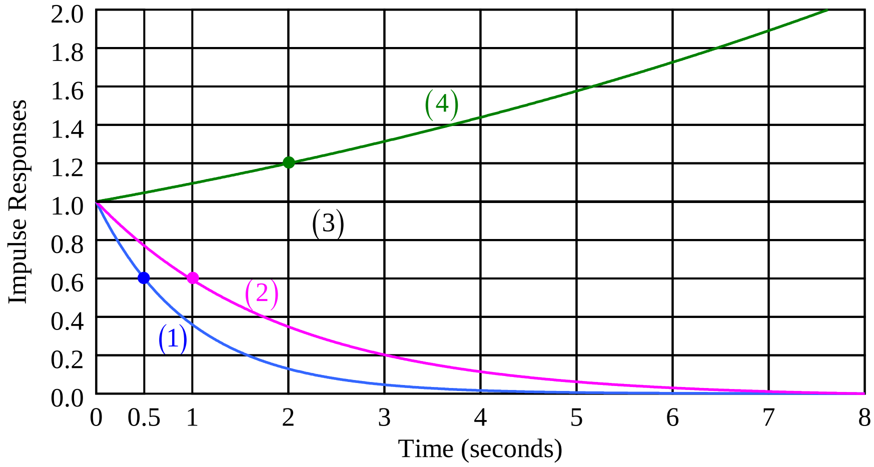
EE2023 TUTORIAL 7 (SOLUTIONS)

Solution to Q.1



(a) Denote impulse response of system i by $y_{\delta,i}(t)$ and step response by $y_{step,i}(t)$ and note the relationship

$$y_{step,i}(t) = \int_{0^-}^{t} y_{\delta,i}(\upsilon)\,d\upsilon .$$

Hence, given the plot for $y_{\delta,i}(t)$, the corresponding step responses, $y_{step,i}(t)$, can be obtained by one of the following two ways:

- Performing graphical integration i.e. summing the area under $y_{\delta,i}(t)$ from 0 to t

or

- Assume first-order dynamics $\boldsymbol{y_{\delta,i} = A_i \exp(-\alpha_i t)u(t)}$ recognizing that $A_i = 1$ for all 4 systems.

$\underbrace{y_{\delta,1}(t) = e^{-\alpha_1 t}u(t)}_{\text{decaying exponential}}$ Point $(0.5, 0.6)$ lies on $y_{\delta,1}(t)$, $\therefore\ 0.6 = e^{-0.5\alpha_1}$ or $\alpha_1 \simeq 1.022$

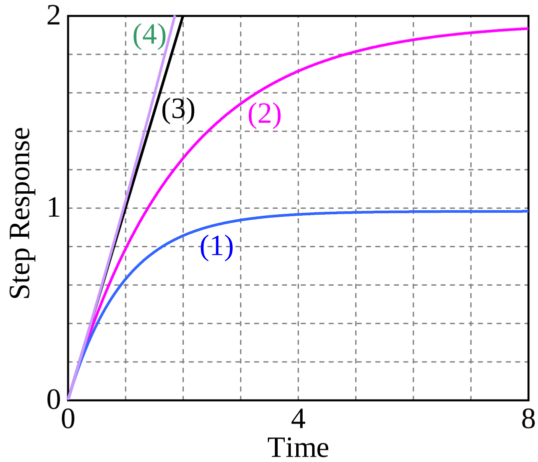
$\underbrace{y_{\delta,2}(t) = e^{-\alpha_2 t}u(t)}_{\text{decaying exponential}}$ Point $(1.0, 0.6)$ lies on $y_{\delta,2}(t)$, $\therefore\ 0.6 = e^{-1.0\alpha_2}$ or $\alpha_2 \simeq 0.511$

$\underbrace{y_{\delta,3}(t) = u(t)}_{\text{unit step}}$ $\alpha_3 = 0$

$\underbrace{y_{\delta,4}(t) = e^{-\alpha_4 t}u(t)}_{\text{growing exponential}}$ Point $(2.0, 1.2)$ lies on $y_{\delta,4}(t)$, $\therefore\ 1.2 = e^{-2.0\alpha_4}$ or $\alpha_4 \simeq -0.091$

and perform integration to get

$$y_{step,1}(t) = \frac{1}{1.022}\left[1 - e^{-1.022t}\right]u(t)$$
$$y_{step,2}(t) = \frac{1}{0.511}\left[1 - e^{-0.511t}\right]u(t)$$
$$y_{step,3}(t) = tu(t)$$
$$y_{step,4}(t) = \frac{1}{0.091}\left[e^{0.091t} - 1\right]u(t)$$

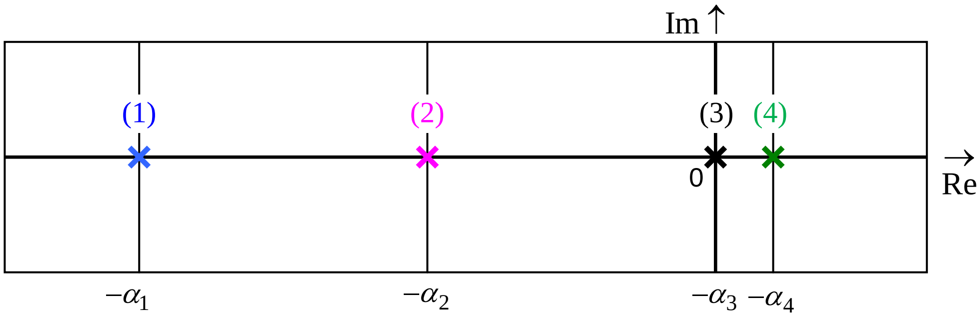


(b) The Laplace transform of $\boldsymbol{y_{\delta,i} = \exp(-\alpha_i t)u(t)}$ is $\boldsymbol{\dfrac{1}{s+\alpha_i}}$ where the pole is located at $\boldsymbol{-\alpha_i}$. This implies that:

- If $\boldsymbol{\alpha_i > 0}$ then the system is stable and a system with a larger α_i will have its pole further to the left of the $j\omega$ axis.
- If $\boldsymbol{\alpha_i = 0}$ then the system is marginally stable and will have its pole on the $j\omega$ axis.
- If $\boldsymbol{\alpha_i < 0}$ then the system is unstable and a system with a smaller α_i will have its pole further to the right of the $j\omega$ axis.

From the given impulse response curves, we make the following observations:

- The impulse responses of Systems 1 and 2 are exponentially decaying which implies that α_1 and α_2 are positive. These two systems are therefore stable with their poles to the left of the $j\omega$ axis. Since $y_{\delta,1}(t)$ decays faster than $y_{\delta,2}(t)$, we conclude that $\alpha_1 > \alpha_2$ and the pole of System 1 is to the left of the pole of System 2.
- The impulse response of System 3 is a unit step which implies that $\alpha_3 = 0$ and thus will have its pole on the $j\omega$ axis.
- The impulse response of System 4 is exponentially growing which implies that α_4 is negative and thus will have its pole to the right of the $j\omega$ axis.



Solution to Q.2

Objective is to find the parameters of the transfer function $G_i(s)=\dfrac{K}{as^2+bs+c}\exp(-sL)$, $(i=1,2,3,4)$, using information about the unit impulse responses and unit step responses.

Concepts needed to formulate solutions: $\begin{cases} G(s) = \text{Laplace transform of the impulse response} \\ \dfrac{G(s)}{s} = \text{Laplace transform of the step response} \end{cases}$

Process 1: Impulse response, $y_{\delta,1}(t)=1.5u(t-1)$

$\therefore G_1(s)=\dfrac{1.5}{s}\exp(-s)$

Compare $G_1(s)$ with $\dfrac{K}{as^2+bs+c}\exp(-sL)$, we have $K=1.5,\ a=0,\ b=1,\ c=0,\ L=1$.

This is an integrator with dead-time $L=1$.

Process 2: Impulse response, $y_{\delta,2}(t)=4(t-0.5)u(t-0.5)$

$\therefore G_2(s)=\dfrac{4}{s^2}\exp(-0.5s)$

Compare $G_2(s)$ with $\dfrac{K}{as^2+bs+c}\exp(-sL)$, we have $K=4,\ a=1,\ b=0,\ c=0,\ L=0.5$.

This is cascade of two integrators with dead-time $L=0.5$.

Process 3: Step response, $y_{step,3}(t)=4(t-0.5)u(t-0.5)$

$\therefore \dfrac{G_3(s)}{s}=\dfrac{4}{s^2}\exp(-0.5s) \ \rightarrow\ G_3(s)=\dfrac{4}{s}\exp(-0.5s)$

Compare $G_3(s)$ with $\dfrac{K}{as^2+bs+c}\exp(-sL)$, we have $K=4,\ a=0,\ b=1,\ c=0,\ L=0.5$.

This is an integrator with dead-time $L=0.5$.

Process 4: As the step response provided is a curve, it is not easy to derive a mathematical equation. A simpler approach is to match the step response in the problem with the step responses of common systems found in the lecture notes. Clearly, this is a stable first-order system with a dead-time.

A stable first order system is generally characterized by:

Transfer function: ······················ $\boldsymbol{G_4(s)=\dfrac{K_4}{sT_4+1}}$

Impulse response: ······ $\boldsymbol{y_{\delta,4}(t)=\dfrac{K_4}{T_4}\exp\left(-\dfrac{t}{T_4}\right)u(t)}$

Step response: ··· $\boldsymbol{y_{step,4}(t)=K_4\left[1-\exp\left(-\dfrac{t}{T_4}\right)\right]u(t)}$

A stable first order system with dead-time $\boldsymbol{L_4}$ is thus characterized by:

Transfer function: ···················· $\boldsymbol{G_4(s)=\frac{K_4}{sT_4+1}\exp(-sL_4)}$

Impulse response: ······ $\boldsymbol{y_{\delta,4}(t)=\frac{K_4}{T_4}\exp\left(-\frac{t-L_4}{T_4}\right)u(t-L_4)}$

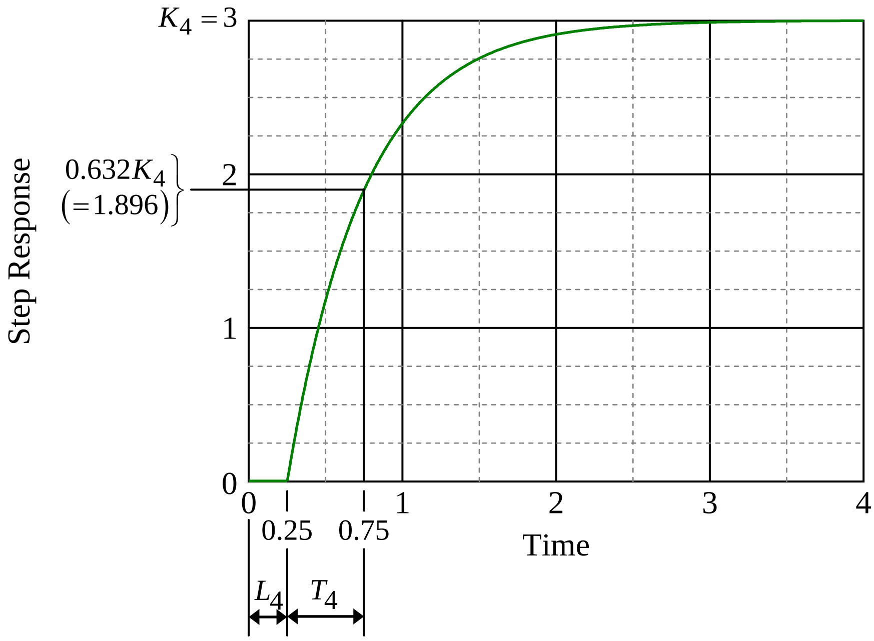
Step response: ··· $\boldsymbol{y_{step,4}(t)=K_4\left[1-\exp\left(-\frac{t-L_4}{T_4}\right)\right]u(t-L_4)}$ ······ (♠)

Comparing (♠) and the given step response graph of $G_4(s)$:

- Since the system is stable and input is a unit step function, the steady-state response of the system given by $\lim_{t\to\infty} y_{step,4}(t)=3$ is equivalent to the system steady-state gain and also the system DC gain. According to $G_4(s)$, the system DC gain is $G(0)=K_4$. Therefore $K_4=3$.

- Dead-time $L_4=0.25$ because the output signal starts to change 0.25 time units after the input is applied.

- Time constant T_4 is the time taken (after the dead-time) for the system to reach 63.2% of the final output value . This occurs when $y_{step,4}(T_4+L_4)=K_4[1-\exp(-1)]=0.632K_4$. From the graph, $T_4+L_4=0.75$ or $T_4=0.5$.



$\therefore\ G_4(s)=\frac{3}{0.5s+1}\exp(-0.25s)$

Compare $G_4(s)$ with $\frac{K}{as^2+bs+c}\exp(-sL)$, we have $K=3,\ a=0,\ b=0.5,\ c=1,\ L=0.25$.

Solution to Q.3

(a) System unit-step response: $y_{step}(t) = 1 - 0.49\exp(-15.1t) - 0.51\cos(1.31t) - 0.97\sin(1.31t)$

Taking Laplace transform on both sides of the equation:

$$\frac{G(s)}{s} = Y_{step}(s) = \frac{1}{s} - 0.49\frac{1}{s+15.1} - 0.51\frac{s}{s^2+1.31^2} - 0.97\frac{1.31}{s^2+1.31^2} = \frac{N(s)}{s(s+15.1)\left(s^2+1.31^2\right)}$$

System transfer function: $G(s) = \dfrac{N(s)}{(s+15.1)\left(s^2+1.31^2\right)}$

System poles are located at: $\boldsymbol{s = -15.1,\ \pm j1.31}$

(b) System transfer function: $G(s) = \dfrac{s^2 - 3s + 4.25}{s^3 + (9+K)s^2 + (20-3K)s + 4.25K}$

System poles are roots of $s^3 + (9+K)s^2 + (20-3K)s + 4.25K = 0$.

Two methods for finding K:

(i) Rearranging $s^3 + (9+K)s^2 + (20-3K)s + 4.25K = 0$ into

$$K = \frac{-s^3 - 9s^2 - 20s}{s^2 - 3s + 4.25}$$

and substituting either $s = -15.1$ or $s = j1.31$ or $s = -j1.31$ into it to get $\boldsymbol{K = 6.1}$

(ii) Expanding $(s+15.1)\left(s^2+1.31^2\right)$ into

$$s^3 + 15.1s^2 + 1.7161s + 25.913$$

and comparing coefficients with

$$s^3 + (9+K)s^2 + (20-3K)s + 4.25K$$

to get $\boldsymbol{K = 6.1}$.

Solution to Q.4

$$\begin{pmatrix}\text{Steady-state response of a stable}\\ \text{system due to a unit step input}\end{pmatrix} = (\text{Steady-state GAIN}) = (\text{DC GAIN})$$

The DC gain of $G(s)=\dfrac{K}{\tau s+1}$ is given by $G(0)=K$.

Since it is given that the system input is a unit step and has unity steady-state gain, we conclude from the above that $K=1$.

(a)

$$\left.\begin{aligned}
G(s)&=\frac{1}{\tau s+1}\\
Y_{step}(s)&=\frac{1}{s}\cdot\frac{1}{\tau s+1}=\frac{1}{s}-\frac{\tau}{\tau s+1}\\
y_{step}(t)&=1-\exp\left(-\frac{t}{\tau}\right)\\
\frac{dy_{step}(t)}{dt}&=\frac{1}{\tau}\exp\left(-\frac{t}{\tau}\right)\\
\left.\frac{dy_{step}(t)}{dt}\right|_{t=0}&=\frac{1}{\tau}=\underbrace{0.025}_{\text{Given}}
\end{aligned}\right\}\quad \therefore\ \boldsymbol{\tau}=\mathbf{40}$$

(b) The steady-state unit step response is 1. Hence, the time taken for the thermometer to indicate 99% of the steady-state value can be obtained by solving

$$y_{step}(t)=1-\exp\left(-\frac{t}{40}\right)=0.99$$

$$\rightarrow\ \boldsymbol{t}=40\ln(100)=\mathbf{184.2}$$