
EE2011 Engineering Electromagnetics

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Prof Yeo Swee Ping
eleyeosp@nus.edu.sg

Lorentz Force

force on charge q moving with velocity \vec{u}

$$\vec{F} = q\vec{E} + q\vec{u} \times \vec{B}$$

notes for $\vec{F}_{\text{magnetic}}$:

(a) normal to $\vec{u} \Rightarrow$ no work done

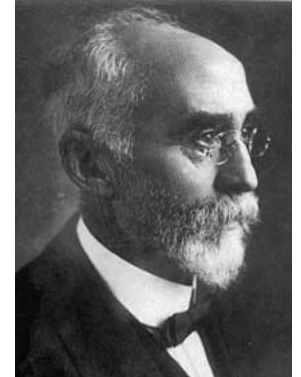
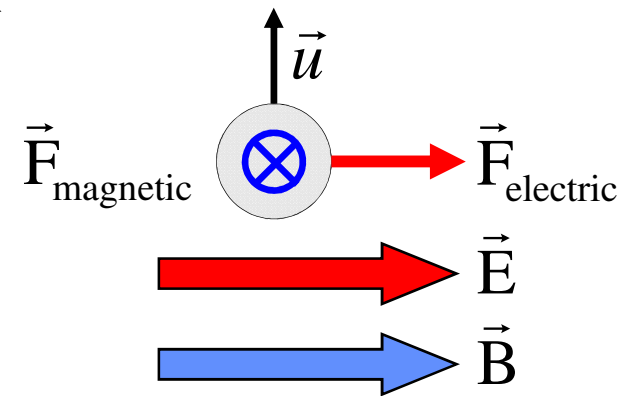
circular trajectory with constant kinetic energy

(b) zero for stationary charge (*i.e* when $|\vec{u}| = 0$)
no magnetic force in electrostatics

(c) independent of $\vec{F}_{\text{electric}}$

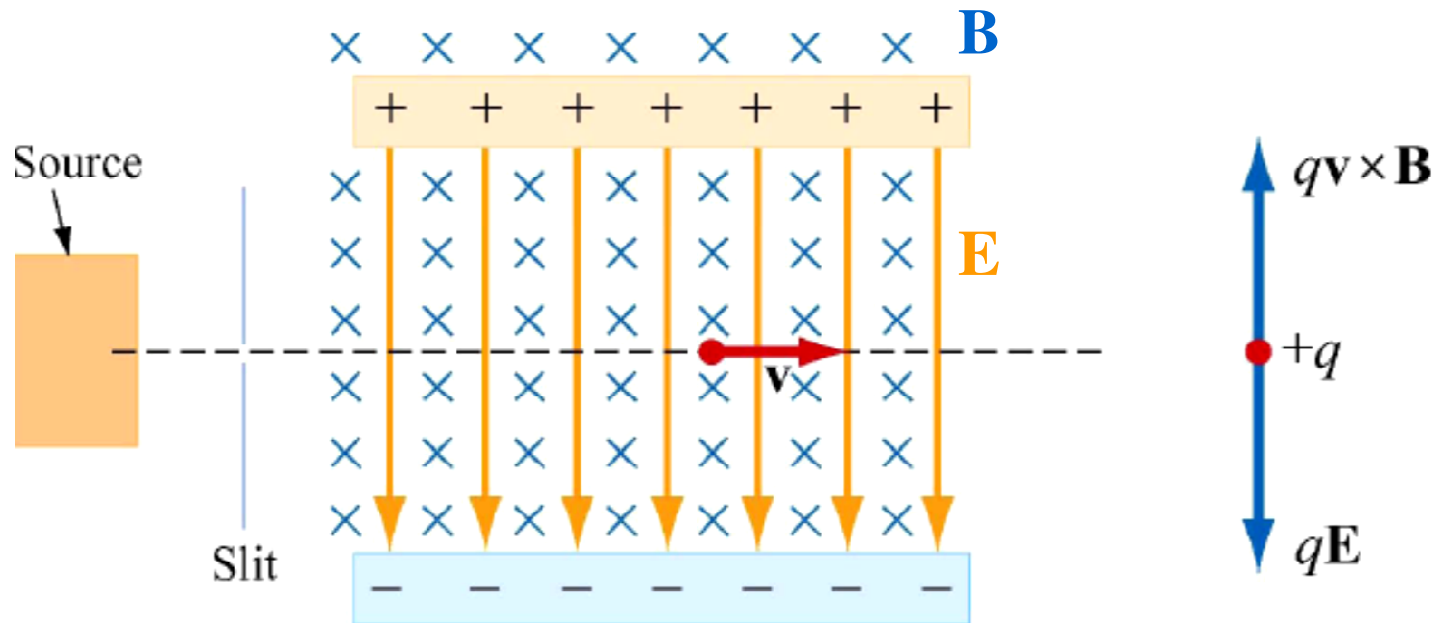
(d) used for defining Tesla = $\frac{\text{N}}{\text{C (m/s)}} = \frac{\text{N}}{(\text{C/s}) \text{ m}} = \frac{\text{N}}{\text{A m}}$

(e) applicable to point charges but can extend to charge systems



Lorentz Force

force on electron flow in vacuum (Thomson's experiment)



adjust \mathbf{E} and \mathbf{B} for electrons to flow in straight path $\rightarrow v = \frac{E}{B}$
used for measuring velocity and charge-to-mass ratio of electron

Magnetic Force

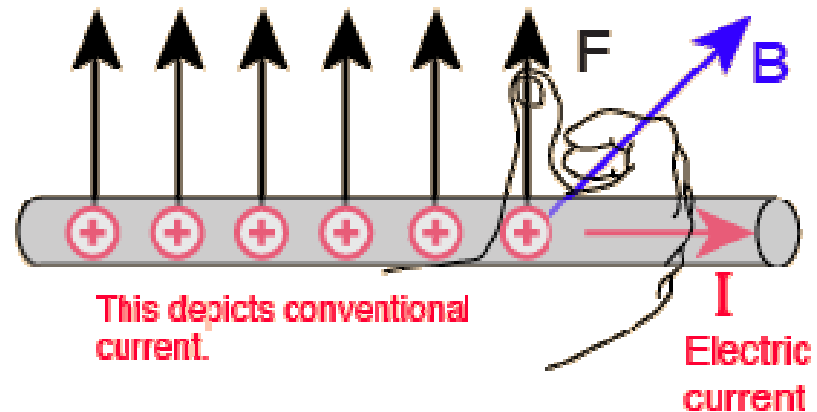
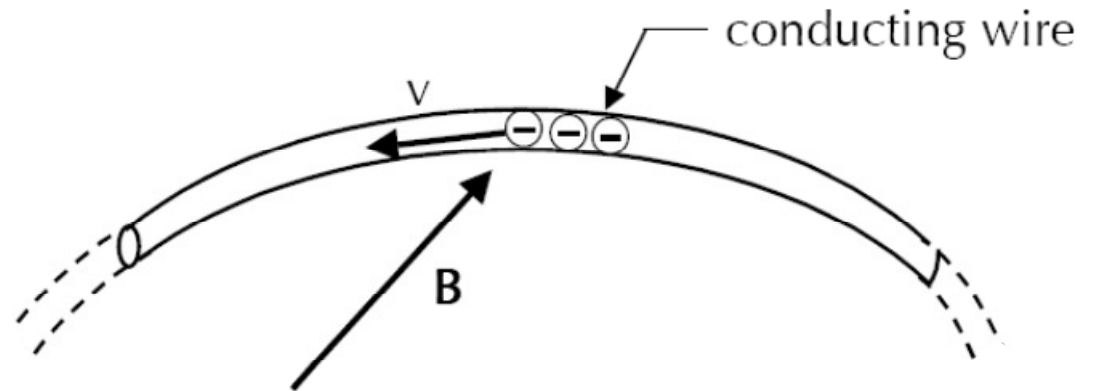
force on electron flow in wire (*i.e.* current)

$$\begin{aligned}d\vec{F} &= \vec{u} \times \vec{B} dq \\&= \vec{u} \times \vec{B} \sigma dV \\&= \vec{J} \times \vec{B} A ds \\&= I d\vec{s} \times \vec{B}\end{aligned}$$

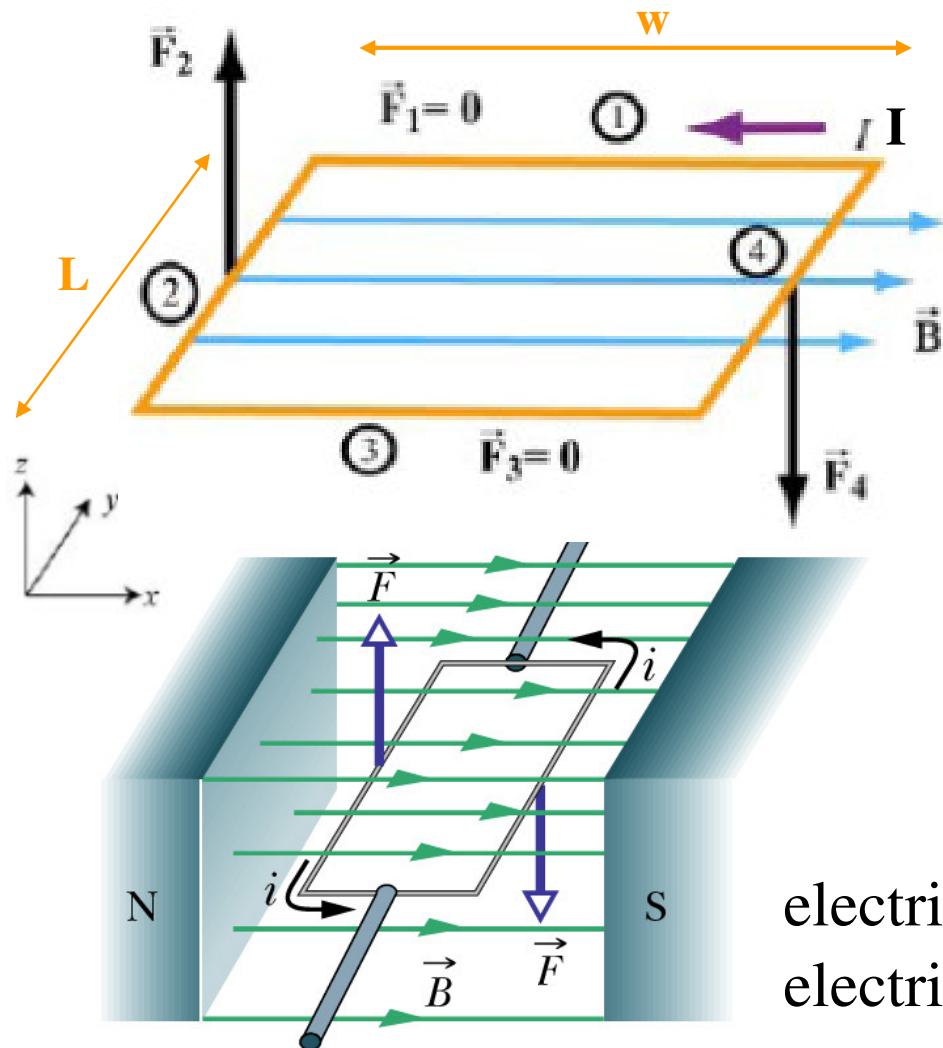
$$\Rightarrow \vec{F} = I \int d\vec{s} \times \vec{B}$$

force on straight wire
(where $d\vec{s} = \hat{u} ds$)

$$\vec{F} = IL \hat{u} \times \vec{B}$$



Magnetic Force



no forces on arms 1 and 3

no net force from arms 2 and 4

because $|\vec{F}_2| = |\vec{F}_4| = |\vec{B}| IL$

resultant torque on wire loop

$$\tau = |\vec{B}| ILw = |\vec{B}| IA$$

torque vector on **any** wire loop

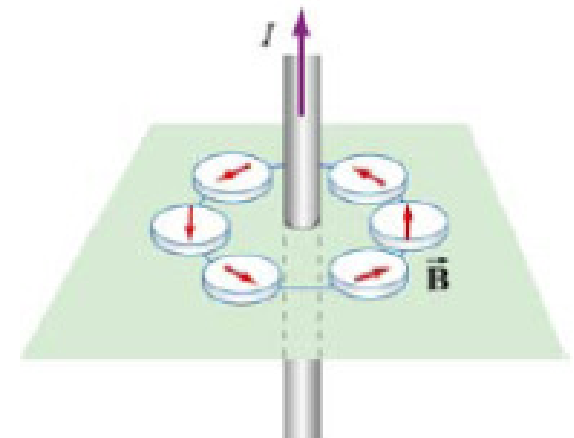
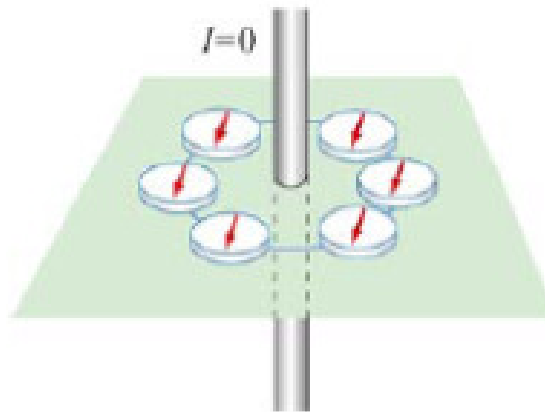
$$\vec{\tau} = I \vec{A} \times \vec{B} = \vec{m} \times \vec{B}$$

define magnetic moment $\vec{m} = I \vec{A}$

electric motor — transducer converting electrical energy into mechanical energy

Magnetic Field

magnetic field around current-carrying wire(s)

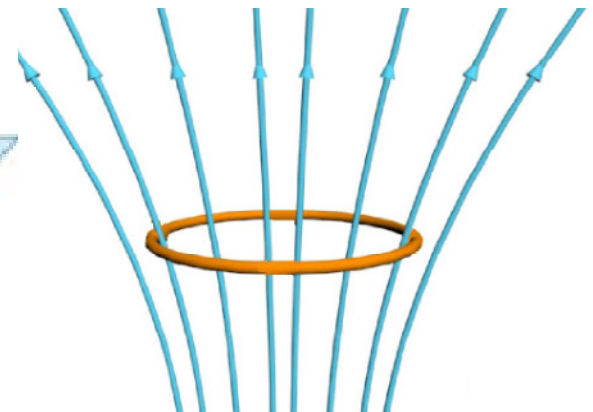
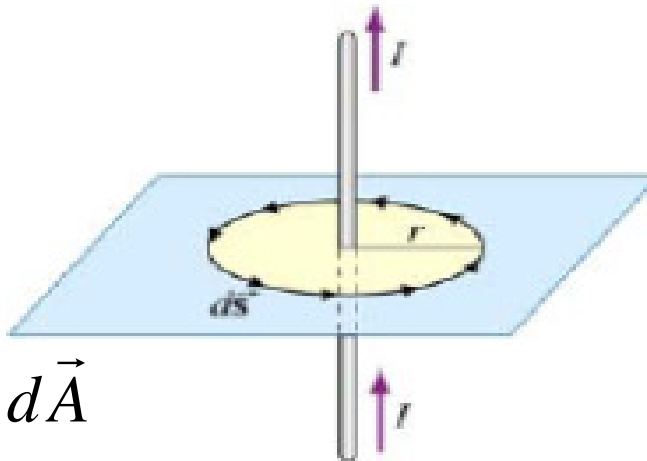


Ampere's Law

$$\oint \vec{B} \cdot d\vec{s} \propto I$$

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 \sum_k I_k$$

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 \iint \vec{J} \cdot d\vec{A}$$



Magnetic Field

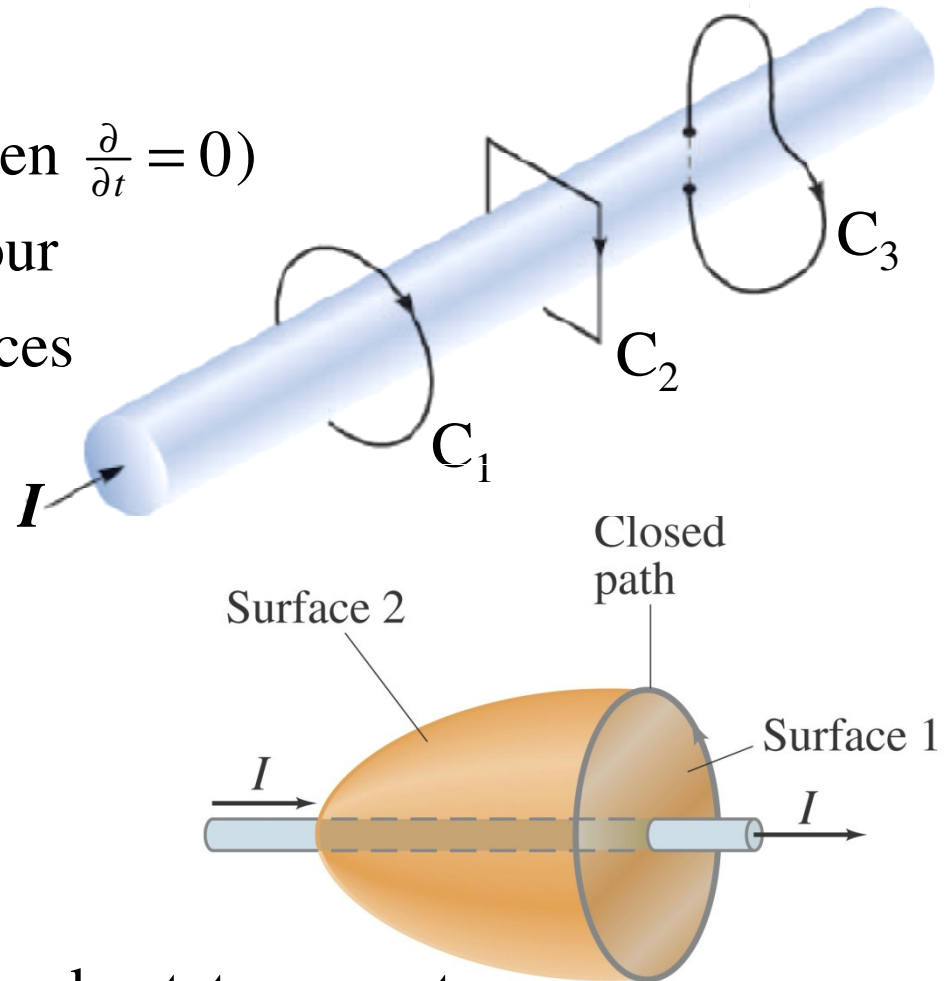
notes on Ampere's Law:

- (a) valid only for DC (*i.e.* when $\frac{\partial}{\partial t} = 0$)
- (b) valid for any closed contour
- (c) valid for non-planar surfaces
- (d) practical utility for cases with symmetry

$$\iint \nabla \times \vec{B} \cdot d\vec{A} = \mu_0 \iint \vec{J} \cdot d\vec{A}$$
$$\Rightarrow \nabla \times \vec{B} = \mu_0 \vec{J}$$

take div of both sides:

$$\nabla \cdot \vec{J} = \frac{1}{\mu_0} \nabla \cdot \nabla \times \vec{B} = 0 \rightarrow \text{steady-state current}$$



Magnetic Field

no isolated magnetic charge *c.f.* Gauss's Law for electric charges

$$\oiint_S \vec{B} \cdot d\vec{A} = 0$$

$$\oiint_S \vec{D} \cdot d\vec{A} = Q_{\text{total in enclosure}}$$

$$\Rightarrow \iiint_V \nabla \cdot \vec{B} \, dV = 0 \quad \text{for application to any volume}$$

PDE version $\nabla \cdot \vec{B} = 0$ for application to particular point

implications:

(a) \vec{B} fields always in closed loops (due to lack of sources/sinks)

(b) basis for introducing magnetic potential (vector)

use null identity $\nabla \cdot (\nabla \times \vec{A}) = 0$

useful to express $\vec{B} = \nabla \times \vec{A}$ during analytical derivation

but not common to leave answer in magnetic-potential form

Magnetic Field

re-visit curl equation in \vec{B} (from Ampere's Law)

PDE in \vec{A} $\nabla \times (\nabla \times \vec{A}) = \mu_0 \vec{J}$

expanding curl curl $\nabla \cdot \vec{A} - \nabla^2 \vec{A} = \mu_0 \vec{J}$

choosing $\nabla \cdot \vec{A} = 0$ $\nabla^2 \vec{A} = -\mu_0 \vec{J}$

decomposing vector Poisson's equation in (x, y, z) coordinates

$$\left. \begin{aligned} \nabla^2 A_x &= -\mu_0 J_x \\ \nabla^2 A_y &= -\mu_0 J_y \\ \nabla^2 A_z &= -\mu_0 J_z \end{aligned} \right\} \text{ c.f. } \nabla^2 V = -\frac{1}{\epsilon_0} \sigma \rightarrow V = \frac{1}{4\pi\epsilon_0} \iiint \frac{\sigma}{r''} dV'$$

$$\Rightarrow \vec{A} = \frac{\mu_0}{4\pi} \iiint \frac{1}{r''} \vec{J} dV' \quad \text{and thereafter obtain } \vec{B} = \nabla \times \vec{A}$$

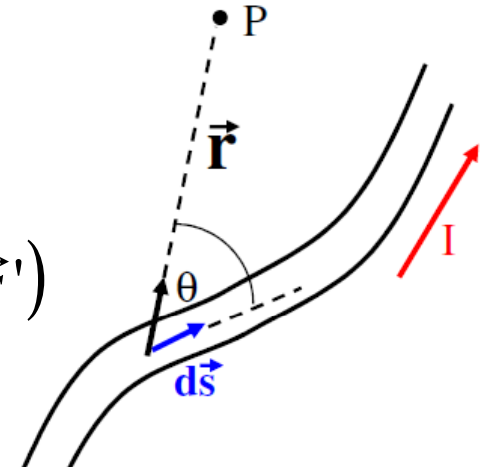
Magnetic Field

short wire (carrying current I) with length $\Delta s \rightarrow 0$ in the limit

$$\vec{J} dV' = \vec{J} A ds' = I d\vec{s}'$$

$$\vec{A} = \frac{\mu_o}{4\pi} \iiint \frac{1}{r''} \vec{J} dV = \frac{\mu_o I}{4\pi} \int \frac{1}{r''} d\vec{s}'$$

$$\begin{aligned} \vec{B} &= \nabla \times \left(\frac{\mu_o I}{4\pi} \int \frac{1}{r''} d\vec{s}' \right) = \frac{\mu_o I}{4\pi} \int \nabla \times \left(\frac{1}{r''} d\vec{s}' \right) \\ &= \frac{\mu_o I}{4\pi} \int \frac{d\vec{s}' \times \hat{u}_{r''}}{(r'')^2} \end{aligned}$$



Biot-Savart's Law to evaluate \vec{B} from wire (for $d\vec{s}$ with $\theta \neq 0$)

check for solenoidal property

$$\nabla \cdot \vec{B} = \frac{\mu_o I}{4\pi} \int \nabla \cdot \left(\frac{d\vec{s}' \times \hat{u}_{r''}}{(r'')^2} \right) = -\frac{\mu_o I}{4\pi} \int d\vec{s}' \cdot \nabla \times \nabla \left(\frac{1}{r''} \right) = 0$$

Magnetic Field

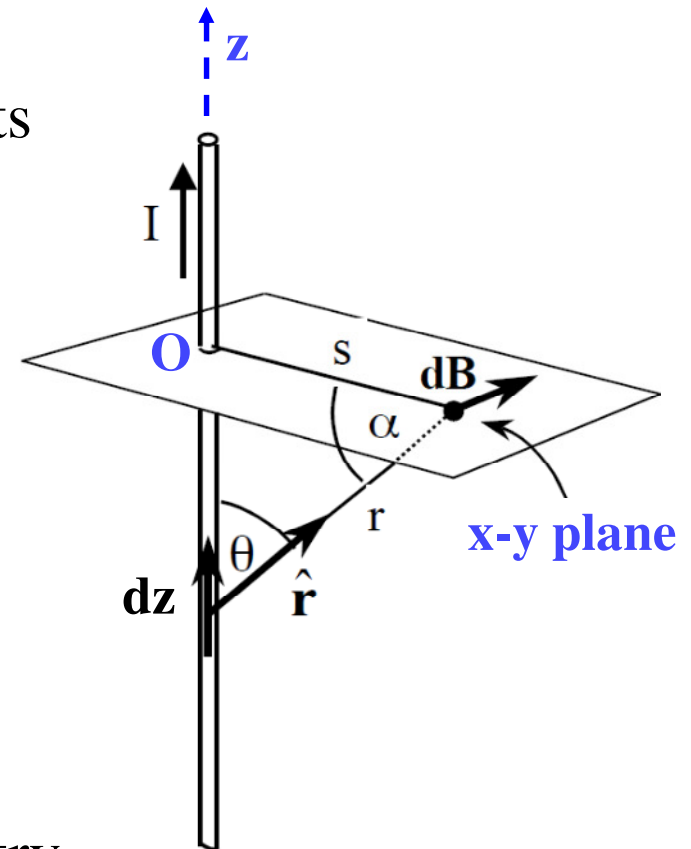
Example #1A: straight wire of length L (with origin at center)

- x-y plane normal to $d\vec{s} = dz \hat{k}$ and \vec{r}
- $d\vec{B}$ in same direction for all dz elements
- r and θ dependent on z

apply Biot-Savart's Law

$$\begin{aligned}\vec{B} &= \frac{\mu_o I}{4\pi} \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{dz \hat{k} \times \hat{r}}{r^2} = \frac{\mu_o I}{4\pi} \hat{u}_\phi \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{\sin \theta dz}{z^2 + s^2} \\ &= \frac{\mu_o I}{4\pi s} \frac{L}{\sqrt{\frac{1}{4}L^2 + s^2}} \hat{u}_\phi \\ &\rightarrow \frac{\mu_o I}{4\pi s} \hat{u}_\phi \text{ when } L \rightarrow \infty\end{aligned}$$

useful tool for structures without symmetry



Magnetic Field

Example #1B: long straight wire of radius R

infer from symmetry:

- azimuthal component ($\vec{B} = B_\phi \hat{u}_\phi$)
- B_ϕ constant of ϕ and z coordinates

(a) apply Ampere's Law to C_1 loop:

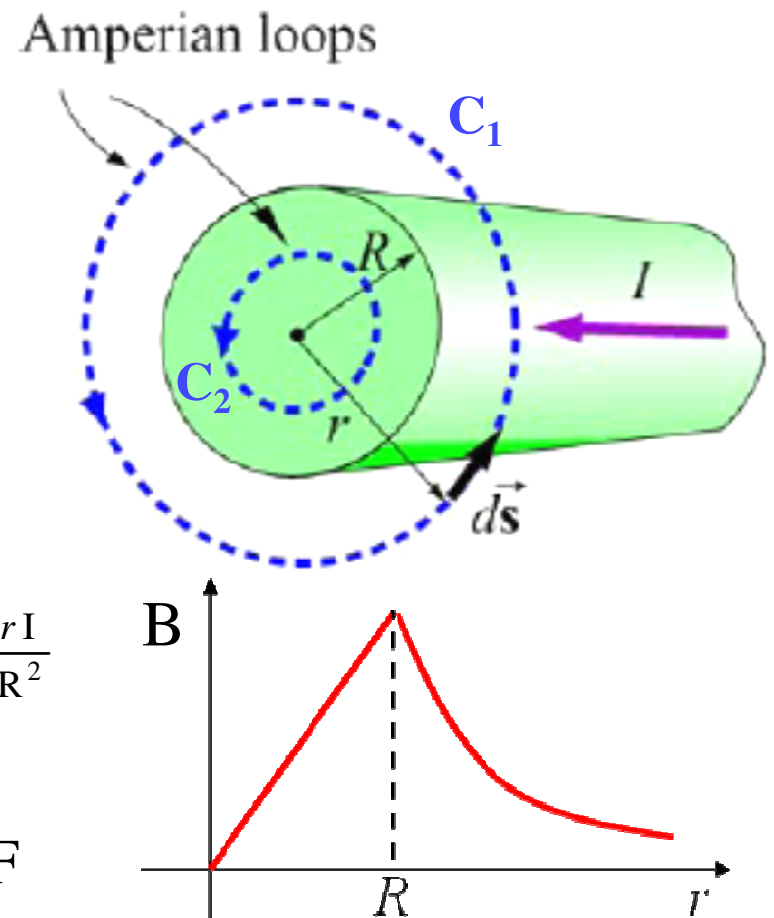
$$2\pi r B_\phi = \mu_o I \quad \Rightarrow \quad B_\phi = \frac{\mu_o I}{2\pi r}$$

(b) apply Ampere's Law to C_2 loop:

$$2\pi r B_\phi = \mu_o \frac{\pi r^2}{\pi R^2} I \quad \Rightarrow \quad B_\phi = \frac{\mu_o r I}{2\pi R^2}$$

valid only for DC (*i.e.* $\frac{dI}{dt} = 0$)

I flowing within skin-depth for RF



Magnetic Field

Example #2: long solenoid (with n turns of wire per meter)

infer from structure:

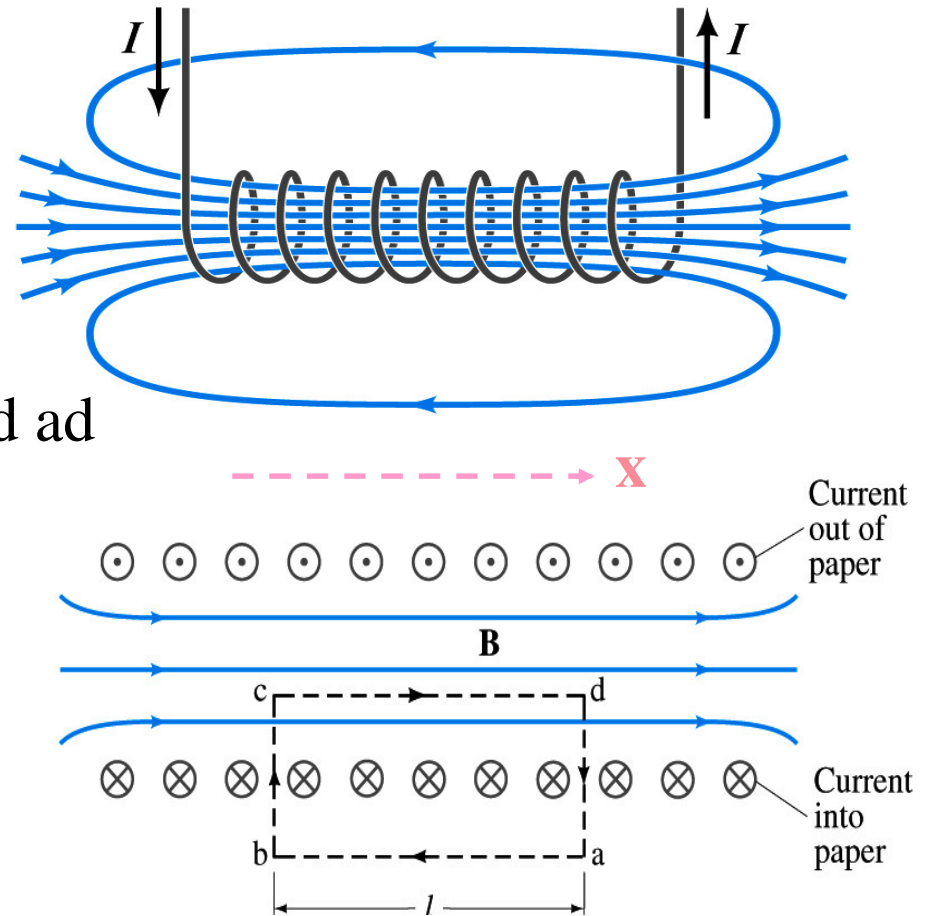
- axial component ($\vec{B} = B_x \hat{u}_x$)
- uniform B within solenoid

apply Ampere's Law to abcd:

(a) no contributions from bc and ad

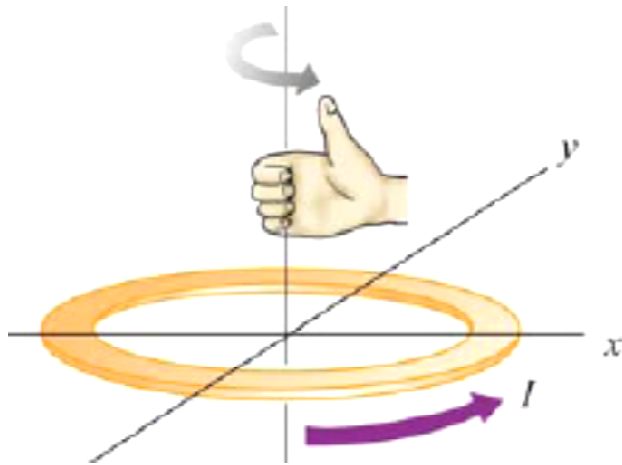
(b) $B \approx 0$ outside solenoid

$$\begin{cases} \oint \vec{B} \cdot d\vec{s} = B_x l + 0 + 0 + 0 \\ \mu_0 \sum_k I_k = \mu_0 (nl) I \\ \Rightarrow \vec{B} = \mu_0 n I \hat{u}_x \end{cases}$$



Magnetic Field

Example #3: small loop (magnetic dipole with $\vec{m} = \pi r_o^2 I \hat{u}_z$)



$$\vec{A} = \frac{\mu_o I}{4\pi} \int \frac{1}{r''} d\vec{s}' = \frac{\mu_o}{4\pi r^2} \vec{m} \times \hat{u}_r \text{ when } r \gg r_o$$

$$\vec{B} = \nabla \times \vec{A} = \frac{\mu_o}{4\pi} \frac{m}{r^3} (2 \cos \theta \hat{u}_r + \sin \theta \hat{u}_\theta)$$

similarity with electric dipole (duality)

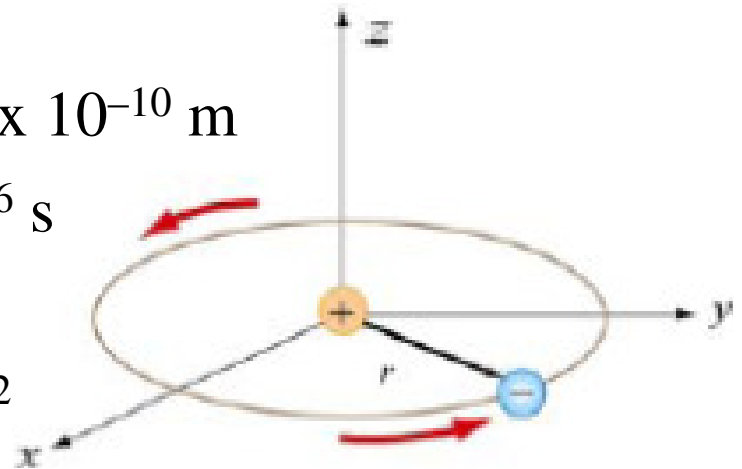
$$\vec{E} = \frac{1}{4\pi\epsilon_o} \frac{p}{r^3} (2 \cos \theta \hat{u}_r + \sin \theta \hat{u}_\theta)$$

extend to electron in orbit with $r_o \approx 0.5 \times 10^{-10} \text{ m}$

travel time for one orbit $\Delta t \approx 1.5 \times 10^{-16} \text{ s}$

\therefore current $I = \frac{e}{\Delta t} \approx 1 \text{ mA}$

moment $m = \pi r_o^2 I \approx 9.3 \times 10^{-24} \text{ A m}^2$



Magnetic Materials

field-matter interactions in magnetic material

(a) vectorial (instead of scalar in dielectrics)

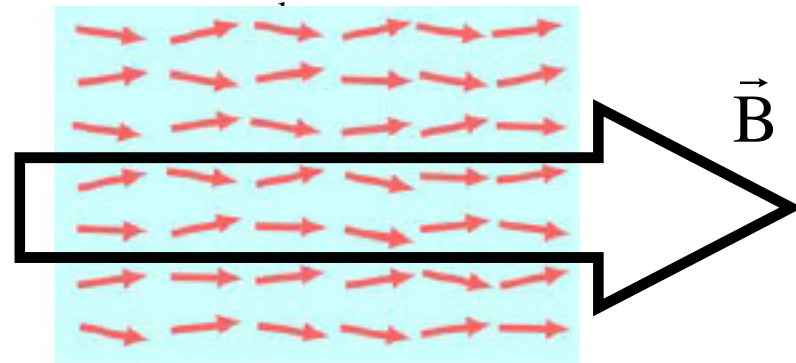
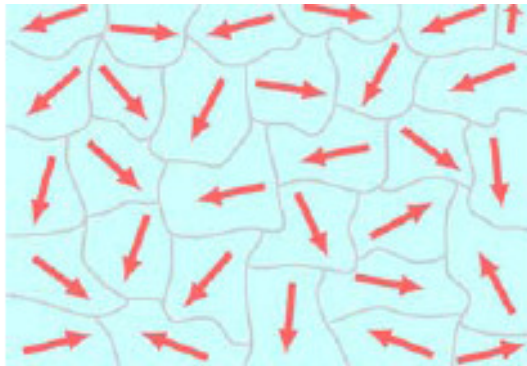
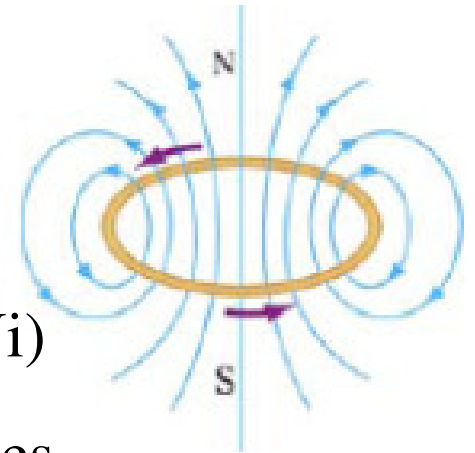
(b) very weak for large majority of materials

very strong in certain materials (*e.g.* Fe, Co, Ni)

not possible to consider individual magnetic dipoles

need to consider average density instead for bulk behaviour

define magnetization vector $\vec{M} = \frac{1}{\text{volume}} \sum \vec{m}_k$



Magnetic Materials

Ampere's Law in differential form

$$\nabla \times \vec{B} = \mu_0 \vec{J} \quad \text{for free space}$$

add magnetization current density $\vec{J}_m = \nabla \times \vec{M}$

$$\nabla \times \vec{B} = \mu_0 (\vec{J} + \vec{J}_m) \quad \text{for magnetic material}$$

$$\Rightarrow \nabla \times \left(\frac{1}{\mu_0} \vec{B} - \vec{M} \right) = \vec{J}$$

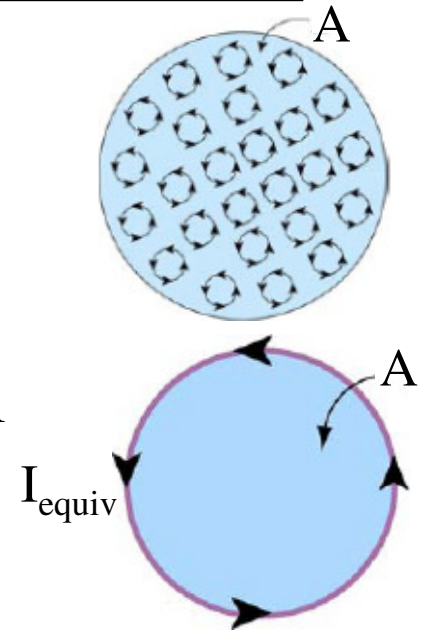
recast as $\nabla \times \vec{H} = \vec{J}$ by introducing another magnetic vector

\vec{H} = magnetic field intensity vector (A m^{-1}) to account for currents

$\vec{B} = \mu_0 (\vec{H} + \vec{M})$ = magnetic flux density vector

$$= \mu_0 (1 + \chi_m) \vec{H} = \mu_0 \mu_r \vec{H}$$

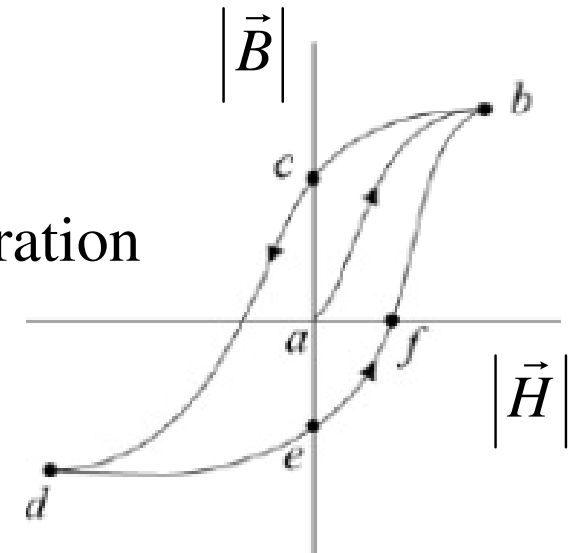
μ_r relative permeability
 χ_m magnetic susceptibility



Magnetic Materials

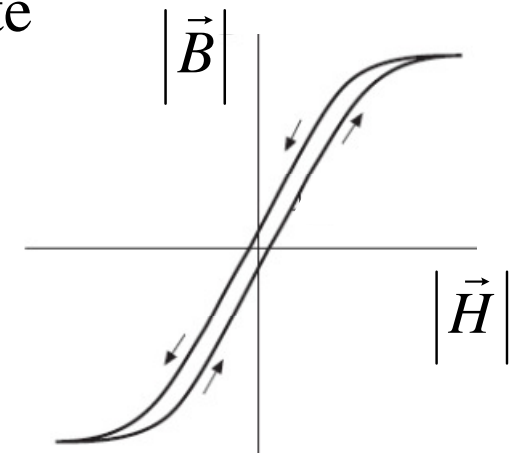
hysteresis behavior

- a: initially unmagnetized state
- b: increasing magnetization till onset of saturation
- c: retention after reducing source to zero
- d: reversing polarity of source to saturation
- e: retention with reversed polarity
- f: not coincident with initial unmagnetized state



implications

- non-linear behavior
- large retentivity for permanent magnets
- reduced hysteresis loop for memory core



Magnetic Materials

relative permeability values:

1.00 for air, water, copper, gold, aluminium, water, *etc*

250 for cobalt

600 for nickel

> 5,000 for iron

> 50,000 for MuMetal[®] at DC

> 35,000 for MuMetal[®] at 50 Hz with low magnetization

200,000 for MuMetal[®] at 50 Hz with high magnetization

- equal either to 1 (for most materials) or very large value
- variable due to hysteresis (check operating conditions)
- anisotropic (use tensor instead of scalar representation)

Magnetic Materials

dual of electric boundary conditions at interface between materials:

(a) from “Gauss’s Law” applied to pill-box structure

$$(B_1)_n = (B_2)_n$$

(b) from Ampere’s Law applied to closed contour

$$(H_1)_t = (H_2)_t \text{ valid for non-conducting materials}$$

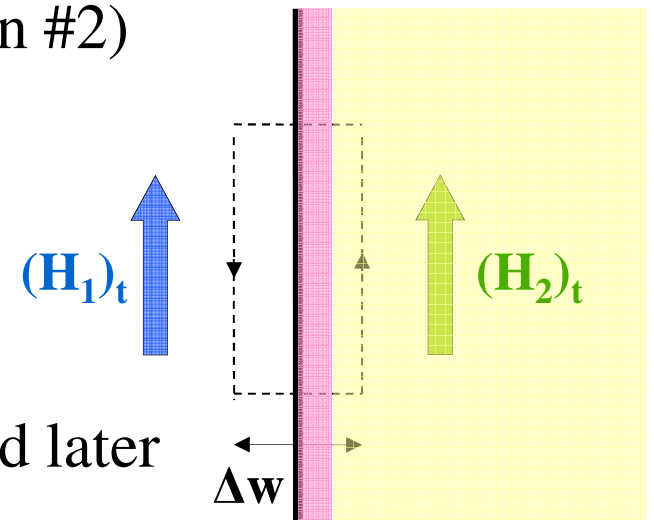
variation for perfect conductor (Region #2)

due to large current densities in RHS

$$\oint \vec{H} \cdot d\vec{s} = \iint \vec{J} \cdot d\vec{A}$$

$$(H_2)_t - (H_1)_t = (J_2)_s$$

linear current density J_s (Am^{-1}) defined later



Electromagnetic Induction

conservative \vec{E} only for electrostatics (*i.e.* when $\frac{\partial}{\partial t} = 0$)

Faraday's Law $V_{\text{emf}} = -\frac{d}{dt}\Phi$

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d}{dt} \iint \vec{B} \cdot d\vec{A}$$

$$\iint \nabla \times \vec{E} \cdot d\vec{A} = -\iint \frac{d}{dt} \vec{B} \cdot d\vec{A}$$

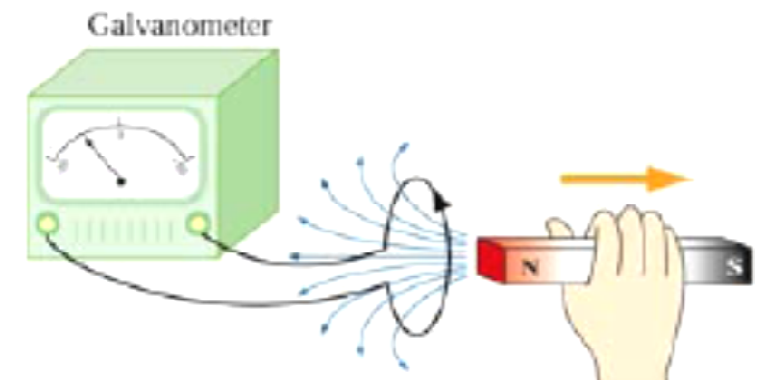
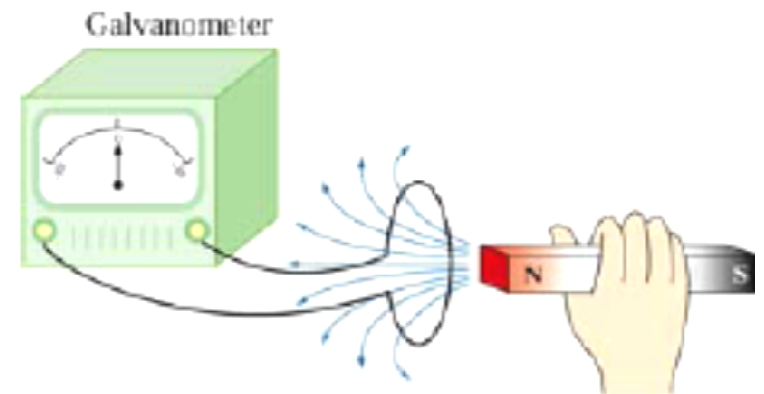
$$\nabla \times \vec{E} = -\frac{d}{dt} \vec{B}$$

apparent: $\frac{d}{dt} \vec{B}$ creates \vec{E}

actual: $\frac{d}{dt} \vec{E}$ creates \vec{B} in turn

→ both inter-coupled

(together with propagation-velocity vector)



Electromagnetic Induction

rate of change of flux $\Phi = \vec{B} \cdot \vec{A}$

Faraday's Law $V_{\text{emf}} = -\frac{d}{dt}(B_n A)$

$$= -\frac{dB_n}{dt} A - B_n \frac{dA}{dt} \quad \text{for time-varying area}$$

example: rod moving along wire rails

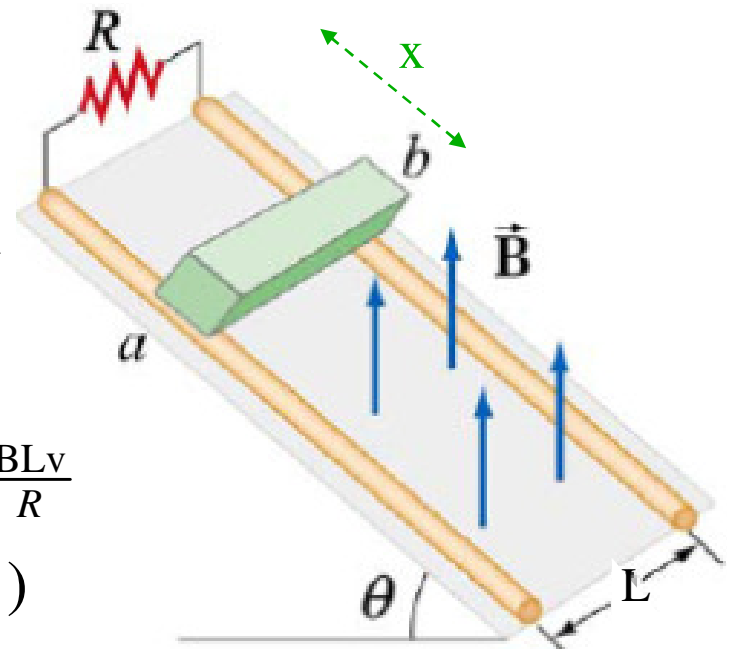
$$\text{with } \frac{dB_n}{dt} = 0$$

$$\text{and } \frac{dA}{dt} = \frac{d(Lx)}{dt} = L \frac{dx}{dt} = Lv$$

induced emf between ab and R

$$\text{current in } R \quad I = \frac{V_{\text{emf}}}{R} = -\frac{B}{R} \frac{dA}{dt} = -\frac{BLv}{R}$$

(note: rod accelerating with $\frac{dv}{dt} = g \sin \theta$)

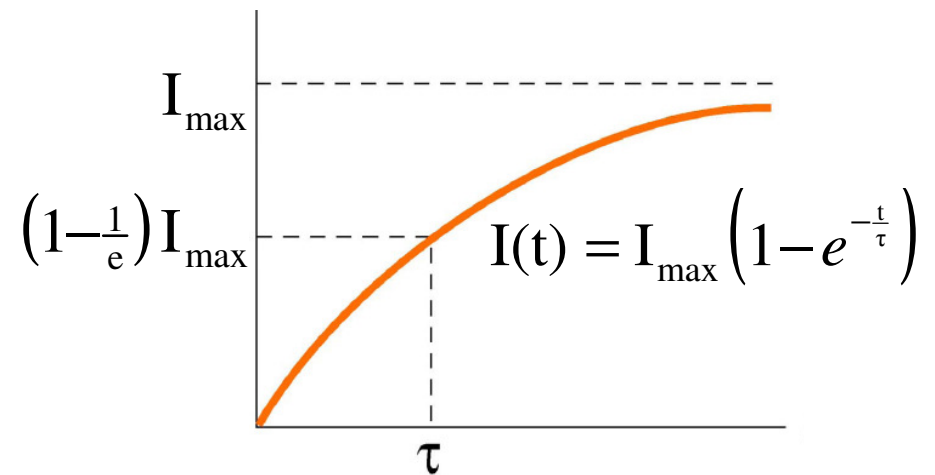
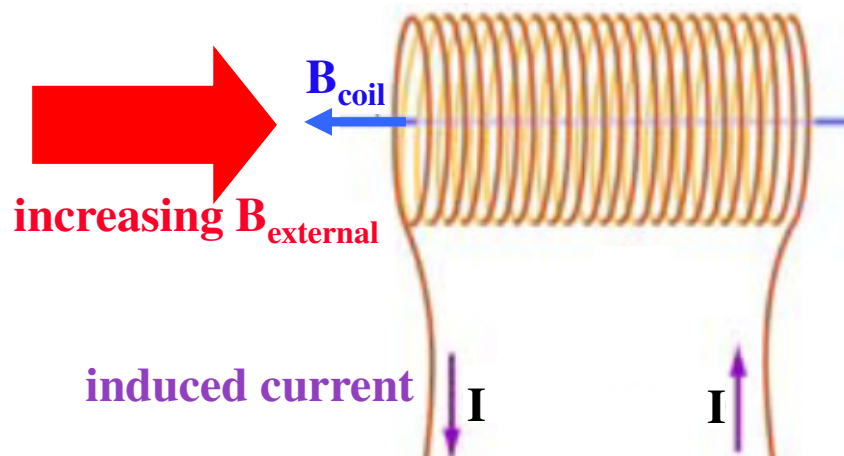


Electromagnetic Induction

Lenz's Law (*i.e.* negative sign in Faraday's Law)

- change of flux \rightarrow induced voltage $V_{\text{emf}} = -\frac{d}{dt}\Phi$
- current induced by back emf $\rightarrow \vec{B}$ to counter change in Φ
(*i.e.* slow down rate of increase/decrease of Φ)

illustration: increasing Φ for solenoid (inductor)



Electromagnetic Induction

mutual inductance between loops

illustration: two coils with N_1 and N_2 turns

- magnetic field from I_1 flow in coil 1
- magnetic flux linkage with coil 2

induced emf $V_2 = -\frac{d}{dt}(N_2 \Phi_{21})$

direction determined by Lenz's Law

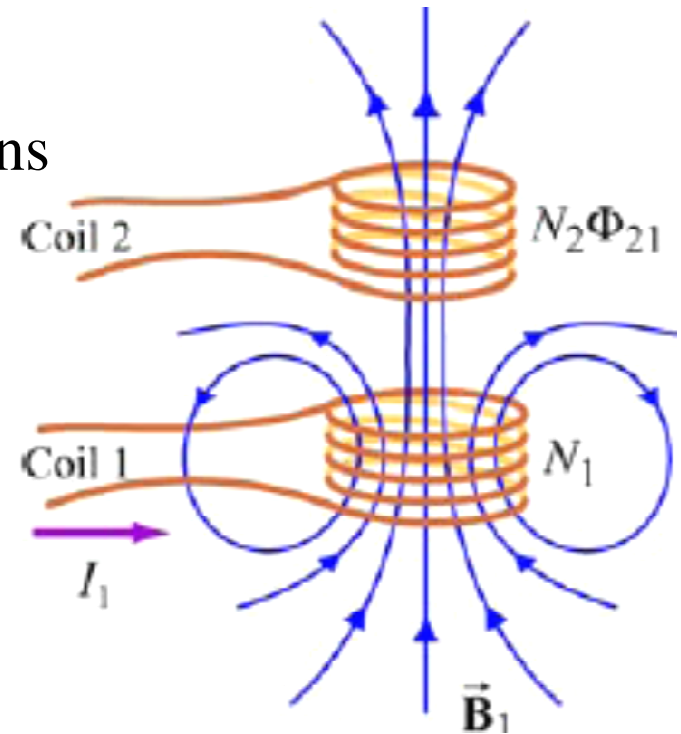
definition: mutual inductance $M = \frac{N_2 \Phi_{21}}{I_1}$

(a) unit of Henry $\left(H = \frac{Tm^2}{A} = \frac{Vs}{A}\right)$

(b) same result if current in other loop $M = \frac{N_1 \Phi_{12}}{I_2}$

(c) disregard N for single-turn loops

(d) use matrix \underline{M} for circuit system with more than two loops



Electromagnetic Induction

mutual-inductance example: coil wrapped around solenoid

recommend current in solenoid (instead of coil)

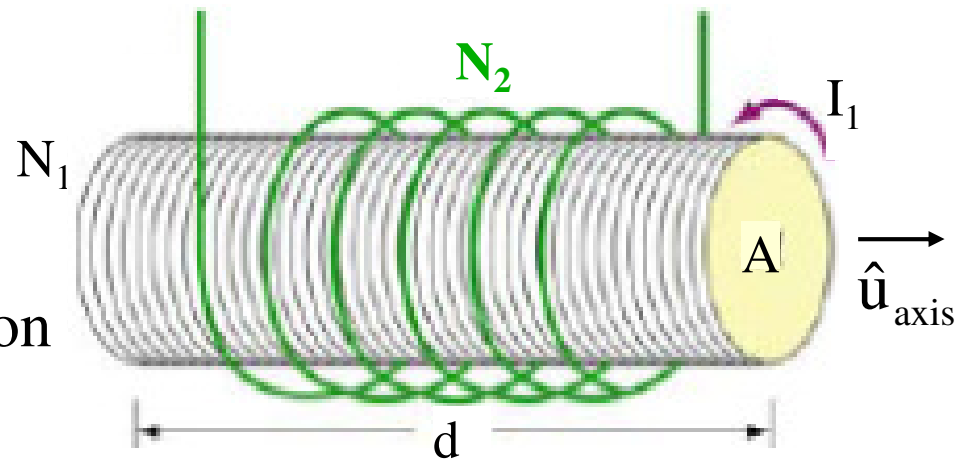
→ magnetic field within solenoid $\vec{B}_1 = \frac{\mu_0 N_1 I_1}{d} \hat{u}_{\text{axis}}$

→ flux linkage with coil $\Phi_{21} = (\vec{B}_1 \bullet \vec{A}_2) N_2 = \frac{\mu_0 N_1 I_1}{d} N_2 A_2$

→ mutual inductance $M = \frac{\Phi_{21}}{I_1} = \frac{\mu_0 N_1 N_2 A}{d}$

starting with current in coil?

- non-uniform \vec{B}_2
- more difficult derivation
- different analytical expression
- but same numerical answer



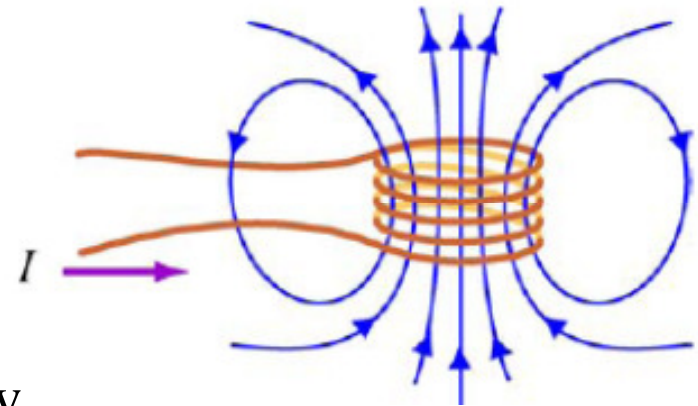
Electromagnetic Induction

self-inductance of current loop

- magnetic field set up by I in coil
- (partial) magnetic flux linkage

$$\text{induced emf } V = -\frac{d}{dt}(N\Phi)$$

direction determined by Lenz's Law



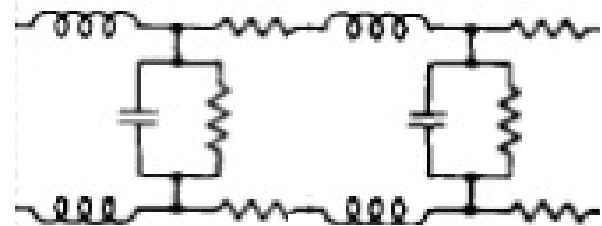
definition: $L = \frac{N\Phi}{I}$ (same unit of Henry as mutual inductance)

(a) interpreted as measure of magnetic flux/energy storage

(b) adopted as series component in transmission-line model

(c) can minimize L by removing loops

(d) may still have to consider L_{internal}
due to magnetic field in conductor



Electromagnetic Induction

self-inductance example: coaxial cable

current in central conductor (need not be uniform)

→ Ampere's Law for deriving \vec{B} expression
in space between conductors ($a < r < b$)

→ flux over narrow strip dr at r

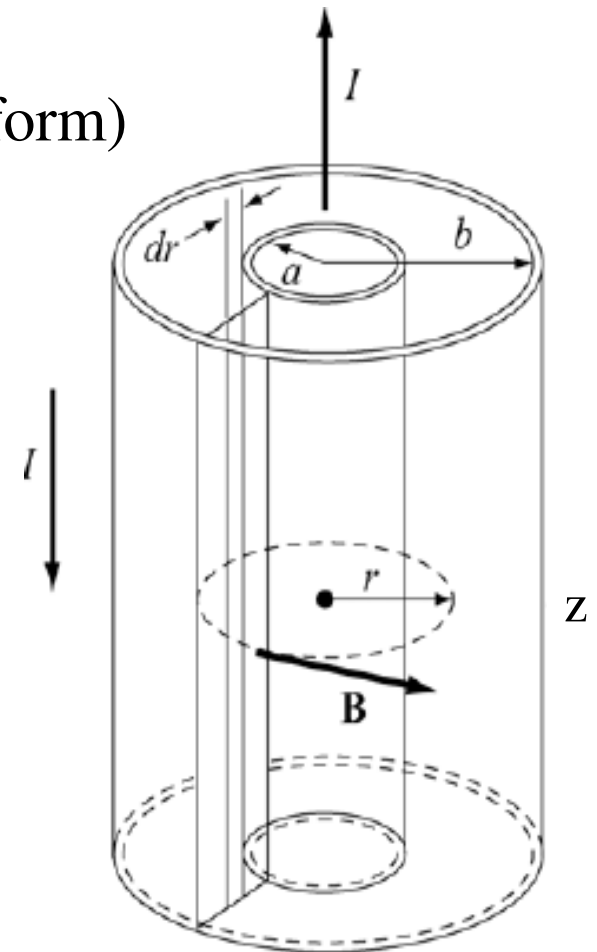
$$d\Phi = \vec{B} \cdot d\vec{A} = \frac{\mu_0 I}{2\pi r} z dr$$

$$\Phi = \frac{\mu_0 I z}{2\pi} \int_a^b \frac{1}{r} dr = \frac{\mu_0 I z}{2\pi} \ln \frac{b}{a}$$

→ inductance per unit length

$$\hat{L} = \frac{L}{z} = \frac{\mu_0}{2\pi} \ln \frac{b}{a}$$

usually disregard L_{interior} due to magnetic
field inside central conductor ($r < a$)



Electromagnetic Induction

transients for circuit with coil

Lenz's Law \rightarrow opposes change in flux

(a) close S_1 (but leave S_2 open)

back emf initially blocking current flow

(b) close S_2 (but leave S_1 open)

back emf initially blocking current decay

