Chapter 3. Integration

3.1 Indefinite Integral

Integration can be considered as the antithesis of differentiation, and they are subtly linked by the **Fundamental Theorem of Calculus**. We first introduce indefinite integration as an "inverse" of differentiation.

3.1.1 Antiderivatives

A (differentiable) function $F(x)$ is an *antiderivative* of a function $f(x)$ if

$$F'(x) = f(x)$$

for all x in the domain of f.

The set of all antiderivatives of f is the *indefinite integral* of f with respect to x, denoted by

$$\int f(x)\,dx.$$

Terminology:

f : *integrand* of the integral x : *variable* of integration

3.1.2 Constant of Integration

Any constant function has zero derivative. Hence the antiderivatives of the zero function are all the constant functions.

If $F'(x) = f(x) = G'(x)$, then $G(x) = F(x) + C$, where C is some constant. So

$$\int f(x)dx = F(x) + C.$$

C here is called the *constant of integration* or an *arbitrary constant.* Thus,

$$\int f(x)\,dx = F(x) + C$$

means the same as

$$\frac{d}{dx}F(x) = f(x).$$

In words,

indefinite integral and antiderivative (of a function) *differ by an arbitrary constant.*

3.1.3 Integral formulas

1. $$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1,\ n \text{ rational}$$

   $$\int 1\,dx = \int dx = x + C \qquad \text{(Special case, } n = 0\text{)}$$

2. $$\int \sin kx\,dx = -\frac{\cos kx}{k} + C$$

3. $$\int \cos kx\,dx = \frac{\sin kx}{k} + C$$

4. $$\int \sec^2 x\,dx = \tan x + C$$

5. $$\int \csc^2 x\,dx = -\cot x + C$$

6. $$\int \sec x \tan x\,dx = \sec x + C$$

7. $$\int \csc x \cot x\,dx = -\csc x + C$$

3.1.4 Rules for indefinite integration

1. $\displaystyle\int kf(x)\,dx = k\int f(x)\,dx,$

   $k =$ constant (independent of x)

2. $\displaystyle\int -f(x)\,dx = -\int f(x)\,dx$

   (Rule 1 with $k = -1$)

3. $\displaystyle\int \big[f(x) \pm g(x)\big]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

3.1.5 Example

Find the curve in the xy-plane which passes through the point $(9, 4)$ and whose slope at each point (x, y) is $3\sqrt{x}$.

Solution. The curve is given by $y = y(x)$, satisfying

(i) $\dfrac{dy}{dx} = 3\sqrt{x}$ and (ii) $y(9) = 4.$

Solving (i), we get

$$y = \int 3\sqrt{x}\,dx = 3\,\frac{x^{3/2}}{3/2} + C = 2x^{3/2} + C.$$

By (ii), $\qquad 4 = (2)9^{3/2} + C = (2)27 + C,$

$C = 4 - 54 = -50.$

Hence $\qquad y = 2x^{3/2} - 50.$

3.2 Riemann Integrals

3.2.1 Area under a curve

Let $f = f(x)$ be a non-negative continuous function $f = f(x)$ on an interval $[a, b]$.

Partition $[a, b]$ into n consecutive sub-intervals $[x_{i-1}, x_i]$ $(i = 1, 2, \ldots, n)$ each of length $\Delta x = \dfrac{b-a}{n}$, where we set $a = x_0$, $b = x_n$, and $x_1, x_2, \cdots, x_{n-1}$ to be successive points between a and b with $x_k - x_{k-1} = \Delta x$.

Let c_k be any intermediate point in the sub-interval $[x_{k-1}, x_k]$.

Then the sum

$$S = \sum_{k=1}^{n} f(c_k)\Delta x$$

gives an approximate area under the curve of $y = f(x)$ from $x = a$ to $x = b$.

The *exact* area A under the curve of $y = f(x)$ is achieved by letting the partition of the interval $[a, b]$ tends to infinity:

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x.$$

3.2.2 Riemann sums

Let $f\colon [a, b] \longrightarrow \mathbb{R}$ be a continuous function, not necessarily nonnegative. Partition $[a, b]$ as in the previous section.

If $f(c_k) > 0$, the product $f(c_k)\,\Delta x$ is the area of the rectangle between the x-axis and the curve over the interval $[x_{k-1}, x_k]$. If $f(c_k) < 0$, it is the negative of that area. Thus it is the *signed area* in general.

The sum

$$S = \sum_{k=1}^{n} f(c_k)\,\Delta x$$

is called a **Riemann sum** for f on $[a, b]$.

It is the *algebraic* (or total *signed*) *area* of the rectangles.

Note that the value of S depends on the choice of the partition P and the points c_k.

As the partition becomes finer, the rectangles will approximate the region between the x-axis and f with increasing accuracy.

3.2.3 Riemann Integral

Let us continue with the notation as in the previous section. Suppose we let the number of partition in P tends to infinity.

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\,\Delta x = I.$$

We call I the **Riemann integral** (or **definite integral**) of f over $[a, b]$ and we write

$$I = \int_a^b f(x)\,dx.$$

3.2.4 Terminology

$$\int_a^b f(x)dx$$

$[a,b]$: the interval of integration

a : lower limit of integration

b : upper limit of integration

x : variable of integration

$f(x)$: the integrand

x is a *dummy* variable, i.e.

$$\int_a^b f(x)\,dx = \int_a^b f(u)\,du = \int_a^b f(t)\,dt, \quad \text{etc.}$$

3.2.5 Rules of algebra for definite integrals

1. $\displaystyle\int_a^a f(x)\,dx = 0$

2. $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3. $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx,$ (any constant k)

   $$\left(\text{In particular, } \int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx\right)$$

4. $\displaystyle\int_a^b \big[f(x) \pm g(x)\big]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5. If $f(x) \geq g(x)$ on $[a, b]$, then

   $$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

6. If $f(x) \geq 0$ on $[a, b]$, then $\displaystyle\int_a^b f(x)\,dx \geq 0$

7. If M and m are maximum and minimum values respectively of f on $[a, b]$,

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

8. If f is continuous on the interval joining a, b and c, then

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

3.2.6 Finding absolute (rather than algebraic) area

When f takes both positive and negative values on $[a, b]$, we can find its absolute area over $[a, b]$ as follows:

1. Find points where $f = 0$.

2. Use these points to partition $[a, b]$ into sub-intervals.

3. Integrate over each sub-interval.

4. Required area is the sum of the absolute values of the results found in 3.

3.2.7 Example

Find the area of the region bounded by the curve $y = x^3 - 4x$ and the x-axis on the interval $[-3, 3]$.

Solution. $x^3 - 4x = x(x^2 - 4) = x(x-2)(x+2)$.

So y is zero when $x = -2, 0$ and 2. i.e. the curve of the function intersects the x-axis at these points.

Moreover, the curve is below the x-axis on the sub-intervals $[-3, -2]$ and $[0, 2]$; and is above the x-axis on the sub-intervals $[-2, 0]$ and $[2, 3]$.

Integrating over each sub-interval:

$$\int_{-3}^{-2} x^3 - 4x\ dx = -25/4, \quad \int_{-2}^{0} x^3 - 4x\ dx = 4,$$
$$\int_{0}^{2} x^3 - 4x\ dx = -4, \quad \int_{2}^{3} x^3 - 4x\ dx = 25/4.$$

So the absolute area is

$$\left|-\frac{25}{4}\right| + 4 + |-4| + \frac{25}{4} = \frac{41}{2}.$$

3.3 The Fundamental Theorem of Calculus

3.3.1 Part 1

If f is continuous on $[a, b]$, then the function

$$F(x) = \int_a^x f(t)\, dt \tag{1}$$

has a derivative at every point of $[a, b]$, and

$$\frac{d}{dx} F(x) = \frac{d}{dx} \int_a^x f(t)\, dt = f(x). \tag{2}$$

3.3.2 Examples

$$\frac{d}{dx}\int_{-\pi}^{x}\cos t\,dt =$$

$$\frac{d}{dx}\int_{0}^{x}\frac{dt}{1+t^2} =$$

$$\frac{d}{dx}\int_{1}^{x^2}\cos t\,dt =$$

3.3.3 Part 2

If f is continuous at every point of $[a,b]$ and F is any antiderivative of f on $[a,b]$,

then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Proof. Set $G(x) = \displaystyle\int_a^x f(t)\,dt.$

By the Fundamental Theorem of Calculus, Part 1, above,

$$G'(x) = \frac{d}{dx}G(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

We also know that $F'(x) = f(x)$. Thus $G'(x) = F'(x)$ for $x \in [a,b]$.

Hence we have $F(x) = G(x) + c$ throughout $[a,b]$ for some constant c. Thus

$$\begin{aligned} F(b) - F(a) &= G(b) + c - \big(G(a) + c\big) \\ &= G(b) - G(a) \\ &= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt \\ &= \int_a^b f(t)\,dt. \end{aligned}$$

3.3.4 Examples

$$\int_0^{\pi} \cos x\,dx =$$

$$\int_0^2 t^2\,dt =$$

$$\int_{-2}^2 (4-u^2)\,du =$$

3.4 Integration by substitution

To evaluate $\displaystyle\int f\big(g(x)\big)g'(x)\,dx$ where f and g' are continuous:

1. Set $u = g(x)$. Then $g'(x) = \dfrac{du}{dx}$, the given integral becomes $\displaystyle\int f(u)\,du$.
2. Integrate with respect to u.
3. Replace u by $g(x)$ in the result of step 2.

3.4.1 Examples

$$\int (x^2 + 2x - 3)^2(x+1)\,dx =$$

$$\int \sin^4 x\,\cos x\,dx =$$

3.4.2 Substitution in definite integrals

The limits change accordingly:

$$\int_a^b f\left(g(x)\right) g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Note that in general we require $g' \geq 0$ or $g' \leq 0$ in

$[a, b]$.

3.4.3 Example

$$I = \int_0^{\pi/4} \tan x \cdot \sec^2 x\,dx =$$

3.5 Integration by parts

Integration by parts is a technique for evaluating integrals of the form

$$\int f(x)g(x)\ dx$$

in which f can be differentiated repeatedly and g can be integrated without difficulty.

Recall the product rule

$$\frac{d}{dx}(uv) = u\,\frac{dv}{dx} + v\frac{du}{dx}$$

In differential form it becomes

$$d(uv) = u\,dv + v\,du$$

or, equivalently,

$$u\,dv = d(uv) - v\,du.$$

Thus we have the **Integration-by-parts Formula**:

$$\int u\,dv = uv - \int v\,du$$

or,

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx.$$

3.5.1 Example

Evaluate $I = \displaystyle\int x\cos x\,dx.$

Solution. (To get workable u and v at first attempt it requires some familiarity of the table of differentiation/integration formulas, plus a keen observation. Needs **practice**.)

Let's look at $\displaystyle\int x\cos x\,dx$. To put it in the form $\displaystyle\int u\,dv$, we have 4 obvious choices:

1. $u = 1,\ dv = x\cos x\,dx$

2. $u = x,\ dv = \cos x\,dx$

3. $u = x\cos x,\ dv = dx$

4. $u = \cos x,\ dv = x\,dx$

The second choice works:

$$\begin{aligned} I &= \int x\cos x\,dx = \int x\,d(\sin x) \\ &= x\sin x - \int \sin x\,dx \\ &= x\sin x + \cos x + C \end{aligned}$$

3.5.2 Summary

To apply the method of integration by parts, the goal is to go from $\displaystyle\int u\,dv$ to $\displaystyle\int v\,du$, which should be **easier to handle**.

The method does not always work. (Try choices 1, 3, 4 above.)

3.5.3 Exercise

Evaluate

(a) $\displaystyle\int \ln\, x\,dx$

(b) $\displaystyle\int x^2 e^x\,dx$

(c) $\displaystyle\int_0^1 x e^x\,dx$

(d) $\displaystyle\int e^x \cos x\,dx$ (*Hint:* Consider also $\displaystyle\int e^x \sin x\,dx$.)

3.6 Area between two curves

If f_1 and f_2 are continuous functions with $f_1(x) \leq f_2(x)$ in the interval $a \leq x \leq b$, then the area of the region between the curves $y = f_1(x)$ and $y = f_2(x)$ from a to b is the integral of $f_2 - f_1$ from a to b, i.e.

$$\text{Area} = \int_a^b \big[f_2(x) - f_1(x)\big]\, dx. \tag{1}$$

This is the basic formula.

If the curves only cross at one or both end points of $[a, b]$, we apply (1) once to find the area. If the curves cross within the interval $[a, b]$, we need to apply (1) more than once. Thus, to find the area of the region between two curves

(i) Sketch the curves and determine the crossing points.

(ii) Evaluate the area(s) using (1). **Or**, integrate

$|f_2 - f_1|$ over $[a, b]$.

3.6.1 Example

Find area enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

3.6.2 Example

Find area of the region in the first quadrant bounded by $y = \sqrt{x}$ and $y = x - 2$.

3.6.3 Remark.

Sometimes we may like to view the curve as $x = g(y)$ (instead of $y = f(x)$) when evaluating area.

The area will be $A = \displaystyle\int_c^d \left[g_2(y) - g_1(y)\right] dy.$

3.6.4 Example

(Example 3.6.2 revisited)

3.7 Volume of solids of revolution

In general, solids of revolutions are solids which are generated by revolving plane regions about x- or y-axis.

3.7.1 Revolution about x-axis

The volume of a solid generated by revolving *about the* x-axis the region between the graph of a continuous function $y = f(x)$ and the x-axis from $x = a$ to $x = b$ is

$$\text{Volume} = \int_a^b \pi \left[f(x)\right]^2 dx.$$

3.7.2 Example

The region between $y = \sqrt{x}$, $0 \le x \le 4$, and the x-axis is revolved about the x-axis. Find the volume of the solid generated.

3.7.3 Example

Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$ and $x = 4$ about the line $y = 1$.

3.7.4 Revolution about y-axis

The volume of a solid generated by revolving about the y-axis the region between the graph of $x = g(y)$ and the y-axis from $y = c$ to $y = d$ is

$$\text{Volume} = \int_c^d \pi \big[g(y)\big]^2 \, dy.$$

3.7.5 Example

The region between the curve $x = \frac{2}{y}$, $1 \leq y \leq 4$ and the y-axis is revolved about the y-axis to generate a solid. Find its volume.