Chapter 6 Instructor Notes

Chapter 6 has also seen substantive revisions with respect to the Third Edition. A section on Fourier analysis has been added (Section 6.2), and the material on Bode plots has been expanded (Section 6.4). The material on Laplace transforms has been moved to the Appendix B. These changes were prompted by comments and suggestions made by numerous users. Chapter 6 can be covered immediately following Chapter 4, or after completing Chapter 5. There is no direct dependence of Chapter 6 on Chapter 5.

After the first section briefly introduces the notion of sinusoidal frequency response and motivates the use of sinusoidal signals, the Fourier Series method of representing signals is described in detail in Section 6.2. Further, the text and examples also illustrate the effect of a multi-components signal propagating through a linear system. Four examples accompany this presentation. Instructors who use this material will find some computing tools available on the website that will assist the students in developing an intuitive understanding of Fourier series.

Section 6.3 introduces filters, and outlines the basic characteristics of low-, high- and band-pass filters. The concept of resonance is treated in greater depth than in the previous edition, and a connection is made with the natural response of second order circuits, which may be useful to those instructors who have already covered transient response of second-order circuits. Four detailed examples are included in this section, Further, the boxes *Focus on Measurements: Wheatstone bridge filter* (pp. 310-303), *Focus on Measurements: AC line interference filter* (pp. 303-305), and *Focus on Measurements: Seismic displacement transducer* (pp. 305-308) touch on additional application examples. The first and last of these boxes can be linked to related material in Chapters2, 3, and 4.

The instructor who has already introduced the operational amplifier as a circuit element will find that section 8.3, on active filters, is an excellent vehicle to reinforce both the op-amp concept and the frequency response ideas. Another alternative (employed by this author) consists of introducing the op-amp at this stage, covering sections 8.1 through 8.3.

Finally, Section 6.4 expands the previous coverage of Bode plots, and illustrates how to create approximate Bode plots using the straight-line asymptotic approximation. The box *Focus on Methodology: Bode Plots* clearly outlines the method, which is further explained in two examples.

The homework problems present several frequency response, Fourier Series, filter and Bode plot exercises of varying difficulty. The instructor who wishes to use one of the many available software aids (e.g., MATLAB® or Electronics Workbench®) to analyze the frequency response of more complex circuits and to exploit more advanced graphics capabilities, will find that several advanced problems lend themselves nicely to such usage. More advanced problems could be used as a vehicle to introduce modern computer aids. The computer aided example solutions found in the Virtual Lab CD-ROM will guide the student in the solution of these more advanced problems.

Learning Objectives

1. Understand the physical significance of frequency domain analysis, and compute the frequency response of circuits using AC circuit analysis tools.
2. Compute the Fourier spectrum of periodic signals using the Fourier series representation, and use this representation in connection with frequency response ideas to compute the response of circuits to periodic inputs.
3. Analyze simple first- and second-order electrical filters, and determine their frequency response and filtering properties.
4. Compute the frequency response of a circuit and its graphical representation in the form of a Bode plot.

Section 6.1: Sinusoidal Frequency Response

Problem 6.1

Solution:

Known quantities:
Resistance and inductance values, in the circuit of Figure P6.1, R = 200 kΩ and L = 0.5 H, respectively.

Find:
a) The frequency response for the circuit of Figure P6.1.
b) Plot magnitude and phase of the circuit using a linear scale for frequency.
c) Repeat part b., using semilog paper.
d) Plot the magnitude response using semilog paper with magnitude in dB.

Analysis:

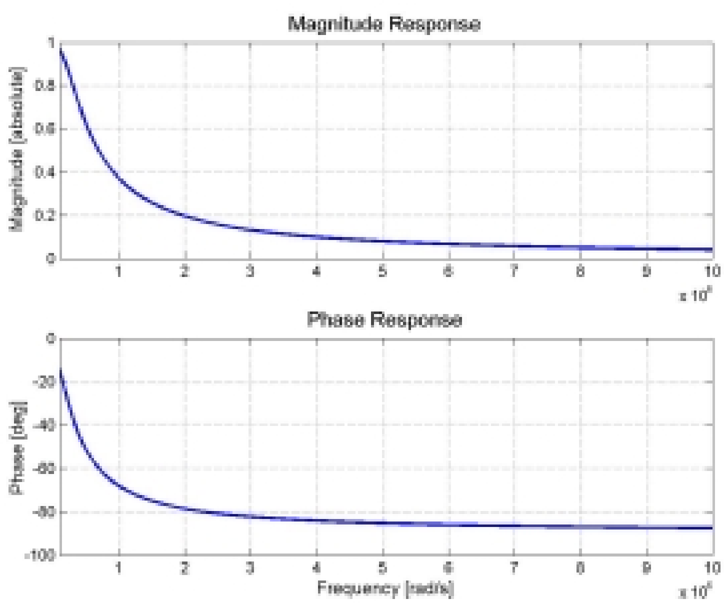
a) $$\frac{\mathrm{v}_{out}}{v_{in}}(j\omega)=\frac{R}{R+j\omega L}=\frac{1}{1+j\omega L/R}=\frac{1}{1+j\left(2.5\times 10^{-6}\omega\right)}$$

$$\left|\frac{V_{out}}{V_{in}}\right| = \frac{1}{\sqrt{1+(\omega L/R)^2}}=\frac{1}{\sqrt{1+6.25\times 10^{-12}\omega^2}}$$

$$\varphi(\omega)=-\arctan\left(2.5\times 10^{-6}\omega\right)$$

The plots obtained using Matlab are shown below:

b)



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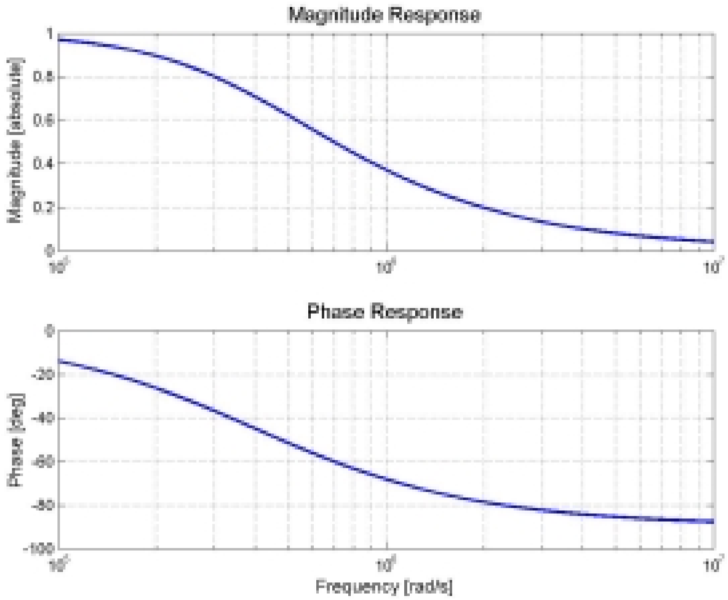

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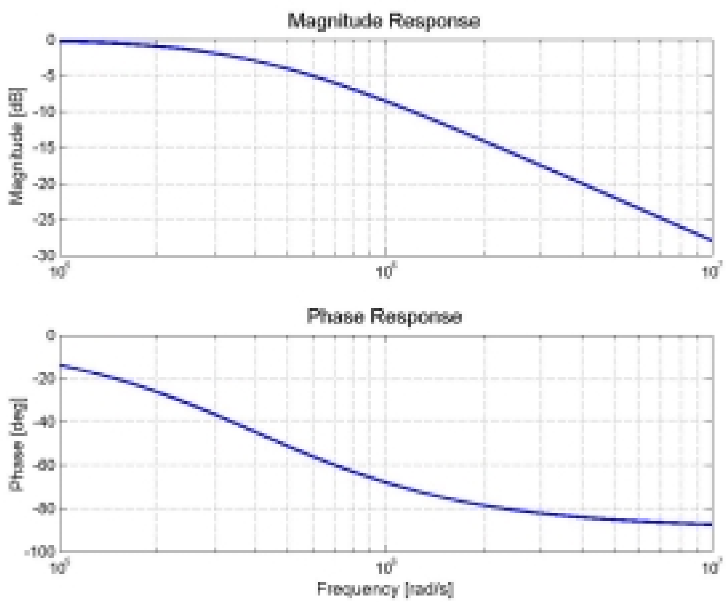


Problem 6.2

Solution:

Known quantities:
Resistance and capacitance values, in the circuit of Figure P6.2.

Find:
a) The frequency response for the circuit of Figure P6.2.
b) Plot magnitude and phase of the circuit using a linear scale for frequency.
c) Repeat part b., using semilog paper.
d) Plot the magnitude response using semilog paper with magnitude in dB.

Analysis:
First, we find the Thévenin equivalent circuit seen by the capacitor:

$$R_T = 500 \| 500 = 250 \ \Omega$$

and

$$v_{OC} = \frac{500}{500+500} v_{in} = \frac{v_{in}}{2}$$

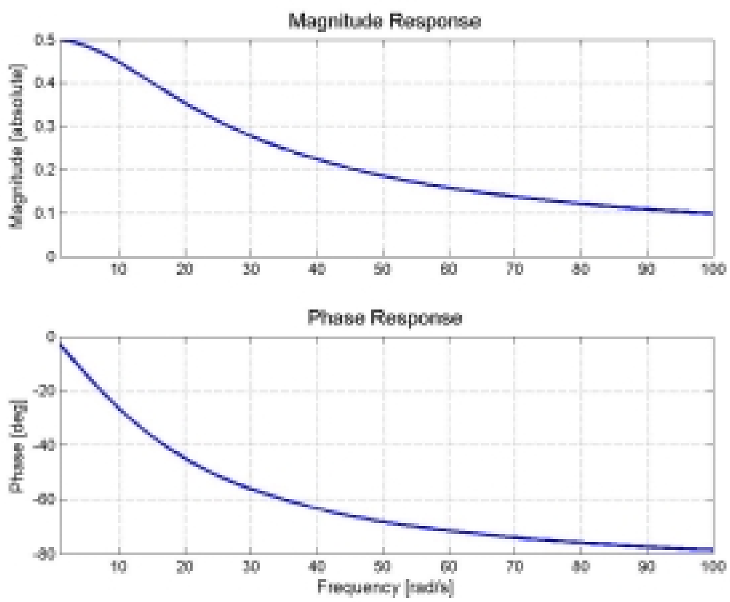
a)
$$\frac{v_{out}}{v_{OC}} = \frac{1/j\omega C}{R_T + 1/j\omega C} = \frac{1}{1 + j\omega R_T C} = \frac{1}{1 + j(0.05\omega)}$$

$$\left|\frac{v_{out}}{v_{OC}}\right| = \frac{1}{\sqrt{1 + 0.0025\omega^2}}$$

$$\left|\frac{v_{out}}{v_{in}}\right| = \frac{1}{2}\left|\frac{v_{out}}{v_{OC}}\right| = \frac{1}{\sqrt{4 + 0.01\omega^2}}$$

$$\varphi(\omega) = -\arctan(0.05\,\omega)$$

b) The plots obtained using Matlab are shown below:



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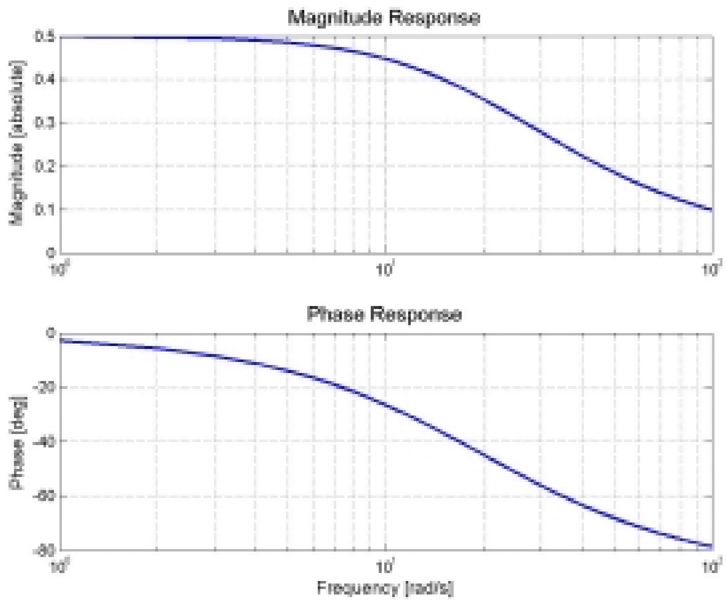

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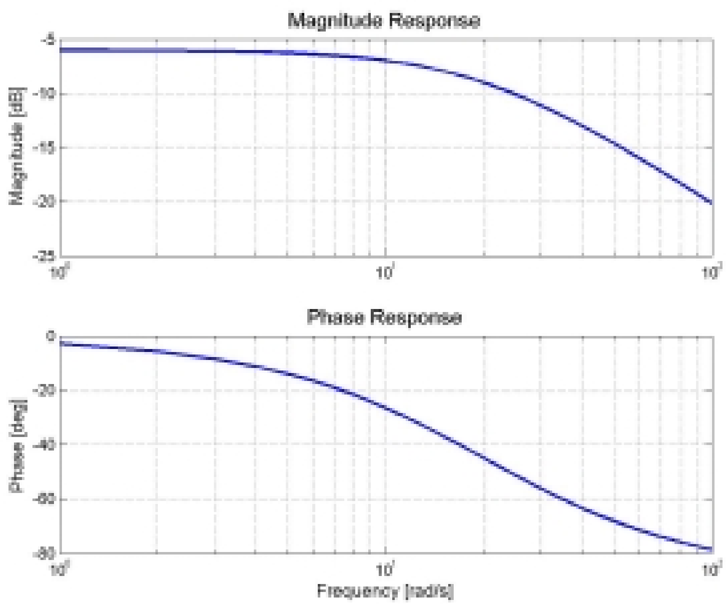


Problem 6.3

Solution:

Known quantities:
Resistance and capacitance values, in the circuit of Figure P6.3.

Find:

a) The frequency response for the circuit of Figure P6.3.
b) Plot magnitude and phase of the circuit using a linear scale for frequency.
c) Repeat part b., using semilog paper.
d) Plot the magnitude response using semilog paper with magnitude in dB.

Analysis:
First, we find the Thévenin equivalent circuit seen by the capacitor:

$$R_T = 2000 \| 2000 + 1000 = 2000 \quad \Omega$$

and

$$v_{OC} = \frac{2000}{2000 + 2000} v_{in} = \frac{v_{in}}{2}$$

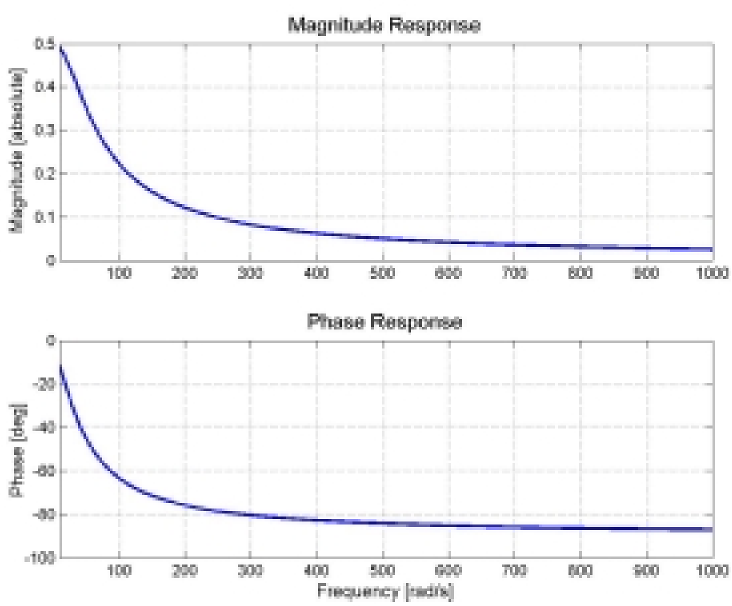
a) $$\frac{v_{out}}{v_{OC}} = \frac{1/j\omega C}{R_T + 1/j\omega C} = \frac{1}{1 + j\omega R_T C} = \frac{1}{1 + j(0.02\omega)}$$

$$\left|\frac{v_{out}}{v_{OC}}\right| = \frac{1}{\sqrt{1 + 4\times10^{-4}\omega^2}}$$

$$\left|\frac{v_{out}}{v_{in}}\right| = \frac{1}{2}\left|\frac{v_{out}}{v_{OC}}\right| = \frac{0.5}{\sqrt{1 + 4\times10^{-4}\omega^2}}$$

$$\varphi(\omega) = -\arctan(0.02\,\omega)$$

b) The plots obtained using Matlab are shown below:



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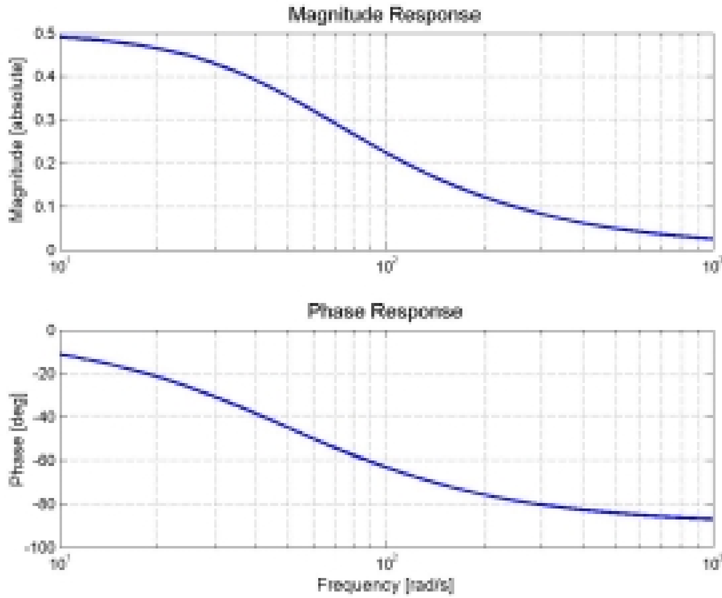

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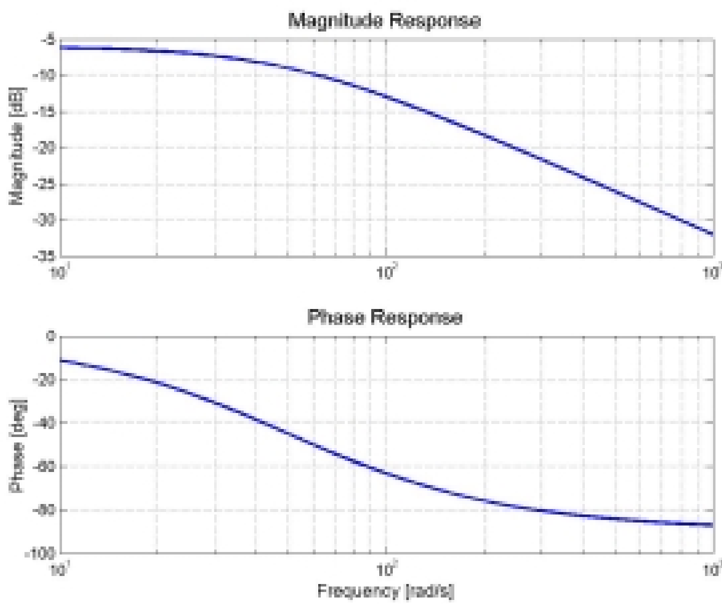


Problem 6.4

Solution:

Known quantities:
Resistance, inductance and capacitance values, in the circuit of Figure P6.4.

Find:

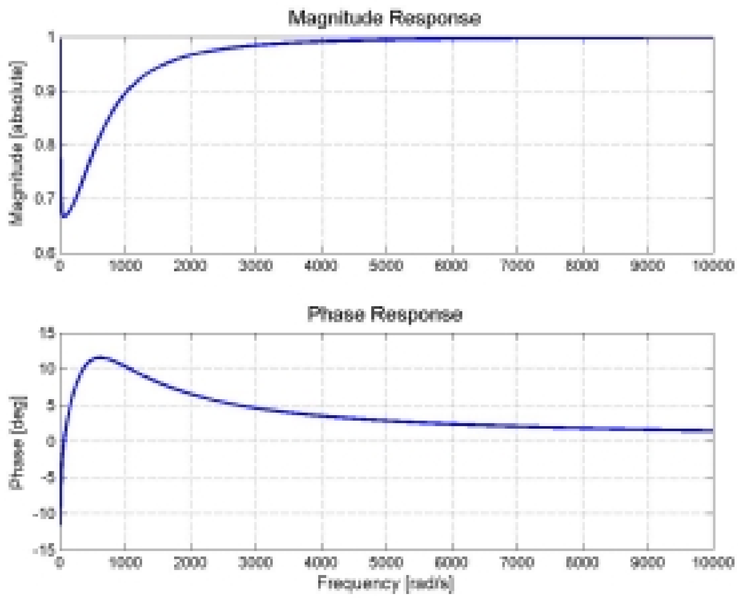
a) The frequency response for the circuit of Figure P6.4.
b) Plot magnitude and phase of the circuit using a linear scale for frequency.
c) Repeat part b., using semilog paper.
d) Plot the magnitude response using semilog paper with magnitude in dB.

Analysis:

a)

$$\frac{v_{out}}{v_{in}}(j\omega)=\frac{R_2+j\omega L+\frac{1}{j\omega C}}{R_1+R_2+j\omega L+\frac{1}{j\omega C}}=\frac{1+j\omega CR_2+(j\omega)^2 LC}{1+j\omega C(R_1+R_2)+(j\omega)^2 LC}=\frac{1-0.0002\omega^2+j(0.1)\omega}{1-0.0002\omega^2+j(0.15)\omega}$$

b) The plots obtained using Matlab are shown below:



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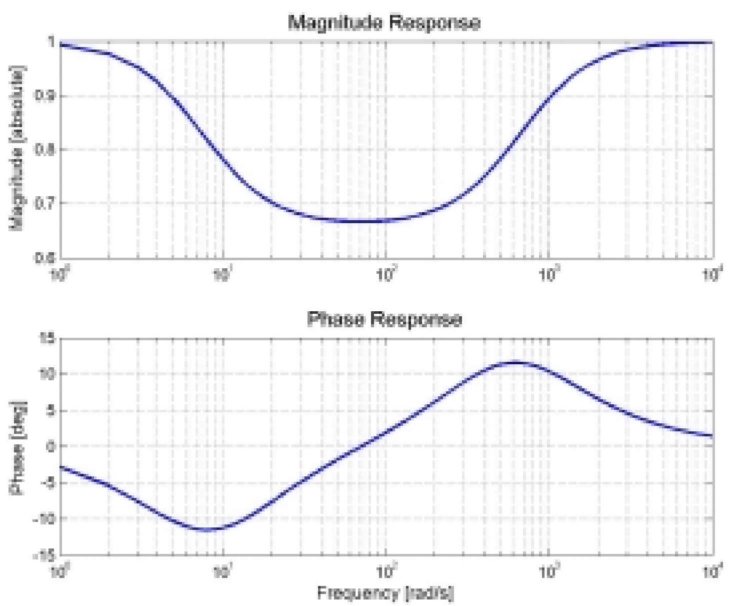

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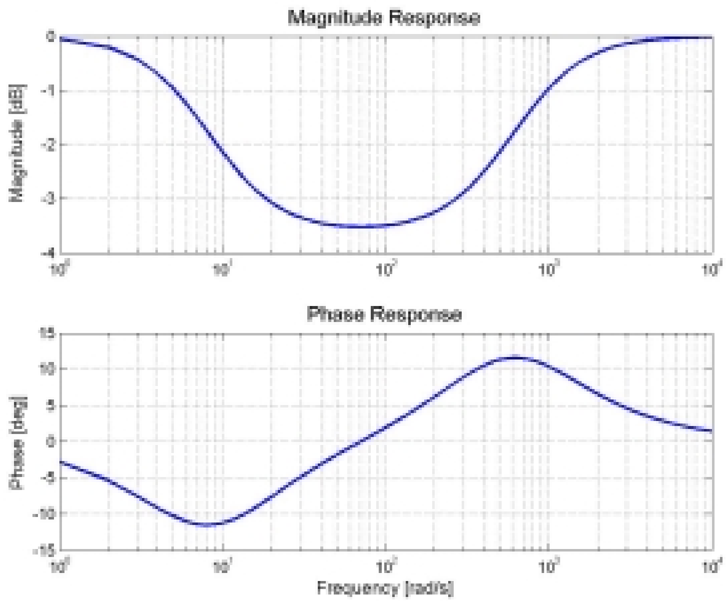


Problem 6.5

Solution:

Known quantities:
Resistance, inductance and capacitance values, in the circuit of Figure P6.5

Find:
a) The frequency response for the circuit of Figure P6.5
b) Plot magnitude and phase of the circuit using a linear scale for frequency.
c) Repeat part b., using semilog paper.
d) Plot the magnitude response using semilog paper with magnitude in dB.

Assume:
Assume that the output voltage is the voltage across the capacitor.

Analysis:

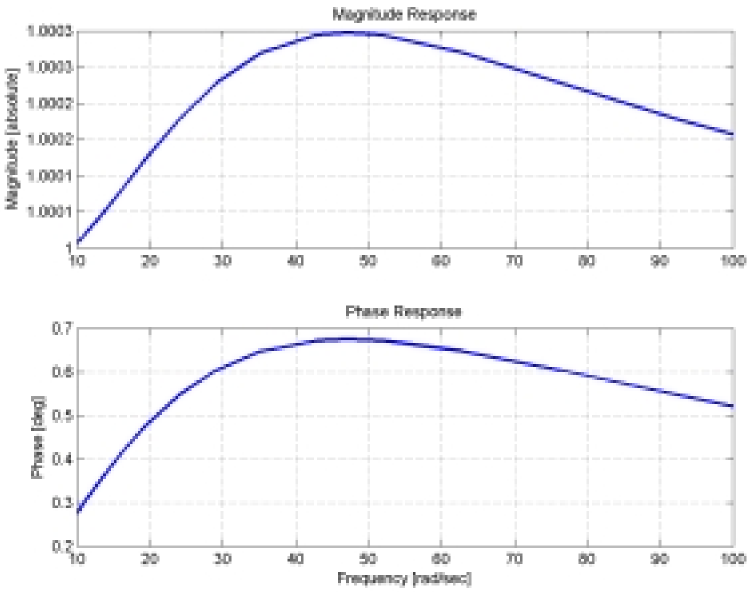
a) $$\frac{\mathbf{v}_{out}}{v_{in}}(j\omega)=\frac{(Z_L \parallel Z_C)+R_2}{(Z_L \parallel Z_C)+R_2+R_1}=\frac{\left(\frac{1}{j\omega L}+j\omega C\right)^{-1}+R_2}{\left(\frac{1}{j\omega L}+j\omega C\right)^{-1}+R_2+R_1}$$

$$\frac{\mathbf{v}_{out}}{v_{in}}(j\omega)=\frac{\left(\frac{j\omega L}{1-CL\omega^2}\right)+R_2}{\left(\frac{j\omega L}{1-CL\omega^2}\right)+R_2+R_1}=\frac{R_2-CLR_2\omega^2+j\omega L}{R_2+R_1-CL(R_2+R_1)\omega^2+j\omega L}$$

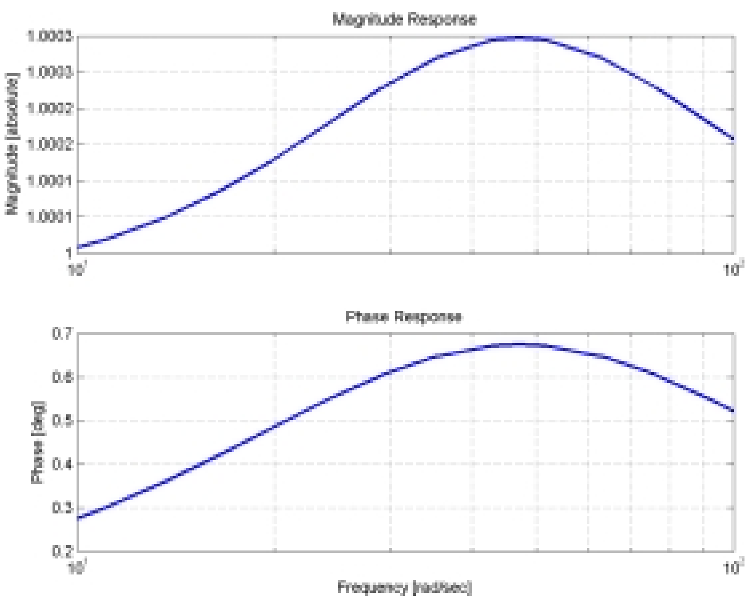
Substituting the numerical values:

$$\frac{\mathbf{v}_{out}}{v_{in}}(j\omega)=\frac{(1-4.5\times10^{-4}\omega^2)+j(0.0015)\omega}{(1-4.5\times10^{-4}\omega^2)+j(0.0010)\omega}$$

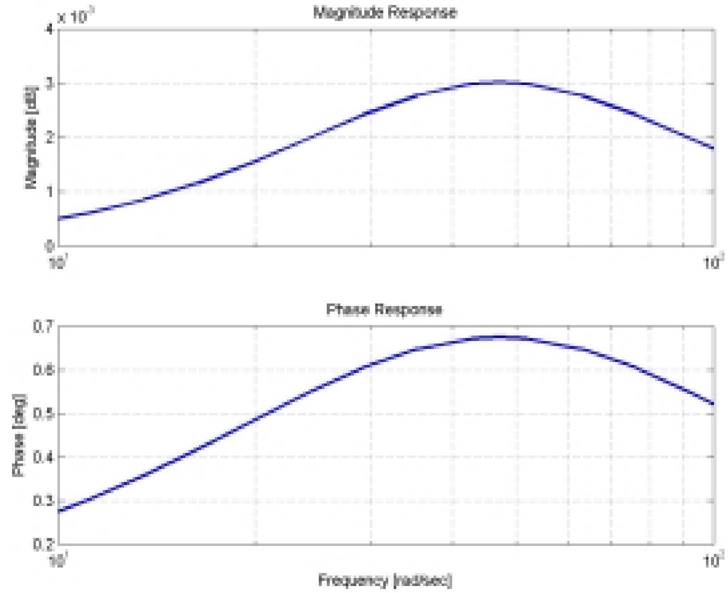
b) The plots obtained using Matlab are shown below:


Magnitude Response
Magnitude [absolute]
Phase Response
Phase [deg]
Frequency [rad/sec]

c)


Magnitude Response
Magnitude [absolute]
Phase Response
Phase [deg]
Frequency [rad/sec]

d)



Problem 6.6

Solution:

Known quantities:
Resistance, inductance and capacitance values, in the circuit of Figure P6.6

Find:
a) The frequency response for the circuit of Figure P6.6
b) Plot magnitude and phase of the circuit using a linear scale for frequency.
c) Repeat part b., using semilog paper.
d) Plot the magnitude response using semilog paper with magnitude in dB.

Analysis:
First, we find the Thévenin equivalent circuit seen by the capacitor:

$$Z_T = Z_{R2} + \left(Z_{C1} \parallel Z_{R1}\right) = R_2 + \frac{R_1 / j\omega C_1}{1/j\omega C_1 + R_1} = R_2 + \frac{R_1}{1 + j\omega C_1 R_1}$$

and

$$v_{OC} = \frac{Z_{R1}}{Z_{R1} + Z_{C1}} v_{in} = \frac{R_1}{R_1 + 1/j\omega C_1} v_{in} = \frac{j\omega C_1 R_1}{1 + j\omega C_1 R_1} v_{in}$$

a) $$\frac{v_{out}}{v_{OC}} = \frac{Z_{C2}}{Z_T + Z_{C2}} = \frac{1/j\omega C_2}{\left(R_2 + \frac{R_1}{1 + j\omega C_1 R_1}\right) + 1/j\omega C_2} = \frac{1}{1 + \left(R_2 + \frac{R_1}{1 + j\omega C_1 R_1}\right) j\omega C_2}$$

Therefore,

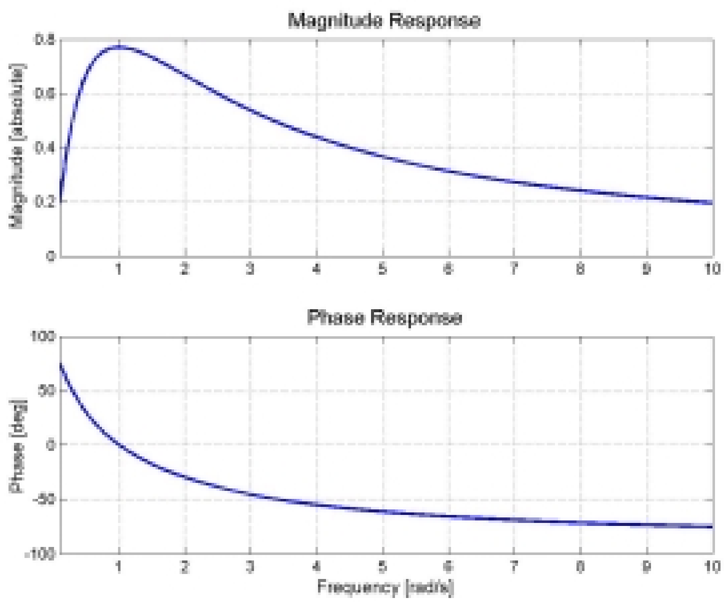
$$\frac{v_{out}}{v_{in}} = \frac{j\omega C_1 R_1}{1 + j\omega C_1 R_1} \cdot \frac{1}{1 + \left(R_2 + \dfrac{R_1}{1 + j\omega C_1 R_1}\right) j\omega C_2}$$

$$= \frac{j\omega C_1 R_1}{1 + j\omega\left[C_1 R_1 + C_2\left(R_1 + R_2\right)\right] + \left(j\omega\right)^2 C_1 C_2 R_1 R_2}$$

Substituting the numerical values:

$$\frac{v_{out}}{v_{in}} = \frac{j(2)\omega}{\left(1 - \omega^2\right) + j(2.6)\omega}$$

b) The plots obtained using Matlab are shown below:



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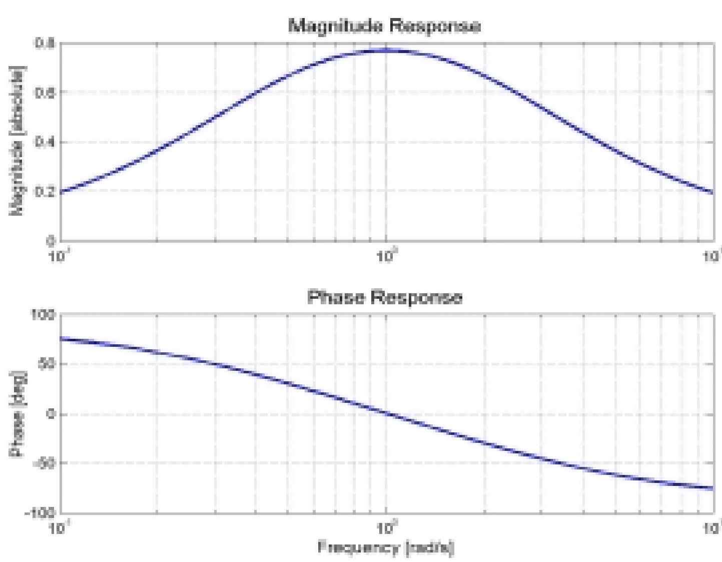
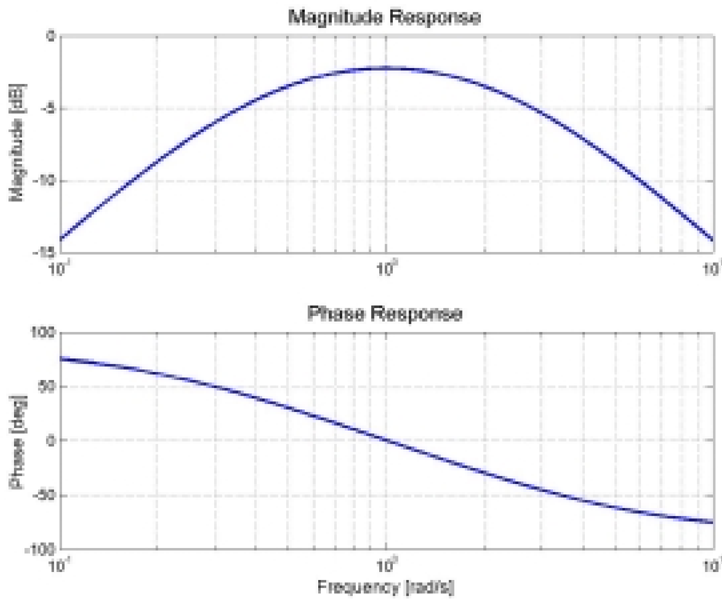


d)



Problem 6.7

Solution:

Known quantities:
Figure P6.7.

Find:

a) How the "driving point" impedance, $Z[j\omega] = \dfrac{V_i[j\omega]}{I_i[j\omega]}$, behaves at extremely high or low frequencies.

b) An expression for the input (or driving point) impedance.

c) Show that this expression can be manipulated into the form:

$$Z[j\omega] = Z_o(\,1 \pm j\ f[\,\omega]\ )$$

$$\text{Where}: \quad Z_o = R \qquad f[\,\omega] = \frac{1}{\omega RC} \qquad C = 0.5\ \ \mu\text{F} \qquad R = \ 2\ \text{k}\Omega$$

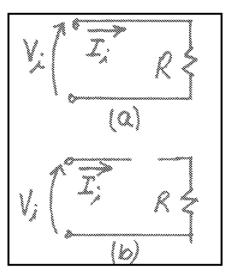
d) Determine the "cutoff" frequency $\omega = \omega_c$ at which $f\,[\omega_c] = 1$.

e) Determine the magnitude and angle of $Z\,[j\omega]$ at $\omega = 100$ rad/s, 1000 rad/s, and 10,000 rad/s.

f) Predict (without computing) the magnitude and angle of $Z\ [j\omega]$ at $\omega = 10$ rad/s and 100,000 rad/s.

Analysis:

a)



$$\text{As}\ \omega \to \infty, \quad Z_C \to 0 \Rightarrow Short \Rightarrow Z \to R$$

$$\text{As}\ \omega \to 0, \quad Z_C \to \infty \Rightarrow Open \Rightarrow Z \to \infty$$

b)

$$KVL: \ -V_i + I_i Z_C + I_i Z_R = 0$$

$$Z[j\omega] = \frac{V_i}{I_i} = Z_C + Z_R = \frac{1}{j\omega C} + R$$

c) $$Z[j\omega] = R + j\frac{1}{j^2\omega C} = R\,[\,1 - j\frac{1}{\omega RC}\,]$$

d) $$f[\omega_c] = \frac{1}{\omega_c RC} = 1 \ \Rightarrow \ \omega_c = \frac{1}{RC} = \frac{1}{[\,2000\,]\,[\,0.5\cdot 10^{-6}\,]} = 1000\ \frac{\text{rad}}{\text{s}}$$

e)

$$Z[100\frac{\text{rad}}{\text{s}}] = R(\,1 - j\frac{1}{[\,100\,]\,[\,2000\,]\,[\,0.5\cdot 10^{-6}\,]}\,)$$

$$= 2000\,[\,1 - j\,10\,] = 20.10\ \text{k}\Omega\angle -84.29^0$$

$$Z[1000\frac{\text{rad}}{\text{s}}] = R(\,1 - j\frac{1}{[\,1000\,]\,[\,2000\,]\,[\,0.5\cdot 10^{-6}\,]}\,)$$

$$= 2000\,[\,1 - j\,1\,] = 2.828\ \text{k}\Omega\angle -45.00^0$$

$$Z[10\,\text{k}\frac{\text{rad}}{\text{s}}] = R(\,1 - j\frac{1}{[\,10000\,]\,[\,2000\,]\,[\,0.5\cdot 10^{-6}\,]}\,)$$

$$= 2000\,[\,1 - j\,0.1\,] = 2.010\ \text{k}\Omega\angle -5.71^0$$

f)

$$Z[10\frac{\text{rad}}{\text{s}}] \approx 200\ \text{k}\Omega\angle -90^0$$

$$Z[100\,\text{k}\frac{\text{rad}}{\text{s}}] \approx 2\ \text{k}\Omega\angle 0^0$$

Problem 6.8

Solution:

Known quantities:
Figure P6.8.

Find:

a) How the "driving point" impedance, $Z[j\omega] = \dfrac{V_i[j\omega]}{I_i[j\omega]}$, behaves at extremely high or low frequencies.

b) An expression for the driving point impedance.
c) Show that this expression can be manipulated into the form:

$$Z[j\omega] = Z_o(1 + j\ f[\omega]\ )$$

$$\text{Where: } Z_0 = R \quad f[\omega] = \frac{\omega L}{R} \quad L = 2\ \text{mH} \quad R = 2\ \text{k}\Omega$$

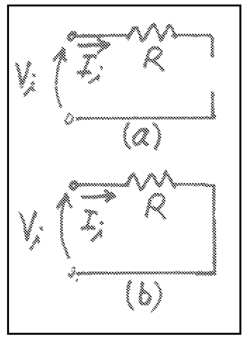
d) Determine the "cutoff" frequency $\omega = \omega_c$ at which $f[\omega_c] = 1$.
e) Determine the magnitude and angle of $Z[j\omega]$ at $\omega = 100$ krad/s, 1,000 krad/s, and 10,000 krad/s.
f) Predict (without computing it) the magnitude and angle of $Z[j\omega]$ at $\omega = 10$ krad/s and 100,000 krad/s.

Analysis:

a)

$$\text{As } \omega \to \infty,\ Z_L \to 0 \Rightarrow Open \Rightarrow Z \to \infty$$

$$\text{As } \omega \to 0,\ Z_L \to 0 \Rightarrow Short \Rightarrow Z \to R$$



b)

$$KVL:\ -V_i + I_i Z_R + I_i Z_L = 0$$

$$Z[j\omega] = \frac{V_i}{I_i} = Z_L + Z_R = j\omega L + R$$

c) In standard form: $Z[j\omega] = R + j\,\omega L = R\,[\,1 + j\frac{\omega L}{R}\,]$

d) $$f[\omega_c] = \frac{\omega_c L}{R} = 1 \Rightarrow \omega_c = \frac{R}{L} = \frac{2000}{2\cdot 10^{-3}} = 1000\ \text{k}\frac{\text{rad}}{\text{s}}$$

e) The standard form can now be rewritten as:

$$Z[j\omega] = R\,[\,1 + j\frac{\omega}{\omega_c}\,] = 2000\,[\,1 + j\frac{\omega}{1\cdot 10^6}\,]$$

$$Z[100\,\text{k}\frac{\text{rad}}{\text{s}}] = R(\,1 + j\frac{100\cdot 10^3}{1\cdot 10^6}\,) = 2000\,[\,1 + j\,0.1\,] = 2.01\,\text{k}\Omega\angle 5.71^0$$

$$Z[1\,\text{M}\frac{\text{rad}}{\text{s}}] = R(\,1 + j\frac{1\cdot 10^6}{1\cdot 10^6}\,) = 2000[1 + j\,1] = 2.82\,\text{k}\Omega\angle 45.00^0$$

$$Z[10\text{M}\frac{\text{rad}}{\text{s}}] = R(\,1 + j\frac{10\cdot 10^6}{1\cdot 10^6}\,) = 2000\,[\,1 + j\,10\,] = 20.10\,\text{k}\Omega\angle 84.29^0$$

Note, in particular, the behavior of the impedance one decade below and one decade above the cutoff frequency.

f)

$$Z[100\ \mathrm{M}\frac{\mathrm{rad}}{\mathrm{s}}] \approx 200\ \mathrm{k}\Omega\angle 90^\circ$$

$$Z[10\ \mathrm{k}\frac{\mathrm{rad}}{\mathrm{s}}] \approx 2\ \mathrm{k}\Omega\angle 0^\circ$$

Problem 6.9

Solution:

Known quantities:

With reference to Figure P6.9:

$$L = 190\ \mathrm{mH} \quad R_1 = 2.3\ \mathrm{k}\Omega$$
$$C = 55\ \mathrm{nF} \quad R_2 = 1.1\ \mathrm{k}\Omega$$

Find:

a) How the "driving point" impedance, $Z[j\omega] = \frac{V_i[j\omega]}{I_i[j\omega]}$, behaves at extremely high or low frequencies.

b) An expression for the driving point impedance in the form:

$$Z[j\omega] = Z_o\left[\frac{1+j\,f_1[\omega]}{1+j\,f_2[\omega]}\right] \qquad Z_o = R_1 + \frac{L}{R_2 C}$$

$$f_1[\omega] = \frac{\omega^2 R_1 LC - R_1 - R_2}{\omega[R_1 R_2 C + L]} \qquad f_2[\omega] = \frac{\omega^2 LC - 1}{\omega C R_2}$$

c) Determine the four cutoff frequencies at which $f_1[\omega] = +1$ or -1 and $f_2[\omega] = +1$ or -1.

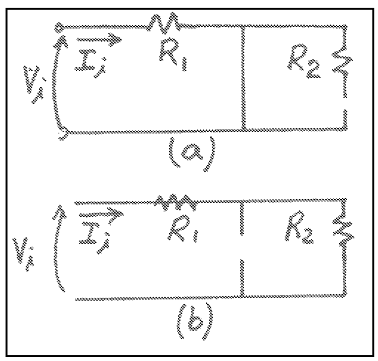
d) Determine the resonant frequency of the circuit.

e) Plot the magnitude of the impedance [in dB] as a function of the Log of the frequency, i.e., a Bode plot.

Analysis:

a)

$$\text{As } \omega \to \infty,\ Z_L \to \infty \Rightarrow Open,\quad Z_C \to 0$$
$$\Rightarrow Short \Rightarrow Z \to R_1$$
$$\text{As } \omega \to 0,\ Z_C \to \infty \Rightarrow Open,\quad Z_L \to 0$$
$$\Rightarrow Short \Rightarrow Z \to R_1 + R_2$$



b)

$$Z[j\omega] = \frac{V[j\omega]}{I[j\omega]} = Z_{R1} + \frac{Z_C[Z_{R2}+Z_L]}{Z_C+[Z_{R2}+Z_L]} = R_1 + \frac{[\frac{1}{j\omega C}][R_2+j\omega L]}{\frac{1}{j\omega C}+[R_2+j\omega L]}\frac{j\omega C}{j\omega C} =$$

$$= R_1 + \frac{R_2 + j\omega L}{1-\omega^2 LC + j\omega R_2 C} = \frac{(R_1[1-\omega^2 LC]+R_2)+j(\omega R_1 R_2 C+\omega L)}{[1-\omega^2 LC]+j\omega R_2 C}\cdot\frac{(-j)}{(-j)} \Rightarrow$$

$$\Rightarrow Z[j\omega] = \frac{\omega(R_1R_2C+L)+j(\omega^2R_1LC-R_1-R_2)}{\omega R_2C+j(\omega^2LC-1)} = \frac{R_1R_2C+L}{R_2C}\cdot\frac{1+j\frac{\omega^2R_1LC-R_1-R_2}{\omega(R_1R_2C+L)}}{1+j\frac{\omega^2LC-1}{\omega R_2C}}$$

c) Both $f_1[\omega]$ and $f_2[\omega]$ can be positive or negative, and therefore equal to plus or minus one depending on the frequency; therefore, both cases must be considered.

$$f_1[\omega_c] = \frac{\omega_c[R_1R_2C+L]}{R_1[1-\omega_c^2LC]+R_2} = \pm 1$$

$$\omega_c^2 \pm [\frac{R_2}{L}+\frac{1}{R_1C}]\omega_c - \frac{R_1+R_2}{R_1LC} = 0$$

$$\frac{R_2}{L}+\frac{1}{R_1C} = \frac{1100}{0.19}+\frac{1}{[2300][55\cdot 10^{-9}]} = 13.69\,\mathrm{k}\frac{\mathrm{rad}}{\mathrm{s}}$$

$$\frac{R_1+R_2}{R_1LC} = \frac{3400}{[2300][0.19][55\cdot 10^{-9}]} = 141.46\,\mathrm{M}\frac{\mathrm{rad}}{\mathrm{s}^2}$$

$$\omega_c = -\frac{1}{2}[\pm 13.69\cdot 10^3] \pm \frac{1}{2}([\pm 13.69\cdot 10^3]^2 - 4[1][-141.5\cdot 10^6])^{1/2}$$

$$= \pm 6.845\cdot 10^3 \pm 13.724\cdot 10^3 \quad \Rightarrow \omega_{c1} = 6.879\,\mathrm{k}\frac{\mathrm{rad}}{\mathrm{s}} \quad \omega_{c4} = 20.569\,\mathrm{k}\frac{\mathrm{rad}}{\mathrm{s}}$$

Where only the positive answers are physically valid, i.e., a negative frequency is physically impossible.

$$f_2[\omega_c] = \frac{\omega_c^2LC-1}{\omega_cR_2C} = \pm 1 \qquad \Rightarrow \omega_c^2 \pm [\frac{R_2}{L}]\omega_c - \frac{1}{LC} = 0$$

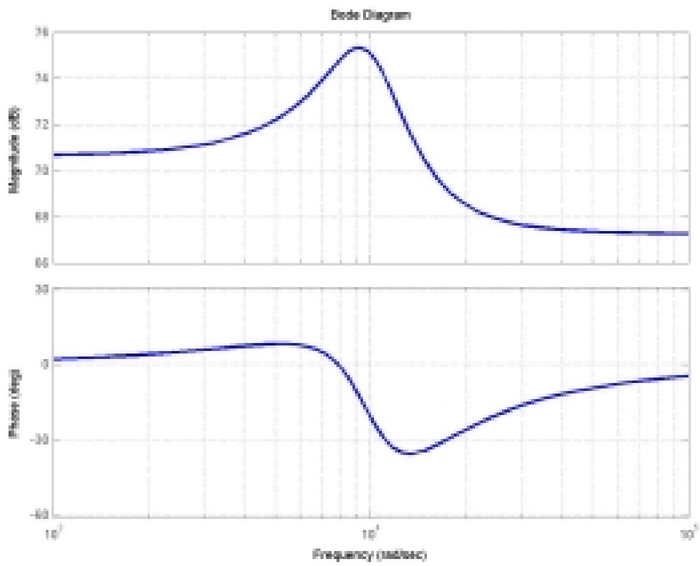
$$\frac{R_2}{L} = \frac{1100}{0.19} = 5.79\,\mathrm{k}\frac{\mathrm{rad}}{\mathrm{s}} \qquad \frac{1}{LC} = \frac{1}{[0.19][55\cdot 10^{-9}]} = 95.69\,\mathrm{M}\frac{\mathrm{rad}}{\mathrm{s}^2}$$

$$\omega_c = -\frac{1}{2}[\pm 5790] \pm \frac{1}{2}([\pm 5790]^2 + 4[1][95.69\cdot 10^6])^{1/2} = \pm 2895 \pm 10201$$

$$\Rightarrow \omega_{c2} = 7.31\,\mathrm{k}\frac{\mathrm{rad}}{\mathrm{s}} \quad \omega_{c3} = 13.09\,\mathrm{k}\frac{\mathrm{rad}}{\mathrm{s}}$$

Again, the negative roots were rejected because they are physically impossible.

d) Plotting the response in a Bode Plot:



Problem 6.10

Solution:

Known quantities:

In the circuit of Figure P6.10:

$$R_1 = 1.3\,\mathrm{k}\Omega \quad R_2 = 1.9\,\mathrm{k}\Omega \quad C = 0.5182\,\mu\mathrm{F}$$

Find:

a) How the voltage transfer function: $H_v[j\omega] = \dfrac{V_o[j\omega]}{V_i[j\omega]}$ behaves at extremes of high and low frequencies.

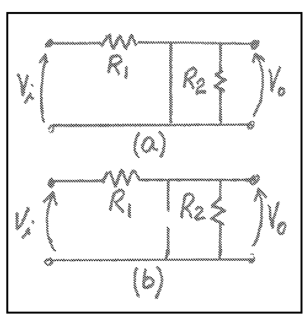
b) An expression for the voltage transfer function, showing that it can be manipulated into the form:

$$H_v[j\omega] = \frac{H_o}{1 + j\ f[\omega]} \quad \text{Where:}\ H_o = \frac{R_2}{R_1 + R_2} \quad f[\omega] = \frac{\omega R_1 R_2 C}{R_1 + R_2}$$

c) The "cutoff" frequency at which $f[\omega] = 1$ and the value of H_o in dB.

Analysis:

a)



$$As\ \omega \to \infty:\ Z_C \to 0\angle -90^0 \Rightarrow Short$$

$$VD:\ H_v \to 0\angle -90^0$$

$$As\ \omega \to 0:\ Z_C \to \infty\angle -90^0 \Rightarrow Open$$

$$VD:\ H_v \to \frac{R_2}{R_1 + R_2}\angle 0^0$$

b) $$Z_{eq} = \frac{Z_C Z_{R2}}{Z_C + Z_{R2}} = \frac{[\frac{1}{j\omega C}][R_2]}{\frac{1}{j\omega C} + R_2} \frac{j\omega C}{j\omega C} = \frac{R_2}{1 + j\omega R_2 C}$$

$$VD: \ H_v[j\omega] = \frac{V_o[j\omega]}{V_i[j\omega]} = \frac{Z_{eq}}{Z_{R1} + Z_{eq}} = \frac{\frac{R_2}{1 + j\omega R_2 C}}{R_1 + \frac{R_2}{1 + j\omega R_2 C}} \frac{1 + j\omega R_2 C}{1 + j\omega R_2 C} =$$

$$= \frac{R_2}{R_1 + R_2 + j\omega R_1 R_2 C} = \frac{R_2}{R_1 + R_2} \frac{1}{1 + j\frac{\omega R_1 R_2 C}{R_1 + R_2}}$$

c)

$$f[\omega_c] = \frac{\omega_c R_1 R_2 C}{R_1 + R_2} = 1 \qquad \omega_c = \frac{1300 + 1900}{[1300][1900][0.5182 \cdot 10^{-6}]} = 2.5\,\text{k}\frac{\text{rad}}{\text{s}}$$

$$H_o = \frac{R_2}{R_1 + R_2} = \frac{1900}{1300 + 1900} = 0.5938 = 20 \cdot \text{Log}[0.5938] = -4.527\ \text{dB}$$

Problem 6.11

Solution:

Known quantities:
Figure P6.11.

Find:

a) The behavior of the voltage transfer function or gain at extremely high and low frequencies.

b) The output voltage V_o if the input voltage has a frequency where:

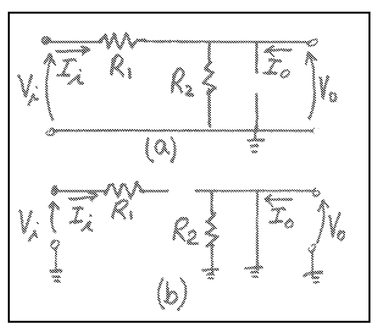
$$V_i = 7.07\ \text{V} \angle 45^o \quad R_1 = 2.2\ \text{k}\Omega \quad R_2 = 3.8\ \text{k}\Omega \quad X_C = 5\ \text{k}\Omega \quad X_L = 1.25\ \text{k}\Omega$$

c) The output voltage if the frequency of the input voltage doubles so that: $X_C = 2.5\ \text{k}\Omega \quad X_L = 2.5\ \text{k}\Omega$

d) The output voltage if the frequency of the input voltage again doubles so that: $X_C = 1.25\ \text{k}\Omega \quad X_L = 5\ \text{k}\Omega$

Analysis:

a)



$$As\ \omega \to 0 \quad Z_C \to \infty \Rightarrow Open$$
$$Z_L \to 0 \Rightarrow Short$$
$$V_o \to 0$$

$$As\ \omega \to \infty \quad Z_C \to 0 \Rightarrow Short$$
$$Z_L \to \infty \Rightarrow Open$$

$$VD: \ V_o = \frac{V_i R_2}{R_1 + R_2} = \frac{[7.07][3800]}{2200 + 3800} = 4.478\ \text{V} \angle 45^0$$

b)

$$Z_{eq1} = Z_{R1} + Z_C = R_1 - jX_C \qquad Z_{eq2} = \frac{Z_{R2} Z_L}{Z_{R2} + Z_L} = \frac{[R_2][jX_L]}{R_2 + jX_L}$$

$$VD: \; V_o = \frac{V_i Z_{eq2}}{Z_{eq1} + Z_{eq2}} = V_i \frac{\dfrac{jR_2 X_L}{R_2 + jX_L}}{R_1 - jX_C + \dfrac{jR_2 X_L}{R_2 + jX_L}} \frac{R_2 + jX_L}{R_2 + jX_L} \Rightarrow$$

$$\Rightarrow V_o = V_i \frac{jR_2 X_L}{[R_1 R_2 + X_C X_L] + j[X_L(R_1 + R_2) - X_C R_2]}$$

$$V_i \cdot [jR_2 X_L] = [7.07\text{ V} \angle 45^0][(3.8\text{ k}\Omega)(1.25\text{ k}\Omega)\angle 90^0] = 33.58 \cdot 10^6 \angle 135^0$$

$$R_1 R_2 + X_C X_L = [2200][3800] + [5000][1250] = 14.61 \cdot 10^6$$

$$X_L[R_1 + R_2] - X_C R_2 = [1250][6000] - [5000][3800] = -11.50 \cdot 10^6$$

$$V_0 = \frac{33.58 \cdot 10^6 \angle 135^0}{14.61 \cdot 10^6 - \text{j } 11.50 \cdot 10^6} = \frac{33.58 \angle 135^0}{18.59 \angle -38.2^0} = 1.806\text{V} \angle 173.2^0$$

c)

$$V_i \cdot [jR_2 X_L] = [7.07\text{ V} \angle 45^0][(3800)(2500)\angle 90^0] = 67.17 \cdot 10^6 \angle 135^0$$

$$R_1 R_2 + X_C X_L = [2200][3800] + [2500][2500] = 14.61 \cdot 10^6$$

$$X_L[R_1 + R_2] + X_C R_2 = [2500][6000] - [2500][3800] = 5.50 \cdot 10^6$$

$$V_o = \frac{67.17 \cdot 10^6 \angle 135^0}{14.61 \cdot 10^6 + \text{j}5.50 \cdot 10^6} = \frac{67.17\text{ V} \angle 135^0}{15.61 \angle 20.6^0} = 4.303\text{ V} \angle 114.4^0$$

d)

$$V_i \cdot [jR_2 X_L] = [7.07\text{ V} \angle 45^0][(3800)(5000)\angle 90^0] = 134.34 \cdot 10^6 \angle 135^0$$

$$R_1 R_2 + X_C X_L = [2200][3800] + [1250][5000] = 14.61 \cdot 10^6$$

$$X_L[R_1 + R_2] + X_C R_2 = [5000][6000] - [1250][3800] = 25.25 \cdot 10^6$$

$$V_o = \frac{134.34 \cdot 10^6 \angle 135^0}{14.61 \cdot 10^6 + \text{j}25.25 \cdot 10^6} = \frac{134.34\text{ V} \angle 135^0}{29.17 \angle 59.94^0} = 4.605\text{ V} \angle 75.05^0$$

Problem 6.12

Solution:

Known quantities:
Figure P6.12.

Find:

a) The voltage transfer function in the form: $$H_v[j\omega] = \frac{V_o[j\omega]}{V_i[j\omega]} = \frac{H_{vo}}{1 \pm j\, f[\omega]}.$$

b) Plot the Bode diagram, i.e., a semilog plot where the magnitude [in dB] of the transfer function is plotted on a linear scale as a function of frequency on a log scale.

Assume:
The values of the resistors and of the capacitor in the circuit of Figure P6.12:

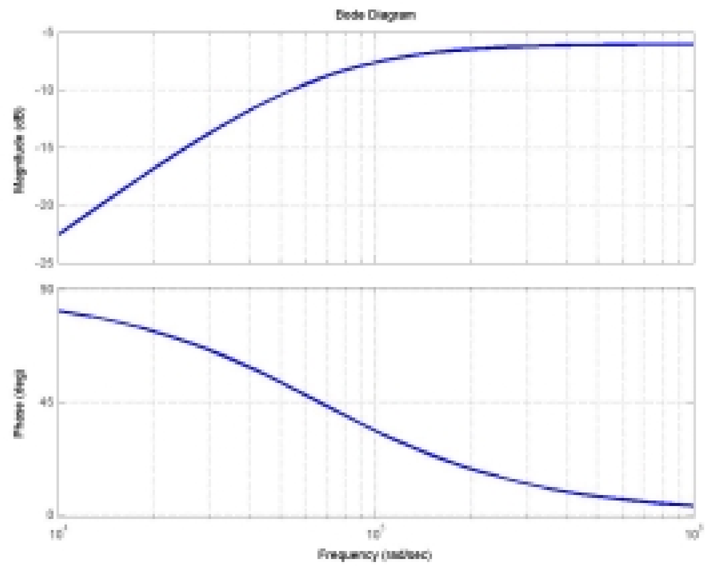
$$R_1 = 16\,\Omega \quad R_2 = 16\,\Omega \quad C = 0.47\,\mu\text{F}$$

Analysis:
a)

$$VD: \; V_o = V_i \frac{Z_{R2}}{Z_{R1} + Z_C + Z_{R2}} = V_i \frac{R_2}{R_1 + \frac{1}{j\omega C} + R_2}$$

$$H_v[j\omega] = \frac{V_o[j\omega]}{V_i[j\omega]} = \frac{R_2}{R_1 + R_2} \frac{1}{1 - j\frac{1}{\omega C\,[\,R_1 + R_2\,]}}$$

b)



Problem 6.13

Solution:

Known quantities:
The values of the resistors and of the capacitor in the circuit of Figure P6.13:

$$R_1 = 100\Omega \quad R_L = 100\Omega \quad R_2 = 50\Omega \quad C = 80\,\text{nF}$$

Find:

Compute and plot the frequency response function.

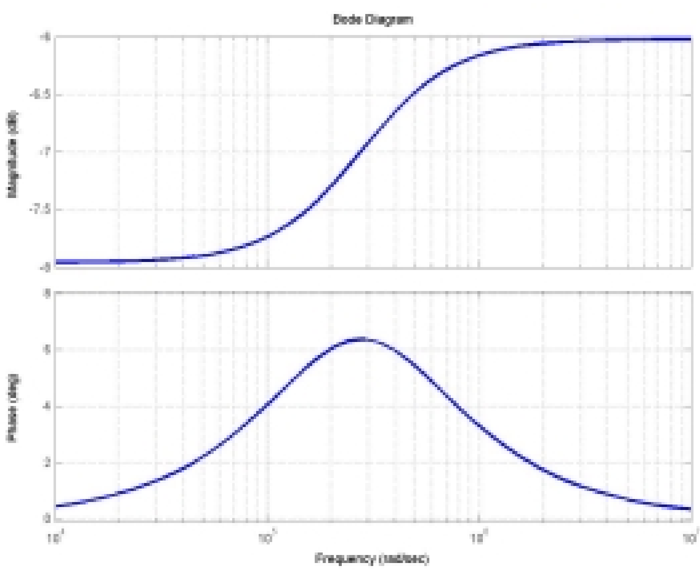
Analysis:
Using voltage division:

$$Z_{eq} = \frac{Z_{R2} Z_C}{Z_{R2} + Z_C} = \frac{R_2 \dfrac{1}{j\omega C}}{R_2 + \dfrac{1}{j\omega C}} \frac{j\omega C}{j\omega C} = \frac{R_2}{1 + j\,\omega R_2 C}$$

$$VD: \; H_v[j\omega] = \frac{V_o[j\omega]}{V_i[j\omega]} = \frac{Z_{RL}}{Z_{R1} + Z_{eq} + Z_{RL}} = \frac{R_L}{R_1 + \dfrac{R_2}{1 + j\omega R_2 C} + R_L} \frac{1 + j\omega R_2 C}{1 + j\omega R_2 C} =$$

$$= \frac{R_L [\, 1 + j\,\omega R_2 C \,]}{R_1 + R_2 + R_L + j\,[\; R_1 + R_L \,]\, \omega R_2 C} = \frac{R_L}{R_1 + R_2 + R_L} \frac{1 + j\,\omega R_2 C}{1 + j\,\dfrac{[\; R_1 + R_L \,]\, \omega R_2 C}{R_1 + R_2 + R_L}}$$

Plotting the response in a Bode Plot:



Section 6.2: Fourier Analysis

Problem 6.14

Solution:

Find:
Use trigonometric identities to show that the equalities in equations 6.16 and 6.17 hold.

Analysis:
Looking at figure 6.8, we can write the following equations:

$$a_n = c_n \sin(\theta_n)$$
$$b_n = c_n \cos(\theta_n)$$

and using the trigonometric identities $\sin^2(\theta_n) + \cos^2(\theta_n) = 1$:

$$a_n^2 + b_n^2 = c_n^2 \sin^2(\theta_n) + c_n^2 \cos^2(\theta_n) = c_n^2 \quad \Rightarrow \quad c_n = \sqrt{a_n^2 + b_n^2}$$

Finally,

$$\frac{b_n}{a_n} = \frac{c_n \cos(\theta_n)}{c_n \sin(\theta_n)} = \cot(\theta_n) = \tan(\psi_n)$$

where,

$$\psi_n = \frac{\pi}{2} - \theta_n .$$

Problem 6.15

Solution:

Known quantities:
The square wave of Figure 6.11(a) in the text.

Find:
A general expression for the Fourier series coefficients.

Assume:
None

Analysis:
The square wave is a function of time as follows:

$$x(t) = \begin{cases} A & (n - \frac{1}{4})T \le t \le (n + \frac{1}{4})T, \quad n = \pm 0, \pm 1, \pm 2, \ldots \\ 0 & (n + \frac{1}{4})T \le t \le (n + \frac{3}{4})T, \quad n = \pm 0, \pm 1, \pm 2, \ldots \end{cases}$$

We can compute the Fourier series coefficient using the integrals in equations (6.20), (6.21) and (6.22):

$$a_0 = \frac{1}{T} \int_0^T x(t)dt = \frac{1}{T} \int_{-T/4}^{T/4} A dt = \frac{A}{2}$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos\left(n\frac{2\pi}{T}t\right)dt = \frac{2}{T}\int_{-T/4}^{T/4} A\cos\left(n\frac{2\pi}{T}t\right)dt =$$

$$= \frac{2A}{T}\left[\sin\left(n\frac{2\pi}{T}t\right)\frac{T}{2n\pi}\right]_{-T/4}^{T/4} = \frac{A}{n\pi}\left[\sin\left(\frac{n\pi}{2}\right) - \sin\left(-\frac{n\pi}{2}\right)\right] = 0 \quad (\forall n)$$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin\left(n\frac{2\pi}{T}t\right)dt = \frac{2}{T}\int_{-T/4}^{T/4} A\sin\left(n\frac{2\pi}{T}t\right)dt =$$

$$= \frac{2A}{T}\left[-\cos\left(n\frac{2\pi}{T}t\right)\frac{T}{2n\pi}\right]_{-T/4}^{T/4} = \frac{A}{n\pi}\left[-\cos\left(\frac{n\pi}{2}\right) + \cos\left(-\frac{n\pi}{2}\right)\right] =$$

$$= \frac{A}{n\pi}\left[-2\cos\left(\frac{n\pi}{2}\right)\right] = \begin{cases} \dfrac{2A}{n\pi} & (n \text{ even}) \\ 0 & (n \text{ odd}) \end{cases}$$

Problem 6.16

Solution:

Known quantities:

The periodic function shown in Figure P6.16 and defined as:

$$x(t) = \begin{cases} A & 0 \le t \le \dfrac{T}{3} \\ 0 & \dfrac{T}{3} \le t \le T \end{cases}$$

Find:

A general expression for the Fourier series coefficients.

Analysis:

We can compute the Fourier series coefficient using the integrals in equations (6.20), (6.21) and (6.22):

$$a_0 = \frac{1}{T}\int_0^T x(t)dt = \frac{1}{T}\int_0^{T/3} Adt = \frac{A}{3}$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos\left(n\frac{2\pi}{T}t\right)dt = \frac{2}{T}\int_0^{T/3} A\cos\left(n\frac{2\pi}{T}t\right)dt =$$

$$= \frac{2A}{T}\left[\sin\left(n\frac{2\pi}{T}t\right)\frac{T}{2n\pi}\right]_0^{T/3} = \frac{A}{n\pi}\left[\sin\left(\frac{2}{3}n\pi\right) - 0\right] = \frac{A}{n\pi}\sin\left(\frac{2}{3}n\pi\right)$$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin\left(n\frac{2\pi}{T}t\right)dt = \frac{2}{T}\int_0^{T/3} A\sin\left(n\frac{2\pi}{T}t\right)dt =$$

$$= \frac{2A}{T}\left[-\cos\left(n\frac{2\pi}{T}t\right)\frac{T}{2n\pi}\right]_0^{T/3} = \frac{A}{n\pi}\left[1 - \cos\left(\frac{2}{3}n\pi\right)\right]$$

Thus, the Fourier series expansion of the function is:

$$x(t) = \frac{A}{3} + \sum_{n=1}^{\infty} \frac{A}{n\pi} \sin\left(\frac{2}{3}n\pi\right)\cos\left(n\frac{2\pi}{T}t\right) + \sum_{n=1}^{\infty} \frac{A}{n\pi}\left[1-\cos\left(\frac{2}{3}n\pi\right)\right]\sin\left(n\frac{2\pi}{T}t\right)$$

Problem 6.17

Solution:

Known quantities:
The periodic function shown in Figure P6.17 and defined as:

$$x(t) = \begin{cases} \cos\left(\frac{2\pi}{T}t\right) & -\frac{T}{4} \le t \le \frac{T}{4} \\ 0 & \text{else} \end{cases}$$

Find:
A general expression for the Fourier series coefficients.

Analysis:
The function in Figure P6.17 is an even function. Thus, we only need to compute the a_n coefficients.
We can compute the Fourier series coefficient using the integrals in equations (6.20) and (6.21):

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)dt = \frac{1}{T}\int_{-T/4}^{T/4}\cos\left(\frac{2\pi}{T}t\right)dt = \frac{1}{2\pi}\left[\sin\left(\frac{2\pi}{\mathrm{T}}t\right)\right]_{-T/4}^{T/4} =$$

$$= \frac{1}{2\pi}\left[\sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right)\right] = \frac{1}{\pi}$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos\left(n\frac{2\pi}{T}t\right)dt = \frac{2}{T}\int_{-T/4}^{T/4}\cos\left(\frac{2\pi}{T}t\right)\cos\left(n\frac{2\pi}{T}t\right)dt =$$

$$= -\frac{2}{\pi}\frac{\cos\left(\frac{n\pi}{2}\right)}{n^2-1} = \begin{cases} (-1)^{\frac{n}{2}-1}\frac{2}{\pi(n^2-1)} & (n \text{ even}) \\ 0 & (n \text{ odd}) \end{cases}$$

Problem 6.18

Solution:

Known quantities:
The periodic function shown in Figure P6.18 and defined as:

$$x(t) = \begin{cases} \frac{2A}{T}\mathrm{t} & 0 \le t \le \frac{T}{2} \\ A & \frac{T}{2} \le t \le T \end{cases}$$

Find:
Compute the Fourier series expansion.

Analysis:
We can compute the Fourier series coefficient using the integrals in equations (6.20), (6.21) and (6.22):

$$a_0 = \frac{1}{T}\int_0^T x(t)dt = \frac{1}{T}\left(\int_0^{T/2}\frac{2A}{T}t\,dt + \int_{T/2}^T A dt\right) = \frac{A}{T}\left(\left[\frac{t^2}{T}\right]_0^{T/2} + T - \frac{T}{2}\right) = \frac{3A}{4}$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos\left(n\frac{2\pi}{T}t\right)dt = \frac{2}{T}\left(\int_0^{T/2}\frac{2A}{T}t\cos\left(n\frac{2\pi}{T}t\right)dt + \int_{T/2}^T A\cos\left(n\frac{2\pi}{T}t\right)dt\right) =$$

$$= \frac{2}{T}\left(\left[\frac{TA}{2(n\pi)^2}\left(\cos\left(n\frac{2\pi}{T}t\right) + n\frac{2\pi}{T}t\sin\left(n\frac{2\pi}{T}t\right)\right)\right]_0^{T/2} + \left[\frac{TA}{2n\pi}\sin\left(n\frac{2\pi}{T}t\right)\right]_{T/2}^T\right) =$$

$$= \frac{A}{(n\pi)^2}\left[\cos(n\pi) - 1 + 2n\pi\sin(n\pi)\cos(n\pi)\right] = \frac{A}{(n\pi)^2}\left[\cos(n\pi) - 1\right]$$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin\left(n\frac{2\pi}{T}t\right)dt = \frac{2}{T}\left(\int_0^{T/2}\frac{2A}{T}t\sin\left(n\frac{2\pi}{T}t\right)dt + \int_{T/2}^T A\sin\left(n\frac{2\pi}{T}t\right)dt\right) =$$

$$= \frac{2}{T}\left(\left[\frac{TA}{2(n\pi)^2}\left(\sin\left(n\frac{2\pi}{T}t\right) - n\frac{2\pi}{T}t\cos\left(n\frac{2\pi}{T}t\right)\right)\right]_0^{T/2} + \left[-\frac{TA}{2n\pi}\cos\left(n\frac{2\pi}{T}t\right)\right]_{T/2}^T\right) =$$

$$= \frac{A}{(n\pi)^2}\left[\sin(n\pi) - 2n\pi\cos^2(n\pi) + n\pi\right] = \frac{A}{n\pi}\left[1 - 2\cos^2(n\pi)\right]$$

Thus, the Fourier series expansion of the function is:

$$x(t) = \frac{3A}{4} + \sum_{n=1}^{\infty}\frac{A}{(n\pi)^2}\left[\cos(n\pi) - 1\right]\cos\left(n\frac{2\pi}{T}t\right) + \sum_{n=1}^{\infty}\frac{A}{n\pi}\left[1 - 2\cos^2(n\pi)\right]\sin\left(n\frac{2\pi}{T}t\right)$$

Problem 6.19

Solution:

Known quantities:

The periodic function shown in Figure P6.19 and defined as:

$$x(t) = \begin{cases} \sin\left(\frac{2\pi}{T}t\right) & 0 \le t \le \frac{T}{2} \\ 0 & \frac{T}{2} \le t \le T \end{cases}$$

Find:

Compute the Fourier series expansion.

Analysis:

The function in Figure P6.19 is an even function. Thus, we only need to compute the a_n coefficients.

We can compute the Fourier series coefficient using the integrals in equations (6.20) and (6.21):

$$a_0 = \frac{1}{T}\int_0^T x(t)dt = \frac{1}{T}\int_0^{T/2} \sin\left(\frac{2\pi}{T}t\right)dt = \frac{1}{2\pi}\left[-\cos\left(\frac{2\pi}{\mathrm{T}}t\right)\right]_0^{T/2} =$$

$$= \frac{1}{2\pi}\left[-\cos(\pi) + \cos(0)\right] = \frac{1}{\pi}$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos\left(n\frac{2\pi}{T}t\right)dt = \frac{2}{T}\int_0^T \sin\left(\frac{2\pi}{T}t\right)\cos\left(n\frac{2\pi}{T}t\right)dt =$$

$$= -\frac{\cos(n\pi)+1}{\pi(n^2-1)} = \begin{cases} -\dfrac{2}{\pi(n^2-1)} & (n \text{ even}) \\ 0 & (n \text{ odd}) \end{cases}$$

Thus, the Fourier series expansion of the function is:

$$x(t) = \frac{1}{\pi} - \sum_{n=1}^{\infty} \frac{\cos(n\pi)+1}{\pi(n^2-1)} \cos\left(n\frac{2\pi}{T}t\right)$$

Problem 6.20

Solution:

Known quantities:
The periodic function shown in Figure P6.20.

Find:

A complete expression for the function $x(t)$ and the Fourier coefficients.

Analysis:
The periodic function shown in Figure P6.20 can be defined as:

$$x(t) = \begin{cases} \dfrac{V}{t_1}t & 0 \le t \le t_1 \\ V & t_1 \le t \le T - t_1 \\ \dfrac{V}{t_1}(T\text{-}t) & T - t_1 \le t \le T + t_1 \\ -V & T + t_1 \le t \le 2T - t_1 \\ \dfrac{V}{t_1}(t - 2T) & 2T - t_1 \le t \le 2T \end{cases}$$

The function in Figure P6.19 is an odd function with period equal to $2T$. Thus, we only need to compute the b_n coefficients.

We can compute the Fourier series coefficient using the integrals in equation (6.22):

$$b_n = \frac{2}{2T}\int_0^{2T} x(t)\sin\left(n\frac{2\pi}{T}t\right)dt = \frac{1}{T}\left(\int_0^{t_1}\frac{V}{t_1}t\sin\left(n\frac{2\pi}{T}t\right)dt + \int_{t_1}^{T-t_1} V\sin\left(n\frac{2\pi}{T}t\right)dt + \right.$$

$$\left. + \int_{T-t_1}^{T+t_1}\frac{V}{t_1}(T-t)\sin\left(n\frac{2\pi}{T}t\right)dt - \int_{T+t_1}^{2T-t_1} V\sin\left(n\frac{2\pi}{T}t\right)dt + \int_{2T-t_1}^{2T}\frac{V}{t_1}(t-2T)\sin\left(n\frac{2\pi}{T}t\right)dt\right)$$

$$= \frac{V}{2(n\pi)^2 t_1}\left[4T\sin\left(n\frac{\pi}{T}t_1\right)\cos\left(n\frac{\pi}{T}t_1\right) - 2\pi n t_1\cos\left(n\frac{\pi}{T}t_1\right) + \pi n t_1\left(1+\cos\left(n\frac{2\pi}{T}t_1\right)\right)\right.$$

$$\left. -12T\cos^2(n\pi)\sin\left(n\frac{\pi}{T}t_1\right)\cos\left(n\frac{\pi}{T}t_1\right) + 8T\cos^4(n\pi)\sin\left(n\frac{\pi}{T}t_1\right)\cos\left(n\frac{\pi}{T}t_1\right)\right]$$

Problem 6.21

Solution:

Known quantities:
The periodic function shown in Figure P6.21.

Find:
A complete expression for the function $x(t)$ and the Fourier coefficients.

Analysis:
The periodic function shown in Figure P6.21 can be defined as:

$$x(t) = \begin{cases} A & 0 \le t \le \frac{T}{4} \\ -A & T - \frac{T}{4} \le t \le T \end{cases}$$

The function in Figure P6.19 is an odd function. Thus, we only need to compute the b_n coefficients.
We can compute the Fourier series coefficient using the integrals in equation (6.22):

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin\left(n\frac{2\pi}{T}t\right)dt = \frac{2}{T}\left(-\int_{-T/4}^{0} A\sin\left(n\frac{2\pi}{T}t\right)dt + \int_0^{T/4} A\sin\left(n\frac{2\pi}{T}t\right)dt\right) =$$

$$= \frac{2A}{n\pi}\left(1-\cos\left(n\frac{\pi}{2}\right)\right) = \frac{2A}{n\pi}$$

Problem 6.22

Solution:

Known quantities:
The periodic function defined as:

$$x(t) = 10\cos(10t + \pi/6)$$

Find:

All Fourier series coefficients.

Analysis:

Using trigonometric identities we can expand the function $x(t)$ in the following way:

$$x(t) = 10\cos(10t + \pi/6) = 10[\cos(10t)\cos(\pi/6) - \sin(10t)\sin(\pi/6)] =$$

$$= 5\sqrt{3}\cos(10t) - 5\sin(10t)$$

Now the function is already in Fourier series form, since it contains only sinusoidal terms! We recognize the following parameters:

$$\omega_0 = 10$$

$$a_0 = 0$$

$$a_1 = 5\sqrt{3}$$

$$b_1 = 5$$

and all other coefficients are equal to zero.

Section 6.3: Filters

Problem 6.23

Solution:

Known quantities:
The resistance of the RC high-pass filter.

Find:

Design an RC high-pass filter with a breakpoint at 200 kHz.

Analysis:
The frequency response of the RC high-pass filter is:

$$\frac{V_o(j\omega)}{V_i(j\omega)} = \frac{j\omega CR}{1 + j\omega CR}$$

The cutoff frequency is:

$$\omega_0 = \frac{1}{RC}$$

Thus,

$$\omega_0 = \frac{1}{RC} = 2\pi \times 200000 \quad \Rightarrow \quad C = \frac{1}{R\omega_0} = \frac{1}{2\pi \cdot 15000 \cdot 200000} \cong 53\,\text{pF}$$

Problem 6.24

Solution:

Known quantities:
The resistance of the RC low-pass filter.

Find:

Design an RC low-pass filter that would attenuate a 120-Hz sinusoidal voltage by 20 dB with respect to the DC gain.

Analysis:
The frequency response of the RC low-pass filter is:

$$H_v(j\omega) = \frac{V_o(j\omega)}{V_i(j\omega)} = \frac{1}{1 + j\omega CR}$$

The response of the circuit to the periodic input $u(t) = A\sin(\hat{\omega}t + \varphi)$ is:

$$y_\infty(t) = \left|H_v(j\hat{\omega})\right| A\sin\left(\hat{\omega}t + \varphi + \angle H_v(j\hat{\omega})\right)$$

In order to attenuate the sinusoidal input by 20 dB (a factor of 10) with respect to the DC gain,

$$\left|H_v(j\hat{\omega})\right| = \frac{1}{\sqrt{1 + (\hat{\omega}CR)^2}} = 0.1 \quad \Rightarrow \quad C = \frac{1}{R\hat{\omega}}\sqrt{10^2 - 1} = \frac{\sqrt{99}}{500 \cdot 2\pi \times 120} \cong 26.4\,\mu\text{F}$$

Problem 6.25

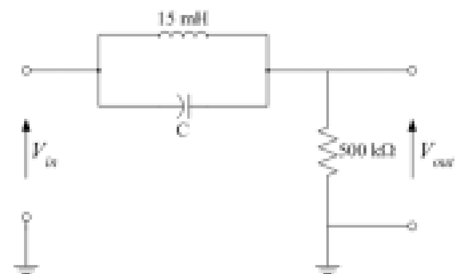
Solution:

Known quantities:
The resistance and the inductance of the parallel LC resonant circuit.

Find:

Design a parallel LC resonant circuit to resonate at 500-kHz.

Analysis:
The frequency response of the parallel LC resonant circuit is:



$$H_v(j\omega) = \frac{V_o(j\omega)}{V_i(j\omega)} = \frac{1+(j\omega)^2 LC}{1 + j\omega\frac{L}{R} + (j\omega)^2 LC}$$

The resonant frequency of the circuit is:

$$\omega_n = \sqrt{\frac{1}{LC}} = 2\pi \times 500\,\text{kHz}$$

Thus,

$$C = \frac{1}{L(\pi^2 \times 10^{12})} \cong 1\,\text{pF}$$

The damping ratio is,

$$\xi = \frac{1/RC}{2\omega_n} = \frac{\pi^2 \times 10^{11}}{500000 \cdot 2(2\pi \times 500000)} \cong 0.3142$$

The quality factor is,

$$Q = \frac{1}{2\xi} \cong 1.5915$$

Problem 6.26

Solution:

Find:

In an RLC circuit, show that $Q = \dfrac{\omega_n}{\Delta\omega}$.

Analysis:
The frequency response of an RLC circuit is:

$$H_v(j\omega) = \frac{V_o(j\omega)}{V_i(j\omega)} = \frac{\left(2\xi/\omega_n\right) j\omega}{1 + j\omega\left(2\xi/\omega_n\right) + \left(j\omega/\omega_n\right)^2}$$

We can compute the half-power frequencies ω_1 and ω_2 by equating the magnitude of the band-pass filter frequency response to $1/\sqrt{2}$ (this will result in a quadratic equation in ω, which can be solved for the two frequencies). Defining $\Omega = \dfrac{\omega}{\omega_n}$, we can write the following equation:

$$\left|H_v(j\omega)\right| = \left|\frac{2\xi(j\Omega)}{1+2\xi(j\Omega)+(j\Omega)^2}\right| = \frac{2\xi\Omega}{\sqrt{(1-\Omega^2)^2+(2\xi\Omega)^2}} = \frac{1}{\sqrt{2}} \quad \Rightarrow$$

$$\Rightarrow \quad \Omega^4 - 2(1+2\xi^2)\Omega^2 + 1 = 0 \quad \Rightarrow \quad \Omega^2 = 1 + 2\xi^2 \pm 2\xi\sqrt{1+\xi^2}$$

Finally, discarding the negative solutions:

$$\Omega_{1,2} = \pm\xi + \sqrt{1+\xi^2} \quad \Rightarrow \quad \omega_{1,2} = \left(\pm\xi + \sqrt{1+\xi^2}\right)\omega_n$$

Thus,

$$\Delta\omega = \omega_2 - \omega_1 = \left(\xi + \sqrt{1+\xi^2}\right)\omega_n - \left(-\xi + \sqrt{1+\xi^2}\right)\omega_n = 2\xi\omega_n$$

and,

$$\frac{\omega_n}{\Delta\omega} = \frac{\omega_n}{2\xi\omega_n} = \frac{1}{2\xi} = Q$$

Problem 6.27

Solution:

Known quantities:

The resistance, inductance and capacitance of a series RLC resonant circuit.

Find:

a) Show that the impedance at the resonance frequency becomes a value of Q times the resistance at the resonance frequency. **NOTE:** The word inductive should not be in the problem statement.

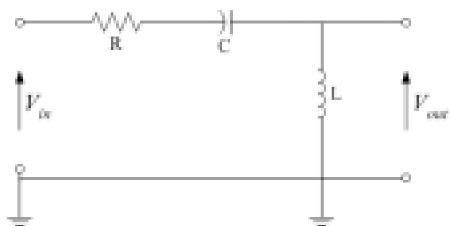
b) Determine the impedance at the resonance frequency, assuming $L = 280 \text{ mH}$, $C = 0.1\ \mu\text{F}$ and $R = 25\ \Omega$.

Assumptions:

The circuit is as shown in the figure below with the output impedance across the inductor of the RLC circuit.

Also, the output impedance is the impedance of interest.

Analysis:



a) The output impedance of the circuit is:

$$Z_{out}(j\omega) = \left(R + \frac{1}{j\omega C}\right) \| \ j\omega L =$$

$$= \frac{1 + j\omega CR}{j\omega C} \| \ j\omega L = \frac{(1 + j\omega CR)j\omega L}{1 + j\omega CR + (j\omega)^2 LC}$$

and the quality factor is:

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{\omega_n L}{R} = \frac{1}{\omega_n RC}$$

Thus, for $\omega \to \omega_n$:

$$Z_{out}(j\omega_n) = \frac{(1+j\omega_n CR)j\omega_n L}{1+j\omega_n CR+(j\omega_n)^2 LC} = \frac{j\omega_n L - \omega_n^2 LRC}{1-\omega_n^2 LC + j\omega_n CR} =$$

$$= R\frac{j\omega_n L/R - (\omega_n L/R)(\omega_n RC)}{1-(\omega_n L/R)(\omega_n RC)+j\omega_n CR} = R\frac{jQ - Q/Q}{1-Q/Q+j/Q} =$$

$$= R\frac{jQ-1}{j/Q} = R\left(Q^2 + jQ\right) = RQ(Q+j)$$

For a high quality factor circuit, we have

$$\left|Z_{out}(j\omega_n)\right| = RQ\sqrt{1+Q^2} \cong RQ^2 = R\left(\omega_n \frac{L}{R}\right)Q = \omega_n LQ$$

Finally, the impedance at the resonance frequency becomes a value of Q times the inductive resistance at the resonance frequency.

b) The quality factor is:

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{25}\sqrt{2.8\times 10^6} \cong 67$$

The impedance at the resonance frequency is:

$$\left|Z_{out}(j\omega_n)\right| = RQ\sqrt{1+Q^2} = 25\cdot 67\sqrt{1+67^2} = 112.01\,\mathrm{k\Omega}$$

while, Q times the inductive resistance at the resonance frequency is:

$$\omega_n LQ = \frac{1}{\sqrt{2.8\times 10^{-8}}}0.28\cdot 67 = 112\,\mathrm{k\Omega}$$

Problem 6.28

Solution:

Known quantities:

Frequency response $H_v(j\omega)$ of the circuit of Example 6.7.

Find:

The frequency at which the phase shift introduced by the circuit is equal to $-10°$.

Analysis:

The frequency response of the circuit is:

$$H_v(j\omega) = \frac{1}{1+j\omega CR}$$

From Example 6.7:

$$\omega_0 = \frac{1}{CR} = 2{,}128\,rad/\sec$$

The phase shift introduced by the circuit is:

$$\angle H_v(j\omega) = -\arctan^{-1}\left(\frac{\omega}{\omega_0}\right)$$

$$\angle H_v(j\omega) = -\arctan^{-1}\left(\frac{\omega}{2128}\right)$$

Thus,

$$\angle H_v(j\overline{\omega}) = -\arctan\left(\frac{\overline{\omega}}{2128}\right) = -10 \quad \Rightarrow \quad \overline{\omega} = 2128\tan(10) = 375.2 \text{ rad/s}$$

Problem 6.29

Solution:

Known quantities:

Frequency response $H_v(j\omega)$ of the circuit of Example 6.7.

Find:

The frequency at which the output of the circuit is attenuated by 10 percent.

Analysis:

The frequency response of the circuit is:

$$H_v(j\omega) = \frac{1}{1 + j\omega CR}$$

From Example 6.7:

$$\omega_0 = \frac{1}{CR} = 2{,}128 rad/\sec$$

The attenuation introduced by the circuit is:

$$|H_v(j\omega)| = \frac{1}{\sqrt{1 + (\omega/\omega_0)^2}}$$

Thus,

$$|H_v(j\overline{\omega})| = \frac{1}{\sqrt{1 + (\overline{\omega}/2128)^2}} = 0.9 \quad \Rightarrow \quad \overline{\omega} = (2128)\sqrt{\left(\frac{1}{0.9}\right)^2 - 1^2} = 1031 \text{ rad/s}$$

Problem 6.30

Solution:

Known quantities:

Frequency response $H_v(j\omega)$ of the circuit of Example 6.11.

Find:

The frequency at which the output of the circuit is attenuated by 10 percent.

Analysis:

The frequency response of the circuit is:

$$H_v(j\omega) = \frac{2}{1 + 0.2j\omega}$$

The attenuation introduced by the circuit is:

$$|H_v(j\omega)| = \frac{2}{\sqrt{1 + (0.2\omega)^2}}$$

Thus,

$$\left|H_v\left(j\overline{\omega}\right)\right| = \frac{2}{\sqrt{1+\left(0.2\overline{\omega}\right)^2}} = 0.9 \quad \Rightarrow \quad \overline{\omega} = \frac{1}{0.2}\sqrt{\left(\frac{2}{0.9}\right)^2 - 1} = 9.9225 \text{ rad/s}$$

Problem 6.31

Solution:

Known quantities:

Frequency response $H_v\left(j\omega\right)$ of the circuit of Example 6.11.

Find:

The frequency at which the phase shift introduced by the circuit is equal to $20°$.

Analysis:

The frequency response of the circuit is:

$$H_v\left(j\omega\right) = \frac{j\omega CR}{1 + j\omega CR + \left(j\omega\right)^2 LC}$$

The phase shift introduced by the circuit is:

$$\angle H_v\left(j\omega\right) = \frac{\pi}{2} - \arctan\left(\frac{\omega CR}{1-\omega^2 LC}\right)$$

Thus,

$$\angle H_v\left(j\overline{\omega}\right) = \frac{\pi}{2} - \arctan\left(\frac{\overline{\omega} CR}{1-\overline{\omega}^2 LC}\right) = 20° \Rightarrow \frac{\overline{\omega} CR}{1-\overline{\omega}^2 LC} = \tan 25°$$

a) $R = 1k\Omega, C = 10\mu F, L = 5mH$:

$\overline{\omega} = 46.6 rad/\sec$

b) $R = 10k\Omega, C = 10\mu F, L = 5mH$:

$\overline{\omega} = 4.66 rad/\sec$

Problem 6.32

Solution:

Known quantities:

The frequency response $H_v\left(j\omega\right)$ of the circuit of P6.1, the period $T = 10\ \mu s$ and the peak amplitude $A = 1$ for the sawtooth waveform.

Find:

Output of system $y(t)$ in response to input $x(t)$.

Analysis:

According to the Fourier series definitions of the previous section, and using the first two terms of the Fourier series expansion of the sawtooth waveform of Example 6.3, we have

$$x(t) = \frac{2A}{\pi}\sin\left(\frac{2\pi}{T}t\right) + \frac{A}{\pi}\sin\left(\frac{4\pi}{T}t\right) = \frac{2}{\pi}\sin\left(2\times 10^5 \pi\, t\right) + \frac{1}{\pi}\sin\left(4\times 10^5 \pi\, t\right)$$

Thus, for this problem,

$$c_1 = \frac{2}{\pi} \qquad \omega_1 = 2\times10^5\pi = 6.28\times10^5 \text{ rad/s}$$

and,

$$c_2 = \frac{1}{\pi} \qquad \omega_2 = 4\times10^5\pi = 12.56\times10^5 \text{ rad/s}$$

The frequency response of the system can be expressed in magnitude and phase form:

$$H_v(j\omega) = \frac{1}{1 + j\left(2.5\times10^{-6}\omega\right)} = \frac{1}{\sqrt{1 + 6.25\times10^{-12}\omega^2}}\angle -\arctan\left(2.5\times10^{-6}\omega\right)$$

At this point, we could evaluate the frequency response of the system at the frequencies ω_1 and ω_2 analytically:

$$\left|H_v(j\omega_1)\right| = \frac{1}{\sqrt{1 + 6.25\times10^{-12}\omega_1^{\ 2}}} = 0.537$$

$$\Phi(j\omega_1) = -\arctan\left(2.5\times10^{-6}\omega_1\right) = -1.0039 \text{ rad} = -57.52^\circ$$

$$\left|H_v(j\omega_2)\right| = \frac{1}{\sqrt{1 + 6.25\times10^{-12}\omega_2^{\ 2}}} = 0.3033$$

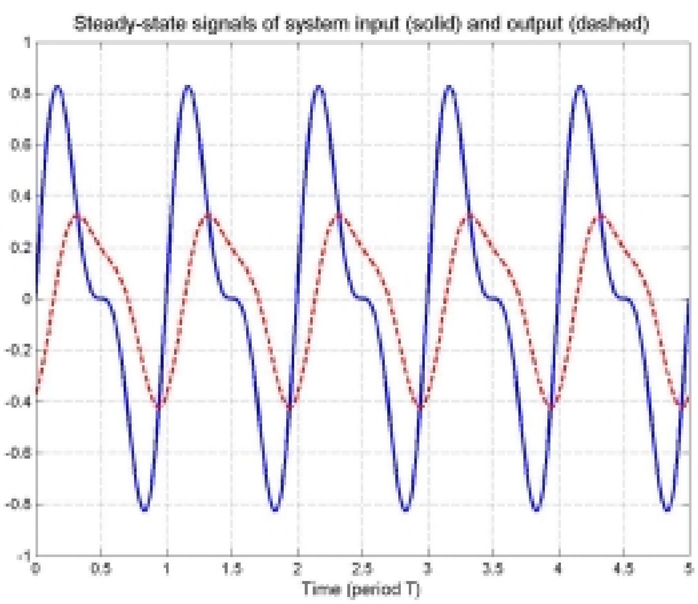
$$\Phi(j\omega_2) = -\arctan\left(2.5\times10^{-6}\omega_2\right) = -1.2626 \text{ rad} = -72.34^\circ$$

Finally, we can compute the steady-state periodic output of the system:

$$y(t) = \sum_{n=1}^{2}\left|H_v(j\omega_n)\right|c_n \sin\left[\omega_n t + \theta_n + \Phi(j\omega_n)\right] =$$

$$= 0.537\frac{2}{\pi}\sin\left(2\times10^5\pi\, t - 1.0039\right) + 0.3033\frac{1}{\pi}\sin\left(4\times10^5\pi\, t - 1.2626\right)$$

The input and output steady state signals plot is shown below:



Problem 6.33

Solution:

Known quantities:

The frequency response $H_v(j\omega)$ of the circuit of P6.1, the period $T = 10\ \mu s$ and the peak amplitude $A = 1$ for the square wave of Figure 6.11(a).

Find:

Output of system $y(t)$ in response to input $x(t)$.

Analysis:

The square wave can be defined as:

$$x(t) = \begin{cases} A & (n-\frac{1}{4})T \le t \le (n+\frac{1}{4})T, \quad n = \pm 0, \pm 1, \pm 2, \ldots \\ 0 & (n+\frac{1}{4})T \le t \le (n+\frac{3}{4})T, \quad n = \pm 0, \pm 1, \pm 2, \ldots \end{cases}$$

We can compute the Fourier series coefficient using the integrals in equations (6.20), (6.21) and (6.22):

$$a_0 = \frac{1}{T}\int_0^T x(t)dt = \frac{1}{T}\int_{-T/4}^{T/4} A dt = \frac{A}{2}$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos\left(n\frac{2\pi}{T}t\right)dt = \frac{2}{T}\int_{-T/4}^{T/4} A\cos\left(n\frac{2\pi}{T}t\right)dt =$$

$$= \frac{2A}{T}\left[\sin\left(n\frac{2\pi}{T}t\right)\frac{T}{2n\pi}\right]_{-T/4}^{T/4} = \frac{A}{n\pi}\left[\sin\left(\frac{n\pi}{2}\right) - \sin\left(-\frac{n\pi}{2}\right)\right] = 0 \quad (\forall n)$$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin\left(n\frac{2\pi}{T}t\right)dt = \frac{2}{T}\int_{-T/4}^{T/4} A\sin\left(n\frac{2\pi}{T}t\right)dt =$$

$$= \frac{2A}{T}\left[-\cos\left(n\frac{2\pi}{T}t\right)\frac{T}{2n\pi}\right]_{-T/4}^{T/4} = \frac{A}{n\pi}\left[-\cos\left(\frac{n\pi}{2}\right) + \cos\left(-\frac{n\pi}{2}\right)\right] =$$

$$= \frac{A}{n\pi}\left[-2\cos\left(\frac{n\pi}{2}\right)\right] = \begin{cases} \dfrac{2A}{n\pi} & (n \text{ even}) \\ 0 & (n \text{ odd}) \end{cases}$$

Using the first two terms of the Fourier series expansion of the square waveform, we have

$$x(t) = \frac{A}{2} + \frac{A}{\pi}\sin\left(\frac{4\pi}{T}t\right) = \frac{1}{2} + \frac{1}{\pi}\sin\left(4\times10^5\pi\, t\right)$$

Thus, for this problem,

$$c_0 = \frac{1}{2} \qquad \omega_0 = 0 \text{ rad/s}$$

$$c_1 = 0 \qquad \omega_1 = 2\times10^5\pi = 6.28\times10^5 \text{ rad/s}$$

and,

$$c_2 = \frac{1}{\pi} \qquad \omega_2 = 4\times10^5\pi = 12.56\times10^5 \text{ rad/s}$$

The frequency response of the system can be expressed in magnitude and phase form:

$$H_v(j\omega) = \frac{1}{1 + j\left(2.5\times10^{-6}\omega\right)} = \frac{1}{\sqrt{1 + 6.25\times10^{-12}\omega^2}} \angle -\arctan\left(2.5\times10^{-6}\omega\right)$$

At this point, we could evaluate the frequency response of the system at the frequencies ω_0 and ω_2 analytically:

$$|H_v(j\omega_0)| = 1$$

$$|H_v(j\omega_2)| = \frac{1}{\sqrt{1 + 6.25\times10^{-12}\omega_2^{\ 2}}} = 0.3033$$

$$\Phi(j\omega_0) = -\arctan\left(2.5\times10^{-6}\omega_0\right) = 0 \text{ rad} = 0^\circ$$

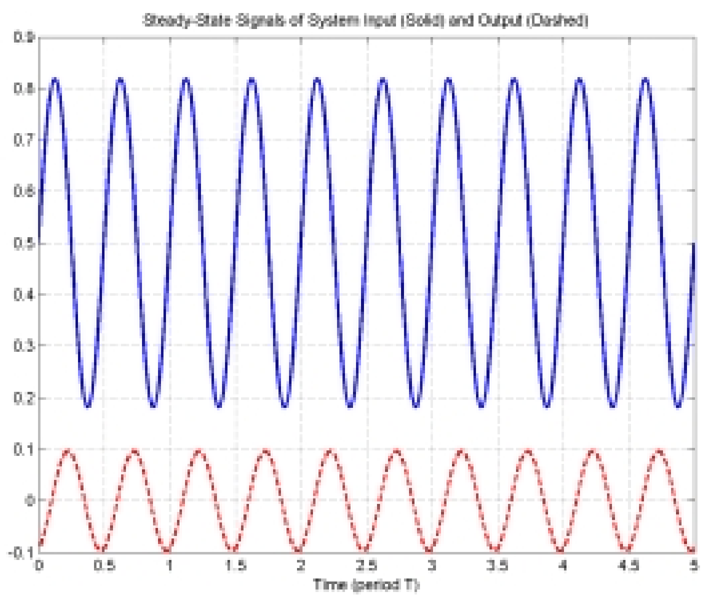
$$\Phi(j\omega_2) = -\arctan\left(2.5\times10^{-6}\omega_2\right) = -1.2626 \text{ rad} = -72.34^\circ$$

Finally, we can compute the steady-state periodic output of the system:

$$y(t) = \sum_{n=1}^{2} |H_v(j\omega_n)| c_n \sin[\omega_n t + \theta_n + \Phi(j\omega_n)] =$$

$$= \frac{1}{2}\sin(0) + \frac{0.3033}{\pi}\sin\left(4\times10^5\pi\, t - 1.2626\right) = \frac{0.3033}{\pi}\sin\left(4\times10^5\pi\, t - 1.2626\right)$$

The input and output steady state signals plot is shown below:



Problem 6.34

Solution:

Known quantities:

The frequency response $H_v(j\omega)$ of the circuit of P6.1, the period $T = 10\ \mu s$ and the peak amplitude $A = 1$ for the pulse waveform.

Find:

Output of system $y(t)$ in response to input $x(t)$.

Analysis:

According to the Fourier series definitions of the previous section, and using the first two terms of the Fourier series expansion of the pulse waveform of Example 6.4, we have

$$x(t) = 0.2 + 0.3027\cos\left(2\times10^5\pi t\right) + 0.2199\sin\left(2\times10^5\pi t\right)$$
$$+ 0.0935\cos\left(4\times10^5\pi t\right) + 0.2879\sin\left(4\times10^5\pi t\right)$$

Thus, for this problem,

$$c_0 = 0.2$$
$$c_1 = 0.3742, \quad \theta_1 = 0.9425 \text{ rad} = 54^\circ \qquad \omega_1 = 2\times10^5\pi = 6.28\times10^5 \text{ rad/s}$$

and,

$$c_2 = 0.3027, \quad \theta_2 = 0.3140 \text{ rad} = 18^\circ \qquad \omega_2 = 4\times10^5\pi = 12.56\times10^5 \text{ rad/s}$$

The frequency response of the system can be expressed in magnitude and phase form:

$$H_v(j\omega) = \frac{1}{1 + j\left(2.5\times10^{-6}\omega\right)} = \frac{1}{\sqrt{1 + 6.25\times10^{-12}\omega^2}}\angle -\arctan\left(2.5\times10^{-6}\omega\right)$$

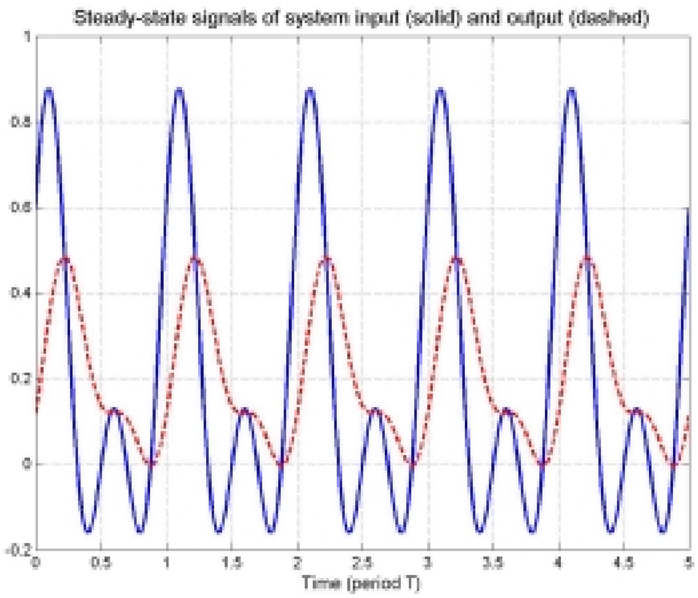
At this point, we could evaluate the frequency response of the system at the frequencies ω_1 and ω_2 analytically:

$$\left|H_v(j\omega_1)\right| = \frac{1}{\sqrt{1 + 6.25\times10^{-12}\omega_1^{\,2}}} = 0.537$$
$$\Phi(j\omega_1) = -\arctan\left(2.5\times10^{-6}\omega_1\right) = -1.0039 \text{ rad} = -57.52^\circ$$
$$\left|H_v(j\omega_2)\right| = \frac{1}{\sqrt{1 + 6.25\times10^{-12}\omega_2^{\,2}}} = 0.3033$$
$$\Phi(j\omega_2) = -\arctan\left(2.5\times10^{-6}\omega_2\right) = -1.2626 \text{ rad} = -72.34^\circ$$

Finally, we can compute the steady-state periodic output of the system:

$$y(t) = c_0\left|H_v(0)\right| + \sum_{n=1}^{2}\left|H_v(j\omega_n)\right|c_n \sin\left[\omega_n t + \theta_n + \Phi(j\omega_n)\right] =$$
$$= 0.2 + 0.2009\sin\left(2\times10^5\pi t - 0.0614\right) + 0.0918\sin\left(4\times10^5\pi t - 0.9484\right)$$

The input and output steady state signals plot is shown below:



Problem 6.35

Solution:

Known quantities:

The frequency response $H_v(j\omega)$ of the circuit of P6.2, the period $T = 0.5\ s$ and the peak amplitude $A = 2$ for the sawtooth waveform.

Find:

Output of system $y(t)$ in response to input $x(t)$.

Analysis:

According to the Fourier series definitions of the previous section, and using the first three terms of the Fourier series expansion of the sawtooth waveform of Example 6.3, we have

$$x(t) = \frac{2A}{\pi}\sin\left(\frac{2\pi}{T}t\right) + \frac{A}{\pi}\sin\left(\frac{4\pi}{T}t\right) + \frac{2A}{3\pi}\sin\left(\frac{6\pi}{T}t\right) =$$

$$= \frac{4}{\pi}\sin(4\pi\, t) + \frac{2}{\pi}\sin(8\pi\, t) + \frac{4}{3\pi}\sin(12\pi\, t)$$

Thus, for this problem,

$$c_1 = \frac{4}{\pi} \qquad \omega_1 = 4\pi = 12.5664 \text{ rad/s}$$

$$c_2 = \frac{2}{\pi} \qquad \omega_2 = 8\pi = 25.1327 \text{ rad/s}$$

$$c_3 = \frac{4}{3\pi} \qquad \omega_3 = 12\pi = 37.6991 \text{ rad/s}$$

The frequency response of the system can be expressed in magnitude and phase form:

$$H_v(j\omega) = \frac{1}{2(1 + 0.05\, j\omega)} = \frac{1}{\sqrt{4 + 0.01\omega^2}} \angle -\arctan(0.05\omega)$$

At this point, we could evaluate the frequency response of the system at the frequencies ω_1, ω_2 and ω_3 analytically:

$$|H_v(j\omega_1)| = \frac{1}{\sqrt{4+0.01\omega_1^2}} = 0.4234$$

$$\Phi(j\omega_1) = -\arctan(0.05\omega_1) = -0.5610 \text{ rad} = -32.14^\circ$$

$$|H_v(j\omega_2)| = \frac{1}{\sqrt{4+0.01\omega_2^2}} = 0.3113$$

$$\Phi(j\omega_2) = -\arctan(0.05\omega_2) = -0.8986 \text{ rad} = -51.49^\circ$$

$$|H_v(j\omega_3)| = \frac{1}{\sqrt{4+0.01\omega_3^2}} = 0.2343$$

$$\Phi(j\omega_3) = -\arctan(0.05\omega_3) = -1.0830 \text{ rad} = -62.05^\circ$$

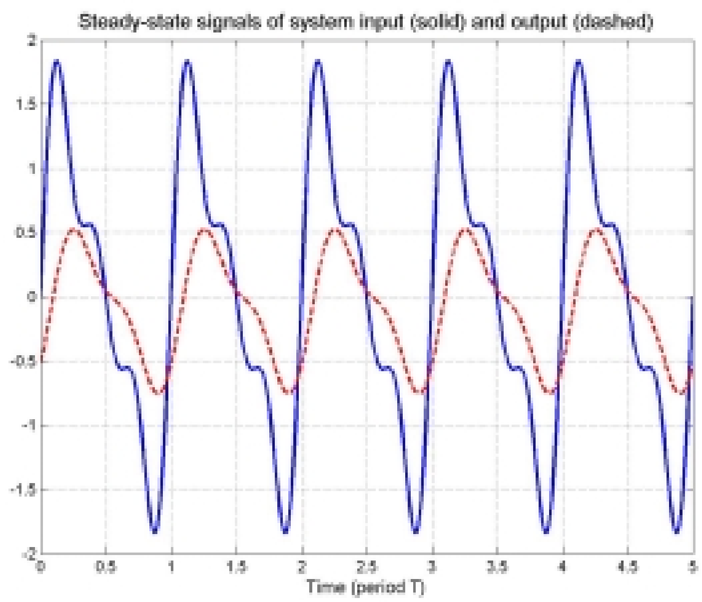
Finally, we can compute the steady-state periodic output of the system:

$$y(t) = \sum_{n=1}^{3} |H_v(j\omega_n)| c_n \sin[\omega_n t + \theta_n + \Phi(j\omega_n)] =$$

$$= 0.4234 \frac{4}{\pi} \sin(4\pi t - 0.561) + 0.3113 \frac{2}{\pi} \sin(8\pi t - 0.8986)$$

$$+ 0.2343 \frac{4}{3\pi} \sin(12\pi t - 1.083)$$

The input and output steady state signals plot is shown below:



Problem 6.36

Solution:

Known quantities:

The frequency response $H_v(j\omega)$ of the circuit of P6.2, the period $T = 0.5\ s$ and the peak amplitude $A = 2$ for the square wave.

Find:

Output of system $y(t)$ in response to input $x(t)$.

Analysis:

According to the Fourier series definitions of the previous section, and using the first three terms of the Fourier series expansion of the square waveform of P6.33, we have

$$x(t) = \frac{4A}{\pi}\sin\left(\frac{2\pi}{T}t\right) + \frac{4A}{3\pi}\sin\left(\frac{6\pi}{T}t\right) = \frac{8}{\pi}\sin(4\pi\, t) + \frac{8}{3\pi}\sin(12\pi\, t)$$

Thus, for this problem,

$$c_1 = \frac{8}{\pi} \qquad \omega_1 = 4\pi = 12.5664 \text{ rad/s}$$

$$c_2 = 0 \qquad \omega_2 = 8\pi = 25.1327 \text{ rad/s}$$

$$c_3 = \frac{8}{3\pi} \qquad \omega_3 = 12\pi = 37.6991 \text{ rad/s}$$

The frequency response of the system can be expressed in magnitude and phase form:

$$H_v(j\omega) = \frac{1}{2(1 + 0.05\, j\omega)} = \frac{1}{\sqrt{4 + 0.01\omega^2}}\angle - \arctan(0.05\omega)$$

At this point, we could evaluate the frequency response of the system at the frequencies ω_1 and ω_3 analytically:

$$|H_v(j\omega_1)| = \frac{1}{\sqrt{4 + 0.01\omega_1^2}} = 0.4234$$

$$\Phi(j\omega_1) = -\arctan(0.05\omega_1) = -0.5610 \text{ rad} = -32.14^\circ$$

$$|H_v(j\omega_3)| = \frac{1}{\sqrt{4 + 0.01\omega_3^2}} = 0.2343$$

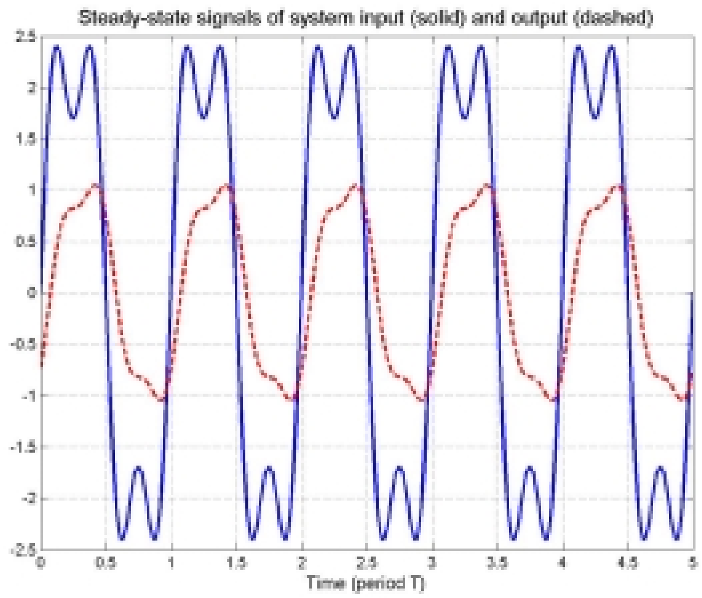
$$\Phi(j\omega_3) = -\arctan(0.05\omega_3) = -1.0830 \text{ rad} = -62.05^\circ$$

Finally, we can compute the steady-state periodic output of the system:

$$y(t) = \sum_{n=1}^{3} |H_v(j\omega_n)| c_n \sin[\omega_n t + \theta_n + \Phi(j\omega_n)] =$$

$$= 0.4234\frac{8}{\pi}\sin(4\pi\, t - 0.561) + 0.2343\frac{8}{3\pi}\sin(12\pi\, t - 1.083)$$

The input and output steady state signals plot is shown below:



Problem 6.37

Solution:

Known quantities:

The frequency response $H_v(j\omega)$ of the circuit of P6.2, the period $T = 0.5\ s$ and the peak amplitude $A = 2$ for the pulse waveform.

Find:

Output of system $y(t)$ in response to input $x(t)$.

Analysis:

According to the Fourier series definitions of the previous section, and using the first three terms of the Fourier series expansion of the pulse waveform of Example 6.4, we have

$$x(t) = 0.4 + 0.6054\cos(4\pi\, t) + 0.4398\sin(4\pi\, t) + 0.1870\cos(8\pi\, t)$$
$$+ 0.5758\sin(8\pi\, t) - 0.1248\cos(12\pi\, t) + 0.3838\sin(12\pi\, t)$$

Thus, for this problem,

$$c_0 = 0.4$$
$$c_1 = 0.7484, \quad \theta_1 = 0.9425 \text{ rad} = 54^\circ \qquad \omega_1 = 4\pi = 12.5664 \text{ rad/s}$$
$$c_2 = 0.6054, \quad \theta_2 = 0.3140 \text{ rad} = 18^\circ \qquad \omega_2 = 8\pi = 25.1327 \text{ rad/s}$$
$$c_3 = 0.4036, \quad \theta_1 = -0.3144 \text{ rad} = -18^\circ \qquad \omega_3 = 12\pi = 37.6991 \text{ rad/s}$$

The frequency response of the system can be expressed in magnitude and phase form:

$$H_v(j\omega) = \frac{1}{2(1 + 0.05\, j\omega)} = \frac{1}{\sqrt{4 + 0.01\omega^2}} \angle -\arctan(0.05\omega)$$

At this point, we could evaluate the frequency response of the system at the frequencies ω_1, ω_2 and ω_3 analytically:

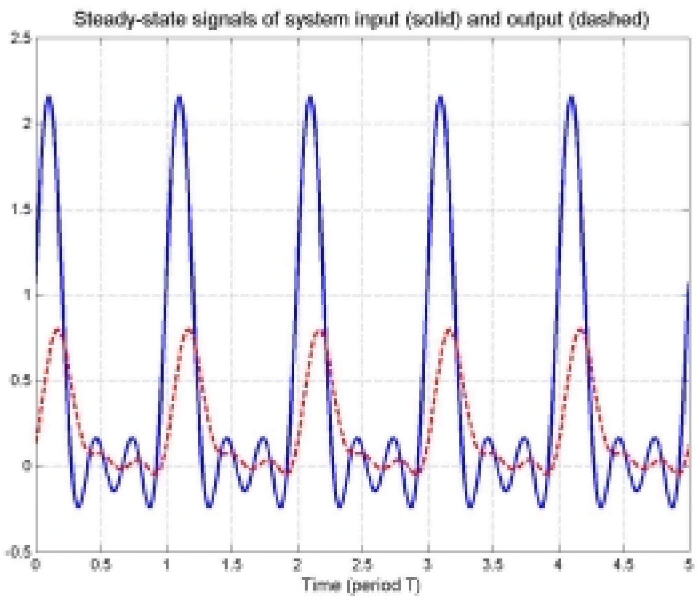
$$|H_v(j\omega_1)| = \frac{1}{\sqrt{4 + 0.01\omega_1^2}} = 0.4234$$

$$\Phi(j\omega_1) = -\arctan(0.05\omega_1) = -0.5610 \text{ rad} = -32.14^\circ$$
$$|H_v(j\omega_2)| = \frac{1}{\sqrt{4+0.01\omega_2^2}} = 0.3113$$
$$\Phi(j\omega_2) = -\arctan(0.05\omega_2) = -0.8986 \text{ rad} = -51.49^\circ$$
$$|H_v(j\omega_3)| = \frac{1}{\sqrt{4+0.01\omega_3^2}} = 0.2343$$
$$\Phi(j\omega_3) = -\arctan(0.05\omega_3) = -1.0830 \text{ rad} = -62.05^\circ$$

Finally, we can compute the steady-state periodic output of the system:

$$y(t) = c_0|H_v(0)| + \sum_{n=1}^{3}|H_v(j\omega_n)|c_n \sin[\omega_n t + \theta_n + \Phi(j\omega_n)] =$$
$$= 0.2 + 0.3168\sin(4\pi\, t + 0.3815) + 0.1885\sin(8\pi\, t - 0.5846) + 0.0946\sin(12\pi\, t - 1.3974)$$

The input and output steady state signals plot is shown below:



Problem 6.38

Solution:

Known quantities:

The frequency response $H_v(j\omega)$ of the circuit of P6.3, the period $T = 0.1\,s$ and the peak amplitude $A = 1$ for the sawtooth waveform.

Find:

Output of system $y(t)$ in response to input $x(t)$.

Analysis:

According to the Fourier series definitions of the previous section, and using the first four terms of the Fourier series expansion of the sawtooth waveform of Example 6.3, we have

$$x(t) = \frac{2A}{\pi}\sin\left(\frac{2\pi}{T}t\right) + \frac{A}{\pi}\sin\left(\frac{4\pi}{T}t\right) + \frac{2A}{3\pi}\sin\left(\frac{6\pi}{T}t\right) + \frac{A}{2\pi}\sin\left(\frac{8\pi}{T}t\right)$$
$$= \frac{2}{\pi}\sin(20\pi\, t) + \frac{1}{\pi}\sin(40\pi\, t) + \frac{2}{3\pi}\sin(60\pi\, t) + \frac{1}{2\pi}\sin(80\pi\, t)$$

Thus, for this problem,

$$c_1 = \frac{2}{\pi} \qquad \omega_1 = 20\pi = 62.8 \text{ rad/s}$$
$$c_2 = \frac{1}{\pi} \qquad \omega_2 = 40\pi = 125.6 \text{ rad/s}$$
$$c_3 = \frac{2}{3\pi} \qquad \omega_3 = 60\pi = 188.4 \text{ rad/s}$$

and,

$$c_4 = \frac{1}{2\pi} \qquad \omega_4 = 80\pi = 251.2 \text{ rad/s}$$

The frequency response of the system can be expressed in magnitude and phase form:

$$H_v(j\omega) = \frac{1}{2(1+0.02\,j\omega)} = \frac{1}{\sqrt{4+0.0016\omega^2}}\angle -\arctan(0.02\omega)$$

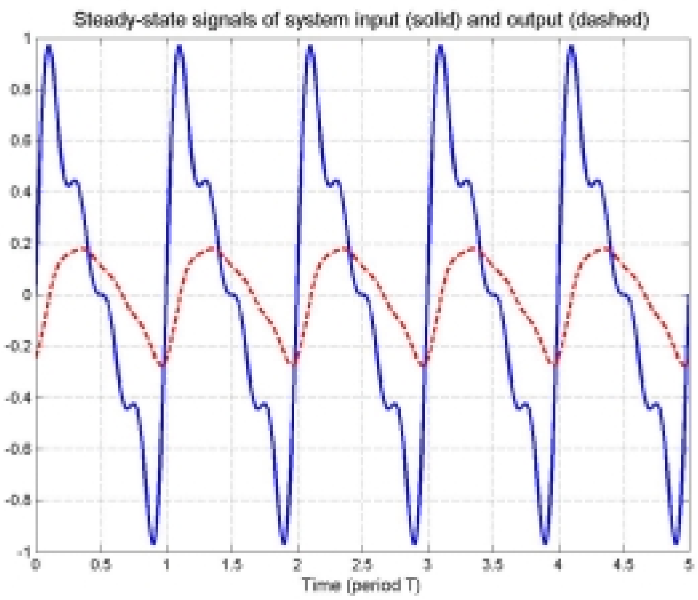
At this point, we could evaluate the frequency response of the system at the frequencies ω_1, ω_2, ω_3 and ω_4 analytically:

$$|H_v(j\omega_1)| = \frac{1}{\sqrt{4+0.0016\omega_1^2}} = 0.3113$$
$$\Phi(j\omega_1) = -\arctan(0.02\omega_1) = -0.8986 \text{ rad} = -51.49^\circ$$
$$|H_v(j\omega_2)| = \frac{1}{\sqrt{4+0.0016\omega_2^2}} = 0.1848$$
$$\Phi(j\omega_2) = -\arctan(0.02\omega_2) = -1.1921 \text{ rad} = -68.3^\circ$$
$$|H_v(j\omega_3)| = \frac{1}{\sqrt{4+0.0016\omega_3^2}} = 0.1282$$
$$\Phi(j\omega_3) = -\arctan(0.02\omega_3) = -1.3115 \text{ rad} = -75.14^\circ$$
$$|H_v(j\omega_4)| = \frac{1}{\sqrt{4+0.0016\omega_4^2}} = 0.0976$$
$$\Phi(j\omega_4) = -\arctan(0.02\omega_4) = -1.3744 \text{ rad} = -78.75^\circ$$

Finally, we can compute the steady-state periodic output of the system:

$$y(t) = \sum_{n=1}^{4} |H_v(j\omega_n)| c_n \sin[\omega_n t + \theta_n + \Phi(j\omega_n)] =$$
$$= 0.1982\sin(20\pi\, t - 0.8986) + 0.0588\sin(40\pi\, t - 1.1921)$$
$$+ 0.0272\sin(60\pi\, t - 1.3115) + 0.0155\sin(80\pi\, t - 1.3744)$$

The input and output steady state signals plot is shown below:



Problem 6.39

Solution:

Known quantities:

The frequency response $H_v(j\omega)$ of the circuit of P6.3, the period $T = 0.1\,s$ and the peak amplitude $A = 1$ for the square waveform.

Find:

Output of system $y(t)$ in response to input $x(t)$.

Analysis:

According to the Fourier series definitions of the previous section, and using the first four terms of the Fourier series expansion of the square waveform of P6.33, we have

$$x(t) = \frac{4A}{\pi}\sin\left(\frac{2\pi}{T}t\right) + \frac{4A}{3\pi}\sin\left(\frac{6\pi}{T}t\right) = \frac{4}{\pi}\sin(20\pi\, t) + \frac{4}{3\pi}\sin(60\pi\, t)$$

Thus, for this problem,

$$c_1 = \frac{4}{\pi} \qquad \omega_1 = 20\pi = 62.8 \text{ rad/s}$$

$$c_3 = \frac{4}{3\pi} \qquad \omega_3 = 60\pi = 188.4 \text{ rad/s}$$

$$c_2 = c_4 = 0$$

The frequency response of the system can be expressed in magnitude and phase form:

$$H_v(j\omega) = \frac{1}{2(1 + 0.02\, j\omega)} = \frac{1}{\sqrt{4 + 0.0016\omega^2}} \angle - \arctan(0.02\omega)$$

At this point, we could evaluate the frequency response of the system at the frequencies ω_1 and ω_3 analytically:

$$|H_v(j\omega_1)| = \frac{1}{\sqrt{4+0.0016\omega_1^2}} = 0.3113$$

$$\Phi(j\omega_1) = -\arctan(0.02\omega_1) = -0.8986 \text{ rad} = -51.49^\circ$$

$$|H_v(j\omega_3)| = \frac{1}{\sqrt{4+0.0016\omega_3^2}} = 0.1282$$

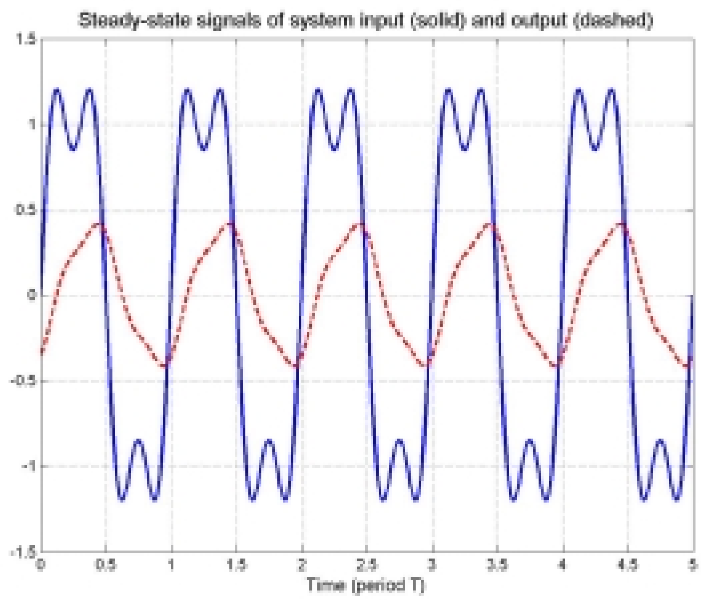
$$\Phi(j\omega_3) = -\arctan(0.02\omega_3) = -1.3115 \text{ rad} = -75.14^\circ$$

Finally, we can compute the steady-state periodic output of the system:

$$y(t) = \sum_{n=1}^{4} |H_v(j\omega_n)| c_n \sin[\omega_n t + \theta_n + \Phi(j\omega_n)] =$$

$$= 0.3964\sin(20\pi\, t - 0.8986) + 0.0544\sin(60\pi\, t - 1.3115)$$

The input and output steady state signals plot is shown below:



Problem 6.40

Solution:

Known quantities:

The frequency response $H_v(j\omega)$ of the circuit of P6.3, the period $T = 0.1\, s$ and the peak amplitude $A = 1$ for the pulse waveform.

Find:

Output of system $y(t)$ in response to input $x(t)$.

Analysis:

According to the Fourier series definitions of the previous section, and using the first four terms of the Fourier series expansion of the pulse waveform of Example 6.4, we have

$$x(t) = 0.2 + 0.3027\cos(20\pi\, t) + 0.2199\sin(20\pi\, t) + 0.0935\cos(40\pi\, t) + 0.2879\sin(40\pi\, t)$$
$$-0.0624\cos(60\pi\, t) + 0.1919\sin(60\pi\, t) - 0.0757\cos(80\pi\, t) + 0.055\sin(80\pi\, t)$$

Thus, for this problem,

$$c_0 = 0.2$$

$$c_1 = 0.3742\,, \quad \theta_1 = 0.9425\,\text{rad} = 54^\circ \qquad \omega_1 = 20\pi = 62.8\,\text{rad/s}$$

$$c_2 = 0.3027\,, \quad \theta_2 = 0.3140\,\text{rad} = 18^\circ \qquad \omega_2 = 40\pi = 125.6\,\text{rad/s}$$

$$c_3 = 0.2018\,, \quad \theta_3 = -0.3144\,\text{rad} = -18^\circ \qquad \omega_3 = 60\pi = 188.4\,\text{rad/s}$$

and,

$$c_4 = 0.0935\,, \quad \theta_1 = -0.9425\,\text{rad} = -54^\circ \qquad \omega_4 = 80\pi = 251.2\,\text{rad/s}$$

The frequency response of the system can be expressed in magnitude and phase form:

$$H_v(j\omega) = \frac{1}{2(1+0.02\,j\omega)} = \frac{1}{\sqrt{4+0.0016\omega^2}} \angle -\arctan(0.02\omega)$$

At this point, we could evaluate the frequency response of the system at the frequencies ω_1, ω_2, ω_3 and ω_4 analytically:

$$\left|H_v(j\omega_1)\right| = \frac{1}{\sqrt{4+0.0016\omega_1^2}} = 0.3113$$

$$\Phi(j\omega_1) = -\arctan(0.02\omega_1) = -0.8986\,\text{rad} = -51.49^\circ$$

$$\left|H_v(j\omega_2)\right| = \frac{1}{\sqrt{4+0.0016\omega_2^2}} = 0.1848$$

$$\Phi(j\omega_2) = -\arctan(0.02\omega_2) = -1.1921\,\text{rad} = -68.3^\circ$$

$$\left|H_v(j\omega_3)\right| = \frac{1}{\sqrt{4+0.0016\omega_3^2}} = 0.1282$$

$$\Phi(j\omega_3) = -\arctan(0.02\omega_3) = -1.3115\,\text{rad} = -75.14^\circ$$

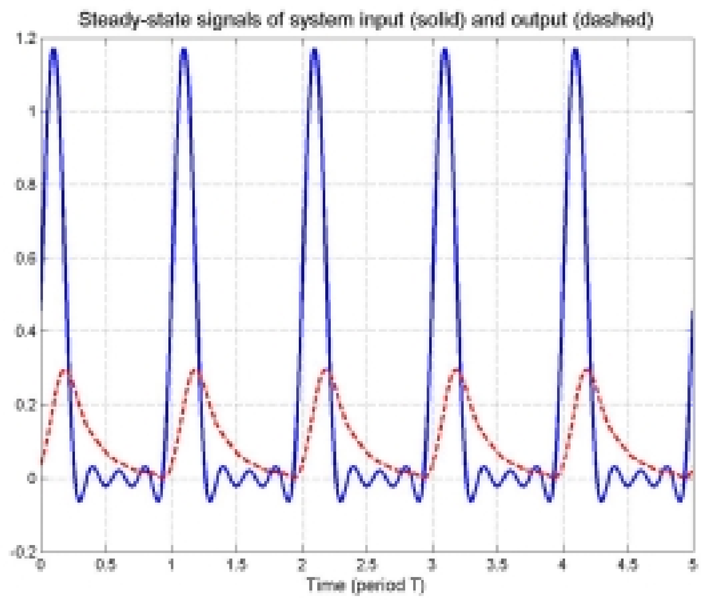
$$\left|H_v(j\omega_4)\right| = \frac{1}{\sqrt{4+0.0016\omega_4^2}} = 0.0976$$

$$\Phi(j\omega_4) = -\arctan(0.02\omega_4) = -1.3744\,\text{rad} = -78.75^\circ$$

Finally, we can compute the steady-state periodic output of the system:

$$\begin{aligned} y(t) &= c_0\left|H_v(0)\right| + \sum_{n=1}^{4}\left|H_v(j\omega_n)\right| c_n \sin\left[\omega_n t + \theta_n + \Phi(j\omega_n)\right] = \\ &= 0.1 + 0.1165\sin(20\pi\, t + 0.0439) + 0.0559\sin(40\pi\, t - 0.8781) \\ &\quad + 0.0259\sin(60\pi\, t - 1.6259) + 0.00915\sin(80\pi\, t - 2.3169) \end{aligned}$$

The input and output steady state signals plot is shown below:



Problem 6.41

Solution:

Known quantities:

The frequency response $H_v(j\omega)$ of the circuit of P6.4, the period $T = 50\ ms$ and the peak amplitude $A = 2$ for the sawtooth waveform.

Find:

Output of system $y(t)$ in response to input $x(t)$.

Analysis:

According to the Fourier series definitions of the previous section, and using the first two terms of the Fourier series expansion of the sawtooth waveform of Example 6.3, we have

$$x(t) = \frac{2A}{\pi}\sin\left(\frac{2\pi}{T}t\right) + \frac{A}{\pi}\sin\left(\frac{4\pi}{T}t\right) = \frac{4}{\pi}\sin(40\pi\, t) + \frac{2}{\pi}\sin(80\pi\, t)$$

Thus, for this problem,

$$c_1 = \frac{4}{\pi} \qquad \omega_1 = 40\pi = 125.6\ \text{rad/s}$$

and,

$$c_2 = \frac{2}{\pi} \qquad \omega_2 = 80\pi = 251.2\ \text{rad/s}$$

The frequency response of the system can be expressed in magnitude and phase form:

$$H_v(j\omega) = \frac{1 - 0.0002\omega^2 + 0.1j\omega}{1 - 0.0002\omega^2 + 0.15j\omega} =$$

$$= \frac{\sqrt{\left[1 - 0.0002\omega^2\right]^2 + 0.01\omega^2}}{\sqrt{\left[1 - 0.0002\omega^2\right]^2 + 0.0225\omega^2}} \angle\left[\arctan\left(\frac{0.1\omega}{1 - 0.0002\omega^2}\right) - \arctan\left(\frac{0.15\omega}{1 - 0.0002\omega^2}\right)\right]$$

At this point, we could evaluate the frequency response of the system at the frequencies ω_1 and ω_2 analytically:

$$|H_v(j\omega_1)| = \frac{\sqrt{\left[1-0.0002\omega_1^2\right]^2 + 0.01\omega_1^2}}{\sqrt{\left[1-0.0002\omega_1^2\right]^2 + 0.0225\omega_1^2}} = 0.6720$$

$$\Phi(j\omega_1) = \arctan\left(\frac{0.1\omega_1}{1-0.0002\omega_1^2}\right) - \arctan\left(\frac{0.15\omega_1}{1-0.0002\omega_1^2}\right) = 0.0561\,\text{rad} = 3.21^\circ$$

$$|H_v(j\omega_2)| = \frac{\sqrt{\left[1-0.0002\omega_2^2\right]^2 + 0.01\omega_2^2}}{\sqrt{\left[1-0.0002\omega_2^2\right]^2 + 0.0225\omega_2^2}} = 0.7020$$

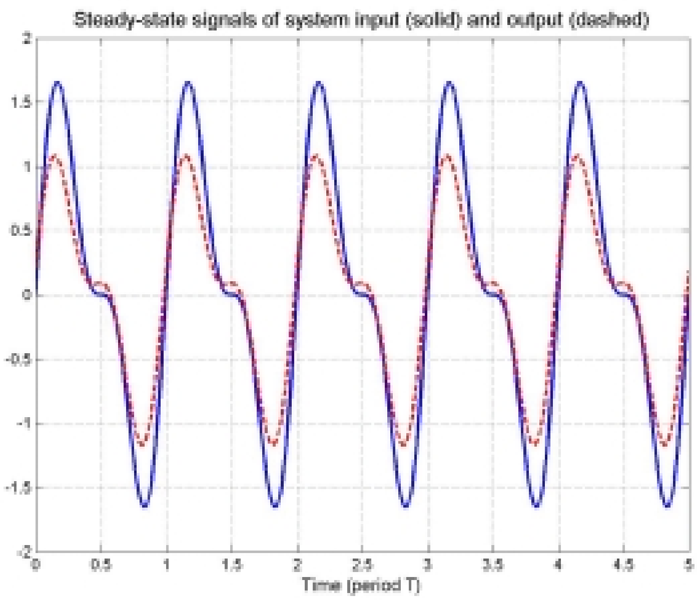
$$\Phi(j\omega_2) = \arctan\left(\frac{0.1\omega_2}{1-0.0002\omega_2^2}\right) - \arctan\left(\frac{0.15\omega_2}{1-0.0002\omega_2^2}\right) = 0.1342\,\text{rad} = 7.69^\circ$$

Finally, we can compute the steady-state periodic output of the system:

$$y(t) = \sum_{n=1}^{2} |H_v(j\omega_n)| c_n \sin\left[\omega_n t + \theta_n + \Phi(j\omega_n)\right] =$$

$$= 0.6720\frac{4}{\pi}\sin(40\pi\, t + 0.0561) + 0.7020\frac{2}{\pi}\sin(80\pi\, t + 0.1342)$$

The input and output steady state signals plot is shown below:



Problem 6.42

Solution:

Known quantities:

The frequency response $H_v(j\omega)$ of the circuit of P6.4, the periods $T_1 = 0.5\ s$ and $T_2 = 5\ ms$ and the peak amplitude $A = 2$ for the sawtooth waveform.

Find:

Output of system $y(t)$ in response to input $x(t)$. Compare the plots with the one obtained in P6.41.

Analysis:

According to the Fourier series definitions of the previous section, and using the first two terms of the Fourier series expansion of the sawtooth waveform of Example 6.3, we have

$$x(t) = \frac{2A}{\pi}\sin\left(\frac{2\pi}{T}t\right) + \frac{A}{\pi}\sin\left(\frac{4\pi}{T}t\right) \quad \Rightarrow \quad \begin{array}{l} x_1(t) = \frac{4}{\pi}\sin(4\pi t) + \frac{2}{\pi}\sin(8\pi t) \\ x_2(t) = \frac{4}{\pi}\sin(400\pi t) + \frac{2}{\pi}\sin(800\pi t) \end{array}$$

Thus, for this problem,

$$c_{1,1} = c_{1,2} = \frac{4}{\pi} \qquad \omega_{1,1} = 4\pi = 12.56 \text{ rad/s} \qquad \omega_{1,2} = 400\pi = 1256 \text{ rad/s}$$

and,

$$c_{2,1} = c_{2,2} = \frac{2}{\pi} \qquad \omega_{2,1} = 8\pi = 25.12 \text{ rad/s} \qquad \omega_{2,2} = 800\pi = 2512 \text{ rad/s}$$

The frequency response of the system can be expressed in magnitude and phase form:

$$H_v(j\omega) = \frac{1 - 0.0002\omega^2 + 0.1j\omega}{1 - 0.0002\omega^2 + 0.15j\omega} =$$

$$= \frac{\sqrt{\left[1 - 0.0002\omega^2\right]^2 + 0.01\omega^2}}{\sqrt{\left[1 - 0.0002\omega^2\right]^2 + 0.0225\omega^2}} \angle\left[\arctan\left(\frac{0.1\omega}{1 - 0.0002\omega^2}\right) - \arctan\left(\frac{0.15\omega}{1 - 0.0002\omega^2}\right)\right]$$

At this point, we could evaluate the frequency response of the system at the frequencies $\omega_{1,1}$, $\omega_{2,1}$, $\omega_{1,2}$ and $\omega_{2,2}$ analytically:

$$\left|H_v(j\omega_{1,1})\right| = 0.7486, \qquad \Phi(j\omega_{1,1}) = -0.1820 \text{ rad} = -10.43^\circ$$

$$\left|H_v(j\omega_{2,1})\right| = 0.6876, \qquad \Phi(j\omega_{2,1}) = -0.1068 \text{ rad} = -6.12^\circ$$

and,

$$\left|H_v(j\omega_{1,2})\right| = 0.9238, \qquad \Phi(j\omega_{1,2}) = 0.1597 \text{ rad} = 9.15^\circ$$

$$\left|H_v(j\omega_{2,2})\right| = 0.9770, \qquad \Phi(j\omega_{2,2}) = 0.0937 \text{ rad} = 5.37^\circ$$

Finally, we can compute the steady-state periodic outputs of the system:

$$y_1(t) = \sum_{n=1}^{2}\left|H_v(j\omega_{n,1})\right| c_{n,1}\sin\left[\omega_{n,1}t + \theta_{n,1} + \Phi(j\omega_{n,1})\right] =$$

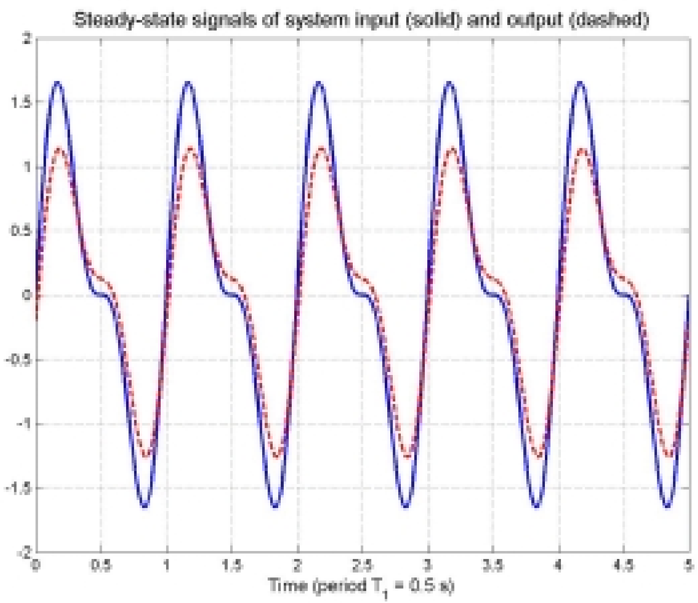
$$= 0.7486\frac{4}{\pi}\sin(4\pi t - 0.182) + 0.6876\frac{2}{\pi}\sin(8\pi t - 0.1068)$$

and,

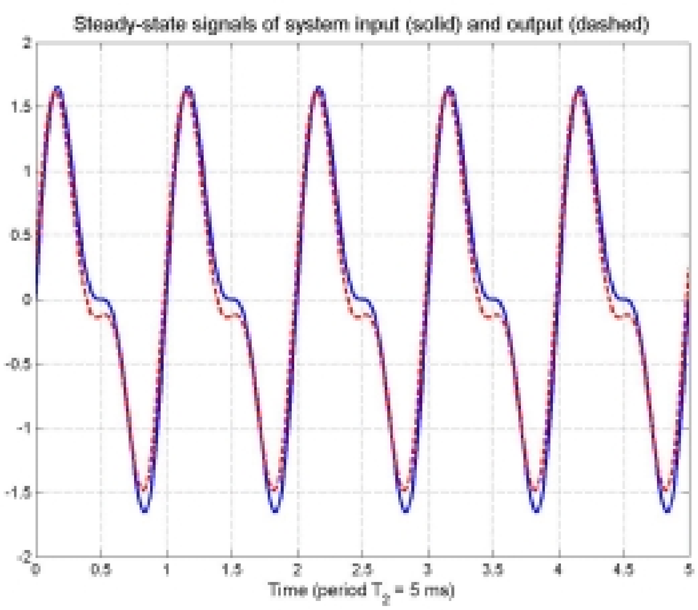
$$y_2(t) = \sum_{n=1}^{2}\left|H_v(j\omega_{n,2})\right| c_{n,2}\sin\left[\omega_{n,2}t + \theta_{n,2} + \Phi(j\omega_{n,2})\right] =$$

$$= 0.9238\frac{4}{\pi}\sin(400\pi t + 0.1597) + 0.977\frac{2}{\pi}\sin(800\pi t + 0.0937)$$

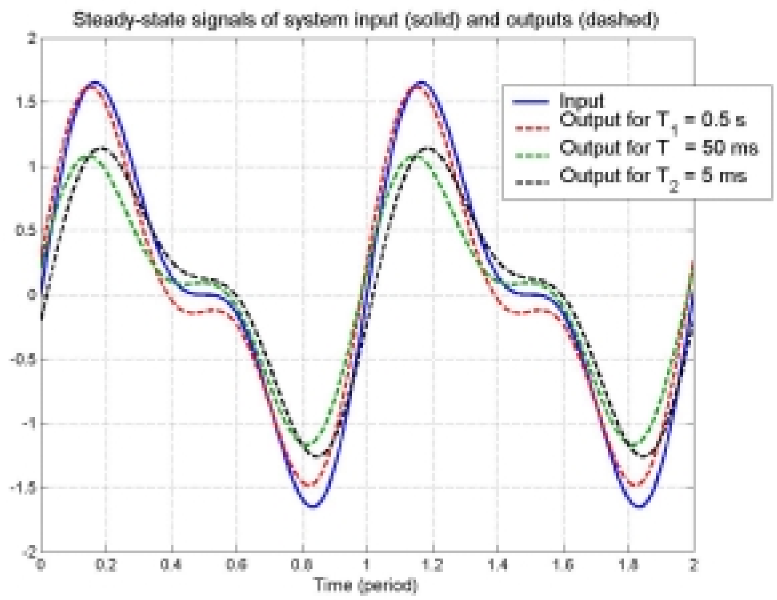
The input and output steady state signals plot for $T_1 = 0.5\ s$ is shown below:



The input and output steady state signals plot for $T_2 = 5\ ms$ is shown below:



Comparing the results with $T = 50\ ms$, we have:



Problem 6.43

Solution:

Known quantities:

The frequency response $H_v(j\omega)$ of the circuit of P6.4, the period $T = 50\ ms$ and the peak amplitude $A = 2$ for the square wave.

Find:

Output of system $y(t)$ in response to input $x(t)$.

Analysis:

According to the Fourier series definitions of the previous section, and using the first two terms of the Fourier series expansion of the square waveform of P6.33, we have

$$x(t) = \frac{4A}{\pi}\sin\left(\frac{2\pi}{T}t\right) = \frac{8}{\pi}\sin(40\pi\, t)$$

Thus, for this problem,

$$c_1 = \frac{8}{\pi} \qquad \omega_1 = 40\pi = 125.6 \text{ rad/s}$$

and,

$$c_2 = 0$$

The frequency response of the system can be expressed in magnitude and phase form:

$$H_v(j\omega) = \frac{1-0.0002\omega^2 + 0.1j\omega}{1-0.0002\omega^2 + 0.15j\omega} =$$

$$= \frac{\sqrt{\left[1-0.0002\omega^2\right]^2 + 0.01\omega^2}}{\sqrt{\left[1-0.0002\omega^2\right]^2 + 0.0225\omega^2}} \angle\left[\arctan\left(\frac{0.1\omega}{1-0.0002\omega^2}\right) - \arctan\left(\frac{0.15\omega}{1-0.0002\omega^2}\right)\right]$$

At this point, we could evaluate the frequency response of the system at the frequency ω_1 analytically:

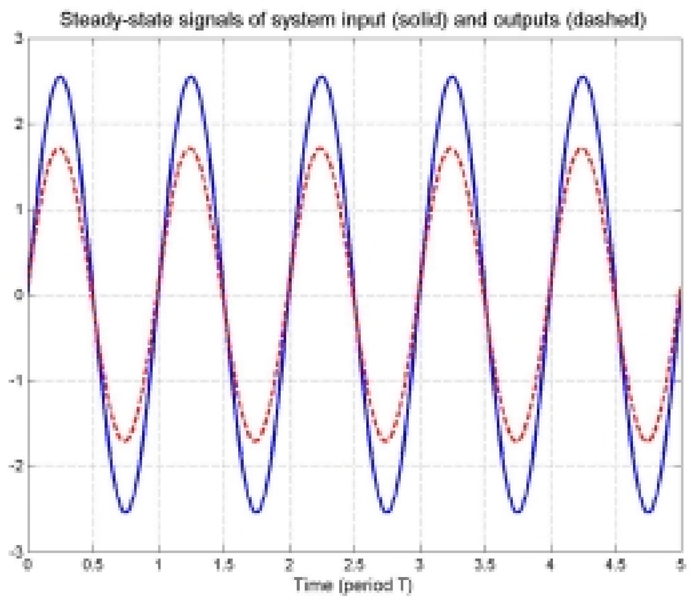
$$\left|H_v(j\omega_1)\right| = \frac{\sqrt{\left[1-0.0002\omega_1^2\right]^2 + 0.01\omega_1^2}}{\sqrt{\left[1-0.0002\omega_1^2\right]^2 + 0.0225\omega_1^2}} = 0.6720$$

$$\Phi(j\omega_1) = \arctan\left(\frac{0.1\omega_1}{1-0.0002\omega_1^2}\right) - \arctan\left(\frac{0.15\omega_1}{1-0.0002\omega_1^2}\right) = 0.0561\,\text{rad} = 3.21^\circ$$

Finally, we can compute the steady-state periodic output of the system:

$$y(t) = \sum_{n=1}^{2}\left|H_v(j\omega_n)\right| c_n \sin\left[\omega_n t + \theta_n + \Phi(j\omega_n)\right] = 0.672\frac{8}{\pi}\sin(40\pi\, t + 0.0561)$$

The input and output steady state signals plot is shown below:



Problem 6.44

Solution:

Known quantities:

The frequency response $H_v(j\omega)$ of the circuit of P6.4, the period $T = 50\ ms$ and the peak amplitude $A = 2$ for the pulse waveform.

Find:

Output of system $y(t)$ in response to input $x(t)$.

Analysis:

According to the Fourier series definitions of the previous section, and using the first two terms of the Fourier series expansion of the pulse waveform of Example 6.4, we have

$$x(t) = 0.4 + 0.6054\cos(40\pi\, t) + 0.4398\sin(40\pi\, t) + 0.1870\cos(80\pi\, t) + 0.5758\sin(80\pi\, t)$$

Thus, for this problem,

$$c_0 = 0.4$$

$$c_1 = 0.7484\,,\quad \theta_1 = 0.9425\,\text{rad} = 54^\circ \qquad \omega_1 = 40\pi = 125.6\,\text{rad/s}$$

and,

$$c_2 = 0.6054, \quad \theta_2 = 0.3140 \text{ rad} = 18^\circ \qquad \omega_2 = 80\pi = 251.2 \text{ rad/s}$$

The frequency response of the system can be expressed in magnitude and phase form:

$$H_v(j\omega) = \frac{1 - 0.0002\omega^2 + 0.1j\omega}{1 - 0.0002\omega^2 + 0.15j\omega} =$$

$$= \frac{\sqrt{\left[1 - 0.0002\omega^2\right]^2 + 0.01\omega^2}}{\sqrt{\left[1 - 0.0002\omega^2\right]^2 + 0.0225\omega^2}} \angle\left[\arctan\left(\frac{0.1\omega}{1 - 0.0002\omega^2}\right) - \arctan\left(\frac{0.15\omega}{1 - 0.0002\omega^2}\right)\right]$$

At this point, we could evaluate the frequency response of the system at the frequencies ω_1 and ω_2 analytically:

$$|H_v(j\omega_1)| = \frac{\sqrt{\left[1 - 0.0002\omega_1^2\right]^2 + 0.01\omega_1^2}}{\sqrt{\left[1 - 0.0002\omega_1^2\right]^2 + 0.0225\omega_1^2}} = 0.6720$$

$$\Phi(j\omega_1) = \arctan\left(\frac{0.1\omega_1}{1 - 0.0002\omega_1^2}\right) - \arctan\left(\frac{0.15\omega_1}{1 - 0.0002\omega_1^2}\right) = 0.0561 \text{ rad} = 3.21^\circ$$

$$|H_v(j\omega_2)| = \frac{\sqrt{\left[1 - 0.0002\omega_2^2\right]^2 + 0.01\omega_2^2}}{\sqrt{\left[1 - 0.0002\omega_2^2\right]^2 + 0.0225\omega_2^2}} = 0.7020$$

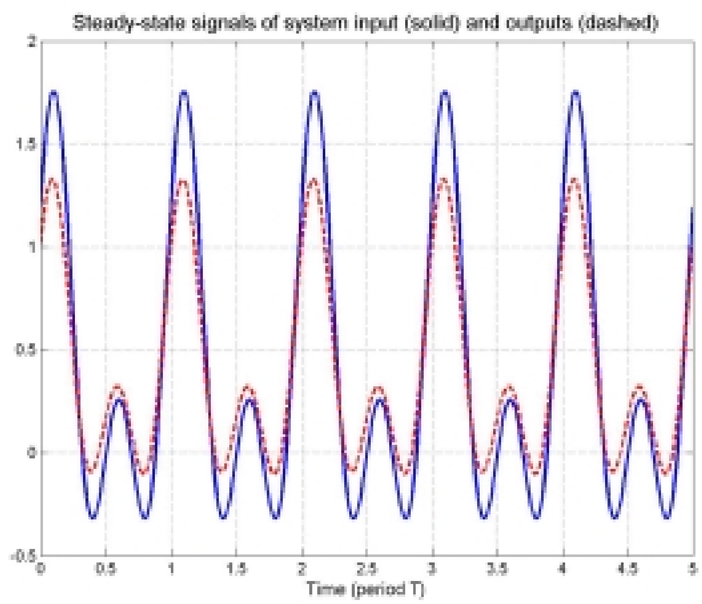
$$\Phi(j\omega_2) = \arctan\left(\frac{0.1\omega_2}{1 - 0.0002\omega_2^2}\right) - \arctan\left(\frac{0.15\omega_2}{1 - 0.0002\omega_2^2}\right) = 0.1342 \text{ rad} = 7.69^\circ$$

Finally, we can compute the steady-state periodic output of the system:

$$y(t) = c_0|H_v(0)| + \sum_{n=1}^{2}|H_v(j\omega_n)|c_n \sin[\omega_n t + \theta_n + \Phi(j\omega_n)] =$$

$$= 0.4 + 0.503\sin(40\pi\, t + 0.9986) + 0.425\sin(80\pi\, t + 0.4482)$$

The input and output steady state signals plot is shown below:



Problem 6.45

Solution:

Known quantities:

The frequency response $H_v(j\omega)$ of the circuit of P6.6, the period $T = 5\ s$ and the peak amplitude $A = 1$ for the sawtooth waveform.

Find:

Output of system $y(t)$ in response to input $x(t)$.

Analysis:

According to the Fourier series definitions of the previous section, and using the first three terms of the Fourier series expansion of the sawtooth waveform of Example 6.3, we have

$$x(t) = \frac{2A}{\pi}\sin\left(\frac{2\pi}{T}t\right) + \frac{A}{\pi}\sin\left(\frac{4\pi}{T}t\right) + \frac{2A}{3\pi}\sin\left(\frac{6\pi}{T}t\right)$$

$$= \frac{2}{\pi}\sin(0.4\pi\, t) + \frac{1}{\pi}\sin(0.8\pi\, t) + \frac{2}{3\pi}\sin(1.2\pi\, t)$$

Thus, for this problem,

$$c_1 = \frac{2}{\pi} \qquad \omega_1 = 0.4\pi = 1.256 \text{ rad/s}$$

$$c_2 = \frac{1}{\pi} \qquad \omega_2 = 0.8\pi = 2.512 \text{ rad/s}$$

and,

$$c_3 = \frac{2}{3\pi} \qquad \omega_3 = 1.2\pi = 3.768 \text{ rad/s}$$

The frequency response of the system can be expressed in magnitude and phase form:

$$H_v(j\omega) = \frac{2j\omega}{(1-\omega^2) + 2.6j\omega} = \frac{2\omega}{\sqrt{(1-\omega^2)^2 + 6.76\omega^2}} \angle\left[90^\circ - \arctan\left(\frac{2.6\omega}{1-\omega^2}\right)\right]$$

At this point, we could evaluate the frequency response of the system at the frequencies ω_1, ω_2 and ω_3 analytically:

$$|H_v(j\omega_1)| = \frac{2\omega_1}{\sqrt{(1-\omega_1^2)^2 + 6.76\omega_1^2}} = 0.7575$$

$$\Phi(j\omega_1) = 90^\circ - \arctan\left(\frac{2.6\omega_1}{1-\omega_1^2}\right) = -0.1750 \text{ rad} = -10^\circ$$

$$|H_v(j\omega_2)| = \frac{2\omega_2}{\sqrt{(1-\omega_2^2)^2 + 6.76\omega_2^2}} = 0.5969$$

$$\Phi(j\omega_2) = 90^\circ - \arctan\left(\frac{2.6\omega_2}{1-\omega_2^2}\right) = -0.6826 \text{ rad} = -39.11^\circ$$

$$|H_v(j\omega_3)| = \frac{2\omega_3}{\sqrt{(1-\omega_3^2)^2 + 6.76\omega_3^2}} = 0.4585$$

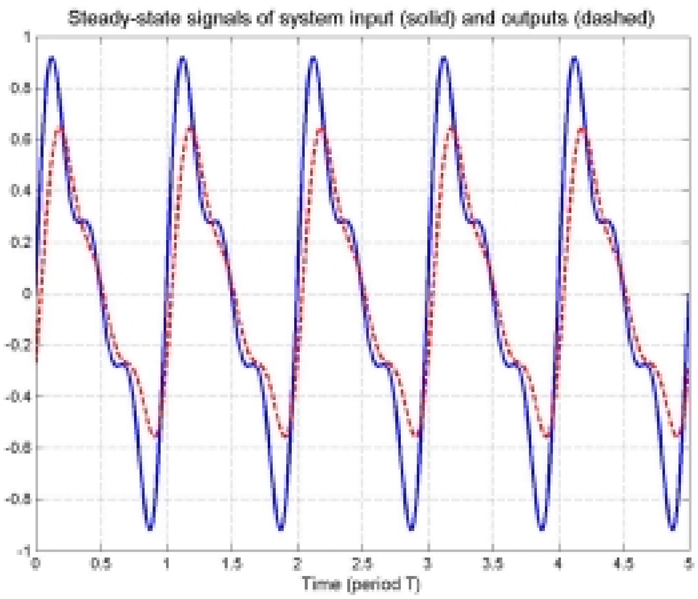
$$\Phi(j\omega_3) = 90^\circ - \arctan\left(\frac{2.6\omega_3}{1-\omega_3^2}\right) = -0.9322 \text{ rad} = -53.41^\circ$$

Finally, we can compute the steady-state periodic output of the system:

$$y(t) = \sum_{n=1}^{3} |H_v(j\omega_n)| c_n \sin[\omega_n t + \theta_n + \Phi(j\omega_n)] =$$

$$= 0.4822\sin(0.4\pi\, t - 0.175) + 0.19\sin(0.8\pi\, t - 0.6826) + 0.0973\sin(1.2\pi\, t - 0.9322)$$

The input and output steady state signals plot is shown below:



Problem 6.46

Solution:

Known quantities:

The frequency response $H_v(j\omega)$ of the circuit of P6.6, the period $T = 50\ s$ and the peak amplitude $A = 1$ for the sawtooth waveform.

Find:

Output of system $y(t)$ in response to input $x(t)$.

Analysis:

According to the Fourier series definitions of the previous section, and using the first three terms of the Fourier series expansion of the sawtooth waveform of Example 6.3, we have

$$x(t) = \frac{2A}{\pi}\sin\left(\frac{2\pi}{T}t\right) + \frac{A}{\pi}\sin\left(\frac{4\pi}{T}t\right) + \frac{2A}{3\pi}\sin\left(\frac{6\pi}{T}t\right)$$

$$= \frac{2}{\pi}\sin(0.04\pi\, t) + \frac{1}{\pi}\sin(0.08\pi\, t) + \frac{2}{3\pi}\sin(0.12\pi\, t)$$

Thus, for this problem,

$$c_1 = \frac{2}{\pi} \qquad \omega_1 = 0.04\pi = 0.1256 \text{ rad/s}$$

$$c_2 = \frac{1}{\pi} \qquad \omega_2 = 0.08\pi = 0.2512 \text{ rad/s}$$

and,

$$c_3 = \frac{2}{3\pi} \qquad \omega_3 = 0.12\pi = 0.3768 \text{ rad/s}$$

The frequency response of the system can be expressed in magnitude and phase form:

$$H_v(j\omega) = \frac{2j\omega}{(1-\omega^2)+2.6j\omega} = \frac{2\omega}{\sqrt{(1-\omega^2)^2+6.76\omega^2}} \angle\left[90^\circ - \arctan\left(\frac{2.6\omega}{1-\omega^2}\right)\right]$$

At this point, we could evaluate the frequency response of the system at the frequencies ω_1, ω_2 and ω_3 analytically:

$$|H_v(j\omega_1)| = \frac{2\omega_1}{\sqrt{(1-\omega_1^2)^2+6.76\omega_1^2}} = 0.2422$$

$$\Phi(j\omega_1) = 90^\circ - \arctan\left(\frac{2.6\omega_1}{1-\omega_1^2}\right) = 1.2504 \text{ rad} = 71.64^\circ$$

$$|H_v(j\omega_2)| = \frac{2\omega_2}{\sqrt{(1-\omega_2^2)^2+6.76\omega_2^2}} = 0.4399$$

$$\Phi(j\omega_2) = 90^\circ - \arctan\left(\frac{2.6\omega_2}{1-\omega_2^2}\right) = 0.9620 \text{ rad} = 55.12^\circ$$

$$|H_v(j\omega_3)| = \frac{2\omega_3}{\sqrt{(1-\omega_3^2)^2+6.76\omega_3^2}} = 0.5787$$

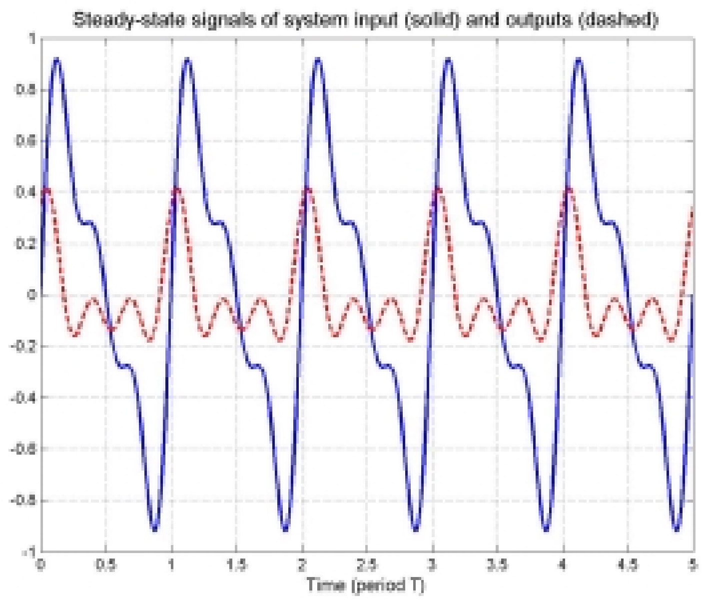
$$\Phi(j\omega_3) = 90^\circ - \arctan\left(\frac{2.6\omega_3}{1-\omega_3^2}\right) = 0.7193 \text{ rad} = 41.21^\circ$$

Finally, we can compute the steady-state periodic output of the system:

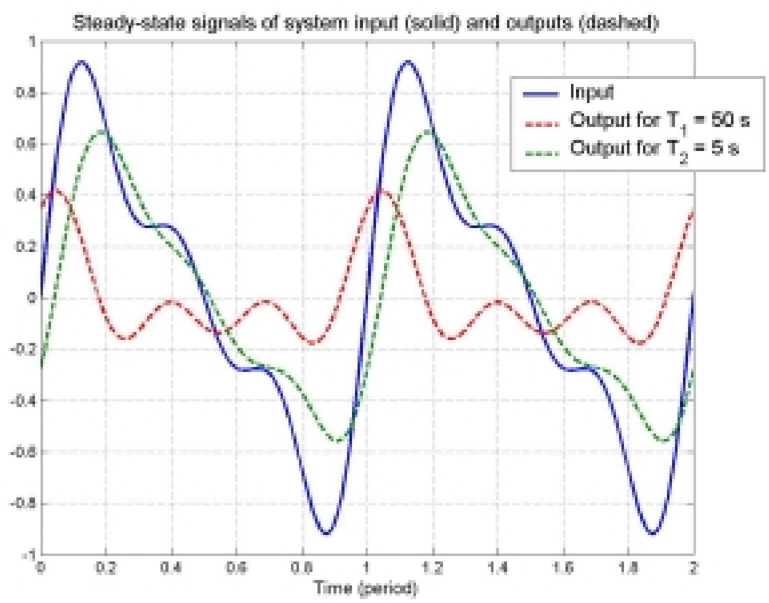
$$y(t) = \sum_{n=1}^{3} |H_v(j\omega_n)| c_n \sin[\omega_n t + \theta_n + \Phi(j\omega_n)] =$$

$$= 0.1542\sin(0.4\pi t + 1.2504) + 0.14\sin(0.8\pi t + 0.9620) + 0.1228\sin(1.2\pi t + 0.7193)$$

The input and output steady state signals plot is shown below:



Comparing the results with $T = 50\ s$, we have:



Problem 6.47

Solution:

Known quantities:

The frequency response $H_v(j\omega)$ of the circuit of P6.6, the period $T = 5\ s$ and the peak amplitude $A = 1$ for the square waveform.

Find:

Output of system $y(t)$ in response to input $x(t)$.

Analysis:

According to the Fourier series definitions of the previous section, and using the first three terms of the Fourier series expansion of the square waveform of P6.33, we have

$$x(t) = \frac{4A}{\pi}\sin\left(\frac{2\pi}{T}t\right) + \frac{4A}{3\pi}\sin\left(\frac{6\pi}{T}t\right) = \frac{4}{\pi}\sin(0.4\pi\, t) + \frac{4}{3\pi}\sin(1.2\pi\, t)$$

Thus, for this problem,

$$c_1 = \frac{4}{\pi} \qquad \omega_1 = 0.4\pi = 1.256 \text{ rad/s}$$

$$c_3 = \frac{4}{3\pi} \qquad \omega_3 = 1.2\pi = 3.768 \text{ rad/s}$$

$$c_2 = 0$$

The frequency response of the system can be expressed in magnitude and phase form:

$$H_v(j\omega) = \frac{2j\omega}{(1-\omega^2)+2.6j\omega} = \frac{2\omega}{\sqrt{(1-\omega^2)^2+6.76\omega^2}}\angle\left[90^\circ - \arctan\left(\frac{2.6\omega}{1-\omega^2}\right)\right]$$

At this point, we could evaluate the frequency response of the system at the frequencies ω_1 and ω_3 analytically:

$$|H_v(j\omega_1)| = \frac{2\omega_1}{\sqrt{(1-\omega_1^2)^2+6.76\omega_1^2}} = 0.7575$$

$$\Phi(j\omega_1) = 90^\circ - \arctan\left(\frac{2.6\omega_1}{1-\omega_1^2}\right) = -0.1750 \text{ rad} = -10^\circ$$

$$|H_v(j\omega_3)| = \frac{2\omega_3}{\sqrt{(1-\omega_3^2)^2+6.76\omega_3^2}} = 0.4585$$

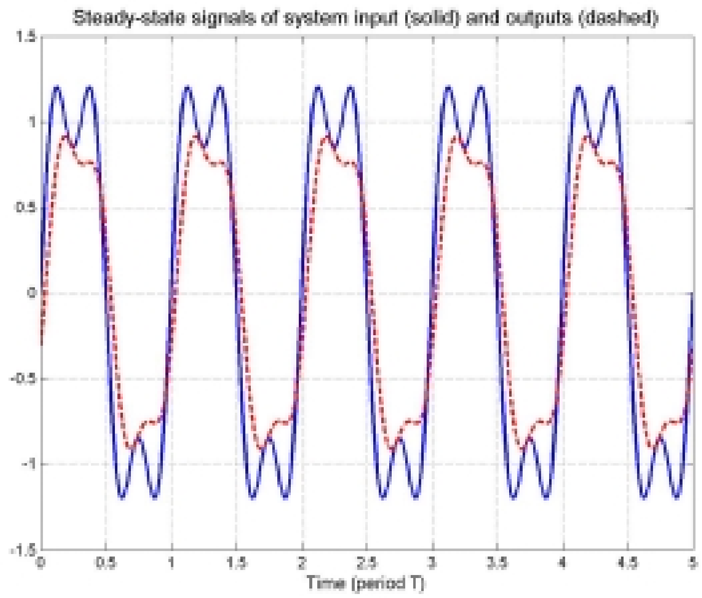
$$\Phi(j\omega_3) = 90^\circ - \arctan\left(\frac{2.6\omega_3}{1-\omega_3^2}\right) = -0.9322 \text{ rad} = -53.41^\circ$$

Finally, we can compute the steady-state periodic output of the system:

$$y(t) = \sum_{n=1}^{3}|H_v(j\omega_n)|c_n \sin[\omega_n t + \theta_n + \Phi(j\omega_n)] =$$

$$= 0.9644\sin(0.4\pi\, t - 0.175) + 0.1946\sin(1.2\pi\, t - 0.9322)$$

The input and output steady state signals plot is shown below:



Problem 6.48

Solution:

Known quantities:

The frequency response $H_v(j\omega)$ of the circuit of P6.6, the period $T = 5\ s$ and the peak amplitude $A = 1$ for the pulse waveform.

Find:

Output of system $y(t)$ in response to input $x(t)$.

Analysis:

According to the Fourier series definitions of the previous section, and using the first three terms of the Fourier series expansion of the pulse waveform of Example 6.4, we have

$$x(t) = 0.2 + 0.3027\cos(0.4\pi\, t) + 0.2199\sin(0.4\pi\, t) + 0.0935\cos(0.8\pi\, t)$$
$$+ 0.2879\sin(0.8\pi\, t) - 0.0624\cos(1.2\pi\, t) + 0.1919\sin(1.2\pi\, t)$$

Thus, for this problem,

$$c_0 = 0.2$$
$$c_1 = 0.3742, \quad \theta_1 = 0.9425\ \text{rad} = 54^\circ \qquad \omega_1 = 0.4\pi = 1.256\ \text{rad/s}$$
$$c_2 = 0.3027, \quad \theta_2 = 0.3140\ \text{rad} = 18^\circ \qquad \omega_2 = 0.8\pi = 2.512\ \text{rad/s}$$

and,

$$c_3 = 0.2018, \quad \theta_3 = -0.3144\ \text{rad} = -18^\circ \qquad \omega_3 = 1.2\pi = 3.768\ \text{rad/s}$$

The frequency response of the system can be expressed in magnitude and phase form:

$$H_v(j\omega) = \frac{2j\omega}{(1-\omega^2) + 2.6j\omega} = \frac{2\omega}{\sqrt{(1-\omega^2)^2 + 6.76\omega^2}} \angle\left[90^\circ - \arctan\left(\frac{2.6\omega}{1-\omega^2}\right)\right]$$

At this point, we could evaluate the frequency response of the system at the frequencies ω_1, ω_2 and ω_3 analytically:

$$\left|H_v(j\omega_1)\right| = \frac{2\omega_1}{\sqrt{\left(1-\omega_1^2\right)^2 + 6.76\omega_1^2}} = 0.7575$$

$$\Phi(j\omega_1) = 90^\circ - \arctan\left(\frac{2.6\omega_1}{1-\omega_1^2}\right) = -0.1750\text{ rad} = -10^\circ$$

$$\left|H_v(j\omega_2)\right| = \frac{2\omega_2}{\sqrt{\left(1-\omega_2^2\right)^2 + 6.76\omega_2^2}} = 0.5969$$

$$\Phi(j\omega_2) = 90^\circ - \arctan\left(\frac{2.6\omega_2}{1-\omega_2^2}\right) = -0.6826\text{ rad} = -39.11^\circ$$

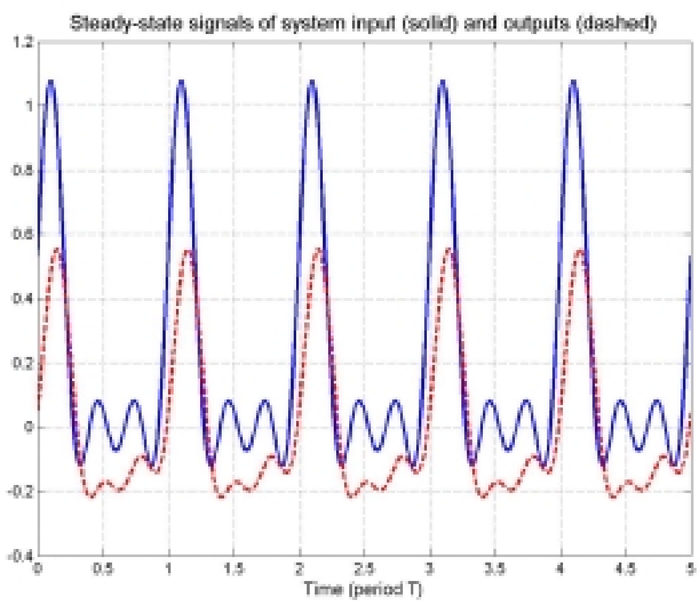
$$\left|H_v(j\omega_3)\right| = \frac{2\omega_3}{\sqrt{\left(1-\omega_3^2\right)^2 + 6.76\omega_3^2}} = 0.4585$$

$$\Phi(j\omega_3) = 90^\circ - \arctan\left(\frac{2.6\omega_3}{1-\omega_3^2}\right) = -0.9322\text{ rad} = -53.41^\circ$$

Finally, we can compute the steady-state periodic output of the system:

$$y(t) = c_0\left|H_v(0)\right| + \sum_{n=1}^{3}\left|H_v(j\omega_n)\right|c_n \sin\left[\omega_n t + \theta_n + \Phi(j\omega_n)\right] =$$
$$= 0.2834\sin(0.4\pi\, t + 0.7675) + 0.1807\sin(0.8\pi\, t - 0.3686) + 0.0925\sin(1.2\pi\, t - 1.2466)$$

The input and output steady state signals plot is shown below:



Problem 6.49

Solution:

Known quantities:
Resistance , capacitance, and inductance values, in the circuit of Figure P6.49.

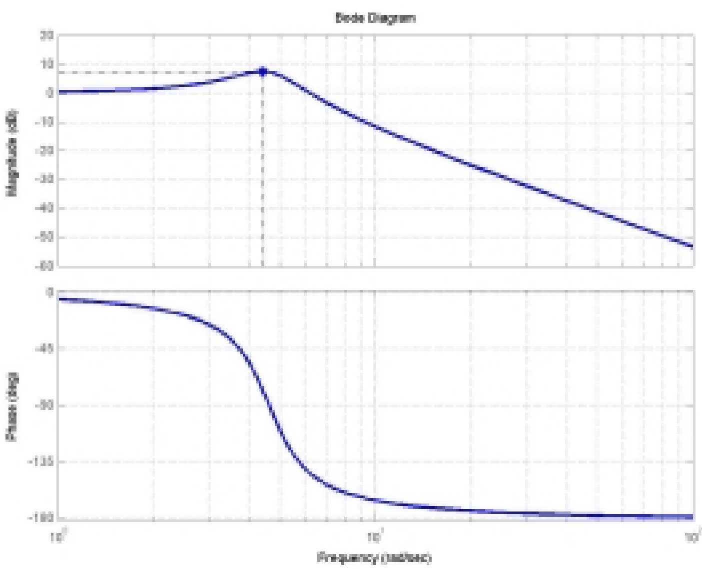
Find:
The resonant frequency and the bandwidth fort the circuit.

Analysis:

Taking the output as the voltage across the parallel R-C subcircuit,

$$\frac{V_o}{V_S} = \frac{1/LC}{(j\omega)^2 + j\omega\frac{1}{RC} + \frac{1}{LC}} = \frac{3/64}{(j\omega)^2 + j\omega 2 + 3/64}\left(= \frac{\omega_n^2 \mu}{(j\omega)^2 + j\omega(2\xi\omega_n) + \omega_n^2}\right)$$

The corresponding Bode diagrams are shown below:



In this circuit, as frequency increases, the impedance of the capacitor decreases and the impedance of the inductor increases. Both effects cause the magnitude of the output voltage to decrease so this is a 2nd order low pass filter.

The resonance frequency is,

$$\omega_n = \sqrt{\frac{1}{LC}} = \sqrt{\frac{64}{3}} \cong 4.6188 \text{ rad/s.}$$

The damping ratio is,

$$\xi = \frac{1/RC}{2\omega_n} = \frac{\sqrt{3}}{8} \cong 0.2165$$

The quality factor is,

$$Q = \frac{1}{2\xi} = \frac{4}{\sqrt{3}} \cong 2.3094$$

The bandwidth is,

$$B = \frac{\omega_n}{Q} = \frac{8}{\sqrt{3}} \frac{1}{4/\sqrt{3}} = 2 \text{ rad/s.}$$

Problem 6.50

Solution:

Known quantities:
Figure P6.50.

Find:
What kind of filters are the ones shown in Figure P6.50.

Analysis:
In a), as frequency increases, the impedance of the capacitor decreases and the impedance of the inductor increases. Both effects cause the magnitude of the output voltage to decrease so this is a 2nd order low pass filter. Note that L and C are connected neither in series nor parallel and do not form a resonant circuit.

In b), L and C are connected in series and form a series resonant circuit with an impedance which is minimum at the resonant frequency and larger above and below the resonant frequency. This series resonant circuit is in series with the output giving, because of voltage division, a maximum output voltage at the resonant frequency and less at higher and lower frequencies. Therefore, b) is a band-pass filter.

In c), L and C are connected in parallel and form a parallel resonant circuit with an impedance which is maximum at the resonant frequency and smaller above and below the resonant frequency. This parallel resonant circuit is in parallel with the output giving, because of voltage division, a maximum output at the resonant frequency and less at higher and lower frequencies. Therefore, c) is a band-pass filter.

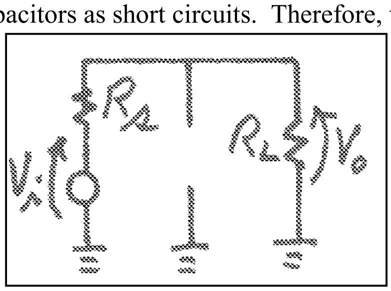
Problem 6.51

Solution:

Known quantities:
Figure P6.51.

Find:
What kind of filters are the ones shown in Figure P6.51.

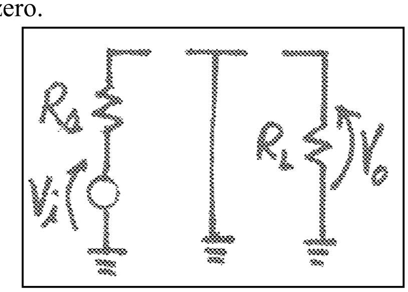
Analysis:
None of the inductors or capacitors is in series or parallel with any other. Therefore, there are no series or parallel resonant circuits and none of the circuits shown is band pass or band stop filters.
Circuits a) and d): As frequency approaches infinity, the inductors can be modeled as open circuits and the capacitors as short circuits. Therefore, the voltage transfer function approaches zero.



As frequency approaches zero, the inductors can be modeled as short circuits and the capacitors as open circuits.

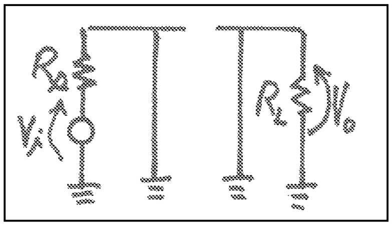
Then: $VD: \; H_v \rightarrow \frac{R_L}{R_s + R_L}$

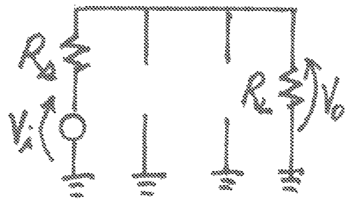
Therefore, circuits a) and d) are low pass filters.



Circuits b) and c) As frequency approaches infinity, the inductors can be modeled as open circuits and the capacitors as short circuits. Then, $VD: \; H_v \rightarrow \frac{R_L}{R_s + R_L}$

Therefore:





As frequency approaches zero, inductors can be modeled as short circuits and the capacitors as open circuits. The voltage transfer function approaches zero. Therefore, circuits b) and c) are high pass filters.

Note: Multiple capacitors and inductors give higher order low and high pass filter. Better performance is obtained outside the pass band where the response for these circuits decreases by 60 dB/decade. In first order filters, the response decreases by only 20 dB/decade.

Problem 6.52

Solution:

Known quantities:
Figure P6.52.

Find:
a) If this is a low-pass, high-pass, band-pass, or band-stop filter.
b) Compute and plot the frequency response function if:

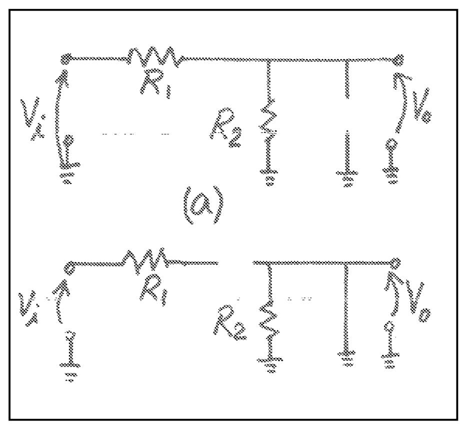
$$L = 11\,\text{mH} \quad C = 0.47\,\text{nF} \quad R_1 = 2.2\,\text{k}\Omega \quad R_2 = 3.8\,\text{k}\Omega$$

Analysis:
a)

$$As\ \omega \rightarrow 0: \qquad Z_L \rightarrow 0 \Rightarrow Short$$
$$Z_C \rightarrow \infty \Rightarrow Open$$
$$\Rightarrow VD: H_v = \frac{V_o}{V_i} \rightarrow \frac{R_2}{R_1 + R_2}$$

$$As\ \omega \rightarrow \infty: \qquad Z_L \rightarrow \infty \Rightarrow Open$$
$$Z_C \rightarrow 0 \Rightarrow Short$$
$$\Rightarrow \quad H_v \rightarrow 0$$

The filter is a low pass filter.



b) First, we find the Thévenin equivalent circuit seen by the capacitor:

$$Z_T = (Z_{R1} + Z_L) \| Z_{R2} = \left(\frac{1}{R_1 + j\omega L} + \frac{1}{R_2} \right)^{-1} = \frac{(R_1 + j\omega L)R_2}{R_1 + j\omega L + R_2}$$

and

$$v_{OC} = \frac{Z_{R2}}{Z_{R1} + Z_L + Z_{R2}} v_{in} = \frac{R_2}{R_1 + j\omega L + R_2} v_{in}$$

$$\frac{v_{out}}{v_{OC}} = \frac{Z_C}{Z_T + Z_C} = \frac{1/j\omega C}{\dfrac{(R_1 + j\omega L)R_2}{R_1 + j\omega L + R_2} + 1/j\omega C} = \frac{R_1 + j\omega L + R_2}{R_1 + j\omega L + R_2 + (R_1 + j\omega L) j\omega C R_2}$$

Therefore,

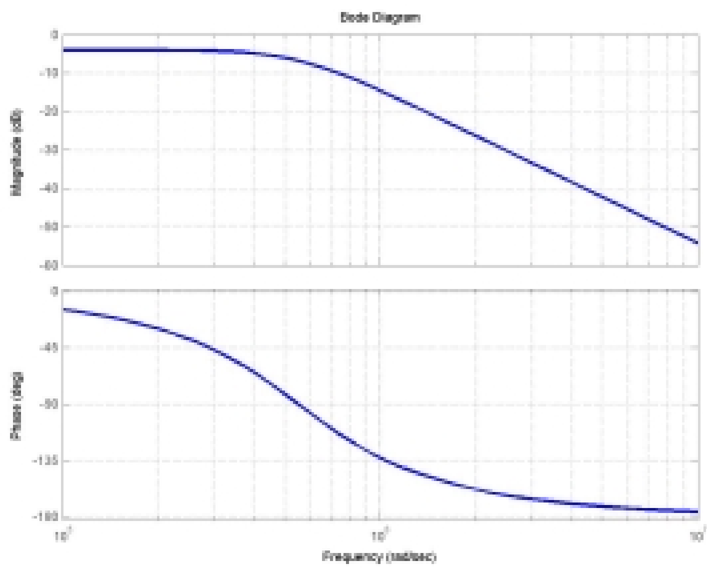
$$\frac{v_{out}}{v_{in}} = \frac{R_2}{R_1 + j\omega L + R_2} \cdot \frac{R_1 + j\omega L + R_2}{R_1 + j\omega L + R_2 + (R_1 + j\omega L) j\omega C R_2}$$

$$= \frac{1}{1 + \dfrac{R_1}{R_2} + j\omega\left(\dfrac{L}{R_2} + CR_1\right) + (j\omega)^2 LC}$$

Substituting the numerical values:

$$\frac{v_{out}}{v_{in}} = \frac{1}{(1.579 - 5.17\times10^{-12}\omega^2) + j(3.929\times10^{-6})\omega}$$

The corresponding Bode diagrams are shown below:



Problem 6.53

Solution:

Known quantities:
The values of the resistors, of the capacitance and of the inductance in the circuit of Figure P6.53.

Find:
Compute and plot the frequency response function. What type of filter is this?

Analysis:
First, we find the Thévenin equivalent circuit seen by the capacitor:

$$Z_T = Z_C \,||\, \left[(Z_{Rs} + Z_L) \,||\, (Z_{Rc} + Z_L)\right] = \left(j\omega C + \frac{1}{R_S + j\omega L} + \frac{1}{R_C + j\omega L} \right)^{-1}$$

$$= \frac{(R_S + j\omega L)(R_C + j\omega L)}{j\omega C(R_S + j\omega L)(R_C + j\omega L) + (R_S + R_C + j2\omega L)}$$

and, by node analysis,

$$v_{OC} = \frac{Z_C \,||\, \left[(Z_{Rs} + Z_L) \,||\, (Z_{Rc} + Z_L)\right]}{Z_{Rs} + Z_L} v_{in} = \frac{(R_C + j\omega L)}{j\omega C(R_S + j\omega L)(R_C + j\omega L) + (R_S + R_C + j2\omega L)} v_{in}$$

$$\frac{v_{out}}{v_{OC}} = \frac{Z_{R_L}}{Z_T + Z_{R_L}} = \frac{R_L}{Z_T + R_L} = \frac{1}{1 + \dfrac{Z_T}{R_L}} =$$

$$\frac{j\omega C(R_S + j\omega L)(R_C + j\omega L) + (R_S + R_C + j2\omega L)}{j\omega C(R_S + j\omega L)(R_C + j\omega L) + (R_S + R_C + j2\omega L) + (R_S + j\omega L)(R_C + j\omega L)/R_L}$$

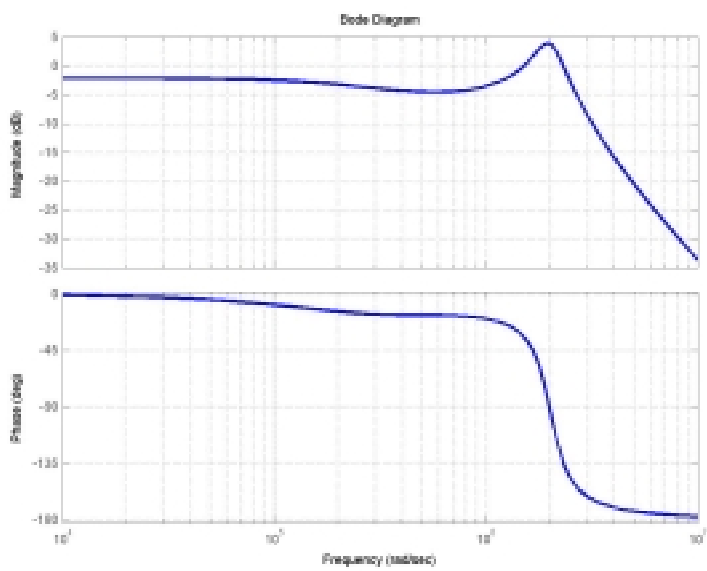
Therefore,

$$\frac{v_{out}}{v_{in}} = \frac{(R_C + j\omega L)}{j\omega C(R_S + j\omega L)(R_C + j\omega L) + (R_S + R_C + j2\omega L) + (R_S + j\omega L)(R_C + j\omega L)/R_L}$$

Substituting the numerical values:

$$\frac{v_{out}}{v_{in}} = \frac{8\times10^{17} + j\left(2\times10^{12}\right)\omega}{(j\omega)^3 + (j\omega)^2 9\times10^5 + j\left(4.24\times10^{12}\right)\omega + 1.016\times10^{18}}$$

The corresponding Bode diagrams are shown below:



The magnitude of the voltage transfer function is highest at the resonant frequency and decreases at higher and lower frequencies. Therefore, this is a band-pass filter.

However, it is not a particularly good filter since the voltage gain [or actually insertion loss] is not very different at the resonant and lower frequencies. This is due to the large inductor losses modeled here as the equivalent resistance R_c. This causes a low "Q" circuit.

Problem 6.54

Solution:

Known quantities:
The values of the resistors, of the capacitance and of the inductance in the circuit of Figure P6.53.

Find:
Compute and plot the frequency response function. What type of filter is this?

Analysis:
First, we find the Thévenin equivalent circuit seen by the capacitor:

$$Z_T = Z_C \,\|\, [(Z_{Rs} + Z_L) \,\|\, (Z_{Rc} + Z_L)] = \left(j\omega C + \frac{1}{R_S + j\omega L} + \frac{1}{R_C + j\omega L} \right)^{-1}$$

$$= \frac{(R_S + j\omega L)(R_C + j\omega L)}{j\omega C(R_S + j\omega L)(R_C + j\omega L) + (R_S + R_C + j2\omega L)}$$

and, by node analysis,

$$v_{OC} = \frac{Z_C \,\|\, [(Z_{Rs} + Z_L) \,\|\, (Z_{Rc} + Z_L)]}{Z_{Rs} + Z_L} v_{in} = \frac{(R_C + j\omega L)}{j\omega C(R_S + j\omega L)(R_C + j\omega L) + (R_S + R_C + j2\omega L)} v_{in}$$

$$\frac{v_{out}}{v_{OC}} = \frac{Z_{R_L}}{Z_T + Z_{R_L}} = \frac{R_L}{Z_T + R_L} = \frac{1}{1 + \dfrac{Z_T}{R_L}} =$$

$$\frac{j\omega C(R_S + j\omega L)(R_C + j\omega L) + (R_S + R_C + j2\omega L)}{j\omega C(R_S + j\omega L)(R_C + j\omega L) + (R_S + R_C + j2\omega L) + (R_S + j\omega L)(R_C + j\omega L)/R_L}$$

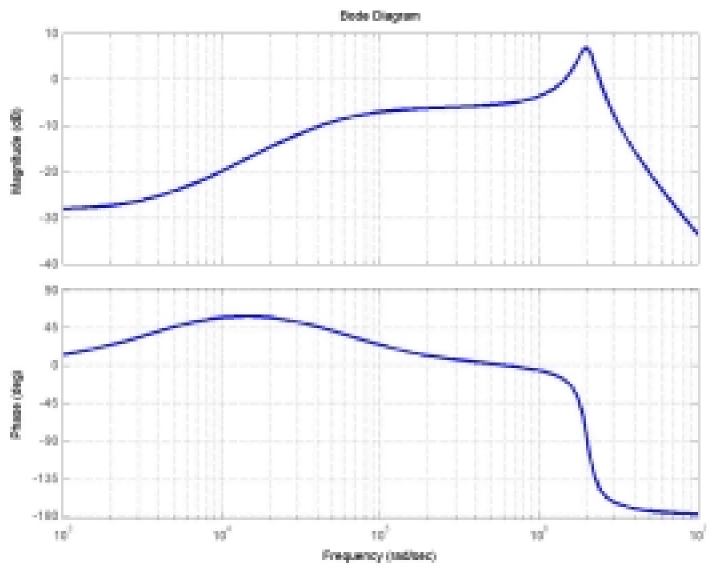
Therefore,

$$\frac{v_{out}}{v_{in}} = \frac{(R_C + j\omega L)}{j\omega C(R_S + j\omega L)(R_C + j\omega L) + (R_S + R_C + j2\omega L) + (R_S + j\omega L)(R_C + j\omega L)/R_L}$$

Substituting the numerical values:

$$\frac{v_{out}}{v_{in}} = \frac{8\times10^{15} + j(2\times10^{12})\omega}{(j\omega)^3 + (j\omega)^2 5.04\times10^5 + j(4.042\times10^{12})\omega + 2.082\times10^{17}}$$

The corresponding Bode diagrams are shown below:



The magnitude of the voltage transfer function is highest at the resonant frequency and decreases at higher and lower frequencies. Therefore, this is a band-pass filter.

Note: The inductor or coil loss is much smaller [$4\ \Omega$] in this circuit, which gives much better band-pass filter performance or a higher "Q" circuit. The magnitude of the voltage transfer ratio or voltage gain [or insertion loss] is much higher at resonance than at higher or lower frequencies.

Problem 6.55

Solution:

Known quantities:

The values of the resistors, of the capacitance and of the inductance in the circuit of Figure P6.55:

$$R_s = 5\,\text{k}\Omega \quad C = 56\,\text{nF} \quad R_L = 100\,\text{k}\Omega \quad L = 9\,\mu\text{H}$$

Find:

a) An expression for the voltage transfer function: $H_v(j\omega) = \dfrac{V_0(j\omega)}{V_i(j\omega)}$

b) The resonant frequency.

c) The half-power frequencies.

d) The bandwidth and Q.

Analysis:

a)

$$Z_{eq} = \frac{1}{\frac{1}{Z_C}+\frac{1}{Z_L}+\frac{1}{Z_{R_L}}} = \frac{1}{j\omega C+\frac{1}{j\omega L}+\frac{1}{R_L}} = \frac{j\omega LR_L}{(j\omega)^2 LCR_L + j\omega L + R_L}$$

$$VD: \ H_v(j\omega) = \frac{V_0(j\omega)}{V_i(j\omega)} = \frac{Z_{eq}}{Z_{R_S}+Z_{eq}} = \frac{\frac{j\omega LR_L}{(j\omega)^2 LCR_L + j\omega L + R_L}}{R_S + \frac{j\omega LR_L}{(j\omega)^2 LCR_L + j\omega L + R_L}}$$

$$H_v(j\omega) = \frac{j\omega LR_L}{(j\omega)^2 LCR_S R_L + j\omega L(R_L+R_S)+R_S R_L} = \frac{1}{R_S}\frac{j\omega L}{(j\omega)^2 LC + j\omega L\left(\frac{R_L+R_S}{R_S R_L}\right)+1}$$

b) The resonance frequency is,

$$\omega_n = \sqrt{\frac{1}{LC}} \cong 1.4086 \text{ Mrad/s.}$$

c)

$$\left|H_v(j\omega_{hp})\right| = \left|\frac{1}{R_S}\frac{j\omega_{hp} L}{(j\omega_{hp})^2 LC + j\omega_{hp} L\left(\frac{R_L+R_S}{R_S R_L}\right)+1}\right| = \frac{1}{\sqrt{2}} \Rightarrow$$

$$\Rightarrow \frac{1}{R_S}\frac{\omega_{hp} L}{\sqrt{\left(1-\omega_{hp}^2 LC\right)^2+\left(\omega_{hp} L\left(\frac{R_L+R_S}{R_S R_L}\right)\right)^2}} = \frac{1}{\sqrt{2}} \Rightarrow$$

$\Rightarrow \omega_{hp1} \cong 1.4069$ Mrad/s and $\omega_{hp2} \cong 1.41028$ Mrad/s.

d) The damping ratio is,

$$\xi = \frac{\omega_n}{2} L\left(\frac{R_L+R_S}{R_S R_L}\right) \cong 0.0013$$

The quality factor is,

$$Q = \frac{1}{2\xi} \cong 375.62$$

The bandwidth is,

$$B = \frac{\omega_n}{Q} \cong 3.75 \text{ Krad/s.}$$

Notes:
1. The absence of coil resistance caused the gain at the resonant frequency to be much higher than at high and low frequencies.
2. The bandwidth is small compared with the resonant frequency and the "Q" is quite large. These are dependent on the "loading" or power dissipation of the source and load resistors and the capacitance.
3. A circuit with a high Q is "selective" since it will pass a very narrow band of frequencies. "High" Q circuits have a Q = 10 or more.

Problem 6.56

Solution:

Known quantities:
The values of the resistors, of the capacitance and of the inductance in the circuit of Figure P6.55:

$$R_s = 5\,\text{k}\Omega \quad C = 0.5\,\text{nF} \quad R_L = 100\,\text{k}\Omega \quad L = 1\,\text{mH}$$

Find:

a) An expression for the voltage transfer function: $H_v(j\omega) = \dfrac{V_0(j\omega)}{V_i(j\omega)}$

b) The resonant frequency.

c) The half-power frequencies.

d) The bandwidth and Q.

Analysis:

a)

$$Z_{eq} = \frac{1}{\frac{1}{Z_C}+\frac{1}{Z_L}+\frac{1}{Z_{R_L}}} = \frac{1}{j\omega C+\frac{1}{j\omega L}+\frac{1}{R_L}} = \frac{j\omega LR_L}{(j\omega)^2 LCR_L + j\omega L + R_L}$$

$$VD:\ H_v(j\omega) = \frac{V_0(j\omega)}{V_i(j\omega)} = \frac{Z_{eq}}{Z_{R_S}+Z_{eq}} = \frac{\dfrac{j\omega LR_L}{(j\omega)^2 LCR_L + j\omega L + R_L}}{R_S + \dfrac{j\omega LR_L}{(j\omega)^2 LCR_L + j\omega L + R_L}}$$

$$H_v(j\omega) = \frac{j\omega LR_L}{(j\omega)^2 LCR_SR_L + j\omega L(R_L+R_S)+R_SR_L} = \frac{1}{R_S}\frac{j\omega L}{(j\omega)^2 LC + j\omega L\left(\dfrac{R_L+R_S}{R_SR_L}\right)+1}$$

b) The resonance frequency is,

$$\omega_n = \sqrt{\frac{1}{LC}} \cong 1.4142 \text{ Mrad/s.}$$

c)

$$\left|H_v(j\omega_{hp})\right| = \left|\frac{1}{R_S}\frac{j\omega_{hp}L}{(j\omega_{hp})^2 LC + j\omega_{hp}L\left(\dfrac{R_L+R_S}{R_SR_L}\right)+1}\right| = \frac{1}{\sqrt{2}} \Rightarrow$$

$$\Rightarrow \frac{1}{R_S}\frac{\omega_{hp}L}{\sqrt{\left(1-\omega_{hp}^2LC\right)^2+\left(\omega_{hp}L\left(\dfrac{R_L+R_S}{R_SR_L}\right)\right)^2}} = \frac{1}{\sqrt{2}} \Rightarrow$$

$$\Rightarrow \omega_{hp1} \cong 1.24 \text{ Mrad/s and } \omega_{hp2} \cong 1.62 \text{ Mrad/s.}$$

d) The damping ratio is,

$$\xi = \frac{\omega_n}{2}L\left(\frac{R_L+R_S}{R_SR_L}\right) \cong 0.1485$$

The quality factor is,

$$Q = \frac{1}{2\xi} \cong 3.3672$$

The bandwidth is,

$$B = \frac{\omega_n}{Q} \cong 420 \text{ Krad/s.}$$

Problem 6.57

Solution:

Known quantities:
The values of the resistors, of the capacitance and of the inductance in the circuit of Figure P6.57.

Find:
Compute and plot the voltage frequency response function. What type of filter is this?

Analysis:
First, we find the Thévenin equivalent circuit seen by the capacitor:

$$Z_T = Z_{Rs} + Z_C \,||\, (Z_{Rc} + Z_L) = R_S + \left(j\omega C + \frac{1}{R_C + j\omega L} \right)^{-1}$$

$$= R_S + \frac{(R_C + j\omega L)}{j\omega C(R_C + j\omega L) + 1} = \frac{(R_C + j\omega L) + R_S[1 + j\omega C(R_C + j\omega L)]}{j\omega C(R_C + j\omega L) + 1}$$

and

$$v_{OC} = v_{in}$$

Therefore,

$$\frac{v_{out}}{v_{in}} = \frac{Z_{R_L}}{Z_T + Z_{R_L}} = \frac{R_L}{Z_T + R_L} = \frac{1}{1 + \frac{Z_T}{R_L}} =$$

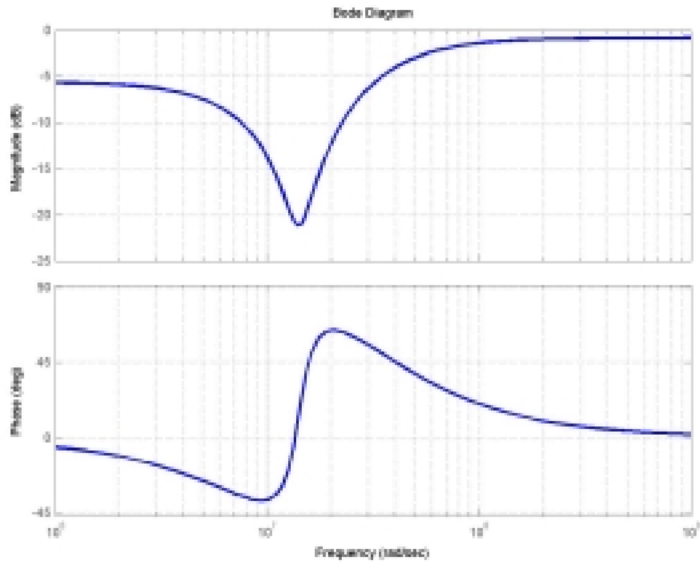
$$= \frac{j\omega C(R_C + j\omega L) + 1}{j\omega C(R_C + j\omega L) + 1 + \{(R_C + j\omega L) + R_S[1 + j\omega C(R_C + j\omega L)]\}/R_L} =$$

$$= \frac{1 + j\omega CR_C + (j\omega)^2 LC}{\left(1 + \frac{R_C + R_S}{R_L}\right) + j\omega\left[CR_C\left(1 + \frac{R_S}{R_L}\right) + \frac{L}{R_L}\right] + (j\omega)^2 LC\left(1 + \frac{R_S}{R_L}\right)}$$

Substituting the numerical values:

$$\frac{v_{out}}{v_{in}} = \frac{1 + j(2\times10^{-8})\omega + (j\omega)^2 5\times10^{-15}}{(j\omega)^2 5.5\times10^{-15} + j(2.22\times10^{-7})\omega + 1.9}$$

The corresponding Bode diagrams are shown below:



The magnitude of the voltage transfer function is lowest at the resonant frequency and increases at higher and lower frequencies. Therefore, this is a band stop or "notch" filter.
At its resonant frequency, a parallel resonant circuit has a high equivalent resistance that is resistive. Connected here in series with the load, this high impedance reduces the magnitude of the voltage transfer function [or voltage gain or insertion loss] at the resonant frequency.
The loading due to the inductor losses, modeled here as an equivalent "coil" resistance, is fairly small giving a substantially lower gain at the resonant frequency compared with the gain at higher or lower frequencies. Therefore this is a high "Q" circuit with good performance and selectivity. The inductor losses also affect only slightly the resonant frequency.
The cutoff frequencies are difficult [but not impossible] to determine in circuits containing a parallel resonant circuit which includes inductor losses, so no attempt was made to do so.

Problem 6.58

Solution:

Known quantities:
The values of the resistors, of the capacitance and of the inductance in the circuit of Figure P6.58.

Find:
Compute and plot the frequency response function.

Analysis:
First, we find the Thévenin equivalent circuit seen by the capacitor:

$$Z_T = Z_{Rs} + Z_C \parallel Z_L = R_S + \left(j\omega C + \frac{1}{j\omega L} \right)^{-1} = \frac{j\omega L + R_S \left[1 + (j\omega)^2 LC \right]}{1 + (j\omega)^2 LC}$$

and

$$v_{OC} = v_{in}$$

Therefore,

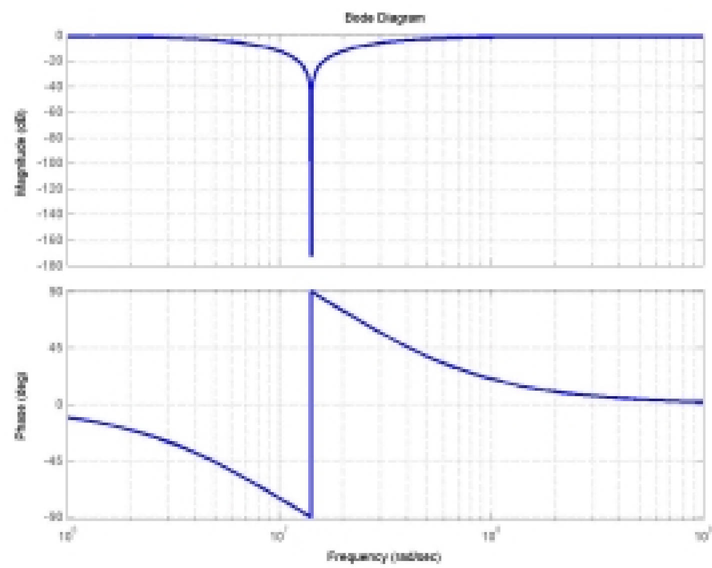
$$\frac{v_{out}}{v_{in}} = \frac{Z_{R_L}}{Z_T + Z_{R_L}} = \frac{R_L}{Z_T + R_L} = \frac{1}{1+\dfrac{Z_T}{R_L}} = \frac{1+(j\omega)^2 LC}{1+(j\omega)^2 LC + \left\{j\omega L + R_S\left[1+(j\omega)^2 LC\right]\right\}/R_L} =$$

$$= \frac{1+(j\omega)^2 LC}{\left(1+\dfrac{R_S}{R_L}\right) + j\omega\dfrac{L}{R_L} + (j\omega)^2 LC\left(1+\dfrac{R_S}{R_L}\right)}$$

Substituting the numerical values:

$$\frac{v_{out}}{v_{in}} = \frac{1+(j\omega)^2 5\times10^{-15}}{(j\omega)^2 5.5\times10^{-15} + j\left(2\times10^{-7}\right)\omega + 1.1}$$

The corresponding Bode diagrams are shown below:



Problem 6.59

Solution:

Known quantities:
The filter circuit shown in Figure P6.58.

Find:
The equation for the voltage transfer function in standard form. Then, if:

$$R_s = 500\,\Omega \quad R_L = 5\,\mathrm{k}\Omega \qquad \omega_n = 12.1278\ \mathrm{M}\frac{\mathrm{rad}}{\mathrm{s}} \quad C = 68\,\mathrm{nF} \quad L = 0.1\,\mu\mathrm{H}$$

determine the cutoff frequencies, the bandwidth, BW, and Q.

Analysis
First, we find the Thévenin equivalent circuit seen by the capacitor:

$$Z_T = Z_{Rs} + Z_C \parallel Z_L = R_S + \left(j\omega C + \frac{1}{j\omega L} \right)^{-1} = \frac{j\omega L + R_S\left[1 + (j\omega)^2 LC\right]}{1 + (j\omega)^2 LC}$$

and

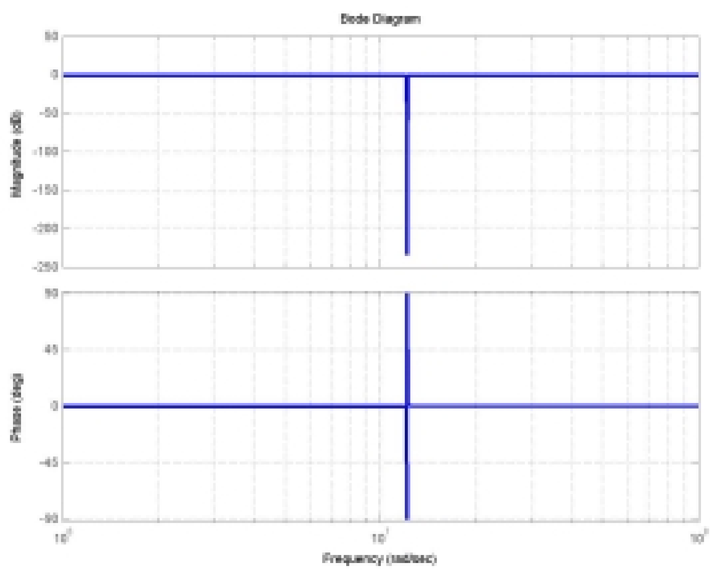
$$v_{OC} = v_{in}$$

Therefore,

$$\frac{v_{out}}{v_{in}} = \frac{Z_{R_L}}{Z_T + Z_{R_L}} = \frac{R_L}{Z_T + R_L} = \frac{1}{1 + \frac{Z_T}{R_L}} = \frac{1 + (j\omega)^2 LC}{\left(1 + \frac{R_S}{R_L}\right) + j\omega \frac{L}{R_L} + (j\omega)^2 LC\left(1 + \frac{R_S}{R_L}\right)}$$

$$= \frac{R_L}{R_S + R_L} \frac{1 - \omega^2 LC}{1 - \omega^2 LC + j\omega \frac{L}{R_S + R_L}} = \frac{R_L}{R_S + R_L} \frac{1}{1 + j\omega \frac{L}{(R_S + R_L)(1 - \omega^2 LC)}}$$

The corresponding Bode diagrams are shown below:



Then, in order to calculate the half-power frequencies, we have to solve:

$$\left|H_v(j\omega_{hp})\right| = \frac{R_L}{R_S + R_L} \frac{1 - \omega_{hp}^2 LC}{\sqrt{\left(1 - \omega_{hp}^2 LC\right)^2 + \left(\omega_{hp} \frac{L}{R_S + R_L}\right)^2}} = \frac{1}{\sqrt{2}} \Rightarrow$$

$\Rightarrow \omega_{hp1} \cong 12.061$ Mrad/s and $\omega_{hp2} \cong 12.194$ Mrad/s.

The damping ratio is,

$$\xi = \frac{\omega_n}{2} \frac{L}{R_L + R_S} \cong 1.1024 \times 10^{-4}$$

The quality factor is,

$$Q = \frac{1}{2\xi} \cong 4535.4$$

The bandwidth is,

$$B = \frac{\omega_n}{Q} \cong 2.6738 \text{ Krad/s.}$$

Problem 6.60

Solution:

Known quantities:
The filter circuit shown in Figure P6.58.

Find:
The equation for the voltage transfer function in standard form. Then, if:

$$R_s = 4.4\,\text{k}\Omega \quad R_L = 60\,\text{k}\Omega \quad \omega_r = 25\ \text{M}\frac{\text{rad}}{\text{s}} \quad C = 0.8\,\text{nF} \quad L = 2\,\mu\text{H}$$

determine the cutoff frequencies, the bandwidth, BW, and Q.

Analysis
First, we find the Thévenin equivalent circuit seen by the capacitor:

$$Z_T = Z_{Rs} + Z_C \parallel Z_L = R_S + \left(j\omega C + \frac{1}{j\omega L} \right)^{-1} = \frac{j\omega L + R_S\left[1 + (j\omega)^2 LC\right]}{1 + (j\omega)^2 LC}$$

and

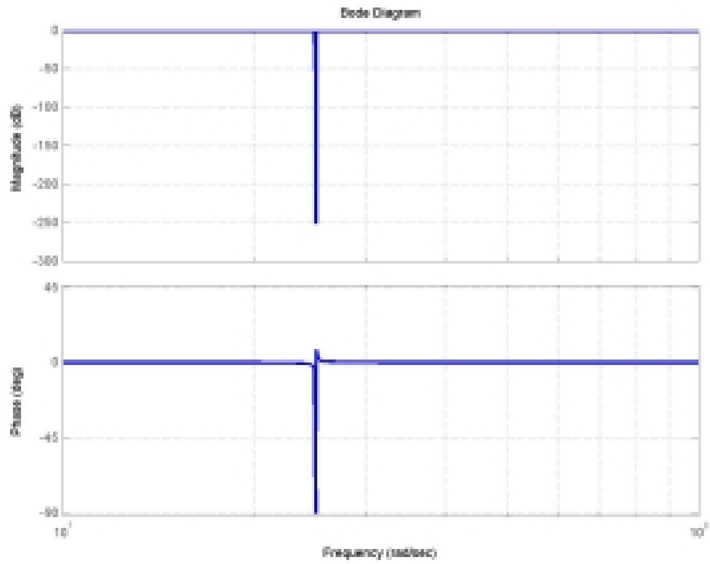
$$v_{OC} = v_{in}$$

Therefore,

$$\frac{v_{out}}{v_{in}} = \frac{Z_{R_L}}{Z_T + Z_{R_L}} = \frac{R_L}{Z_T + R_L} = \frac{1}{1 + \frac{Z_T}{R_L}} = \frac{1 + (j\omega)^2 LC}{\left(1 + \frac{R_S}{R_L}\right) + j\omega\frac{L}{R_L} + (j\omega)^2 LC\left(1 + \frac{R_S}{R_L}\right)}$$

$$= \frac{R_L}{R_S + R_L}\frac{1 - \omega^2 LC}{1 - \omega^2 LC + j\omega\frac{L}{R_S + R_L}} = \frac{R_L}{R_S + R_L}\frac{1}{1 + j\omega\frac{L}{(R_S + R_L)(1 - \omega^2 LC)}}$$

The corresponding Bode diagrams are shown below:



Then, in order to calculate the half-power frequencies, we have to solve:

$$\left|H_v\left(j\omega_{hp}\right)\right| = \frac{R_L}{R_S+R_L}\frac{1-\omega_{hp}^2 LC}{\sqrt{\left(1-\omega_{hp}^2 LC\right)^2+\left(\omega_{hp}\frac{L}{R_S+R_L}\right)^2}} = \frac{1}{\sqrt{2}} \Rightarrow$$

$\Rightarrow \omega_{hp1} \cong 24.8609$ Mrad/s and $\omega_{hp2} \cong 24{,}066$ Mrad/s.

The damping ratio is,

$$\xi = \frac{\omega_n}{2}\frac{L}{R_L+R_S} \cong 3.882\times 10^{-4}$$

The quality factor is,

$$Q = \frac{1}{2\xi} = 1288$$

The bandwidth is,

$$B = \frac{\omega_n}{Q} \cong 19.41 \text{ Krad/s.}$$

Problem 6.61

Solution:

Known quantities:

The bandstop filter circuit shown in Figure P6.61, where:

$$L = 0.4\,\text{mH} \quad R_c = 100\Omega \quad C = 1\,\text{pF} \quad R_s = R_L = 3.8\,\text{k}\Omega$$

Find:

a) An expression for the voltage transfer function or gain in the form:

$$H_v[j\omega] = \frac{V_o[j\omega]}{V_i[j\omega]} = H_o\frac{1+j\,f_1[\omega]}{1+j\,f_2[\omega]}$$

b) The magnitude of the function at high and low frequencies and at the resonant frequency.
c) The resonant frequency.
d) The half power frequencies.

Analysis

a)

$$Z_{eq1} = Z_{Rc} + Z_L + Z_C = R_c + j\left[\omega L - \frac{1}{\omega C}\right]$$

$$Z_{eq} = \frac{Z_{eq1} Z_{RL}}{Z_{eq1} + Z_{RL}} = \frac{\left(R_c + j\left[\omega L - \frac{1}{\omega C}\right]\right) R_L}{\left(R_c + j\left[\omega L - \frac{1}{\omega C}\right]\right) + R_L} = \frac{R_c R_L + j\left[\omega L - \frac{1}{\omega C}\right]}{R_c + R_L + j R_L\left[\omega L - \frac{1}{\omega C}\right]} = \frac{N}{D}$$

$$VD: \quad H_v[j\omega] = \frac{V_o[j\omega]}{V_i[j\omega]} = \frac{Z_{eq}}{Z_{Rs} + Z_{eq}} = \frac{\frac{N}{D}}{R_s + \frac{N}{D}} \frac{D}{D} = \frac{N}{R_s D + N} \qquad (TRICKY)$$

$$\Rightarrow \quad H_v[j\omega] = \frac{R_c R_L + j R_L\left[\omega L - \frac{1}{\omega C}\right]}{R_s\left(R_c + R_L + j\left[\omega L - \frac{1}{\omega C}\right]\right) + \left(R_c R_L + j R_L\left[\omega L - \frac{1}{\omega C}\right]\right)} =$$

$$= \frac{R_c R_L + j R_L\left[\omega L - \frac{1}{\omega C}\right]}{R_s\left[R_c + R_L\right] + R_c R_L + j\left[R_s + R_L\right]\left[\omega L - \frac{1}{\omega C}\right]}$$

$$H_v[j\omega] = \frac{R_c R_L}{R_s\left[R_c + R_L\right] + R_c R_L} \frac{1 + j\frac{1}{R_c}\left[\omega L - \frac{1}{\omega C}\right]}{1 + j\frac{\left[R_s + R_L\right]\left[\omega L - \frac{1}{\omega C}\right]}{R_c\left[R_s + R_L\right] + R_s R_L}}$$

b)

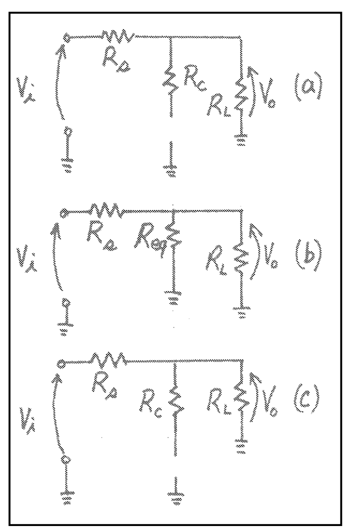
$$As\ \omega \to \infty: \quad Z_L \to \infty \Rightarrow Open$$
$$Z_C \to 0 \Rightarrow Short$$

$$VD: \quad H_v[j\omega] \to \frac{R_L}{R_s + R_L} = 0.5 = -6.021\,\text{dB}$$

$$As\ \omega \to 0: \quad Z_C \to \infty \Rightarrow Open$$
$$Z_L \to 0 \Rightarrow Short$$

$$VD: \quad H_v[j\omega] \to \frac{R_L}{R_s + R_L} = 0.5 = -6.021\,\text{dB}$$

At resonance:



$$H[\ j\omega_r\ ] = H_0 = \frac{R_c R_L}{R_s\ [\ R_c + R_L\ ] + R_c R_L} =$$

$$= \frac{[100][3800]}{[3800][100+3800]+[100][3800]} = 0.025 = -32.00\ \text{dB}$$

c) At the resonant frequency, the transfer function is real. This requires the two functions of frequency to be equal:

$$f_1[\omega_r] = f_2[\omega_r] \qquad \Rightarrow \frac{1}{R_c}[\ \omega_r L - \frac{1}{\omega_r C}\ ] = \frac{[\ R_s + R_L\ ]}{R_c[\ R_s + R_L\ ] + R_s R_L}[\ \omega_r L - \frac{1}{\omega_r C}\ ]$$

$$\Rightarrow \omega_r L - \frac{1}{\omega_r C} = 0 \quad \Rightarrow \quad \omega_r = [\ LC\ ]^{-1/2} = ([\,0.4 \cdot 10^{-3}\,][\,1 \cdot 10^{-12}\,])^{-1/2} = 50\,\text{M}\frac{\text{rad}}{\text{s}}$$

d) Then, in order to calculate the half-power frequencies, we have to solve:

$$\left|H_v\left(j\omega_{hp}\right)\right| = \frac{R_c R_L}{R_s\left(R_c + R_L\right) + R_c R_L} \frac{\sqrt{1+\left(\frac{1}{R_c}\left(\omega L - \frac{1}{\omega C}\right)\right)^2}}{\sqrt{1+\left(\frac{\left(R_c + R_L\right)\left(\omega L - \frac{1}{\omega C}\right)}{R_c\left(R_c + R_L\right) + R_s R_L}\right)^2}} = \frac{1}{\sqrt{2}} \Rightarrow$$

$\Rightarrow \omega_{hp1} \cong 44.9922$ Mrad/s and $\omega_{hp2} \cong 55.565$ Mrad/s.

Problem 6.62

Solution:

Known quantities:
The values of the resistors, of the capacitance and of the inductance in the circuit of Figure P6.55:

$$R_s = 5\,\text{k}\Omega \quad C = 5\,\text{nF} \quad R_L = 50\,\text{k}\Omega \quad L = 2\,\text{mH}$$

Find:

a) An expression for the voltage transfer function: $H_v(j\omega) = \dfrac{V_0(j\omega)}{V_i(j\omega)}$

b) The resonant frequency.
c) The half-power frequencies.
d) The bandwidth and Q.
e) Plot $H_v(j\omega)$.

Analysis:
a)

$$Z_{eq} = \frac{1}{\frac{1}{Z_C} + \frac{1}{Z_L} + \frac{1}{Z_{R_L}}} = \frac{1}{j\omega C + \frac{1}{j\omega L} + \frac{1}{R_L}} = \frac{j\omega L R_L}{(j\omega)^2 LCR_L + j\omega L + R_L}$$

$$VD: \; H_v(j\omega) = \frac{V_0(j\omega)}{V_i(j\omega)} = \frac{Z_{eq}}{Z_{R_S} + Z_{eq}} = \frac{\dfrac{j\omega L R_L}{(j\omega)^2 LCR_L + j\omega L + R_L}}{R_L + \dfrac{j\omega L R_L}{(j\omega)^2 LCR_L + j\omega L + R_L}}$$

$$H_v(j\omega) = \frac{j\omega L R_L}{(j\omega)^2 LCR_S R_L + j\omega L(R_L + R_S) + R_S R_L} = \frac{1}{R_S} \frac{j\omega L}{(j\omega)^2 LC + j\omega L\left(\frac{R_L + R_S}{R_S R_L}\right) + 1}$$

b) The resonance frequency is,

$$\omega_n = \sqrt{\frac{1}{LC}} \cong 316.23 \text{ Krad/s.}$$

c)

$$\left|H_v(j\omega_{hp})\right| = \left|\frac{1}{R_S} \frac{j\omega_{hp} L}{(j\omega_{hp})^2 LC + j\omega_{hp} L\left(\frac{R_L + R_S}{R_S R_L}\right) + 1}\right| = \frac{1}{\sqrt{2}} \Rightarrow$$

$$\Rightarrow \frac{1}{R_S} \frac{\omega_{hp} L}{\sqrt{\left(1 - \omega_{hp}^2 LC\right)^2 + \left(\omega_{hp} L\left(\frac{R_L + R_S}{R_S R_L}\right)\right)^2}} = \frac{1}{\sqrt{2}} \Rightarrow$$

$$\Rightarrow \omega_{hp1} \cong 1.4069 \text{ Mrad/s and } \omega_{hp2} \cong 1.41028 \text{ Mrad/s.}$$

d) The damping ratio is,

$$\xi = \frac{\omega_n}{2} L\left(\frac{R_L + R_S}{R_S R_L}\right) \cong 0.0696$$

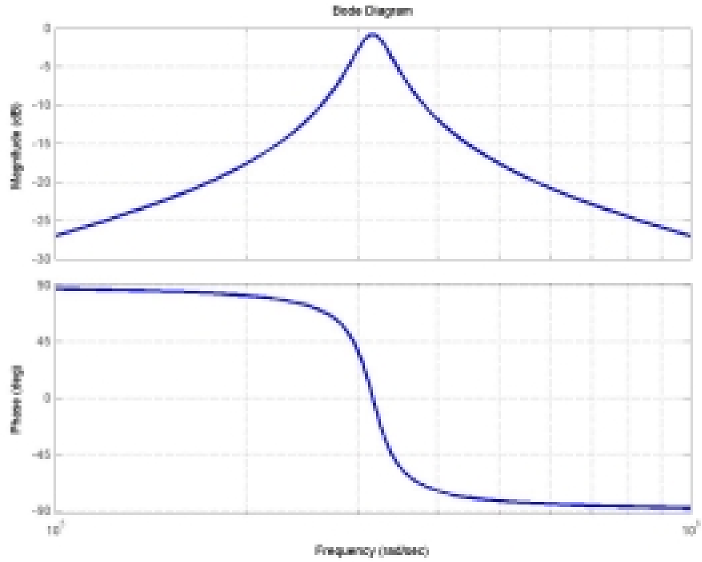
The quality factor is,

$$Q = \frac{1}{2\xi} \cong 7.187$$

The bandwidth is,

$$B = \frac{\omega_n}{Q} \cong 40 \text{ Krad/s.}$$

e) The Bode diagrams are shown below:


Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)

Section 6.4: Bode Plots

Focus on Methodology

Bode plots

This box illustrates the Bode plot asymptotic approximation construction procedure. The method assumes that there are no complex conjugate factors in the response, and that both the numerator and denominator can be factored into first-order terms with real roots.

1. Express the frequency response function in factored form, resulting in an expression similar to equation 6.57:

$$H\ j \qquad \frac{K\ \frac{j}{\ _1}\quad 1 \ldots \frac{j}{\ _m}\quad 1}{\frac{j}{\ _{m\ 1}}\quad 1 \ldots \frac{j}{\ _n}\quad 1}$$

2. Select the appropriate frequency range for the semi-logarithmic plot, extending at least a decade below the lowest 3-dB frequency and a decade above the highest 3-dB frequency.
3. Sketch the magnitude and phase response asymptotic approximations for each of the first-order factors using the techniques illustrated in Figures 6.36 and 6.37.
4. Add, graphically, the individual terms to obtain a composite response.
5. If desired, apply the correction factors of Table 6.2.

Problem 6.63

Solution:

Known quantities:

The values of the resistors, of the capacitance and of the inductance in the circuit of Figure P6.63:

$$R_1 = R_2 = 1\,\text{k}\Omega \quad C_1 = 1\,\mu\text{F} \quad C_2 = 1\,m\text{F} \quad L = 1\,\text{H}$$

Find:

a) The frequency response function $H_v(j\omega) = \dfrac{V_{out}(j\omega)}{V_{in}(j\omega)}$ for the circuit of Figure P6.63.

b) Manually sketch a magnitude and phase Bode plot of the system, using a five-cycle semilog paper.

c) Use Matlab and the Bode command to generate the same plot.

Analysis:

First, we find the Thévenin equivalent circuit seen by the capacitor:

a) $$Z_T = Z_{R_2} + Z_{C_1} \,||\, Z_L \,||\, Z_{R_1} = R_2 + \left(j\omega C_1 + \frac{1}{j\omega L} + \frac{1}{R_1}\right)^{-1} = \frac{(j\omega)^2 LC_1R_2 + j\omega L\left(1+\frac{R_2}{R_1}\right) + R_2}{(j\omega)^2 LC_1 + j\omega\frac{L}{R_1} + 1}$$

and

$$V_{OC} = \frac{j\omega\frac{L}{R_1}}{(j\omega)^2 LC_1 + j\omega\frac{L}{R_1} + 1}V_{in}$$

$$\frac{V_{out}}{V_{OC}} = \frac{1 / j\omega C_2}{Z_T + 1 / j\omega C_2} = \frac{(j\omega)^2 LC_1 + j\omega \frac{L}{R_1} + 1}{(j\omega)^3 LC_1C_2R_2 + (j\omega)^2 L\left[C_1 + C_2\left(1 + \frac{R_2}{R_1}\right)\right] + j\omega\left(\frac{L}{R_1} + C_2R_2\right) + 1}$$

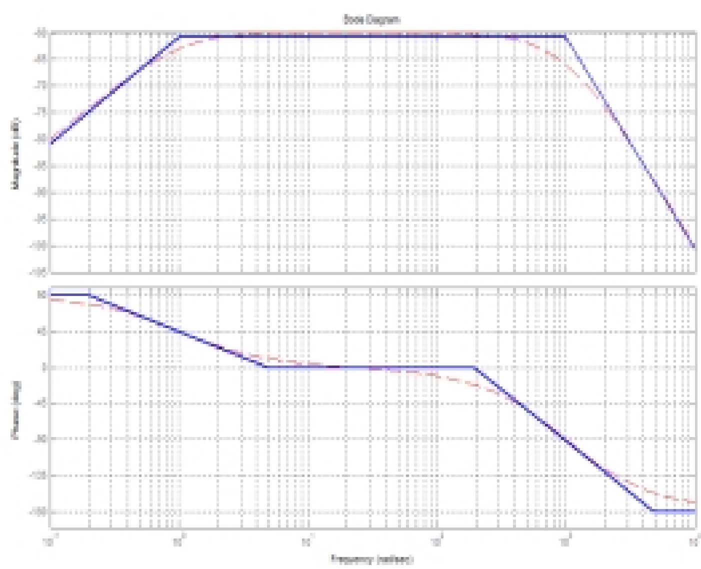
Thus,

$$\frac{V_{out}}{V_{in}} = \frac{j\omega \frac{L}{R_1}}{(j\omega)^3 LC_1C_2R_2 + (j\omega)^2 L\left[C_1 + C_2\left(1 + \frac{R_2}{R_1}\right)\right] + j\omega\left(\frac{L}{R_1} + C_2R_2\right) + 1}$$

b) Substituting the numerical values and expressing the frequency response function in factored form, we have:

$$H_v(j\omega) = 10^{-3} \frac{j\omega}{(j\omega + 1)\left(\frac{j\omega}{968.361} + 1\right)\left(\frac{j\omega}{1031.638} + 1\right)}$$

c) The sketch plots and the ones obtained using Matlab are shown below:



Problem 6.64

Solution:

Known quantities:

The values of the resistors, of the capacitance and of the inductance in the circuit of Figure P6.63:

$$R_1 = R_2 = 1\,\text{k}\Omega \quad C_1 = 1\,\mu\text{F} \quad C_2 = 1\,m\text{F} \quad L = 1\,\text{H}$$

Find:

a) The frequency response function $H_v(j\omega) = \dfrac{I_{out}(j\omega)}{V_{in}(j\omega)}$ for the circuit of Figure P6.63.

b) Manually sketch a magnitude and phase Bode plot of the system, using a five-cycle semilog paper.
c) Use Matlab and the Bode command to generate the same plot.

Analysis:

a) The frequency response function $H_v(j\omega) = \dfrac{V_{out}(j\omega)}{V_{in}(j\omega)}$ is (see P6.63 for details):

$$\frac{V_{out}}{V_{in}} = \frac{j\omega \dfrac{L}{R_1}}{(j\omega)^3 LC_1C_2R_2 + (j\omega)^2 L\left[C_1 + C_2\left(1+\dfrac{R_2}{R_1}\right)\right] + j\omega\left(\dfrac{L}{R_1} + C_2R_2\right) + 1}$$

and,

$$I_{out} = j\omega C_2 V_{out}$$

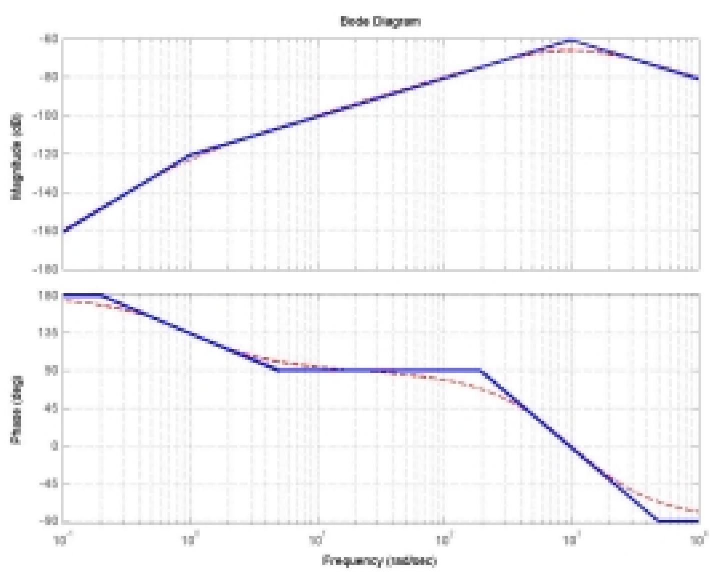
Thus,

$$\frac{I_{out}}{V_{in}} = \frac{(j\omega)^2 \dfrac{LC_2}{R_1}}{(j\omega)^3 LC_1C_2R_2 + (j\omega)^2 L\left[C_1 + C_2\left(1+\dfrac{R_2}{R_1}\right)\right] + j\omega\left(\dfrac{L}{R_1} + C_2R_2\right) + 1}$$

b) Substituting the numerical values and expressing the frequency response function in factored form, we have:

$$H(j\omega) = 10^{-6}\frac{(j\omega)^2}{(j\omega+1)\left(\dfrac{j\omega}{968.361}+1\right)\left(\dfrac{j\omega}{1031.638}+1\right)}$$

c) The sketch plots and the ones obtained using Matlab are shown below:



Problem 6.65

Solution:

Known quantities:
The values of the resistors, of the capacitance and of the inductance in the circuit of Figure P6.65:

$$R_1 = R_2 = 1\,\text{k}\Omega \quad C = 1\,m\text{F} \quad L = 1\,\text{H}$$

Find:

a) The frequency response function $H_v(j\omega) = \dfrac{V_{out}(j\omega)}{I_{in}(j\omega)}$ for the circuit of Figure P6.65.

b) Manually sketch a magnitude and phase Bode plot of the system, using a five-cycle semilog paper.

c) Use Matlab and the Bode command to generate the same plot.

Analysis:
First, we find the Thévenin equivalent circuit seen by the capacitor:

a)

$$Z_T = \left(Z_{R_1} + Z_{C_1}\right) \| Z_L = \left(R_1 + \frac{1}{j\omega C}\right) \| j\omega L = \left(\frac{j\omega C}{1 + j\omega CR_1} + \frac{1}{j\omega L}\right)^{-1} =$$

$$= \frac{(j\omega)^2 LCR_1 + j\omega L}{(j\omega)^2 LC + j\omega CR_1 + 1}$$

and

$$\frac{V_{OC}}{I_{in}} = \frac{j\omega L}{(j\omega)^2 LC + j\omega CR_1 + 1}$$

$$\frac{V_{out}}{V_{OC}} = \frac{R_2}{Z_T + R_2} = \frac{(j\omega)^2 LC + j\omega CR_1 + 1}{(j\omega)^2 LC\left(1 + \frac{R_1}{R_2}\right) + j\omega\left(CR_1 + \frac{L}{R_2}\right) + 1}$$

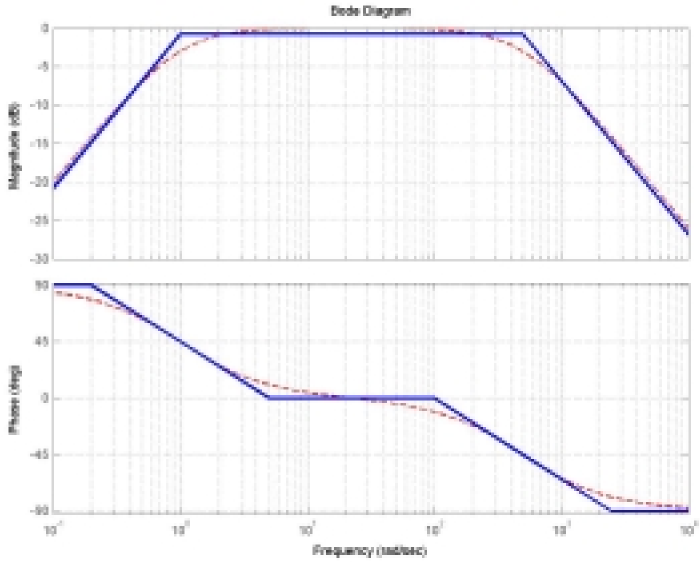
Thus,

$$\frac{V_{out}}{I_{in}} = \frac{j\omega L}{(j\omega)^2 LC\left(1 + \frac{R_1}{R_2}\right) + j\omega\left(CR_1 + \frac{L}{R_2}\right) + 1}$$

b) Substituting the numerical values and expressing the frequency response function in factored form, we have:

$$H_v(j\omega) = \frac{j\omega}{(j\omega + 1)\left(\frac{j\omega}{499.5} + 1\right)}$$

c) The sketch plots and the ones obtained using Matlab are shown below:



Problem 6.66

Solution:

Known quantities:
The values of the resistors, of the capacitance and of the inductance in the circuit of Figure P6.65:

$$R_1 = R_2 = 1\,\text{k}\Omega \qquad C = 1\,m\text{F} \qquad L = 1\,\text{H}$$

Find:

a) The frequency response function $H_v(j\omega) = \dfrac{I_{out}(j\omega)}{I_{in}(j\omega)}$ for the circuit of Figure P6.65.

b) Manually sketch a magnitude and phase Bode plot of the system, using a five-cycle semilog paper.

c) Use Matlab and the Bode command to generate the same plot.

Analysis:

a) The frequency response function $H_v(j\omega) = \dfrac{V_{out}(j\omega)}{I_{in}(j\omega)}$ is (see P6.65 for details):

$$\frac{V_{out}}{I_{in}} = \frac{j\omega L}{(j\omega)^2 LC\left(1+\frac{R_1}{R_2}\right) + j\omega\left(CR_1 + \frac{L}{R_2}\right) + 1}$$

and,

$$I_{out} = \frac{V_{out}}{R_2}$$

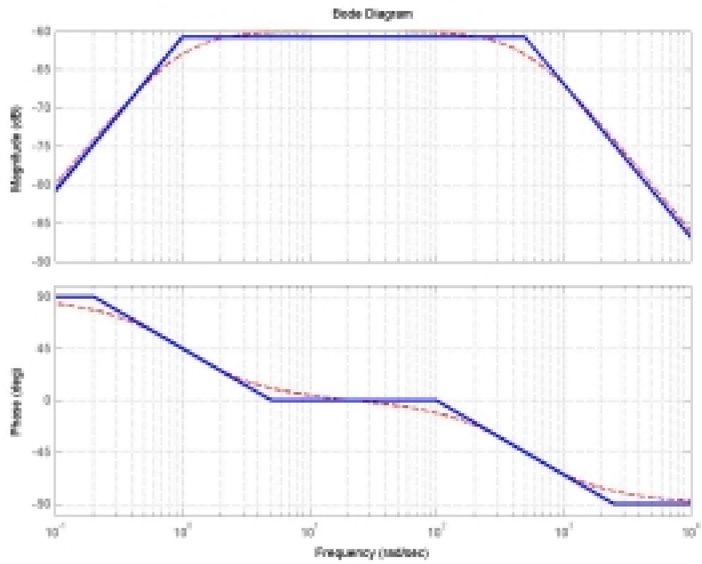
Thus,

$$\frac{I_{out}}{I_{in}} = \frac{j\omega\frac{L}{R_2}}{(j\omega)^2 LC\left(1+\frac{R_1}{R_2}\right) + j\omega\left(CR_1 + \frac{L}{R_2}\right) + 1}$$

b) Substituting the numerical values and expressing the frequency response function in factored form, we have:

$$H_v(j\omega)=10^{-3}\frac{j\omega}{(j\omega+1)\left(\frac{j\omega}{499.5}+1\right)}$$

c) The sketch plots and the ones obtained using Matlab are shown below:



Problem 6.67

Solution:

Known quantities:
The values of the resistors and of the capacitances in the circuit of Figure P6.67:

$$R_1 = R_2 = 1\,\text{k}\Omega \qquad C_1 = 1\,\mu\text{F} \qquad C_2 = 1\,\text{mF}$$

Find:

a) The frequency response function $H_v(j\omega) = \dfrac{V_{out}(j\omega)}{I_{in}(j\omega)}$ for the circuit of Figure P6.65.

b) Manually sketch a magnitude and phase Bode plot of the system, using a five-cycle semilog paper.
c) Use Matlab and the Bode command to generate the same plot.

Analysis:
First, we find the Thévenin equivalent circuit seen by the capacitor:

a) $$Z_T = Z_{R_2} + Z_{C_1} \,||\, Z_{R_1} = R_2 + \left(j\omega C_1 + \frac{1}{R_1}\right)^{-1} = R_2 + \frac{R_1}{1+j\omega C_1 R_1} = \frac{j\omega C_1 R_1 R_2 + R_1 + R_2}{1+j\omega C_1 R_1}$$

and

$$\frac{V_{OC}}{I_{in}} = \frac{R_1}{1+j\omega C_1 R_1}$$

$$\frac{V_{out}}{V_{OC}} = \frac{1/j\omega C_2}{Z_T + 1/j\omega C_2} = \frac{j\omega C_2}{1 + j\omega C_2\left[\dfrac{j\omega C_1 R_1 R_2 + R_1 + R_2}{1 + j\omega C_1 R_1}\right]} =$$

$$= \frac{(1 + j\omega C_1 R_1) j\omega C_2}{(j\omega)^2 C_1 C_2 R_1 R_2 + j\omega[C_1 R_1 + C_2(R_1 + R_2)] + 1}$$

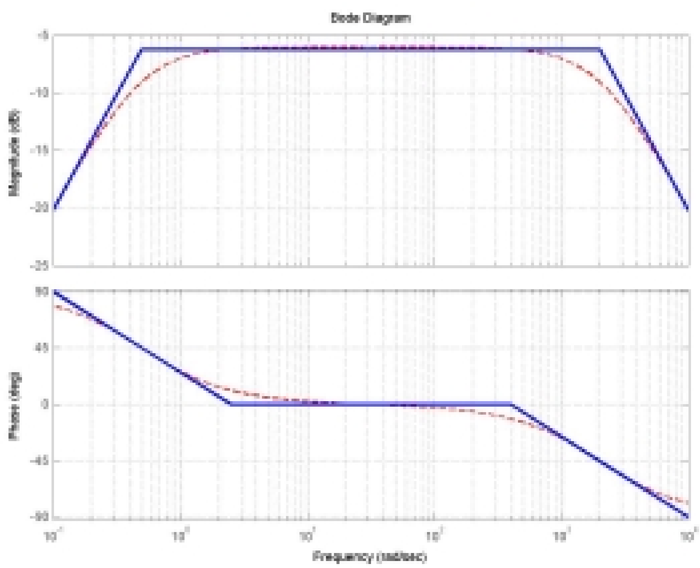
Thus,

$$\frac{V_{out}}{I_{in}} = \frac{j\omega C_2 R_1}{(j\omega)^2 C_1 C_2 R_1 R_2 + j\omega[C_1 R_1 + C_2(R_1 + R_2)] + 1}$$

b) Substituting the numerical values and expressing the frequency response function in factored form, we have:

$$H_v(j\omega) = \frac{j\omega}{\left(\dfrac{j\omega}{0.5} + 1\right)\left(\dfrac{j\omega}{2000} + 1\right)}$$

c) The sketch plots and the ones obtained using Matlab are shown below:



Problem 6.68

Solution:

Known quantities:

The values of the resistors and of the capacitances in the circuit of Figure P6.67:

$$R_1 = R_2 = 1\,\text{k}\Omega \quad C_1 = 1\,\mu\text{F} \quad C_2 = 1\,\text{mF}$$

Find:

a) The frequency response function $H_v(j\omega) = \dfrac{V_{out}(j\omega)}{I_{in}(j\omega)}$ for the circuit of Figure P6.65.

b) Manually sketch a magnitude and phase Bode plot of the system, using a five-cycle semilog paper.

c) Use Matlab and the Bode command to generate the same plot.

Analysis:

a) The frequency response function $H_v(j\omega) = \dfrac{V_{out}(j\omega)}{I_{in}(j\omega)}$ is (see P6.67 for details):

$$\frac{V_{out}}{I_{in}} = \frac{j\omega C_2 R_1}{(j\omega)^2 C_1 C_2 R_1 R_2 + j\omega[C_1 R_1 + C_2(R_1 + R_2)] + 1}$$

and,

$$I_{out} = j\omega C_2 V_{out}$$

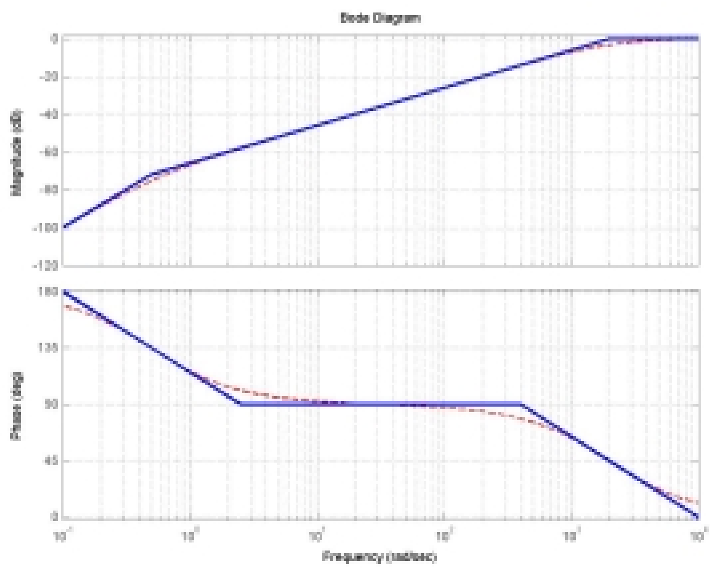
Thus,

$$\frac{I_{out}}{I_{in}} = \frac{(j\omega C_2)^2 R_1}{(j\omega)^2 C_1 C_2 R_1 R_2 + j\omega[C_1 R_1 + C_2(R_1 + R_2)] + 1}$$

b) Substituting the numerical values and expressing the frequency response function in factored form, we have:

$$H_v(j\omega) = 10^{-3} \frac{(j\omega)^2}{\left(\dfrac{j\omega}{0.5} + 1\right)\left(\dfrac{j\omega}{2000} + 1\right)}$$

c) The sketch plots and the ones obtained using Matlab are shown below:



Problem 6.69

Solution:

Known quantities:
Resistance, inductance and capacitance values, in the circuit of Figure P6.4.

Find:
a) Manually sketch a magnitude and phase Bode plot of the system, using a five-cycle semilog paper.
b) Use Matlab and the Bode command to generate the same plot.

Analysis:

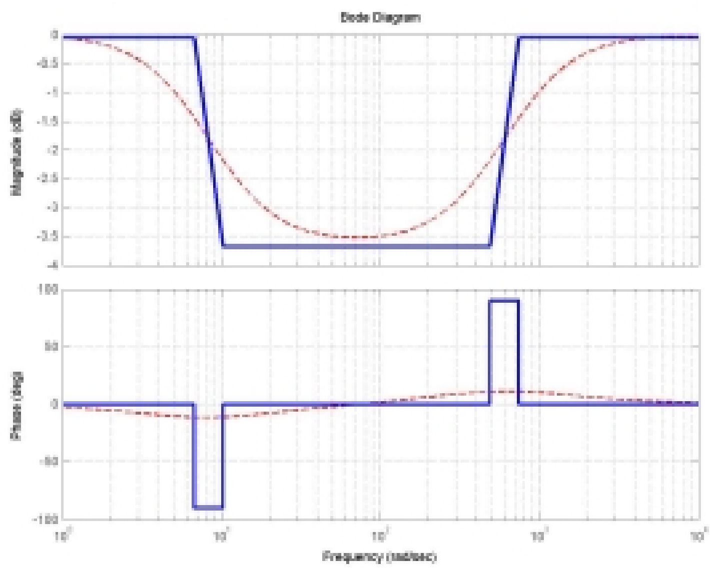
The frequency response function $H_v(j\omega) = \frac{V_{out}(j\omega)}{V_{in}(j\omega)}$ is (see P6.4 for details):

$$\frac{V_{out}}{V_{in}}(j\omega) = \frac{1 + j\omega CR_2 + (j\omega)^2 LC}{1 + j\omega C(R_1 + R_2) + (j\omega)^2 LC}$$

a) Substituting the numerical values and expressing the frequency response function in factored form, we have:

$$H_v(j\omega) = \frac{\left(\frac{j\omega}{489.79} + 1\right)\left(\frac{j\omega}{10.21} + 1\right)}{\left(\frac{j\omega}{743.27} + 1\right)\left(\frac{j\omega}{6.72} + 1\right)}$$

b) The sketch plots and the ones obtained using Matlab are shown below:



Problem 6.70

Solution:

Known quantities:
Resistance, inductance and capacitance values, in the circuit of Figure P6.5.

Find:
a) Manually sketch a magnitude and phase Bode plot of the system, using a five-cycle semilog paper.
b) Use Matlab and the Bode command to generate the same plot.

Assume:
Assume that the output voltage is the voltage across the resistor.

Analysis:

The frequency response function $H_V(j\omega) = \dfrac{V_{out}(j\omega)}{V_{in}(j\omega)}$ is (see P6.5 for details):

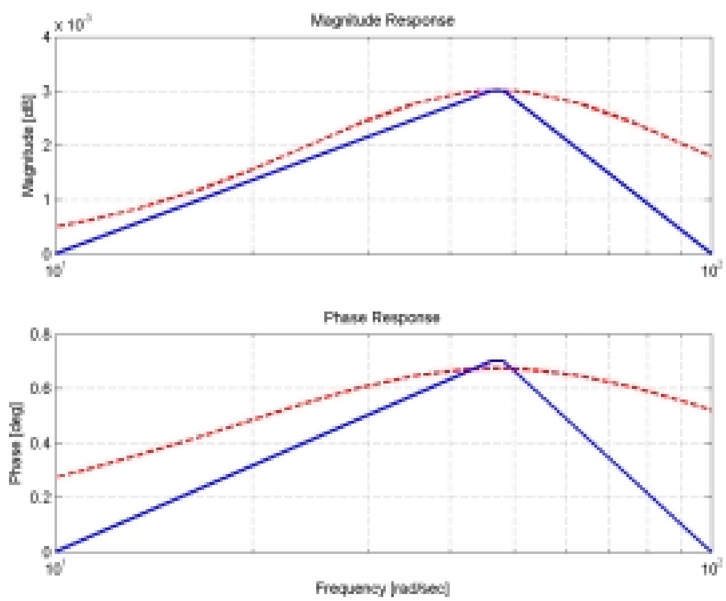
$$H_V(j\omega) = \frac{V_{out}(j\omega)}{V_{in}(j\omega)} = \frac{R_2 - CLR_2\omega^2 + j\omega L}{R_2 + R_1 - CL(R_2 + R_1)\omega^2 + j\omega L}$$

a) Substituting the numerical values and expressing the frequency response function in factored form, we have:

$$H_V(j\omega) = \frac{\left(1 - 4.5\times10^{-4}\omega^2\right) + j(0.0015)\omega}{\left(1 - 4.5\times10^{-4}\omega^2\right) + j(0.0010)\omega}$$

$$H_V(j\omega) = \frac{-1.004\left(\dfrac{j\omega}{48.84}+1\right)\left(\dfrac{j\omega}{45.54}+1\right)}{\left(\dfrac{j\omega}{48.26}+1\right)\left(\dfrac{j\omega}{46.04}+1\right)}$$

b) The sketch plots and the ones obtained using Matlab are shown below:



Problem 6.71

Solution:

Known quantities:
Resistance, inductance and capacitance values, in the circuit of Figure P6.5.

Find:
a) Manually sketch a magnitude and phase Bode plot of the system, using a five-cycle semilog paper.
b) Use Matlab and the Bode command to generate the same plot.

Analysis:

The frequency response function $H_v(j\omega) = \dfrac{V_{out}(j\omega)}{V_{in}(j\omega)}$ is (see P6.5 for details):

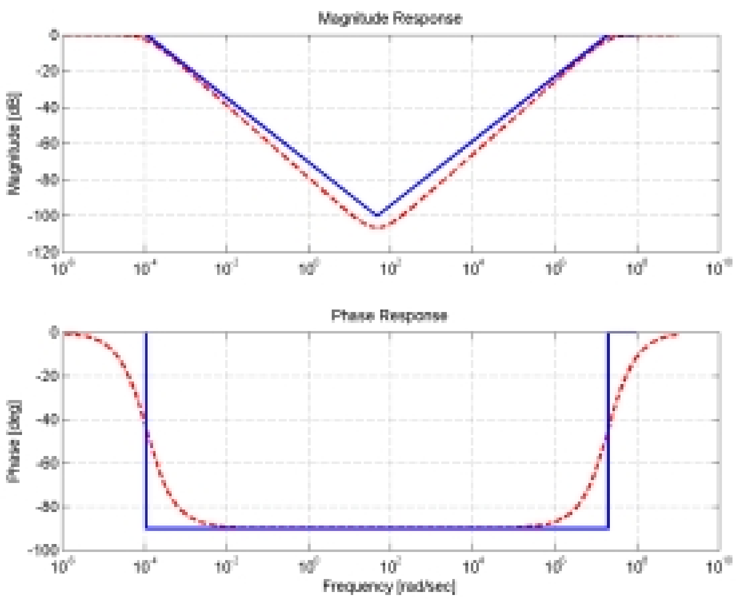
$$\frac{V_{out}}{V_{in}}(j\omega) = \frac{V_C}{V_{in}}(j\omega) = \frac{1 - CL\omega^2}{1 - CL\omega^2 + j\omega L(R_2 + R_1)}$$

a) Substituting the numerical values and expressing the frequency response function in factored form, we have:

$$H_V(j\omega) = \frac{\left(1 - 4.5\times10^{-4}\omega^2\right)}{\left(1 - 4.5\times10^{-4}\omega^2\right) + j(9000)\omega}$$

$$H_V(j\omega) = \frac{\left(\dfrac{j\omega}{47.14}+1\right)\left(\dfrac{j\omega}{47.14}+1\right)}{\left(\dfrac{j\omega}{2e7}+1\right)\left(\dfrac{j\omega}{1.11e-4}+1\right)}$$

b) The sketch plots and the ones obtained using Matlab are shown below:



Problem 6.72

Solution:

Known quantities:
Resistance, inductance and capacitance values, in the circuit of Figure P6.4.

Find:

a) Manually sketch a magnitude and phase Bode plot of the system, using a five-cycle semilog paper.
b) Use Matlab and the Bode command to generate the same plot.

Analysis:

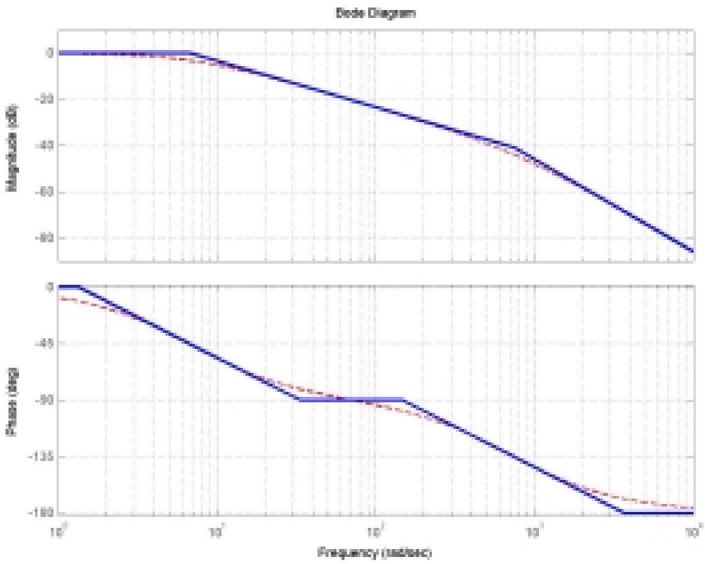
The frequency response function is:

$$\frac{V_{out}}{V_{in}}(j\omega) = \frac{1}{1 + j\omega C(R_1 + R_2) + (j\omega)^2 LC}$$

a) Substituting the numerical values and expressing the frequency response function in factored form, we have:

$$H_v(j\omega) = \frac{1}{\left(\frac{j\omega}{743.27}+1\right)\left(\frac{j\omega}{6.72}+1\right)}$$

b) The sketch plots and the ones obtained using Matlab are shown below:



Problem 6.73

Solution:

Known quantities:
Resistance and capacitance values, in the circuit of Figure P6.6.

Find:
a) Manually sketch a magnitude and phase Bode plot of the system, using a five-cycle semilog paper.
b) Use Matlab and the Bode command to generate the same plot.

Analysis:

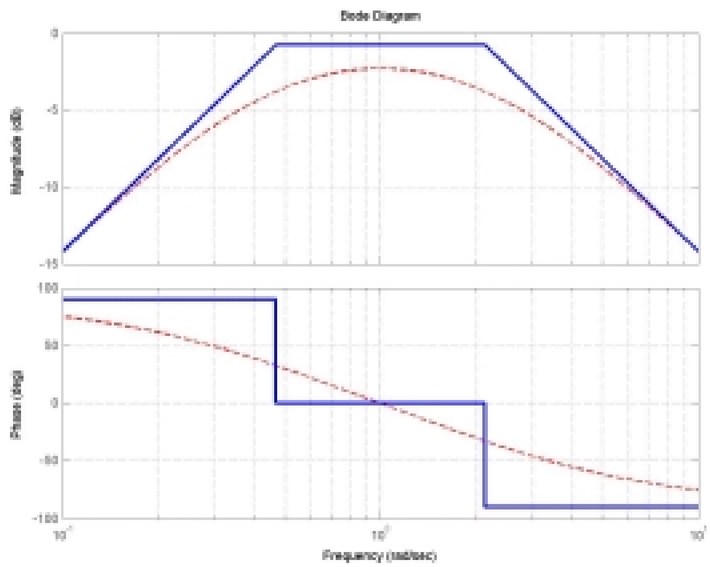
The frequency response function $H_v(j\omega) = \frac{V_{out}(j\omega)}{V_{in}(j\omega)}$ is (see P6.6 for details):

$$\frac{V_{out}}{V_{in}}(j\omega) = \frac{j\omega C_1 R_1}{1 + j\omega[C_1R_1 + C_2(R_1+R_2)] + (j\omega)^2 C_1C_2R_1R_2}$$

a) Substituting the numerical values and expressing the frequency response function in factored form, we have:

$$H_v(j\omega) = \frac{2j\omega}{(j\omega)^2 + 2.6j\omega + 1} = 2\frac{j\omega}{\left(\frac{j\omega}{2.13}+1\right)\left(\frac{j\omega}{0.47}+1\right)}$$

b) The sketch plots and the ones obtained using Matlab are shown below:



Problem 6.74

Solution:

Known quantities:

Ratio of output amplitude to input amplitude, being proportional to $1/\omega^3$ in a certain frequency range.

Find:

The slope of the Bode plot in this frequency range, expressed in dB per decade.

Analysis:

If $\left|\dfrac{v_{out}}{v_{in}}\right| \propto \dfrac{1}{\omega^3}$, it is seen that the amplitude is reduced by a factor of 1000, or multiplied by $\dfrac{1}{1000}$, every time the frequency increases by a factor of 10. Since $\dfrac{1}{1000}$ is a -60 dB gain, we speak of the transfer function rolling of at a $-60\ \dfrac{\text{dB}}{\text{decade}}$ slope. The term "decade" refers to a frequency factor of 10.

Problem 6.75

Solution:

The output amplitude of a given circuit as a function of frequency:

$$|V| = \frac{A\omega + B}{\sqrt{C + D\omega^2}}$$

Find:

a) The break frequency.
b) The slope of the Bode plot (in dB per decade) above the break frequency.

c) The slope of the Bode plot below the break frequency.
d) The high-frequency limit of V.

Analysis:

a) Given $|V| = \dfrac{A\omega + B}{\sqrt{C + D\omega^2}}$, this is seen to rise from $\dfrac{B}{\sqrt{C}}$ at zero frequency to $\dfrac{A}{\sqrt{D}}$ at high frequencies. The corresponding complex phasor function is:

$$V = \frac{j\omega A + B}{\sqrt{C} + j\sqrt{D}\omega} = \frac{\frac{B}{\sqrt{C}} + j\omega\frac{A}{\sqrt{C}}}{1 + j\sqrt{\frac{D}{C}}\omega} = \frac{\frac{B}{\sqrt{C}} + j\omega\frac{A}{\sqrt{C}}}{1 + \frac{j\omega}{\sqrt{\frac{C}{D}}}} = \frac{\left(1 + j\omega\frac{A}{B}\right)\frac{B}{\sqrt{C}}}{1 + \frac{j\omega}{\sqrt{\frac{C}{D}}}}$$

which we recognize to have a break frequency (or cut-off frequency, or half-power frequency) of:

$$\omega_{CO} = \sqrt{\frac{C}{D}}$$

b) At high frequencies the slope is zero and the magnitude is equal to $\dfrac{A}{\sqrt{D}}$

c) At low frequencies the slope is zero and the magnitude is equal to $\dfrac{B}{\sqrt{C}}$

d) At high frequencies, $|V| \to \dfrac{A}{\sqrt{D}}$

Problem 6.76

Solution:

Known quantities:
Figures P6.76a and P6.76b.

Find:
An expression for the equivalent impedance in Figure P6.76a in standard form. Choose the Bode plot, from Figure P6.76b, that best describes the behavior of the impedance as a function of frequency and describe how (a simple one line statement and no analysis is sufficient) you would obtain the resonant and cutoff frequencies and the magnitude of the impedance where it is constant over some frequency range. Label the Bode plot to indicate which feature you are discussing.

Analysis:
In standard form.

$$Z[j\omega] = \frac{Z_C[Z_{Rc} + Z_L]}{Z_C + Z_{Rc} + Z_L} = \frac{\frac{1}{j\omega C}[R_c + j\omega L]}{\frac{1}{j\omega C} + R_c + j\omega L}\,\frac{j\omega C}{j\omega C} =$$

$$= \frac{R_c + j\omega L}{[1 - \omega^2 LC] + j\omega R_c C} = \frac{R_c}{1 - \omega^2 LC}\,\frac{1 + j\frac{\omega L}{R_c}}{1 + j\frac{\omega R_c C}{1 - \omega^2 LC}} = Z_o\frac{1 + j f_1[\omega]}{1 + j f_2[\omega]}$$

Bode Plot [b]. The circuit is a parallel resonant circuit and should exhibit maxima of impedance and minima of impedance at low and high frequencies.

1. At the resonant frequency, the impedance is real, i.e., the reactive part is zero.
2. The magnitude of the impedance at the resonant frequency is Z_o evaluated at the resonant frequency.
3. There are three cutoff frequencies [the 3 dB] frequencies evaluated by making the functions of frequency equal to +1 or - 1.
4. The magnitude of the impedance when the frequency is low can be determined in two ways. First, the circuit can be modeled at low frequencies by replacing the inductor with a short circuit and the capacitor with an open circuit. Under these conditions the impedance is equal to that of the resistor. Or the limit of the impedance as the frequency approaches zero can be determined.

1. $f_1[\omega_r] = f_2[\omega_r]$ ==> Solve for ω_r

2. $Z_o = \dfrac{R_c}{1 - \omega_r^2 LC}$

3. $f_1[\omega_c] = 1$ Gives ω_{c3}.
$f_2[\omega_c] = \pm 1$ Gives ω_{c1} and ω_{c2}.

4. $Z_o = \lim Z[j\omega]_{\omega \to 0} = R_c$