Chapter 4
Sequences & Series

Overview

- Infinite Sequences
  - Limits of Sequences
  - Sequences & Functions

- Infinite Series
  - Partial Sums
  - Geometric Series
  - Ratio Test

Overview

- Power Series
  - Convergence of Power Series
  - Radius of Convergence
  - Differentiation and Integration of Power Series

- Taylor Series
  - Definition
  - Taylor Polynomials
  - An Application of Taylor Polynomials

Infinite Sequences

Infinite Sequences

- A sequence of real numbers:

$$a_1,\ a_2,\ \cdots\ ,a_n,\ \cdots$$

a_n : general term of the sequence

We use $\{a_n\}$ to denote an infinite sequence

Infinite Sequences

When we look at an infinite sequence $\{a_n\}$, we are looking at an infinite list of numbers:

$$a_1,\ a_2,\ \cdots\ ,a_n,\ \cdots$$

Infinite Sequences - Examples

(i) $a_n = n - 1$

$$a_1,\ a_2,\ a_3,\ \cdots,\ a_n,\ \cdots$$

$$0,\ 1,\ 2,\ \cdots,\ n-1,\ \cdots$$

(ii) $a_n = \dfrac{1}{n}$

$$1,\ \frac{1}{2},\ \frac{1}{3},\ \cdots,\ \frac{1}{n},\ \cdots$$

Infinite Sequences - Examples

(ii) $a_n = \dfrac{1}{n}$

$$1,\ \frac{1}{2},\ \frac{1}{3},\ \ldots,\ \frac{1}{n},\ \ldots$$

(iii) $a_n = (-1)^{n+1}\left(\dfrac{1}{n}\right)$

$$1,\ -\frac{1}{2},\ \frac{1}{3},\ -\frac{1}{4},\ \frac{1}{5},\ \ldots$$

Alternate sign

Infinite Sequences - Examples

Alternate sign

$(-1)^{n+1}$ gives $+,\ -,\ +,\ -,\ +,\ \cdots$

$(-1)^{n}$ gives $-,\ +,\ -,\ +,\ -,\ +,\ \cdots$

Infinite Sequences - Examples

(iv) $a_n = \dfrac{n-1}{n}$

$$0,\ \frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \ldots$$

(v) $a_n = (-1)^{n+1}$

$$1, -1, 1, -1, \cdots$$

Infinite Sequences - Examples

(vi) $a_n = k$ (where k is a constant)

$$k, k, k, k, \cdots$$

constant sequence with every term having value k

Limits of Sequences

A number L is called the ***limit*** of a sequence $\{a_n\}$, if for sufficiently large n, we can get a_n as close as we want to a number L.

We write

$$\lim_{n\to\infty} a_n = L$$

or

$$a_n \to L$$

Limits of Sequences

A number L is called the ***limit*** of a sequence $\{a_n\}$, if for sufficiently large n, we can get a_n as close as we want to a number L.

$$\lim_{n\to\infty} a_n = L$$

or

$$a_n \to L$$

When we look at the limit of a sequence $\{a_n\}$, we are interested to know what happen to the value of a_n when n is sufficiently large.

Limits of Sequences

The limit of $\{a_n\}$, if it ***exists***, is ***unique***.

If $\lim_{n\to\infty} a_n = L$, we say that $\{a_n\}$ is ***convergent***

and $\{a_n\}$ ***converges*** to L.

If $\lim_{n\to\infty} a_n$ ***does not exist***, we say that $\{a_n\}$ is ***divergent***.

Limits of Sequences - Examples

A number L is called the ***limit*** of a sequence $\{a_n\}$, if for sufficiently large n, we can get a_n as close as we want to a number L.

When we look at the limit of a sequence $\{a_n\}$, we are interested to know what happen to the value of a_n when n is sufficiently large.

(i) $a_n = n - 1$

$$0,\ 1,\ 2,\ \cdots,\ n-1,\ \cdots$$

Divergent

The value of a_n gets bigger and bigger. So when n is large, the value of a_n is not near to any finite number L.

Limits of Sequences - Examples

A number L is called the ***limit*** of a sequence $\{a_n\}$, if for sufficiently large n, we can get a_n as close as we want to a number L.

When we look at the limit of a sequence $\{a_n\}$, we are interested to know what happen to the value of a_n when n is sufficiently large.

(ii) $a_n = \dfrac{1}{n}$

$$1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots$$

Convergent, $a_n \to 0$

Note that when n is large, $\frac{1}{n} \to 0$. So when n is large, the value of a_n is near to 0.

Limits of Sequences - Examples

A number L is called the ***limit*** of a sequence $\{a_n\}$, if for sufficiently large n, we can get a_n as close as we want to a number L.

When we look at the limit of a sequence $\{a_n\}$, we are interested to know what happen to the value of a_n when n is sufficiently large.

(iii) $a_n = (-1)^{n+1}\left(\frac{1}{n}\right)$

$$1,\ -\frac{1}{2},\ \frac{1}{3},\ -\frac{1}{4},\ \frac{1}{5},\ \cdots$$

Convergent, $a_n \to 0$

Note that when n is large, $(-1)^{n+1}\frac{1}{n} \to 0$. So when n is large, the value of a_n is near to 0.

Limits of Sequences - Examples

A number L is called the ***limit*** of a sequence $\{a_n\}$, if for sufficiently large n, we can get a_n as close as we want to a number L.

When we look at the limit of a sequence $\{a_n\}$, we are interested to know what happen to the value of a_n when n is sufficiently large.

(iv) $a_n = \dfrac{n-1}{n}$

$$0,\ \frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \cdots$$

Convergent, $a_n \to 1$

Note that:

$$a_n = \frac{n-1}{n} = \frac{n}{n} - \frac{1}{n} = 1 - \frac{1}{n}$$

Note that when n is large, $\frac{1}{n} \to 0$. So when n is large, the value of a_n is near to 1.

Limits of Sequences - Examples

A number L is called the ***limit*** of a sequence $\{a_n\}$, if for sufficiently large n, we can get a_n as close as we want to a number L.

When we look at the limit of a sequence $\{a_n\}$, we are interested to know what happen to the value of a_n when n is sufficiently large.

(v) $a_n = (-1)^{n+1}$

$$1, -1, 1, -1, \cdots$$

Divergent

Note that when n is even, $a_n = -1$
when n is odd, $a_n = 1$.

So when n is large,
the value of a_n is not near to any finite number L.

Limits of Sequences - Examples

A number L is called the ***limit*** of a sequence $\{a_n\}$, if for sufficiently large n, we can get a_n as close as we want to a number L.

When we look at the limit of a sequence $\{a_n\}$, we are interested to know what happen to the value of a_n when n is sufficiently large.

(vi) $a_n = k$ (where k is a constant)

$$k, k, k, k, \cdots$$

Convergent, $a_n \to k$

constant sequence with every term having value k

So when n is large,
the value of a_n is always equal to k.

Some Rules on Limits

Let $\lim\limits_{n\to\infty} a_n = A$ and $\lim\limits_{n\to\infty} b_n = B$, with A and B real numbers.

(1) Sum rule: $\lim\limits_{n\to\infty}(a_n + b_n) = A + B.$

(2) Difference rule: $\lim\limits_{n\to\infty}(a_n - b_n) = A - B.$

(3) Product rule: $\lim\limits_{n\to\infty}(a_n b_n) = AB.$

(4) Quotient rule: $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = \dfrac{A}{B}$, if $B \neq 0$.

Sequences & Function

Let $\{a_n\}$ be a sequence.

Suppose there is a function $f(x)$ such that

$$a_n = f(n).$$

If $\lim\limits_{x\to\infty} f(x) = L,$ then $\lim\limits_{n\to\infty} a_n = L$

What is true for the function $f(x)$ is also true for the sequence a_n.

Sequences & Function

Let $\{a_n\}$ be a sequence.

Suppose there is a function $f(x)$ such that

$$a_n = f(n).$$

If $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} a_n = L$

Pause and Think !!!

What is the difference between

$f(x)$ and $f(n)$???

Pause and Think !!!

What is the difference between

$f(x)$ and $f(n)$???

When we look at $f(n)$, we consider

$f(1),\ f(2),\ f(3),\ \ldots\ldots$

since n is an integer.

When we look at $f(x)$, we consider

x to be a real number.

What is true for the function $f(x)$

is also true for the sequence a_n.

Sequences & Function - Example

Evaluate $\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$.

Consider the sequence $\{a_n\}$ where $a_n = \sqrt{n+1}-\sqrt{n}$.

Then $a_n = f(n)$, where $f(x)$ is the function $\sqrt{x+1}-\sqrt{x}$.

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$$

$$= \lim_{x\to\infty}\left(\sqrt{x+1}-\sqrt{x}\right)$$

$$= 0$$

$$\lim_{x\to\infty}\left(\sqrt{x+1}-\sqrt{x}\right)\times\frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}+\sqrt{x}}$$

$$= \lim_{x\to\infty}\frac{(x+1)-x}{\sqrt{x+1}+\sqrt{x}}$$

$$= \lim_{x\to\infty}\frac{1}{\sqrt{x+1}+\sqrt{x}}$$

$$= 0.$$

Sequences & Function - Example

Show that $\lim_{n\to\infty} \frac{\ln n}{n} = 0.$

Consider the sequence $\{a_n\}$ where $a_n = \frac{1n\, n}{n}$.

Then $a_n = f(n)$, where $f(x)$ is the function $\frac{\ln x}{x}$ (defined for $x > 0$).

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$$

$$\begin{aligned}\lim_{n\to\infty} \frac{\ln n}{n} &= \lim_{x\to\infty} \frac{\ln x}{x} \\ &= \lim_{x\to\infty} \frac{\frac{1}{x}}{1} \\ &= 0.\end{aligned}$$

apply L'Hopital's rule

Infinite Series

Infinite Series

When we look at an infinite sequence $\{a_n\}$, we are looking at an infinite list of numbers:

$$a_1,\ a_2,\ \cdots\ ,\ a_n,\ \cdots$$

If we consider the ***sum***

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots,$$

we get an ***infinite series***.

In notation, we write

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n.$$

The term a_n is called the n-th term of the series.

Infinite Series

From the sequence

- $$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \ldots\ldots$$

We obtain the finite series,

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\cdots$$

The n-th term is $a_n = \dfrac{1}{2^n}$.

$$\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \cdots$$

$$a_1,\ a_2,\ a_3,\ \cdots\ ,\ a_n,\ \cdots$$

Note

$$\text{Infinite Sequence}: a_1, a_2, a_3, \cdots, a_n, \cdots$$

$$\text{Infinite Series}: \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

Infinite Series

Pause and Think !!!

A bit of Advanced Mathematics:

$$1-1+1-1+1-1+1-1+\cdots=?$$

Which of the following is true?

(a) $$(1-1)+(1-1)+(1-1)+(1-1)+\cdots$$
$$=0+0+0+0+\cdots=0.$$

(b) $$1+(-1+1)+(-1+1)+(-1+1)+\cdots$$
$$=1+0+0+0+\cdots=1.$$

Infinite Series

$$\text{Infinite Series}: \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

What is

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = ?$$

Infinite Series

$$\text{Infinite Series}: \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

What is $a_1 + a_2 + a_3 + \cdots + a_n + \cdots = ?$

We are adding infinite number of terms, so the sum might not be finite.

If the sum if finite, we may or may not be able to find the finite sum

We want to know when an infinite series gives a finite sum.

If we get a finite sum, we say the infinite series is convergent, otherwise, we say the infinite series is divergent.

Partial Sums

$$\text{Infinite Series}: \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

Consider:

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_n &= a_1 + a_2 + \cdots + a_n \\ &\vdots \end{aligned}$$

$s_1, s_2, s_3, \cdots, s_n, \cdots$ is called the sequence of ***partial sums***.

The term s_n is called the ***n - th partial sum***.

Given an infinite series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$$

we add 'all' the terms of the sequence $a_1,\ a_2,\ a_3,\ \cdots,\ a_n,\ \cdots$

So, we are adding infinite number of terms.

Consider the ***partial sum*** $s_n = a_1 + a_2 + \cdots + a_n$.

If we let $n \to \infty$, we have

$$\lim_{n\to\infty} s_n = \underbrace{a_1 + a_2 + \cdots + \cdots}_{\substack{\text{add infinite number} \\ \text{of terms since } n\to\infty}}$$

Thus, we can think that $\sum_{n=1}^{\infty} a_n$ is the same as $\lim_{n\to\infty} s_n$.

Consider the ***partial sum*** $s_n = a_1 + a_2 + \cdots + a_n$.

If we let $n \to \infty$, we have

$$\lim_{n\to\infty} s_n = \underbrace{a_1 + a_2 + \cdots + \cdots}_{\substack{\text{add infinite number} \\ \text{of terms since } n\to\infty}}$$

Thus, we can think that $\sum_{n=1}^{\infty} a_n$ is the same as $\lim_{n\to\infty} s_n$.

So if $\lim_{n\to\infty} s_n = L$, then we can say that

$$a_1 + a_2 + \cdots + a_n + \cdots = L$$

$$\sum_{n=1}^{\infty} a_n = L$$

If the sequence of partial sums $\{s_n\}$ is such that

$s_n \to L$, then we say that the series $\sum_{n=1}^{\infty} a_n$ is ***convergent***

and $\sum_{n=1}^{\infty} a_n = L$ (we say that the sum is L).

If the sequence of partial sums $\{s_n\}$ ***does not converge***, then

we say that the series $\sum_{n=1}^{\infty} a_n$ is divergent.

If the sequence of partial sums $\{s_n\}$ ***does not converge***, then we say that the series $\sum_{n=1}^{\infty} a_n$ is divergent.

If the sequence of partial sums $\{s_n\}$ do not converge, it could be because:

(i) $\lim_{n\to\infty} s_n$ does not exist, or

(ii) $\lim_{n\to\infty} s_n = \infty$ or $-\infty$

Pause and Think !!!

A bit of Advanced Mathematics :

$$1-1+1-1+1-1+1-1+\cdots=?$$

Sequence of [artial sums:

$$1,\ 0,\ 1,\ 0,\ 1,\ 0,\ 1,\ 0,\ \cdots$$

$$S_1, S_2, S_3, \cdots, S_n, \cdots$$

The sequence is divergent and so the infinite sum

$$1-1+1-1+1-1+1-1+\cdots$$

is divergent.

Pause and Think !!!

A bit of Advanced Mathematics :

$$1-1+1-1+1-1+1-1+\cdots=?$$

$$s_1=1$$

$$s_2=1-1=0$$

$$s_3=1-1+1=1$$

$$s_4=1-1+1-1=0$$

$\therefore$ the sequence of partial sums alternates between 0 and 1.

So $\lim_{n\to\infty} s_n$ does not exist.

Thus the infinite sum

$$1-1+1-1+1-1+1-1+\cdots$$

is divergent.