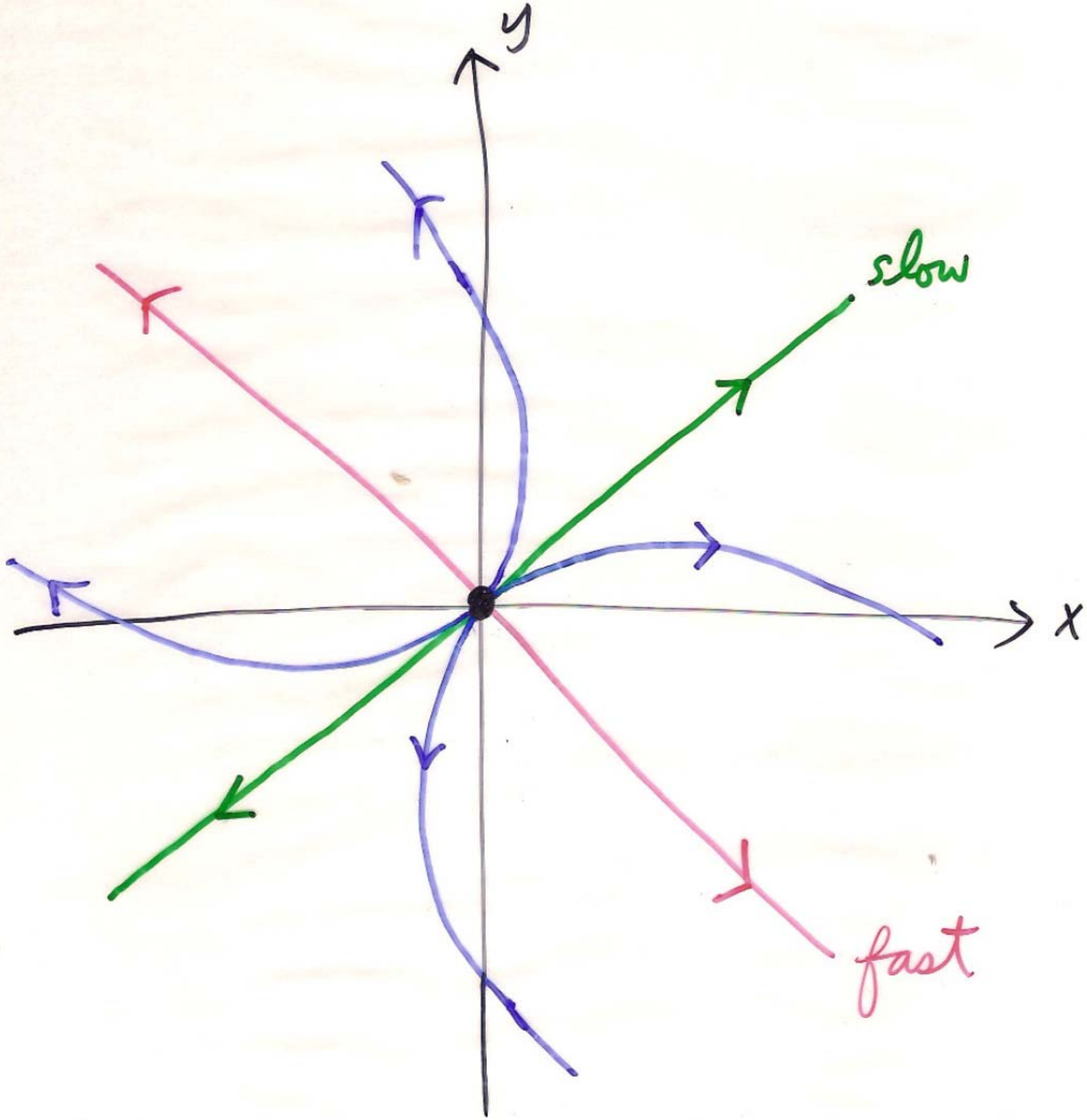
$\therefore$ when t is near $-\infty$

$$\begin{pmatrix} x \\ y \end{pmatrix} \approx c_1 e^{\lambda_1 t} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = c_1 e^{\lambda_1 t} \vec{u}_1 .$$

$$\because t \to -\infty \Rightarrow (C) \longrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore (C)$ approches to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ in the direction (approximately) $\vec{u}_1$.

Now we can draw a typical phase plane diagram of (*) for Case 1(i) (two real $0 < \lambda_1 < \lambda_2$):



This is called a NODAL SOURCE.

Example Draw the phase plane diagram of

$$\begin{cases} \dfrac{dx}{dt} = 2x + 2y \\ \dfrac{dy}{dt} = x + 3y \end{cases}$$

Solution

$$\mathrm{Tr}B = 2+3 = 5 > 0$$

$$\det B = 6-2 = 4 > 0$$

$$(\mathrm{Tr}B)^2 - 4(\det B) = 25-16 = 9 > 0$$

$\therefore$ We have a NODAL SOURCE.

$$\lambda = \frac{1}{2}\left\{ (\mathrm{Tr}B) \pm \sqrt{(\mathrm{Tr}B)^2 - 4(\det B)} \right\}$$

$$= \frac{1}{2}\left\{ 5 \pm 3 \right\} = 1, 4.$$

Let $\lambda_1 = 1 < \lambda_2 = 4$.

For $\lambda_1 = 1$, we have

$$B - \lambda_1 I = B - I = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

$$\therefore \quad x + 2y = 0$$

$$\therefore \quad x = 2t \Rightarrow y = -t$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2t \\ -t \end{pmatrix} = t \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\therefore \quad \vec{u}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad \text{(slow)}$$

For $\lambda_2 = 4$, we have

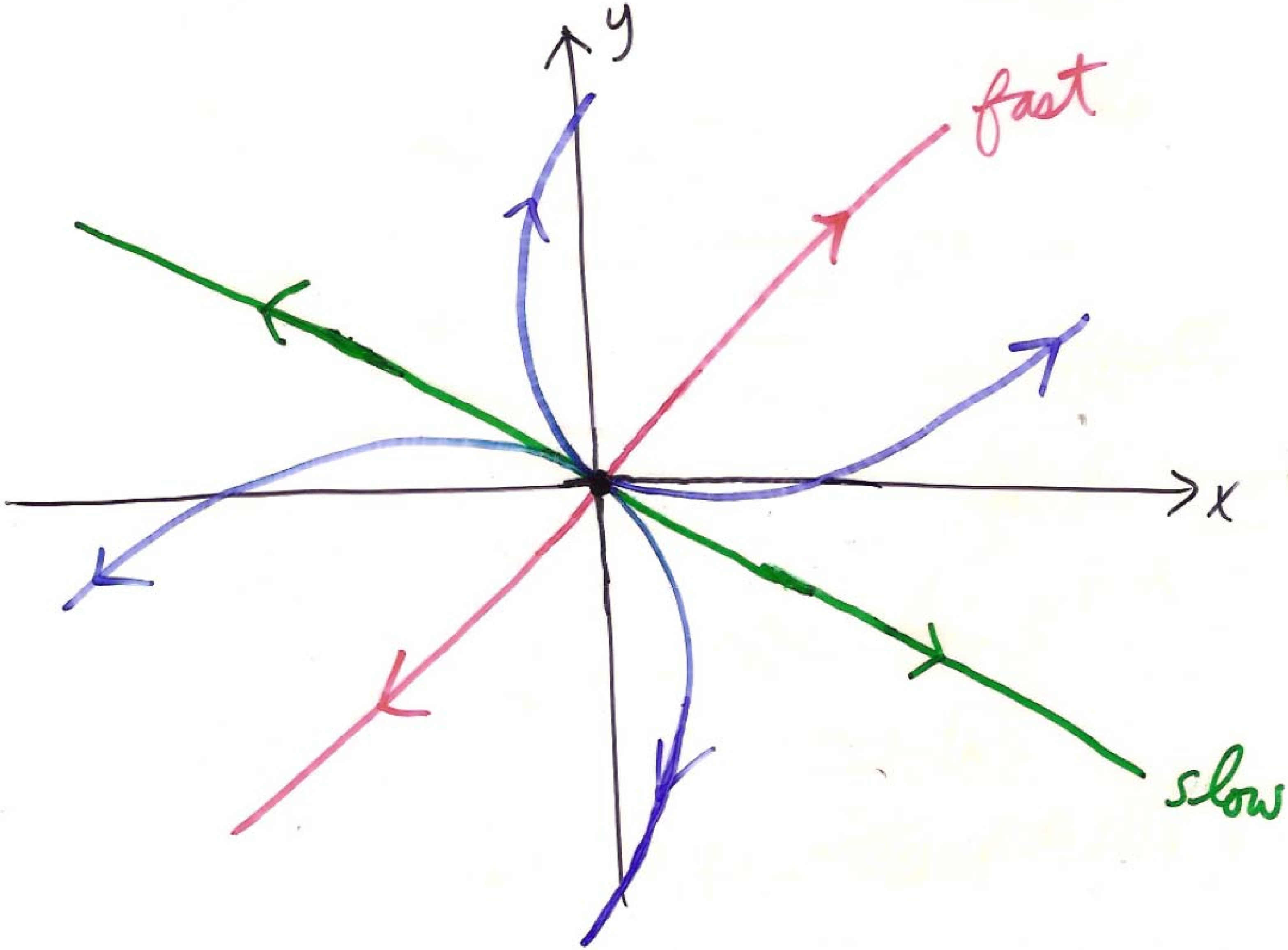
$$B - \lambda_2 I = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}$$

$$\therefore \quad x - y = 0$$

$$\therefore x = t \Rightarrow y = t$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ t \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\therefore \vec{u}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{(fast)}$$


y
fast
x
slow

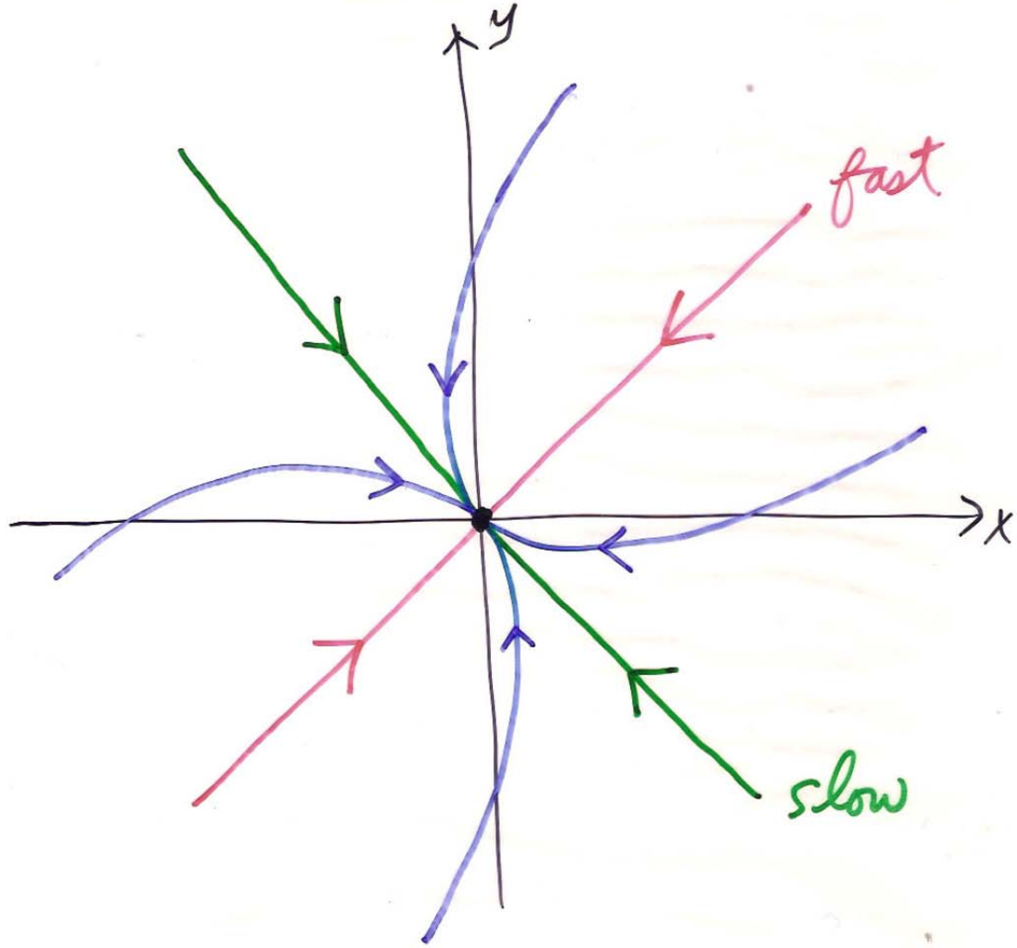
Case 1 (ii) Both real, distinct and -ve.

i.e. $\lambda_1 < \lambda_2 < 0$.

i.e. $\begin{cases} (\mathrm{Tr} B)^2 > 4(\det B) \\ \det B > 0 \\ \mathrm{Tr}\, B < 0 \end{cases}$

This is similar to case 1 (i), but now the arrows are pointing towards $\binom{0}{0}$ and $\vec{u}_1$ is now the fast direction, $\vec{u}_2$ the slow direction ($\because \lambda_1 < \lambda_2 < 0$
$\therefore$ when $t \to \infty$, $e^{\lambda_1 t} \to 0$ faster than $e^{\lambda_2 t} \to 0$, i.e. when $t \to \infty$, $e^{\lambda_1 t}$ is much smaller than $e^{\lambda_2 t}$ and so $e^{\lambda_2 t}$ will dominate,
$\therefore (C) \to \binom{0}{0}$ approximately $\parallel$ to $\vec{u}_2$).

A typical phase plane diagram looks like :



This is called a <u>NODAL SINK</u>.

<u>Example</u> Draw the phase plane diagram of

$$\begin{cases} \dfrac{dx}{dt} = -2x + y \\ \\ \dfrac{dy}{dt} = x - 2y \end{cases}$$

Solution

$$\mathrm{Tr}B = -2-2 = -4 < 0$$

$$\det B = 4-1 = 3 > 0$$

$$(\mathrm{Tr}B)^2 - 4(\det B) = 16-12 = 4 > 0$$

$\therefore$ We have a NODAL SINK.

$$\lambda = \frac{1}{2}\left\{ (\mathrm{Tr}B) \pm \sqrt{(\mathrm{Tr}B)^2 - 4(\det B)} \right\}$$

$$= \frac{1}{2}\left\{ -4 \pm 2 \right\} = -1, -3$$

Let $\lambda_1 = -3 < \lambda_2 = -1$.

For $\lambda_1 = -3$, we have

$$\begin{pmatrix} -2-(-3) & 1 \\ 1 & -2-(-3) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x + y = 0$$

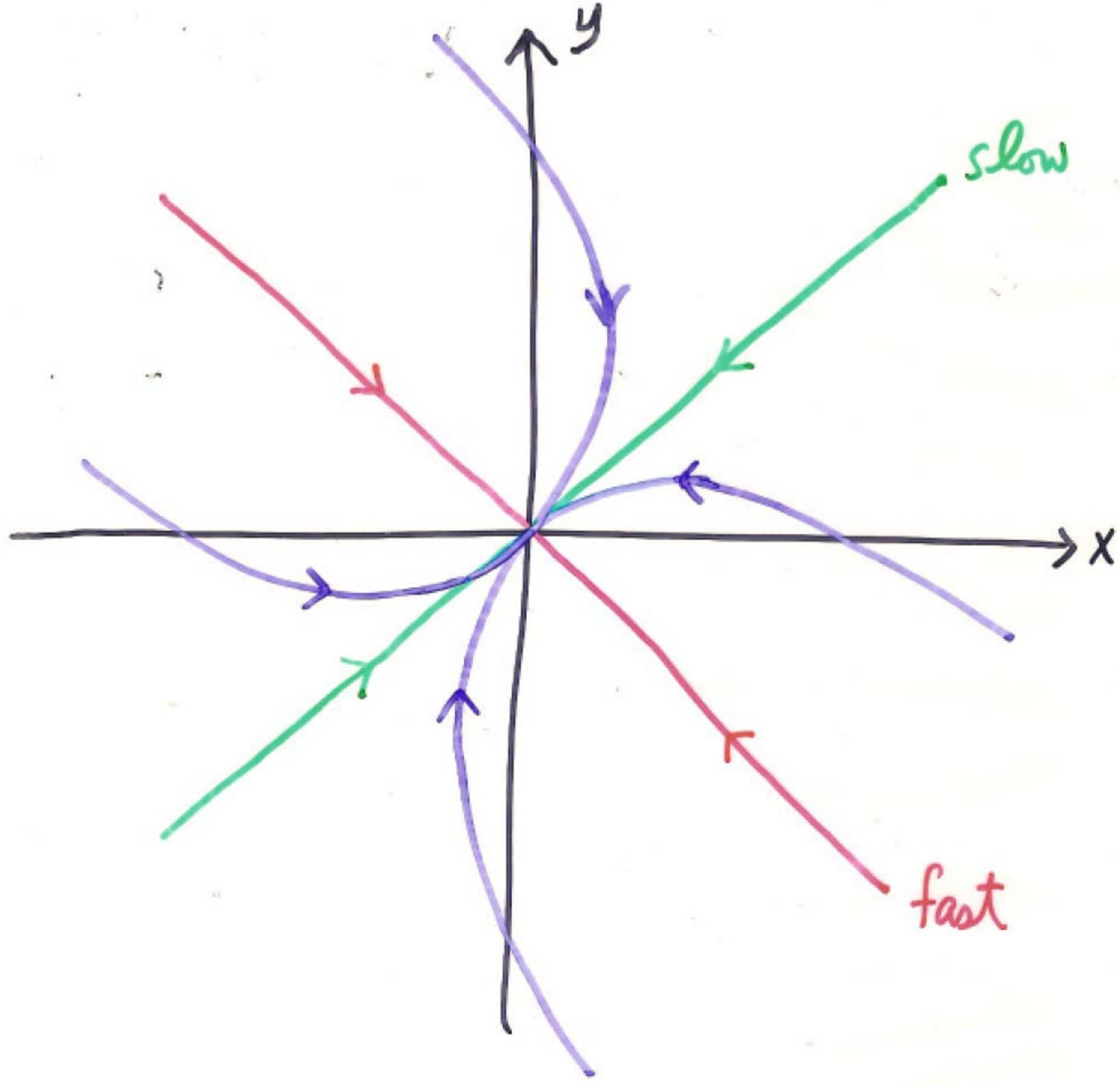
Take $\vec{u}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ (fast)

For $\lambda_2 = -1$, we have

$$\begin{pmatrix} -2-(-1) & 1 \\ 1 & -2-(-1) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-x + y = 0$$

Take $\vec{u}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (slow)


y
x
slow
fast

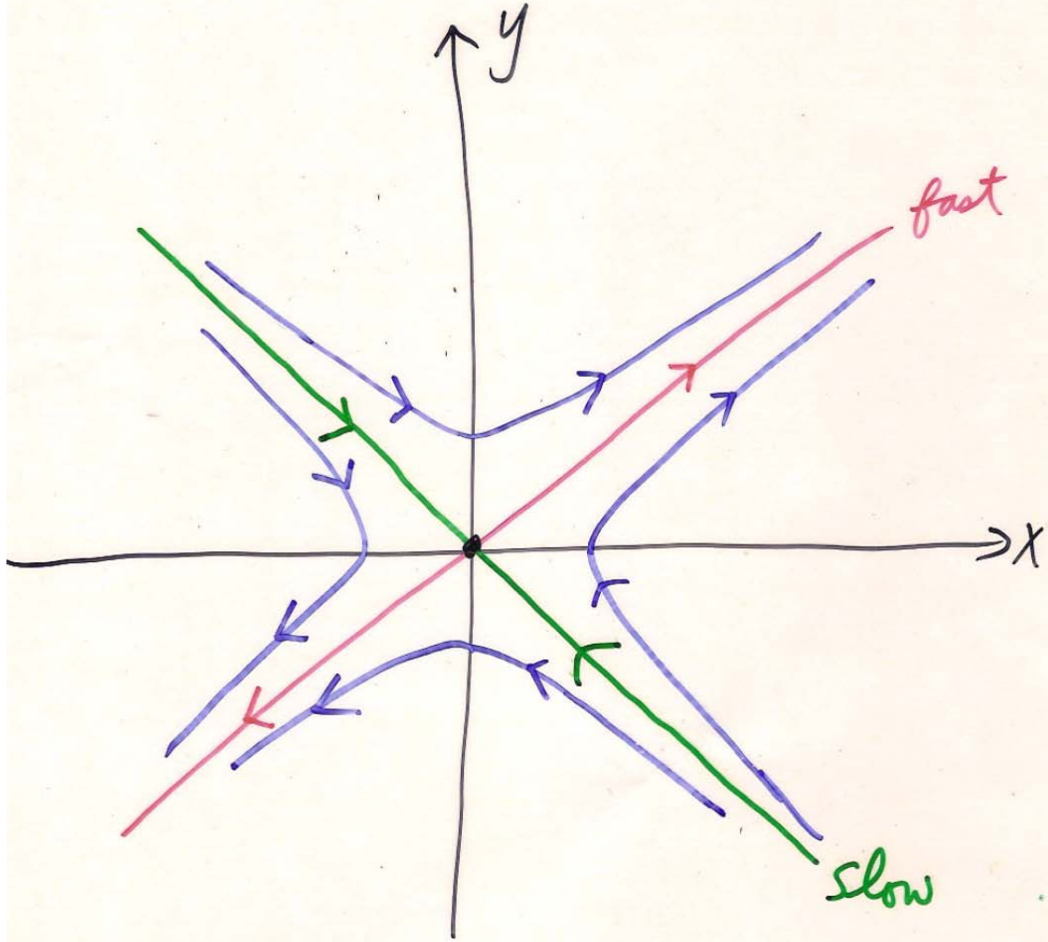
Case 1 (iii) $\lambda_1 < 0 < \lambda_2$

i.e. $\begin{cases} (\operatorname{Tr} B)^2 > 4(\det B) \\ \det B < 0 \end{cases}$

In this case, the arrows on L_1 points towards $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and the arrows on L_2 points away from $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Clearly $e^{\lambda_2 t}$ dominates when $t \to \infty$,
we call $\vec{u}_2$ the fast direction.
when $t \to -\infty$, $e^{\lambda_1 t}$ will dominate,
we call $\vec{u}_1$ the slow direction.

a typical phase plane diagram
for $\lambda_1 < 0 < \lambda_2$:



This is called a SADDLE.

Example Draw the phase plane diagram of

$$\begin{cases} \dfrac{dx}{dt} = x + y \\[2ex] \dfrac{dy}{dt} = 4x + y \end{cases}$$

<u>Solution</u>:

$$\text{Tr}\,B = 1+1 = 2 \quad > 0$$

$$\det B = 1-4 = -3 \quad < 0$$

$\therefore$ We have a SADDLE.

$$\lambda = \frac{1}{2}\left\{ (\text{Tr}B) \pm \sqrt{(\text{Tr}B)^2 - 4(\det B)} \right\}$$

$$= \frac{1}{2}\left\{ 2 \pm 4 \right\} = -1,\ 3$$

Let $\lambda_1 = -1$, $\lambda_2 = 3$

For $\lambda_1 = -1$, we have

$$\begin{pmatrix} 1-(-1) & 1 \\ 4 & 1-(-1) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$2x + y = 0$$

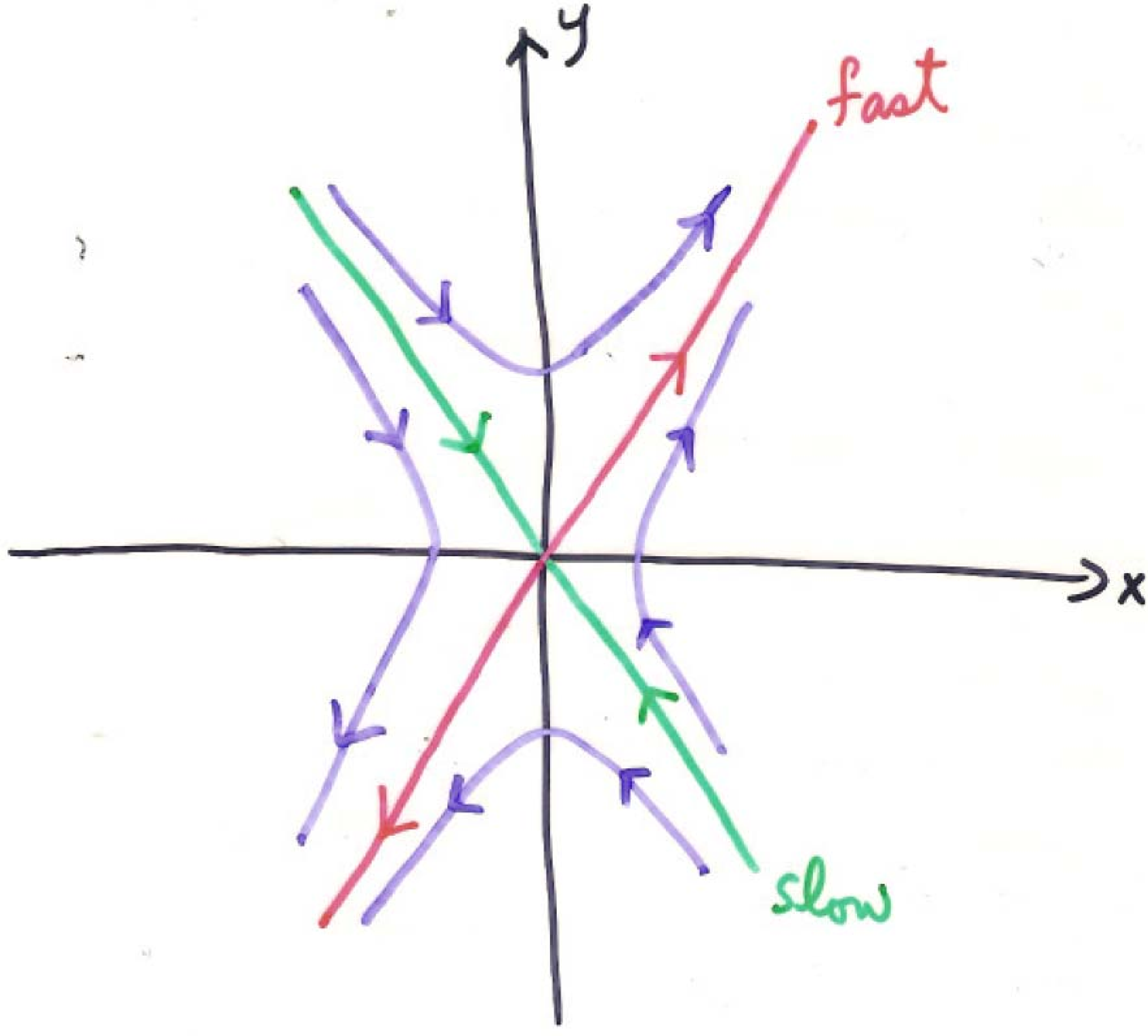
Take $\vec{u}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ (slow)

For $\lambda_2 = 3$, we have

$$\begin{pmatrix} 1-3 & 1 \\ 4 & 1-3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-2x + y = 0$$

Take $\vec{u}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ (fast)


y
fast
x
slow

Case 2: B has two complex eigenvalues

$$\lambda_1 = p + iq$$

$$\lambda_2 = p - iq$$

$q \neq 0.$

i.e. $(\mathrm{Tr} B)^2 < 4(\det B)$

The parametric equation of a typical phase trajectory looks like

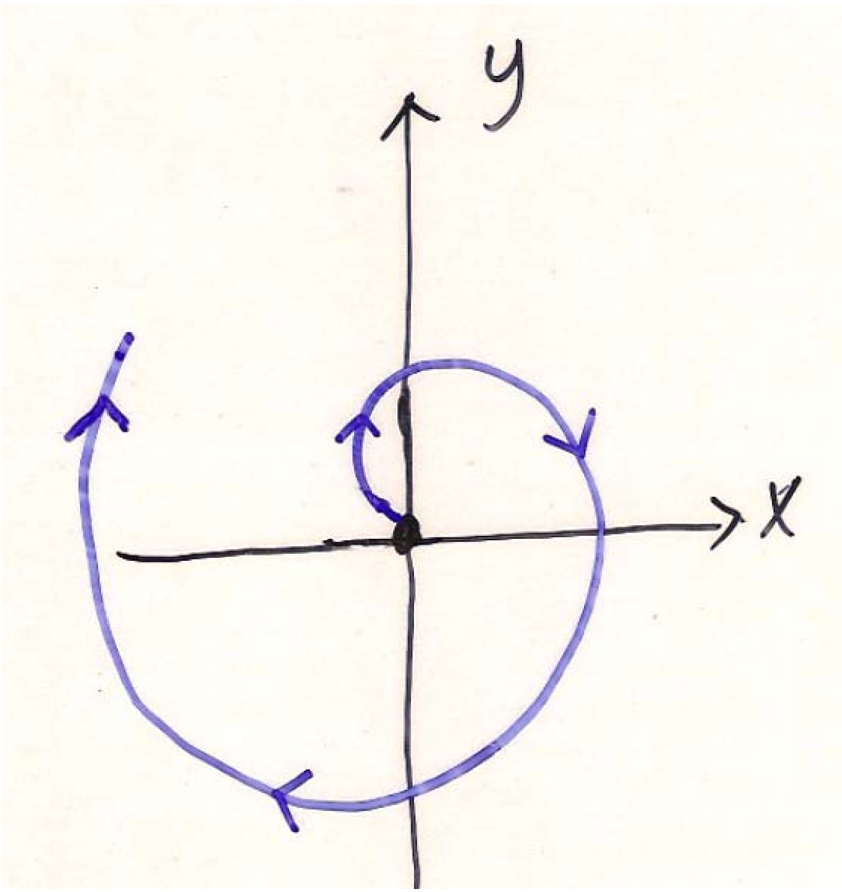
$$\begin{cases} x = \alpha e^{pt} \cos qt + \beta e^{pt} \sin qt \\ y = \gamma e^{pt} \cos qt + \delta e^{pt} \sin qt \end{cases}$$

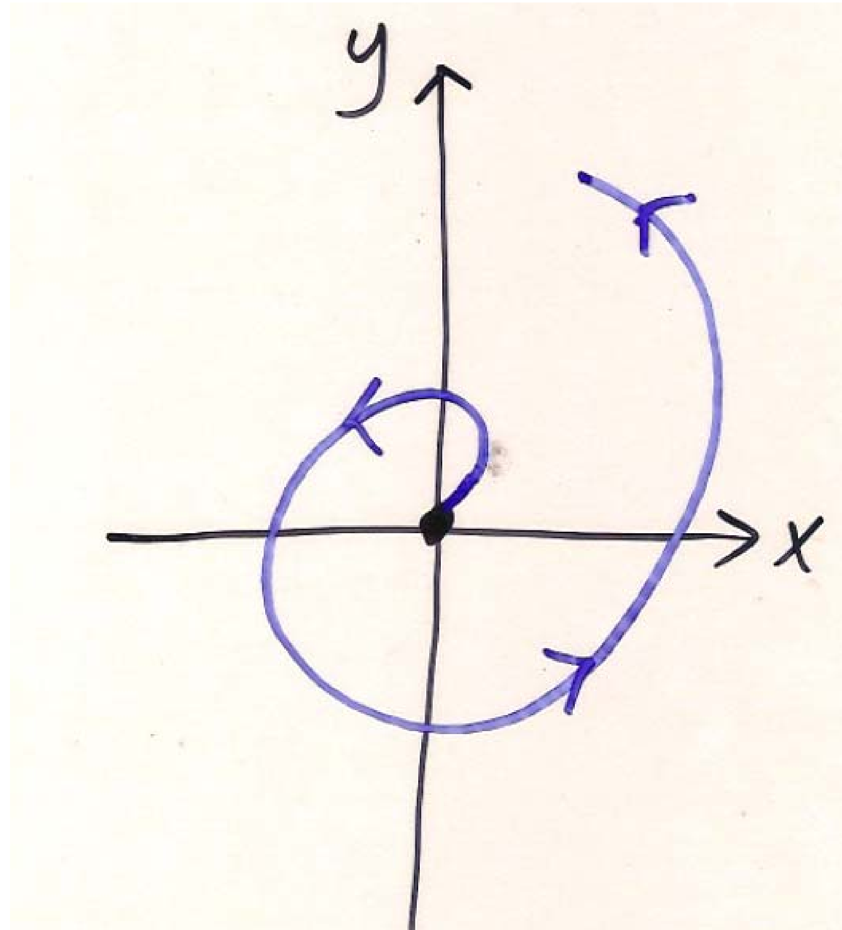
(i) $p > 0$

i.e. $\mathrm{Tr}B > 0$

$$\lim_{t \to \infty} e^{pt} = \infty \quad , \quad \lim_{t \to -\infty} e^{pt} = 0$$

SPIRAL SOURCE



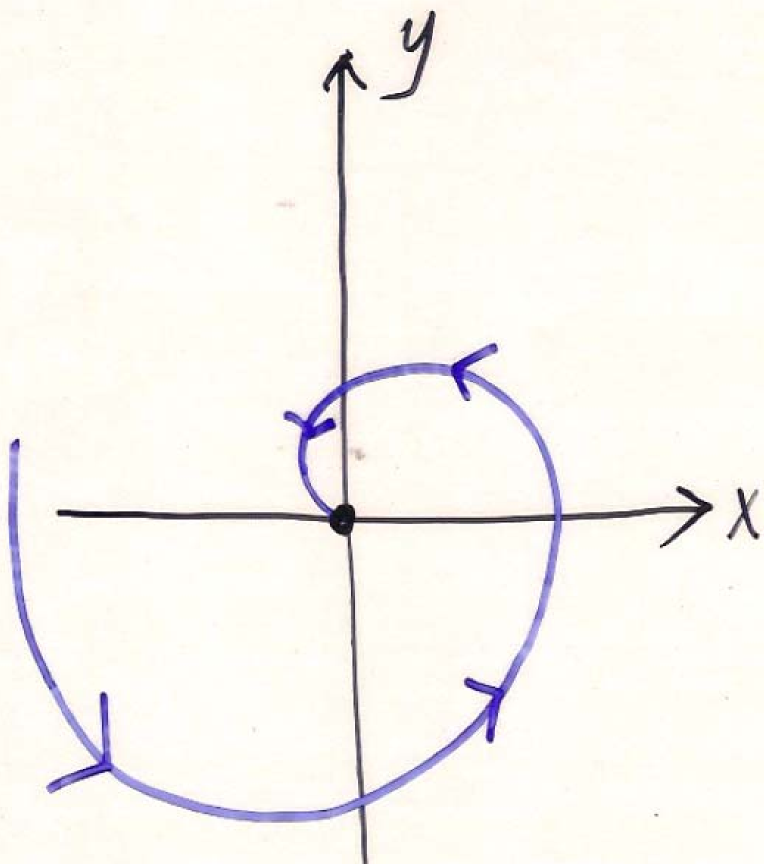


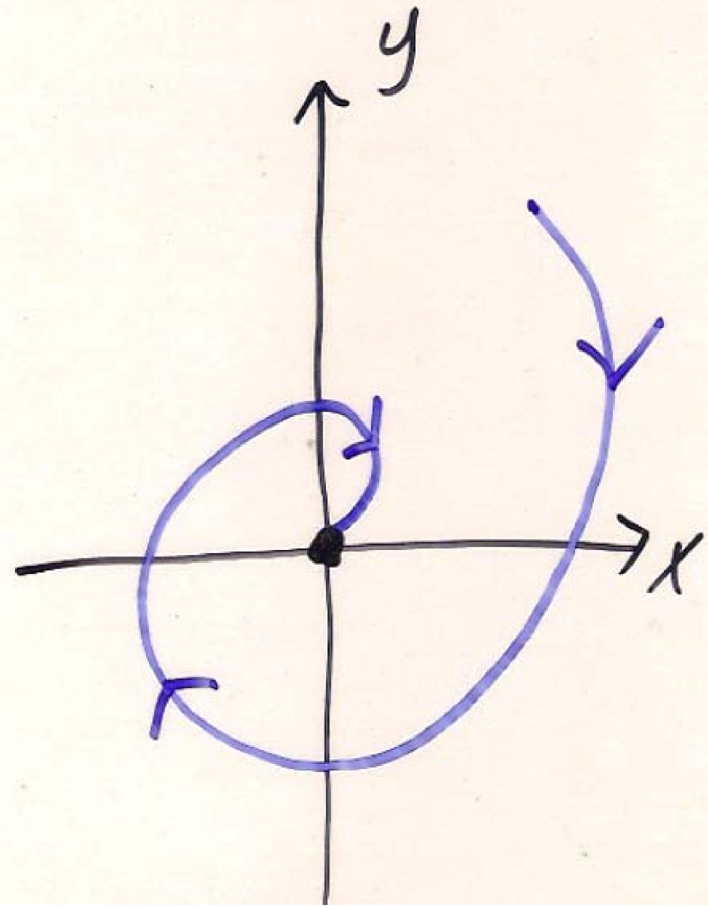
(ii) $p < 0$

i.e. $\operatorname{Tr} B < 0$

$$\lim_{t \to \infty} e^{pt} = 0 \quad , \quad \lim_{t \to -\infty} e^{pt} = \infty$$

SPIRAL SINK

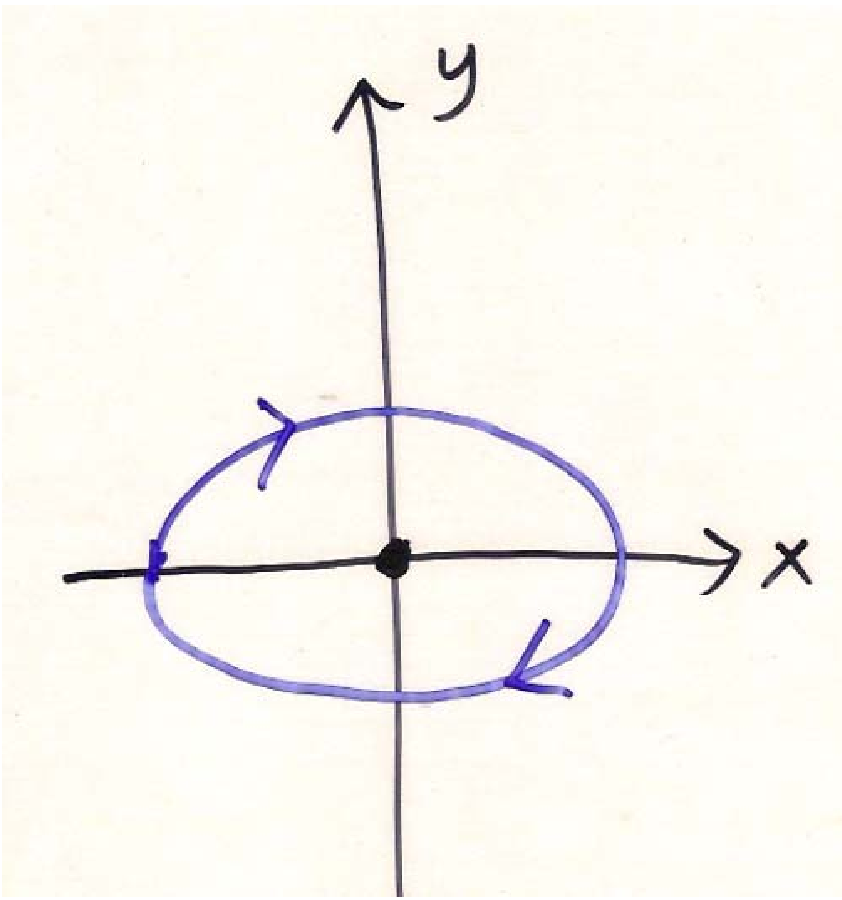


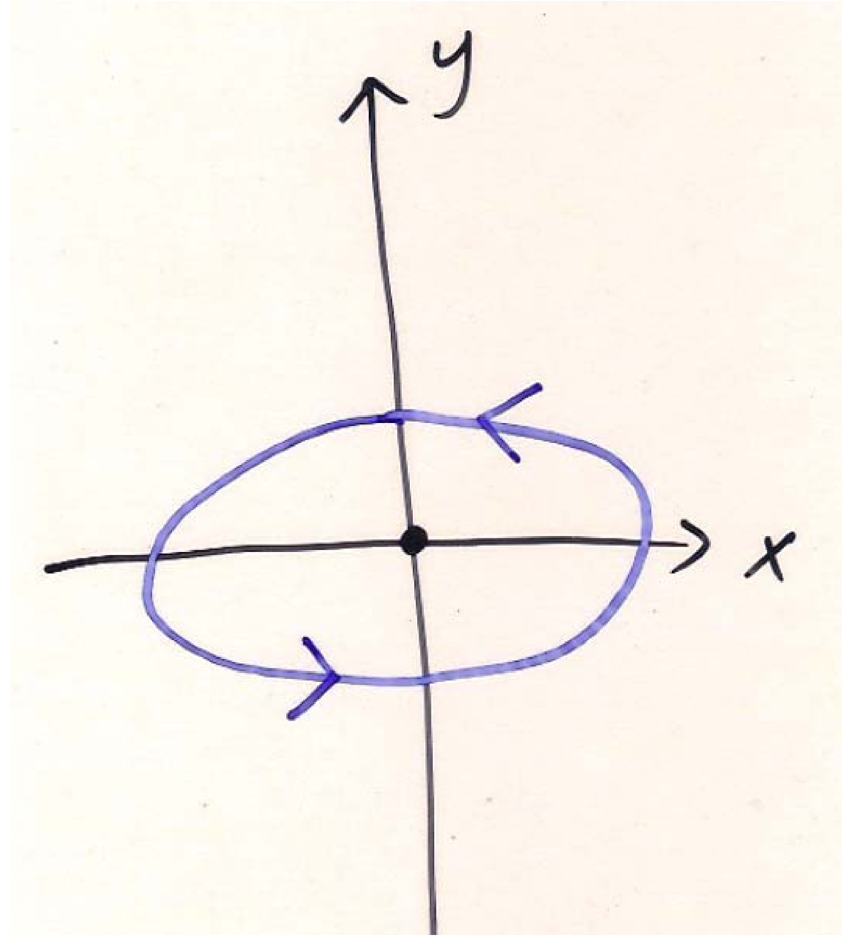


(iii) $p = 0$

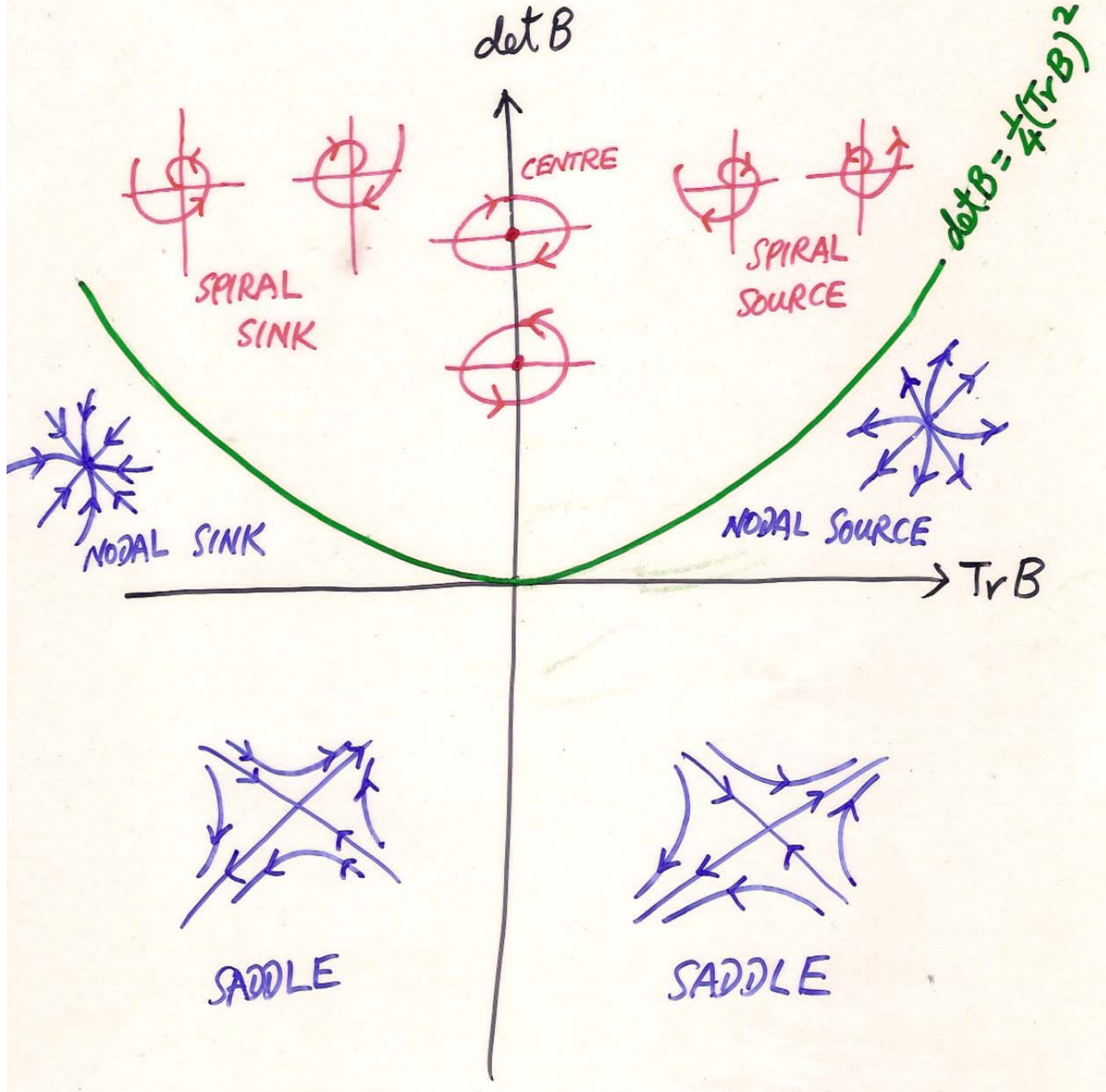
i.e. $\mathrm{Tr} B = 0$

CENTRE





SUMMARY


det B
Tr B
$\det B = \frac{1}{4}(\mathrm{Tr}\, B)^2$
CENTRE
SPIRAL SINK
SPIRAL SOURCE
NODAL SINK
NODAL SOURCE
SADDLE
SADDLE

Example (P. 38)

Romeo & Juliet again.

$$\begin{cases} \dfrac{dR}{dt} = aJ \\[2ex] \dfrac{dJ}{dt} = -bR \end{cases} \qquad a, b > 0$$

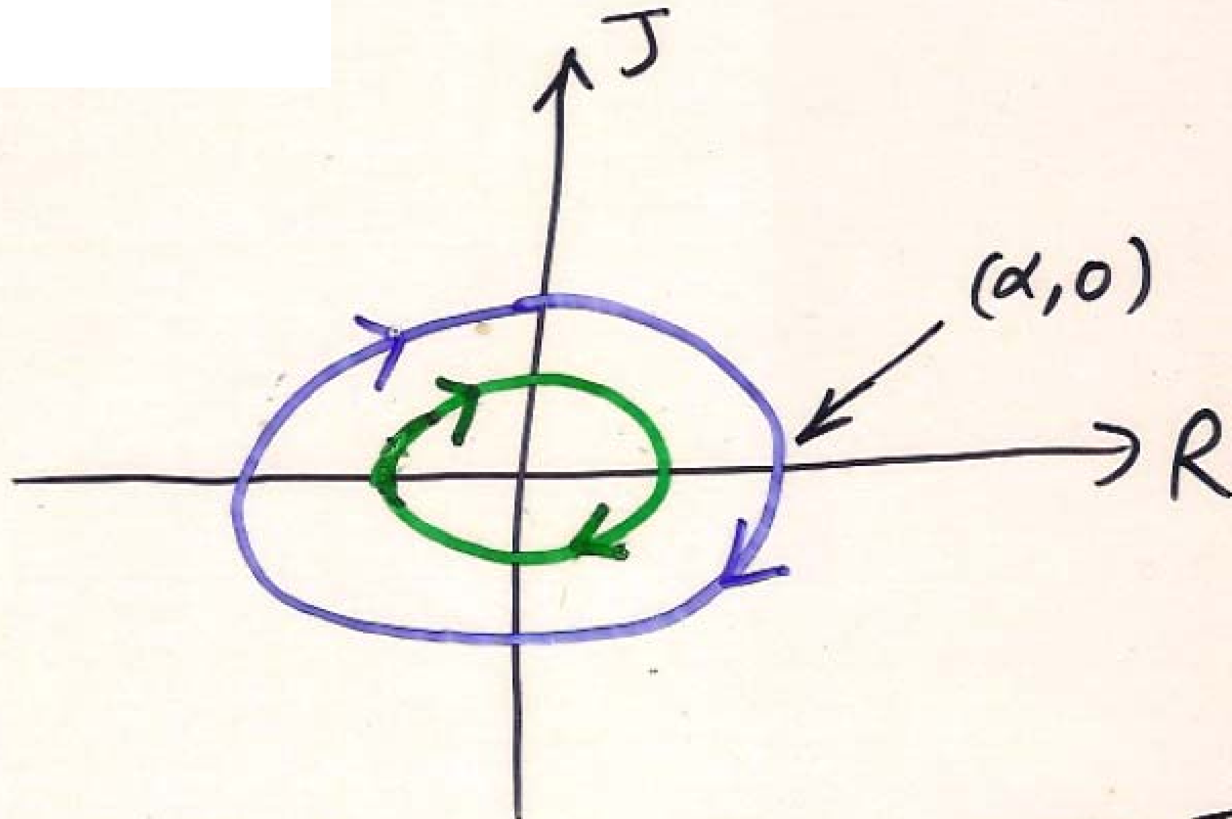
$$B = \begin{pmatrix} 0 & a \\ -b & 0 \end{pmatrix}$$

$$\det B = ab > 0$$

$$\operatorname{Tr} B = 0$$

$$(\operatorname{Tr} B)^2 - 4 \det B = -ve$$

$\Rightarrow$ CENTRE.



at $(\alpha, 0)$, $\frac{dJ}{dt} = -b\alpha < 0 \Rightarrow J \downarrow$

$\therefore$ the arrow there points down.

But now suppose we modify the story a little. [To be concrete, let's suppose that $a = 5/\text{hour}$ and $b = 3/\text{hour}$.] The original equations mean that, for example, if Juliet feels nothing for Romeo, then his feelings for her will remain static. But in reality his patience is not infinite: if she persists in feeling nothing for him then his affection will decay away. So a better equation might be $dR/dt = -0.8\,R + 5\,J$. This means that, if Juliet feels nothing for him, then his feeling for her will decay exponentially, as you know from the Malthus model. OK, he's not very patient.

Juliet, on the other hand, being somewhat eccentric and perhaps a little confused, might become more and more attached to Romeo even if he is unaware of her existence, eg in the way some girls fall in love with members of boy bands.

So perhaps we really have $dJ/dt = -3\ R + 0.7\ J$. This time we see that, even if $R = 0$, Juliet's affections will grow exponentially.

The matrix is now $\begin{pmatrix} -0.8 & 5 \\ -3 & 0.7 \end{pmatrix}$.

$$B = \begin{pmatrix} -0.8 & 5 \\ -3 & 0.7 \end{pmatrix}$$

$$\operatorname{Tr} B = -0.8 + 0.7 = -0.1$$

$$\det B = -0.56 + 15 = 14.44$$

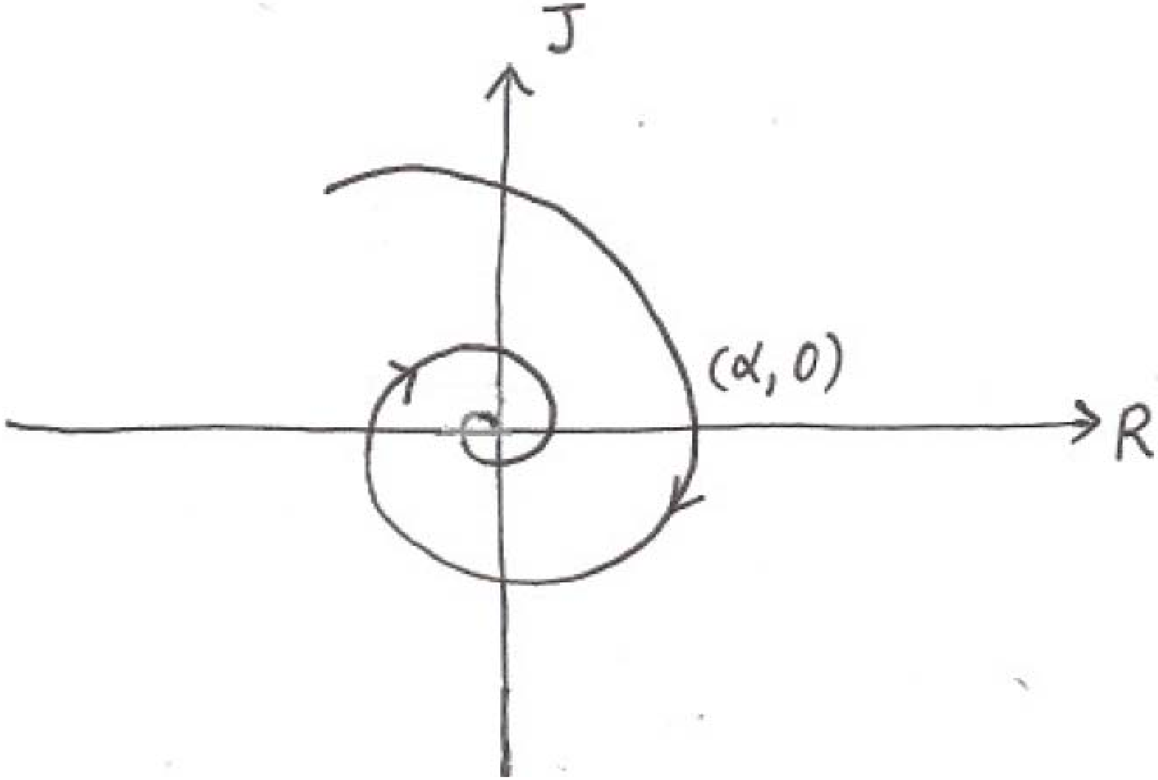
$$\therefore \det B > \frac{1}{4}(\operatorname{Tr} B)^2$$

$\therefore$ <u>SPIRAL SINK</u>

at a point $(\alpha, 0)$, $\alpha = +ve$, we have

$$\frac{dJ}{dt} = -3R + 0.7J = -3\alpha = -ve$$

$\therefore$ graph goes $\downarrow$ at $(\alpha, 0)$.



Example (P.42)

Continue from P. 13.

$$B = \begin{pmatrix} -4 & 3 \\ -2 & 1 \end{pmatrix}$$

$$\det B = -4 + 6 = 2 > 0$$

$$\mathrm{Tr}\, B = -4 + 1 = -3 < 0$$

$$(\mathrm{Tr}\, B)^2 - 4(\det B) = 9 - 8 = 1 > 0$$

$\Longrightarrow$ NODAL SINK.

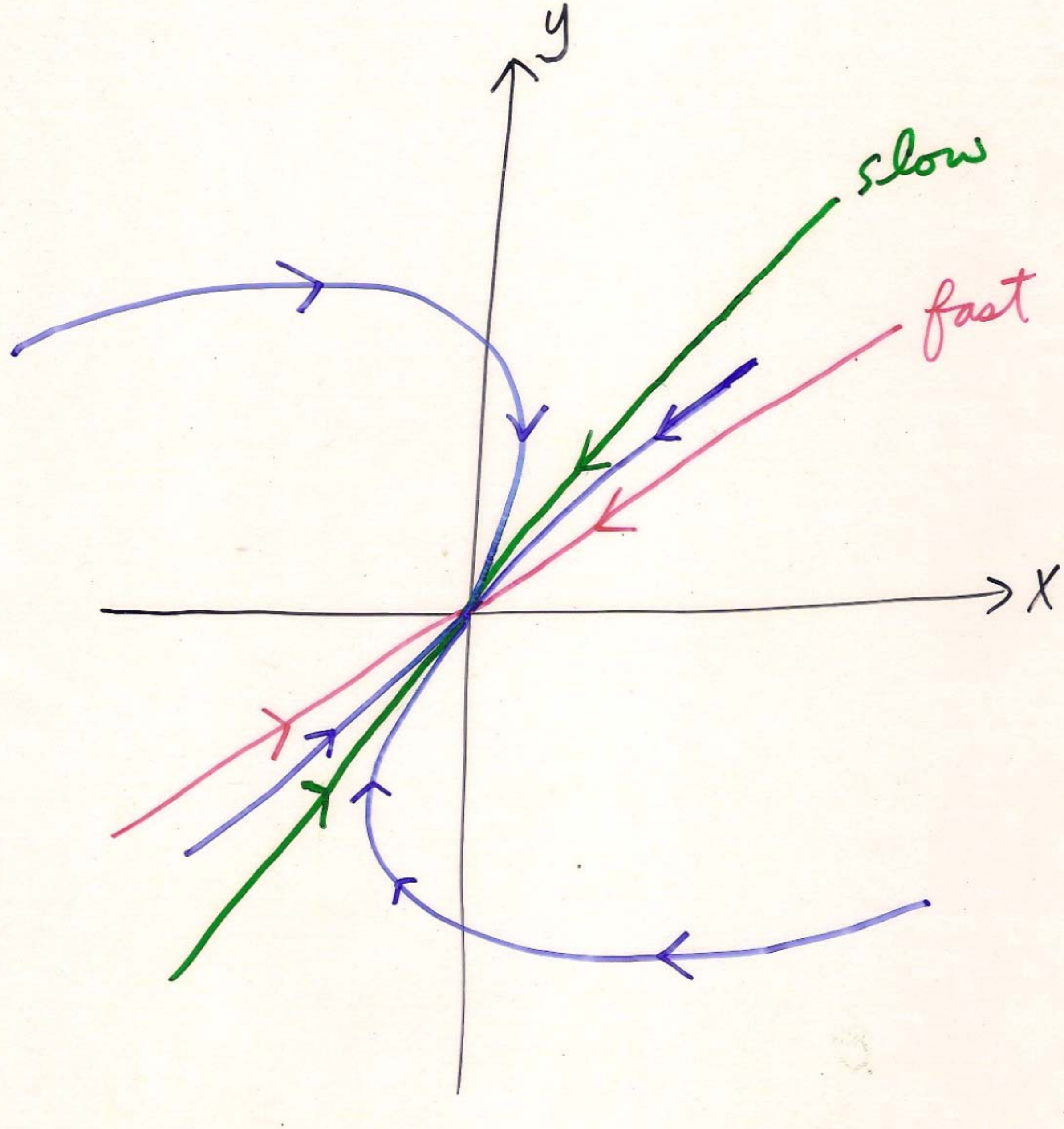
Recall from P. 13

$$\lambda_1 = -2 < \lambda_2 = -1 < 0$$

$$\vec{u}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

fast direction

slow direction


y
slow
fast
X

Example

Continue from P.14.

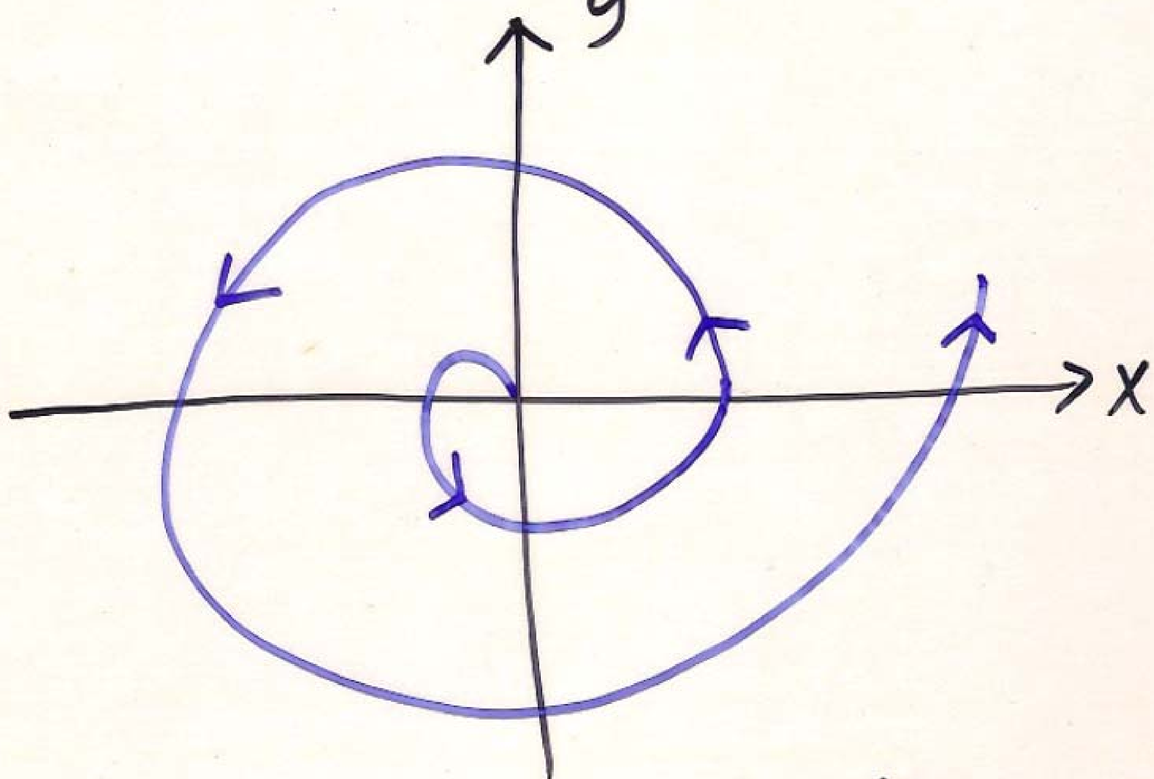
$$B = \begin{pmatrix} 4 & -5 \\ 2 & -2 \end{pmatrix}$$

$$\det B = -8 + 10 = 2 > 0$$

$$\mathrm{Tr}\, B = 4 - 2 = 2 > 0$$

$$(\mathrm{Tr} B)^2 - 4 \det B = 4 - 8 < 0$$

$\Rightarrow$ SPIRAL SOURCE



Check at $(\alpha, 0)$; $\alpha > 0 \Rightarrow \dfrac{dy}{dt} = 2\alpha > 0 \Rightarrow y \uparrow$

$\therefore$ the arrow points up at $(\alpha, 0)$.

7.5. WARFARE

A long and bitter battle is being fought on the slopes of Mount Doom between 15,000 Men of Gondor and 11,000 Orcs of Mordor. The Men die at a rate proportional to the number of Orcs, and also from a dread disease spread among them by the servants of Sauron, while the Orcs only die at a rate proportional to the number of Men – Orcs never get sick.

Let G(t) denote the number of Gondorians and M(t) denote the number of Mordor citizens in the battle. Then the above information tells us that we have a pair of differential equations which might have this form:

$$\frac{d}{dt}\begin{pmatrix} G \\ M \end{pmatrix} = \begin{pmatrix} -1 & -0.75 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} G \\ M \end{pmatrix},$$

where the -1 in the top left corner is the death rate per capita of the Gondorians due to disease, the -0.75 describes the rate at which Mordorians kill Gondorians, and the -1 in the second row describes the rate at which Gondorians kill Mordorians. [So the Gondorians are better soldiers.]

ODEs have actually been used to study real battles in this way! Of course, AS USUAL, you would want to make the model a lot more complicated than this model if you are really serious.

$$
\begin{cases}
\dfrac{dG}{dt} = -G - 0.75M \\[2ex]
\dfrac{dM}{dt} = -G \\[2ex]
G(0) = 15000 \\[1ex]
M(0) = 11000
\end{cases}
$$

Here $B = \begin{pmatrix} -1 & -0.75 \\ -1 & 0 \end{pmatrix}$

$$\det B = -0.75 < 0$$

$\Rightarrow$ SADDLE

To find eigenvalues, set

$$B - \lambda I = \begin{pmatrix} -1-\lambda & -0.75 \\ -1 & -\lambda \end{pmatrix}$$

$$\begin{vmatrix} -1-\lambda & -0.75 \\ -1 & -\lambda \end{vmatrix} = 0$$

$$\lambda + \lambda^2 - 0.75 = 0$$

$$4\lambda^2 + 4\lambda - 3 = 0$$

$$(2\lambda + 3)(2\lambda - 1) = 0$$

$$\lambda_1 = -\frac{3}{2} < 0 < \lambda_2 = \frac{1}{2}$$

When $\lambda_1 = -\frac{3}{2}$

$$B - \lambda I = \begin{pmatrix} \frac{1}{2} & -\frac{3}{4} \\ -1 & \frac{3}{2} \end{pmatrix}$$

$$\therefore \ \frac{1}{2}x - \frac{3}{4}y = 0$$

$$\therefore \ 2x - 3y = 0$$

$$x = 3t \Rightarrow y = 2t$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3t \\ 2t \end{pmatrix} = t\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

let $\vec{u}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ (slow direction)

When $\lambda_2 = \frac{1}{2}$

$$B - \lambda I = \begin{pmatrix} -\frac{3}{2} & -\frac{3}{4} \\ -1 & -\frac{1}{2} \end{pmatrix}$$

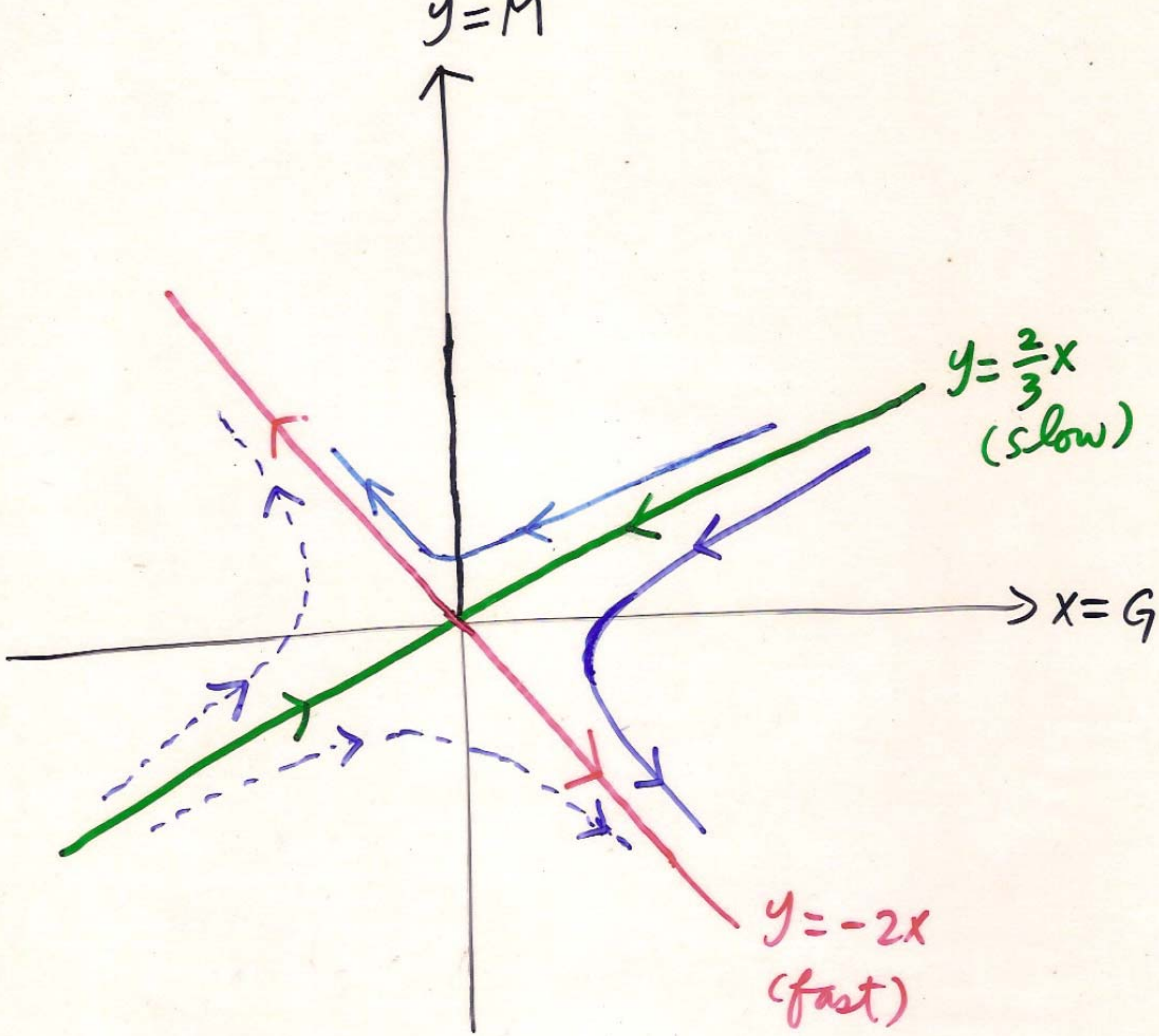
$$\therefore -x - \frac{1}{2}y = 0$$

$$2x + y = 0$$

$$x = t \Rightarrow y = -2t$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ -2t \end{pmatrix} = t\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

let $\vec{u_2} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ (fast direction)



y = M
x = G
y = 2/3 x
(slow)
y = -2x
(fast)

Observe: If we start below $M = \frac{2}{3} G$

i.e. if $M < \frac{2}{3} G$

then the trajectory will hit

the G-axis at some $(\alpha, 0)$; $\alpha > 0$

$\therefore$ $G = \alpha$ and $M = 0$ at some time

i.e. G wins.

If we start above $M = \frac{2}{3} G$

i.e. if $M > \frac{2}{3} G$

then the trajectory will hit

the M-axis at some $(0, \beta)$;

$\beta > 0$

$\therefore$ $G = 0$ and $M = \beta$ at some time

i.e. M wins.

Conclusion :

$$M(0) = 11000$$

$$\frac{2}{3} G(0) = \frac{2}{3}(15000) = 10000$$

$$\therefore \; M(0) > \frac{2}{3} G(0)$$

$$\Rightarrow \underline{\underline{M \text{ wins}}}$$