Q7

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

use http://wims.unice.fr/wims

eigenvectors are

$$\lambda_1 = -\frac{1}{2}\left(3\sqrt{33} - 15\right)$$

$$\lambda_2 = \frac{1}{2}\left(3\sqrt{33} + 15\right)$$

$$\lambda_3 = 0$$

Corresponding eigenvectors are

$$u_1 = \begin{pmatrix} 1 \\ -\frac{1}{16}\left(3\sqrt{33} - 19\right) \\ -\frac{1}{8}\left(3\sqrt{33} - 11\right) \end{pmatrix} \qquad u_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$u_2 = \begin{pmatrix} 1 \\ \frac{1}{16}\left(3\sqrt{33} + 19\right) \\ \frac{1}{8}\left(3\sqrt{33} + 11\right) \end{pmatrix}$$

These three eigenvectors are nonparallel (linearly indep.)

$\therefore$ Any vector in $\mathbb{R}^3$ can be written as

$$\alpha u_1 + \beta u_2 + \gamma u_3$$

Now

$$A(\alpha u_1 + \beta u_2 + \gamma u_3)$$

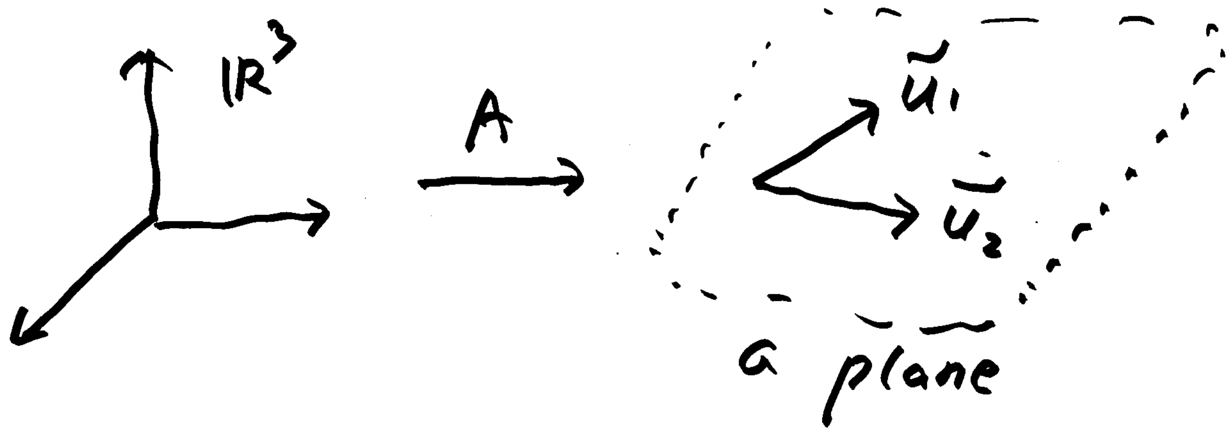
$$= \alpha A u_1 + \beta A u_2 + \gamma A u_3$$

$$= \alpha \lambda_1 u_1 + \beta \lambda_2 u_2 + \gamma \lambda_3 u_3$$

$$= (\alpha \lambda_1) u_1 + (\beta \lambda_2) u_2$$

$\therefore$ A maps every pt in $\mathbb{R}^3$ to a plane induced by u_1 and u_2

$\therefore$ A maps $\mathbb{R}^3$ to a plane induced by $\vec{u_1}$ and $\vec{u_2}$



$\therefore$ A is rank 2

$$(u_1 \times u_2)\cdot w = 0$$

$\Leftrightarrow$ w lies in the plane

We can check that

$$(u_1 \times u_2)\cdot \begin{pmatrix}1\\2\\4\end{pmatrix} \neq 0$$

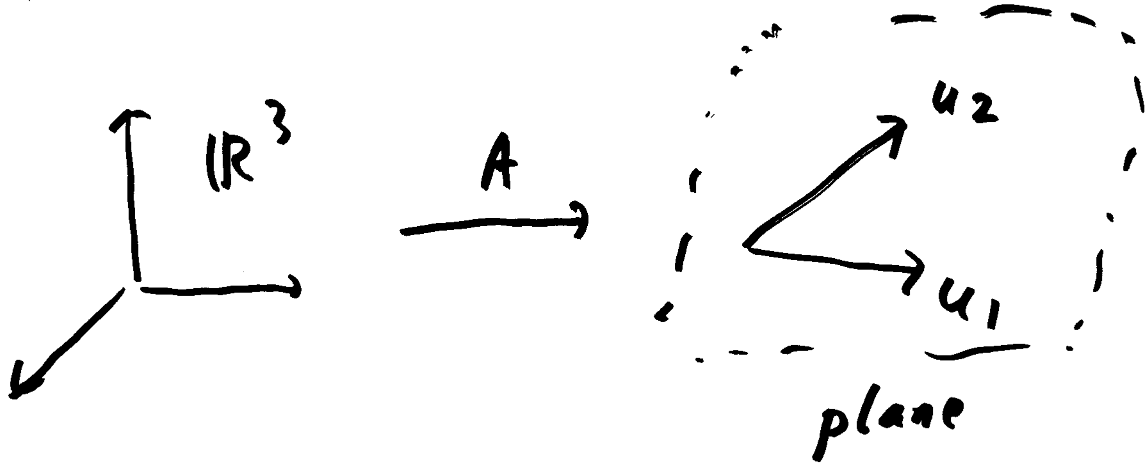
$\therefore \begin{pmatrix}1\\2\\4\end{pmatrix}$ does not lie in the plane

$$(u_1 \times u_2)\cdot \begin{pmatrix}1\\2\\3\end{pmatrix} = 0$$

$\therefore \begin{pmatrix}1\\2\\3\end{pmatrix}$ lies in the plane

Recall $\left[\begin{pmatrix}11\\12\\13\end{pmatrix}\times\begin{pmatrix}14\\15\\16\end{pmatrix}\right]\cdot\begin{pmatrix}17\\18\\19\end{pmatrix}$

$= \det\begin{pmatrix}11 & 12 & 13\\14 & 15 & 16\\17 & 18 & 19\end{pmatrix}$



$\begin{pmatrix}1\\2\\4\end{pmatrix}$ <u>not</u> in the plane

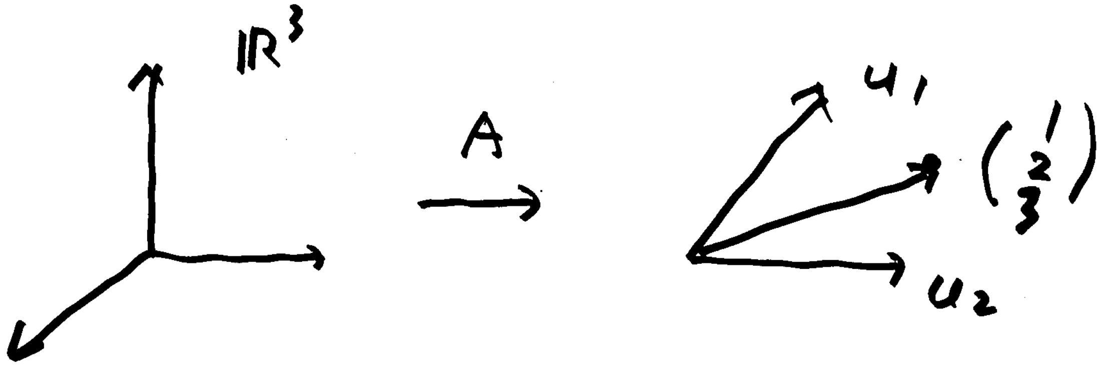
Hence no vector in $\mathbb{R}^3$ maps to $\begin{pmatrix}1\\2\\4\end{pmatrix}$

$$A\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\2\\4\end{pmatrix}$$

can't find such $\begin{pmatrix}x\\y\\z\end{pmatrix}$

$\therefore$ No solution

Since $\begin{pmatrix}1\\2\\3\end{pmatrix}$ lies on the plane



$\therefore$ We can find $\begin{pmatrix}x_0\\y_0\\z_0\end{pmatrix}$ in $\mathbb{R}^3$

such that $$A\begin{pmatrix}x_0\\y_0\\z_0\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix}$$

On the other hand eigenvalue $= 0$

$$A\begin{pmatrix}1\\-2\\1\end{pmatrix} = \lambda_3\begin{pmatrix}1\\-2\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\therefore \quad A\,c\begin{pmatrix}1\\-2\\1\end{pmatrix} = c\,A\begin{pmatrix}1\\-2\\1\end{pmatrix} = c\begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

c any real number

$\therefore$

$$A\left[\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + c\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\right]$$ c any real number

$$= A\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + cA\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 0 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$\therefore$ many solutions