CHAPTER 6

LINEAR TRANSFORMATIONS

6.1 WHAT IS A LINEAR TRANSFORMATION?

You know what a function is - it's a RULE which turns NUMBERS INTO OTHER NUMBERS: $f(x) = x^2$ means "please turn 3 into 9, 12 into 144 and so on".

Similarly a TRANSFORMATION is a rule which turns VECTORS into other VECTORS. For example, "please rotate all 3-dimensional vectors through an angle of 90° clockwise around the z-axis". A LINEAR TRANSFORMATION T is one that ALSO satisfies these rules: if c is any scalar, and $\overrightarrow{u}$ and $\overrightarrow{v}$ are vectors, then

$$T(c\overrightarrow{u}) = cT(\overrightarrow{u}) \text{ and } T(\overrightarrow{u} + \overrightarrow{v}) = T(\overrightarrow{u}) + T(\overrightarrow{v}).$$

EXAMPLE: Let I be the rule $I\overrightarrow{u} = \overrightarrow{u}$ for all $\overrightarrow{u}$.

You can check that I is linear! Called IDENTITY Linear Transformation.

EXAMPLE : Let D be the rule $D\overrightarrow{u} = 2\overrightarrow{u}$ for all $\overrightarrow{u}$.

$D(c\overrightarrow{u}) = 2(c\overrightarrow{u}) = c(2\overrightarrow{u}) = cD\overrightarrow{u}$

$D(\overrightarrow{u} + \overrightarrow{v}) = 2(\overrightarrow{u} + \overrightarrow{v}) = 2\overrightarrow{u} + 2\overrightarrow{v} = D\overrightarrow{u} + D\overrightarrow{v} \rightarrow$ LINEAR!

Note: Usually we write $D(\overrightarrow{u})$ as just $D\overrightarrow{u}$.

Note: A linear transformation T always maps $\vec{0}$ to $\vec{0}$.

<u>Proof</u>: $T(\vec{0}) = T(\vec{0}+\vec{0})$

$= T(\vec{0}) + T(\vec{0})$

$\therefore T(\vec{0}) = \vec{0}$. //

6.2. THE BASIC BOX, AND THE MATRIX OF A LINEAR TRANSFORMATION

The usual vectors $\hat{i}$ and $\hat{j}$ define a square:

Let's call this the BASIC BOX in two dimensions.

Similarly, $\hat{i}, \hat{j}$, and $\hat{k}$ define the BASIC BOX in 3 dimensions.

Matrix representation of a linear transformation T.

Let $\vec{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Suppose $T(\vec{i}) = a\vec{i} + c\vec{j} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$

and $T(\vec{j}) = b\vec{i} + d\vec{j} = b\begin{pmatrix} 1 \\ 0 \end{pmatrix} + d\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$

For any $\vec{u} = \begin{pmatrix} x \\ y \end{pmatrix} = x\vec{i} + y\vec{j}$

$$T(\vec{u}) = xT(\vec{i}) + yT(\vec{j})$$

$$= x\begin{pmatrix} a \\ c \end{pmatrix} + y\begin{pmatrix} b \\ d \end{pmatrix}$$

$$= \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

We say that the matrix of T relative to $\vec{i}, \vec{j}$ is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Example The identity transformation

$$I(\vec{u}) = \vec{u} \text{ for all vectors } \vec{u}.$$

In two dimensions:

$$I(\vec{i}) = \vec{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$I(\vec{j}) = \vec{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\therefore \quad I \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

<u>Example</u> In two dimensions, $T(\vec{u}) = k\vec{u}$

where k = fixed constant.

Then $T(\vec{i}) = k\vec{i} = \begin{pmatrix} k \\ 0 \end{pmatrix}$

$$T(\vec{j}) = k\vec{j} = \begin{pmatrix} 0 \\ k \end{pmatrix}$$

$$\therefore \; T \sim \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

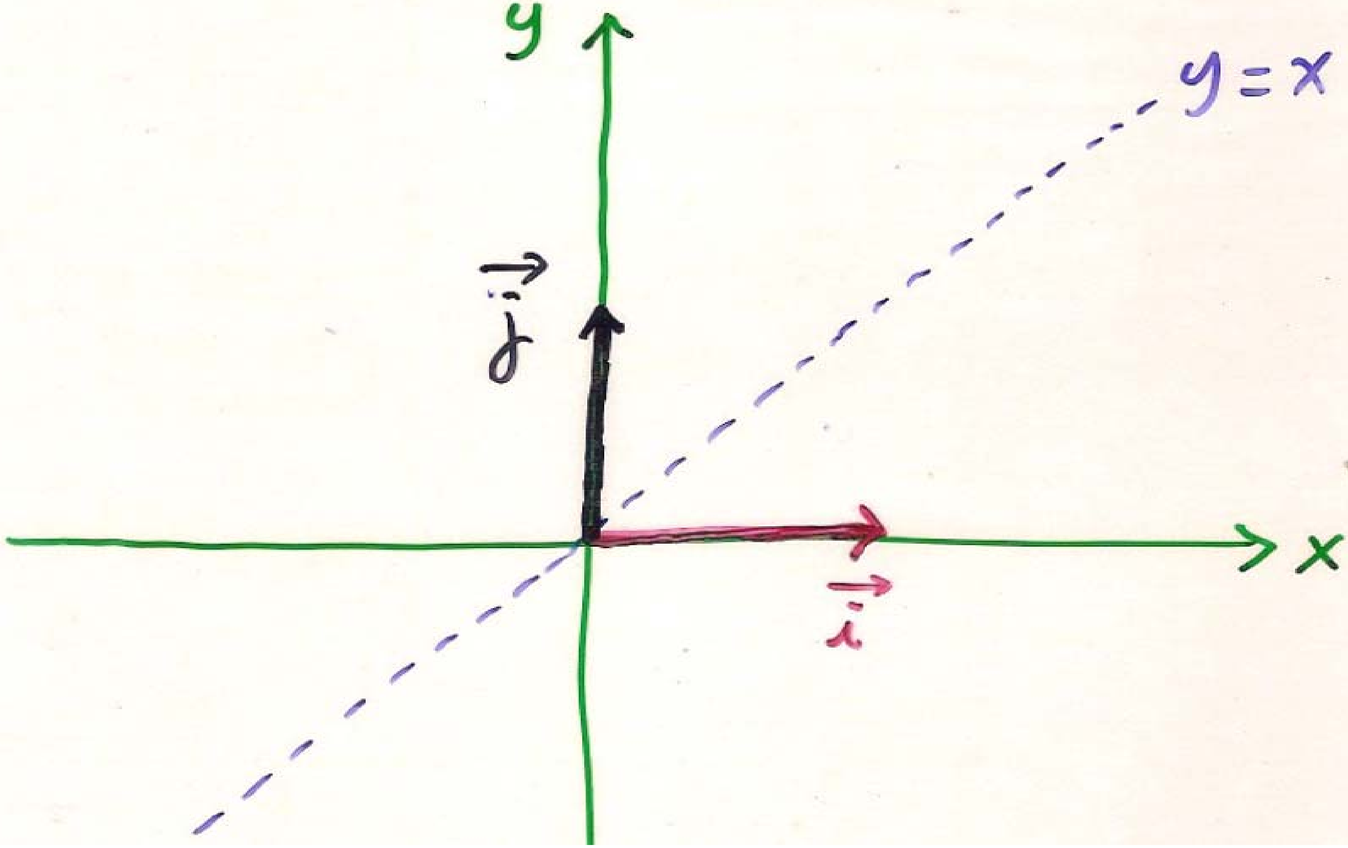
Example In two dimensions,

$$T(\vec{i}) = \vec{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$T(\vec{j}) = \vec{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\therefore \; T \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Note: This is a reflection about the line $y = x$.



<u>Example</u> In two dimensions,

$$T(\vec{i}) = i + \frac{1}{4}\vec{j} = \begin{pmatrix} 1 \\ 1/4 \end{pmatrix}$$

$$T(\vec{j}) = \frac{1}{4}\vec{i} + \vec{j} = \begin{pmatrix} 1/4 \\ 1 \end{pmatrix}$$

$$\therefore T \sim \begin{pmatrix} 1 & 1/4 \\ 1/4 & 1 \end{pmatrix}$$

Example In three dimensions,

$$T\vec{i} = \vec{i} + 4\vec{j} + 7\vec{k} = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}$$

$$T\vec{j} = 2\vec{i} + 5\vec{j} + 8\vec{k} = \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}$$

$$T\vec{k} = 3\vec{i} + 6\vec{j} + 9\vec{k} = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}$$

$$\therefore T \sim \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

<u>Example</u> $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$T\vec{i} = \vec{i} + \vec{j} + 2\vec{k} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$T\vec{j} = \vec{i} - 3\vec{k} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$$

$$\therefore\ T \sim \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 2 & -3 \end{pmatrix}$$

e.g. What is $T(-\vec{i}+\vec{j})$?

Solution: $T(-\vec{i}+\vec{j}) = T\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -1 \\ -5 \end{pmatrix}$$

$$= \underline{\underline{-\vec{j} - 5\vec{k}}}$$

<u>Example</u> $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$,

$$T\vec{i} = 2\vec{i} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$T\vec{j} = \vec{i} + \vec{j} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$T\vec{k} = \vec{i} - \vec{j} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\therefore T \sim \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

e.g. What is $T(\vec{i} - \vec{j} + 2\vec{k})$?

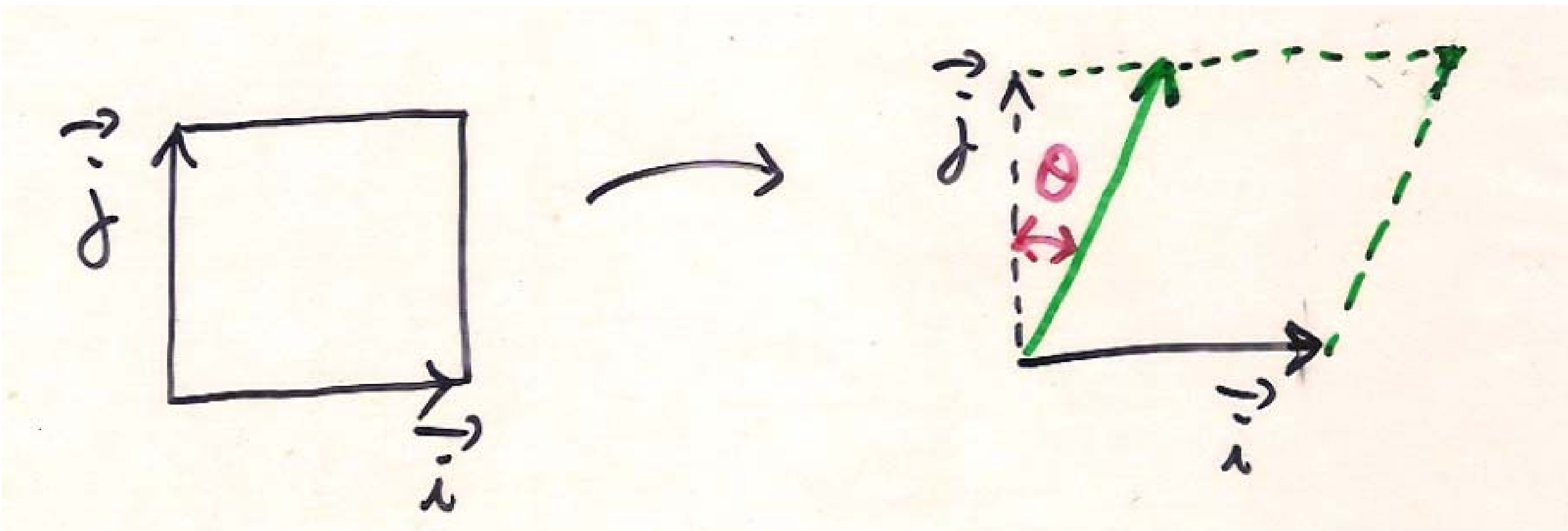
Solution: $T(\vec{i} - \vec{j} + 2\vec{k}) = T\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$$= \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ -3 \end{pmatrix}$$

$$= \underline{\underline{3\vec{i} - 3\vec{j}}}$$

EXAMPLE: Suppose, in solid mechanics, you take a flat square of rubber and SHEAR it, as shown.



A two dimensional <u>shear</u>

$$S\vec{i} = \vec{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$S\vec{j} = (\tan\theta)\,\vec{i} + \vec{j} = \begin{pmatrix} \tan\theta \\ 1 \end{pmatrix}$$

$$\therefore \quad S \sim \begin{pmatrix} 1 & \tan\theta \\ 0 & 1 \end{pmatrix}$$

e.g. A shearing force, parallel to the x-axis, with shearing angle 30°, is applied to the vector $2\vec{i} - 3\vec{j}$. What is its image?

Solution: $S \sim \begin{pmatrix} 1 & \tan 30^\circ \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1/\sqrt{3} \\ 0 & 1 \end{pmatrix}$

$$\therefore S(2\vec{i} - 3\vec{j}) = S\begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1/\sqrt{3} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} 2-\sqrt{3} \\ -3 \end{pmatrix}$$

$$= (2-\sqrt{3})\,\vec{i} - 3\,\vec{j}$$

Example In two dimensions,

$$T\vec{i} = \vec{i} + \vec{j} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$T\vec{j} = \vec{i} + \vec{j} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

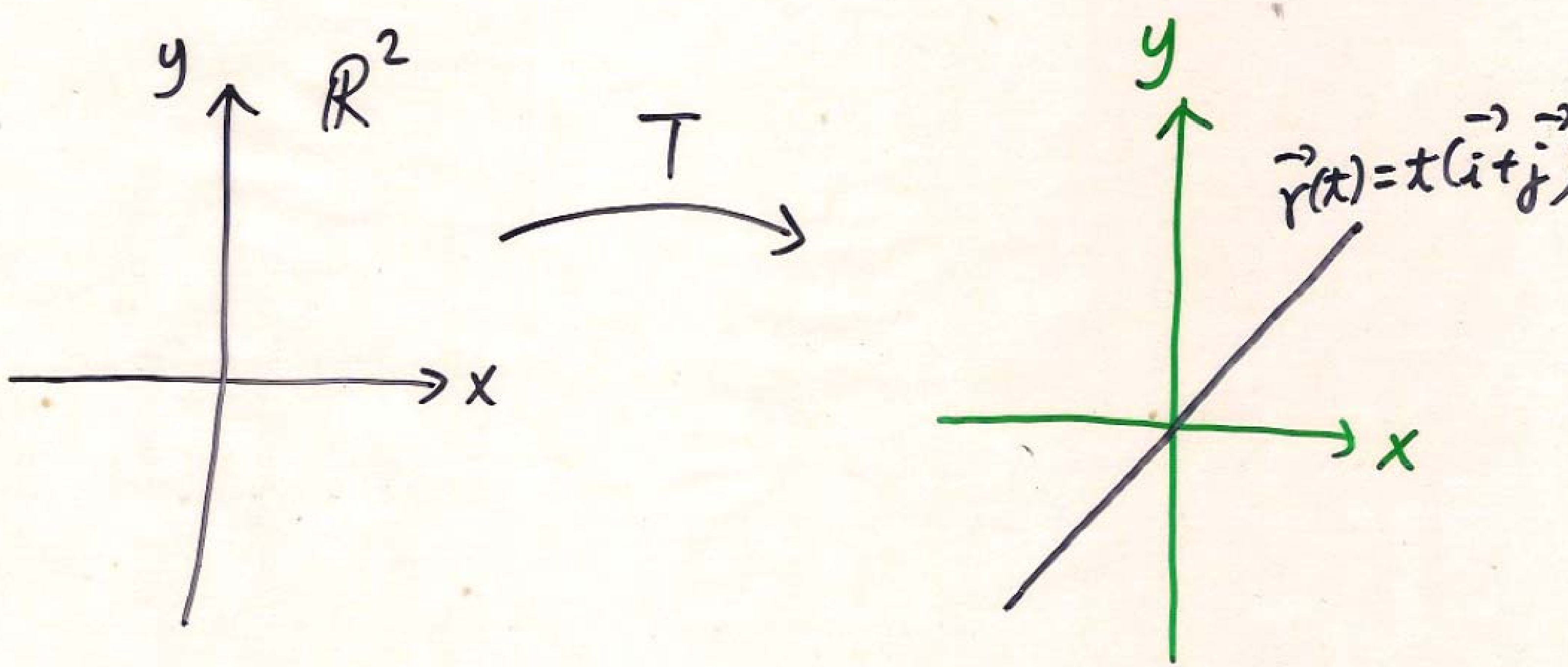
$$\therefore T \sim \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Take any vector $\vec{u} = a\vec{i} + b\vec{j} = \begin{pmatrix} a \\ b \end{pmatrix}$

Then $T\vec{u} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+b \\ a+b \end{pmatrix}$

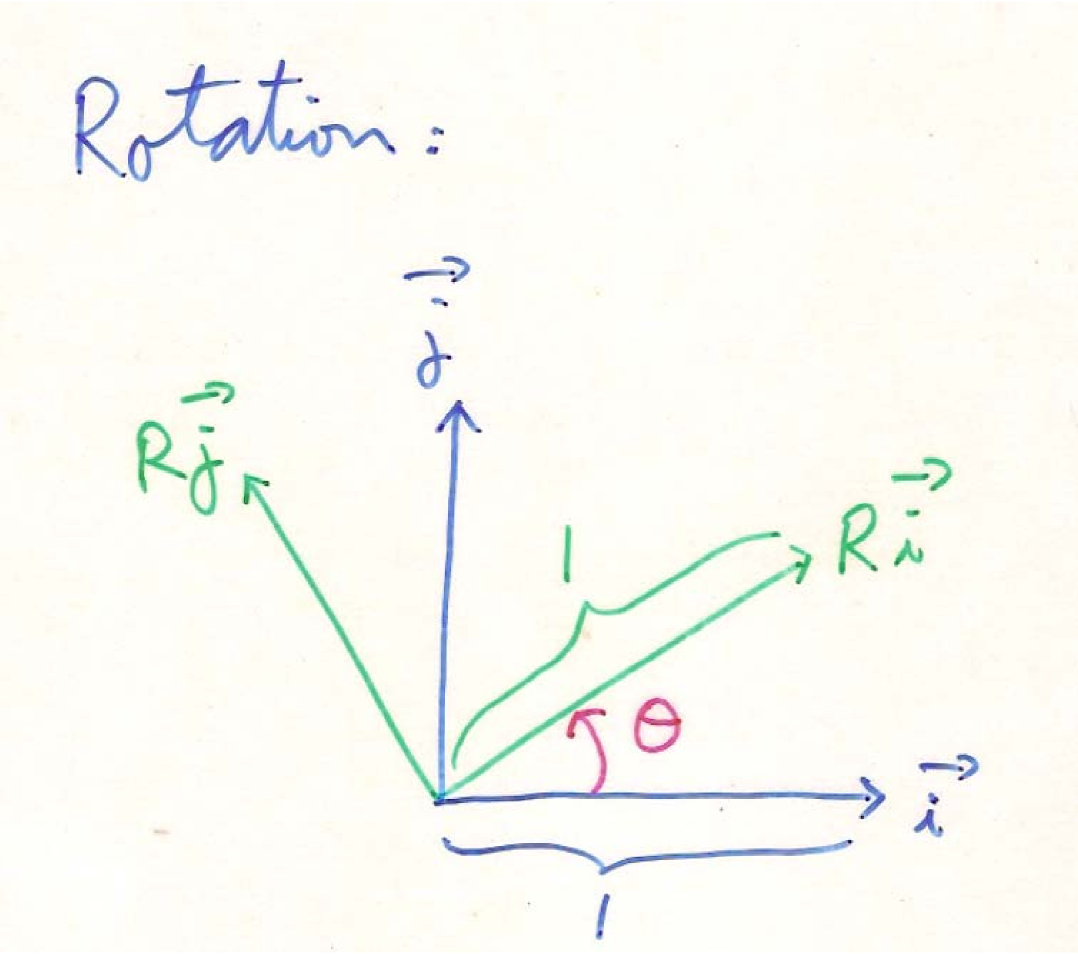
$$= (a+b)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = (a+b)\{\vec{i} + \vec{j}\}$$

$\therefore$ T maps $\mathbb{R}^2$ to the line $\vec{r}(t) = t(\vec{i} + \vec{j})$



y
R^2
x
T
y
$\vec{r}(t) = t(\vec{i} + \vec{j})$
x

EXAMPLE: Rotations in the plane.



$$R\vec{i} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$R\vec{j} = \cos\left(\frac{\pi}{2}+\theta\right)\vec{i} + \sin\left(\frac{\pi}{2}+\theta\right)\vec{j}$$

$$= -\sin\theta\,\vec{i} + \cos\theta\,\vec{j} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

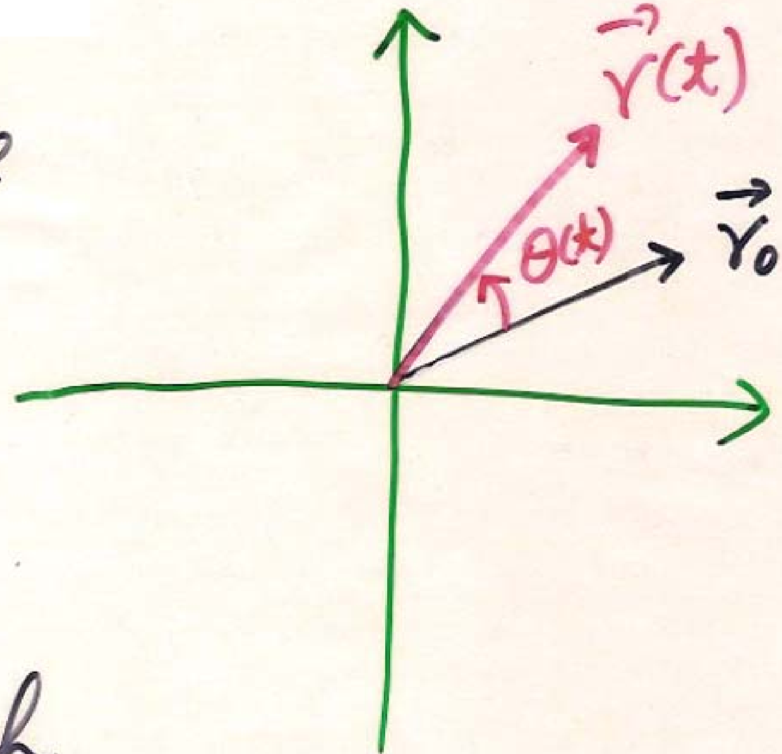
The matrix of R relative to $\vec{i}, \vec{j}$ is

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Application: Suppose an object is moving on a circle at constant angular speed ω. What is its acceleration?

at $t=0$, the particle is at $\vec{r}_0$.

at t, the particle has rotated through an angle $\theta(t) = \omega t$



$$\therefore \ \vec{r}(t) = \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix} \vec{r}_0$$

$$= \begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix} \vec{r}_0$$

$$\dot{\vec{r}}(t) = \frac{d}{dt}\vec{r}(t) = \begin{pmatrix} -\omega\sin\omega t & -\omega\cos\omega t \\ \omega\cos\omega t & -\omega\sin\omega t \end{pmatrix} \vec{r_0}$$

$$\ddot{\vec{r}} = \begin{pmatrix} -\omega^2\cos\omega t & \omega^2\sin\omega t \\ -\omega^2\sin\omega t & -\omega^2\cos\omega t \end{pmatrix} \vec{r_0}$$

$$= -\omega^2 \begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix} \vec{r_0}$$

$$= -\omega^2 \vec{r}$$

$$\frac{d^2 \overrightarrow{r}}{dt^2} = -\omega^2 \overrightarrow{r},$$

which is formula you know from physics.

6.3. COMPOSITE TRANSFORMATIONS AND MATRIX MULTIPLICATION.

You know what it means to take the COMPOSITE of two functions: if $f(u) = \sin(u)$, and $u(x) = x^2$, then $f \circ u$ means: "please do u FIRST, THEN f, so $f \circ u(x) = \sin(x^2)$. NOTE THE ORDER!!

$u \circ f(x) = \sin^2(x)$, NOT the same!

Similarly if A and B are linear transformations, then AB means "do B FIRST, then A".

Consider an example

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "B"] \arrow[dr, dashed, "A \circ B"'] & \mathbb{R}^3 \arrow[d, "A"] \\ & \mathbb{R}^2 \end{tikzcd}$$

Suppose

$$\left.\begin{aligned} B\vec{i} &= \vec{i} + \vec{j} + \vec{k} \\ B\vec{j} &= \vec{i} - \vec{j} + \vec{k} \end{aligned}\right\} \Rightarrow B \sim \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\left.\begin{aligned} A\vec{i} &= \vec{i} + \vec{j} \\ A\vec{j} &= \vec{i} - \vec{j} \\ A\vec{k} &= -\vec{i} + \vec{j} \end{aligned}\right\} \Rightarrow A \sim \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

Then $A \circ B(\vec{i}) = A(B\vec{i})$

$$= A(\vec{i} + \vec{j} + \vec{k})$$

$$= (\vec{i} + \vec{j}) + (\vec{i} - \vec{j}) + (-\vec{i} + \vec{j})$$

$$= \vec{i} + \vec{j}$$

$$A \circ B(\vec{j}) = A(B\vec{j})$$

$$= A(\vec{i} - \vec{j} + \vec{k})$$

$$= (\vec{i} + \vec{j}) - (\vec{i} - \vec{j}) + (-\vec{i} + \vec{j})$$

$$= -\vec{i} + 3\vec{j}$$

$$\therefore \; A \circ B \sim \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$$

$$\therefore \; A \circ B \sim AB$$

IMPORTANT FACT: Suppose a_{ij} is the matrix of a linear transformation A relative to $\hat{i}\hat{j}\hat{k}$, and suppose b_{ij} is the matrix of the Linear Transformation B relative to $\hat{i}\hat{j}\hat{k}$. Suppose that AB makes sense. Then the matrix of AB relative to $\hat{i}\hat{j}$ or $\hat{i}\hat{j}\hat{k}$ is just the matrix product of a_{ij} and b_{ij}.

EXAMPLE: What happens to the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ if we shear 45° parallel to the x axis and then rotate 90° anticlockwise? What if we do the same in the reverse order?

S = shear $45°$, // to x-axis.

R = Rotate $90°$ anticlockwise

$$\therefore S \sim \begin{pmatrix} 1 & \tan 45° \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$R \sim \begin{pmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(i) Shear first, then rotate:

$$R \circ S \sim \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

Apply to $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, we have

$$R \circ S \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$= -2\vec{i} + 3\vec{j}$$

(ii) Rotate first, then shear:

$$S \circ R \sim \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

Apply to $\begin{pmatrix}1\\2\end{pmatrix}$, we have

$$S \circ R \begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix}$$

$$= \begin{pmatrix}-1\\1\end{pmatrix}$$

$$= -\vec{i} + \vec{j}$$

Example

$$B \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{pmatrix}$$

$$A \sim \begin{pmatrix} 0 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

(i) $A \circ B \sim \begin{pmatrix} 0 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{pmatrix}$

$$= \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}$$

(ii) $B \circ A \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$$

EXAMPLE: Suppose you take a piece of rubber in 2 dimensions and shear it parallel to the x axis by θ degrees, and then shear it again by ϕ degrees. What happens?

S_θ = shear by θ°, $\parallel$ to x-axis.

S_ϕ = shear by ϕ°, $\parallel$ to x-axis.

S_θ first and then S_ϕ:

$$S_\phi \circ S_\theta \sim \begin{pmatrix} 1 & \tan\phi \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \tan\theta \\ 0 & 1 \end{pmatrix}$$

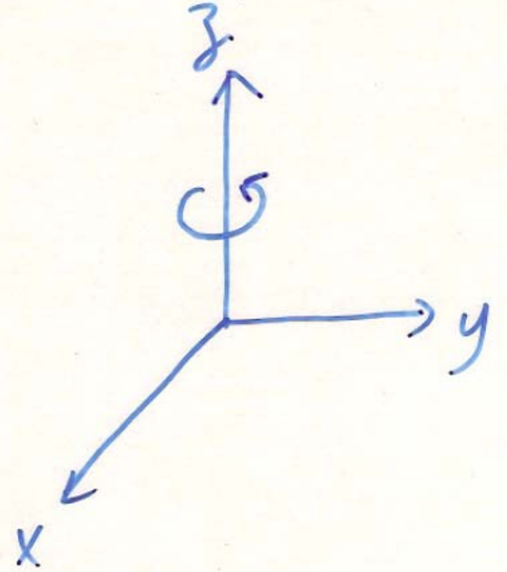
$$= \begin{pmatrix} 1 & \tan\phi + \tan\theta \\ 0 & 1 \end{pmatrix}$$

This is also a shear by $\alpha°$, // to x-axis, where α satisfies

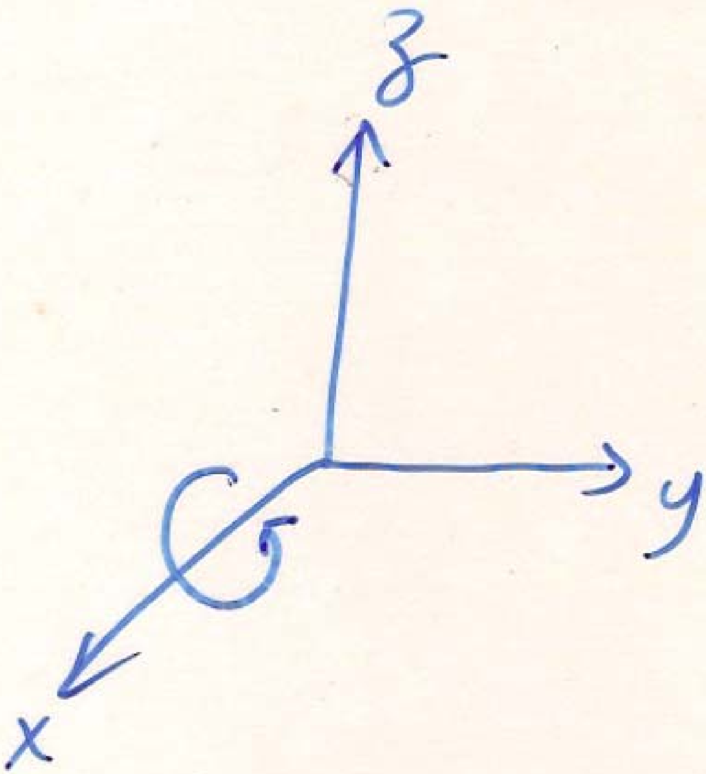
$$\tan\alpha = \tan\phi + \tan\theta$$

The shear angles don't add up, since $\tan\phi + \tan\theta \neq \tan(\theta + \phi)$.

EXAMPLE: Rotate 90° around z-axis, then rotate 90° around x-axis in 3 dimensions. [Always anti-clockwise unless otherwise stated.] Is it the same if we reverse the order?



$$A: \quad \begin{aligned} \vec{i} &\longmapsto \vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ \vec{j} &\longmapsto -\vec{i} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \\ \vec{k} &\longmapsto \vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{aligned}$$



$$B: \quad \begin{aligned} \vec{i} &\longmapsto \vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ \vec{j} &\longmapsto \vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ \vec{k} &\longmapsto -\vec{j} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \end{aligned}$$

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

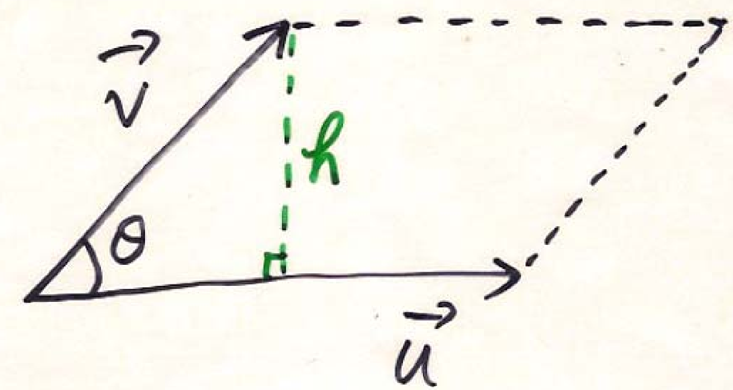
A then B $\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$

B then A $\Rightarrow \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

so the answer is NO!

6.4 DETERMINANTS

Area of a parallelogram



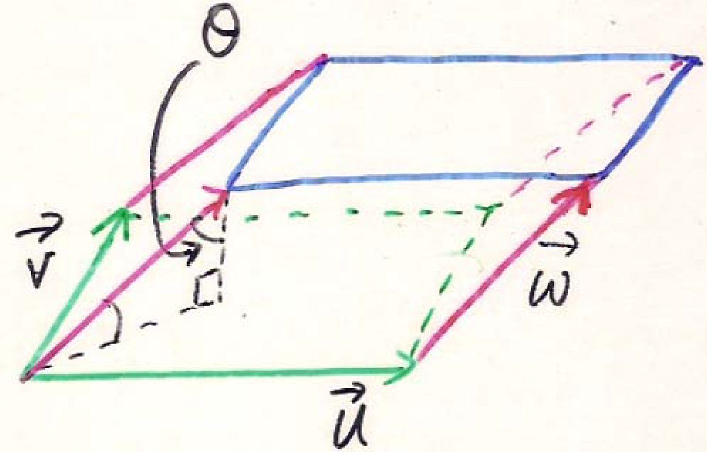
Area of parallelogram

$$= (\|\vec{u}\|)h$$

$$= \|\vec{u}\|\,\|\vec{v}\| \sin\theta$$

$$= \|\vec{u} \times \vec{v}\|$$

Volume of a parallelopiped



Base area = area of green parallelogram

$$= \|\vec{u} \times \vec{v}\|$$

height $= \|\vec{w}\| |\cos\theta|$

Volume $= \|\vec{u} \times \vec{v}\| \cdot \|\vec{w}\| |\cos\theta|$

$$= |(\vec{u} \times \vec{v}) \cdot \vec{w}|$$

Determinant

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation.

$\therefore T\vec{i}$ and $T\vec{j}$ are two vectors in $\mathbb{R}^2$.

Then $\vec{T_i} \times \vec{T_j} = \begin{cases} \vec{0} & \text{if } \vec{T_i} \text{ is } \parallel \text{ to } \vec{T_j} \\ \text{a vector } \perp \text{ to } \mathbb{R}^2 & \\ \quad \text{if } \vec{T_i} \nparallel \vec{T_j} \end{cases}$

$\therefore \vec{T_i} \times \vec{T_j} = \text{multiple of } \vec{k}$

where this multiple is zero if $\vec{T_i} \parallel \vec{T_j}$.

The <u>determinant</u> of T is defined by

$$T\vec{i} \times T\vec{j} = (\det T)\,\vec{k}.$$

Note: $\operatorname{Det} T = 0 \iff T\vec{i} \parallel T\vec{j}$.

Example I = identity map.

$$I\vec{i} \times I\vec{j} = \vec{i} \times \vec{j} = (1)\vec{k}$$

$$\therefore \det I = \underline{\underline{1}}$$

Example $T\vec{u} = c\vec{u}$, $c = \text{constant}$.

$$T\vec{i} \times T\vec{j} = c\vec{i} \times c\vec{j}$$

$$= (c^2)\vec{k}$$

$$\therefore \det T = c^2$$

Example

$$T\vec{i} = \vec{i} + \frac{1}{4}\vec{j}$$

$$T\vec{j} = \frac{1}{4}\vec{i} + \vec{j}$$

$$T\vec{i} \times T\vec{j} = \left(\vec{i} + \frac{1}{4}\vec{j}\right) \times \left(\frac{1}{4}\vec{i} + \vec{j}\right)$$

$$= \vec{i} \times \vec{j} + \frac{1}{16}\vec{j} \times \vec{i}$$

$$= \vec{i} \times \vec{j} - \frac{1}{16}\vec{i} \times \vec{j}$$

$$= \frac{15}{16}\vec{k}$$

$$\therefore \det T = \underline{\underline{\frac{15}{16}}}$$

EXAMPLE: $T\hat{i} = \hat{j}$, $T\hat{j} = \hat{i}$,

$$T\hat{i} \times T\hat{j} = \hat{j} \times \hat{i} = -\hat{k} \to \det T = -1$$

EXAMPLE: Shear, $S\hat{i} = \hat{i}$, $S\hat{j} = \hat{i}\tan\theta + \hat{j}$,

$$S\hat{i} \times S\hat{j} = \hat{k} \rightarrow \det S = 1.$$

EXAMPLE: $T\hat{i} = \hat{i} + \hat{j} = T\hat{j}$,

$$T\hat{i} \times T\hat{j} = \overrightarrow{0} \rightarrow \det T = 0.$$

EXAMPLE: Rotation

$$\begin{aligned} R\hat{i} \times R\hat{j} &= (\cos\theta\hat{i} + \sin\theta\hat{j}) \times (-\sin\theta\hat{i} + \cos\theta\hat{j}) \\ &= (\cos^2\theta - -\sin^2\theta)\hat{k} = \hat{k} \to \det(R) = 1. \end{aligned}$$

$$T\vec{i} \times T\vec{j} = (\det T)\,\vec{k}$$

$$\det T = \left(T\vec{i} \times T\vec{j}\right) \cdot \vec{k}$$

$$T \sim \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$T\vec{i} = a\vec{i} + c\vec{j}$$

$$T\vec{j} = b\vec{i} + d\vec{j}$$

$$T\vec{i} \times T\vec{j} = (a\vec{i} + c\vec{j}) \times (b\vec{i} + d\vec{j})$$

$$= 0 + ad\vec{k} - bc\vec{k} + 0$$

$$= (ad - bc)\vec{k}$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} \vec{k}$$

For 3×3 determinant:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$- a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$+ a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example

$$\begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 2 & 0 & 0 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix} - (-1)\begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} + 0\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix}$$

$$= 1(0) + 1(2) + 0$$

$$= \underline{\underline{2}}$$

Some properties:

(a) $\det(AB) = (\det A)(\det B)$

$\therefore$ By Induction

$\det(A_1 A_2 \cdots A_n) = (\det A_1)(\det A_2)\cdots(\det A_n)$

(b) $\det(A^T) = \det A$

(c) $\det(cA) = c^n \det A$

where c = constant

and A = $n \times n$ matrix

Example

Let M be an orthogonal matrix

$$\therefore \quad MM^T = I$$

$$\therefore \quad \det(MM^T) = \det I = 1$$

$$(\det M)(\det M^T) = 1$$

$$(\det M)(\det M) = 1$$

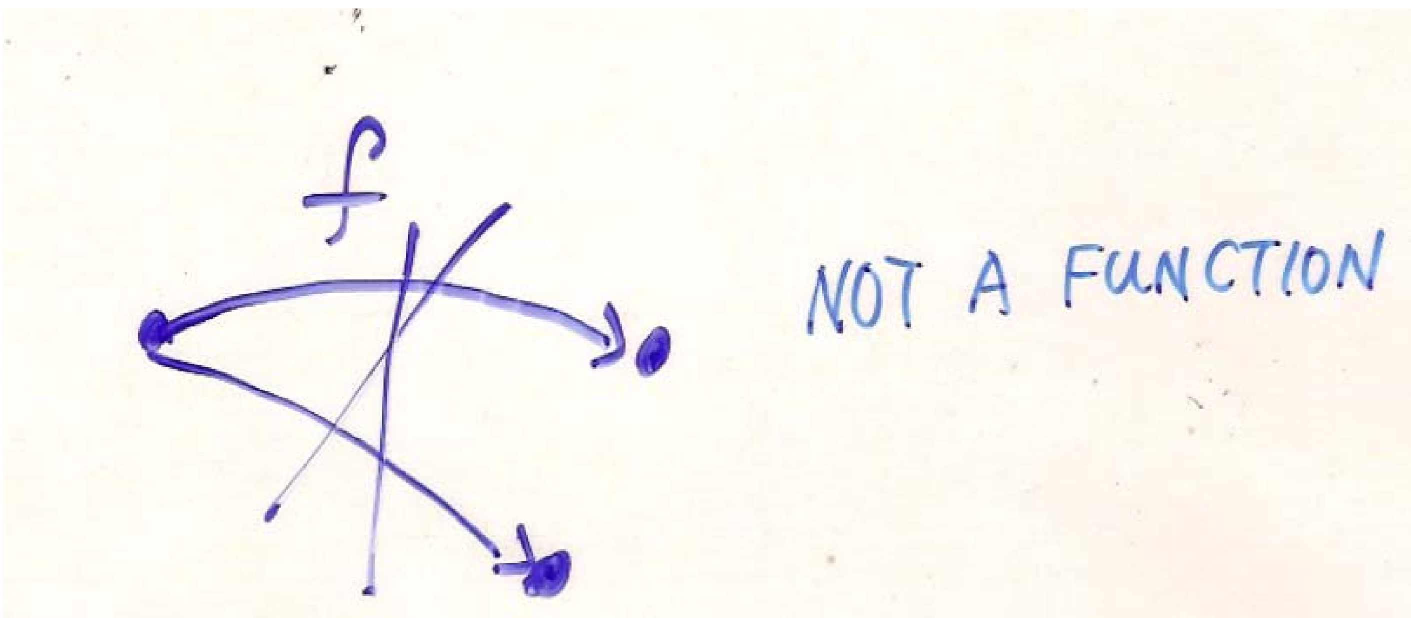
$$(\det M)^2 = 1$$

$$\therefore \quad \det M = \underline{\underline{\pm 1}}$$

6.5. INVERSES.

If I give you a 3-dimensional vector $\overrightarrow{u}$ and a 3-dimensional linear transformation T, then T sends

$\overrightarrow{u}$ to a particular vector, it never sends $\overrightarrow{u}$ to two DIFFERENT VECTORS! So this picture is impossible:



$$\begin{tikzcd} 1 \arrow[dr, "f(x)=x^2"] & \\ & 1 \\ -1 \arrow[ur] & \end{tikzcd}$$

Function? Yes.

NOT INJECTIVE

$\bullet \xrightarrow{f} \bullet$

INJECTIVE

f^{-1} exists.

Suppose T is not injective.

$\therefore$ We can find $\vec{u} \neq \vec{v}$ with

$$T(\vec{u}) = T(\vec{v})$$

Let $\vec{w} = \vec{u} - \vec{v}$

$\therefore \vec{w} \neq \vec{0}$

and $T(\vec{w}) = T(\vec{u} - \vec{v})$

$= T\vec{u} - T\vec{v} = \vec{0}$

Clearly, we have either $\vec{w} \nparallel \vec{i}$ or $\vec{w} \nparallel \vec{j}$ (or may be both).

Say $\vec{w} \nparallel \vec{i}$.

Then we can write

$$\vec{j} = a\vec{i} + b\vec{w}$$

where a, b are constants.

$$\therefore Tj = aT\vec{i} + bT\vec{w}$$

$$= aT\vec{i} \qquad (\because T\vec{w} = \vec{0})$$

$$\therefore T\vec{i} \times T\vec{j} = T\vec{i} \times aT\vec{i}$$

$$= a\{T\vec{i} \times T\vec{i}\} = 0$$

$$\therefore \det T = 0.$$

Conversely, it can be shown that if $\det T = 0$, then T is not injective.

<u>Definition</u>: T is <u>singular</u> if $\det T = 0$.

<u>Note</u>: T is non-singular

$\iff T^{-1}$ exists.

Example

$$T \sim \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\det T = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 1 - 1 = 0$$

$\therefore$ T is singular

i.e. T^{-1} does not exist.

Example

$$T \sim \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\det T = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = 0 - (-1) = 1 \neq 0$$

$\therefore$ T is non-singular

i.e. T^{-1} exists.

How to FIND THE INVERSE.

$$\boxed{\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

For example, when we needed to find the matrix S in Section 4 of Chapter 5, we needed to find a way of solving

$$S\begin{bmatrix} 0.7 & -0.4 \\ -0.5 & 0.7 \end{bmatrix} = I.$$

This just means that we need to inverse of $\begin{bmatrix} 0.7 & -0.4 \\ -0.5 & 0.7 \end{bmatrix}$, and the above formula does the job for us.

For bigger square matrices there are many tricks for finding inverses. A general [BUT NOT VERY PRACTICAL] method is as follows:

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Define: cofactor of $a_{11} = A_{11}$

$$= + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

cofactor of $a_{12} = A_{12}$

$$= - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$\vdots$

cofactor of $a_{33} = A_{33}$

$$= + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Form the adjoint of A :

$$\text{adjoint}(A) = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}^T$$

If A is non-singular,

$\therefore \det A \neq 0$

Then

$$\boxed{A^{-1} = \frac{1}{\det(A)} \text{adjoint}(A)}$$

Example on page 33

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then $\det A = 1 \neq 0$

$A_{11} = 1, \quad A_{12} = 0, \quad A_{13} = 0$

$A_{21} = 0, \quad A_{22} = 1, \quad A_{23} = 0$

$A_{31} = -1, \quad A_{32} = 0, \quad A_{33} = 1$

$$\therefore \text{adjoint}(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}^T$$

$$= \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\therefore \; A^{-1} = \frac{1}{\det(A)} \, \text{adjoint}(A) = \underline{\underline{\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}}$$

Check:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Inverse of a product:

A, B are $n \times n$ matrices

and $\det A \neq 0$, $\det B \neq 0$

Then $\det(AB) = (\det A)(\det B) \neq 0$

$\therefore (AB)^{-1}$ exists.

Note: $\boxed{(AB)^{-1} = B^{-1} A^{-1}}$

Proof: $(B^{-1}A^{-1})(AB)$

$$= B^{-1}(A^{-1}A)B$$

$$= B^{-1}B$$

$$= \underline{I}$$

$$\therefore \; B^{-1}A^{-1} = (AB)^{-1}.$$

APPLICATION: SOLVING LINEAR SYSTEMS.

System of linear equations:

$$(*) \quad \begin{cases} a_{11}x + a_{12}y + a_{13}z = c_1 \\ a_{21}x + a_{22}y + a_{23}z = c_2 \\ a_{31}x + a_{32}y + a_{33}z = c_3 \end{cases}$$

(*) can be written in matrix form as

$$A\vec{u} = \vec{C}$$

where $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$$\vec{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

Case 1: If $\det A \neq 0$,

$\therefore$ A is non-singular

$\therefore$ A^{-1} exists.

$\therefore$ $A^{-1}(A\vec{u}) = A^{-1}\vec{c}$

i.e. $\underline{\underline{\vec{u} = A^{-1}\vec{c}}}$

$\therefore$ (*) has an unique solution.

Case 2: If $\det A = 0$,

$\therefore$ A is singular

$\therefore$ A^{-1} does not exist.

In this case, it can be shown that

(*) has either no solution

or infinitely many solutions.

Suppose you want to solve

$$\begin{aligned} x + 2y + 3z &= 1 \\ 4x + 5y + 6z &= 2 \\ 7x + 8y + 9z &= 4. \end{aligned}$$

One way is to write it as

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}.$$

Now actually $\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 0$, and you can see why: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

So this transformation destroys everything in the direction of $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. In fact it squashes 3-dimensional space down to a certain 2-dimensional space. [We say that the matrix has RANK 2. If it had squashed everything down to a 1-dimensional space, we would say that it had RANK 1.] Now actually $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$. DOES

NOT lie in that two-dimensional space. Since $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ squashes EVERYTHING into that two-dimensional space, it is IMPOSSIBLE for $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ to be equal to $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$. Hence the system has NO solutions.

If we change $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, this vector DOES lie in the special 2-dimensional space, and the system $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ DOES have a solution – in fact it has infinitely many!

SUMMARY:

Any system of linear equations can be written as

$$M\overrightarrow{r} = \overrightarrow{a}$$

where M is a matrix, $\overrightarrow{r}$ = the vector of variables, and $\vec{a}$ is a given vector. Suppose M is square.

[a] If $\det M \neq 0$, there is exactly one solution,

$$\vec{r} = M^{-1}\vec{a}\,.$$

[b] If $\det M = 0$, there is probably no solution. But if there is one, then there will be many.

PRACTICAL ENGINEERING PERSPECTIVE:

In the REAL world, NOTHING IS EVER EXACTLY EQUAL TO ZERO! So if $\det M = 0$, either [a] you have made a mistake, OR [b] you are pretending that your data are more accurate than they really are!

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \text{ REALLY means } \begin{bmatrix} 1.01 & 2.08 & 3.03 \\ 3.99 & 4.97 & 6.02 \\ 7.01 & 7.96 & 8.98 \end{bmatrix}$$

and of course the determinant of THIS is non-zero! Actually, $\det = 0.597835$!

6.6 EIGENVECTORS AND EIGENVALUES.

Remember we said that a linear transformation USUALLY changes the direction of a vector. But there may be some special vectors which DON'T have their direction changed!

EXAMPLE: $\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$ clearly DOES change the direction of $\hat{i}$ and $\hat{j}$, since $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is not parallel to $\hat{i}$ and $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$ is not parallel to $\hat{j}$. BUT

$$\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

which IS parallel to $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

In general if a transformation T does not change the direction of a vector $\overrightarrow{u}$, that is

$$T\overrightarrow{u} = \lambda\overrightarrow{u}$$

for some λ (SCALAR), then $\overrightarrow{u}$ is called an EIGEN-VECTOR of T. The scalar λ is called the EIGEN-VALUE of $\overrightarrow{u}$.

Note: If $\vec{u}$ is an eigenvector, then $k\vec{u}$ is also an eigenvector for any constant k, with same eigenvalue.

Proof: Let $T\vec{u} = \lambda\vec{u}$

$$\therefore T(k\vec{u}) = kT\vec{u}$$
$$= k\lambda\vec{u}$$
$$= \lambda(k\vec{u}).$$

6.7 FINDING EIGENVALUES AND EIGENVECTORS.

Let $\vec{u} \neq \vec{0}$ and

$$T\vec{u} = \lambda \vec{u}$$

$$= \lambda I \vec{u}$$

$$\therefore \quad T\vec{u} - \lambda I \vec{u} = \vec{0}$$

$$\text{i.e.} \quad (T - \lambda I)\vec{u} = \vec{0}.$$

$$\because \quad \vec{u} \neq \vec{0}$$

$\therefore$ $T - \lambda I$ must be singular.

$$\therefore \quad \boxed{\det(T - \lambda I) = 0}$$

Example (page 41.)

To find the eigenvalues and eigenvectors of $T = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$

Solution

$$T - \lambda I = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(-2-\lambda) - 4 = 0$$

$$\Rightarrow \lambda^2 + \lambda - 6 = 0$$

$$\Rightarrow (\lambda+3)(\lambda-2) = 0$$

$$\Rightarrow \lambda = -3 \text{ or } \lambda = 2.$$

Case 1: $\lambda = -3$.

$$T - \lambda I = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$$

We solve

$$\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

i.e. $\begin{cases} 4x + 2y = 0 \\ 2x + y = 0 \end{cases}$

Note that the two equations are proportional (as expected) since $\det(T-\lambda I)=0$.

We can take any one of these two equations. Say we take

$$2x+y=0.$$

Put $y = t$

$$\Rightarrow x = -\frac{y}{2} = -\frac{1}{2}t.$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}t \\ t \end{pmatrix} = t\begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}.$$

$\therefore \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = -3$.

Case 2: $\lambda = 2$

$$T - \lambda I = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}$$

We solve $-x + 2y = 0$

Put $y = t$

$\Rightarrow x = 2y = 2t$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2t \\ t \end{pmatrix} = t \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$\therefore \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda = 2$.

Example (Page 43)

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$T - \lambda I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix}$$

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 1 = 0$$

$$\Rightarrow \lambda = \pm i$$

Case 1: $\lambda = i$

Solve $-ix - y = 0$

Put $x = t \Rightarrow y = -it$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ -it \end{pmatrix} = t \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$\therefore \begin{pmatrix} 1 \\ -i \end{pmatrix}$ is an eigenvector for $\lambda = i$.

Case 2 : $\lambda = -i$

Solve $ix - y = 0$

Put $x = t \Rightarrow y = it$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ it \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix} t$$

$\therefore \begin{pmatrix} 1 \\ i \end{pmatrix}$ is an eigenvector for $\lambda = -i$.

Note that a REAL matrix can have COMPLĒX̄ eigenvalues and eigenvectors! This is happening simply because $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is a ROTATION through $90°$, and of course such a transformation leaves NO [real] vector's direction unchanged (apart from the zero vector).

6.8. DIAGONAL FORM OF A LINEAR TRANSFORMATION.

Let T be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Let
$$T(\vec{i}) = P_{11}\vec{i} + P_{21}\vec{j} = \begin{pmatrix} P_{11} \\ P_{21} \end{pmatrix} = \vec{u}$$

$$T(\vec{j}) = P_{12}\vec{i} + P_{22}\vec{j} = \begin{pmatrix} P_{12} \\ P_{22} \end{pmatrix} = \vec{v}$$

$\{\vec{u}, \vec{v}\}$ is called a BASIS for $\mathbb{R}^2$

if $\vec{u}$ and $\vec{v}$ are not parallel,

i.e. $\vec{u} \times \vec{v} \neq \vec{0}$

i.e. $\det T \neq 0$

i.e. $\begin{vmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{vmatrix} \neq 0$

<u>Example</u> (Page 46)

Does $\left\{ \vec{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ form a basis?

Solution :

$$\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

$\therefore$ Yes.

Change basis for vectors

The vector $\begin{pmatrix}1\\2\end{pmatrix}$ relative to the basis $(\vec{i}, \vec{j})$ means the vector $\vec{i} + 2\vec{j}$.

What are the coordinates of the same vector relative to the basis $(\vec{u}, \vec{v})$, where as in page 46,

$$\vec{u} = \vec{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{(\vec{i}, \vec{j})}$$

$$\vec{v} = \vec{i} + \vec{j} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}_{(\vec{i}, \vec{j})} \quad ?$$

ie.

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}_{(\vec{i},\vec{j})} = \begin{pmatrix} ? \\ ? \end{pmatrix}_{(\vec{u},\vec{v})}$$

Let $\begin{pmatrix} 1 \\ 2 \end{pmatrix}_{(\vec{i}, \vec{j})} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}_{(\vec{u}, \vec{v})}$

$\therefore \vec{i} + 2\vec{j} = \alpha \vec{u} + \beta \vec{v}$ ---- ①

Let $P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$$\left.\begin{aligned} \text{Then} \quad P\vec{i} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{(\vec{i},\vec{j})} = \vec{u} \\ P\vec{j} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}_{(\vec{i},\vec{j})} = \vec{v} \end{aligned}\right\} \cdots (2)$$

$$① \,\&\, ② \Rightarrow \vec{i} + 2\vec{j} = \alpha P\vec{i} + \beta P\vec{j}$$

$$= P(\alpha\vec{i} + \beta\vec{j})$$

$$\Rightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix} = P\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = P^{-1}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\det P = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

$$\therefore \; P^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 1 \\ 2 \end{pmatrix}_{(\vec{i}, \vec{j})} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}_{(\vec{u}, \vec{v})}$$

Summary:

To find the coordinates of $\vec{w} = \begin{pmatrix} a \\ b \end{pmatrix}_{(\vec{i}, \vec{j})}$

relative to the basis $(\vec{u}, \vec{v})$:

1° Write $\vec{u} = \begin{pmatrix} p_{11} \\ p_{21} \end{pmatrix}_{(\vec{i}, \vec{j})}$

$$\vec{v} = \begin{pmatrix} p_{12} \\ p_{22} \end{pmatrix}_{(\vec{i}, \vec{j})}$$

2° Let $P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$

3° Then $\vec{w} = \left(P^{-1} \begin{pmatrix} a \\ b \end{pmatrix} \right)_{(\vec{u}, \vec{v})}$.

Change of basis for matrices

$$\vec{u} = \begin{pmatrix} p_{11} \\ p_{21} \end{pmatrix}_{(\vec{i}, \vec{j})}$$

$$\vec{v} = \begin{pmatrix} p_{12} \\ p_{22} \end{pmatrix}_{(\vec{i}, \vec{j})}$$

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}, \quad \det P \neq 0.$$

T = linear transformation

$\vec{w}$ = a vector $= \begin{pmatrix} a \\ b \end{pmatrix}_{(\vec{i},\vec{j})}$

$T : \vec{w} \longmapsto T\vec{w} = \begin{pmatrix} c \\ d \end{pmatrix}_{(\vec{i},\vec{j})}$

Let $T_{(\vec{i},\vec{j})}$ = matrix of T in the basis $\{\vec{i},\vec{j}\}$

$T_{(\vec{u},\vec{v})}$ = matrix of T in the basis $\{\vec{u},\vec{v}\}$.

Q: How is $T_{(\vec{u},\vec{v})}$ related to $T_{(\vec{i},\vec{j})}$?

In the basis $\{\vec{i}, \vec{j}\}$,

$$T_{(\vec{i},\vec{j})} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix} \quad \text{------} \quad (1)$$

In the basis $\{\vec{u}, \vec{v}\}$,

$$T_{(\vec{u},\vec{v})} \vec{w}_{(\vec{u},\vec{v})} = (T\vec{w})_{(\vec{u},\vec{v})}$$

Recall : $\vec{w}_{(\vec{u},\vec{v})} = P^{-1}\begin{pmatrix} a \\ b \end{pmatrix}$, $(T\vec{w})_{(\vec{u},\vec{v})} = P^{-1}\begin{pmatrix} c \\ d \end{pmatrix}$

$$\therefore \quad T_{(\vec{u},\vec{v})} P^{-1}\begin{pmatrix} a \\ b \end{pmatrix} = P^{-1}\begin{pmatrix} c \\ d \end{pmatrix}$$

$$\therefore \quad P\, T_{(\vec{u},\vec{v})} P^{-1}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix} \text{ ------ } ②$$

$$① \,\&\, ② \Rightarrow \boxed{T_{(\vec{i},\vec{j})} = P\, T_{(\vec{u},\vec{v})} P^{-1}}$$

$$\text{or} \quad \boxed{T_{(\vec{u},\vec{v})} = P^{-1} T_{(\vec{i},\vec{j})} P}$$

Summary:

To find the matrix of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{(\vec{i},\vec{j})}$

relative to the basis $(\vec{u},\vec{v})$:

1° Write $\vec{u} = \begin{pmatrix} p_{11} \\ p_{21} \end{pmatrix}_{(\vec{i}, \vec{j})}$

$\vec{v} = \begin{pmatrix} p_{12} \\ p_{22} \end{pmatrix}_{(\vec{i}, \vec{j})}$

2° Let $P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$

3° The answer is

$$\boxed{\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix}_{(\vec{u},\vec{v})} = P^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix}_{(\vec{i},\vec{j})} P}$$

<u>Example</u> (P. 48 of notes)

$$T_{(\vec{i},\vec{j})} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$$

$$\vec{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{(\vec{i},\vec{j})} \qquad \vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}_{(\vec{i},\vec{j})}$$

What is $T_{(\vec{u},\vec{v})}$?

Solution: $P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\det P = 1 \neq 0$.

$\therefore P^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

$$T_{(\vec{u},\vec{v})} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} 1 & 4 \\ 0 & -1 \end{pmatrix}}}$$

Note: Very often, we may be able to choose a basis $(\vec{u}, \vec{v})$ such that

$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix}_{(\vec{u}, \vec{v})}$$ takes a simple form,

then we use

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{(\vec{i},\vec{j})} = P \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix}_{(\vec{u},\vec{v})} P^{-1}}$$

to express $\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{(\vec{i},\vec{j})}$ in terms

of another matrix of simpler form.

Typical case

Let $T \sim \begin{pmatrix} a & b \\ c & d \end{pmatrix}_{(\vec{i},\, \vec{j})}$.

Suppose $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_{(\vec{i},\vec{j})}$

and $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}_{(\vec{i},\vec{j})}$

are eigenvectors of T and

$$T\vec{u} = \lambda_1 \vec{u}$$

$$T\vec{v} = \lambda_2 \vec{v}$$

Suppose $\begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} \neq 0$

so that $(\vec{u}, \vec{v})$ is a basis.

$\therefore\ T\vec{u} = \lambda_1 \vec{u} = \begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix}_{(\vec{u}, \vec{v})}$

$T\vec{v} = \lambda_2 \vec{v} = \begin{pmatrix} 0 \\ \lambda_2 \end{pmatrix}_{(\vec{u}, \vec{v})}$

i.e. $T \sim \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}_{(\vec{u}, \vec{v})}$ relative to the basis $(\vec{u}, \vec{v})$.

$$\therefore \quad \boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}^{-1}}$$

Note: If $A = PBP^{-1}$,

then $A^n = PB^nP^{-1}$.

Proof

$$A^n = (PBP^{-1})^n$$

$$= \underbrace{(PBP^{-1})(PBP^{-1})\cdots(PBP^{-1})}_{n \text{ terms}}$$

$$= PB^nP^{-1}$$

<u>Note</u>: If $B = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ - & - & - & \cdots \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}$

Then $B^k = \begin{pmatrix} \lambda_1^k & & & 0 \\ & \lambda_2^k & & \\ & & \ddots & \\ 0 & & & \lambda_n^k \end{pmatrix}$

Typical application

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a given matrix with eigenvectors $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ for λ_1 and $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ for λ_2 such that $\begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} \neq 0$.

Then

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix}^n = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}^{-1}}$$

Example (Page 52)

$T = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$ on page 41.

We have found that

$\begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$ is an eigenvector for $\lambda = -3$

and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector for $\lambda = 2$.

$$\therefore \; P = \begin{pmatrix} -\frac{1}{2} & 2 \\ 1 & 1 \end{pmatrix}$$

and $\begin{vmatrix} -\frac{1}{2} & 2 \\ 1 & 1 \end{vmatrix} = -\frac{1}{2} - 2 = -\frac{5}{2}$

$$\therefore \; P^{-1} = -\frac{2}{5} \begin{pmatrix} 1 & -2 \\ -1 & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{2}{5} & \frac{4}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix}$$

Then

$$\begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -\frac{2}{5} & \frac{4}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix}$$

Example (page 53)

The shear matrix $\begin{pmatrix} 1 & \tan\theta \\ 0 & 1 \end{pmatrix}$.

$$\begin{pmatrix} 1 & \tan\theta \\ 0 & 1 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\lambda & \tan\theta \\ 0 & 1-\lambda \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & \tan\theta \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = 0 \Rightarrow \lambda = 1, \text{ double root.}$$

When $\lambda = 1$, we have

$$\begin{pmatrix} 0 & \tan\theta \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow (\tan\theta)\, y = 0$$

Assume $\theta \neq 0 \Rightarrow x = t,\ y = 0$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ 0 \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

We have only one eigenvector $\begin{pmatrix}1\\0\end{pmatrix}$ correspond to the (double root) $\lambda = 1$.

$\therefore$ The shear matrix is NOT diagonalizable.

6.9 APPLICATION – MARKOV CHAINS.

From the Markov Chain example in chapter 5, the transition matrix is

$$M = \begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{pmatrix}$$

where $P(R \to R) = 0.6$

$P(R \to S) = 0.4$

$P(S \to R) = 0.3$

$P(S \to S) = 0.7$

R = Rainy day

S = Sunny day

$$M - \lambda I = \begin{pmatrix} 0.6-\lambda & 0.3 \\ 0.4 & 0.7-\lambda \end{pmatrix}$$

$$\begin{vmatrix} 0.6-\lambda & 0.3 \\ 0.4 & 0.7-\lambda \end{vmatrix} = 0$$

$$\Rightarrow 0.42 - 1.3\lambda + \lambda^2 - 0.12 = 0$$

$$\therefore \lambda^2 - 1.3\lambda + 0.3 = 0$$

$$\therefore (\lambda - 1)(\lambda - 0.3) = 0$$

$$\therefore \lambda = 1 \text{ or } \lambda = 0.3$$

Case 1: $\lambda = 1$

$$\Rightarrow \begin{pmatrix} -0.4 & 0.3 \\ 0.4 & -0.3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore \; -0.4x + 0.3y = 0$$

$$3y = 4x$$

$$x = 3t \Rightarrow y = 4t$$

Take $t = 1 \Rightarrow \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ is an eigenvector for the eigenvalue 1.

Case 2: $\lambda = 0.3$

$$\Rightarrow \begin{pmatrix} 0.3 & 0.3 \\ 0.4 & 0.4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad x = -y$$

$$x = t \Rightarrow y = -t$$

Take $t = 1 \Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector

for the eigenvalue 0.3.

Let $P = \begin{pmatrix} 3 & 1 \\ 4 & -1 \end{pmatrix}$

$\therefore \det P = -3 - 4 = -7$

$\therefore P^{-1} = -\frac{1}{7}\begin{pmatrix} -1 & -1 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{4}{7} & -\frac{3}{7} \end{pmatrix}$

$$\therefore \; M = P \begin{pmatrix} 1 & 0 \\ 0 & 0.3 \end{pmatrix} P^{-1}$$

$$\text{i.e.} \; \begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0.3 \end{pmatrix} \begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{4}{7} & -\frac{3}{7} \end{pmatrix}$$

$$\therefore \; \begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{pmatrix}^n = \begin{pmatrix} 3 & 1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (0.3)^n \end{pmatrix} \begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{4}{7} & -\frac{3}{7} \end{pmatrix}$$

e.g. $n = 30$

$$\begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{pmatrix}^{30} = \begin{pmatrix} 3 & 1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (0.3)^{30} \end{pmatrix} \begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{4}{7} & -\frac{3}{7} \end{pmatrix}$$

$$\approx \begin{pmatrix} 0.42857 & 0.42857 \\ 0.57143 & 0.57143 \end{pmatrix}$$

If it rains today, then the probability that it is a rainy day 30 days from now is 0.42857.

e.g. $n \longrightarrow \infty$

$$\text{R.H.S.} \longrightarrow \begin{pmatrix} 3 & 1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{4}{7} & -\frac{3}{7} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{7} & \frac{3}{7} \\ \frac{4}{7} & \frac{4}{7} \end{pmatrix}$$

6.10 THE TRACE OF A MATRIX.

For $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ - & - & - & - \\ a_{n1} & - & - & a_{nn} \end{pmatrix}$

The trace of A is defined by

$$\operatorname{Tr} A = a_{11} + a_{22} + a_{33} + \cdots + a_{nn}$$
$$= \sum_{i=1}^{n} a_{ii}.$$

Example

$$\mathrm{Tr}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1+1 = 2$$

$$\mathrm{Tr}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = 1+5+9 = 15$$

Proposition

A, B two $n \times n$ matrices.

Then $\operatorname{Tr} AB = \operatorname{Tr} BA$

Proof. Let $C = AB$, $D = BA$.

Put $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ - & - & - \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ - & - & - \\ b_{n1} & \cdots & b_{nn} \end{pmatrix}$

$$C = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ - & - & - \\ c_{n1} & \cdots & c_{nn} \end{pmatrix}, \quad D = \begin{pmatrix} d_{11} & \cdots & d_{1n} \\ - & - & - \\ d_{n1} & \cdots & d_{nn} \end{pmatrix}$$

Then $c_{ii} = \sum_{j=1}^{n} a_{ij} b_{ji}$

$$d_{jj} = \sum_{i=1}^{n} b_{ji} a_{ij}$$

$$\mathrm{Tr}\,AB = \mathrm{Tr}\,C = \sum_{i=1}^{n} c_{ii} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, b_{ji}$$

$$\mathrm{Tr}\,BA = \mathrm{Tr}\,D = \sum_{j=1}^{n} d_{jj} = \sum_{j=1}^{n} \sum_{i=1}^{n} b_{ji}\, a_{ij}$$

$$\therefore \mathrm{Tr}\,AB = \mathrm{Tr}\,BA \quad //$$

Note : Suppose $A = PBP^{-1}$

Then $\operatorname{Tr} A = \operatorname{Tr}\{(PB)(P^{-1})\}$

$= \operatorname{Tr}\{(P^{-1})(PB)\}$

$= \operatorname{Tr} B.$

∴ If A is diagonalizable with eigenvalues $\lambda_1, \cdots, \lambda_n$.

Then $$A = P \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} P^{-1}$$

and $\operatorname{Tr} A = \operatorname{Tr}\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}$

$= \lambda_1 + \dots + \lambda_n$ //

and $\det A = \det P \det\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix} \det P^{-1}$

$= \underline{\underline{\lambda_1 \cdots \lambda_n}}$