National University of Singapore
Department of Electrical & Computer Engineering

EE2023 Signals & Systems
Tutorial 6 Solutions

Section I

1. (a) Differential equation can be obtained simply by inspecting the transfer function and using the following properties :

   - Under the assumption that all initial conditions are zero, the derivative of transform rule reduces to
   $$\mathcal{L}\left\{\frac{\mathrm{d}^n x(t)}{\mathrm{d}t^n}\right\} = s^n X(s)$$
   - Numerator polynomial is formed using coefficients of the input function and its derivative(s)
   - Denominator polynomial is formed using coefficients of the output signal and its derivative(s)

   Hence, the differential equation is

   $$\ddot{y}(t) + 6\dot{y}(t) + 13y(t) = \dot{u}(t) + 9u(t)$$

   Transfer function assumes that all initial conditions are zero. As the output signal of a dynamic system must be continuous and the given system is 2nd order, the initial conditions needed are $y(0) = \dot{y}(0) = 0$.

   (b) Question provides the input signal and the system transfer function. Solution required is the steady-state output signal of a dynamic system i.e. $\lim\limits_{t\to\infty} y(t)$. Hence, concept needed to solve problem is the definition of a transfer function, $Y(s) = G(s)U(s)$.

   - Given that input is a step function of magnitude 2, $U(s) = \dfrac{2}{s}$.
   - $Y(s) = G(s)U(s) = \dfrac{s+9}{s^2+6s+13}\dfrac{2}{s}$
   - Performing inverse Laplace Transform, the time-domain expression of the output signal is

   $$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{2(s+9)}{s\left(s^2+6s+13\right)}\right\} \\ &= \frac{18}{13} - \frac{18}{13}e^{-3t}\cos 2t - \frac{14}{13}e^{-3t}\sin 2t \end{aligned}$$

   $$\therefore \text{ Steady-state value of } y(t) = \lim_{t\to\infty}\left\{\frac{18}{13} - \frac{18}{13}e^{-3t}\cos 2t - \frac{14}{13}e^{-3t}\sin 2t\right\} = \frac{18}{13}$$

- Final Value Theorem state that $\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$. Substituting $Y(s) = G(s)U(s) = \dfrac{s+9}{s^2+6s+13}\dfrac{2}{s}$ into the Final Value Theorem,

$$\begin{aligned}\lim_{t\to\infty} y(t) &= \lim_{s\to 0} sY(s) \\ &= \lim_{s\to 0}\left\{\frac{2(s+9)}{s^2+6s+13}\right\} \\ &= \frac{18}{13}\end{aligned}$$

$\Longrightarrow$ Final value theorem correctly predicts the steady-state value of $y(t)$.

2. The problem provides information about the convolution integral and asks for the transfer function. To formulate the solution, a concept that links the functions in a convolution integral to the transfer function is needed. The following concepts are covered in class :

   - The output signal, $y(t)$, of a system may be obtained by convolving the impulse response, $h(t)$, with a general input/forcing signal, $u(t)$.
   - Transfer function is the Laplace transform of the impulse function.

   Hence, the required transfer function is

$$\begin{aligned}G(s) &= \mathcal{L}\left\{150e^{-0.5\tau}\sin(0.5\tau)\right\} \\ &= \frac{150\times 0.5}{(s+0.5)^2+0.5^2} \\ &= \frac{75}{s^2+s+0.5}\end{aligned}$$

Section II

1. Circuit has two independent input sources but question only requires the transfer function relating $i_1(t)$ and $i(t)$. Hence, the first step is to apply the principle of the superposition and "kill" the voltage source (short the voltage source) so only the input signal needed to solve the problem is left in the circuit.

   - Applying Kirchoff current law and Kirchoff voltage law, the following differential equation relating $i_1(t)$ and $i(t)$.

$$LC\frac{d^2 i_1(t)}{dt^2} + R_1 C\frac{di_1(t)}{dt} + i_1(t) = i(t)$$

   Then, assuming that the initial conditions $i_1(0) = i_1'(0) = 0$, apply Laplace Transform to derive the transfer function.

- Transfer functions assume that all initial conditions are zero. In this problem, the initial conditions are needed to ensure voltage continuity for the capacitor and current continuity for the inductor. Hence, assuming $i_1(0) = 0$ and $v_c(0) = 0$,
  Voltage-Current relationship for the capacitor is $V(s) = \frac{1}{sC}I(s)$
  Voltage-Current relationship for an inductor is $V(s) = sLI(s)$
  Using the current division rule,

$$\begin{aligned} I_1(s) &= \frac{\frac{1}{sC}}{\frac{1}{sC} + R_1 + sL} I(s) \\ \frac{I_1(s)}{I(s)} &= \frac{1}{LCs^2 + R_1Cs + 1} \end{aligned}$$

2. (a) Question states that the temperature reading is allowed to stabilise i.e. reach steady state. Hence, the question requires the steady-state input signal to be derived using the differential equation and the steady-state output signal.

   When the temperature reading stabilises, $\dfrac{\mathrm{d}y(0)}{\mathrm{d}t} = 0$ so the differential equation reduces to

$$y(t) = 0.99u(t)$$

   Given that $y(0) = 24.75$, the temperature of the heat bath is $u(0) = \dfrac{y(0)}{0.99} = 25^o\mathrm{C}$.

   (b) Since the bath temperature increases at a steady rate of 1^oC/second, the input signal $u(t)$ is a straight line. Using the general form of a straight line $y = mx + c$ where m is the gradient and c is the y-intercept,

$$u(t) = [25 + t]\, U(t)$$

   Substituting $u(t)$ into the differential equation, the time-domain expression for the measured temperature can be found by solving

$$5\frac{\mathrm{d}y(t)}{\mathrm{d}t} + y(t) = 0.99u(t) \qquad \text{where } y(0) = 24.75$$

   Taking Laplace Transform on both sides of the equation,

$$\begin{aligned} 5[sY(s) - y(0)] + Y(s) &= 0.99U(s) \\ Y(s) &= \frac{0.99U(s)}{5s+1} + \frac{5y(0)}{5s+1} \\ &= \frac{0.99}{5s+1}\left[\frac{25}{s} + \frac{1}{s^2}\right] + \frac{5y(0)}{5s+1} \\ y(t) &= 19.8 + 0.99t + 4.95e^{-\frac{t}{5}} \end{aligned}$$

   (c) Same method as Section I Q1a.

(d) Transfer functions are defined under the assumption that the system is initially at rest. In this problem, $y(0) \neq 0$ so the zero initial conditions assumption is violated. As $\dfrac{\mathrm{d}y(0)}{\mathrm{d}t} = 0$, the system is initially at rest so transfer function is applicable by shifting the input and output axes. Let

$$\begin{aligned}
y_1(t) &= y(t) - y(0) = y(t) - 24.75 \\
u_1(t) &= u(t) - u(0) = u(t) - 25 = t + 25 - 25 = t \\
\therefore\ U_1(s) &= \frac{1}{s^2} \\
Y_1(s) &= \frac{0.99}{5s+1} U_1(s) \\
y_1(t) &= -4.95 + 0.99t + 4.95e^{-\frac{t}{5}}
\end{aligned}$$

By definition, $y_1(t) = y(t) - y(0)$. The time-domain expression for the measured temperature is

$$y(t) = y_1(t) + y(0) = y_1(t) + 24.75 = 19.8 + 0.99t + 4.95e^{-\frac{t}{5}}$$

3. Stability of a dynamic system depends on whether the transient response is bound or on the location of the system poles.

(a) Transient response is $e^{-t} + e^{2t}$ for $t \geq 0$.
System is unstable because the presence of e^{2t} cause the transient response to grow without bound.

(b) Transient response is $\sin 2t$ for $t \geq 0$.
System is marginally stable because the transient response oscillate with constant amplitude.

(c) Transient response is $e^{-t} \sin 2t$ for $t \geq 0$.
System is stable because the transient response decays to zero when $t \to \infty$.

(d) Differential equation is $\ddot{y}(t) - \dot{y}(t) - 6y(t) = 4u(t)$. As stability is depends on the characteristics of the transient response, the first step is to derive the transient response or the system poles.

$$s^2 - s - 6 = 0 \qquad \text{i.e. } s = 3, -2$$

Transient response, $y_{tr}(t) = A_1 e^{3t} + A_2 e^{-2t}$. Since $\lim_{t\to\infty} y_{tr}(t) \neq 0$, system is unstable.

(e) Transfer function is $\dfrac{s+3}{s^2+3}$. The roots of the $s^2 + 3 = 0$ or the system poles are

$$\text{System poles are located at } s = \pm j\sqrt{3}$$

Since system poles lie on the imaginary axis, transient response is a sinusoid so the system is marginally stable.

(f) Transfer function is $\dfrac{4}{(s^2+4)^2}$

$$\text{Poles are located at } s = \pm 2j, \pm 2j$$

There is one pair of *repeated* poles on the imaginary axis. To determine if such a system is stable, consider the case where the input is a step function (bounded input signal). The step response is

$$\begin{aligned} y_{step}(t) &= \mathcal{L}^{-1}\left\{\frac{4}{(s^2+4)^2}\frac{1}{s}\right\} \\ &= \frac{1}{4} - \frac{1}{4}\cos 2t - \frac{1}{4}t\sin 2t \end{aligned}$$

When $t \to \infty$, $\lim_{t\to\infty} y_{step}(t)$ is unbounded because of the $\frac{1}{4}t\sin 2t$ term.
$\therefore$ the system is unstable because a bounded input signal resulted in an unbounded output signal.

(g) Transfer function is $\dfrac{2s-1}{s^2+2s+4}$

$$\text{Poles are located at } s = -1 \pm j\sqrt{3}$$

Since system poles are in the LHP, system is stable.
Note that system zeros does not influence stability.

(h) Consider a system whose output is $2t - \frac{2}{5} + \frac{2}{5}e^{-5t}$ when the input signal is the ramp function, t. Although the output signal is unbounded, conclusions about stability cannot be made directly because the input is also unbounded. In cases where "rules" cannot be applied directly, it is best to revert to first principle by examining the transient response. It is difficult to divide the output signal by inspection into the transient response and the steady-state response so one option is to examine the location of the system poles. Applying Laplace Transform to $y(t) = 2t - \frac{2}{5} + \frac{2}{5}e^{-5t}$,

$$Y(s) = \frac{10}{s^2(s+5)}$$

From the definition of transfer function, $Y(s) = G(s)U(s)$ where $G(s)$ is the system transfer function and $U(s) = \mathcal{L}\{t\}$. Comparing with the above equation,

$$G(s) = \frac{10}{s+5}$$

Since the system pole $s = -5$ lies in the LHP, the system is stable.

4. (a) Transfer function of the air heating system is

$$\frac{\theta_o(s)}{H(s)} = \frac{R}{RCs+1}$$

Hence, impulse response of the heating system is

$$\theta_o(t) = \mathcal{L}^{-1}\left\{\frac{R}{RCs+1}\right\} = \frac{1}{C}e^{-\frac{t}{RC}}$$

(b) Substitute two points from the graph into $\theta_o(t) = \frac{1}{C}e^{-\frac{t}{RC}}$, and solve simultaneously for R and C. Of the 5 points provided, the simultaneous equations can be solved most easily if the following data points are used to formulate the equations

- At $t = 0$, $\theta_o(t) = \frac{1}{C} = 10$
- At $t = RC$, $\theta_o(t) = \frac{1}{C}e^{-1} = \frac{0.36788}{C} = 3.6788$

Section III

1. Method similar to Q1 in Section II.

2. Assuming that there are no system zeros, $Y(s) = \frac{K}{D(s)}$

   - Given that the poles of $Y(s)$ (roots of $D(s)$) are $s = 0, -3, -7 \pm 5j$ so $D(s) = s(s + 3)(s + 7 + 5j)(s + 7 - 5j) = s(s + 3)(s^2 + 14 + 74)$
   - Value of K can be found from $\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s\dfrac{K}{s(s+3)(s^2+14+74)} = 8.$
   - Since the input signal is a step function of magnitude 4, $Y(s) = G(s) \times \frac{4}{s} = \frac{K}{D(s)}$. The system transfer function, $G(s)$, then be found.
   - Note that the answer given in the tutorial problem sheet is not unique, because the question does not provide any information about zeros. Hence, all answers that satisfy the two given constraints are acceptable.