2008/2009 SEMESTER 1 MID-TERM TEST

MA1505 MATHEMATICS I

29 September 2008

8:30pm to 9:30pm

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY:

1. This test paper consists of **TEN (10)** multiple choice questions and comprises **Twelve (12)** printed pages.
2. Answer all 10 questions. 1 mark for each correct answer. No penalty for wrong answers. Full mark is 10.
3. All answers (Choices A, B, C, D, E) are to be submitted using the pink form (FORM CC1).
4. Use **only 2B pencils** for FORM CC1.
5. On FORM CC1 (section B), **write** your **matriculation number** and **shade** the corresponding numbered circles **completely**. Your FORM CC1 will be graded by a computer and it will record a **ZERO** for your score if your matriculation number is not correct.
6. Write your full name in section A of FORM CC1.
7. Only circles for answers 1 to 10 are to be shaded.
8. For each answer, the circle corresponding to your choice should be **properly** and **completely** shaded. If you change your answer later, you must make sure that the original answer is properly erased.
9. For each answer, **do not shade more than one circle**. The answer for a question with more than one circle shaded will be marked wrong.
10. **Do not fold** FORM CC1.
11. Submit FORM CC1 before you leave the test hall.

Formulae List

1. The **Taylor series** of f at a is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

2.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

3.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

4.

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

5.

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$$

6.

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

1. Let $f(x)$ be a differentiable function which satisfies $f(1) = \sqrt{3}$ and $f'(1) = 10$. Find the value of the expression $\frac{d}{dx}\left[\sqrt{1+[f(x)]^2}\right]$ at the point $x = 1$.

(A) $5\sqrt{3}$

(B) $\frac{5}{2}\sqrt{3}$

(C) 5

(D) $\frac{5}{2}$

(E) $\sqrt{3}$

2. Consider the curve $y = (\ln x)^{(\ln x)}$, which is defined on $x > 1$. Let L denote the tangent line to this curve at the point where $x = e^2$. Find the y-coordinate of the point of intersection of L with the y-axis.

(A) $3 - 4\ln 2$

(B) $1 - 4\ln 2$

(C) $-4\ln 2$

(D) $\ln 2 - 3$

(E) $-\frac{8}{3}\ln 2$

3. Find the limit

$$\lim_{x \to +\infty} \left(x + e^{x} + e^{2x}\right)^{1/x}$$

if it exists.

(A) 1

(B) 2

(C) e

(D) e^2

(E) The limit does not exist

4. A wire 10 m long is cut into two pieces. One piece is used to form a square. The other piece is used to form a rectangle with length twice as long as its width. If the total area enclosed by the two figures is minimum, then this minimum area in square metres equals

(A) $\frac{73}{25}$

(B) $\frac{50}{17}$

(C) $\frac{47}{16}$

(D) $\frac{99}{34}$

(E) $\frac{103}{35}$

5. Evaluate

$$\int_{\frac{1}{e}}^{e} |\ln x| \, dx$$

(A) $2\,(1+e)$

(B) $2\,(e-1)$

(C) $2\left(1+e^{-1}\right)$

(D) $2\left(e-e^{-1}\right)$

(E) $2\left(1-e^{-1}\right)$

6. Find the area of the finite region bounded by the straight line $y = x - 2$ and the curve $y^2 = x$.

(A) $\frac{14}{3}$

(B) $\frac{11}{3}$

(C) $\frac{19}{4}$

(D) $\frac{9}{2}$

(E) $\frac{30}{7}$

7. Let n be a positive integer which is bigger than 1505. Then

$$\int_1^2 \frac{1}{x\left(1+x^n\right)} dx =$$

(A) $\ln 2+\frac{1}{n} \ln \left(1+2^n\right)-\frac{1}{n} \ln 2$

(B) $\ln 2-\frac{1}{n} \ln \left(1+2^n\right)+\frac{1}{n} \ln 2$

(C) $\ln 2-\frac{1}{n} \ln \left(1+2^n\right)-\frac{1}{n} \ln 2$

(D) $-\ln 2+\frac{1}{n} \ln \left(1+2^n\right)-\frac{1}{n} \ln 2$

(E) $-\ln 2-\frac{1}{n} \ln \left(1+2^n\right)+\frac{1}{n} \ln 2$

8. A finite region R is bounded by the curve $y = 1 - x^2$ and the x-axis. Find the volume of the solid formed by revolving R one complete round about the x-axis.

(A) $\frac{16\pi}{15}$

(B) $\frac{17\pi}{16}$

(C) $\frac{15\pi}{14}$

(D) $\frac{4\pi}{3}$

(E) $\frac{6\pi}{5}$

9. Evaluate the sum

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+2)}.$$

(Hint: Integrate the Taylor series of xe^{-x}.)

(A) $\frac{1}{e}$

(B) $\frac{3-e}{e}$

(C) 1

(D) $\frac{e-1}{2e}$

(E) $\frac{e-2}{e}$

10. Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \left(\frac{1}{3^n + (-2)^n} \right) \frac{x^n}{(n+1)}.$$

(A) $\frac{1}{3}$

(B) $\frac{2}{3}$

(C) $\frac{3}{2}$

(D) 2

(E) 3

END OF PAPER

National University of Singapore

Department of Mathematics

2008-2009 Semester 1 MA1505 Mathematics I Mid-Term Test Answers

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Question	1	2	3	4	5	6	7	8	9	10
Answer	A	C	D	B	E	D	B	A	E	E

2008/2009 Test Solutions

1). $\frac{d}{dx}\sqrt{1+(f(x))^2}\Big|_{x=1} = \frac{f(x)\,f'(x)}{\sqrt{1+(f(x))^2}}\Big|_{x=1} = \frac{(\sqrt{3})(10)}{\sqrt{1+(\sqrt{3})^2}} = \underline{\underline{5\sqrt{3}}}$ 　 A

2). $\ln y = (\ln x)\ln(\ln x)$

$$\frac{y'}{y} = \frac{1}{x}\ln(\ln x) + (\ln x)\frac{1}{(\ln x)}\frac{1}{x} = \frac{\ln(\ln x)+1}{x}$$

at $x=e^2$, $y' = \left[(\ln e^2)^{(\ln e^2)}\right]\frac{\ln(\ln e^2)+1}{e^2}$

$$= \frac{4(\ln 2+1)}{e^2}$$

$\therefore L: \quad y-4 = \frac{4(\ln 2+1)}{e^2}(x-e^2)$

at $x=0$, $\quad y = 4-4(\ln 2+1) = \underline{\underline{-4\ln 2}}$ 　 C

3). Let $y = (x+e^x+e^{2x})^{1/x}$

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{\ln(x+e^x+e^{2x})}{x}$$

$$= \lim_{x\to\infty}\frac{\left(\frac{1}{x+e^x+e^{2x}}\right)(1+e^x+2e^{2x})}{1}$$

$$= \lim_{x\to\infty}\frac{e^{-2x}+e^{-x}+2}{xe^{-2x}+e^{-x}+1} = 2$$

$\therefore \lim_{x\to\infty} y = \underline{\underline{e^2}}$ 　 D

4). Let x = side of the square.

$\therefore$ Total area $= A = x^2 + \left[2\left(\frac{5-2x}{3}\right)\right]\left(\frac{5-2x}{3}\right)$, $0 \le x \le \frac{5}{2}$

$$A = x^2 + \frac{50 - 40x + 8x^2}{9} = \frac{17x^2 - 40x + 50}{9}$$

$$\frac{dA}{dx} = 0 \Rightarrow 34x - 40 = 0 \Rightarrow x = \frac{20}{17} \in \left[0, \frac{5}{2}\right]$$

$\frac{dA}{dx}$	$[0, \frac{20}{17})$	$(\frac{20}{17}, \frac{5}{2}]$
	$-$	$+$

$0 \qquad \frac{20}{17} \qquad \frac{5}{2}$

$\uparrow$ absolute minimum at $\frac{20}{17}$

at $x = \frac{20}{17}$, $A = \dfrac{17\left(\frac{20}{17}\right)^2 - 40\left(\frac{20}{17}\right) + 50}{9} = \underline{\underline{\frac{50}{17}}}$ (B)

5). $$\int_{1/e}^{e} |\ln x|\, dx = \int_{1/e}^{1} -\ln x\, dx + \int_{1}^{e} \ln x\, dx$$

$$= -x\ln x\Big|_{1/e}^{1} + \int_{1/e}^{1} dx + x\ln x\Big|_{1}^{e} - \int_{1}^{e} dx$$

$$= -\frac{1}{e} + 1 - \frac{1}{e} + e - e + 1$$

$$= 2 - \frac{2}{e} = \underline{\underline{2(1 - e^{-1})}}$$ (E)

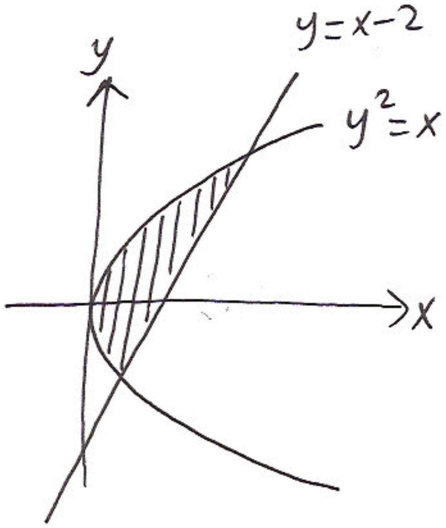
6) $x = y+2$ and $x = y^2$

$\Rightarrow y^2 = y+2$

$\Rightarrow y^2 - y - 2 = 0$

$\Rightarrow (y-2)(y+1) = 0$

$\Rightarrow y = -1, 2$



$$\text{Area} = \int_{-1}^{2} (y+2-y^2)\,dy$$

$$= \left[\frac{1}{2}y^2 + 2y - \frac{1}{3}y^3\right]_{-1}^{2}$$

$$= \frac{9}{2}$$

D

7). Let $u = 1 + x^n$

$\therefore du = nx^{n-1}\,dx$, $\quad x=1 \Rightarrow u=2$, $\quad x=2 \Rightarrow u = 1+2^n$

$$\therefore \int_1^2 \frac{1}{x(1+x^n)}\,dx = \int_2^{1+2^n} \frac{1}{x(u)} \frac{du}{nx^{n-1}}$$

$$= \frac{1}{n}\int_2^{1+2^n} \frac{du}{(u-1)u} = \frac{1}{n}\int_2^{1+2^n} \left(\frac{1}{u-1} - \frac{1}{u}\right) du$$

$$= \frac{1}{n}\Big[\ln|u-1| - \ln|u|\Big]_2^{1+2^n}$$

$$= \frac{1}{n}\left\{ n\ln 2 - \ln(1+2^n) + \ln 2 \right\}$$

$$= \ln 2 - \frac{1}{n}\ln(1+2^n) + \frac{1}{n}\ln 2$$

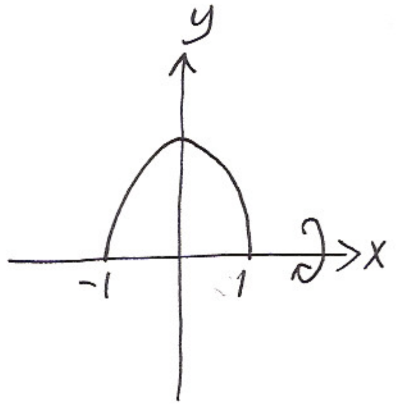
B

8). $$\text{Volume} = \int_{-1}^{1} \pi (1-x^2)^2 \, dx$$

$$= 2\pi \int_0^1 (1-2x^2+x^4)\, dx$$

$$= 2\pi \left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5 \right]_0^1$$

$$= \underline{\underline{\frac{16\pi}{15}}}$$



A

9). $$\int_0^1 x e^{-x} dx = \int_0^1 x \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 x^{n+1} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+2)}$$

$$\| $$

$$-\int_0^1 x\, d(e^{-x})$$

$$\| $$

$$-xe^{-x}\Big|_0^1 + \int_0^1 e^{-x} dx = -e^{-1} - e^{-x}\Big|_0^1 = -2e^{-1}+1 = \underline{\underline{\frac{e-2}{e}}}$$

E

10). $$\left| \frac{\dfrac{x^{n+1}}{(n+2)\{3^{n+1}+(-2)^{n+1}\}}}{\dfrac{x^n}{(n+1)\{3^n+(-2)^n\}}} \right| = \frac{(n+1)\{1+(\frac{-2}{3})^n\}}{(n+2)\{3+(-2)(\frac{-2}{3})^n\}} |x| \longrightarrow \frac{1}{3}|x|$$

$$\frac{1}{3}|x| < 1 \Rightarrow \underline{\underline{|x| < 3}}$$

E