Chapter 2
Differentiation

Outline

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  - Definitions
  - Rules of Differentiation

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  - Implicit Differentiation
  - Higher Order Derivatives

- Maxima and Minima
  - Local and absolute extremes
  - Finding Extreme Values
  - Critical Points
  - Increasing and Decreasing Functions

- Derivative Test
  - First Derivative Test for Local Extremes
  - Concavity and Points of Inflection
  - Second Derivative Test for Local Extremes

- Optimization Problems
  - Absolute Extreme Values

- <u>Indeterminate Form</u> (Limits)
  - L'Hospital's Rule
  - Other Indeterminate Forms

Derivative

Derivative

$y = x^n$	$f(x) = x^n$
$\frac{dy}{dx} = nx^{n-1}$	$f'(x) = nx^{n-1}$
The derivative of y with respect to x.	
The derivative of $f(x)$ with respect to x.	

with respect to --- w.r.t

Some results

$\frac{d}{dx}(c) = 0$, where c is any constant.

$\frac{d}{dx}(x^n) = nx^{n-1}$, where n is any constant.

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad \frac{d}{dx}\left(e^x\right) = e^x$$

Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

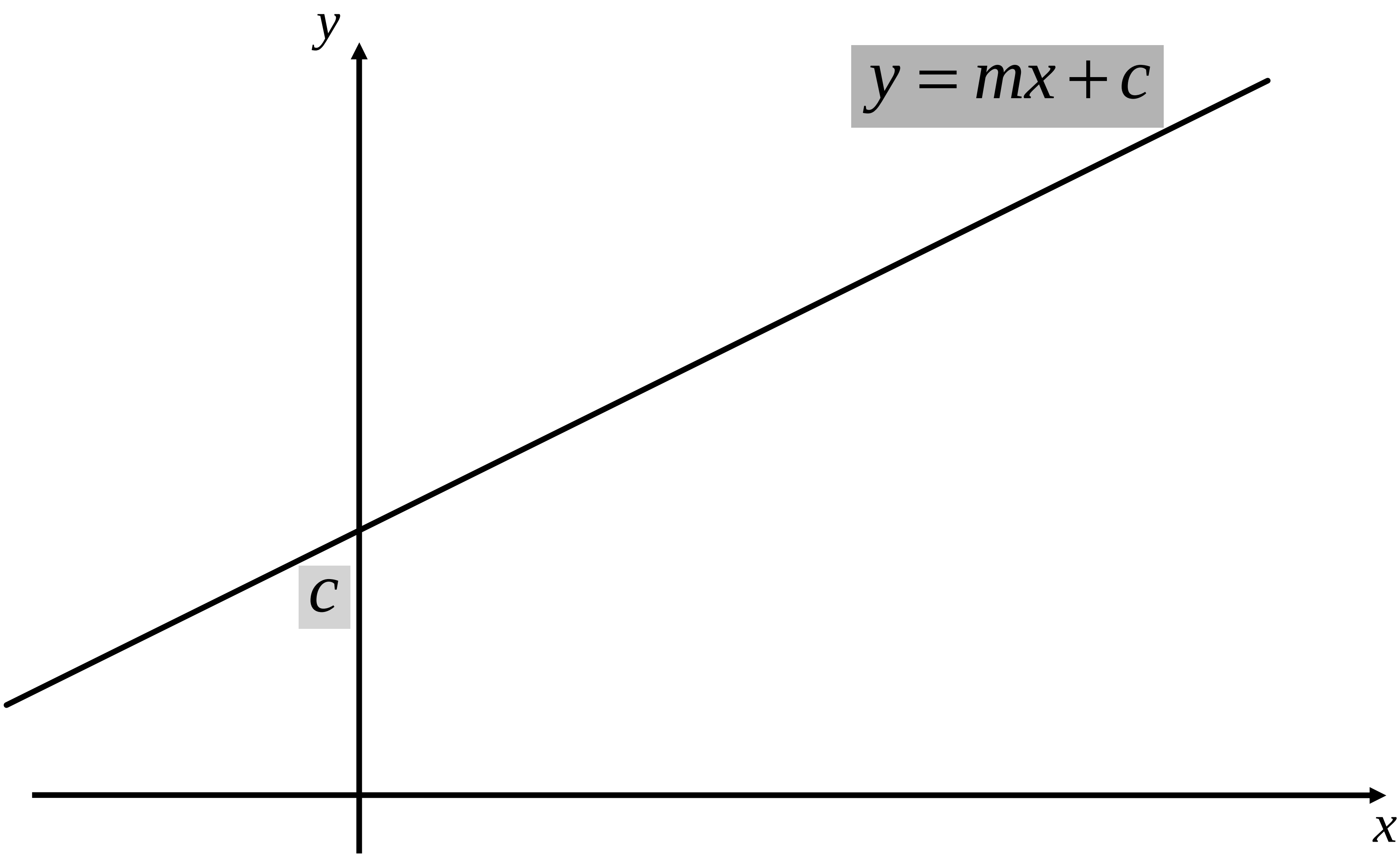
Question: How to derive these results? Using limits

1. Derivative --- using the concept of limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

2. Derivative --- **(Geometrically) Gradient (Slope) of the tangent line**

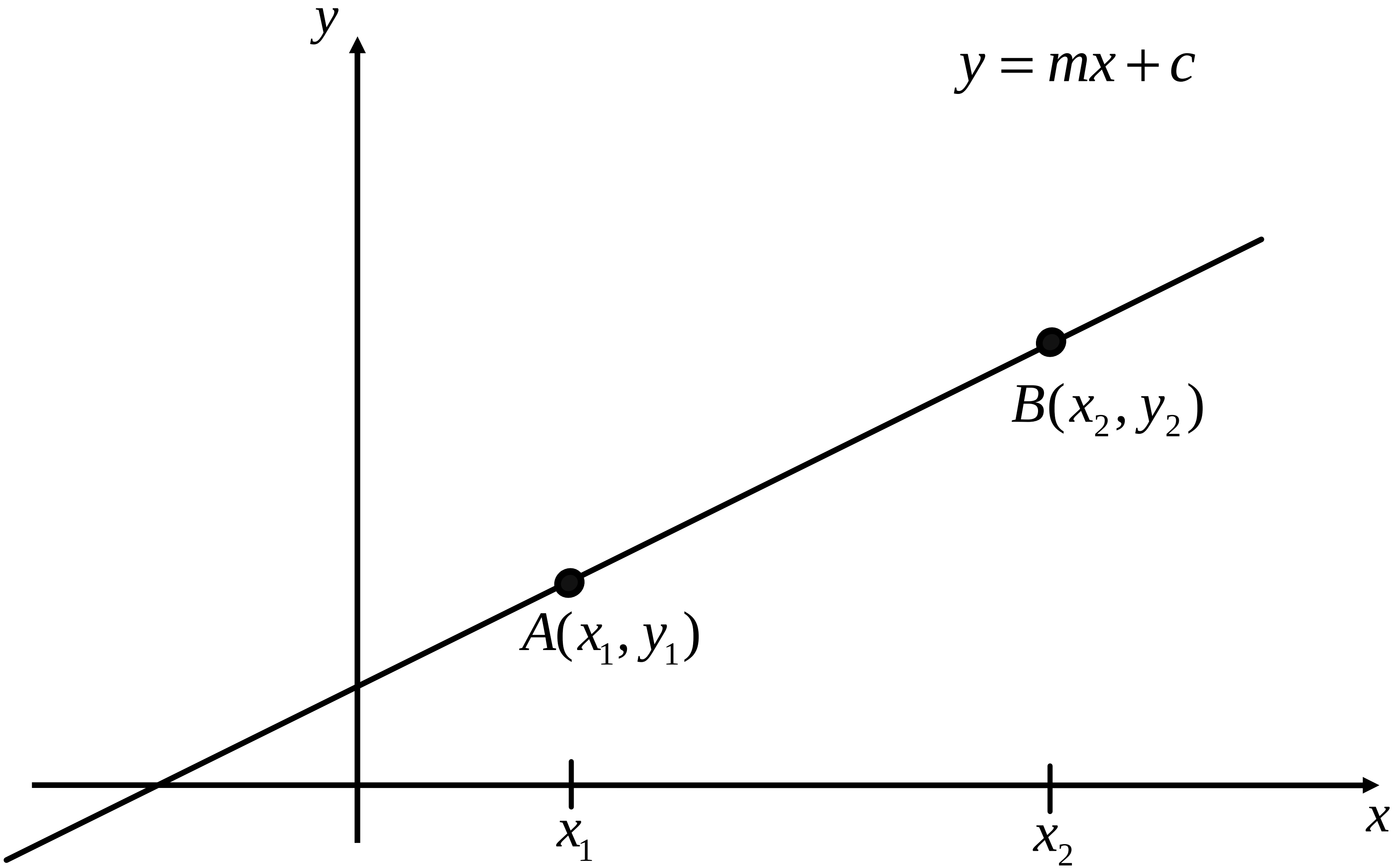
3. Derivative --- **instantaneous rate of change of the function**



m ------ gradient (slope) of the line

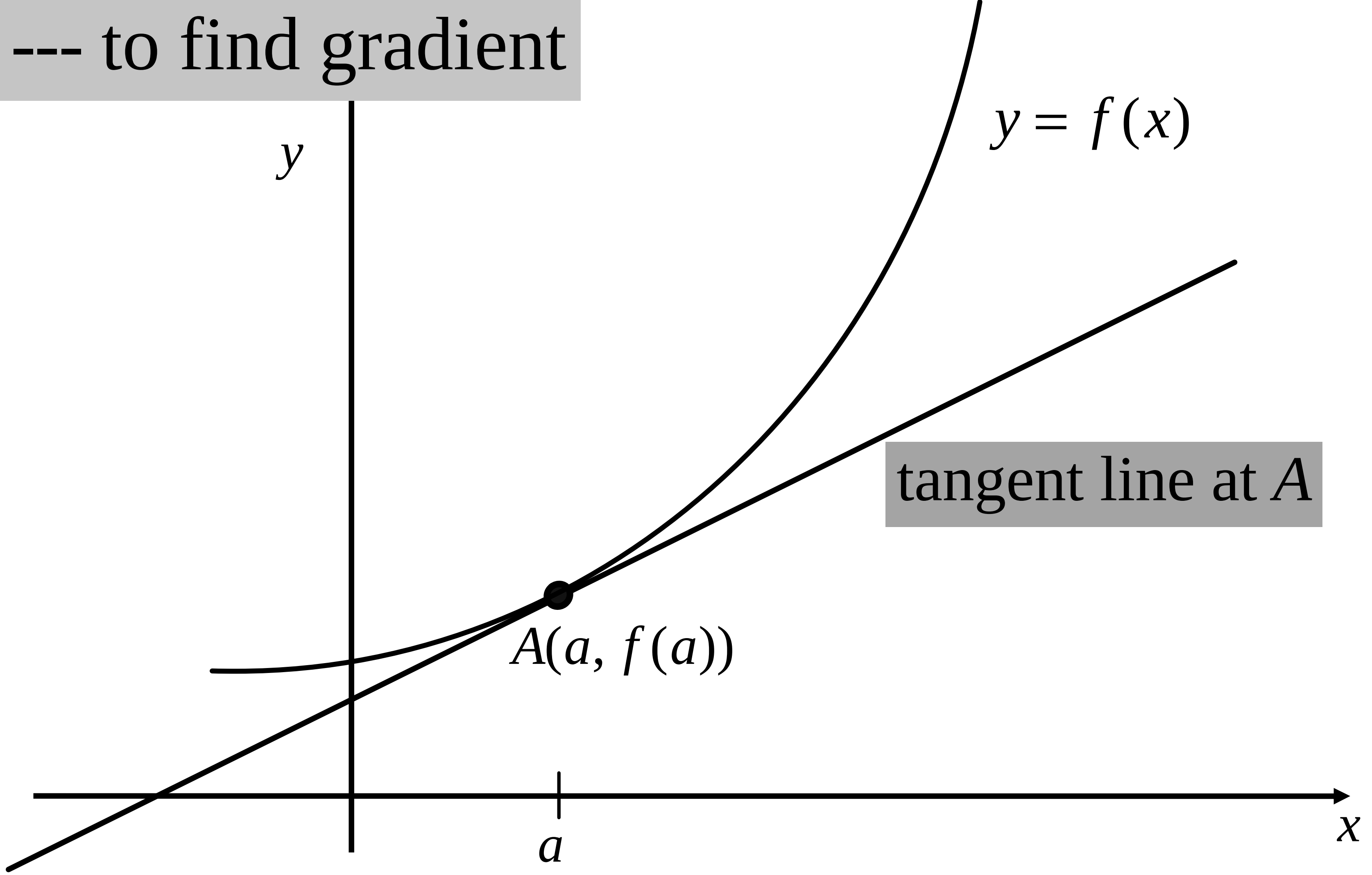
c ------ $y-$intercept

Straight line --- find gradient



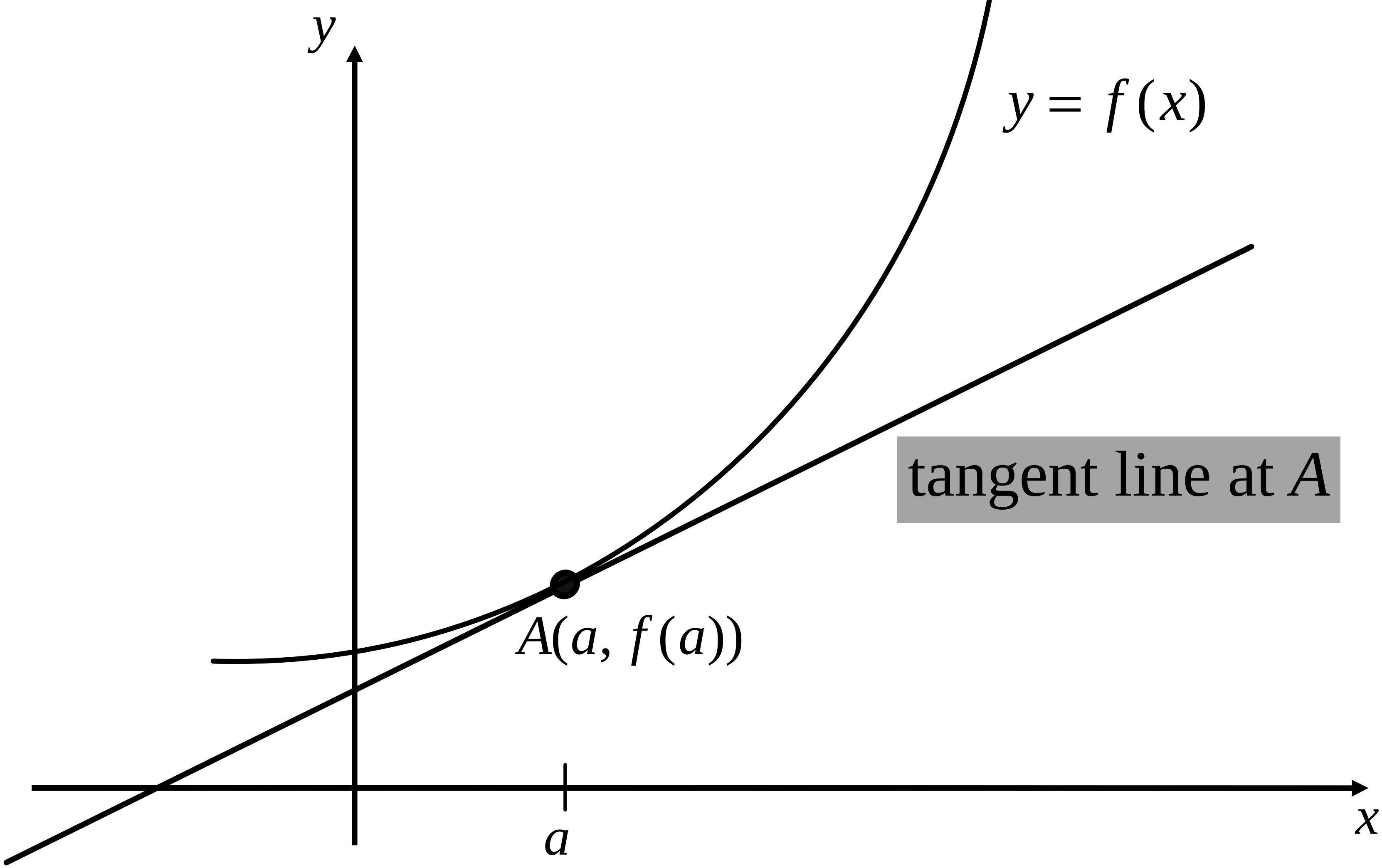
$$\text{gradient (slope)} = m = \frac{y_2 - y_1}{x_2 - x_1}$$

Curve --- to find gradient



gradient at A = gradient of tangent line at A

$$= f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$



Question: Why we define

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

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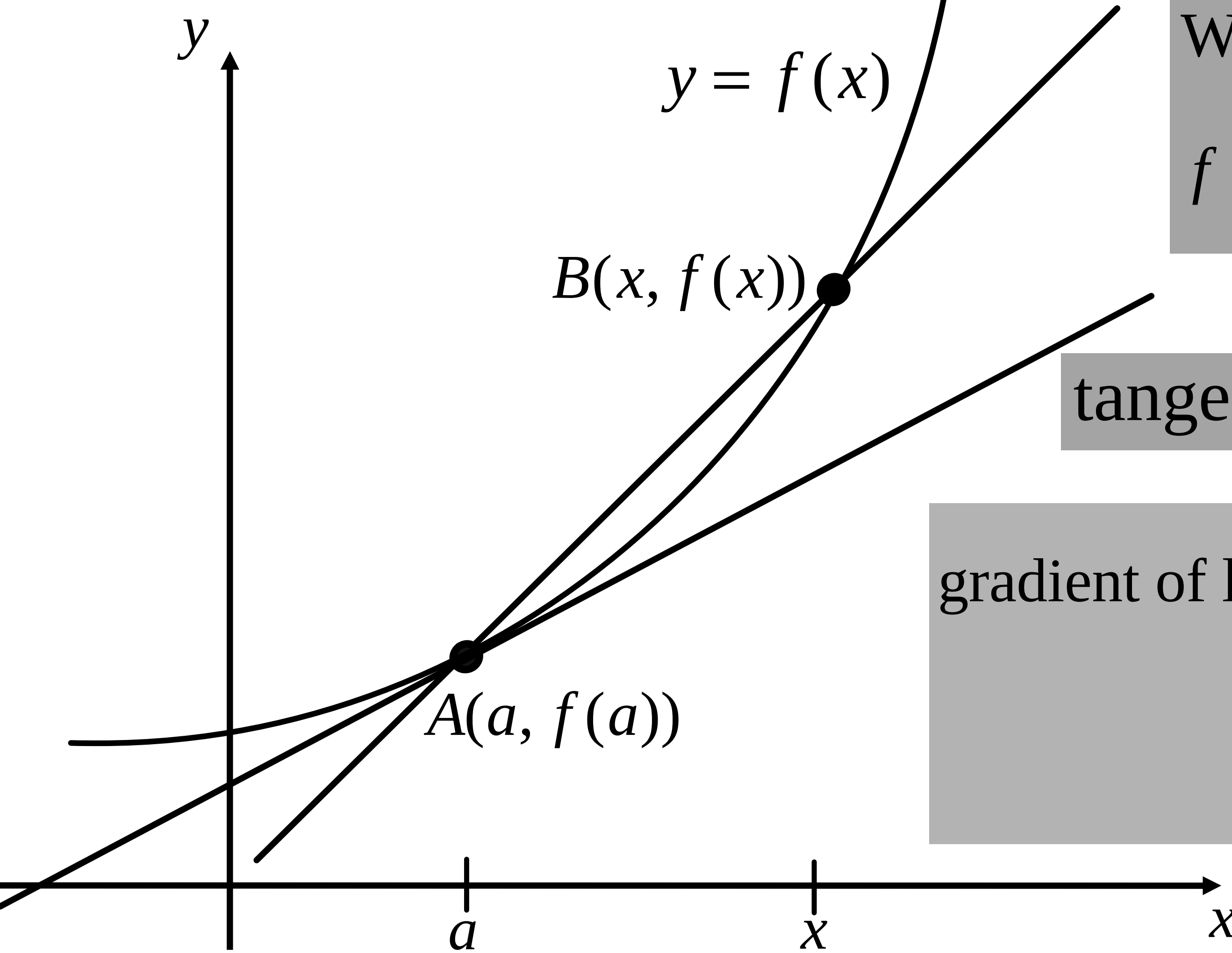


tangent line at A

$$\text{gradient of line } AB = \frac{y_2 - y_1}{x_2 - x_1}$$

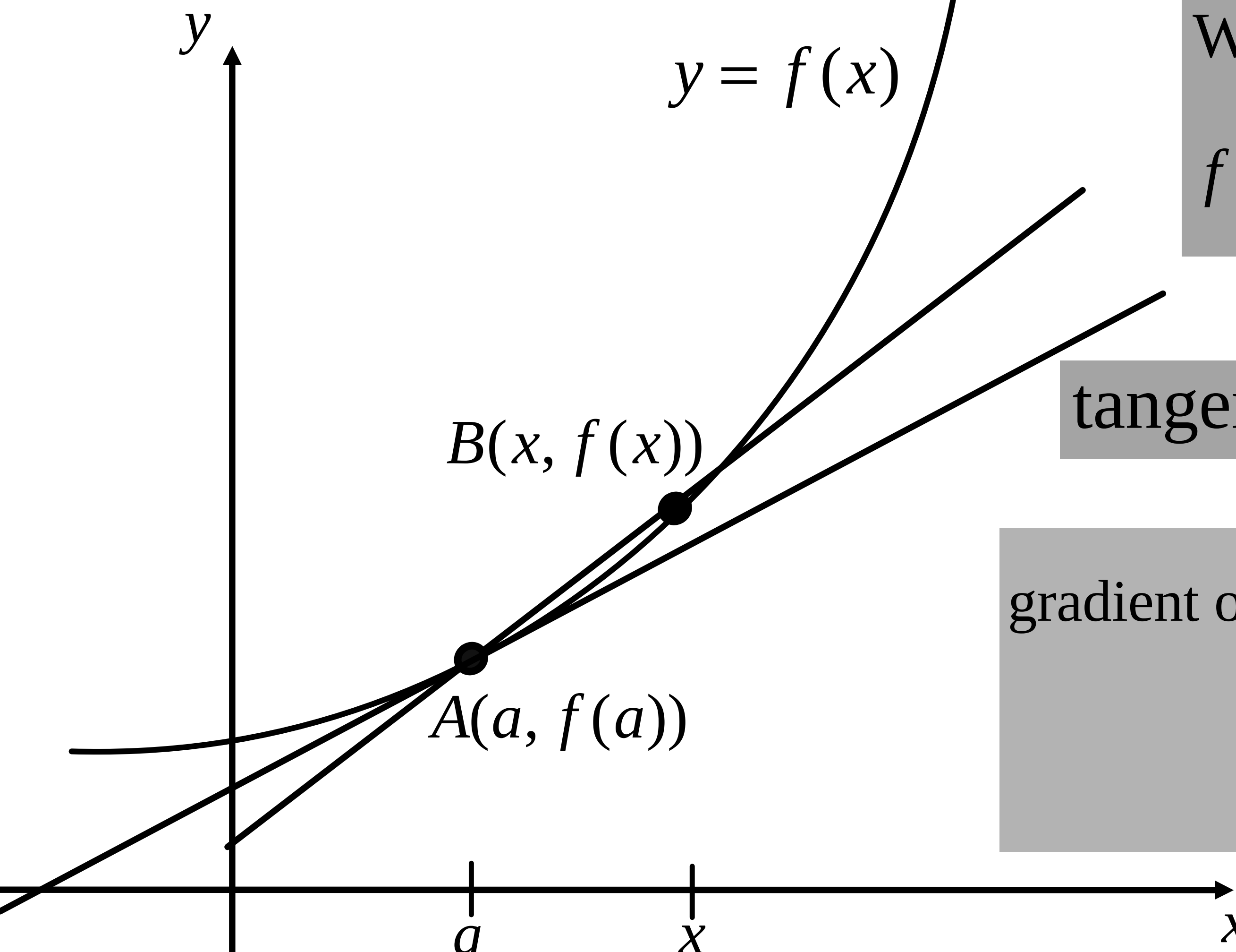
$$= \frac{f(x) - f(a)}{x - a}$$

gradient at A = gradient of tangent line at A

gradient at $A \neq$ gradient of line AB

Why we define

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$



$$\text{gradient of line } AB = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{f(x) - f(a)}{x - a}$$

If we choose B close to A, then

gradient of line $AB \simeq$ gradient at A

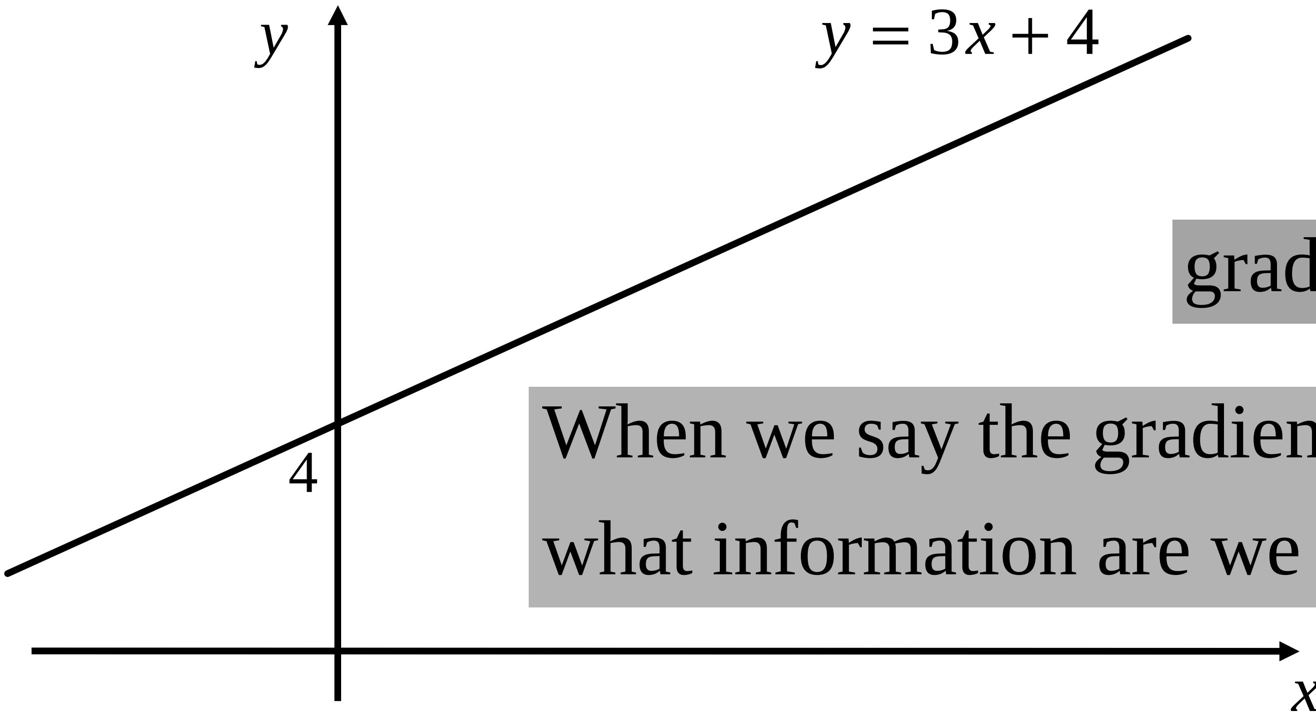
Choosing B closer and closer to A

is the same as letting x approaches a.

Taking limit, we have, $\text{gradient at } A = \lim_{x \to a} (\text{gradient of } AB)$

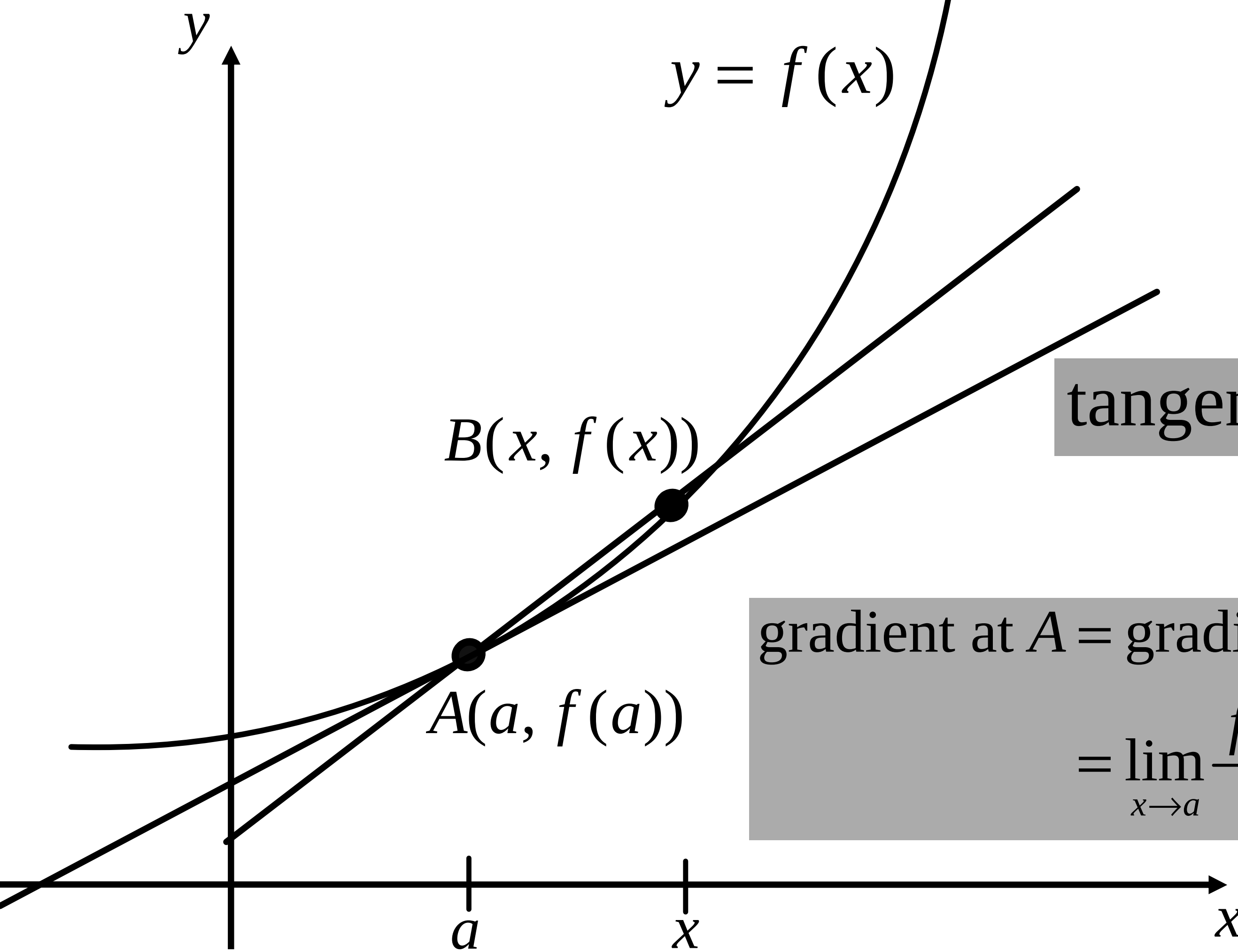
$$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Pause and Think !!!



$\text{gradient} = 3$

When we say the gradient is 3,
what information are we providing?



tangent line at A

$$\text{gradient at } A = \text{gradient of tangent line at } A$$
$$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Similarly, gradient at A gives

instantaneous rate of change of the function $f(x)$.

Derivative

Let $f(x)$ be a function.

The derivative of f at a is defined to be

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided the limit exists.

If $f'(a)$ exists, we say that f is ***differentiable*** at $x = a$.

If f is ***differentiable*** at any point a in the domain of f, we say that f is ***differentiable***.

Derivative

Let $f(x)$ be a function.

The derivative of f at a is defined to be

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided the limit exists.

The value of $\frac{dy}{dx}$ at $x = a$.

$$f'(a) = \frac{dy}{dx}\Big|_{x=a} = \frac{dy}{dx}(a)$$

Theorem (to decide when limit exists)

$\lim_{x\to a} f(x)$ exists if and only if $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ both exist and are equal.

For $\lim_{x\to a} f(x)$ to exist,

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$$

Left limit = Right limit.

For $\lim_{x\to a}(\text{ expression })$ to exist,

$$\lim_{x\to a^+}(\text{ expression }) = \lim_{x\to a^-}(\text{ expression })$$

Left limit = Right limit.

For $\lim\limits_{x \to a} f(x)$ to exist,

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For $\lim\limits_{x \to a} (\text{ expression })$ to exist,

$$\lim_{x \to a^+} (\text{ expression }) = \lim_{x \to a^-} (\text{ expression })$$

Left limit = Right limit.

For $f'(a) = \lim\limits_{x \to a} \left(\dfrac{f(x) - f(a)}{x - a}\right)$ to exist,

$$\lim_{x \to a^+} \left(\frac{f(x) - f(a)}{x - a}\right) = \lim_{x \to a^-} \left(\frac{f(x) - f(a)}{x - a}\right)$$

Left limit = Right limit.

Derivative

Let $f(x)$ be a function.

The derivative of f at a is defined to be

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided the limit exists.

An equivalent formulation is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Let $\boxed{x = a + h.}$

Since $x \to a,$ so $a + h \to a.$

Therefore, we have

$h \to 0.$

Thus,

$$\lim_{x \to a} \frac{f(\boxed{x}) - f(a)}{\boxed{x} - a} = \lim_{h \to 0} \frac{f(\boxed{a+h}) - f(a)}{\boxed{a+h} - a}$$

$$= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Pause and Think !!!

(1) $$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

(2) $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Question : When to use which one ?

Question : Which one is easier to use?

Evaluate the derivative of $f(x) = x^2$ at the point $x = 1$.

Using $f'(a) = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$,

$$\begin{aligned}
f'(1) &= \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} \\
&= \lim_{x \to 1} \frac{x^2 - 1^2}{x - 1} \\
&= \lim_{x \to 1} \frac{(x-1)(x+1)}{x - 1} \\
&= \lim_{x \to 1} x + 1 \\
&= 1 + 1 \\
&= 2.
\end{aligned}$$

Recalled that

$a^2 - b^2 = (a-b)(a+b).$

Derivative - Example

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Let $f(x) = x^3$. Show that $f'(x) = 3x^2$.

- $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\boxed{(x+h)^3} - x^3}{h}$$

Binomial expansion

$$= \lim_{h \to 0} \frac{\boxed{x^3 + 3x^2h + 3xh^2 + h^3} - x^3}{h}$$

$$= \lim_{h \to 0} (3x^2 + 3xh + h^2)$$

$$= 3x^2$$

Let $f(x) = |x|$.

Show that f is differentiable for $x \neq 0$ and has no derivative at $x = 0$.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Note that : $|x| \geq 0$. ($|x|$ is always positive)

When $x = 3$, $|x| = |3| = 3$. (x is positive)

When $x = -4$, $|x| = |-4| = 4$

$-x = -(-4) = 4$

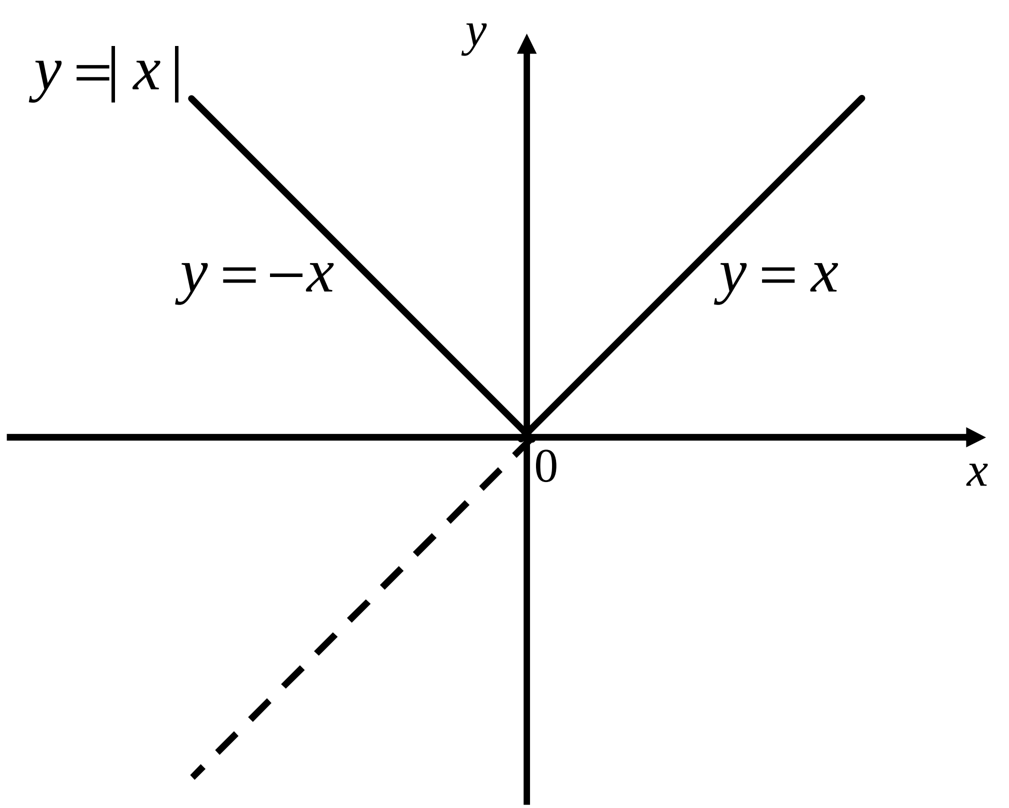
Therefore, $|x| = -x$.

(x is negative and so $-x$ is positive)

Let $f(x) = |x|$.

Show that f is differentiable for $x \neq 0$ and has no derivative at $x = 0$.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



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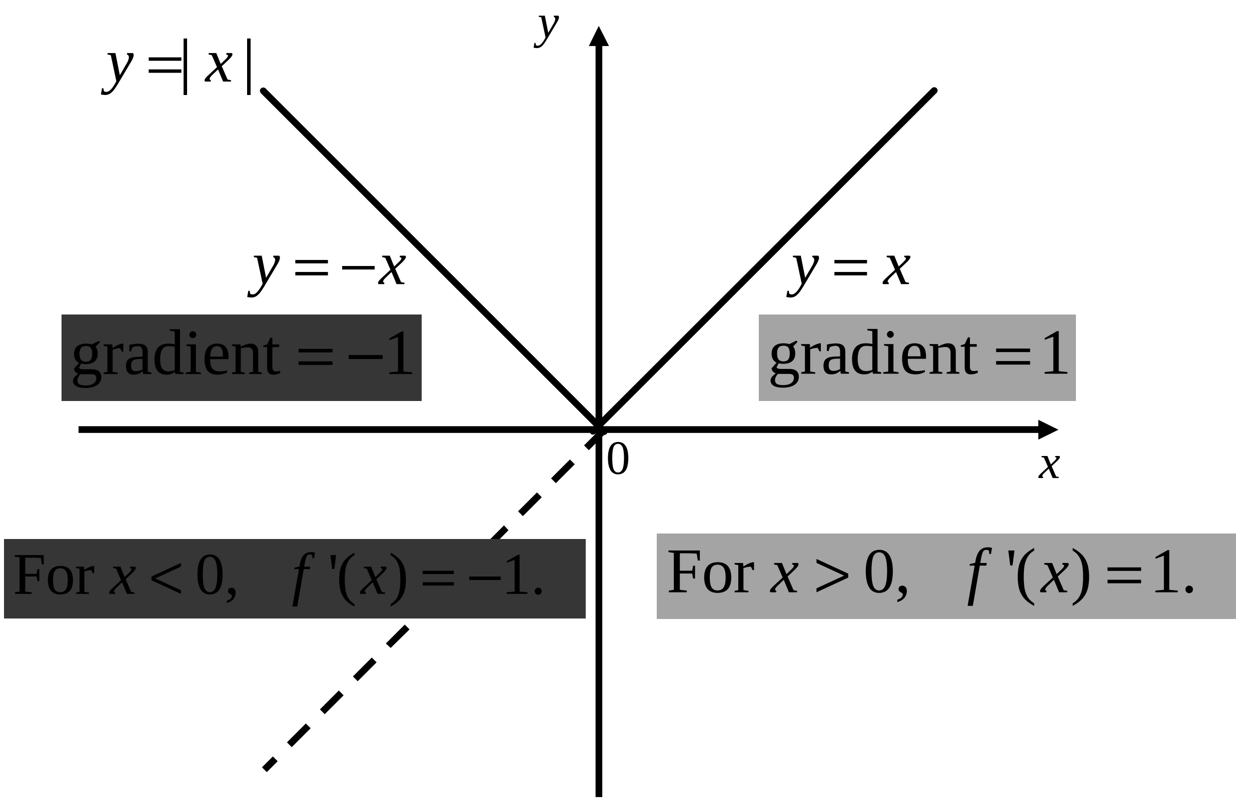


Question: At $x = 0$, $f'(0) = 1$ or -1 ??

No derivative at $x = 0$.

To show f has No derivative at $x=0$,

need to show $f'(0)$ does not exist.

Since $f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}$, we shall show that

$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}$ does not exist.

For $\lim_{x\to a}(\frac{f(x)-f(a)}{x-a})$ to exist,

$$\lim_{x\to a^+}(\frac{f(x)-f(a)}{x-a})=\lim_{x\to a^-}(\frac{f(x)-f(a)}{x-a})$$

Left limit = Right limit.

Thus, we show that,

$$\lim_{x\to 0^+}(\frac{f(x)-f(0)}{x-0})\neq\lim_{x\to a^-}(\frac{f(x)-f(0)}{x-0})$$

Left limit $\neq$ Right limit.

Thus, we show that,

$$\lim_{x\to 0^+}\left(\frac{f(x)-f(0)}{x-0}\right) \neq \lim_{x\to a^-}\left(\frac{f(x)-f(0)}{x-0}\right)$$

Left limit $\neq$ Right limit.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$x \to 0^+ \quad x > 0$

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x} = \lim_{x\to 0^+}\frac{|x|}{x} = \lim_{h\to 0^+}\frac{x}{x} = 1$$

$x \to 0^- \quad x < 0$

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x} = \lim_{x\to 0^-}\frac{|x|}{x} = \lim_{h\to 0^+}\frac{-x}{x} = -1$$

Therefore,

$$\lim_{x\to 0^+}\left(\frac{f(x)-f(0)}{x-0}\right) \neq \lim_{x\to a^-}\left(\frac{f(x)-f(0)}{x-0}\right)$$

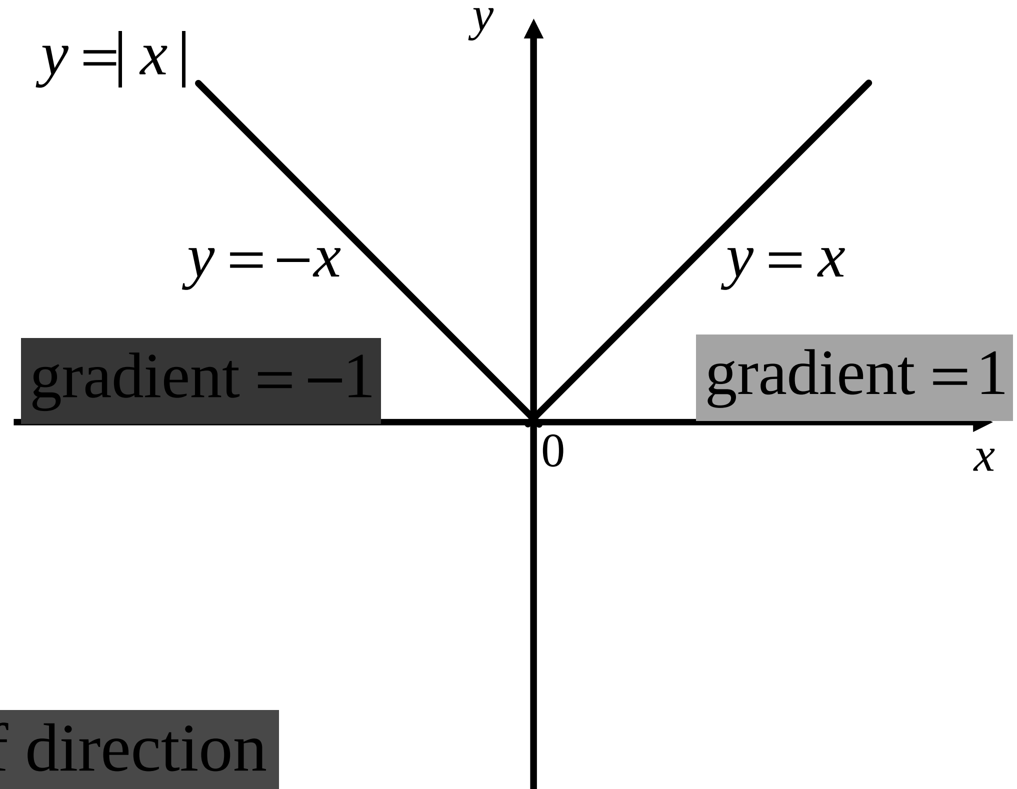
Left limit $\neq$ Right limit.

No derivative at $x = 0$.

Let $f(x) = |x|$.

Show that f is differentiable for $x \neq 0$ and has no derivative at $x = 0$.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



No derivative at $x = 0$.

Note that :

1. There is a sudden change of direction for the graph of $f(x) = |x|$ at $x = 0$.

2. The graph of $f(x) = |x|$ at $x = 0$ is a "sharp point".

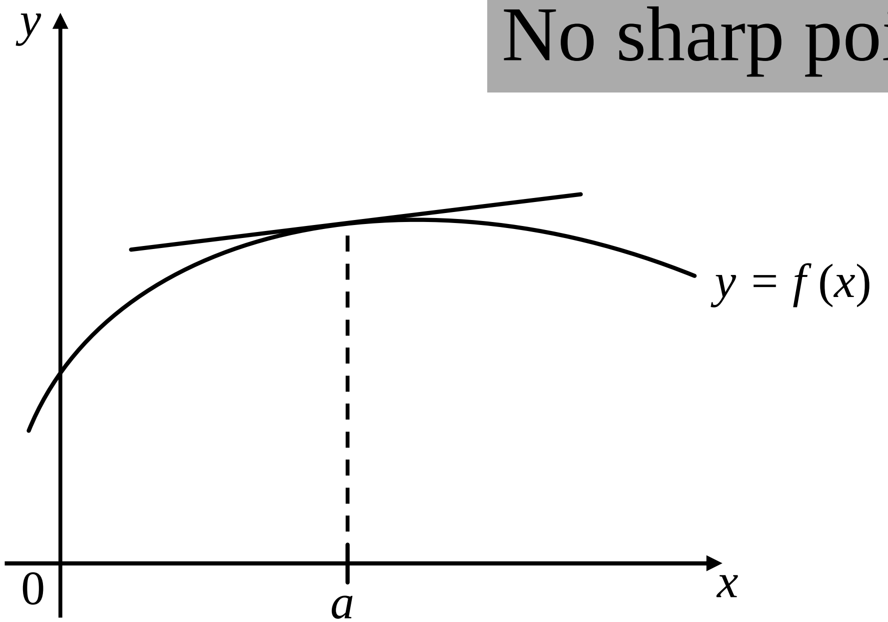
Derivative

The ***existence*** of $f\,'(a)$ is a ***smoothness*** condition on the

- curve $y = f(x)$ at "$x = a$".

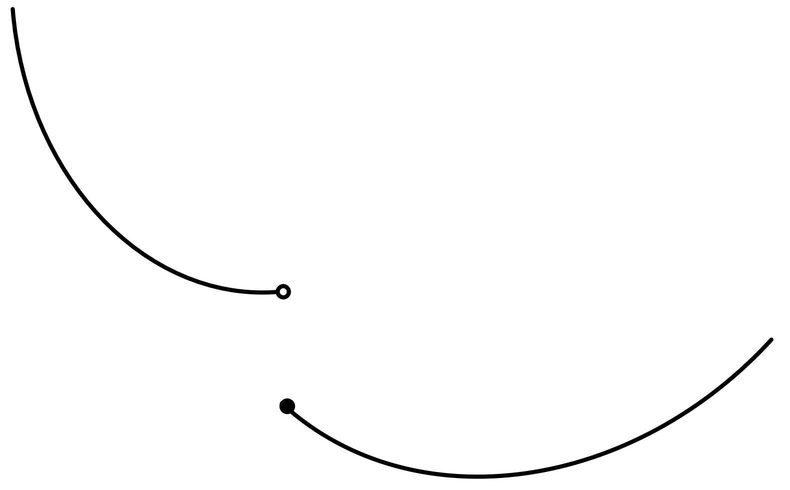
No sudden change of direction.

No sharp point.



Derivative – Non-existence of $f'(a)$

- Discontinuity

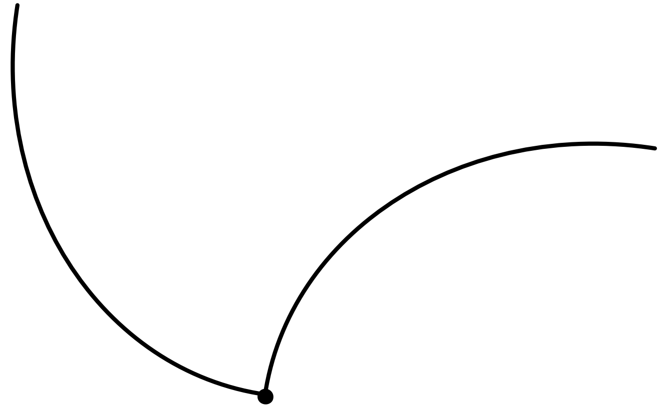


Indeed we have:

$$\boxed{(*) \qquad \text{If } f'(a)\ \textbf{\textit{exists}}, \text{ then } f \text{ is } \textbf{\textit{continuous}} \text{ at } 'x = a'.}$$

Derivative – Non-existence of $f'(a)$

- Corner

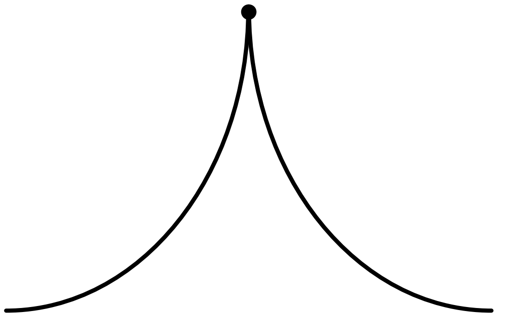


Remark: The ***converse*** of (*) is ***not*** necessarily true.

(*)	If $f'(a)$ ***exists***, then f is ***continuous*** at '$x = a$'.

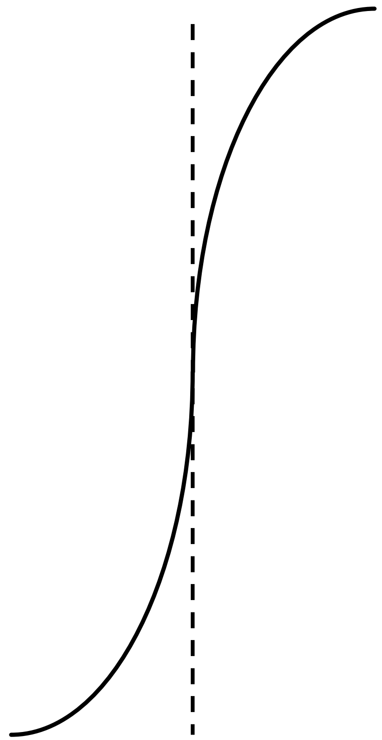
Derivative – Non-existence of $f'(a)$

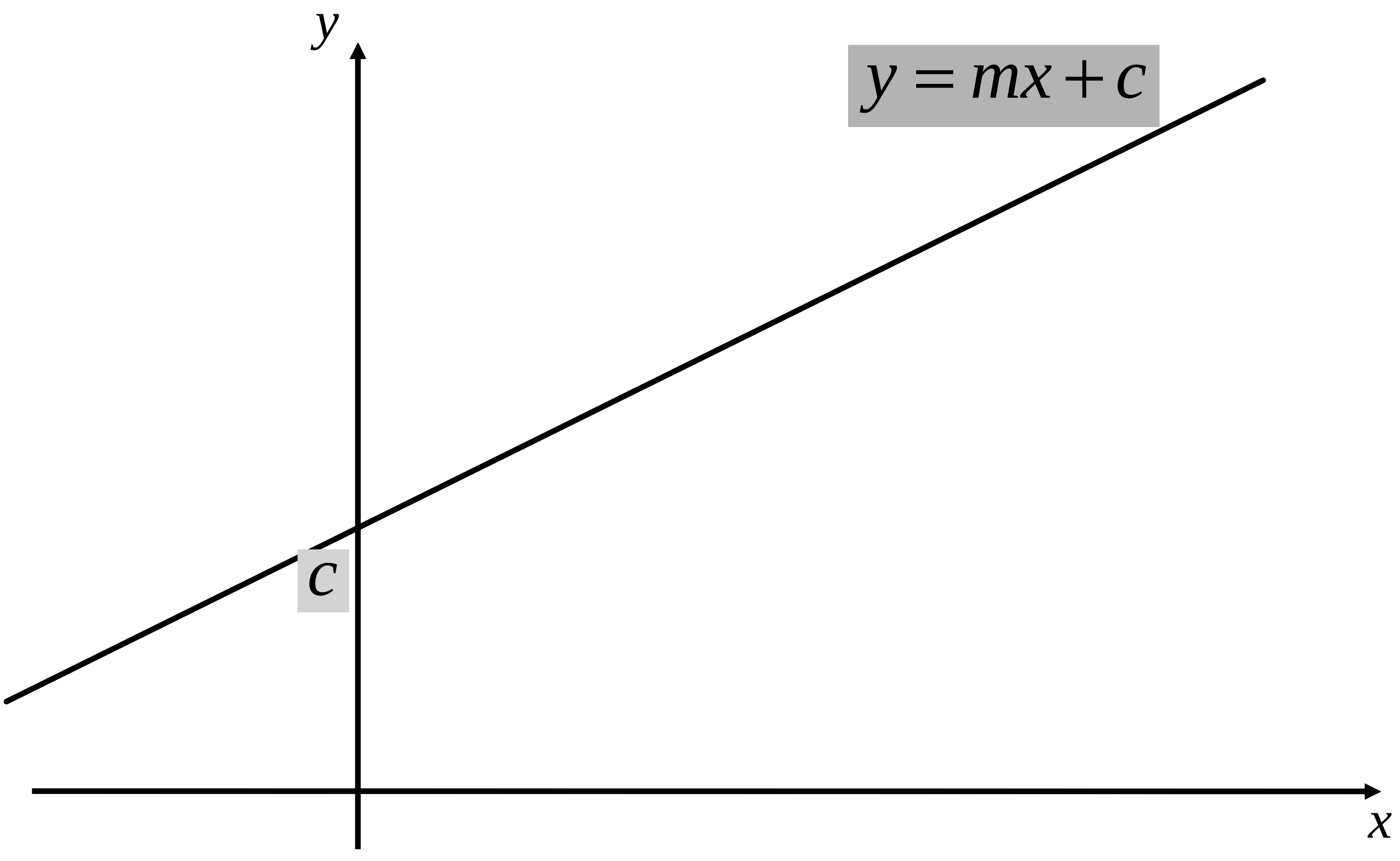
- Cusp



Derivative – Non-existence of *f'*(*a*)

- Vertical tangent

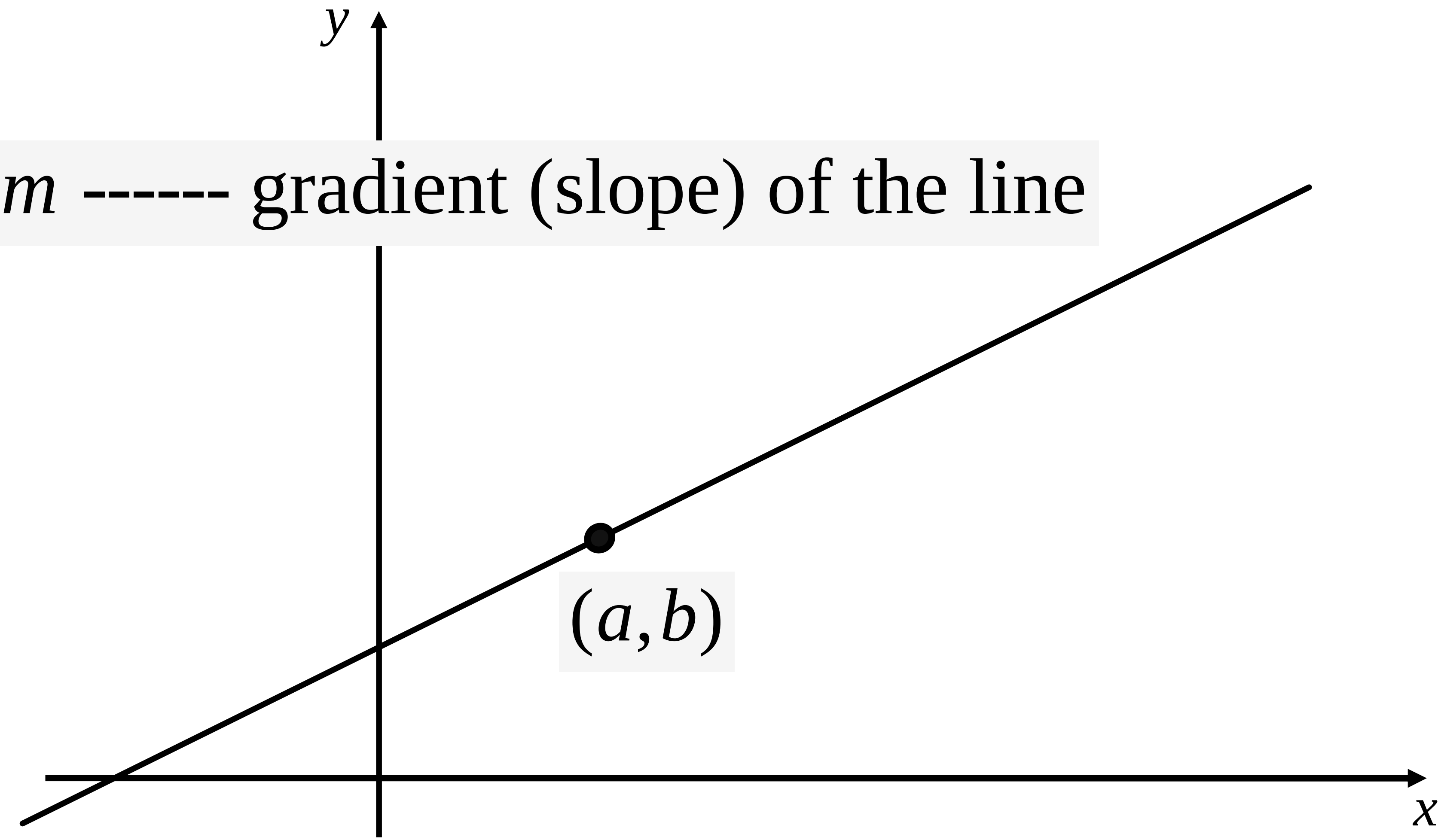




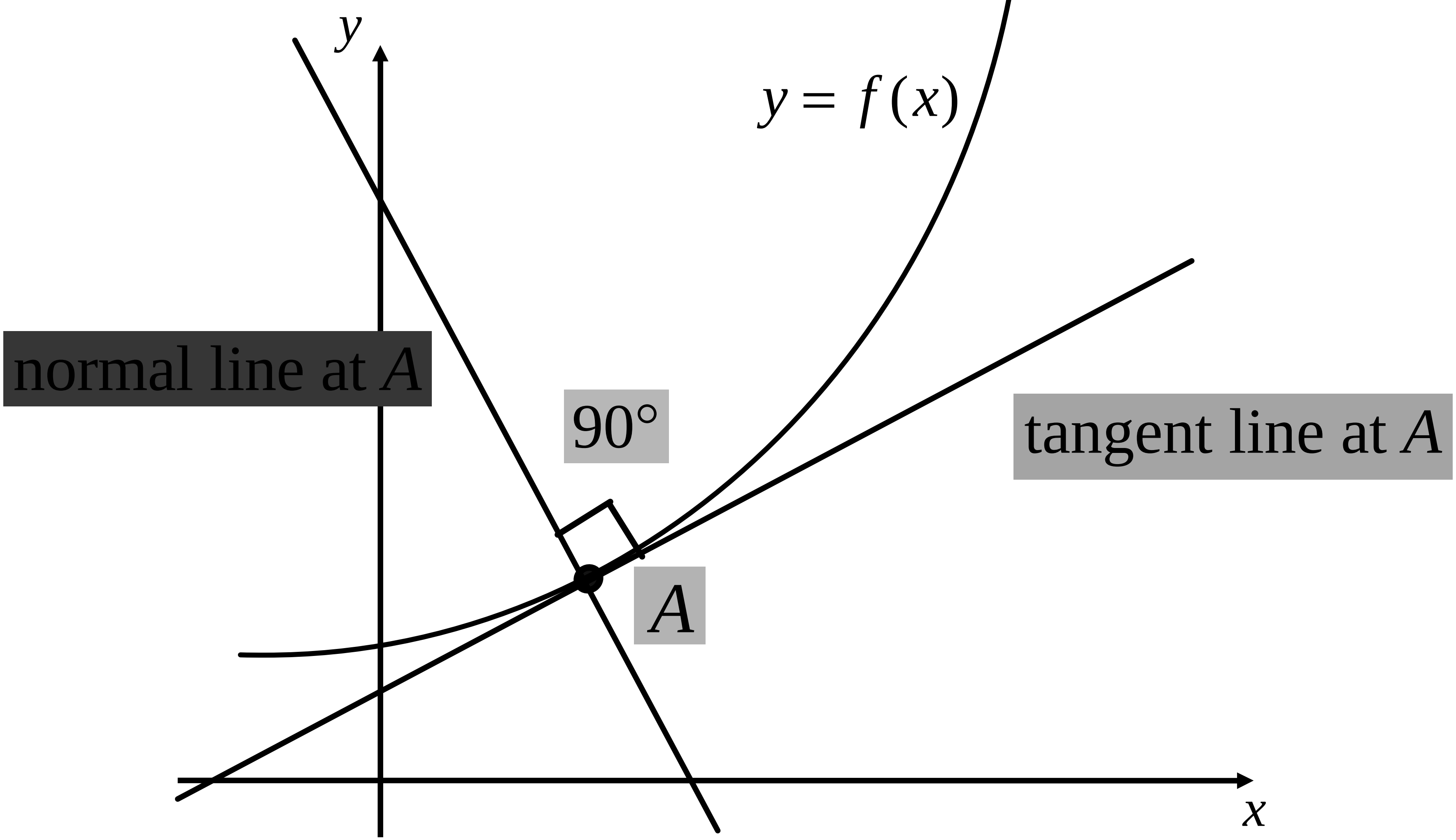
m ------ gradient (slope) of the line

c ------ $y-$intercept

Point – Slope form



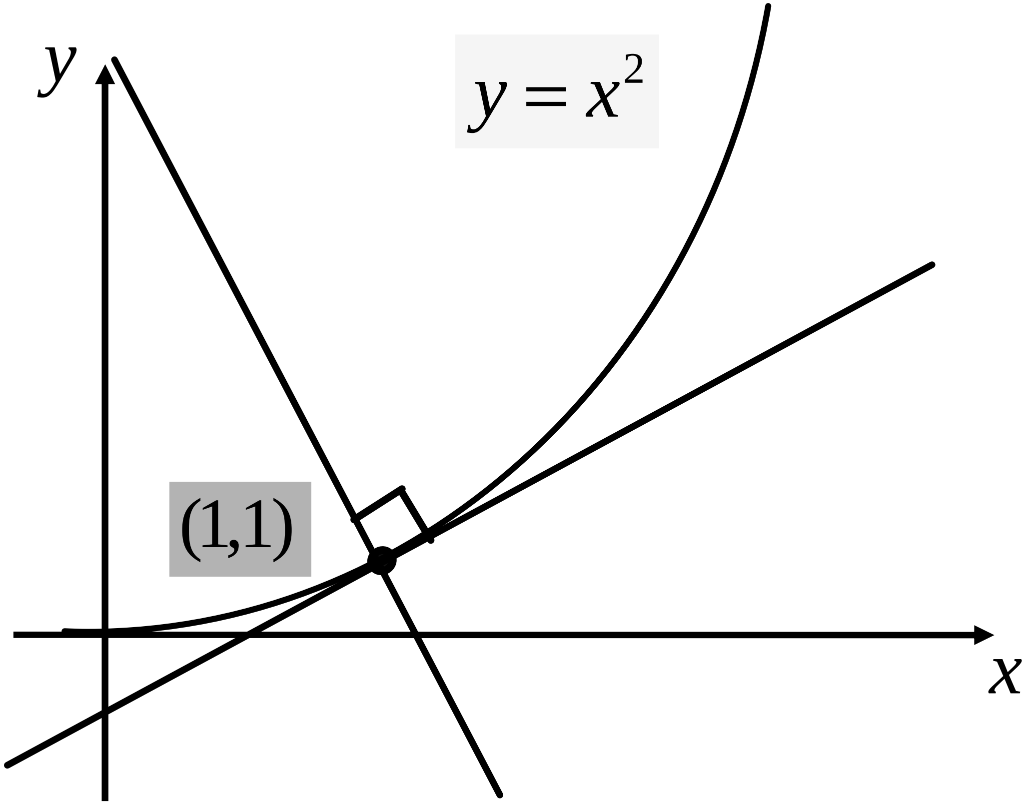
$$y - b = m(x - a)$$



Result :

(gradient of tangent line)×(gradient of normal line) = −1

Find equation of lines which are tangent and normal to the curve $y = x^2$ at $x = 1$ respectively.



$$\frac{dy}{dx} = 2x$$

$$x = 1, \ \frac{dy}{dx} = 2$$

gradient of tangent $= 2$

gradient of normal $= -\frac{1}{2}$

Equation of tangent : $y - 1 = 2(x - 1)$

Equation of normal : $y - 1 = -\frac{1}{2}(x - 1)$

Derivative – Rules of Differentiation

Product Rule

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

$$\frac{d}{dx} f(x)g(x) = \frac{df(x)}{dx} g(x) + f(x)\frac{dg(x)}{dx}$$

$$\frac{d}{dx}(uv) = \frac{du}{dx} v + u\frac{dv}{dx}$$

Derivative – Rules of Differentiation

Quotient Rule

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Quotient Rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

- Show that $\frac{d}{dx}\tan x = \sec^2 x$.

$$u = \sin x$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$v = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x}$$

$$= \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

$$\cos^2 x + \sin^2 x = 1$$

Derivative – Rules of Differentiation

Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

The Chain Rule - Example

- Find $\dfrac{d}{dx}\sin(x^3)$.

$$\frac{d}{dx}\sin(x^3)$$

$$= \cos(x^3)\cdot\frac{d}{dx}x^3$$

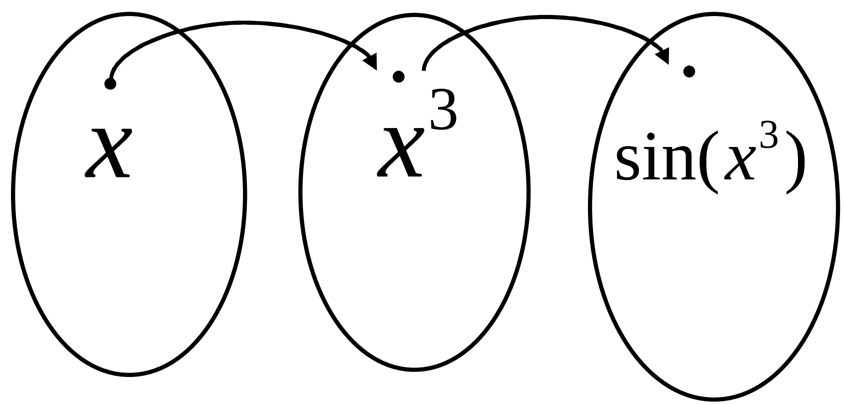
$$= 3x^2\cos(x^3)$$

Fix a value for x

Let $x = p$

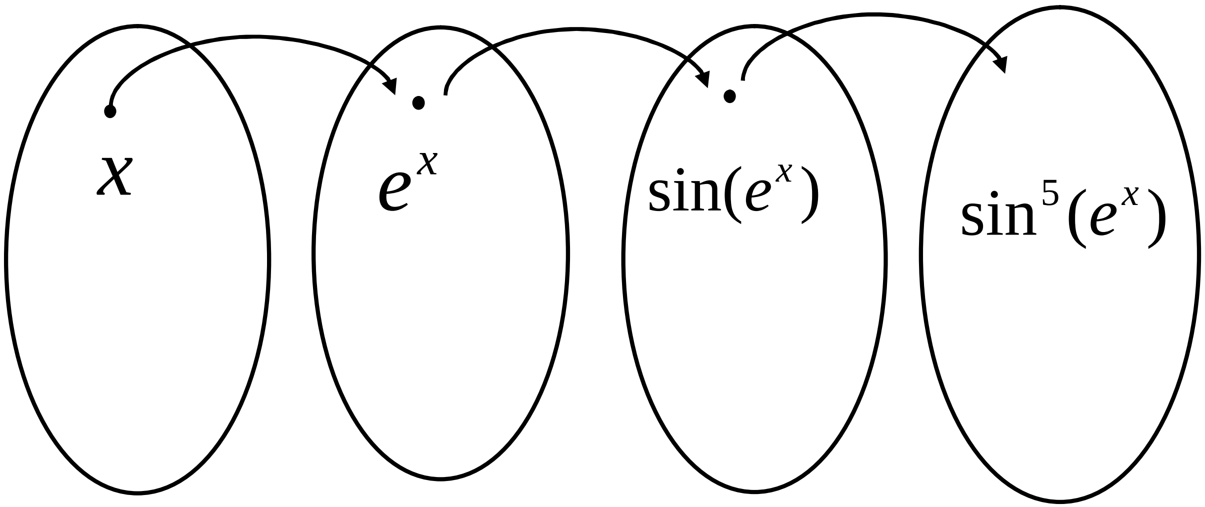
Step 1. p^3

Step 2. $\sin(p^3)$



The Chain Rule - Example

- Find $\dfrac{d}{dx}\sin^5(e^x)$.



The Chain Rule - Example

- Let $y = (x^5 + \cos(3x^2))^9$. Find $\dfrac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{d}{dx}(x^5 + \cos(3x^2))^9$$

$$= 9(x^5 + \cos(3x^2))^8 . \frac{d}{dx}(x^5 + \cos(3x^2))$$

$$= 9(x^5 + \cos(3x^2))^8 (5x^4 - \sin(3x^2) \cdot 6x)$$

$$= 9x(x^5 + \cos(3x^2))^8 (5x^3 - 6\sin(3x^2))$$