Chapter 8
Multiple Integrals

Overview

- Double Integrals
  - Properties of Double Integrals
- Evaluation
  - Rectangular Regions
  - Type A region
  - Type B region

Overview

- Double Integrals in Polar Coordinates
  - Circle
  - Ring
  - Sector of a Circle
  - Polar Rectangular
  - Change of Variables

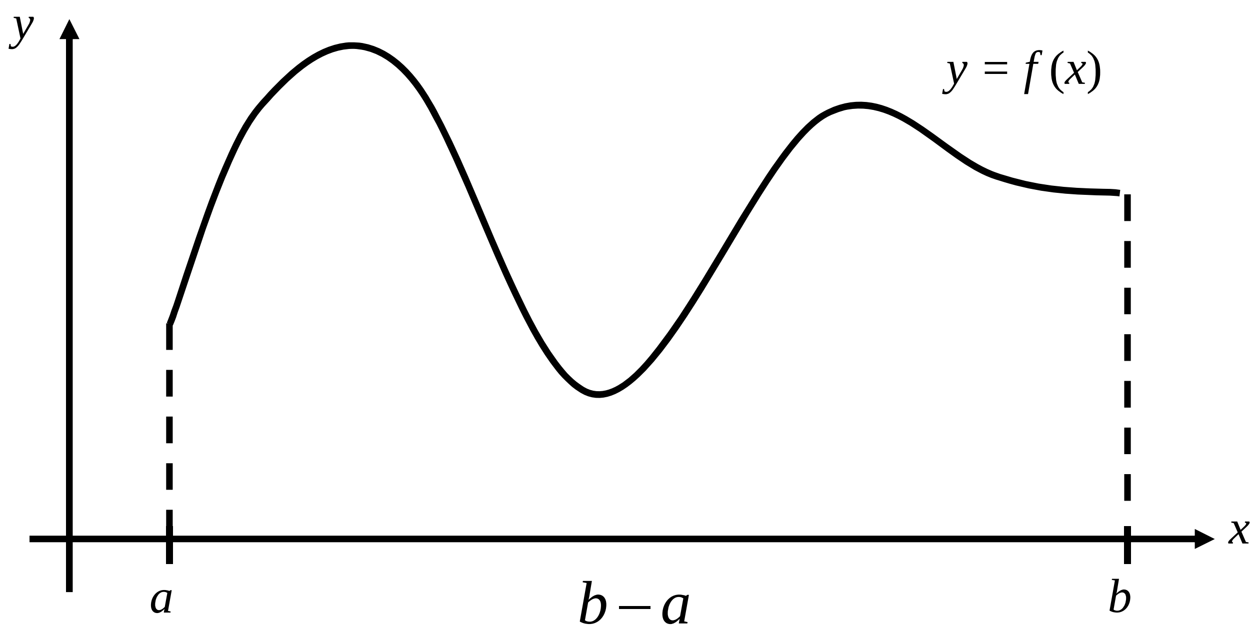
Overview

- <u>Application of Double Integrals</u>
  - Volume
  - Surface Area
  - Mass and Center of Gravity

- <u>Triple Integral</u>
  - Physical Meaning
  - Rectangular Region

Double Integrals

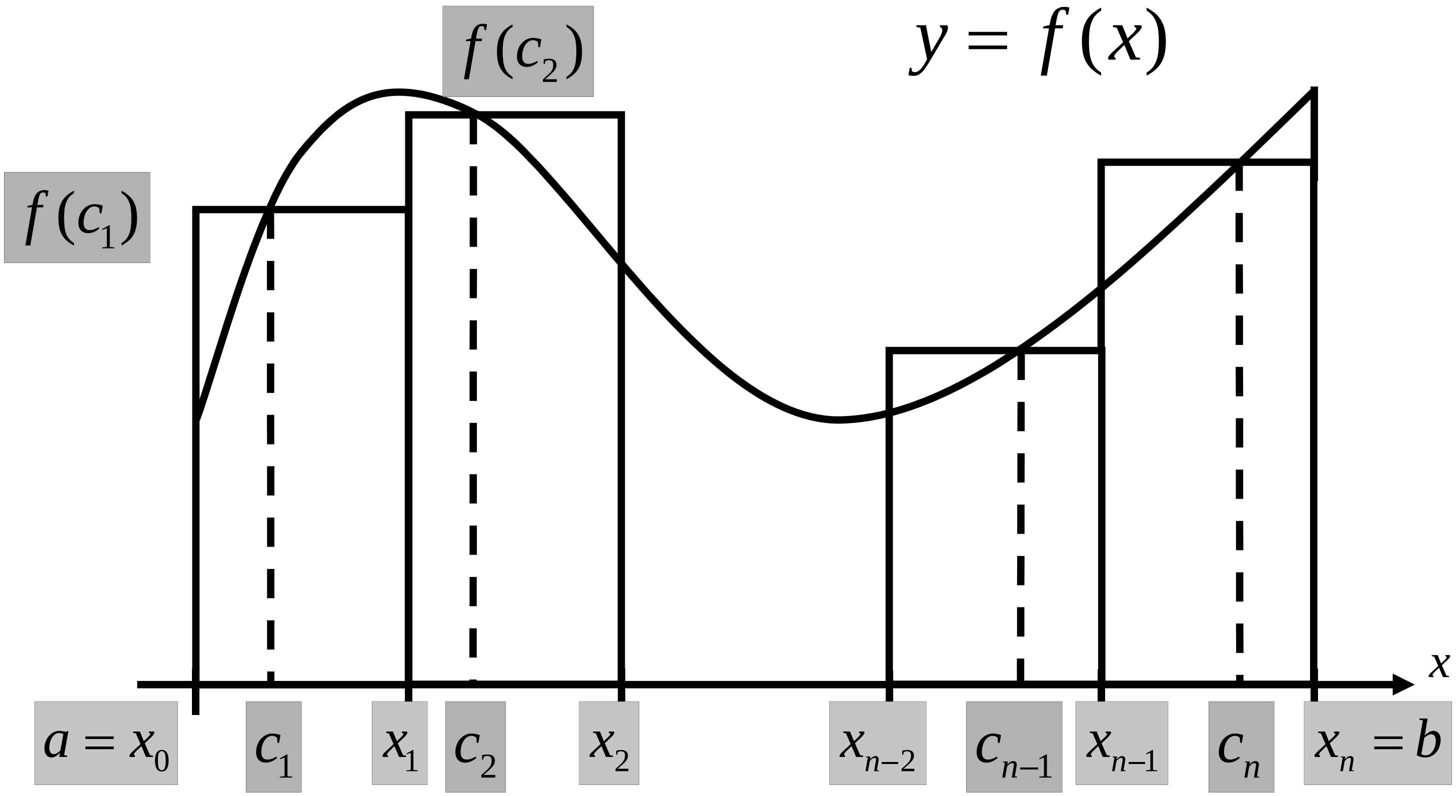
Integrals



Area under curve

$$A = \int_a^b f(x)\,dx$$

Riemann Integrals

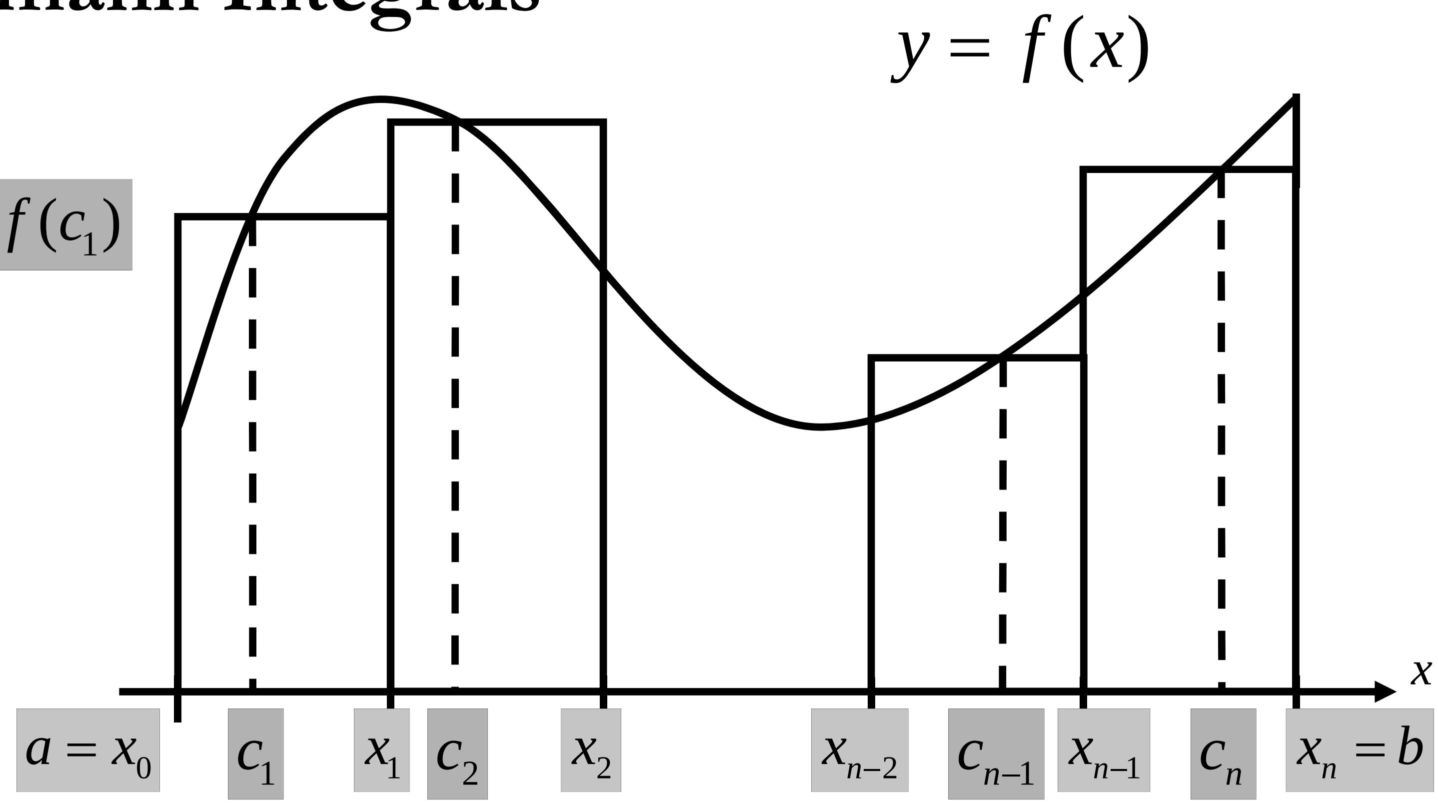


Divide $[a,b]$ into n equal intervals

$$\text{Length of each interval} = \Delta x = \frac{b-a}{n}$$

$$\text{Area of rectangles } = f(c_1)\Delta x + f(c_2)\Delta x + \cdots + f(c_n)\Delta x$$

Riemann Integrals

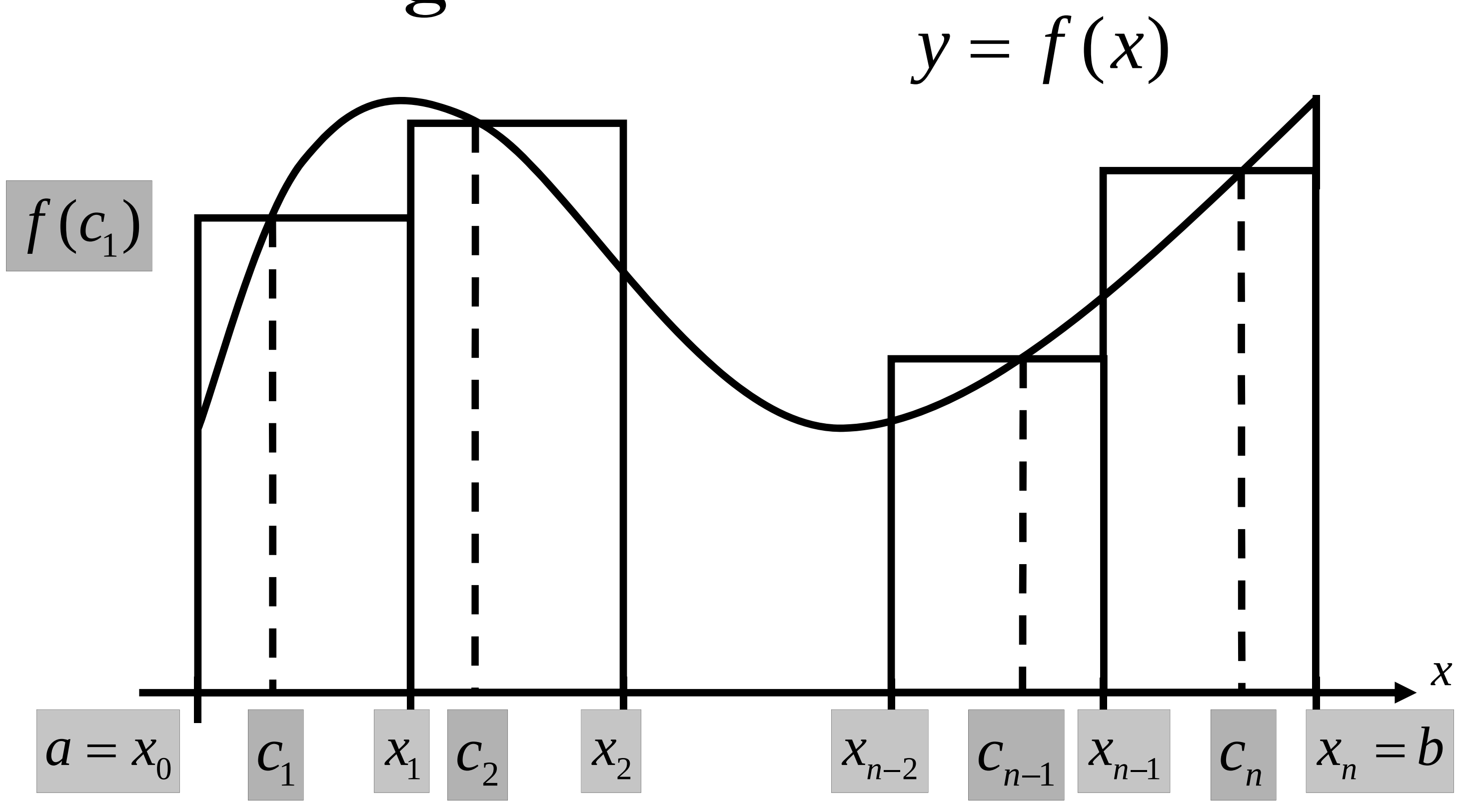


The ***area*** under the curve of $y = f(x)$ from a to b

$$\approx \sum_{k=1}^{n} f(c_k)\,\Delta x$$

Riemann sum of f on $[a,b]$

Riemann Integrals

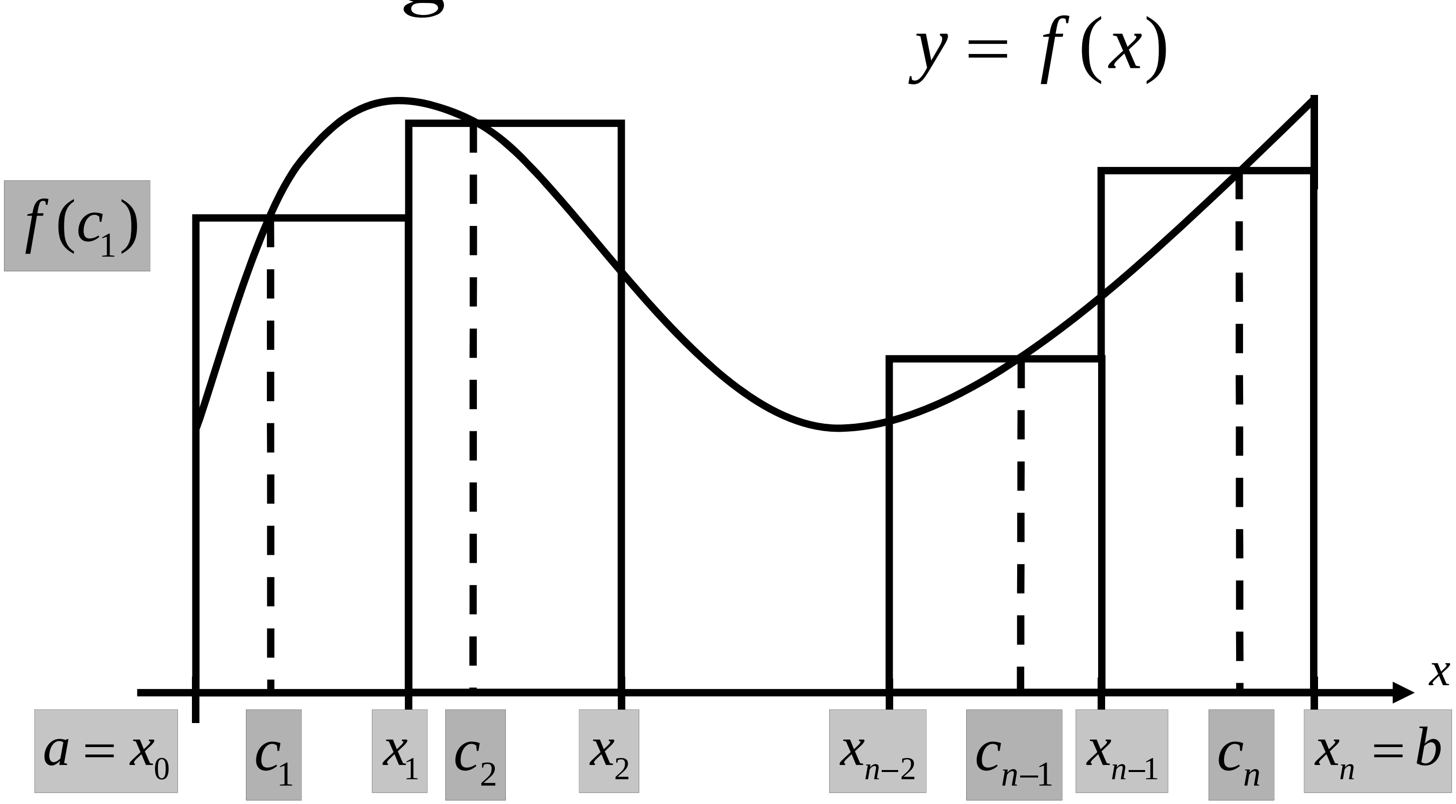


Length of each interval $= \Delta x = \dfrac{b-a}{n}$

When $n \to \infty$, we have

Area of rectangles $\to$ Area under the curve $f(x)$ from $x = a$ to $x = b$.

Riemann Integrals



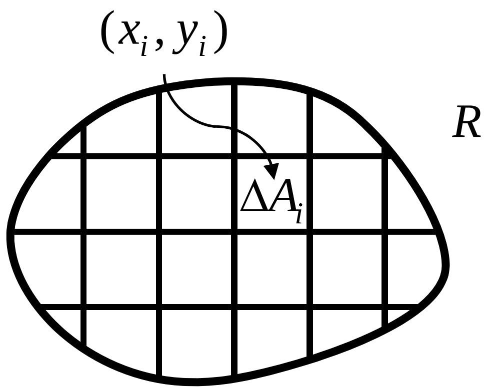
Let $n \to \infty$

The exact area *A* is given by $\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\,\Delta x$

$$A = \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\,\Delta x$$

Double Integrals

$$\iint_R f(x,y)\,dA = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i)\Delta A_i$$

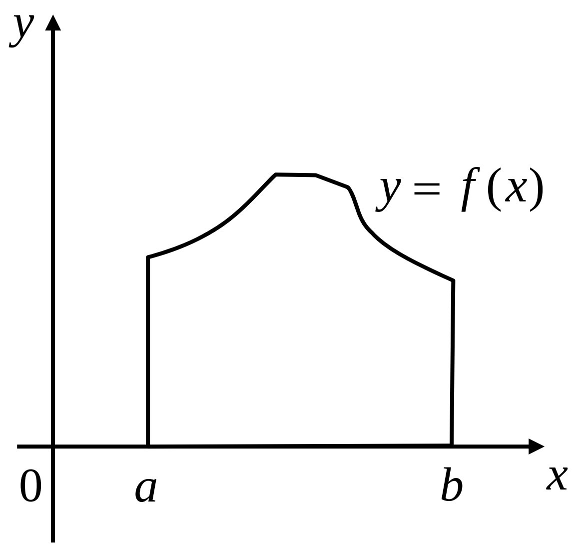


Note: $\iint_R$

double integral sign means we are integrating over a two-dimensional region.

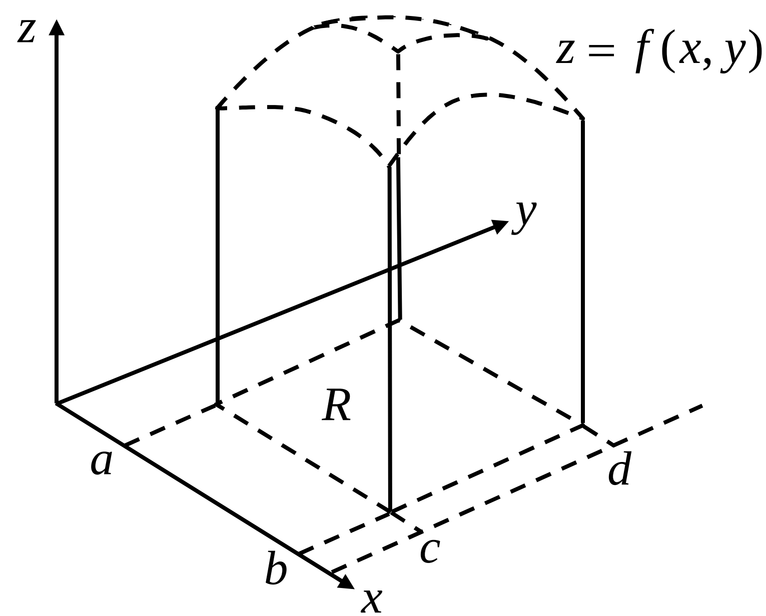
Geometrical Meaning

$\int_a^b f(x)\,dx$ $\boxed{f(x) \geq 0}$



area under the curve over the interval $[a, b]$

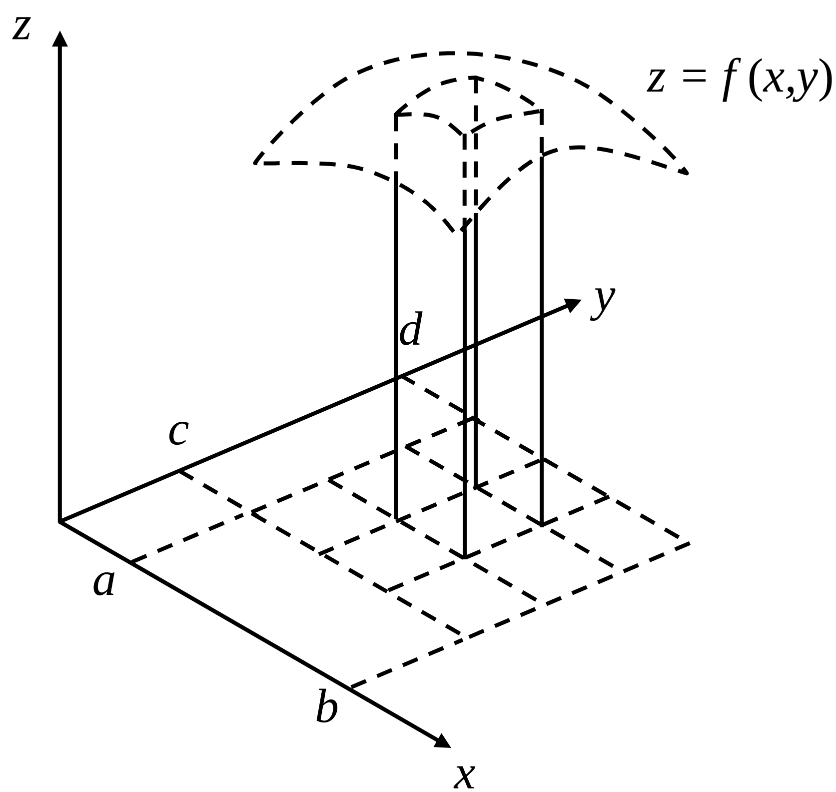
$\iint_R f(x, y)\,dA$ $\boxed{f(x, y) \geq 0}$



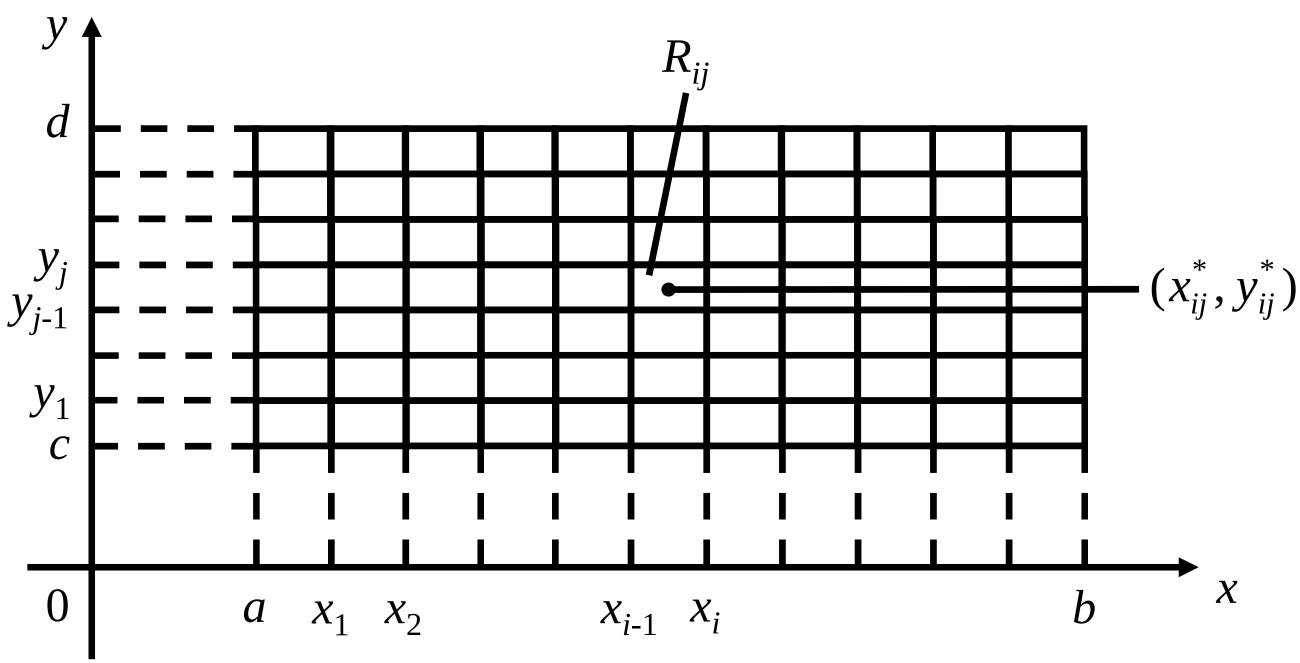
volume under the surface over the region R

Double Integrals (Geometrical meaning)

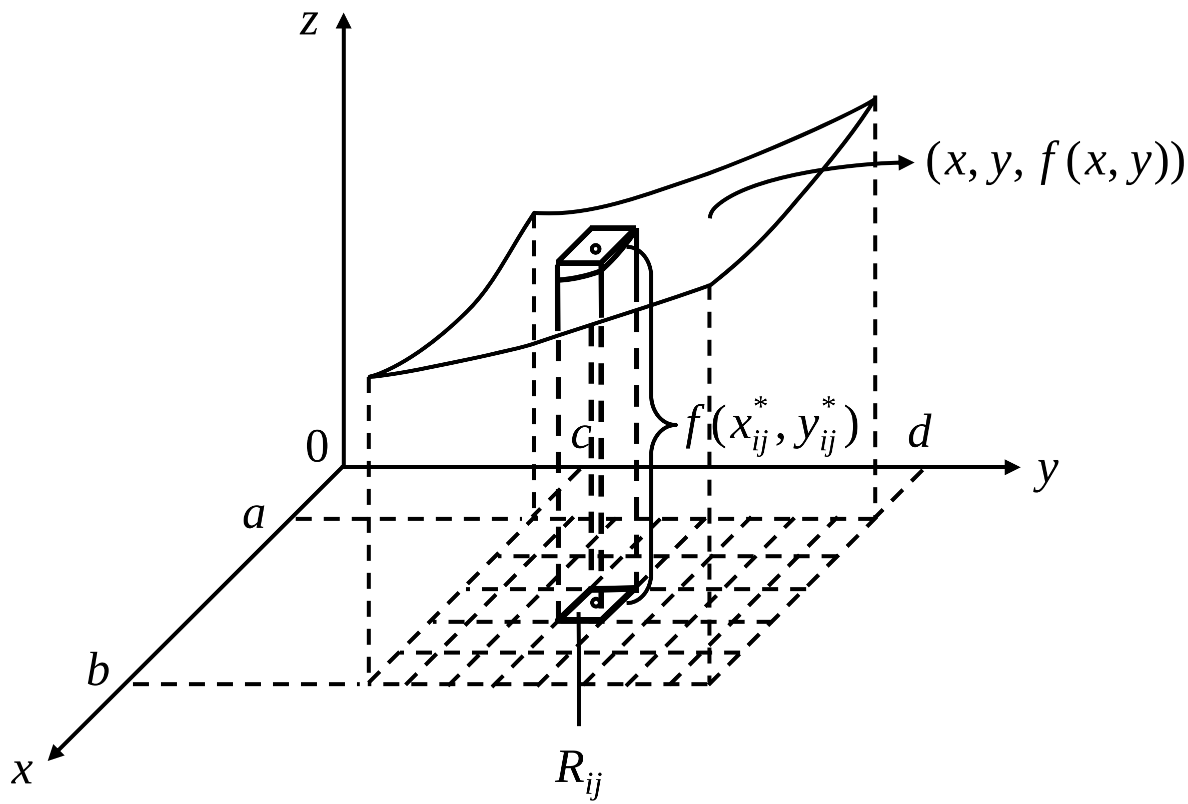
If $f(x,y) \geq 0$ for all points (x,y) in $R,$ the definite integral $\iint_R f(x,y)\, dA$ is equal to the volume under the surface $z = f(x,y)$ and above the $xy-$plane over the region $R.$



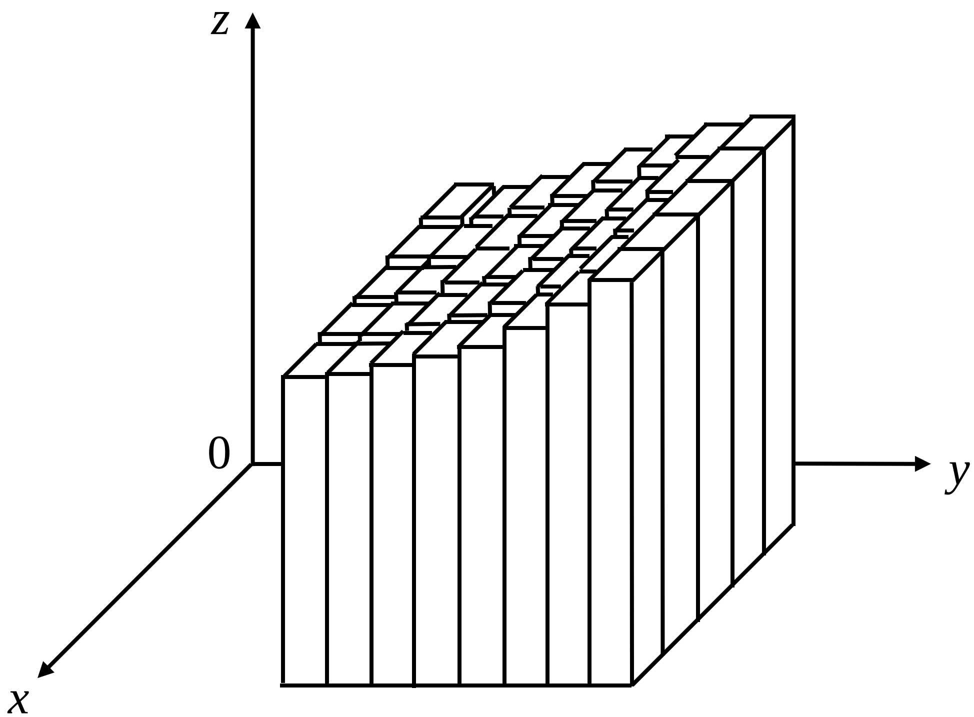
Double Integrals



Double Integrals



Double Integrals



$$\iint_R f(x, y)\, dA = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta A_i$$

Properties of Double Integrals

$$\iint_R [f(x,y)+g(x,y)]\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$$

$$\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA, \text{ where } c \text{ is a constant.}$$

If $f(x,y) \geq g(x,y)$ for all points (x,y), then

$$\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA.$$

$$\iint_R dA = \iint_R 1\,dA = \text{ the area of } R.$$

Double Integrals

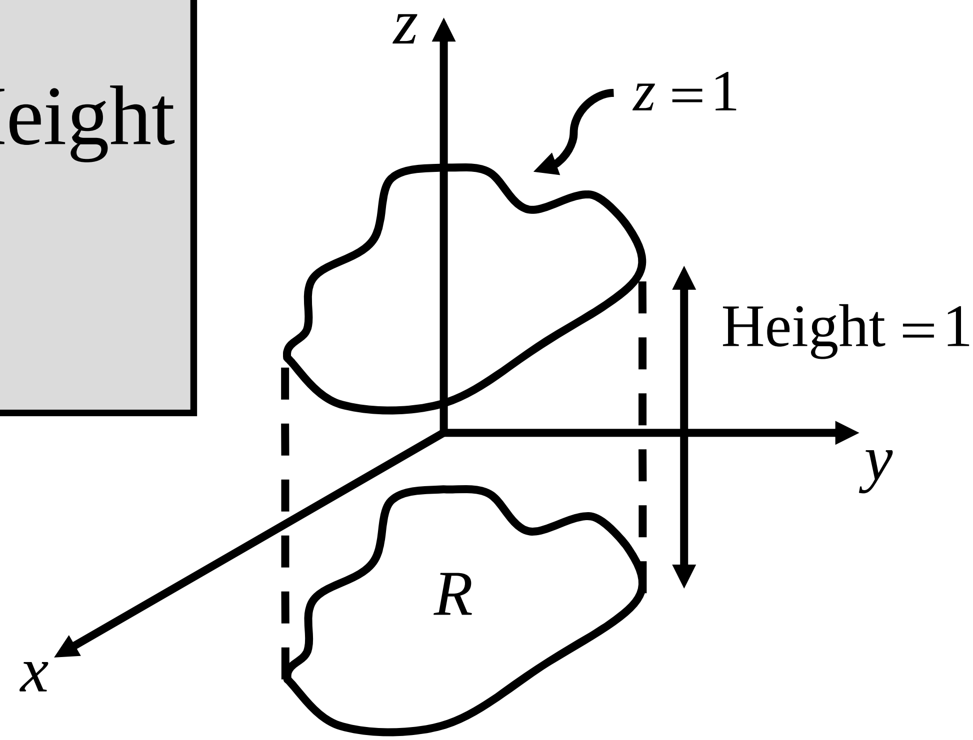
$$\boxed{\iint_R dA = \iint_R 1\, dA = \text{ the area of } R.}$$

$$\iint_R 1\, dA = \text{Volume of solid}$$

$$= \text{Base Area} \times \text{Height}$$

$$= \text{Area of } R \times 1$$

$$= \text{Area of } R$$



Properties of Double Integrals

If $m \le f(x, y) \le M$ for all points (x, y) in R, then

$$\iint_R m\, dA \le \iint_R f(x, y)\, dA \le \iint_R M\, dA$$

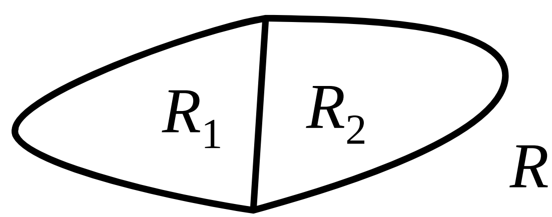
$$m \iint_R 1\, dA \le \iint_R f(x, y)\, dA \le M \iint_R 1\, dA$$

$$m(\text{Area } R) \le \iint_R f(x, y)\, dA \le M(\text{Area } R)$$

Properties of Double Integrals

$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA,$$

where $R = R_1 \cup R_2$ and R_1, R_2 do not overlap except perhaps on their boundary.



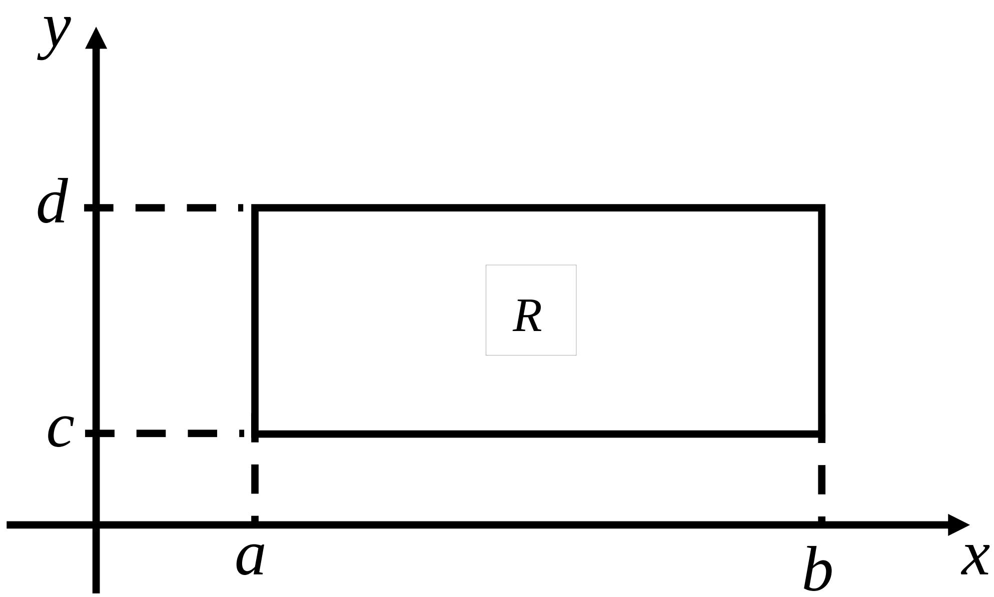
Evaluation

Evaluation

How to evaluate $\iint_R f(x, y)\, dA$ efficiently?

Rectangular Regions

$$\iint_R f(x, y)\, dA$$



$$a \le x \le b \quad \text{and} \quad c \le y \le d$$

Rectangular Regions

$$a \le x \le b \quad \text{and} \quad c \le y \le d$$

[Treat y terms as constant]

$$\iint_R f(x,y)\, dA = \int_c^d \int_a^b f(x,y)\, dx\, dy$$

[Treat x terms as constant]

$$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx$$

Note : We can perform dx first then dy or dy first then dx.

Evaluate the iterated integrals:

(a) $\int_0^3 \int_1^2 (x+2y)\, dy\, dx$

(b) $\int_1^2 \int_0^3 (x+2y)\, dx\, dy$

Note : In part (a), we can perform dy first then dx and in part (b), we perform dx first then dy.

Note : We should get the same answer for part (a) and (b).

(a) $\int_0^3 \int_1^2 x + 2y \; dy \; dx.$

[Treat x terms as constant]

$$\int_0^3 \int_1^2 x + 2y \; dy \; dx = \int_0^3 \left[xy + y^2 \right]_{y=1}^{y=2} dx$$

$$= \int_0^3 (2x+4) - (x+1) \; dx$$

$$= \int_0^3 x + 3 \; dx$$

$$= \left[\frac{x^2}{2} + 3x \right]_0^3 = \frac{27}{2}$$

$$(a) \quad \int_0^3 \int_1^2 (x + 2y) \; dy \; dx = \frac{27}{2}$$

$$(b)\ \int_1^2\int_0^3 (x+2y)\,dx\,dy$$

[Treat y terms as constant]

$$\int_1^2\int_0^3 x+2y\ dx\ dy=\int_1^2\left[\frac{x^2}{2}+2xy\right]_{x=0}^{x=3} dy$$

$$=\int_1^2 (4.5+6y)-(0+0)\ dy$$

$$=\int_1^2 4.5+6y\ dy$$

$$=\left[4.5y+3y^2\right]_1^2=\frac{27}{2}$$

$$\boxed{(b)\quad \int_1^2\int_0^3 (x+2y)\ dx\ dy=\frac{27}{2}}$$

$$\boxed{(a)\quad \int_0^3\int_1^2 (x+2y)\ dy\ dx=\frac{27}{2}}$$

$$(a) \quad \int_0^3 \int_1^2 (x+2y)\, dy\, dx = \frac{27}{2}$$

$$(\mathrm{b}) \quad \int_1^2 \int_0^3 (x+2y)\, dx\, dy = \frac{27}{2}$$

Note : We get the same answer for part (a) and (b).

Let R be the rectangular region $0 \le x \le 4,\ 1 \le y \le 2$.

Evaluate $\iint_R x^2 y\, dA$.

$$\iint_R x^2 y\, dA = \int_0^4 \int_1^2 x^2 y\, dy\, dx$$

Recall that $\int kf(y)\,dy = k\int f(y)\,dy$

[constant can bring out]

$\int_1^2 y\,dy$ is a constant.

$$\int_0^4 \int_1^2 x^2 y\,dy\,dx = \int_0^4 x^2 \left(\int_1^2 y\,dy\right) dx$$

x^2 treated as constant when integrating with respect to y.

$$= \int_1^2 y\,dy \int_0^4 x^2 dx$$

$$\int_0^4 \int_1^2 x^2 y\,dy\,dx = \int_1^2 y\,dy \int_0^4 x^2 dx$$

Product of two integrals,

one in x only and one in y only

Let R be the rectangular region $0 \le x \le 4,\ 1 \le y \le 2.$

Evaluate $\iint_R x^2 y \, dA.$

$$\iint_R x^2 y \, dA = \int_0^4 \int_1^2 x^2 y \, dy \, dx$$

$$= \left(\int_0^4 x^2 \, dx \right) \left(\int_1^2 y \, dy \right)$$

$$= \left[\frac{x^3}{3} \right]_0^4 \left[\frac{y^2}{2} \right]_0^4$$

$$= \frac{64}{3} \times \frac{3}{2}$$

$$= 32.$$

Evaluation - Remark

In general, if $f(x, y) = g(x)h(y),$ then

$$\iint_R g(x)h(y)\, dA = \left(\int_a^b g(x)\, dx\right)\left(\int_c^d h(y)\, dy\right)$$

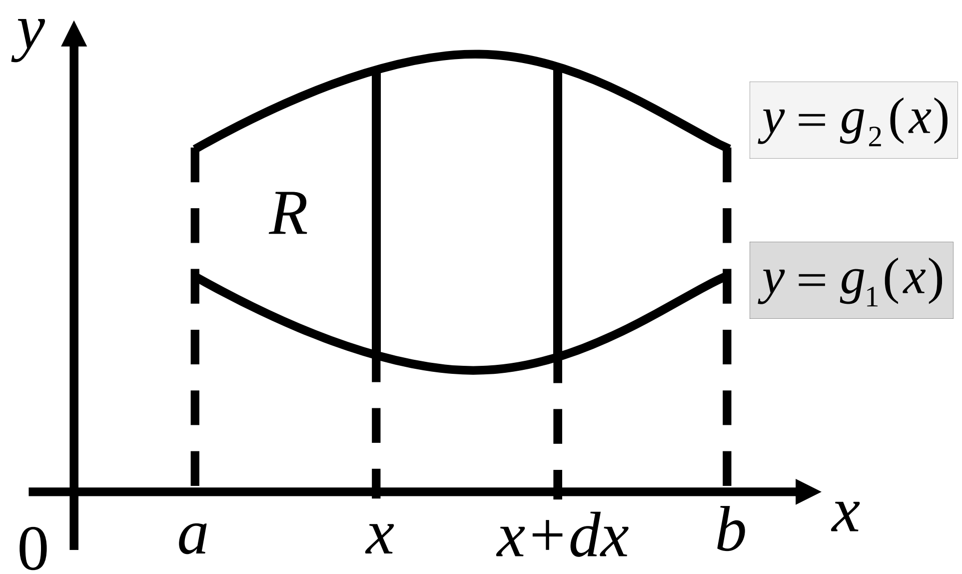
where R is the rectangular region $a \le x \le b, \quad c \le y \le d.$

Only true for rectangular region

Evaluation

Type A : Perform dy first

$$R: \ g_1(x) \le y \le g_2(x), \ \ a \le x \le b.$$



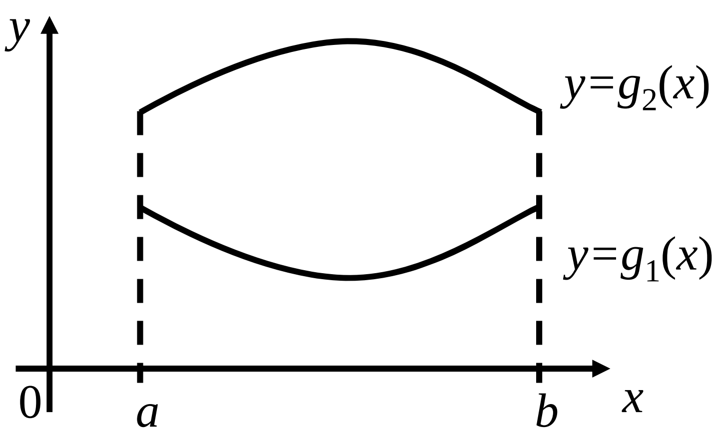
Type A : Vertical line meets top and bottom boundaries

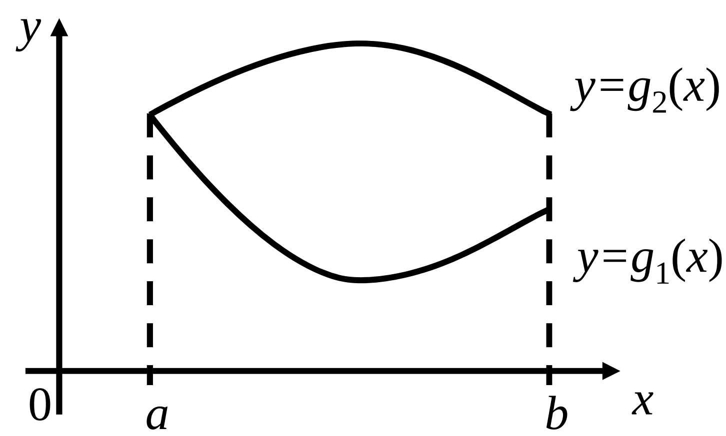
$$\boxed{\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx}$$

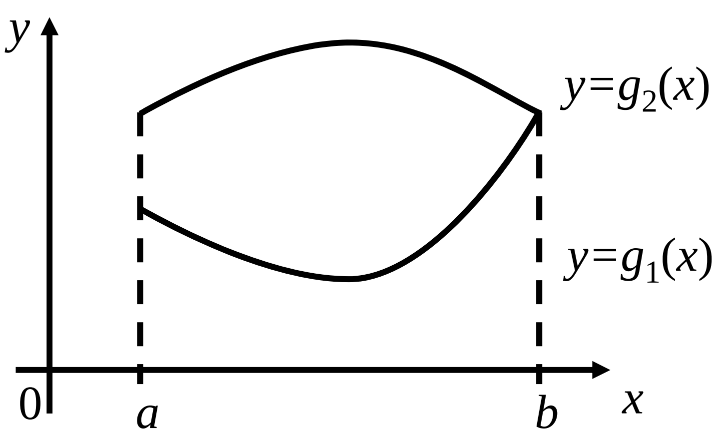
Type A : Perform dy first

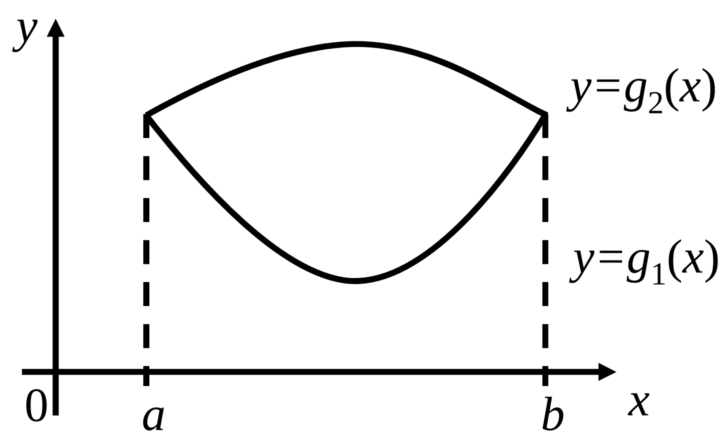
$R: \; g_1(x) \le y \le g_2(x), \;\; a \le x \le b.$

Type A : Vertical line meets top and bottom boundaries





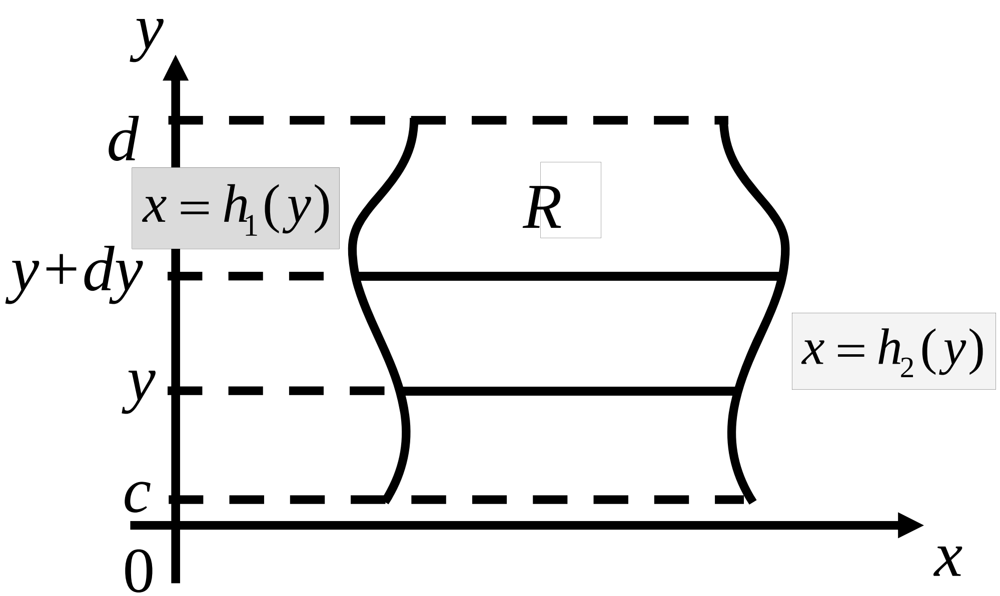




Evaluation

Type B : Perform dx first

$$R: \quad h_1(y) \le x \le h_2(y), \quad c \le y \le d.$$

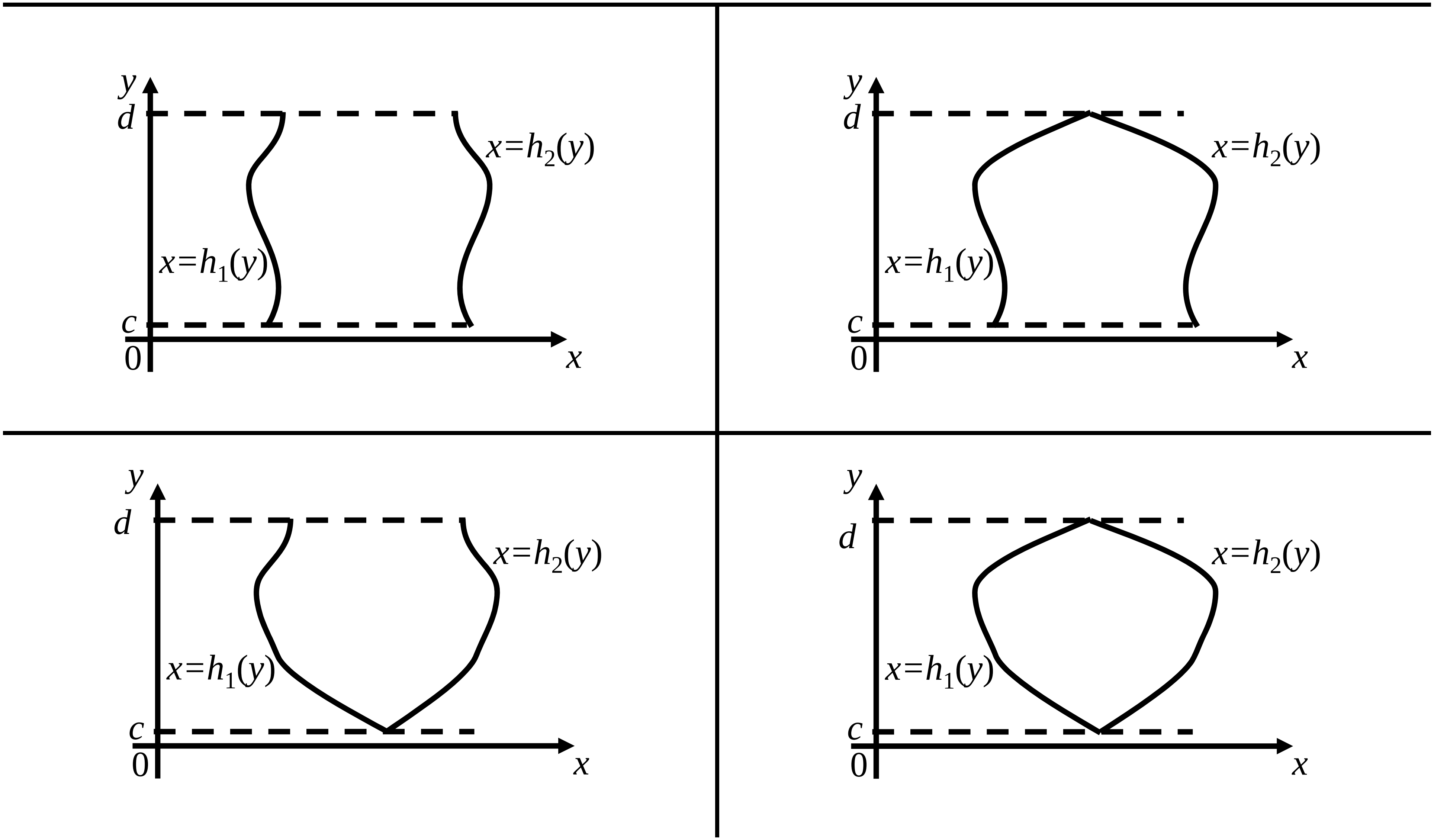


Type B : Horizontal line meets left and right boundaries

$$\boxed{\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy}$$

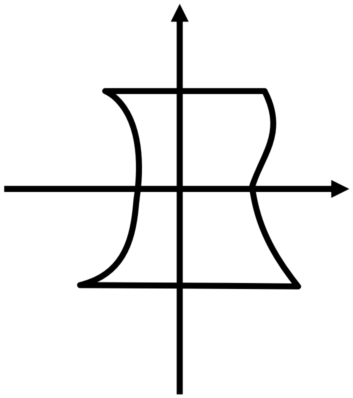
Type B : Perform dx first $R: \quad h_1(y) \le x \le h_2(y), \quad c \le y \le d.$

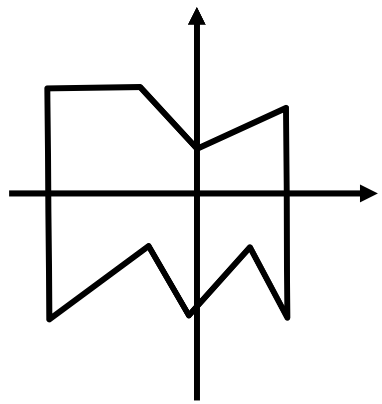
Type B : Horizontal line meets left and right boundaries

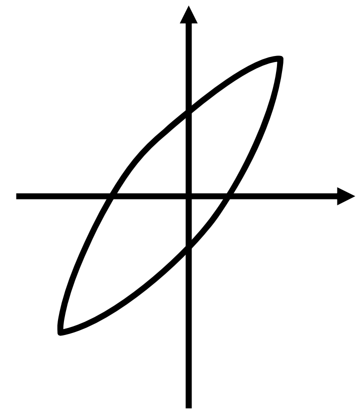


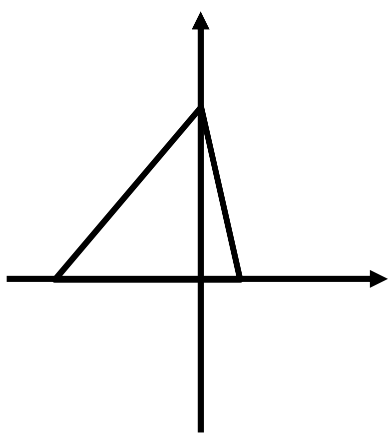
Pause and Think !!!

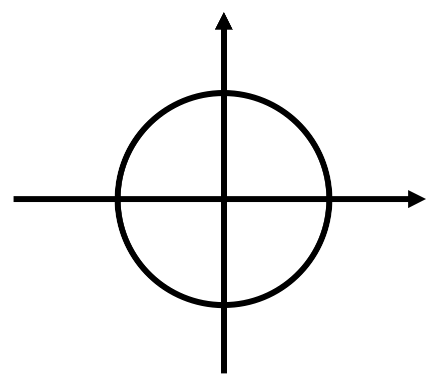
Quiz (Type A or Type B) ???











Example

If R is bounded by $y = x$ and $y = x^2$, find $\iint_R xy\, dA$.

Type A : Vertical line meets top and bottom boundaries

Type A : Perform dy first

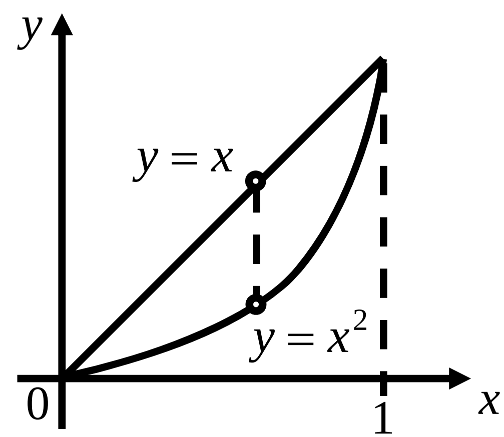
$R: \quad g_1(x) \le y \le g_2(x), \quad a \le x \le b.$

$$\iint_R xy\, dA = \int_0^1 \int_{x^2}^{x} xy\, dy\, dx$$

$$= \int_0^1 \left[\frac{xy^2}{2} \right]_{y=x^2}^{y=x} dx$$

$$= \frac{1}{2} \int_0^1 (x^3 - x^5)\, dx$$

$$= \frac{1}{24}.$$



Treat R as a Type A region.

1. y-limits 2. x-limits

$R: \quad x^2 \le y \le x, \quad 0 \le x \le 1.$

Example

If R is bounded by $y = x$ and $y = x^2$, find $\iint_R xy\, dA.$

Type B : Perform dx first

$R: \quad h_1(y) \le x \le h_2(y), \quad c \le y \le d.$

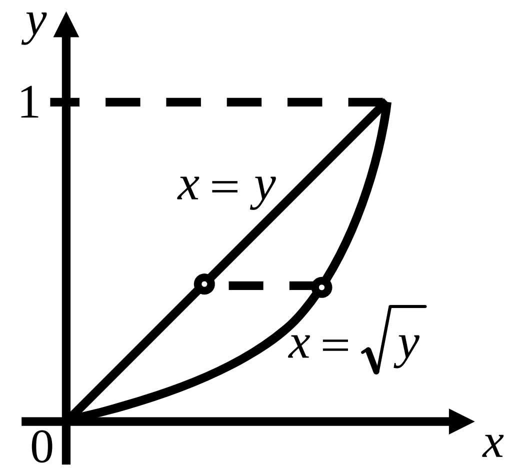
Type B : Horizontal line meets left and right boundaries

$$\iint_R xy\, dA = \int_0^1 \int_y^{\sqrt{y}} xy\, dx\, dy$$

$$= \int_0^1 \left[\frac{x^2 y}{2} \right]_{x=y}^{x=\sqrt{y}} dy$$

$$= \frac{1}{2} \int_0^1 (y^2 - y^3)\, dy$$

$$= \frac{1}{24}.$$

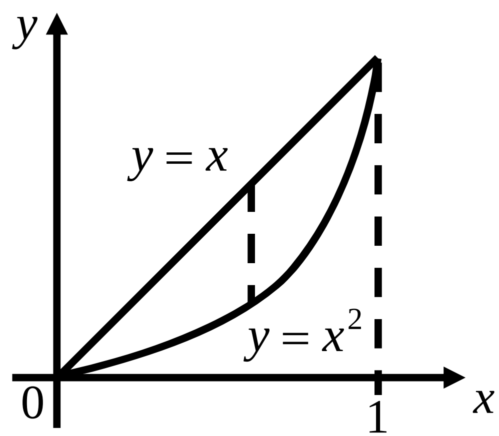


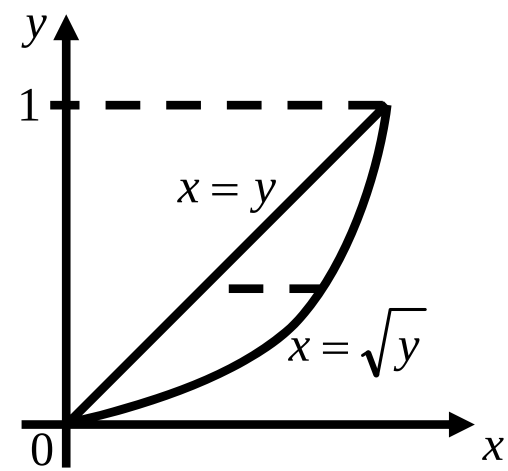
Treat R as a Type B region.

1. x-limits 2. y-limits

$y \le x \le \sqrt{y}, \quad 0 \le y \le 1.$

If R is bounded by $y = x$ and $y = x^2$, find $\iint_R xy\ dA$.





Treat R as a Type A region.

$R: \quad x^2 \le y \le x, \quad 0 \le x \le 1.$

$$\iint_R xy\ dA = \int_0^1 \int_{x^2}^{x} xy\ dy\ dx$$

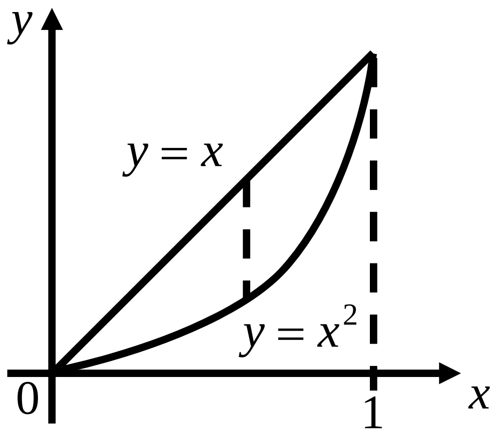
Treat R as a Type B region.

$y \le x \le \sqrt{y}, \quad 0 \le y \le 1.$

$$\iint_R xy\ dA = \int_0^1 \int_{y}^{\sqrt{y}} xy\ dx\ dy$$

In this example, R is both type A and type B.

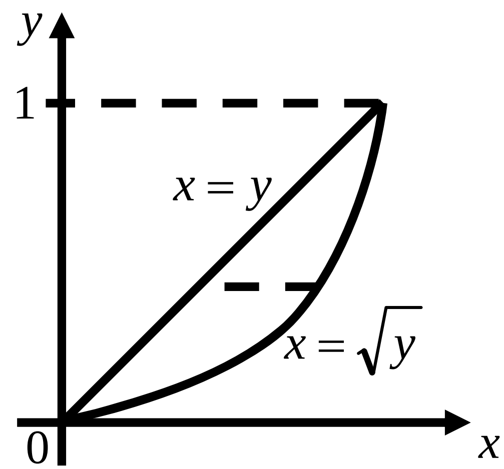
We may find $\iint_R xy\ dA$, by treating R as either type A or type B.



Treat R as a Type A region.

$R: \quad x^2 \le y \le x, \quad 0 \le x \le 1.$

$$\iint_R xy\ dA = \int_0^1 \int_{x^2}^{x} xy\ dy\ dx$$



Treat R as a Type B region.

$y \le x \le \sqrt{y}, \quad 0 \le y \le 1.$

$$\iint_R xy\ dA = \int_0^1 \int_{y}^{\sqrt{y}} xy\ dx\ dy$$

If a region is both type A and type B,

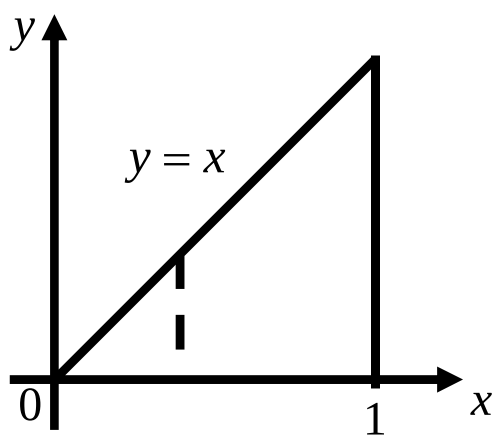
the order of integration might make a difference,

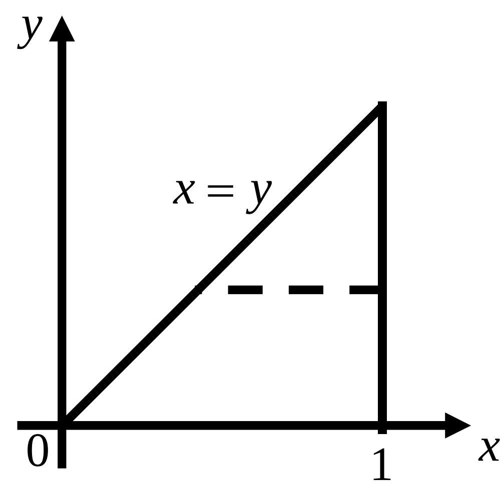
sometimes type A is easier,

sometimes type B is easier.

Example

Calculate $\iint_R \frac{\sin x}{x}\, dA$, where R is the triangle in the $xy-$plane bound by the $x-$axis, the line $y = x$ and the line $x = 1$.





Treat R as a Type A region.

$R: 0 \le y \le x, \quad 0 \le x \le 1.$

$$\iint_R \frac{\sin x}{x}\, dA = \int_0^1 \int_0^x \frac{\sin x}{x}\, dy\, dx$$

Treat R as a Type B region.

$R: \ y \le x \le 1, \quad 0 \le y \le 1.$

$$\iint_R \frac{\sin x}{x}\, dA = \int_0^1 \int_y^1 \frac{\sin x}{x}\, dx\, dy$$

Example

Calculate $\iint_R \frac{\sin x}{x}\, dA,$ where R is the triangle in the $xy-$plane bound by the $x-$axis, the line $y = x$ and the line $x = 1$.

Pause and Think !!!

Which one is easier ???

$(a) \quad \iint_R \frac{\sin x}{x}\, dA = \int_0^1 \int_0^x \frac{\sin x}{x}\, dy\; dx$

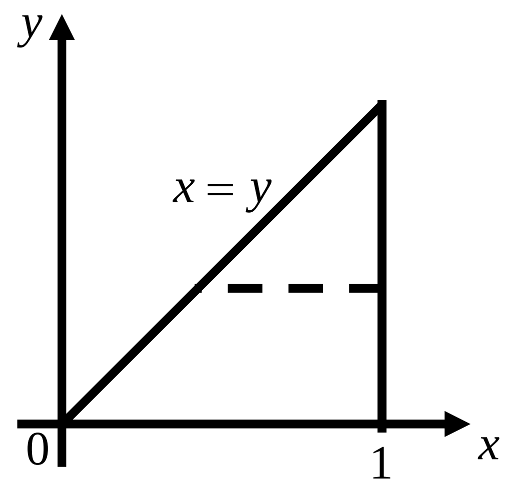
$(b) \quad \iint_R \frac{\sin x}{x}\, dA = \int_0^1 \int_y^1 \frac{\sin x}{x}\, dx\; dy$

Example

Calculate $\iint_R \frac{\sin x}{x}\, dA,$ where R is the triangle in the $xy-$plane bound by the $x-$axis, the line $y = x$ and the line $x = 1.$

Treat R as a Type B region.

$R: \; y \le x \le 1, \quad 0 \le y \le 1.$



$$\iint_R \frac{\sin x}{x}\, dA = \int_0^1 \int_y^1 \frac{\sin x}{x}\, dx\, dy$$

$\int_y^1 \frac{\sin x}{x}\, dx$ cannot be evaluated by elementary means !!!

Calculate $\iint_R \frac{\sin x}{x}\, dA$, where R is the triangle in the $xy-$plane bound by the $x-$axis, the line $y = x$ and the line $x = 1$.

treated as constant since doing dy

$$\iint_R \frac{\sin x}{x}\, dA = \int_0^1 \int_0^x \frac{\sin x}{x}\, dy\, dx$$

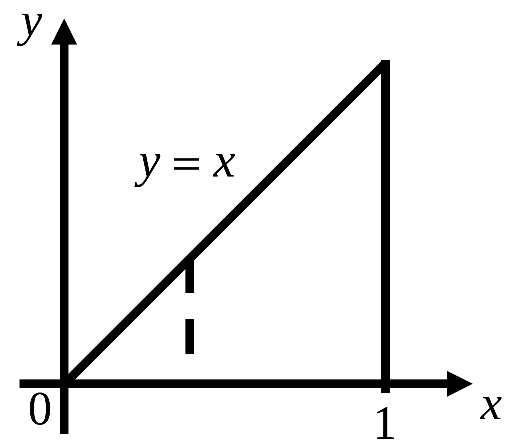
$$= \int_0^1 \left[\frac{\sin x}{x} y\right]_{y=0}^{y=x} dx$$

$$= \int_0^1 \frac{\sin x}{x} x - \frac{\sin x}{x} 0\, dx$$

$$= \int_0^1 (\sin x)\, dx$$

$$= \left[-\cos x\right]_0^1$$

$$= 1 - \cos 1.$$



Treat R as a Type A region.

$$R: 0 \le y \le x,\quad 0 \le x \le 1.$$

$$\iint_R \frac{\sin x}{x}\, dA = \int_0^1 \int_0^x \frac{\sin x}{x}\, dy\, dx$$

Pause and Think !!!

What should you do if you are ask to evaluate

$$\int_0^1 \int_y^1 \frac{\sin x}{x}\, dx\, dy \quad ???$$

Pause and Think !!!

What should you do if you are ask to evaluate

$$\int_0^1 \int_y^1 \frac{\sin x}{x}\, dx\, dy \quad ???$$

Change the order of integration from

$dxdy$ to $dydx$

Question :

How to change the order of integration ??

Pause and Think !!!

Question :

How to change the order of integration ??

Evaluate $\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3} \, dy \, dx.$

$$\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3} \, dy \, dx = \int_{\sqrt{x/3}}^1 \int_0^3 e^{y^3} \, dx \, dy \quad ??$$

Is it correct ??

Pause and Think !!!

Question :

How to change the order of integration ??

To change the order of integration,

we need to consider

the region of integration.

Example (change order of integration)

Evaluate $\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3} \, dy \, dx.$

To change the order of integration,
we need to consider
the region of integration.

Note that : $\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3} \, dy \, dx \neq \int_{\sqrt{x/3}}^1 \int_0^3 e^{y^3} \, dx \, dy$

Example (change order of integration)

Evaluate $\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3}\, dy\, dx.$

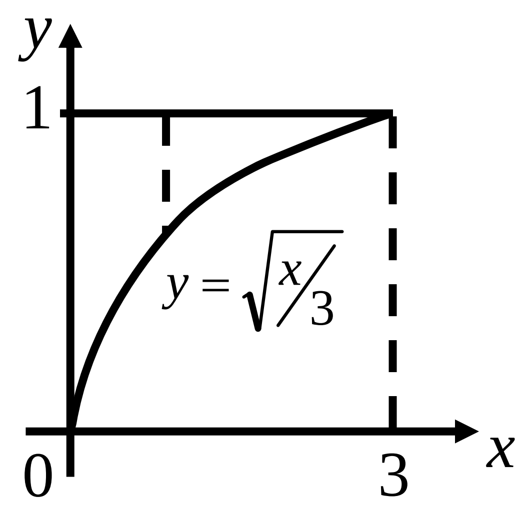
It is difficult to integrate e^{y^3} directly.

Example (change order of integration)

Evaluate $\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3}\, dy\, dx.$

Type A region

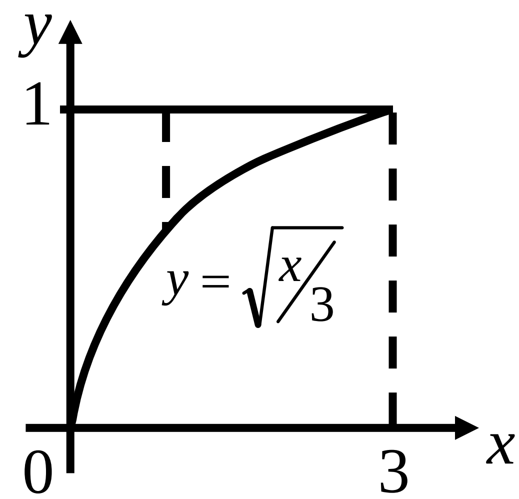
$R: \quad \sqrt{x/3} \le y \le 1, \;\; 0 \le x \le 3.$



First sketch and identify the region R

Example (change order of integration)

Evaluate $\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3}\, dy\, dx.$

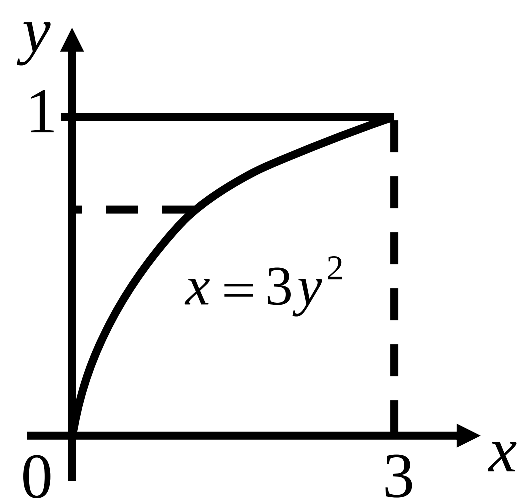


Type *A* region

$R: \quad \sqrt{x/3} \le y \le 1, \;\; 0 \le x \le 3.$

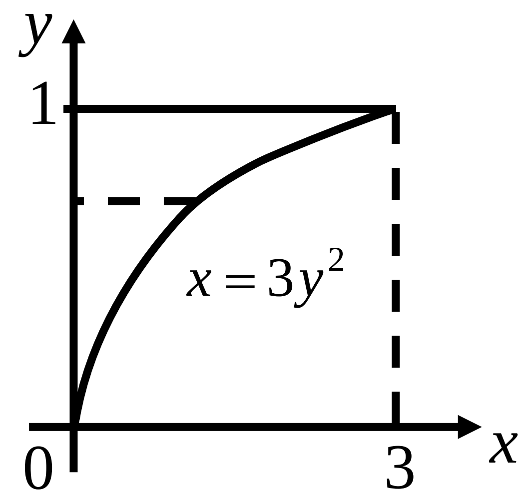
Type *B* region

$R: \quad 0 \le x \le 3y^2, \;\; 0 \le y \le 1.$



Note that $y = \sqrt{x/3} \Rightarrow x = 3y^2.$

Evaluate $\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3}\, dy\, dx.$



Type B region

$$R: \quad 0 \le x \le 3y^2, \ \ 0 \le y \le 1.$$

$$\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3}\, dy\, dx = \int_0^1 \int_0^{3y^2} e^{y^3}\, dx\, dy$$

$$= \int_0^1 \left[x e^{y^3} \right]_{x=0}^{x=3y^2} dy$$

$$= \int_0^1 3y^2 e^{y^3}\, dy$$

$$= \int_0^1 e^u\, du \quad (\text{Let } u = y^3.)$$

$$= \left[e^u \right]_{u=0}^{u=1} = e - 1.$$

Past Exam Question

Evaluate

$$\iint_D (4e^{x^2} - 5\sin y)\, dx\, dy$$

where D is the region in the first quadrant bounded by the graphs of $y = x$, $y = 0$, and $x = 4$.

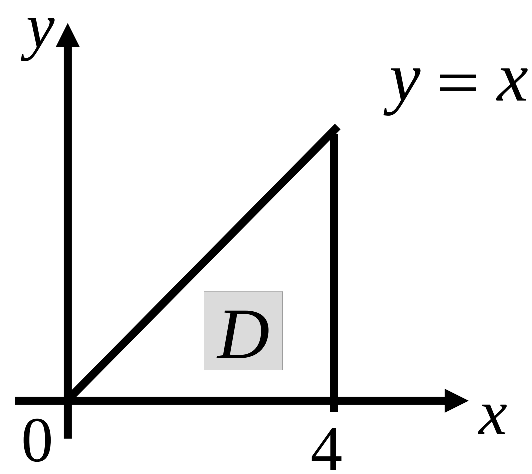
Past Exam Question

Evaluate

$$\iint_D (4e^{x^2} - 5\sin y)\, dx\, dy$$

where D is the region in the first quadrant bounded by the graphs of $y = x$, $y = 0$, and $x = 4$.

$$\iint_D (4e^{x^2} - 5\sin y)\, dx\, dy$$
$$= \int_0^4 \int_0^x (4e^{x^2} - 5\sin y)\, dy\, dx$$

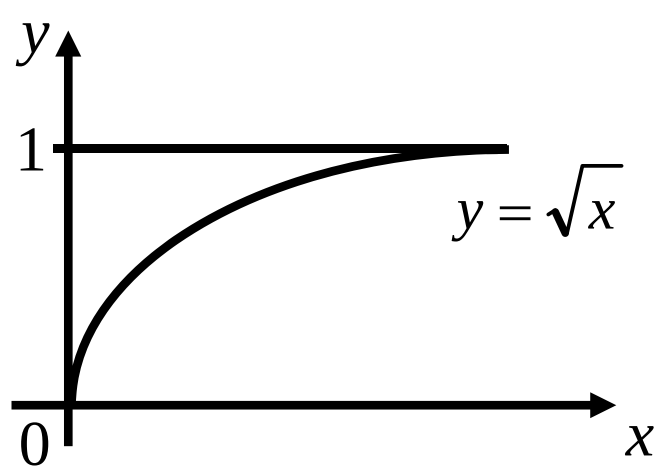


$\int e^{x^2} dx$??

Should do dy first !!

Past Exam Question

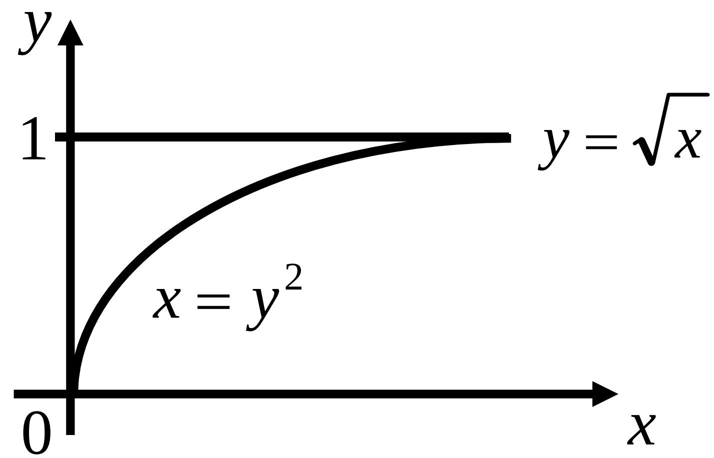
Evaluate $\displaystyle \int_0^1 \left[\int_{\sqrt{x}}^1 \sin\left(\frac{y^3+1}{2}\right) dy \right] dx.$



$\displaystyle \int \sin\left(\frac{y^3+1}{2}\right) dy$???

Should do dx first !!

$$y=\sqrt{x} \quad \rightarrow \quad x=y^2$$



$$\int_0^1\left[\int_{\sqrt{x}}^1 \sin\left(\frac{y^3+1}{2}\right)\,dy\right]dx=\int_0^1\left[\int_0^{y^2} \sin\left(\frac{y^3+1}{2}\right)\,dx\right]dy$$

$$=\int_0^1\left[x\ \sin\left(\frac{y^3+1}{2}\right)\right]_0^{y^2}\,dy$$

$$=\int_0^1 y^2\sin\left(\frac{y^3+1}{2}\right)\,dy$$

$$=\frac{2}{3}\left[-\cos\left(\frac{y^3+1}{2}\right)\right]_0^1$$

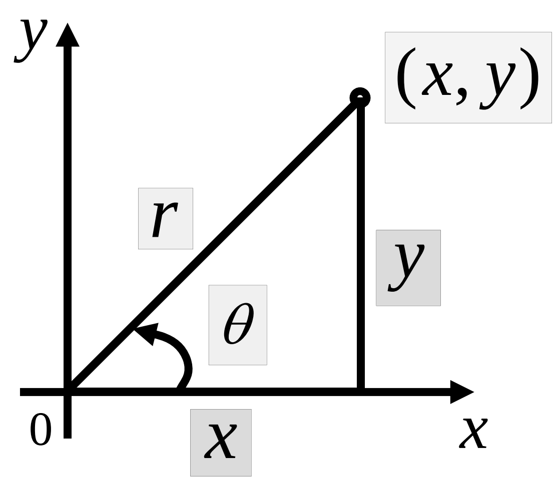
$$=\frac{2}{3}\left(\cos\frac{1}{2}-\cos 1\right)$$

Double Integrals in Polar Coordinates

Double Integrals in Polar Coordinates

Polar coordinates

In Cartesian coordinates,
to specify a point,
we need to give x and y.



In polar coordinates,
to specify a point,
we need to give r and θ.

$$\frac{x}{r} = \cos\theta$$

$$x = r\cos\theta$$

$$\frac{y}{r} = \sin\theta$$

$$y = r\sin\theta$$

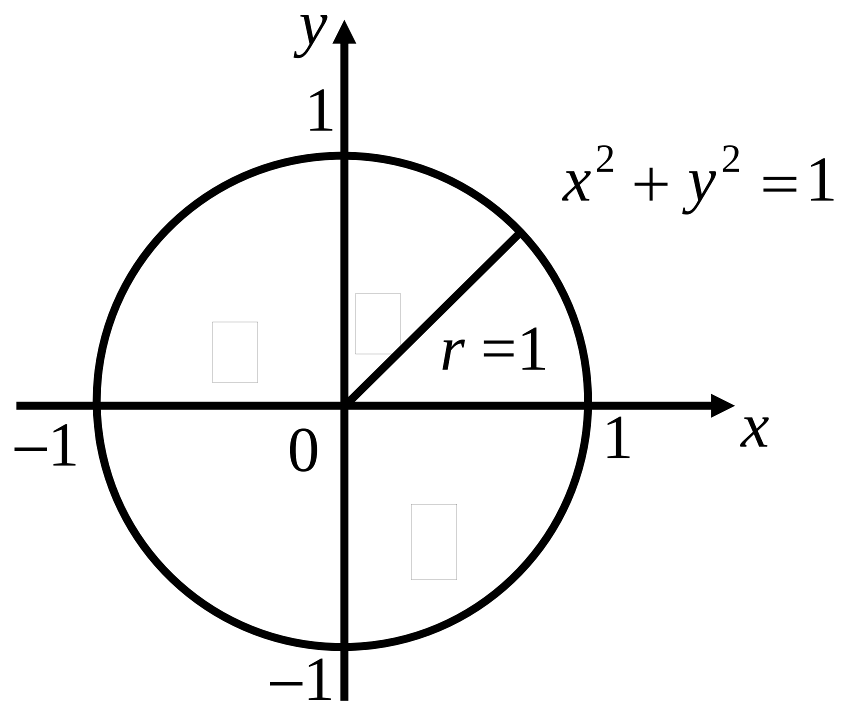
Double Integrals in Polar Coordinates

Circle center (0,0) with radius r

$x = r\cos\theta$

$y = r\sin\theta$

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$$
$$= r^2$$

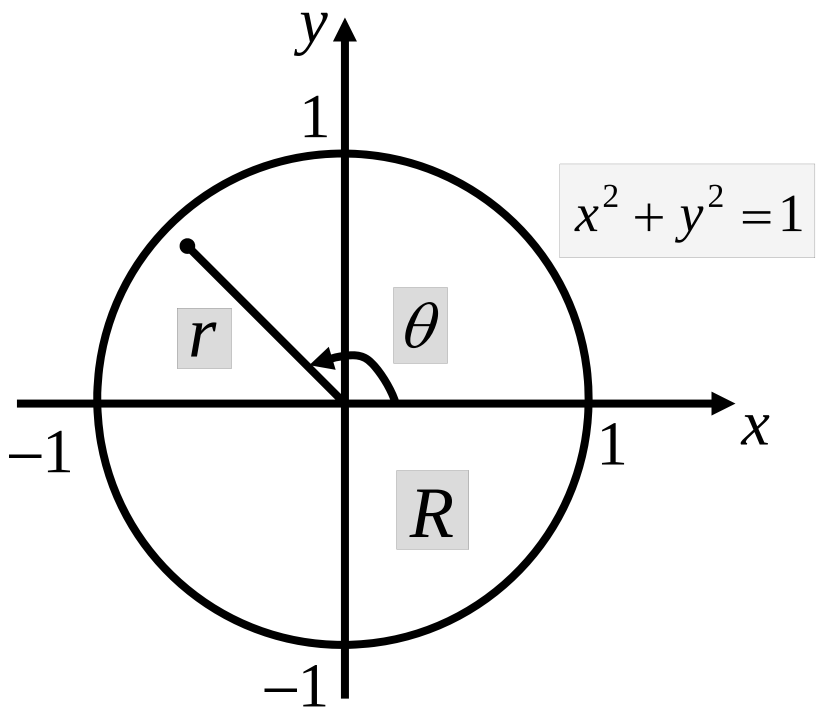


In Polar coordinates,
equation of circle
becomes very simple !!

$r = 1,\ 0 \le \theta \le 2\pi$

Double Integrals in Polar Coordinates

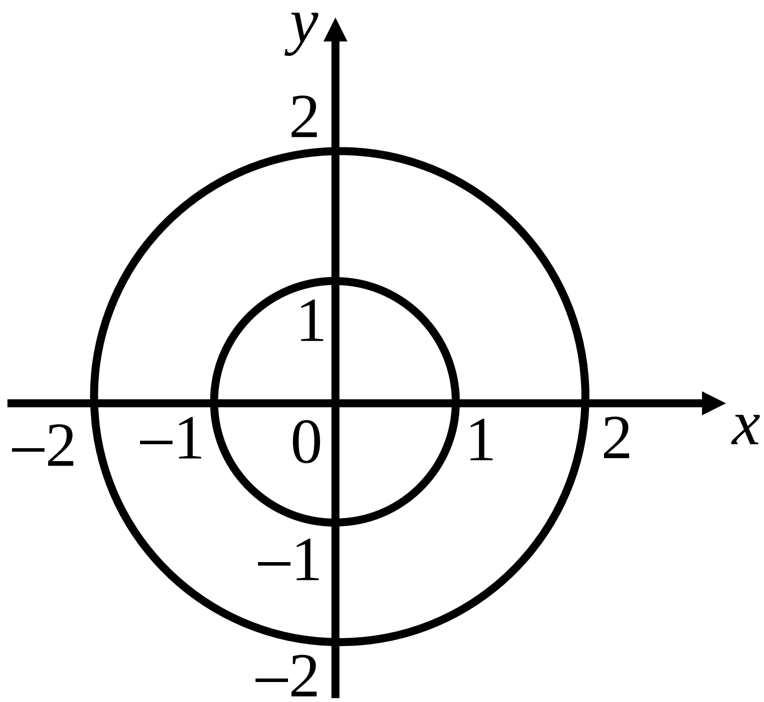
Circle



$$R: \quad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi$$

Double Integrals in Polar Coordinates

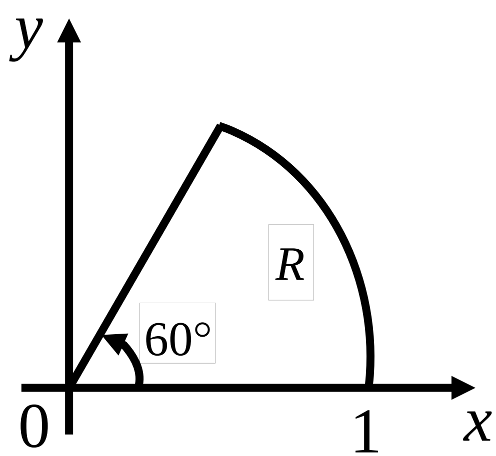
Ring



$$\boxed{R:\ \ 1 \le r \le 2,\ \ \ 0 \le \theta \le 2\pi}$$

Double Integrals in Polar Coordinates

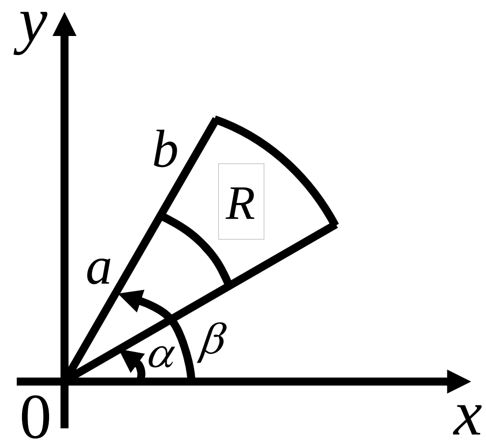
Sector of a Circle



$$R: \quad 0 \le r \le 1, \quad 0 \le \theta \le \frac{\pi}{3}$$

Double Integrals in Polar Coordinates

Polar Rectangular



$$R: \quad a \le r \le b, \quad \alpha \le \theta \le \beta$$

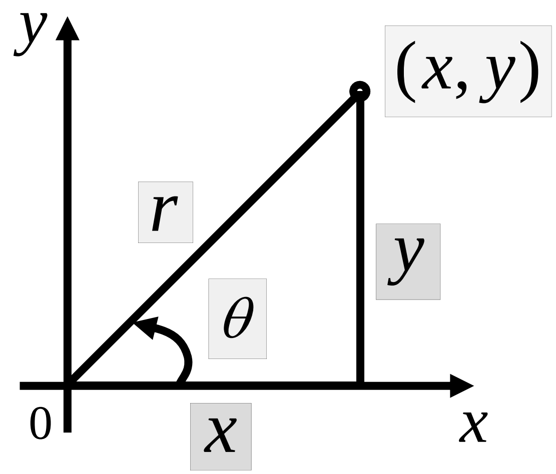
Double Integrals in Polar Coordinates

Change of Variable

To change $(x, y) \rightarrow (r, \theta)$

How to change

$$\iint_R f(x,y)\,dA \rightarrow \ ??$$



Double Integrals in Polar Coordinates

Change of Variable to polar coordinates

$x = r\cos\theta$

$y = r\sin\theta$

$$dA = dx\,dy \;\rightarrow\; r\;dr\;d\theta$$

If $R: a \le r \le b,\ \alpha \le \theta \le \beta$, then we have

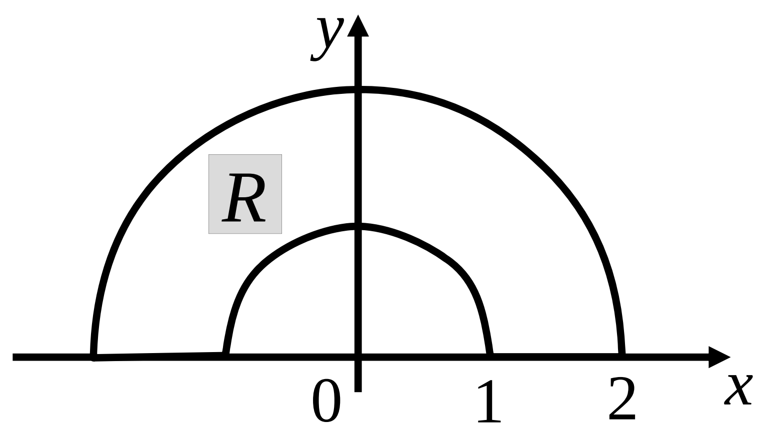
$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r\,dr\,d\theta.$$

Example

Evaluate

$$\iint_R (3x + 4y^2)\, dA,$$

where R is the semicircular ring in the upper half-plane between the semi-circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.



$$\boxed{R: \quad 1 \le r \le 2, \quad 0 \le \theta \le \pi}$$

$x = r\cos\theta$ $y = r\sin\theta$ $R: \ 1 \le r \le 2, \ \ 0 \le \theta \le \pi$

If $R: a \le r \le b, \ \alpha \le \theta \le \beta$, then we have

$$\iint_R f(x,y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r\, dr\, d\theta.$$

$$\iint_R (3x + 4y^2)\, dA = \int_0^\pi \int_1^2 (3r\cos\theta + 4r^2\sin^2\theta) r\, dr\, d\theta$$

$$= \int_0^\pi \Big[r^3\cos\theta + r^4\sin^2\theta \Big]_{r=1}^{r=2} d\theta$$

$$= \int_0^\pi (7\cos\theta + 15\sin^2\theta)\, d\theta$$

$$= \int_0^\pi \left[7\cos\theta + \frac{15}{2}(1 - \cos 2\theta) \right] d\theta$$

$$= \left[7\sin\theta + \frac{15}{2}\left(\theta - \frac{\sin 2\theta}{2} \right) \right]_{\theta=0}^{\theta=\pi} = \frac{15\pi}{2}$$

Past Exam Question

Let k be a positive constant. Evaluate

$$\iint_D x^2 e^{xy}\, dx\, dy$$

where D is the plane region given by $D:\ 0 \le x \le 2k$ and $0 \le y \le \dfrac{1}{2k}$.

Pause and Think !!!

Should you do dx first or dy first ???

Question :

dx first easier or dy first easier ???

Let k be a positive constant. Evaluate

$$\iint_D x^2 e^{xy}\,dx\,dy$$

where D is the plane region given by $D:\ 0 \le x \le 2k$ and $0 \le y \le \dfrac{1}{2k}$.

Treated as constant

$$\iint_D x^2 e^{xy}\,dx\,dy = \int_0^{2k}\int_0^{1/2k} x^2 e^{xy}\ dy\ dx$$

$$= \int_0^{2k}\Big[x e^{xy}\Big]_{y=0}^{y=1/2k} dx$$

$$= \int_0^{2k}\Big[x e^{x/2k} - x\Big]\ dx$$

$$\int e^{xy}\ dy = \frac{1}{x}e^{xy}$$

Needs integration by parts

Integration by parts

$$\int_0^{2k} xe^{x/2k}\,dx = 2k\left[xe^{x/2k}\right]_0^{2k} - 2k\int_0^{2k} e^{x/2k}\,dx$$

$$= (2k)(2ke) - (2k)2k\left[e^{x/2k}\right]_0^{2k}$$

$$= 4k^2$$

$$\boxed{\int_0^{2k} xe^{x/2k}\,dx = 4k^2}$$

Integration by parts

$$\boxed{\int_0^{2k} xe^{x/2k}\, dx = 4k^2}$$

$$\begin{aligned}
\iint_D x^2 e^{xy}\,dx\,dy &= \int_0^{2k}\int_0^{1/2k} x^2 e^{xy}\,dy\,dx \\
&= \int_0^{2k}\left[xe^{xy}\right]_{y=0}^{y=1/2k} dx \\
&= \int_0^{2k}\left[xe^{x/2k} - x\right] dx \\
&= 4k^2 - \left[\frac{1}{2}x^2\right]_0^{2k} \\
&= 4k^2 - 2k^2 \\
&= 2k^2
\end{aligned}$$

Applications of Double Integrals

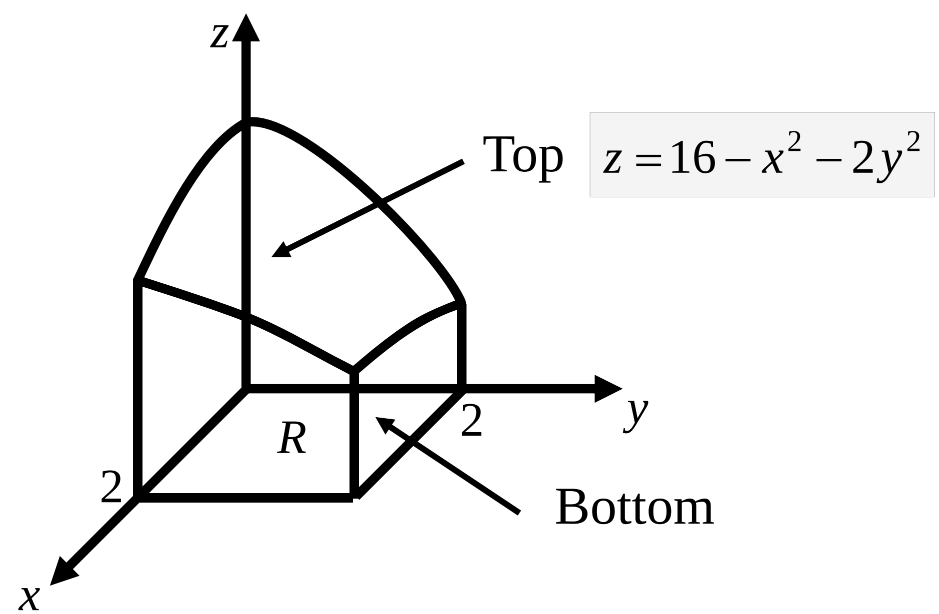
Volume

Suppose D is a solid region under the surface of a function $f(x, y)$ over a plane region R. Then the volume of D is given by

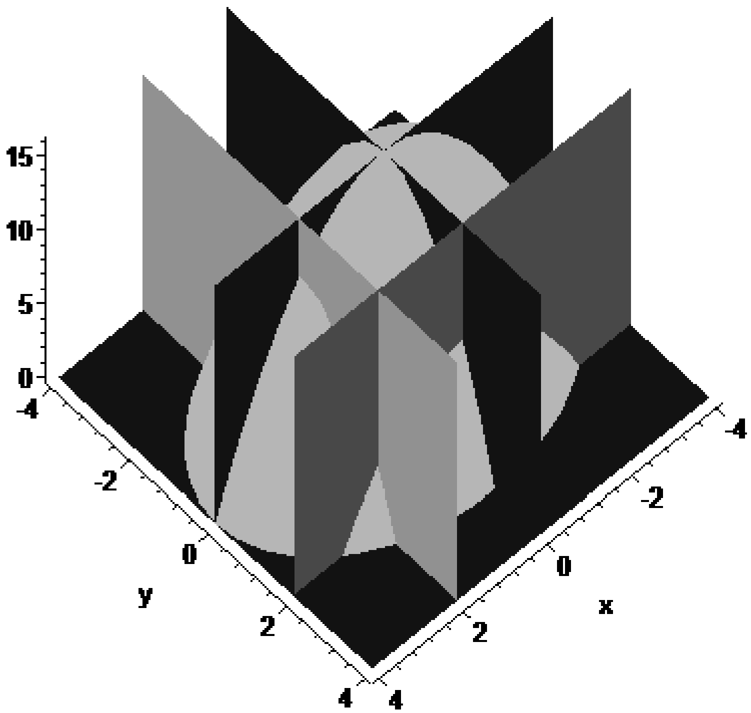
$$\iint_R f(x, y)\, dA$$

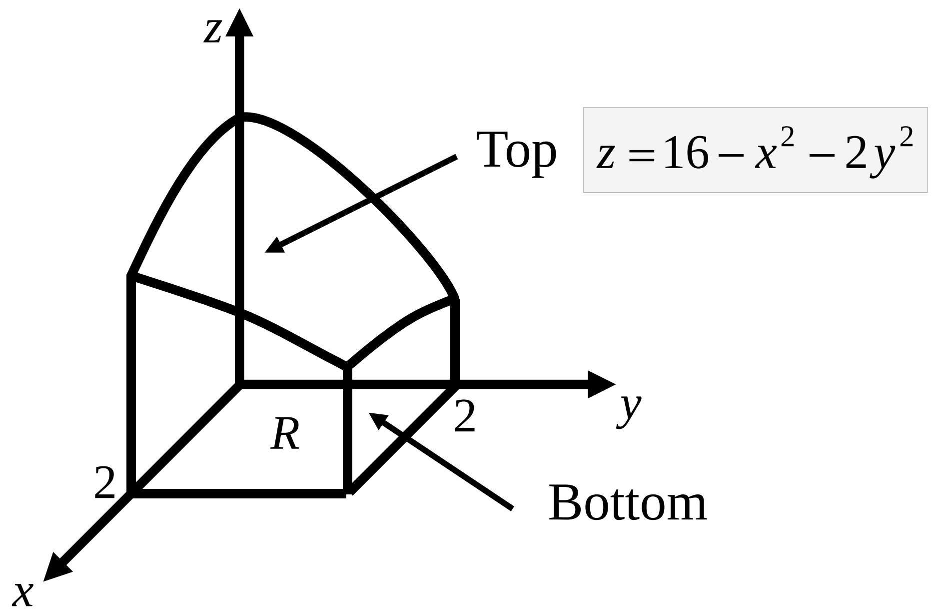
Volume - Example

Find the volume of the solid D that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$, $y = 2$ and the 3-coordinate planes.



Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$, $y = 2$, the 3 coordinate planes. (x = 0, y = 0, z = 0)

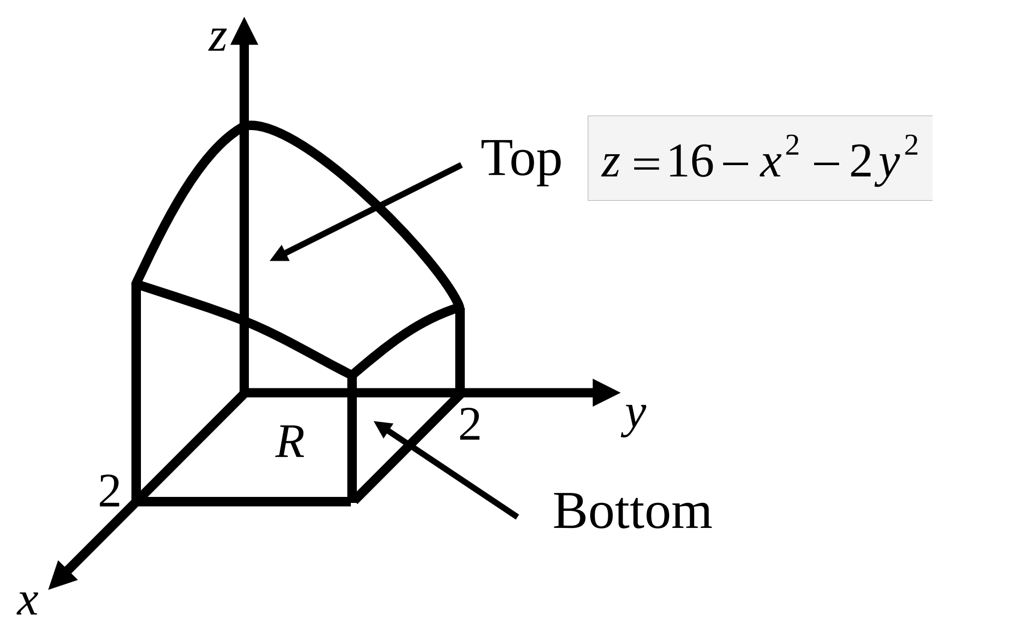




The solid region D is under the surface represented by the function $f(x, y) = 16 - x^2 - 2y^2$ and is above the rectangular region $R: 0 \le x \le 2,\ \ 0 \le y \le 2.$

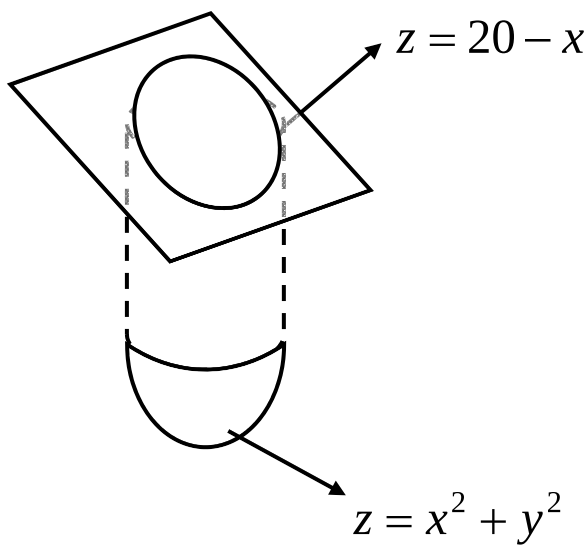
So the volume of D is

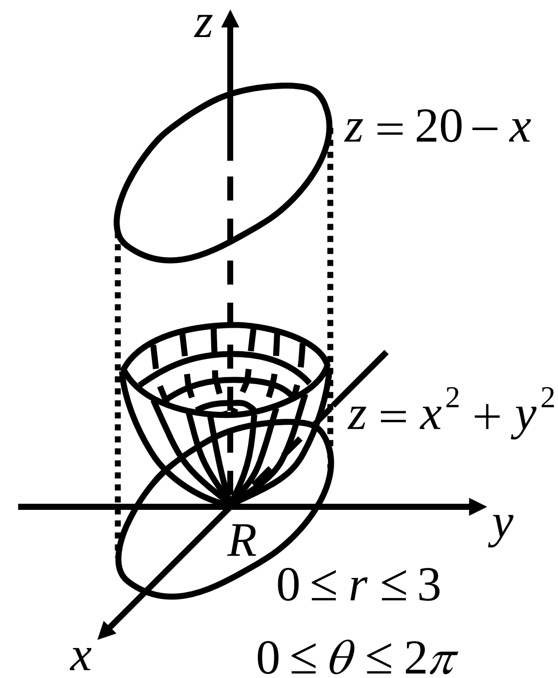
$$\iint_R (16 - x^2 - 2y^2)\, dA = \int_0^2 \int_0^2 (16 - x^2 - 2y^2)\, dx\, dy$$
$$= 48 \text{ units}^3$$



Volume - Example

Find the volume of the solid enclosed laterally by the circular cylinder about z-axis of radius 3 and bounded on top by the plane $x + z = 20$ and below by the paraboloid $z = x^2 + y^2$.





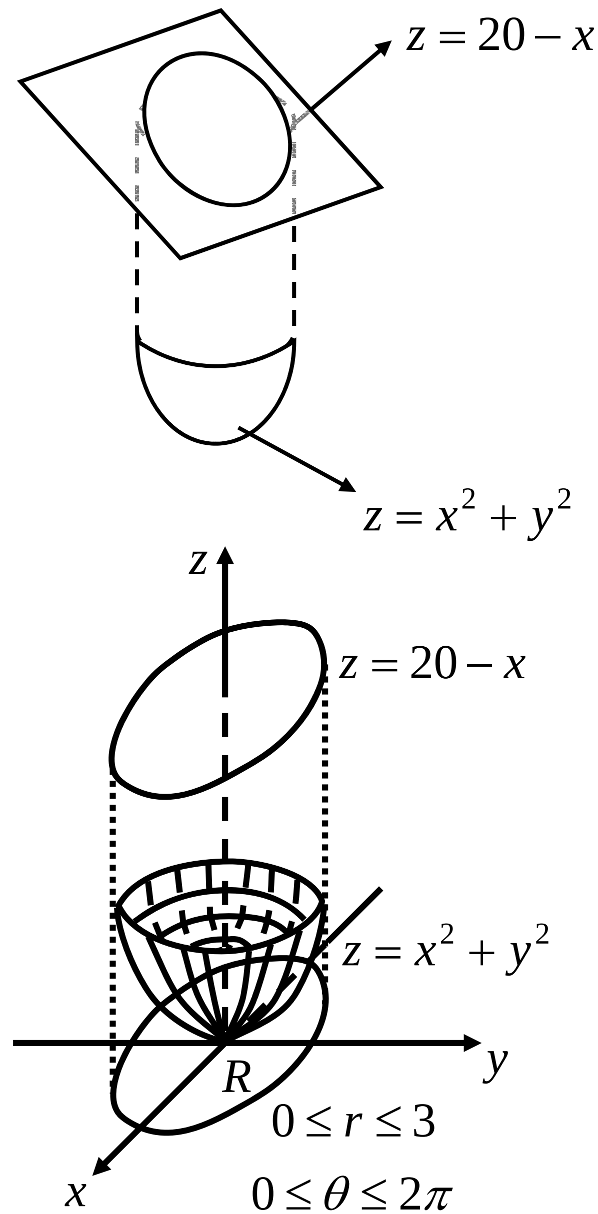
The volume can be computed as

$$V = \iint_R f_1(x,y)\, dA - \iint_R f_2(x,y)\, dA$$

where $f_1(x,y) = 20 - x$ and $f_2(x,y) = x^2 + y^2$,
and $R: \quad 0 \le r \le 3, \quad 0 \le \theta \le 2\pi.$

So the volume of the solid is

$$V = \iint_R (20 - x)\, dA - \iint_R x^2 + y^2 \; dA$$



$$V = \iint_R (20 - x)\, dA - \iint_R x^2 + y^2 \, dA$$

$$R: \quad 0 \le r \le 3, \quad 0 \le \theta \le 2\pi.$$

$$V = \int_0^{2\pi}\int_0^3 (20 - r\cos\theta) r \, dr \, d\theta - \int_0^{2\pi}\int_0^3 (r^2) r \, dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^3 (20r - r^2\cos\theta - r^3)\, dr \, d\theta$$

$$= \int_0^{2\pi} \left[10r^2 - \frac{r^3}{3}\cos\theta - \frac{r^4}{4} \right]_0^3 d\theta$$

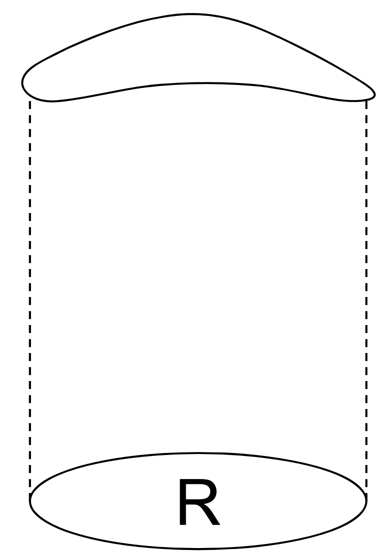
$$= \int_0^{2\pi} \left(90 - 9\cos\theta - \frac{81}{4} \right) d\theta$$

$$= \left[\frac{279}{4}\theta - 9\sin\theta \right]_0^{2\pi} = \frac{279}{2}\pi \text{ units}^3$$

Surface Area

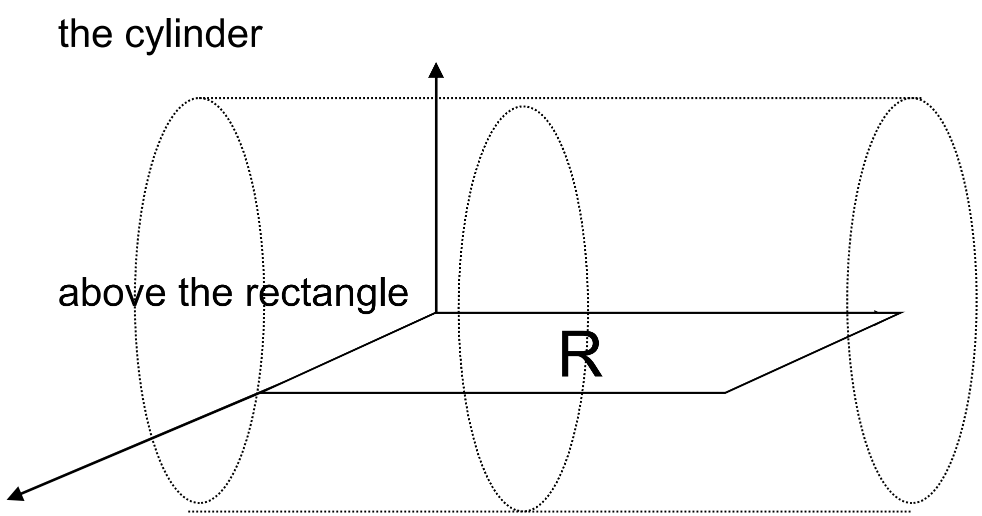
If f has continuous first partial derivatives on a closed region R of the xy-plane, then the area S of that portion of the surface $z = f(x, y)$ that projects onto R is

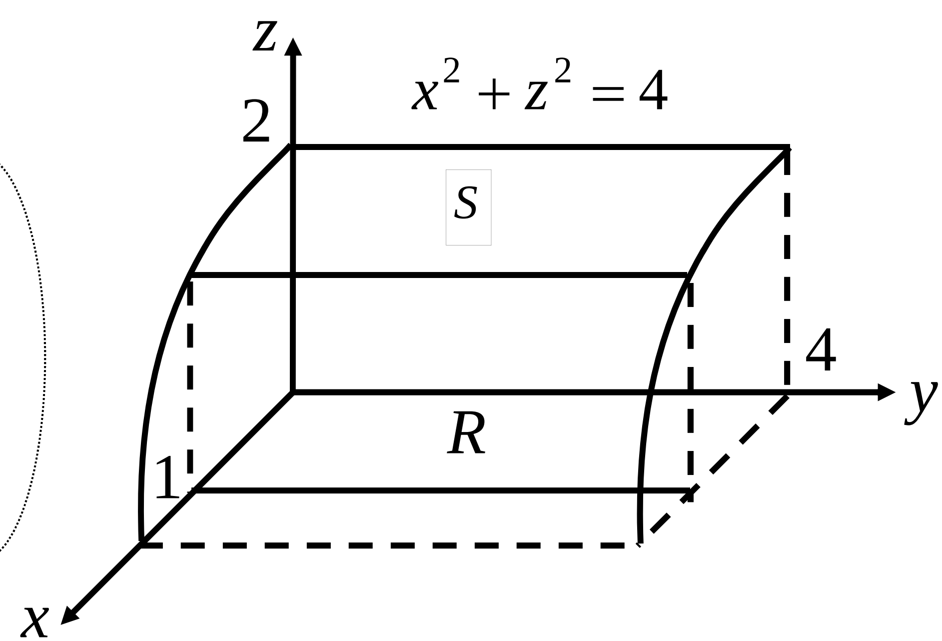
$$S = \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA.$$



Surface Area - Example

Find the surface area of the portion of the cylinder $x^2 + z^2 = 4$ above the rectangle $R:\ 0 \le x \le 1,\ \ 0 \le y \le 4.$

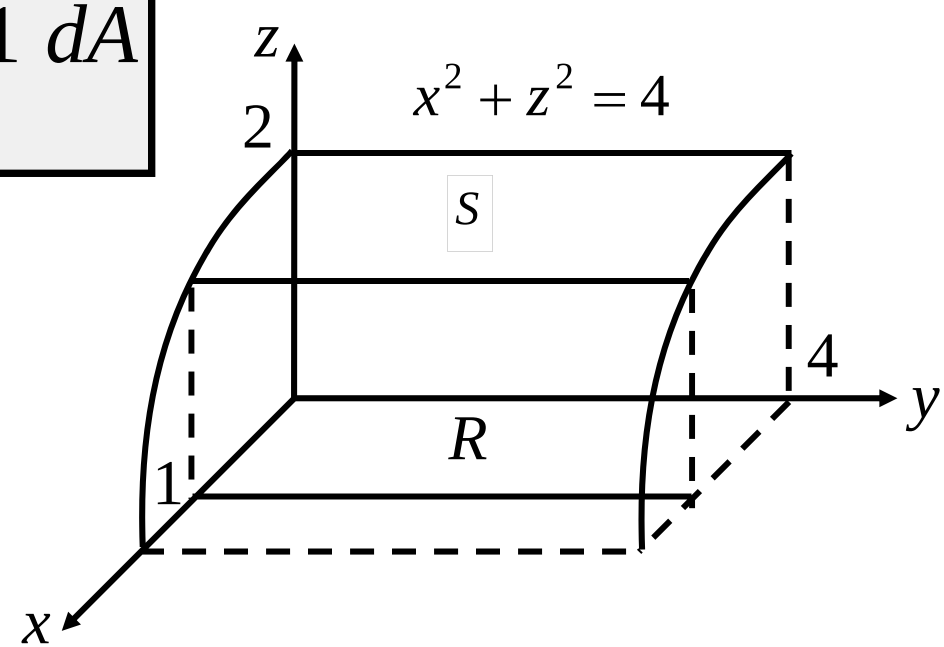




The portion of the cylinder $x^2 + z^2 = 4$ that lies above the xy-plane has the equation $z = \sqrt{4 - x^2}$.

So the surface is given by the function $f(x, y) = \sqrt{4 - x^2}$.

$$\boxed{S = \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA}$$



$$S = \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\; dA$$

$$z = \sqrt{4 - x^2}$$

Note that

$$\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} = \sqrt{\left(-\frac{x}{\sqrt{4 - x^2}}\right)^2 + 0^2 + 1}$$

$$= \frac{2}{\sqrt{4 - x^2}}.$$

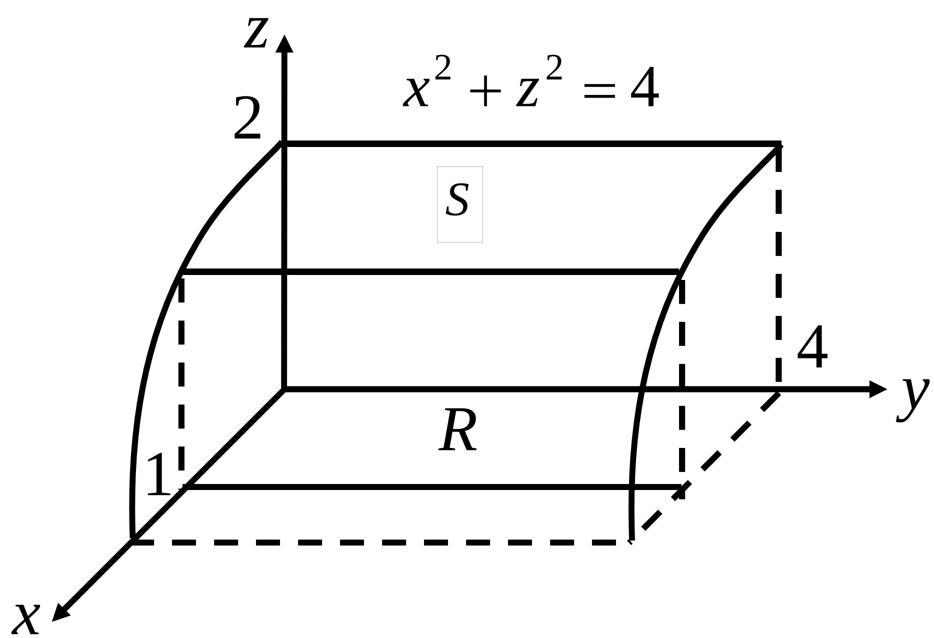
$$S = \int_0^4 \left[\int_0^1 \frac{2}{\sqrt{4 - x^2}} \, dx \right] dy$$

$$= 2\int_0^4 \left[\sin^{-1}\left(\frac{x}{2} \right) \right]_{x=0}^{x=1} dy$$

$$= 2\int_0^4 \frac{\pi}{6} \, dy$$

$$= \frac{4\pi}{3} \text{ units}^2.$$

$$R: \; 0 \le x \le 1, \quad 0 \le y \le 4$$



$$\boxed{\sin \frac{\pi}{6} = \frac{1}{2}}$$

Recall that $\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1}\left(\frac{x}{a} \right) + C.$

Mass and Center of Gravity

If a lamina with a continuous ***density function*** $\delta(x, y)$ occupies a region R in the xy-plane, its ***total mass*** M is given by the double integral

$$M = \iint_R \delta(x, y)\, dA$$

and its ***center of gravity*** $(\bar{x}, \bar{y})$ is

$$\bar{x} = \frac{\iint_R x\delta(x, y)\, dA}{\iint_R \delta(x, y)\, dA} = \frac{\iint_R x\delta(x, y)\, dA}{M},$$

$$\bar{y} = \frac{\iint_R y\delta(x, y)\, dA}{\iint_R \delta(x, y)\, dA} = \frac{\iint_R y\delta(x, y)\, dA}{M}.$$

Mass and Center of Gravity

Note that if $\delta(x, y)$ is a constant, then the center of gravity of the lamina is

$$\bar{x} = \frac{\iint_R x \, dA}{\iint_R 1 \, dA} = \frac{\iint_R x \, dA}{\text{Area of } R},$$

$$\bar{y} = \frac{\iint_R y \, dA}{\iint_R 1 \, dA} = \frac{\iint_R y \, dA}{\text{Area of } R}.$$

$$\boxed{\bar{x} = \frac{\iint_R x\delta(x, y) \, dA}{\iint_R \delta(x, y) \, dA} = \frac{\iint_R x dA}{\iint_R 1 \, dA}}$$

Mass and Center of Gravity - Example

Find the center of gravity of the triangular lamina with vertices (0,0), (0,1) and (1,0) and density function $\delta(x, y) = xy.$

The triangular lamina has boundaries

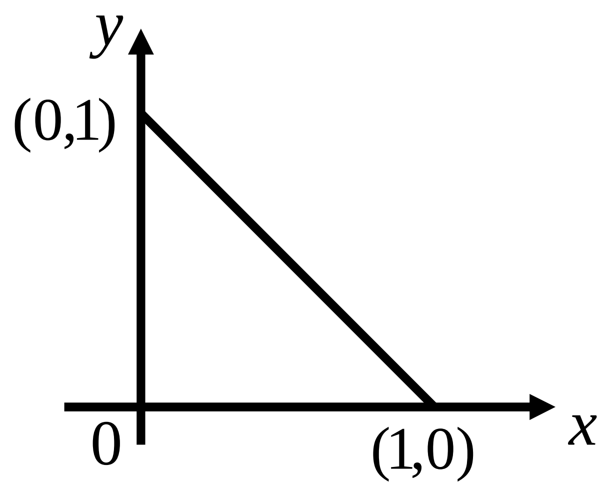
$$x = 0, y = 0, \text{ and } y = -x + 1.$$

It is described by

$$R: \quad 0 \le y \le -x + 1, \quad 0 \le x \le 1.$$

$$\bar{x} = \frac{\iint_R x\delta(x, y)\, dA}{\iint_R \delta(x, y)\, dA} = \frac{\iint_R x\delta(x, y)\, dA}{M},$$

$$\bar{y} = \frac{\iint_R y\delta(x, y)\, dA}{\iint_R \delta(x, y)\, dA} = \frac{\iint_R y\delta(x, y)\, dA}{M}$$



Find the center of gravity of the triangular lamina with vertices (0,0), (0,1) and (1,0) and density function $\delta(x, y) = xy.$

$R: \quad 0 \le y \le -x+1, \quad 0 \le x \le 1.$

The mass of the lamina is

$$M = \iint_R \delta(x, y)\, dA = \iint_R xy\, dA$$

$$= \int_0^1 \int_0^{-x+1} xy\, dy\, dx$$

$$= \int_0^1 \left[\frac{1}{2} xy^2 \right]_{y=0}^{y=-x+1} dx$$

$$= \int_0^1 \left[\frac{1}{2} x^3 - x^2 + \frac{1}{2} x \right] dx = \frac{1}{24}.$$

$$\begin{aligned}\iint_R x\delta(x,y)\, dA &= \iint_R x^2 y\, dA \\ &= \int_0^1 \int_0^{-x+1} x^2 y\, dy\, dx \\ &= \int_0^1 \left[\frac{1}{2} x^2 y^2 \right]_{y=0}^{y=-x+1} dx \\ &= \int_0^1 \left[\frac{1}{2} x^4 - x^3 + \frac{1}{2} x^2 \right] dx = \frac{1}{60}.\end{aligned}$$

$$\begin{aligned}\iint_R y\delta(x,y)\, dA &= \iint_R xy^2\, dA \\ &= \int_0^1 \int_0^{-x+1} xy^2\, dy\, dx \\ &= \int_0^1 \left[\frac{1}{3} xy^3 \right]_{y=0}^{y=-x+1} dx \\ &= \int_0^1 \left[-\frac{1}{3} x^4 + x^3 - x^2 + \frac{1}{3} x \right] dx = \frac{1}{60}.\end{aligned}$$

$$M = \iint_R \delta(x,y)\, dA = \frac{1}{24}$$

$$\iint_R x\delta(x,y)\, dA = \frac{1}{60}$$

$$\iint_R y\delta(x,y)\, dA = \frac{1}{60}$$

$$\bar{x} = \frac{\iint_R x\delta(x,y)\, dA}{\iint_R \delta(x,y)\, dA} = \frac{1/60}{1/24} = \frac{2}{5}$$

$$\bar{y} = \frac{\iint_R y\delta(x,y)\, dA}{\iint_R \delta(x,y)\, dA} = \frac{1/60}{1/24} = \frac{2}{5}$$

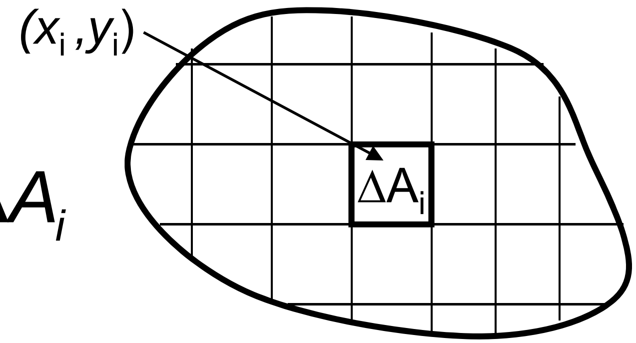
Triple Integral

Triple Integral

We can also define integration on functions of three variables over solid region in xyz-space.

Triple integral

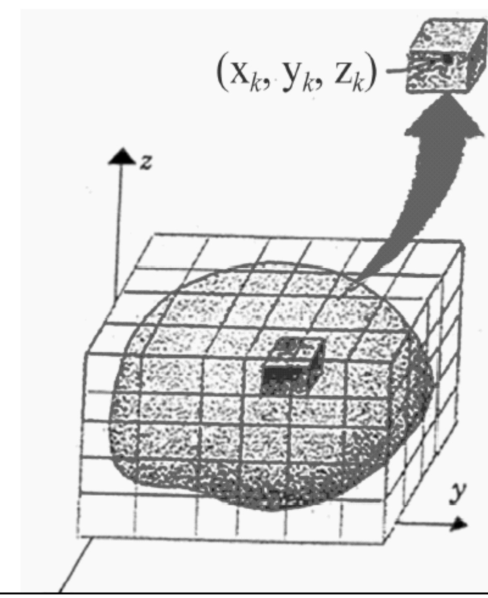
Function of two variables



$$\iint_R f(x,y)\,dA = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i, y_i)\Delta A_i$$

double integral sign indicate we are integrating over a two-dimensional region

Function of three variables



$$\iiint_D f(x,y,z)\,dV = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta V_i$$

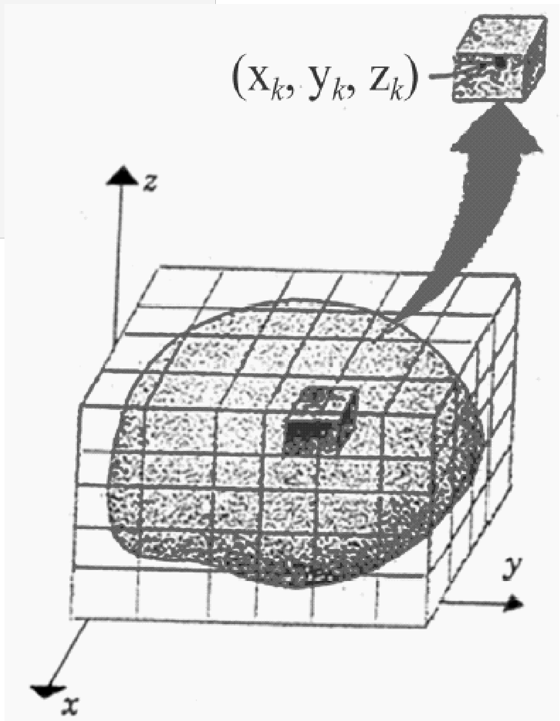
triple integral sign indicate we are integrating over a three-dimensional solid region

Triple Integral

Let D be a solid region in the xyz space. Subdivide D into smaller cubic region D_i for $i = 1, \cdots, n.$

Let ΔV_i be the volume of V_i and (x_i, y_i, z_i) be a point in D_i. Let $f(x, y, z)$ be a function of three variables. Then the triple integral of f over D is

$$\iiint_D f(x, y, z)\, dV = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta V_i.$$



Physical Meaning

No direct geometrical meaning for $\iiint_D f(x, y, z)\, dV$.

If the function f represents certain physical quantity, then $\iiint_D f(x, y, z)\, dV$ may have some physical meaning.

When f is the constant function 1, then

$$\iiint_D 1\, dV = \text{volume of } D.$$

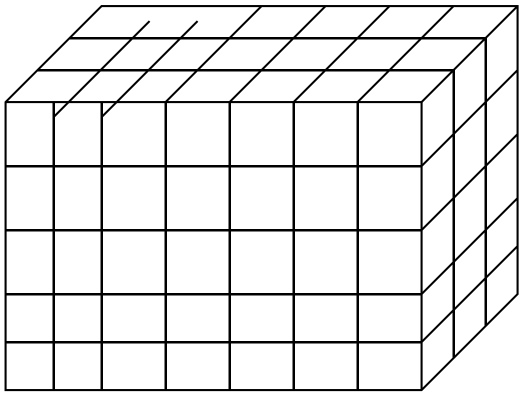
Geometrical meaning? None

Physical meaning depends on what physical quantity f(x,y,z) represents

Volume = V constant density = δ

What is the mass M?

$$M = \delta \times V$$



Geometrical meaning? None

Physical meaning depends on what physical quantity f(x,y,z) represents

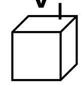
Volume = V

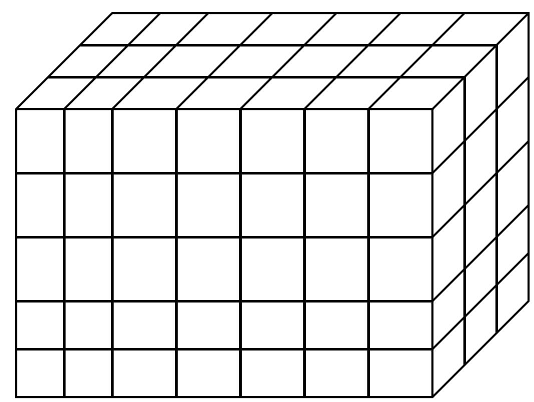
variable density = δ(x,y,z)

What is the mass M?

$$M_i = \delta(x_i, y_i, z_i) \times \Delta V_i$$

V_i





$$M \approx \sum_{i=1}^{n} \delta(x_i, y_i, z_i)\Delta V_i$$

$$M = \lim_{n\to\infty} \sum_{i=1}^{n} \delta(x_i, y_i, z_i)\Delta V_i = \iiint_D \delta(x, y, z)\,dV$$

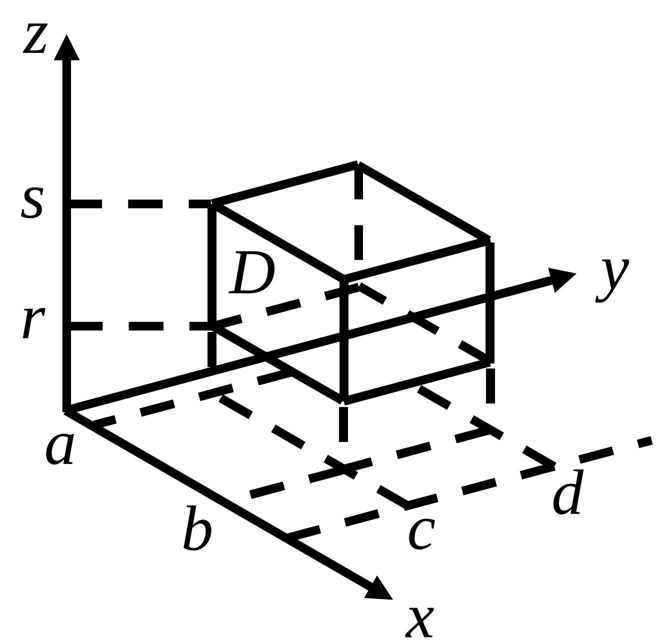
Let M be the mass of a solid object D with volume V and density function $\delta(x, y, z)$. Then

$$M = \iiint_D \delta(x, y, z)\, dV.$$

How to evaluate $\iiint_D \delta(x, y, z)\, dV$?

Rectangular Region

Suppose D is the rectangular box consisting of points (x, y, z) such that $D: a \le x \le b,\ c \le y \le d,\ r \le z \le s.$



$$\iiint_D f(x,y,z)\, dV = \int_a^b \int_c^d \int_r^s f(x,y,z)\, dz\, dy\, dx.$$

As in the case of double integrals, the order of integration with respect to the three variables does not affect the answer of the triple integrals.

Rectangular Region $D: a \le x \le b,\;\; c \le y \le d,\;\; r \le z \le s.$

$$\iiint_D f(x,y,z)\,dV = \int_a^b \int_c^d \int_r^s f(x,y,z)\,dz\,dy\,dx.$$

$$\iiint_D f(x,y,z)\,dV = \int_c^d \int_a^b \int_r^s f(x,y,z)\,dz\,dx\,dy.$$

$$\iiint_D f(x,y,z)\,dV = \int_a^b \int_r^s \int_c^d f(x,y,z)\,dy\,dz\,dx.$$

$$\iiint_D f(x,y,z)\,dV = \int_r^s \int_a^b \int_c^d f(x,y,z)\,dy\,dx\,dz.$$

$$\iiint_D f(x,y,z)\,dV = \int_c^d \int_r^s \int_a^b f(x,y,z)\,dx\,dz\,dy.$$

$$\iiint_D f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz.$$

Rectangular Region - Example

Evaluate $\iiint_D \frac{1}{xyz}\,dV$, where $D: 1 \le x \le e,\ 1 \le y \le e,\ 1 \le z \le e.$

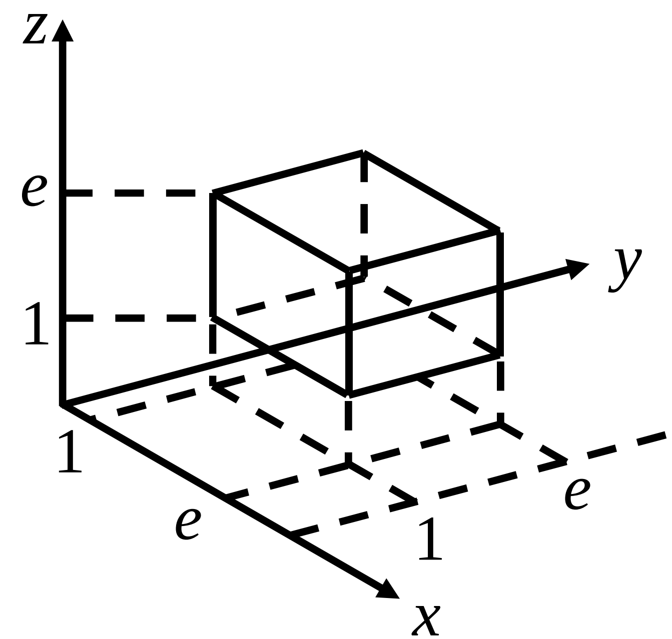
$$\iiint_D \frac{1}{xyz}\,dV = \int_1^e \int_1^e \int_1^e \frac{1}{xyz}\,dz\,dy\,dx$$

$$= \int_1^e \int_1^e \left[\frac{\ln z}{xy}\right]_{z=1}^{z=e} dy\,dx$$

$$= \int_1^e \int_1^e \frac{1}{xy}\,dy\,dx$$

$$= \cdots$$

$$= 1$$



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