NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2002-2003

Solutions to MA1505 MATHEMATICS I

November 2002 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided below.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **TEN (10)** questions and comprises **FORTY ONE (41)** printed pages.
3. Answer **ALL** questions. For each question, write your answer and your working in the space provided inside the booklet following that question.
4. The marks for each question are indicated at the beginning of the question.
5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Matriculation Number:

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Question	1	2	3	4	5	6	7	8	9	10
Marks										

Question 1 (b) [5 marks]

Find

$$\lim_{x\to 1} \frac{\sin(\ln\sqrt{x})^3}{(x-1)^3}.$$

Answer.

$$\lim_{x\to 1} \frac{\sin(\ln\sqrt{x})^3}{(x-1)^3} = \lim_{x\to 1} \frac{\sin(\ln\sqrt{x})^3}{(\ln\sqrt{x})^3} \cdot \left(\frac{\frac{1}{2}\ln x}{x-1}\right)^3$$

$$= \left(\lim_{\theta\to 0} \frac{\sin\theta}{\theta}\right) \frac{1}{8} \left(\lim_{x\to 1} \frac{\ln x}{x-1}\right)^3 \qquad \left(\text{let } \theta = (\ln\sqrt{x})^3\right)$$

$$= \frac{1}{8}\left(\lim_{x\to 1} \frac{1/x}{1}\right)^3$$

$$= \underline{\underline{\frac{1}{8}}}$$

Question 2 (a) [5 marks]

Let L denote the tangent line to the curve $y = x \sin \frac{1}{x}$ at the point $\left(\frac{1}{3\pi}, 0\right)$. Find the y-coordinate of the point of intersection of L and the y-axis.

Answer.

$$\frac{dy}{dx} = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}$$

$$\therefore \left.\frac{dy}{dx}\right|_{x=\frac{1}{3\pi}} = \sin 3\pi - 3\pi \cos 3\pi = 3\pi$$

$\therefore$ L is given by

$$y = 3\pi\left(x - \frac{1}{3\pi}\right)$$

$$\therefore x = 0 \Rightarrow y = 3\pi\left(-\frac{1}{3\pi}\right)$$

$$= \underline{\underline{-1}}$$

Question 3 (a) [5 marks]

Find the value of
$$\int_0^{\frac{\pi}{2}} x^2 \sin(2x)\, dx.$$

Answer.

$$\int_0^{\frac{\pi}{2}} x^2 \sin 2x\, dx = -\frac{1}{2}\int_0^{\frac{\pi}{2}} x^2\, d(\cos 2x)$$

$$= -\frac{1}{2}x^2 \cos 2x \Big|_0^{\frac{\pi}{2}} + \frac{1}{2}\int_0^{\frac{\pi}{2}} (\cos 2x)\, 2x\, dx$$

$$= \frac{\pi^2}{8} + \frac{1}{2}\int_0^{\frac{\pi}{2}} x\, d(\sin 2x)$$

$$= \frac{\pi^2}{8} + \frac{1}{2} x \sin 2x \Big|_0^{\frac{\pi}{2}} - \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 2x\, dx$$

$$= \frac{\pi^2}{8} + \frac{1}{2}\, \frac{\cos 2x}{2} \Big|_0^{\frac{\pi}{2}}$$

$$= \underline{\underline{\frac{\pi^2}{8} - \frac{1}{2}}}$$

Question 3 (b) [5 marks]

Find the value of

$$\int_{\frac{1}{2\pi}}^{\frac{2}{\pi}} \frac{1}{x^2}\left(\sin\frac{1}{x}\right)^2\left(\cos\frac{1}{x}\right)dx.$$

Answer.

$$\int_{\frac{1}{2\pi}}^{\frac{2}{\pi}} \frac{1}{x^2}\left(\sin\frac{1}{x}\right)^2\left(\cos\frac{1}{x}\right)dx$$

$$= -\int_{\frac{1}{2\pi}}^{\frac{2}{\pi}} \left(\sin\frac{1}{x}\right)^2 d\left(\sin\frac{1}{x}\right)$$

$$= -\frac{1}{3}\left(\sin\frac{1}{x}\right)^3\Big|_{\frac{1}{2\pi}}^{\frac{2}{\pi}}$$

$$= \underline{\underline{-\frac{1}{3}}}$$

Question 4 (a) [5 marks]

By using the Ratio Test, or otherwise, determine whether the series

$$\sum_{n=1}^{\infty} \frac{n!}{5^n}$$

is convergent or divergent. Show clearly all your steps.

Answer.

$$\lim_{n\to\infty} \frac{\frac{(n+1)!}{5^{n+1}}}{\frac{n!}{5^n}} = \lim_{n\to\infty} \frac{n+1}{5} = \infty > 1$$

By the Ratio Test,

the series is divergent.

Question 5 (a) [5 marks]

Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{6^n}(3x-8)^n.$$

Answer.

$$\sum_{n=1}^{\infty} \frac{1}{6^n}(3x-8)^n = \sum_{n=1}^{\infty} \frac{1}{2^n}\left(x-\frac{8}{3}\right)^n$$

$$\lim_{n\to\infty} \frac{\frac{1}{2^{n+1}}\left|x-\frac{8}{3}\right|^{n+1}}{\frac{1}{2^n}\left|x-\frac{8}{3}\right|^n} = \frac{1}{2}\left|x-\frac{8}{3}\right|$$

$$\therefore \frac{1}{2}\left|x-\frac{8}{3}\right| < 1 \iff \left|x-\frac{8}{3}\right| < 2$$

$\therefore$ radius of convergence $= \underline{\underline{2}}$

Question 5 (b) [5 marks]

Let

$$f(x) = \frac{5x-13}{x^2-5x+6}.$$

By using the Taylor Series of $f(x)$ centered at $a=1$, or otherwise, find the value of $f^{(4)}(1)$.

Answer.

$$f(x) = \frac{5x-13}{x^2-5x+6} = \frac{3}{x-2} + \frac{2}{x-3}$$

$$= \frac{3}{(x-1)-1} + \frac{2}{(x-1)-2}$$

$$= \frac{-3}{1-(x-1)} - \frac{1}{1-\left(\frac{x-1}{2}\right)}$$

$$= -3\sum_{n=0}^{\infty}(x-1)^n - \sum_{n=0}^{\infty}\left(\frac{x-1}{2}\right)^n$$

$$= \sum_{n=0}^{\infty}\left(-3-\frac{1}{2^n}\right)(x-1)^n = \sum_{n=0}^{\infty}\frac{f^{(n)}(1)}{n!}(x-1)^n$$

$$\therefore \quad -3-\frac{1}{2^4} = \frac{f^{(4)}(1)}{4!}$$

$$\therefore \quad f^{(4)}(1) = -\frac{147}{2}$$

Question 6 (a) [5 marks]

Let

$$f(x) = \begin{cases} -\frac{\pi}{2} - \frac{x}{2}, & -\pi < x < 0; \\ \frac{\pi}{2} - \frac{x}{2}, & 0 < x < \pi. \end{cases}$$

Find the Fourier Series for $f(x)$.

You may use the following formulae: $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$, $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$, $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$.

Answer.

Note that for $0 < x < \pi$, we have $0 > -x > -\pi$

$$\therefore f(-x) = -\frac{\pi}{2} - \frac{(-x)}{2} = -\frac{\pi}{2} + \frac{x}{2} = -f(x)$$

$\therefore f$ is an odd function.

$\therefore a_n = 0 \quad \forall n = 0, 1, 2, \cdots$.

Next $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx$

$$= \frac{2}{\pi} \int_0^{\pi} \left(\frac{\pi}{2} - \frac{x}{2}\right) \sin nx dx$$

$$= \frac{2}{\pi} \left\{ -\frac{\pi}{2} \frac{1}{n} \cos nx \Big|_0^{\pi} + \frac{1}{2n} \int_0^{\pi} x\, d(\cos nx) \right\}$$

$$= \frac{2}{\pi} \left\{ -\frac{\pi}{2n} (\cos n\pi - 1) + \frac{1}{2n} x \cos nx \Big|_0^{\pi} - \frac{1}{2n} \int_0^{\pi} \cos nx\, dx \right\}$$

$$= \frac{2}{\pi} \left\{ -\frac{\pi}{2n} (\cos n\pi - 1) + \frac{\pi}{2n} \cos n\pi \right\}$$

$$= \frac{1}{n}$$

Answer: $\underline{\underline{\sum_{n=1}^{\infty} \frac{1}{n} \sin nx}}$

Question 6 (b) [5 marks]

Let $f(x) = \sin x$ for $0 < x < \pi$. Let $a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ be the Fourier Cosine Series which represents $f(x)$. Find the value of the coefficient a_4.
You may use the formulae given in Question 6 (a).

Answer.

We extend f to an even function on $(-\pi, \pi)$.

$$\therefore a_4 = \frac{2}{\pi} \int_0^{\pi} f(x) \cos 4x \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \sin x \cos 4x \, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} (\sin 5x - \sin 3x) \, dx$$

$$= \frac{1}{\pi} \left\{ -\frac{1}{5} \cos 5x + \frac{1}{3} \cos 3x \right\} \Big|_0^{\pi}$$

$$= \frac{1}{\pi} \left\{ \frac{1}{5} - \frac{1}{3} + \frac{1}{5} - \frac{1}{3} \right\}$$

$$= \underline{\underline{-\frac{4}{15\pi}}}$$

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Marks										

Question 1 (a) [5 marks]

For what value of m is the line $y = mx + c$ perpendicular to the tangent line of the graph of the function $y = \sqrt{x^2 + 16}$ at the point $(3, 5)$?

Answer 1(a)	$-\frac{5}{3}$

(Show your working below and on the next page.)

$$\because \quad \frac{dy}{dx}(x) = \frac{x}{\sqrt{x^2+16}}$$

$$\therefore \quad \frac{dy}{dx}(3) = \frac{3}{5}$$

$$\therefore \quad m = -\frac{1}{3/5} = \underline{\underline{-\frac{5}{3}}}$$

Question 1 (b) [5 marks]

Find $f'\left(\sqrt{3}\right)$ if $f\left(x\right)=\frac{x\left(1-x^2\right)^2}{\sqrt{1+x^2}}$.

Answer 1(b)	$\frac{25}{2}$

(Show your working below and on the next page.)

$$\ln f(x) = \ln x + \ln(1-x^2)^2 - \frac{1}{2}\ln(1+x^2)$$

$$\therefore \frac{f'(x)}{f(x)} = \frac{1}{x} - \frac{4x}{1-x^2} - \frac{x}{1+x^2}$$

$$\therefore f'(x) = \frac{(1-x^2)^2}{\sqrt{1+x^2}} - \frac{4x^2(1-x^2)}{\sqrt{1+x^2}} - \frac{x^2(1-x^2)^2}{(1+x^2)^{3/2}}$$

$$\therefore f'(\sqrt{3}) = 2 + 12 - \frac{3}{2}$$

$$= \underline{\underline{\frac{25}{2}}}$$

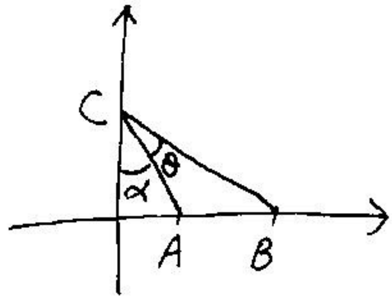
Question 2 (a) [5 marks]

Two points A and B start at time $t = 0$ at the origin and move along the positive x-axis with B moving 3 times as fast as A. Let C denote the fixed point $(0, 1)$ on the y-axis. Let θ denote the value of the angle $\angle$ACB at any time t later. What is the maximum value of $\tan\theta$?

Answer 2(a)	$\frac{1}{\sqrt{3}}$

(Show your working below and on the next page.)

Let A be at the point $(x, 0)$ at time t.



$\therefore$ B is at $(3x, 0)$ at time t.

$$\therefore \frac{3x}{1} = \tan(\theta + \alpha) = \frac{\tan\theta + \tan\alpha}{1 - \tan\theta\tan\alpha} = \frac{\tan\theta + x}{1 - x\tan\theta}$$

$$\therefore 3x - 3x^2\tan\theta = x + \tan\theta$$

$$\therefore \tan\theta = \frac{2x}{3x^2+1}$$

$$\therefore \frac{d}{dx}(\tan\theta) = \frac{2(3x^2+1) - 2x(6x)}{(3x^2+1)^2} = \frac{2(1-\sqrt{3}x)(1+\sqrt{3}x)}{(3x^2+1)^2}$$

$\because x \geqslant 0$, $\therefore$ $x < \frac{1}{\sqrt{3}} \Rightarrow \frac{d}{dx}(\tan\theta) > 0$

and $x > \frac{1}{\sqrt{3}} \Rightarrow \frac{d}{dx}(\tan\theta) < 0$

$\therefore$ max. of $\tan\theta = \underline{\underline{\frac{1}{\sqrt{3}}}}$

Question 2 (b) [5 marks]

Find the value of

$$\lim_{x \to 3^+} \frac{x^2 \int_3^x \sqrt{t^3 + 9}\,dt}{|3 - x|}.$$

Answer 2(b)	54

(Show your working below and on the next page.)

$$\lim_{x \to 3^+} \frac{x^2 \int_3^x \sqrt{t^3+9}\,dt}{|3-x|}$$

$$= \lim_{x \to 3^+} \frac{x^2 \int_3^x \sqrt{t^3+9}\,dt}{x-3}$$

$$= \lim_{x \to 3^+} \frac{2x \int_3^x \sqrt{t^3+9}\,dt + x^2 (\sqrt{x^3+9})}{1}$$

$$= \underline{\underline{54}}$$

Question 3 (a) [5 marks]

Let θ be the angle in the first quadrant such that $\sin\theta = \frac{\sqrt{15}}{4}$. Find the value of

$$\int_0^{\theta} \frac{80\sin^3 x}{\sqrt{\cos x}}dx.$$

Answer 3(a)	49

(Show your working below and on the next page.)

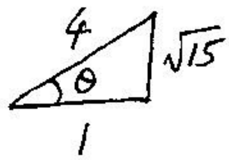
$$\int_0^{\theta} \frac{80\sin^3 x}{\sqrt{\cos x}}dx = -\int_0^{\theta} \frac{80\sin^2 x}{\sqrt{\cos x}}d(\cos x)$$

$$= -80\int_0^{\theta}\left\{\frac{1}{\sqrt{\cos x}} - (\cos x)^{3/2}\right\}d(\cos x)$$

$$= -80\left[2\sqrt{\cos x} - \frac{2}{5}(\cos x)^{5/2}\right]_0^{\theta}$$

$$= -80\left\{1 - \frac{1}{80} - 2 + \frac{2}{5}\right\}$$

$$= \underline{\underline{49}}$$



Question 3 (b) [5 marks]

Let $x > 1$. Find

$$\int \left(\frac{1}{\ln x} - \frac{1}{(\ln x)^2}\right) dx.$$

Answer 3(b)	$\frac{x}{\ln x} + C$

(Show your working below and on the next page.)

$$\int\left(\frac{1}{\ln x} - \frac{1}{(\ln x)^2}\right) dx = \int \frac{1}{\ln x} dx - \int \frac{1}{(\ln x)^2} dx$$

$$= \frac{x}{\ln x} + \int x \frac{1}{(\ln x)^2} \frac{1}{x} dx - \int \frac{1}{(\ln x)^2} dx$$

$$= \underline{\underline{\frac{x}{\ln x} + C}}$$

Question 4 (a) [5 marks]

By using the Ratio Test, or otherwise, determine whether the series

$$\sum_{n=1}^{\infty} \frac{6^n (n!)^2}{(2n)!}$$

is convergent or divergent. Show clearly all your steps.

Answer 4(a)	Divergent

(Show your working below and on the next page.)

$$\lim_{n \to \infty} \frac{6^{n+1} [(n+1)!]^2}{(2n+2)!} \cdot \frac{(2n)!}{6^n (n!)^2}$$

$$= \lim_{n \to \infty} \frac{6 (n+1)^2}{(2n+2)(2n+1)} = \frac{3}{2} > 1$$

$\therefore$ Ratio Test $\Rightarrow$ Divergence.

Question 5 (b) [5 marks]

Let

$$f(x) = \frac{1}{x^2 + x + 1}.$$

Let $f(x) = \sum_{n=0}^{\infty} c_n x^n$ be the Maclaurin series representation for $f(x)$. Find the value of $c_{36} - c_{37} + c_{38}$.

Answer 5(b)	2

(Show your working below and on the next page.)

$$f(x) = \frac{1}{x^2+x+1} = \frac{1-x}{1-x^3}$$

$$= (1-x)\sum_{n=0}^{\infty} x^{3n} = \sum_{n=0}^{\infty} x^{3n} - \sum_{n=0}^{\infty} x^{3n+1}$$

$$\therefore\ C_{36} = C_{3\times 12} = 1$$

$$C_{37} = C_{3\times 12+1} = -1$$

$$C_{38} = C_{3\times 12+2} = 0$$

$$\therefore\ C_{36} - C_{37} + C_{38} = 1-(-1)+0 = \underline{\underline{2}}$$

Question 6 (a) [5 marks]

Let

$$f(x) = \begin{cases} 0 & \text{if } -1 < x < 0 \\ 1 & \text{if } 0 < x < 1. \end{cases}$$

Let $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$ be the Fourier Series representation for $f(x)$. Find the value of

$$a_0 - \pi a_3 + \pi b_5.$$

Answer 6(a)	$\frac{9}{10}$

(Show your working below and on the next page.)

$$a_0 = \frac{1}{2}\int_{-1}^{1} f(x)\,dx = \frac{1}{2}\int_0^1 1\,dx = \frac{1}{2}$$

$$a_3 = \frac{1}{1}\int_{-1}^{1} f(x)\cos 3\pi x\,dx = \int_0^1 \cos 3\pi x\,dx = \left.\frac{\sin 3\pi x}{3\pi}\right|_0^1 = 0$$

$$b_5 = \frac{1}{1}\int_{-1}^{1} f(x)\sin 5\pi x\,dx = \int_0^1 \sin 5\pi x\,dx$$

$$= -\left.\frac{\cos 5\pi x}{5\pi}\right|_0^1 = \frac{2}{5\pi}$$

$$\therefore\ a_0 - \pi a_3 + \pi b_5 = \frac{1}{2} + \frac{2}{5} = \underline{\underline{\frac{9}{10}}}$$

Question 6 (b) [5 marks]

Let $f(x) = x(\pi - x)$ for $0 < x < \pi$. Let $\sum_{n=1}^{\infty} b_n \sin nx$ be the Fourier Sine Series which represents $f(x)$. Find the value of the coefficient b_3.
Give your answer in terms of π.

Answer 6(b)	$\dfrac{8}{27\pi}$

(Show your working below and on the next page.)

$$b_3 = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin 3x\,dx$$

$$= 2\int_0^{\pi} x\sin 3x\,dx - \frac{2}{\pi}\int_0^{\pi} x^2 \sin 3x\,dx$$

$$= 2\int_0^{\pi} -\frac{1}{3}x\,d(\cos 3x) + \frac{2}{\pi}\int_0^{\pi}\frac{1}{3}x^2\,d(\cos 3x)$$

$$= -\frac{2}{3}x\cos 3x\Big|_0^{\pi} + \frac{2}{3}\int_0^{\pi}\cos 3x\,dx$$

$$+ \frac{2}{3\pi}x^2\cos 3x\Big|_0^{\pi} - \frac{2}{3\pi}\int_0^{\pi} 2x\cos 3x\,dx$$

$$= \frac{2\pi}{3} + \frac{2}{9}\sin 3x\Big|_0^{\pi} - \frac{2\pi}{3} - \frac{2}{3\pi}\int_0^{\pi}\frac{2}{3}x\,d(\sin 3x)$$

$$= -\frac{4}{9\pi}x\sin 3x\Big|_0^{\pi} + \frac{4}{9\pi}\int_0^{\pi}\sin 3x\,dx$$

$$= -\frac{4}{27\pi}\cos 3x\Big|_0^{\pi} = \underline{\underline{\frac{8}{27\pi}}}$$

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Question 1 (a) [5 marks]

Find the slope of the tangent line at the point $(2, -2)$ on the graph of $x^2y^2 - 2x = 4 - 4y$.

Answer 1(a)	$\frac{7}{6}$

(Show your working below and on the next page.)

$$x^2y^2 - 2x = 4 - 4y$$

$$\Rightarrow 2xy^2 + 2x^2yy' - 2 = -4y'$$

$$x = 2,\ y = -2 \Rightarrow 16 - 16y' - 2 = -4y'$$

$$\Rightarrow 12y' = 14$$

$$\Rightarrow y' = \frac{7}{6}$$

Question 1 (b) [5 marks]

Find $\frac{1}{\pi}\left(f'(1) - \frac{1}{2\sqrt{3}}\right)$ if $f(x) = x\sin^{-1}\frac{x}{x+1}$.

Answer 1(b)	$\frac{1}{6}$

(Show your working below and on the next page.)

$$f'(x) = \sin^{-1}\frac{x}{x+1} + x\frac{1}{\sqrt{1-\frac{x^2}{(x+1)^2}}}\frac{x+1-x}{(x+1)^2} \quad \text{for } x>0$$

$$= \sin^{-1}\frac{x}{x+1} + \frac{x}{(x+1)\sqrt{2x+1}} \quad \text{for } x>0$$

$$f'(1) = \sin^{-1}\frac{1}{2} + \frac{1}{2\sqrt{3}}$$

$$= \frac{\pi}{6} + \frac{1}{2\sqrt{3}}$$

$$\frac{1}{\pi}\left(f'(1) - \frac{1}{2\sqrt{3}}\right) = \underline{\underline{\frac{1}{6}}}$$

Question 2 (a) [5 marks]

Given that the function $f(x) = \frac{x(3x-2)}{(x-1)(x-2)}$, where $x \in (1,2)$, attains its absolute maximum value at the point $C \in (1,2)$. Find the value of $(3-\sqrt{2})\,C$.

Answer 2(a)	2

(Show your working below and on the next page.)

$$f(x) = \frac{3x^2-2x}{x^2-3x+2}$$

$$f'(x) = \frac{(x^2-3x+2)(6x-2)-(3x^2-2x)(2x-3)}{(x^2-3x+2)^2}$$

$$= \frac{-7x^2+12x-4}{(x^2-3x+2)^2}$$

$$f'(x)=0 \Rightarrow 7x^2-12x+4=0$$

$$\Rightarrow x = \frac{12 \pm \sqrt{144-112}}{14} = \frac{6 \pm \sqrt{8}}{7} = \frac{2}{7}(3 \pm \sqrt{2})$$

$$\therefore C = \frac{2}{7}(3+\sqrt{2}) \quad (\because C \in (1,2))$$

$$\therefore (3-\sqrt{2})\,C = \frac{2}{7}(3+\sqrt{2})(3-\sqrt{2}) = \underline{\underline{2}}$$

Question 2 (b) [5 marks]

Find the value of

$$\lim_{x\to 0}\frac{\cos^2 8x - \cos^2 5x}{x^2}.$$

Answer 2(b)	-39

(Show your working below and on the next page.)

$$\lim_{x\to 0}\frac{\cos^2 8x - \cos^2 5x}{x^2}$$

$$=\left(\lim_{x\to 0}\frac{\cos 8x-\cos 5x}{x^2}\right)\left(\lim_{x\to 0}(\cos 8x+\cos 5x)\right)$$

$$= 2\lim_{x\to 0}\frac{-8\sin 8x+5\sin 5x}{2x}$$

$$= \lim_{x\to 0}\left(-64\cos 8x+25\cos 5x\right)$$

$$= \underline{\underline{-39}}$$

Question 3 (a) [5 marks]

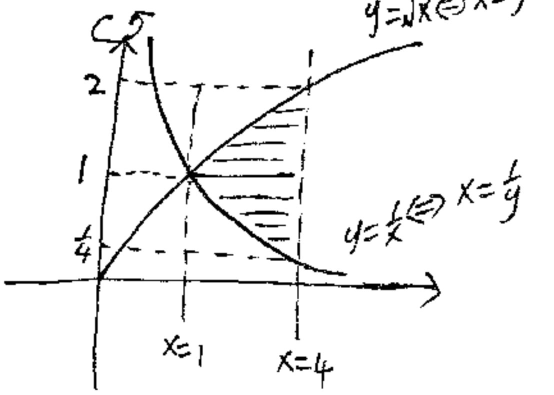
Find the volume of the solid obtained by revolving the region bounded by

$$y = \sqrt{x},\ y = \frac{1}{x},\ x = 1 \text{ and } x = 4$$

about the y-axis. Give your answer in terms of π.

Answer 3(a)	$\frac{94}{5}\pi$

(Show your working below and on the next page.)



$$\text{Volume} = \int_{\frac{1}{4}}^{1} \pi\left(16 - \frac{1}{y^2}\right)dy + \int_{1}^{2} \pi(16 - y^4)\,dy$$

$$= \pi\left[16y + \frac{1}{y}\right]_{\frac{1}{4}}^{1} + \pi\left[16y - \frac{1}{5}y^5\right]_{1}^{2}$$

$$= \pi(16 + 1 - 4 - 4) + \pi\left(32 - \frac{32}{5} - 16 + \frac{1}{5}\right)$$

$$= 9\pi + \frac{49}{5}\pi$$

$$= \underline{\underline{\frac{94}{5}\pi}}$$

Question 3 (b) [5 marks]

Find the value of

$$\int_0^{\pi/3} (\sin^3 x)(\cos x)\, dx.$$

Answer 3(b)	$\frac{9}{64}$

(Show your working below and on the next page.)

$$\int_0^{\frac{\pi}{3}} \sin^3 x \cos x dx$$

$$= \int_0^{\frac{\pi}{3}} \sin^3 x \; d(\sin x)$$

$$= \frac{1}{4} \sin^4 x \Big|_0^{\frac{\pi}{3}}$$

$$= \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)^4$$

$$= \underline{\underline{\frac{9}{64}}}$$

Question 5 (b) [5 marks]

Let

$$f(x) = \int_0^{x^2} \tan^{-1} t \, dt.$$

Let $f(x) = \sum_{n=0}^{\infty} c_n (x-1)^n$ be the Taylor series representation for $f(x)$ about the point $a = 1$. Find the value of c_2.

Answer 5(b)	$\frac{\pi}{4} + 1$

(Show your working below and on the next page.)

$$f'(x) = 2x \tan^{-1}(x^2)$$

$$f''(x) = 2\tan^{-1}(x^2) + \frac{2x}{1+x^4}(2x)$$

$$f''(1) = 2\left(\frac{\pi}{4}\right) + 2 = \frac{\pi}{2} + 2$$

$$\therefore C_2 = \frac{f''(1)}{2!} = \underline{\underline{\frac{\pi}{4} + 1}}$$

Question 6 (a) [5 marks]

Let

$$f(x) = \begin{cases} 0 & \text{if } -2\pi < x < 0 \\ x^2 & \text{if } 0 < x < 2\pi. \end{cases}$$

Find the coefficient of $\cos x$ in the Fourier Series representation for $f(x)$.

Answer 6(a)	2

(Show your working below and on the next page.)

Let $2L =$ period of f.

$\therefore 2L = 4\pi \Rightarrow L = 2\pi$

$\cos\frac{n\pi x}{L} = \cos x \Rightarrow \frac{n\pi}{L} = 1 \Rightarrow n = 2$

$a_2 = \frac{1}{L}\int_{-L}^{L} f(x)\cos x\,dx = \frac{1}{2\pi}\int_0^{2\pi} x^2\cos x\,dx$

$= \frac{1}{2\pi}\int_0^{2\pi} x^2\,d(\sin x) = \frac{1}{2\pi}\left\{ x^2\sin x\Big|_0^{2\pi} - 2\int_0^{2\pi} x\sin x\,dx \right\}$

$= \frac{1}{2\pi}\left\{ 2\int_0^{2\pi} x\,d(\cos x) \right\} = \frac{1}{\pi}\left\{ x\cos x\Big|_0^{2\pi} - \int_0^{2\pi}\cos x\,dx \right\}$

$= \underline{\underline{2}}$

Question 6 (b) [5 marks]

Let $f(x) = \cos x$ for $0 < x < \pi$. Let $\sum_{n=1}^{\infty} b_n \sin nx$ be the Fourier Sine Series which represents $f(x)$. Find the value of

$$b_1 + b_2.$$

Answer 6(b)	$\frac{8}{3\pi}$

(Show your working below and on the next page.)

$$b_1 = \frac{2}{\pi}\int_0^{\pi} \cos x \sin x \, dx = \frac{1}{\pi}\int_0^{\pi} \sin 2x \, dx$$

$$= \frac{1}{\pi}\left[-\frac{1}{2}\cos 2x\right]_0^{\pi} = 0$$

$$b_2 = \frac{2}{\pi}\int_0^{\pi} \cos x \sin 2x \, dx = \frac{1}{\pi}\int_0^{\pi}(\sin 3x + \sin x)\,dx$$

$$= \frac{1}{\pi}\left[-\frac{1}{3}\cos 3x - \cos x\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left\{\frac{1}{3} + 1 + \frac{1}{3} + 1\right\} = \frac{8}{3\pi}$$

$$\therefore\ b_1 + b_2 = \frac{8}{3\pi}$$

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2005-2006

MA1505 MATHEMATICS I

November 2005 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided below.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **EIGHT (8)** questions and comprises **THIRTY THREE (33)** printed pages.
3. Answer **ALL** questions. For each question, write your answer in the box and your working in the space provided inside the booklet following that question.
4. The marks for each question are indicated at the beginning of the question.
5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Matriculation Number:

For official use only. Do not write below this line.

Question	1	2	3	4	5	6	7	8
Marks								

Question 1 (a) [5 marks]

Given that $y^2 - 4x = 4 - 4y$. Find the value of $\frac{dy}{dx}$ at the point $(2, 2)$.

Answer 1(a)	$\frac{1}{2}$

(Show your working below and on the next page.)

$$2yy' - 4 = -4y'$$

$$(2y+4)y' = 4$$

$$y' = \frac{4}{2y+4}$$

$$\therefore y'(2,2) = \frac{4}{2(2)+4}$$

$$= \frac{1}{2}$$

Question 1 (b) [5 marks]

Let $f(x) = (\sin x)^{\sin x}$ for all $x \in \left(0, \frac{\pi}{2}\right)$. Given that f has a critical point at $c \in \left(0, \frac{\pi}{2}\right)$ .Find the value of $\sin c$.

Answer 1(b)	$\frac{1}{e}$

(Show your working below and on the next page.)

$$\ln f = (\sin x) \ln(\sin x)$$

$$\frac{1}{f} f' = (\cos x) \ln(\sin x) + \frac{\sin x}{\sin x} \cos x$$

$$= \cos x \{ \ln(\sin x) + 1 \}$$

$$f'(c) = 0 \text{ for } c \in \left(0, \frac{\pi}{2}\right)$$

$$\Rightarrow \ln(\sin c) + 1 = 0$$

$$\Rightarrow \ln(\sin c) = -1$$

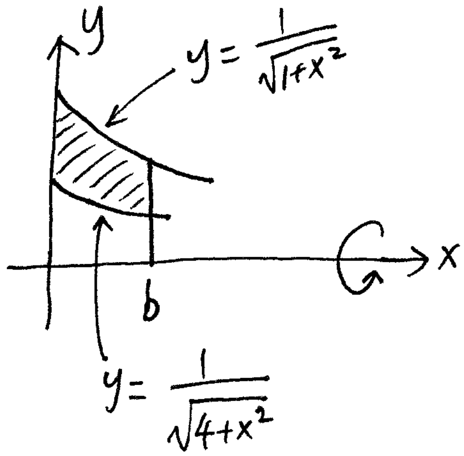
$$\Rightarrow \sin c = \frac{1}{e}$$

Question 2 (a) [5 marks]

The region bounded by the graphs of $y = \frac{1}{\sqrt{1+x^2}}$, $y = \frac{1}{\sqrt{4+x^2}}$, $x = 0$ and $x = b$ where b denotes a positive constant is rotated about the x-axis to generate a solid of revolution. Let $V(b)$ denote the volume of this solid of revolution. Find the value of $\lim_{b\to\infty} V(b)$.

Answer 2(a)	$\frac{\pi^2}{4}$

(Show your working below and on the next page.)



$$V(b) = \int_0^b \pi \left\{ \left(\frac{1}{\sqrt{1+x^2}}\right)^2 - \left(\frac{1}{\sqrt{4+x^2}}\right)^2 \right\} dx$$

$$= \pi \int_0^b \left(\frac{1}{1+x^2} - \frac{1}{4+x^2}\right) dx$$

$$= \pi \left[\tan^{-1} x - \frac{1}{2} \tan^{-1} \frac{x}{2} \right]_0^b$$

$$= \pi \left\{ \tan^{-1} b - \frac{1}{2} \tan^{-1} \frac{b}{2} \right\}$$

$$\therefore \lim_{b\to\infty} V(b) = \pi \left\{ \frac{\pi}{2} - \frac{1}{2}\left(\frac{\pi}{2}\right) \right\}$$

$$= \underline{\underline{\frac{\pi^2}{4}}}$$

Question 2 (b) [5 marks]

Find the value of

$$\frac{\int_{-\frac{\pi}{2}}^{0} \cos^{10} x \, \mathrm{d}x}{\int_{-\frac{\pi}{2}}^{0} \cos^{8} x \, \mathrm{d}x}.$$

Answer 2(b)	$\frac{9}{10}$

(Show your working below and on the next page.)

$$\int_{-\frac{\pi}{2}}^{0} \cos^{10} x \, dx = \int_{-\frac{\pi}{2}}^{0} \cos^{9} x \; d(\sin x)$$

$$= \cos^{9} x \sin x \Big|_{-\frac{\pi}{2}}^{0} + \int_{-\frac{\pi}{2}}^{0} 9 \cos^{8} x \sin^{2} x \, dx$$

$$= 9 \int_{-\frac{\pi}{2}}^{0} \cos^{8} x \, (1 - \cos^{2} x) \, dx$$

$$= 9 \int_{-\frac{\pi}{2}}^{0} \cos^{8} x \, dx - 9 \int_{-\frac{\pi}{2}}^{0} \cos^{10} x \, dx$$

$$\therefore \; 10 \int_{-\frac{\pi}{2}}^{0} \cos^{10} x \, dx = 9 \int_{-\frac{\pi}{2}}^{0} \cos^{8} x \, dx$$

$$\therefore \; \frac{\int_{-\frac{\pi}{2}}^{0} \cos^{10} x \, dx}{\int_{-\frac{\pi}{2}}^{0} \cos^{8} x \, dx} = \frac{9}{10}$$

Question 3 (a) [5 marks]

Find the radius of covergence of the power series

$$\sum_{n=0}^{\infty} \frac{8^n + (-9)^n}{n+1} (x+2)^{2n}.$$

Answer 3(a)	$\frac{1}{3}$

(Show your working below and on the next page.)

$$\lim_{n\to\infty} \left| \frac{\frac{8^{n+1}+(-9)^{n+1}}{n+2}(x+2)^{2n+2}}{\frac{8^n+(-9)^n}{n+1}(x+2)^{2n}} \right|$$

$$= \lim_{n\to\infty} \left\{ \frac{n+1}{n+2} \left| \frac{8\left(\frac{8}{9}\right)^n + (-1)^n 9}{\left(\frac{8}{9}\right)^n + (-1)^n} \right| |x+2|^2 \right\}$$

$$= 9|x+2|^2 \qquad \left(\because \lim_{n\to\infty}\left(\frac{8}{9}\right)^n = 0\right)$$

$$\therefore 9|x+2|^2 < 1 \Rightarrow |x+2| < \frac{1}{3}$$

$$\Rightarrow |x-(-2)| < \underline{\underline{\frac{1}{3}}}$$

Question 3 (b) [5 marks]

Let $f(x) = \tan^{-1}\left(\frac{1+x}{1-x}\right)$ where $-\frac{1}{2} \le x \le \frac{1}{2}$. Find the value of

$$f^{(2005)}(0).$$

Give your answer in terms of factorials.

Answer 3(b)	$2004!$

(Show your working below and on the next page.)

$$f'(x) = \frac{1}{1+\left(\frac{1+x}{1-x}\right)^2}\left\{\frac{(1-x)(1)-(1+x)(-1)}{(1-x)^2}\right\} = \frac{2}{(1-x)^2+(1+x)^2}$$

$$= \frac{2}{2+2x^2} = \frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

$$\therefore \int_0^x f'(t)\,dt = \sum_{n=0}^{\infty}(-1)^n \int_0^x t^{2n}\,dt$$

$$\therefore f(x) - f(0) = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}$$

$$\therefore f(x) = \frac{\pi}{4} + \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1} \qquad \left(\because f(0) = \tan^{-1}(1) = \frac{\pi}{4}\right)$$

$$\therefore \frac{f^{(2005)}(0)}{(2005)!} = \frac{(-1)^{\frac{2005-1}{2}}}{2005}$$

$$\therefore f^{(2005)}(0) = \underline{\underline{(2004)!}}$$

Question 4 (a) [5 marks]

Let $f(x) = \cos\frac{x}{2}$ for all $x \in (0, \pi)$. Let

$$a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

be the Fourier Cosine Series which represents $f(x)$. Find the value of $a_0 + a_1$. Give your answer in terms of π.

Answer 4(a)	$\frac{10}{3\pi}$

(Show your working below and on the next page.)

$$a_0 = \frac{1}{\pi}\int_0^{\pi} \cos\frac{x}{2}\,dx = \frac{2}{\pi}\sin\frac{x}{2}\Big|_0^{\pi} = \frac{2}{\pi}$$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \cos\frac{x}{2}\cos x\,dx = \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\left\{\cos\frac{x}{2} + \cos\frac{3x}{2}\right\}dx$$

$$= \frac{1}{\pi}\left[2\sin\frac{x}{2} + \frac{2}{3}\sin\frac{3x}{2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left(2 - \frac{2}{3}\right) = \frac{4}{3\pi}$$

$$\therefore a_0 + a_1 = \frac{2}{\pi} + \frac{4}{3\pi} = \underline{\underline{\frac{10}{3\pi}}}$$

Question 4 (b) [5 marks]

Let $f(x) = 2x + 1$ for all $x \in (-\pi, \pi)$ and $f(x) = f(x + 2\pi)$. Let

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be the Fourier Series which represents $f(x)$. Find the value of $a_0 + a_5 + b_5$.

Answer 4(b)	$\frac{9}{5}$

(Show your working below and on the next page.)

The function $g(x) = x$ is an odd function on $(-\pi, \pi)$.

$\therefore\ g(x) \sim \sum_{n=1}^{\infty} C_n \sin nx$

and $C_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx\, dx = \frac{-2}{\pi}\int_0^{\pi} \frac{1}{n} x\, d(\cos nx)$

$= -\frac{2}{\pi}\left\{ \frac{1}{n} x \cos nx \Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi} \cos nx\, dx \right\}$

$= -\frac{2}{n}\left\{\cos n\pi\right\} = (-1)^{n+1}\frac{2}{n}$

$\therefore\ f(x) = 2x+1 = 2g(x)+1 \sim 1 + \sum_{n=1}^{\infty} (-1)^{n+1}\frac{4}{n}\sin nx$

$\therefore\ a_0 + a_5 + b_5 = 1 + 0 + (-1)^6 \frac{4}{5} = \underline{\underline{\frac{9}{5}}}$