CHAPTER 4

Exercises

E4.1 The voltage across the circuit is given by Equation 4.8:

$$v_C(t) = V_i \exp(-t / RC)$$

in which V_i is the initial voltage. At the time $t_{1\%}$ for which the voltage reaches 1% of the initial value, we have

$$0.01 = \exp(-t_{1\%} / RC)$$

Taking the natural logarithm of both sides of the equation, we obtain

$$\ln(0.01) = -4.605 = -t_{1\%} / RC$$

Solving and substituting values, we find $t_{1\%}$ = 4.605RC = 23.03 ms.

E4.2 The exponential transient shown in Figure 4.4 is given by

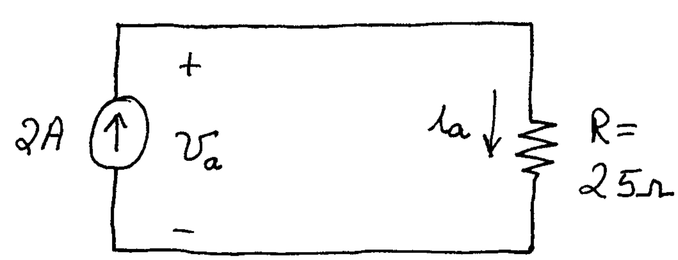
$$v_C(t) = V_s - V_s \exp(-t / \tau)$$

Taking the derivative with respect to time, we have

$$\frac{dv_C(t)}{dt} = \frac{V_s}{\tau} \exp(-t / \tau)$$

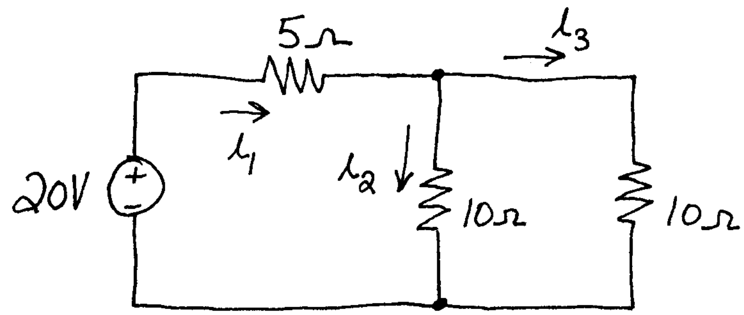
Evaluating at t = 0, we find that the initial slope is V_S / τ. Because this matches the slope of the straight line shown in Figure 4.4, we have shown that a line tangent to the exponential transient at the origin reaches the final value in one time constant.

E4.3 (a) In dc steady state, the capacitances act as open circuits and the inductances act as short circuits. Thus the steady-state (i.e., t approaching infinity) equivalent circuit is:



From this circuit, we see that $i_a = 2$ A. Then ohm's law gives the voltage as $v_a = Ri_a = 50$ V.

(b) The dc steady-state equivalent circuit is:



Here the two 10-Ω resistances are in parallel with an equivalent resistance of 1/(1/10 + 1/10) = 5 Ω. This equivalent resistance is in series with the 5-Ω resistance. Thus the equivalent resistance seen by the source is 10 Ω, and $i_1 = 20/10 = 2\ A$. Using the current division principle, this current splits equally between the two 10-Ω resistances, so we have $i_2 = i_3 = 1\ A$.

E4.4 (a) $\tau = L/R_2 = 0.1/100 = 1\ \text{ms}$

(b) Just before the switch opens, the circuit is in dc steady state with an inductor current of $V_s/R_1 = 1.5\ A$. This current continues to flow in the inductor immediately after the switch opens so we have $i(0+) = 1.5\ A$. This current must flow (upward) through R_2 so the initial value of the voltage is $v(0+) = -R_2 i(0+) = -150\ \text{V}$.

(c) We see that the initial magnitude of $v(t)$ is ten times larger than the source voltage.

(d) The voltage is given by

$$v(t) = -\frac{V_s L}{R_1 \tau}\exp(-t/\tau) = -150\exp(-1000t)$$

Let us denote the time at which the voltage reaches half of its initial magnitude as t_H. Then we have

$$0.5 = \exp(-1000 t_H)$$

Solving and substituting values we obtain

$$t_H = -10^{-3}\ln(0.5) = 10^{-3}\ln(2) = 0.6931\ \text{ms}$$

E4.5 First we write a KCL equation for $t \geq 0$.

$$\frac{v(t)}{R} + \frac{1}{L}\int_0^t v(x)dx + 0 = 2$$

Taking the derivative of each term of this equation with respect to time and multiplying each term by R, we obtain:

$$\frac{dv(t)}{dt} + \frac{R}{L}v(t) = 0$$

The solution to this equation is of the form:

$$v(t) = K\exp(-t/\tau)$$

in which $\tau = L/R = 0.2\text{ s}$ is the time constant and K is a constant that must be chosen to fit the initial conditions in the circuit. Since the initial (t = 0+) inductor current is specified to be zero, the initial current in the resistor must be 2 A and the initial voltage is 20 V:

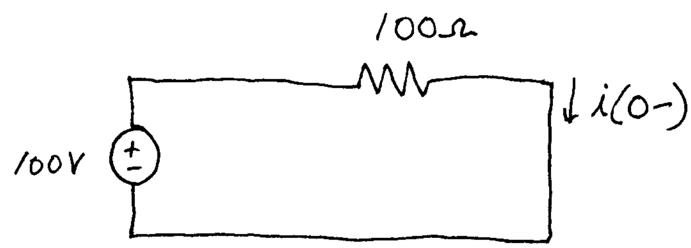
$$v(0+) = 20 = K$$

Thus, we have

$$v(t) = 20\exp(-t/\tau) \quad i_R = v/R = 2\exp(-t/\tau)$$

$$i_L(t) = \frac{1}{L}\int_0^t v(x)dx = \frac{1}{2}\left[-20\tau\exp(-x/\tau)\right]_0^t = 2 - 2\exp(-t/\tau)$$

E4.6 Prior to t = 0, the circuit is in DC steady state and the equivalent circuit is



Thus we have i(0-) = 1 A. However the current through the inductor cannot change instantaneously so we also have i(0+) = 1 A. With the switch open, we can write the KVL equation:

$$\frac{di(t)}{dt} + 200i(t) = 100$$

The solution to this equation is of the form

$$i(t) = K_1 + K_2\exp(-t/\tau)$$

in which the time constant is $\tau = 1/200 = 5\text{ ms}$. In steady state with the switch open, we have $i(\infty) = K_1 = 100/200 = 0.5\text{ A}$. Then using the initial current, we have $i(0+) = 1 = K_1 + K_2$, from which we determine that $K_2 = 0.5$. Thus we have

$$\begin{aligned} i(t) &= 1.0 \ \text{ A for } t < 0 \\ &= 0.5 + 0.5\exp(-t/\tau) \text{ for t} > 0. \end{aligned}$$

$$\begin{aligned} v(t) &= L\frac{di(t)}{dt} \\ &= 0 \ \text{ V for } t < 0 \\ &= -100\exp(-t/\tau) \text{ for t} > 0. \end{aligned}$$

E4.7 As in Example 4.4, the KVL equation is

$$Ri(t) + \frac{1}{C}\int_0^t i(x)dx + v_C(0+) - 2\cos(200t) = 0$$

Taking the derivative and multiplying by C, we obtain

$$RC\frac{di(t)}{dt} + i(t) + 400C\sin(200t) = 0$$

Substituting values and rearranging the equation becomes

$$5 \times 10^{-3}\frac{di(t)}{dt} + i(t) = -400 \times 10^{-6}\sin(200t)$$

The particular solution is of the form

$$i_p(t) = A\cos(200t) + B\sin(200t)$$

Substituting this into the differential equation and rearranging terms results in

$$5 \times 10^{-3}\left[-200A\sin(200t) + 200B\cos(200t)\right] + A\cos(200t) + B\sin(200t)$$
$$= -400 \times 10^{-6}\sin(200t)$$

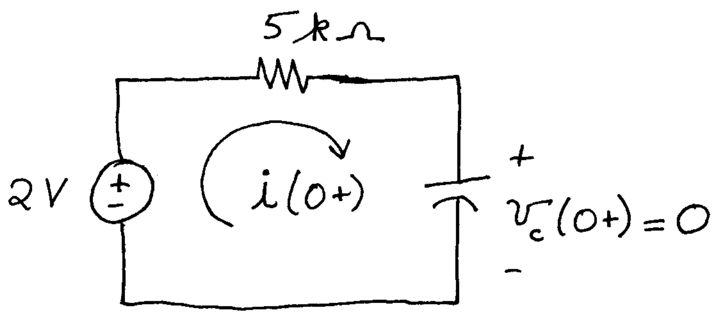
Equating the coefficients of the cos and sin terms gives the following equations:

$$-A + B = -400 \times 10^{-6} \quad \text{and } B + A = 0$$

from which we determine that $A = 200 \times 10^{-6}$ and $B = -200 \times 10^{-6}$. Furthermore, the complementary solution is $i_C(t) = K\exp(-t/\tau)$, and the complete solution is of the form

$$i(t) = 200\cos(200t) - 200\sin(200t) + K\exp(-t/\tau) \ \ \mu A$$

At t = 0+, the equivalent circuit is



from which we determine that $i(0+) = 2/5000 = 400\ \mu A$. Then evaluating our solution at t = 0+, we have $i(0+) = 400 = 200 + K$, from which we determine that $K = 200$ μA. Thus the complete solution is

$$i(t) = 200\cos(200t) - 200\sin(200t) + 200\exp(-t/\tau)\ \ \mu A$$

E4.8 The KVL equation is

$$Ri(t) + \frac{1}{C}\int_0^t i(x)dx + v_C(0+) - 10\exp(-t) = 0$$

Taking the derivative and multiplying by C, we obtain

$$RC\frac{di(t)}{dt} + i(t) + 10C\exp(-t) = 0$$

Substituting values and rearranging, the equation becomes

$$2\frac{di(t)}{dt} + i(t) = -20\times 10^{-6}\exp(-t)$$

The particular solution is of the form

$$i_p(t) = A\exp(-t)$$

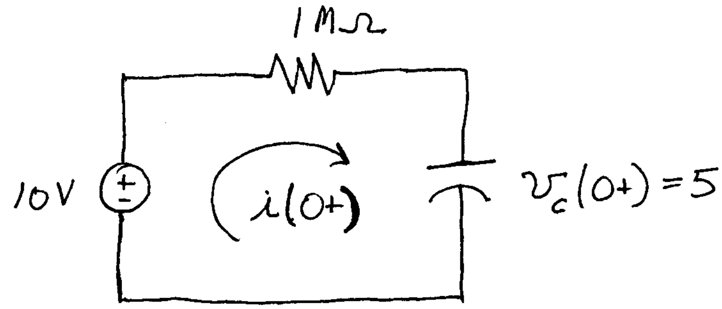
Substituting this into the differential equation and rearranging terms results in

$$-2A\exp(-t) + A\exp(-t) = -20\times 10^{-6}\exp(-t)$$

Equating the coefficients gives $A = 20\times 10^{-6}$. Furthermore, the complementary solution is $i_C(t) = K\exp(-t/2)$, and the complete solution is of the form

$$i(t) = 20\exp(-t) + K\exp(-t/2)\ \ \mu A$$

At t = 0+, the equivalent circuit is



from which we determine that $i(0+) = 5/10^6 = 5\ \mu A$. Then evaluating our solution at t = 0+, we have $i(0+) = 5 = 20 + K$, from which we determine that $K = -15\ \mu A$. Thus the complete solution is

$$i(t) = 20\exp(-t) - 15\exp(-t/2)\ \ \mu A$$

E4.9 (a) $\omega_0 = \dfrac{1}{\sqrt{LC}} = \dfrac{1}{\sqrt{10^{-3} \times 10^{-7}}} = 10^5 \quad \alpha = \dfrac{1}{2RC} = 2 \times 10^5 \quad \zeta = \dfrac{\alpha}{\omega_0} = 2$

(b) At t = 0+, the KCL equation for the circuit is

$$0.1 = \frac{v(0+)}{R} + i_L(0+) + Cv'(0+) \qquad (1)$$

However, $v(0+) = v(0-) = 0$, because the voltage across the capacitor cannot change instantaneously. Furthermore, $i_L(0+) = i_L(0-) = 0$, because the current through the inductance cannot change value instantaneously. Solving Equation (1) for $v'(0+)$ and substituting values, we find that $v'(0+) = 10^6$ V/s.

(c) To find the particular solution or forced response, we can solve the circuit in steady-state conditions. For a dc source, we treat the capacitance as an open and the inductance as a short. Because the inductance acts as a short $v_p(t) = 0$.

(d) Because the circuit is overdamped ($\zeta > 1$), the homogeneous solution is the sum of two exponentials. The roots of the characteristic solution are given by Equations 4.72 and 4.73:

$$s_1 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -373.2 \times 10^3$$

$$s_2 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -26.79 \times 10^3$$

Adding the particular solution to the homogeneous solution gives the general solution:

$$v(t) = K_1 \exp(s_1 t) + K_2 \exp(s_2 t)$$

Now using the initial conditions, we have

$$v(0+) = 0 = K_1 + K_2 \qquad v'(0+) = 10^6 = K_1 s_1 + K_2 s_2$$

Solving we find $K_1 = -2.887$ and $K_2 = 2.887$. Thus the solution is:

$$v(t) = 2.887[\exp(s_2 t) - \exp(s_1 t)]$$

E4.10 (a) $\omega_0 = \dfrac{1}{\sqrt{LC}} = \dfrac{1}{\sqrt{10^{-3} \times 10^{-7}}} = 10^5 \quad \alpha = \dfrac{1}{2RC} = 10^5 \quad \zeta = \dfrac{\alpha}{\omega_0} = 1$

(b) The solution for this part is the same as that for Exercise 4.9b in which we found that $v'(0+) = 10^6$ V/s.

(c) The solution for this part is the same as that for Exercise 4.9c in which we found $v_p(t) = 0$.

(d) The roots of the characteristic solution are given by Equations 4.72 and 4.73:

$$s_1 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -10^5 \qquad s_2 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -10^5$$

Because the circuit is critically damped $(\zeta = 1)$, the roots are equal and the homogeneous solution is of the form:

$$v(t) = K_1 \exp(s_1 t) + K_2 t \exp(s_1 t)$$

Adding the particular solution to the homogeneous solution gives the general solution:

$$v(t) = K_1 \exp(s_1 t) + K_2 t \exp(s_1 t)$$

Now using the initial conditions we have

$$v(0+) = 0 = K_1 \qquad v'(0+) = 10^6 = K_1 s_1 + K_2$$

Solving we find $K_2 = 10^6$ Thus the solution is:

$$v(t) = 10^6 t \exp(-10^5 t)$$

E4.11 (a) $\omega_0 = \dfrac{1}{\sqrt{LC}} = \dfrac{1}{\sqrt{10^{-3} \times 10^{-7}}} = 10^5 \quad \alpha = \dfrac{1}{2RC} = 2 \times 10^4 \quad \zeta = \dfrac{\alpha}{\omega_0} = 0.2$

(b) The solution for this part is the same as that for Exercise 4.9b in which we found that $v'(0+) = 10^6$ V/s.

(c) The solution for this part is the same as that for Exercise 4.9c in which we found $v_p(t) = 0$.

(d) Because we have $(\zeta < 1)$, this is the underdamped case and we have

$$\omega_n = \sqrt{\omega_0^2 - \alpha^2} = 97.98 \times 10^3$$

Adding the particular solution to the homogeneous solution gives the general solution:

$$v(t) = K_1 \exp(-\alpha t)\cos(\omega_n t) + K_2 \exp(-\alpha t)\sin(\omega_n t)$$

Now using the initial conditions we have

$$v(0+) = 0 = K_1 \qquad v'(0+) = 10^6 = -\alpha K_1 + \omega_n K_2$$

Solving we find $K_2 = 10.21$ Thus the solution is:

$$v(t) = 10.21\exp(-2\times 10^4 t)\sin(97.98\times 10^3 t)\ \mathrm{V}$$

E4.12 The commands are:

```
syms ix t R C vCinitial w
ix = dsolve('(R*C)*Dix + ix = (w*C)*2*cos(w*t)', 'ix(0)=-vCinitial/R');
ians =subs(ix,[R C vCinitial w],[5000 1e-6 1 200]);
pretty(vpa(ians, 4))
ezplot(ians,[0 80e-3])
```

An m-file named Exercise_4_12 containing these commands can be found in the MATLAB folder on the OrCAD disk.

E4.13 The commands are:

```
syms vc t
% Case I R = 300:
vc = dsolve('(1e-8)*D2vc + (1e-6)*300*Dvc+ vc =10', ...
            'vc(0) = 0','Dvc(0)=0');
vpa(vc,4)
ezplot(vc, [0 1e-3])
hold on % Turn hold on so all plots are on the same axes
% Case II R = 200:
vc = dsolve('(1e-8)*D2vc + (1e-6)*200*Dvc+ vc =10',...
            'vc(0) = 0','Dvc(0)=0');
vpa(vc,4)
ezplot(vc, [0 1e-3])
% Case III R = 100:
vc = dsolve('(1e-8)*D2vc + (1e-6)*100*Dvc+ vc =10',...
            'vc(0) = 0','Dvc(0)=0');
vpa(vc,4)
```

ezplot(vc, [0 1e-3])

An m-file named Exercise_4_13 containing these commands can be found in the MATLAB folder on the OrCAD disk.

Answers for Selected Problems

P4.2* The leakage resistance must be greater than 11.39 MΩ.

P4.3* $v_C(t) = 10 - 20\exp\left(-t/(2\times 10^{-3})\right)$ V $\qquad t_0 = 2\ln(2) = 1.386$ ms

P4.4* $t_2 = 0.03466$ seconds

P4.5* $R = 4.328\ \mathrm{M\Omega}$

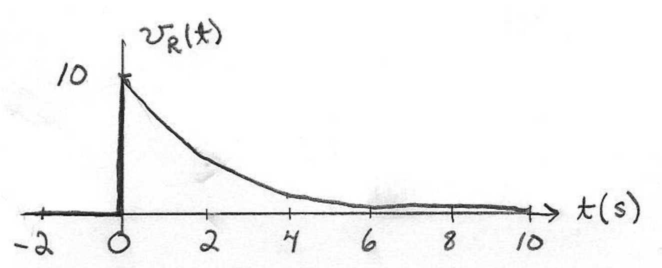
P4.6* $v(t) = V_1 \exp[-(t - t_1)/RC]$ for $t \ge t_1$

P4.21* $i_1 = 0 \qquad i_3 = i_2 = 2$ A

P4.22* $v_{C,\text{steady state}} = 10$ V $\qquad t_{99} = 46.05$ ms

P4.23*

$$v_R(t) = 0 \quad t < 0$$
$$= 10\exp(-t/\tau) \quad \text{V} \quad \text{for } t \ge 0$$



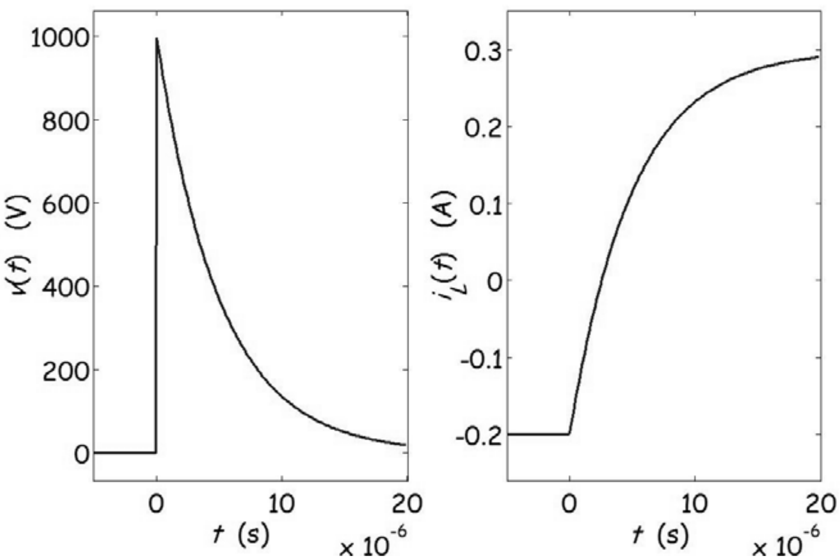
P4.33*

$$i(t) = 0 \qquad \text{for } t < 0$$
$$= 1 - \exp(-20t)\ \text{A} \quad \text{for } t \ge 0$$

P4.34* $i_L(t) = 0.3 - 0.5\exp(-2\times10^5 t)$ A for $t > 0$

$v(t) = 0$ for $t < 0$

$= 1000\exp(-2\times10^5 t)$ A for $t > 0$



P4.36* $R \le 399.6\ \mu\Omega$

P4.45* $i_L(t) = -\exp(-t) + \exp(-Rt/L)$ for $t \ge 0$

P4.46* $v_C(t) = 10^6\exp(-t) - 10^6\exp(-3t)$ $t > 0$

P4.47* $v(t) = 25\exp(-t/\tau) + 25\cos(10t) - 25\sin(10t)$ $t \ge 0$

P4.61* $s_1 = -0.2679\times10^4$

$s_2 = -3.732\times10^4$

$v_C(t) = 50 - 53.87\exp(s_1 t) + 3.867\exp(s_2 t)$

P4.62* $s_1 = -10^4$

$v_C(t) = 50 - 50\exp(s_1 t) - (50\times10^4)t\exp(s_1 t)$

P4.63* $\alpha = 0.5\times10^4$

$\omega_n = 8.660\times10^3$

$$v_C(t) = 50 - 50\exp(-\alpha t)\cos(\omega_n t) - (28.86)\exp(-\alpha t)\sin(\omega_n t)$$

Practice Test

T4.1 (a) Prior to the switch opening, the circuit is operating in DC steady state, so the inductor acts as a short circuit, and the capacitor acts as an open circuit.

$$i_1(0-) = 10/1000 = 10 \text{ mA} \qquad i_2(0-) = 10/2000 = 5 \text{ mA}$$

$$i_3(0-) = 0 \qquad i_L(0-) = i_1(0-) + i_2(0-) + i_3(0-) = 15 \text{ mA}$$

$$v_C(0-) = 10 \text{ V}$$

(b) Because infinite voltage or infinite current are not possible in this circuit, the current in the inductor and the voltage across the capacitor cannot change instantaneously. Thus, we have $i_L(0+) = i_L(0-) = 15$ mA and $v_C(0+) = v_C(0-) = 10$ V. Also, we have $i_1(0+) = i_L(0+) = 15$ mA, $i_2(0+) = v_C(0+)/5000 = 2$ mA, and $i_3(0+) = -i_2(0+) = -2$ mA.

(c) The current is of the form $i_L(t) = A + B\exp(-t/\tau)$. Because the inductor acts as a short circuit in steady state, we have

$$i_L(\infty) = A = 10/1000 = 10 \text{ mA}$$

At $t = 0+$, we have $i_L(0+) = A + B = 15$ mA, so we find $B = 5$ mA.
The time constant is $\tau = L/R = 2\times10^{-3}/1000 = 2\times10^{-6}$ s.
Thus, we have $i_L(t) = 10 + 5\exp(-5\times10^5 t)$ mA.

(d) This is a case of an initially charged capacitance discharging through a resistance. The time constant is $\tau = RC = 5000\times10^{-6} = 5\times10^{-3}$ s. Thus we have $v_C(t) = V_i\exp(-t/\tau) = 10\exp(-200t)$ V.

T4.2 (a) $2\dfrac{di(t)}{dt} + i(t) = 5\exp(-3t)$

(b) The time constant is $\tau = L/R = 2$ s and the complementary solution is of the form $i_c(t) = A\exp(-0.5t)$.

(c) The particular solution is of the form $i_p(t) = K\exp(-3t)$. Substituting into the differential equation produces

$$-6K\exp(-3t) + K\exp(-3t) \equiv 5\exp(-3t)$$

from which we have $K = -1$.

(d) Adding the particular solution and the complementary solution, we have

$$i(t) = A\exp(-0.5t) - \exp(-3t)$$

However, the current must be zero in the inductor prior to $t = 0$ because of the open switch, and the current cannot change instantaneously in this circuit, so we have $i(0+) = 0$. This yields $A = 1$. Thus, the solution is

$$i(t) = \exp(-0.5t) - \exp(-3t) \text{ A}$$

T4.3 (a) Applying KVL to the circuit, we obtain

$$L\frac{di(t)}{dt} + Ri(t) + v_C(t) = 15 \qquad (1)$$

For the capacitance, we have

$$i(t) = C\frac{dv_C(t)}{dt} \qquad (2)$$

Using Equation (2) to substitute into Equation (1) and rearranging, we have

$$\frac{d^2v_C(t)}{dt^2} + (R/L)\frac{dv_C(t)}{dt} + (1/LC)v_C(t) = 15/LC \qquad (3)$$

$$\frac{d^2v_C(t)}{dt^2} + 2000\frac{dv_C(t)}{dt} + 25\times10^6 v_C(t) = 375\times10^6$$

(b) We try a particular solution of the form $v_{Cp}(t) = A$, resulting in $A = 15$. Thus, $v_{Cp}(t) = 15$. (An alternative method to find the particular solution is to solve the circuit in dc steady state. Since the capacitance acts as an open circuit, the steady-state voltage across it is 15 V.)

(c) We have

$$\omega_0 = \frac{1}{\sqrt{LC}} = 5000 \text{ and } \alpha = \frac{R}{2L} = 1000$$

Since we have $\alpha < \omega_0$, this is the underdamped case. The natural frequency is given by:

$$\omega_n = \sqrt{\omega_0^2 - \alpha^2} = 4899$$

The complementary solution is given by:

$$v_{Cc}(t) = K_1\exp(-1000t)\cos(4899t) + K_2\exp(-1000t)\sin(4899t)$$

(d) The complete solution is

$$v_C(t) = 15 + K_1 \exp(-\alpha t)\cos(\omega_n t) + K_2 \exp(-\alpha t)\sin(\omega_n t)$$

The initial conditions are

$$v_C(0) = 0 \quad \text{and} \quad i(0) = 0 = C\frac{dv_C(t)}{dt}\Big|_{t=0}$$

Thus, we have

$$v_C(0) = 0 = 15 + K_1$$

$$\frac{dv_C(t)}{dt}\Big|_{t=0} = 0 = -\alpha K_1 + \omega_n K_2$$

Solving, we find $K_1 = -15$ and $K_2 = -3.062$. Finally, the solution is

$$v_C(t) = 15 - 15\exp(-1000t)\cos(4899t) - (3.062)\exp(-1000t)\sin(4899t) \text{ V}$$

T4.4 One set of commands is

```
syms vC t
S = dsolve('D2vC + 2000*DvC + (25e6)*vC = 375e6',...
            'vC(0) = 0, DvC(0) = 0');
simple(vpa(S,4))
```

These commands are stored in the m-file named T_4_4 on the OrCAD disk.