MA1506 TUTORIAL 2

1. Solve the following differential equations:

(a) $xy' + (1+x)y = e^{-x}, \qquad x > 0$

(b) $y' - (1+\frac{3}{x})y = x + 2, \qquad y(1) = e - 1, \qquad x > 0$

(c) $y' + y + \frac{x}{y} = 0$ (d) $2xyy' + (x-1)y^2 = x^2 e^x, \qquad x > 0$

2. If a cable is held up at two ends at the same height, then it will sag in the middle, making a U-shaped curve called a **catenary**. This is the shape seen in electricity cables suspended between poles, in countries less advanced than Singapore, such as Japan and the US. It can be shown using simple physics that if the shape is given by a function y(x), then this function satisfies

$$\frac{dy}{dx} = \frac{\mu}{T}\int_0^x \sqrt{\left(\frac{dy}{dt}\right)^2 + 1}\, dt,$$

where x = 0 at the lowest point of the catenary and y(0) = 0, where μ is the weight per unit length of the cable, and where T is the horizontal component of its tension; this horizontal component is a constant along the cable. Find a formula for the shape of the cable. [Hint: use the Fundamental Theorem of Calculus, and think of the resulting equation as a **first-order** ODE.]

3. Psychologists talk about something called a **Performance Curve**. Suppose an MA1506 student is solving mathematics problems. She starts with differential equations. Let P(t) be a non-negative function that measures her performance, that is, her success rate at solving DEs. Her performance increases rapidly at first, but then the rate of increase slows down as she becomes more expert. Let M, a positive constant, be the best possible performance; then one can suppose that P satisfies

$$\frac{dP}{dt} = C[M - P],$$

where C is a constant. What are the units of this constant? What does this constant measure? Solve this equation assuming that she is completely incompetent at t = 0 [that is, P(0) = 0].

Now the student turns to another kind of problem, say in linear algebra. Again her performance is low at first but gets better in accordance with this equation. Now as the years go by, her overall ability to solve mathematics problems gradually gets better, so C, instead of being a constant, is really a slowly increasing function of time. Suppose that C(t) = K tanh(t/T), t $\geq$ 0, where K and T are positive constants. [Is this reasonable? Why? What are the meanings of the constants K and T?] Replacing C with C(t), solve for P, again assuming that P(0) = 0. Sketch P(t), the performance curve. [Choose values of the constants for yourself, and use Graphmatica [www.graphmatica.com] if necessary.]

4. A certain MA1506 student starts a rumour to the effect that one of the lecturers has been seen dating a famous movie actress. The number of students who have heard the rumour, R(t), is given

by

$$\frac{dR}{dt} = KR[1300 - R],$$

where K is a positive constant, and 1300 is the number of students taking MA1506. What is the meaning of K? Is this equation reasonable? [Hint: surely the rumour will spread slowly both when hardly anyone has heard it yet, **but also** when nearly everyone has already heard it!] By regarding this equation as a Bernoulli equation, find R(t).

5. The half-life of Thorium 230 is about 75000 years, while that of Uranium 234 is about 245000 years. A certain sample of ancient coral has a Thorium/Uranium ratio of 10 percent. How old is the coral?