MA1506

Mathematics II

Chapter 3

Basic Mathematical Modelling

What is Modelling?

- Art of using mathematics to analyze simple situations
- To *approximate* complicated realistic situation

Mathematical Modelling

Practical Problems

- Improve efficiency of chemical reactor
- Maximize audio signal output
- Analyze impact of raising taxi fares

Develop a model

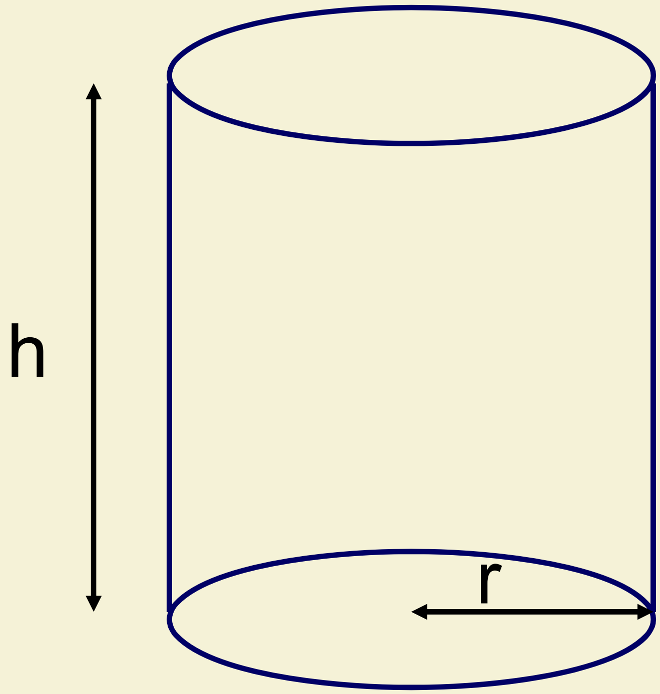
Equations

Validate / Implement

How should cans be made?

- Aim: Minimize Costs
- Costs depends on
  - Raw material
  - Labour
  - Production costs
- Minimize amount of tin (aluminium)

Model 1 : Minimize amount of tin



$$A = 2\pi r^2 + 2\pi r h$$

A is min when *r = 0*

Add Condition: Fixed Volume

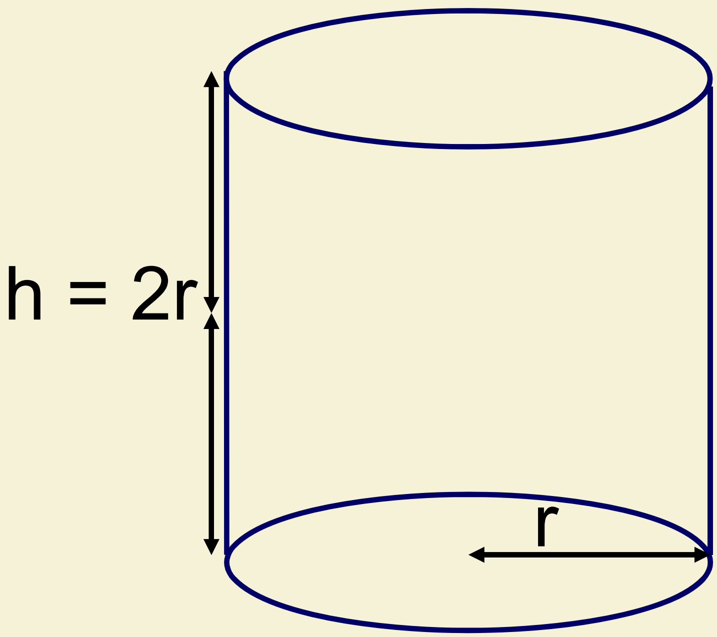
$$V = \pi r^2 h$$

$$A = 2\pi r^2 + \frac{2V}{r}$$

$$A' = 4\pi r - \frac{2V}{r^2} = 0$$

$$V = 2\pi r^3 \quad \Rightarrow \quad h = 2r$$

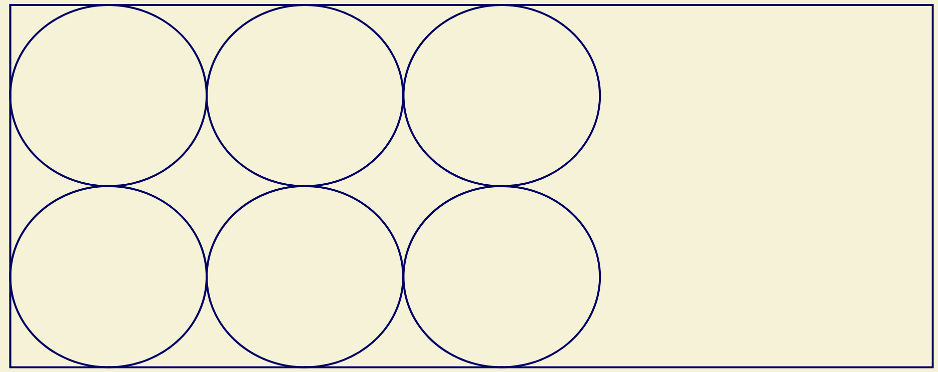
Model 1 : h = 2r



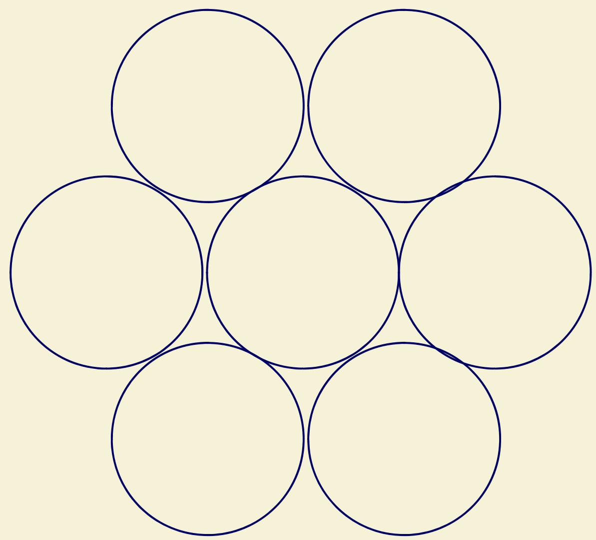
Something is not quite right!

Model 2 : Model 1 + min wastage

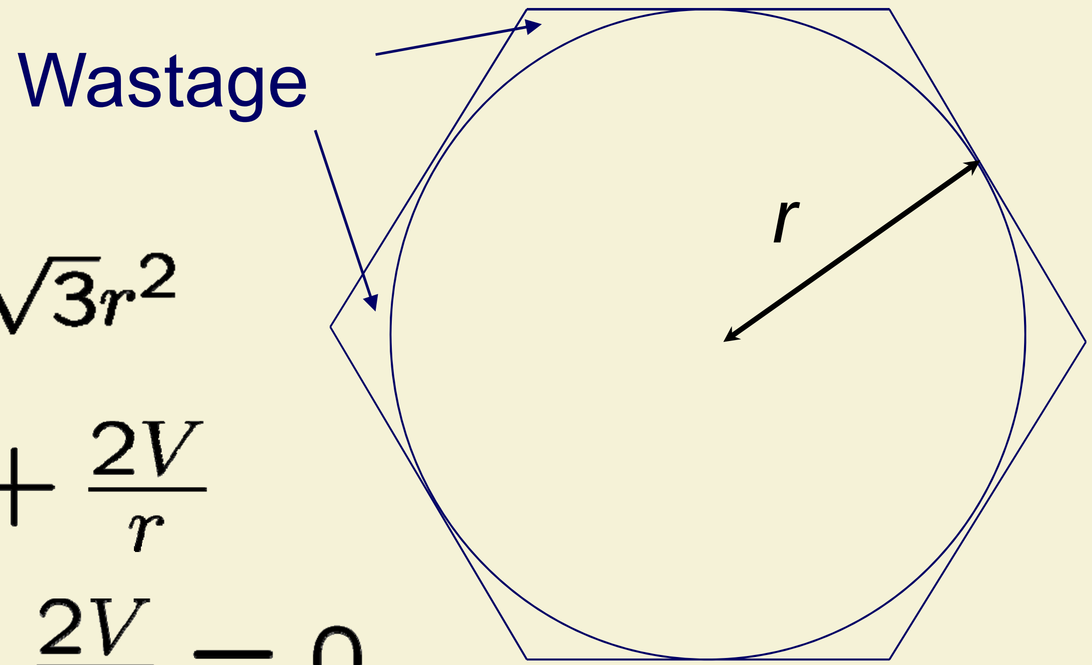
Tops and bottoms



Better packing



Model 2 : Model 1 + min wastage



Area of hexagon = $2\sqrt{3}r^2$

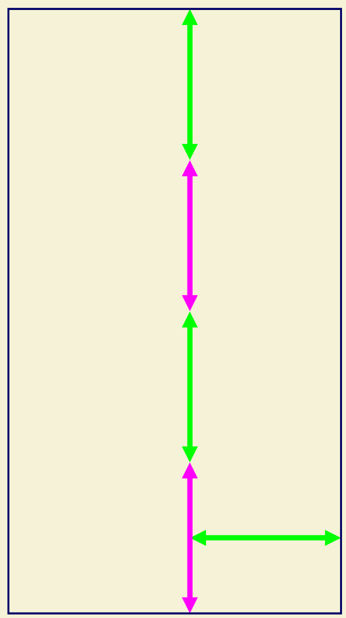
$$A = 4\sqrt{3}r^2 + \frac{2V}{r}$$

$$A' = 8\sqrt{3}r - \frac{2V}{r^2} = 0$$

$$\pi r^2 h = V = 4\sqrt{3}r^3$$

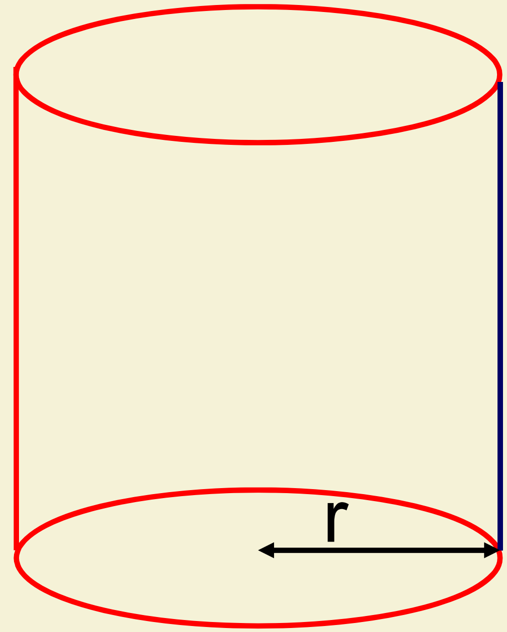
$$\Rightarrow \quad h = \frac{4\sqrt{3}}{\pi}r \approx 2.21r$$

Model 2 : h = 2.2r



Still not quite right!

Model 3 : Model 2 +
Manufacturing Process



Welding the top and bottom and side

\$/cm^2 \$/cm h

$$C(r) = J\left(4\sqrt{3}r^2 + \frac{2V}{r}\right) + K\left(4\pi r + \frac{V}{\pi r^2}\right)$$

$$C'(r) = J\left(8\sqrt{3}r - \frac{2V}{r^2}\right) + K\left(4\pi - \frac{2V}{\pi r^3}\right) = 0$$

$$\Rightarrow \frac{h}{r} = \frac{4\sqrt{3} + \frac{2\pi K}{rJ}}{\pi + \frac{K}{rJ}}$$

units : K/J = cm

Model 3 : Model 2 + Manufacturing Process

K/J sets the scale

$$\frac{h}{r} = \frac{4\sqrt{3} + \frac{2\pi K}{rJ}}{\pi + \frac{K}{rJ}}$$

$\boxed{K/J \ll r}$

$$\frac{h}{r} \approx \frac{4\sqrt{3}}{\pi}$$

Model 2

$\boxed{K/J \gg r}$

$$\frac{h}{r} \approx \frac{\frac{2\pi K}{rJ}}{\frac{K}{rJ}} = 2\pi$$

http://www.yankeecontainers.com/Tin-Cans-Paint-Cans.html

Summary

- We have been constructing models, that is, very simple versions of a real problem. The real problem is very complicated, the model is just an <u>approximation</u>, but it is easier to understand.

- **Basic Principle:** begin with simple models, understand their weaknesses, and only then make them more complicated!

3.2 Malthus Model of Population

Total Population: $N(t)$

Per Capita Birth-Rate, B

$$\#\text{ babies born in } \delta t = BN\,\delta t$$

Per Capita Death-Rate, D

$$\#\text{ deaths in } \delta t = DN\,\delta t$$

$$\begin{aligned}\delta N &= \#\text{ births} - \#\text{ deaths}\\ &= (B-D)N\delta t\end{aligned}$$

Thomas Malthus
1766 -1834

3.2 Malthus Model of Population

$$\boxed{\frac{dN}{dt} = (B - D)N = kN}$$

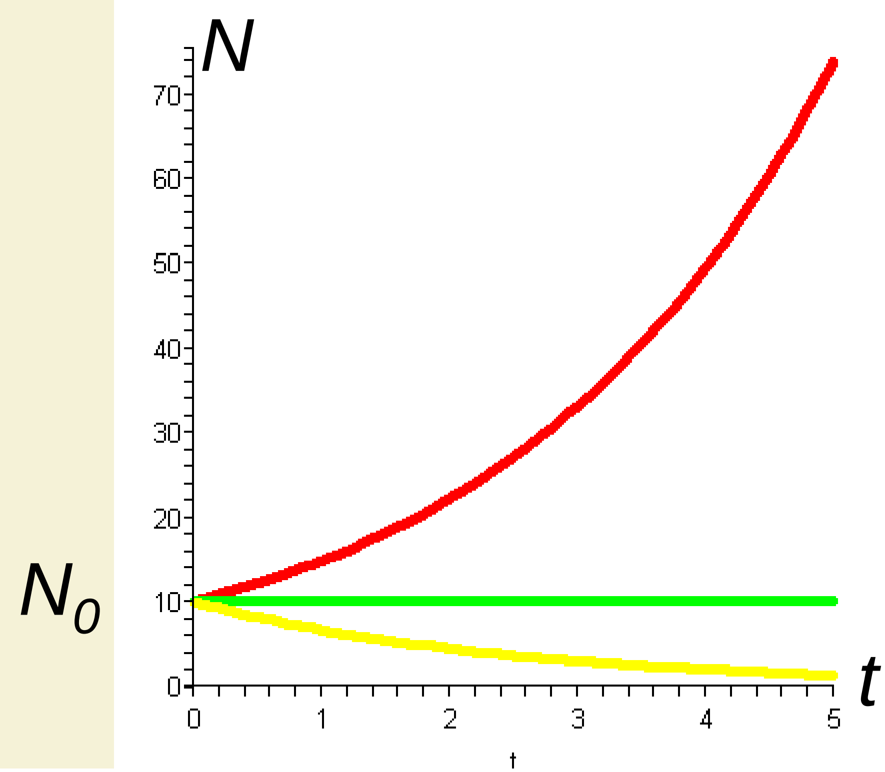
$$\int \frac{dN}{N} = \int k dt = k \int dt = kt + c$$

$$N(t) = Ae^{kt} = N_0 e^{kt}$$

k > 0 : population explosion

k = 0 : stable

k < 0 : extinction



3.3 Improving on Malthus

e^{kt} Grows too quickly

Competition

Death rate is a function of population

$\boxed{D = sN}$ (logistic) Assumption

Simple models before complicated ones!

3.3 Improving on Malthus

$$\boxed{D = sN}$$ (logistic) Assumption

$$\text{Unit of } D = \frac{\text{\# death per sec}}{\text{\# population}} = \text{sec}^{-1}$$

Unit of s = sec^{-1}

3.3 Improving on Malthus

$$\boxed{D = sN}$$ (logistic) Assumption

$$\frac{dN}{dt} = BN - DN = BN - sN^2$$

logistic equation

Small N $\Rightarrow$ $$\frac{dN}{dt} \approx BN$$ (sN^2 small)

$$\Downarrow$$

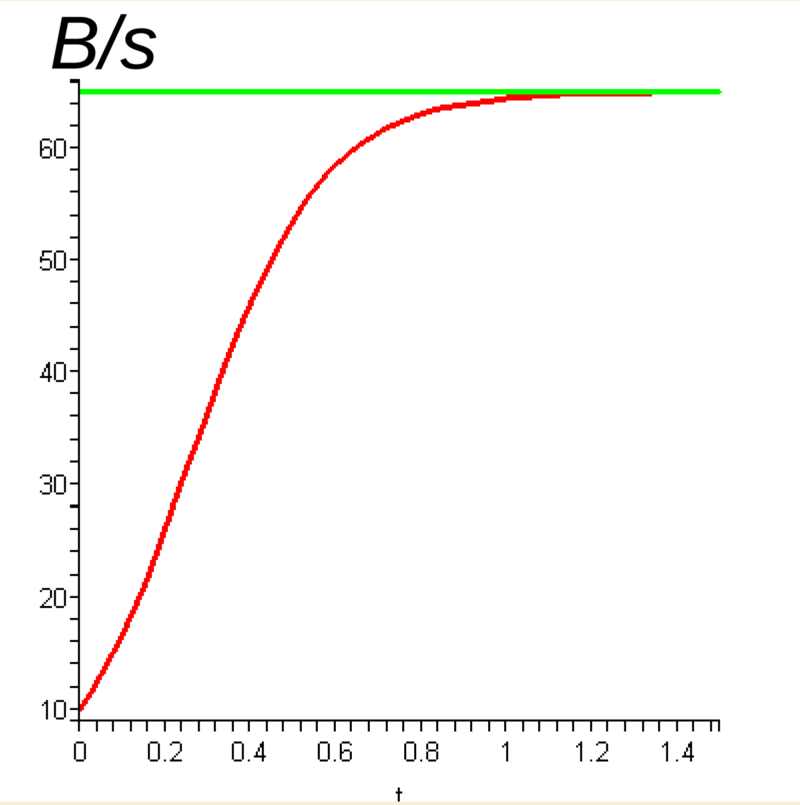
$$N(t) \approx \hat{N} e^{Bt}$$

Initially grows exponentially

3.3 Improving on Malthus

$$\frac{dN}{dt} = BN - sN^2$$

Small N ➡ $\frac{dN}{dt} \approx BN$ ➡ $N(t) \approx \hat{N}e^{Bt}$



as N increases,

N^2 grows faster

Grow rate decreases!

until $BN = s\,N^2$

3.3 Improving on Malthus

$$\frac{dN}{dt} = BN - sN^2$$

$$t = \int \frac{dN}{N(B - sN)} + c$$

non zero

Use partial fraction

$$\frac{1}{N(B - sN)} = \frac{\alpha}{N} + \frac{\beta}{B - sN}$$

$$1 = \alpha(B - sN) + \beta N = \alpha B + (\beta - \alpha s)N$$

$$1 = \alpha B \ , \ \beta = \alpha s$$

Unknown

3.3 Improving on Malthus

$$\frac{dN}{dt} = BN - sN^2$$

$$t = \int \frac{dN}{N(B - sN)} + c$$

$$\begin{aligned}\int \frac{dN}{N(B-sN)} &= \frac{1}{B}\int \frac{dN}{N} + \frac{s}{B}\int \frac{dN}{B-sN} \\ &= \frac{1}{B}\ln N - \frac{1}{B}\ln|B-sN| + c.\end{aligned}$$

assume $B - sN > 0$

$$\frac{N}{B - sN} = Ke^{Bt}$$

3.3 Improving on Malthus

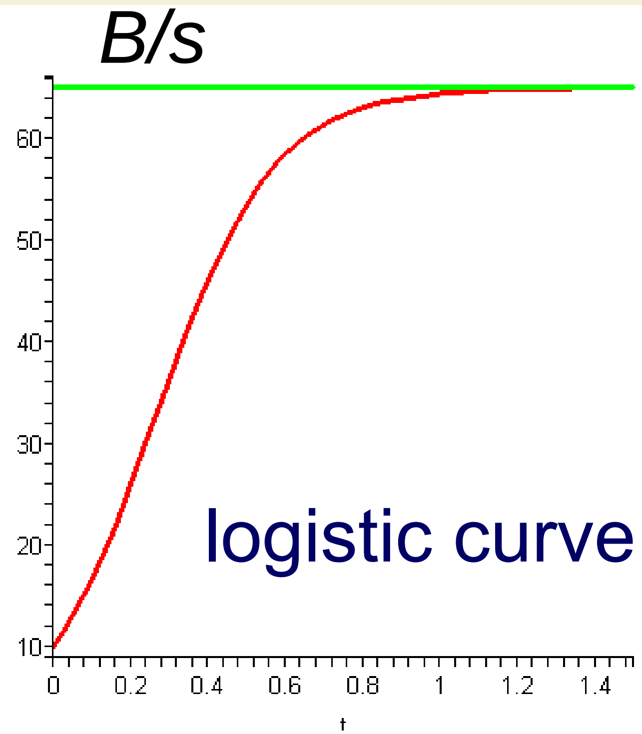
$$\frac{dN}{dt} = BN - sN^2$$

$$\frac{N}{B - sN} = Ke^{Bt}$$

$$N(0) = \hat{N}$$

$$\frac{N}{B - sN} = \frac{\hat{N}}{B - s\hat{N}} e^{Bt}$$

$$\boxed{N(t) = \frac{B}{s + \left(\frac{B}{\hat{N}} - s\right) e^{-Bt}}}$$



assumed $B - sN > 0$

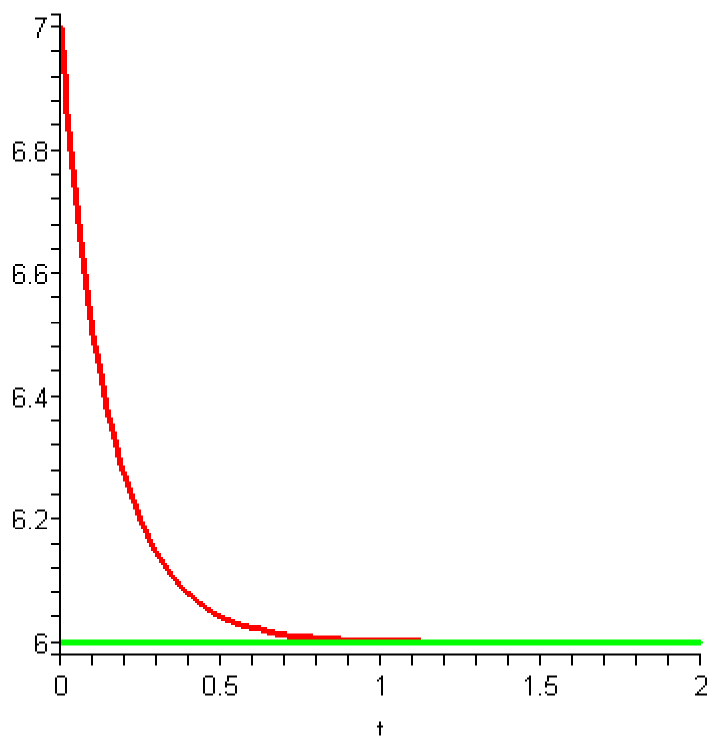
3.3 Improving on Malthus

$$\frac{dN}{dt} = BN - sN^2$$

assume $B - sN < 0$

$$t = \frac{1}{B}\ln N - \frac{1}{B}\ln|B - sN| + c$$

$$= \frac{1}{B}\ln\frac{N}{sN - B} + c$$



$$N(t) = \frac{B}{s - \left(s - \frac{B}{\hat{N}}\right)e^{-Bt}}$$

3.3 Improving on Malthus

$$\frac{dN}{dt} = BN - sN^2$$

B/s is the carrying capacity or sustainable population

set $N_\infty = B/s$

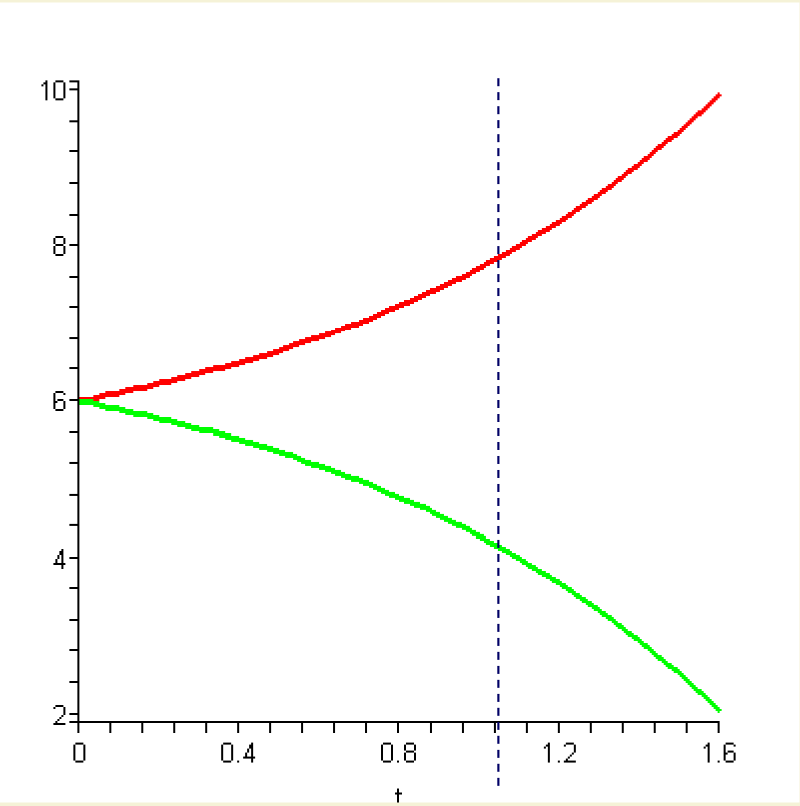
$$N(t) = \frac{N_\infty}{1 + \left(\frac{N_\infty}{\hat{N}} - 1\right) e^{-Bt}} \qquad \hat{N} < N_\infty$$

$$N(t) = \frac{N_\infty}{1 - \left(1 - \frac{N_\infty}{\hat{N}}\right) e^{-Bt}} \qquad \hat{N} > N_\infty$$

$$N(t) = N_\infty \qquad \hat{N} = N_\infty$$

Remark

Will such a situation happen?

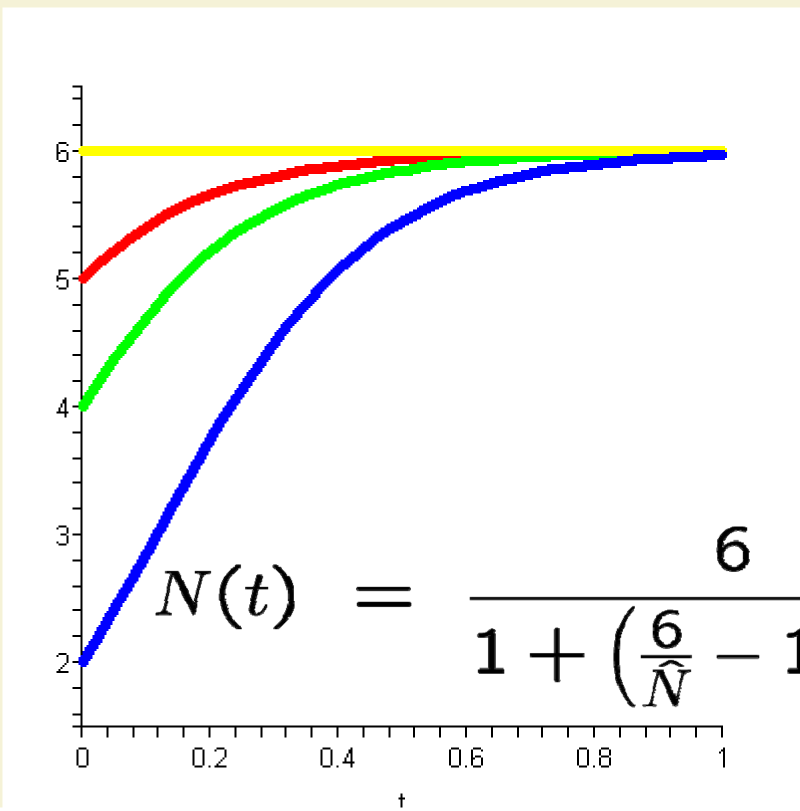


Not for real world situation

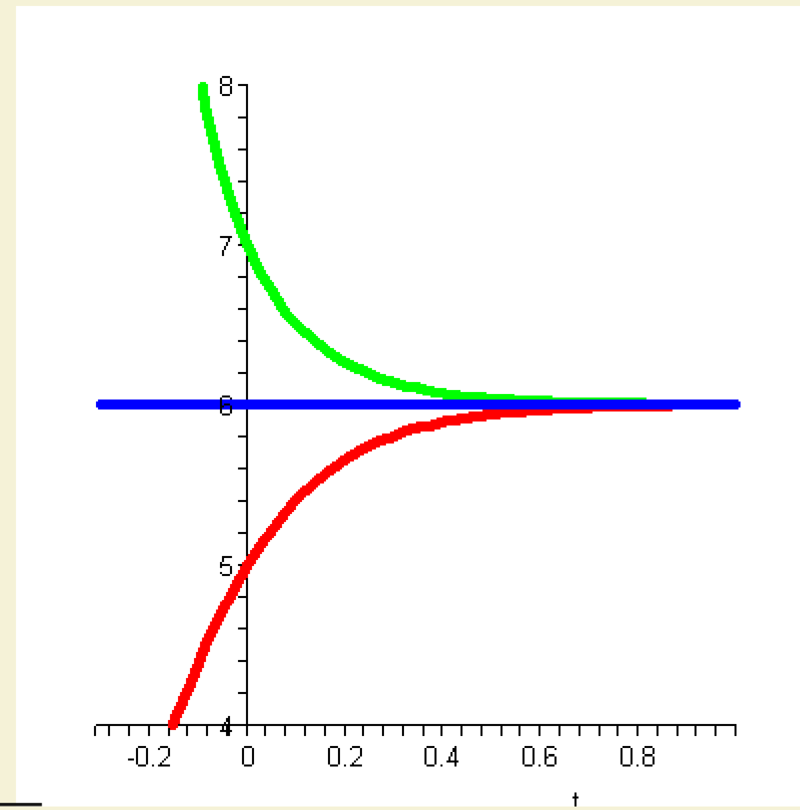
- choose a solution
- use a new model

Solution curves of a first order ode should never cross each other.

If *N* > *B/s* initially, it will never cross *B/s*



$$N(t) = \frac{6}{1 + \left(\frac{6}{\hat{N}} - 1\right) e^{-6t}}$$



Assumptions

- Fish population : *N* - Logistic model
- Constant rate of fishing, *E*

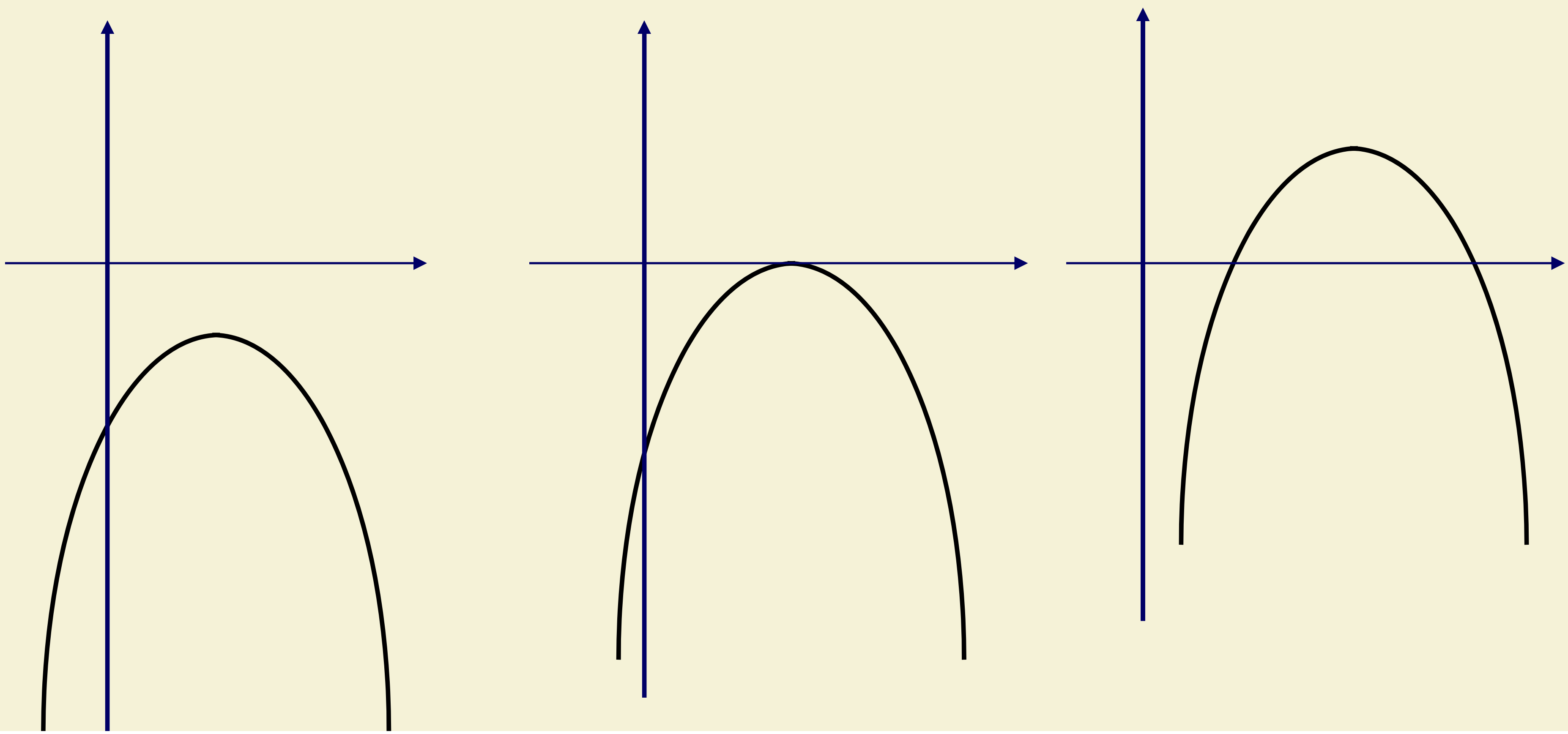
$$\frac{dN}{dt} = (B - sN)N - E$$

Basic Harvesting Model

3.5 Harvesting

$$\frac{dN}{dt} \leftarrow F(N) = -sN^2 + BN - E$$

$$\text{disc} = B^2 - 4(-s)(-E)$$

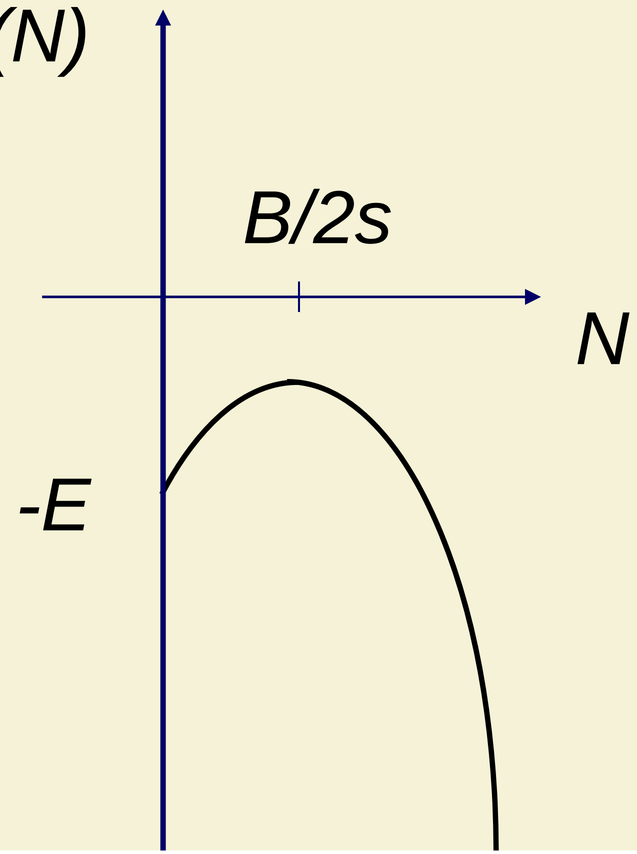


3.5 Harvesting Rate

$$E > \frac{B^2}{4s}$$

$$\boxed{F(N) = -sN^2 + BN - E}$$

$$\frac{dN}{dt} = F(N)$$



$$B^2 - 4sE < 0$$

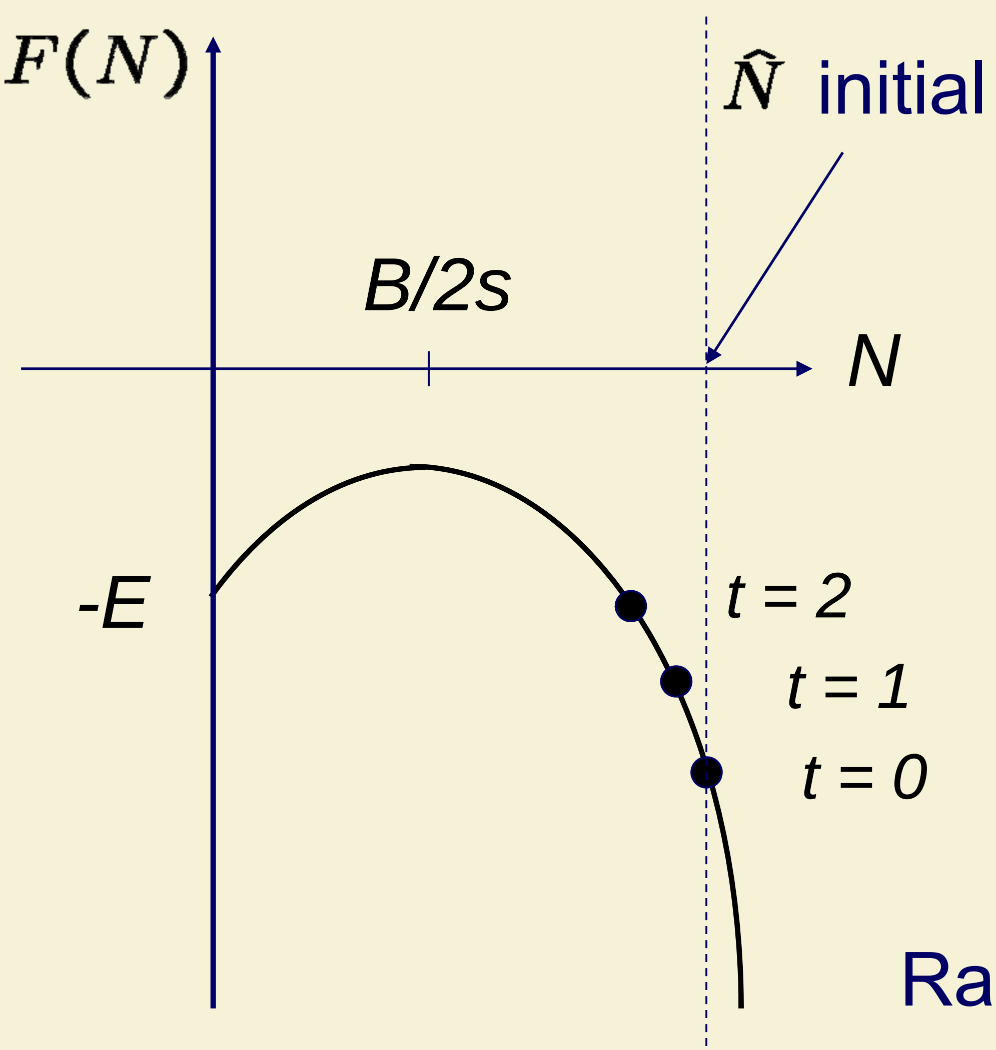
$$E > \frac{B^2}{4s}$$

Where is t ?

3.5 Harvesting Rate

$$E > \frac{B^2}{4s}$$

$$\frac{dN}{dt} = F(N)$$



$$\frac{dN}{dt} < 0$$

N

Case $N > \dfrac{B}{2s}$

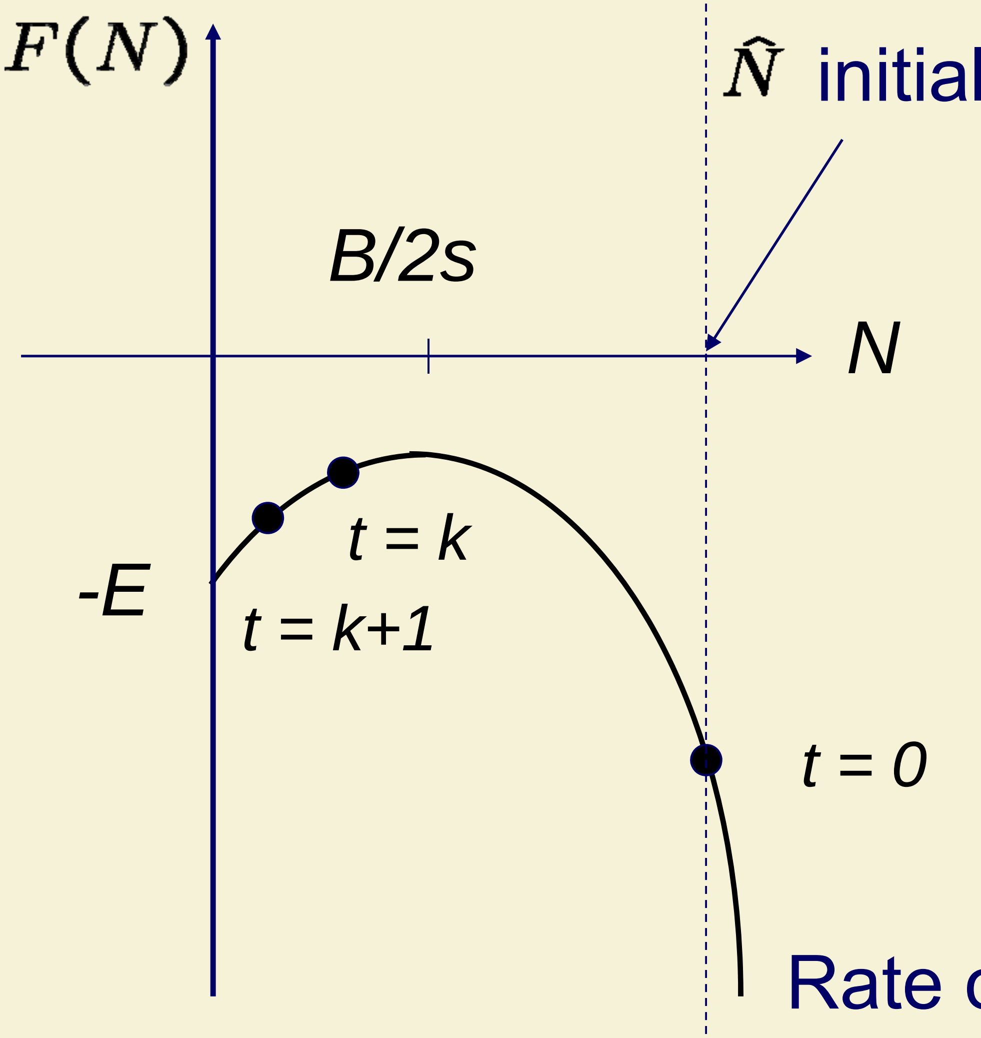
$\dfrac{dN}{dt}$ $\quad \left|\dfrac{dN}{dt}\right|$

Rate of decline slows

3.5 Harvesting Rate

$$E > \frac{B^2}{4s}$$

$$\frac{dN}{dt} = F(N)$$



$$\frac{dN}{dt} < 0$$

N ↓

Case $N < \frac{B}{2s}$

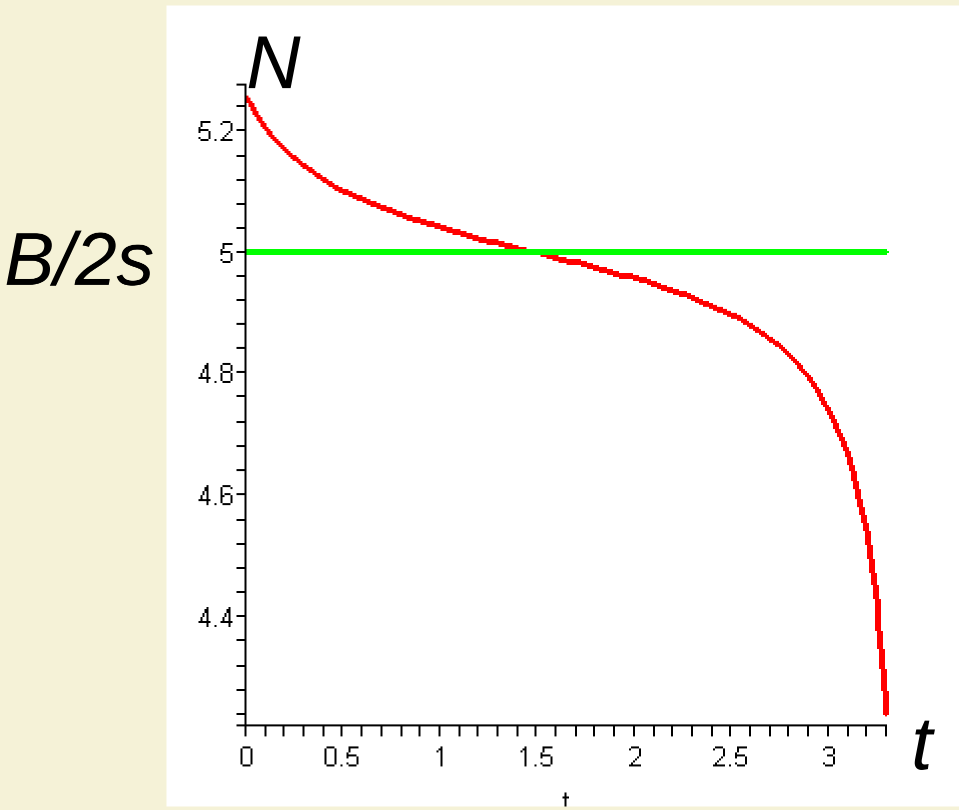
$\frac{dN}{dt}$ ↓

Rate of decline increases

3.5 Harvesting Rate

$$E > \frac{B^2}{4s}$$

What is the equation of this graph?

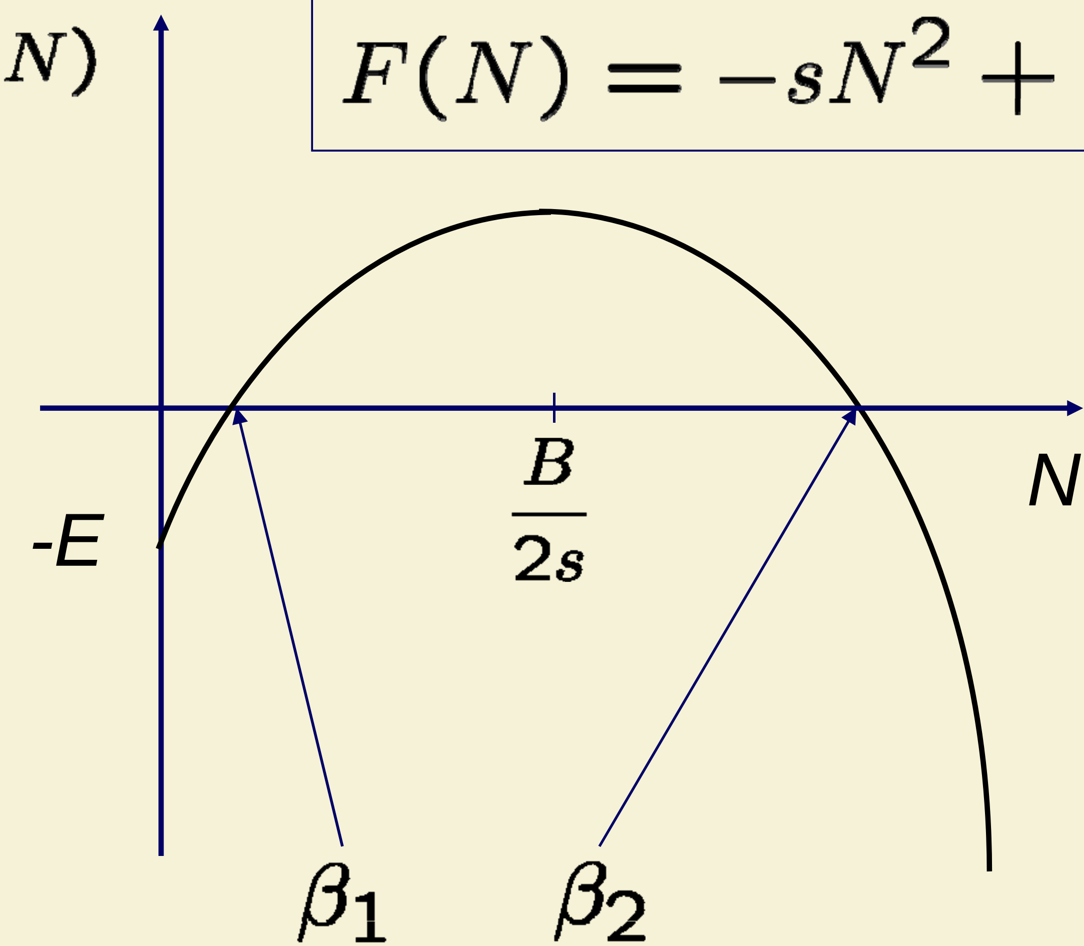


3.5 Harvesting Rate

$$E < \frac{B^2}{4s}$$

$$\frac{dN}{dt} = F(N)$$

$$F(N) = -sN^2 + BN - E$$



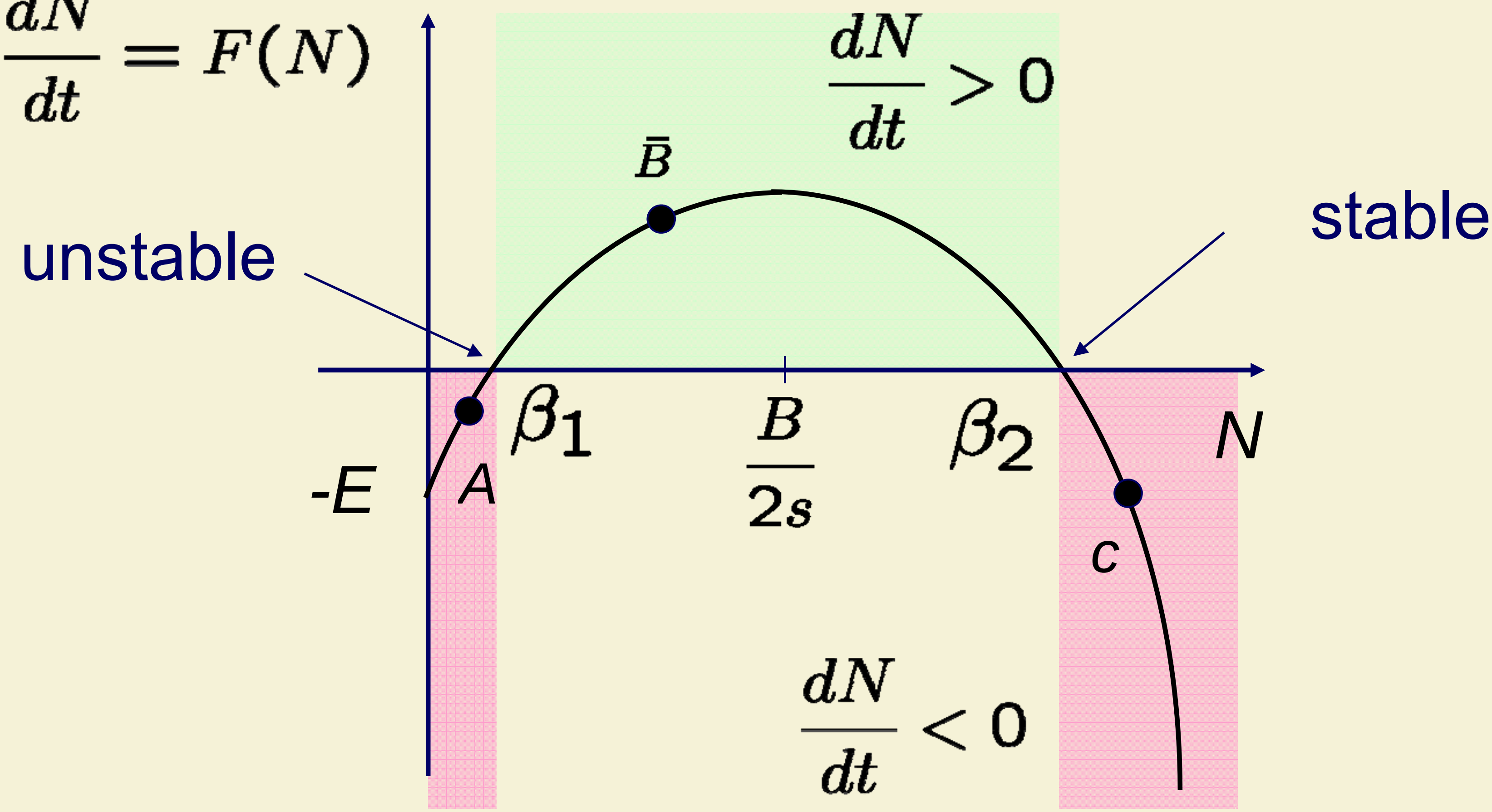
Roots

= Equilibriums (Why?)

3.5 Harvesting Rate

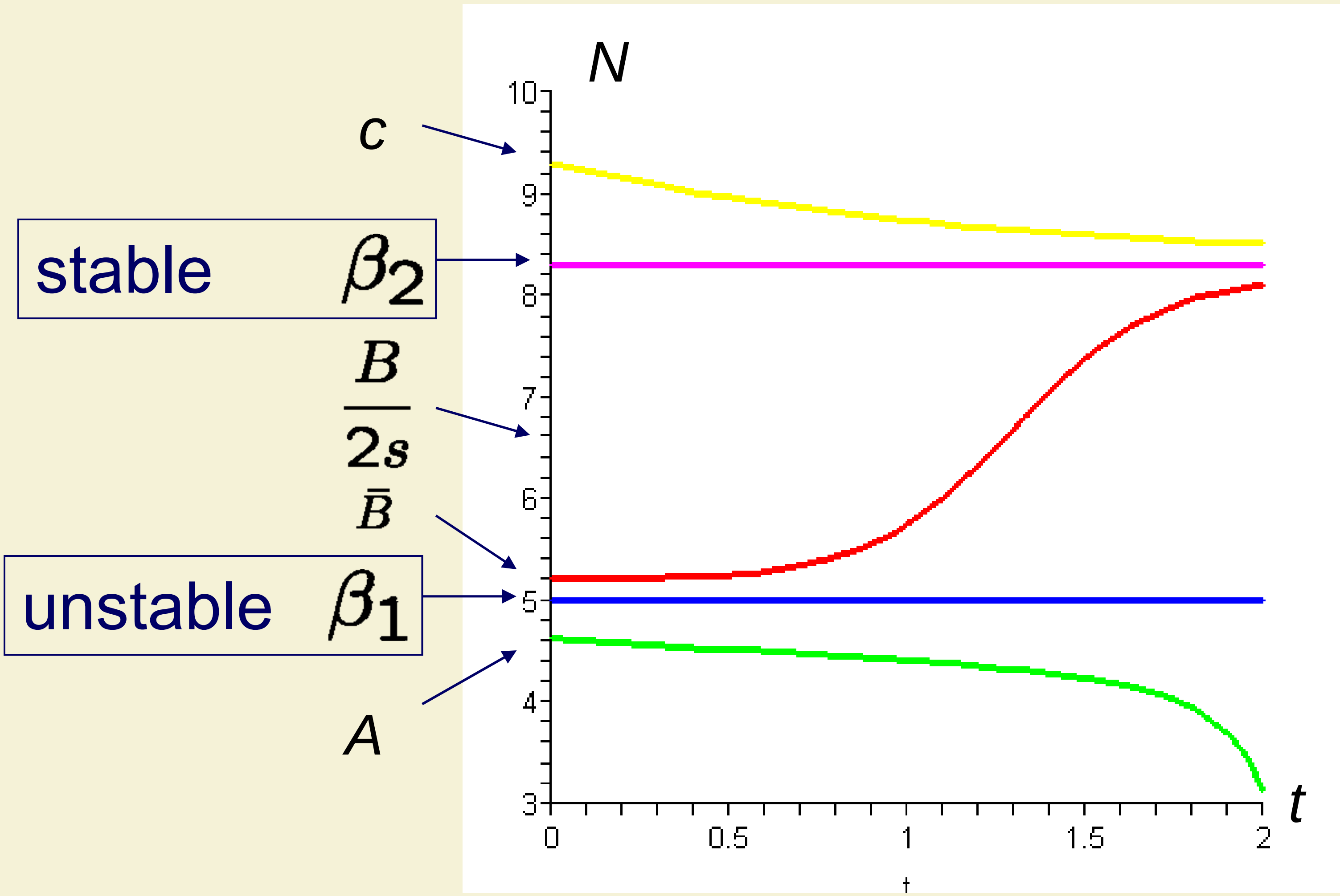
$$E < \frac{B^2}{4s}$$

$$\frac{dN}{dt} = F(N)$$



3.5 Harvesting Rate

$$E < \frac{B^2}{4s}$$



3.5 Extinction Time

$$\frac{dN}{dt} = -sN^2 + BN - E$$

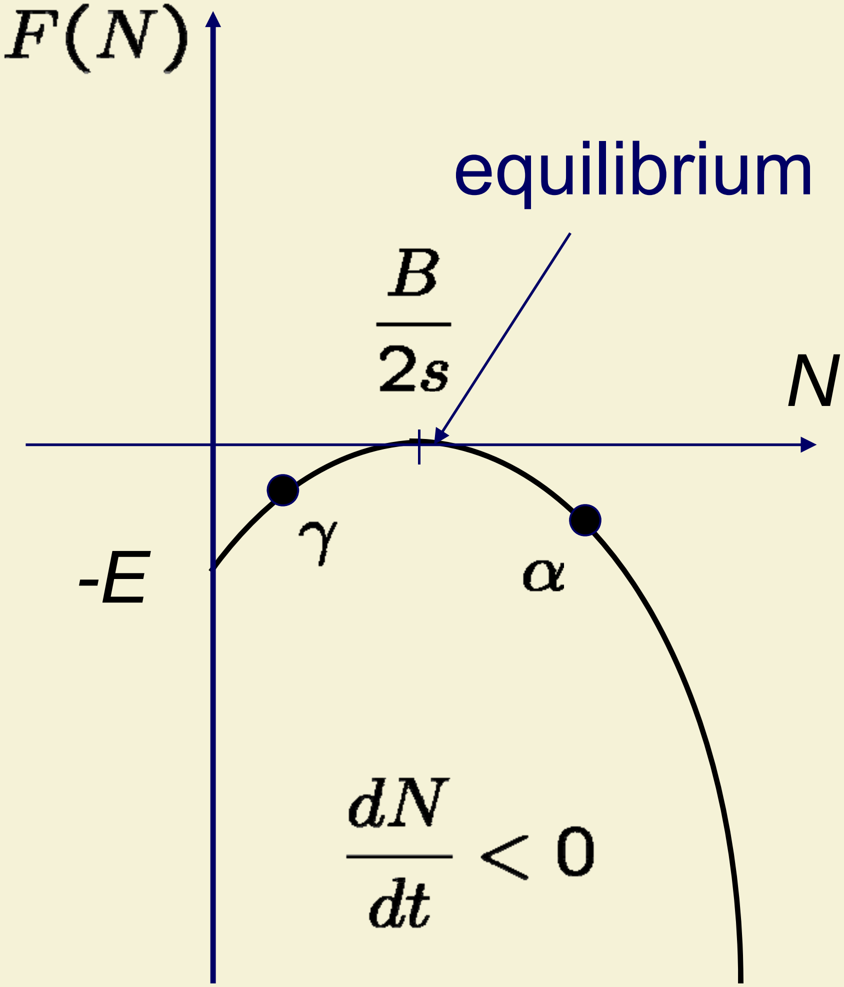
$$\int_0^T dt = T = \int_{\hat{N}}^0 \frac{dN}{N(B - sN) - E}$$

How to integrate?

3.5 Harvesting Rate

$$E = \frac{B^2}{4s}$$

$$\frac{dN}{dt} = F(N)$$

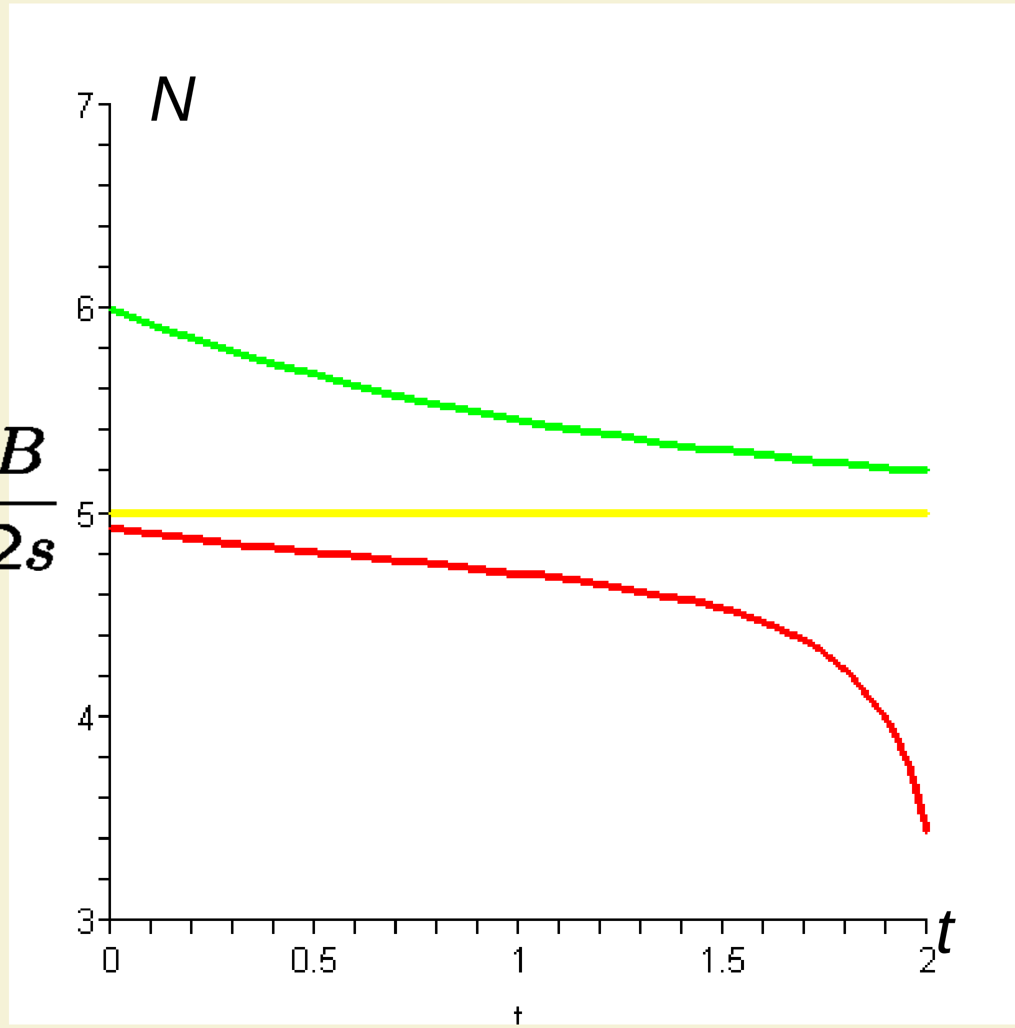


An equilibrium is stable only if it is stable to perturbations from both sides

3.5 Harvesting Rate

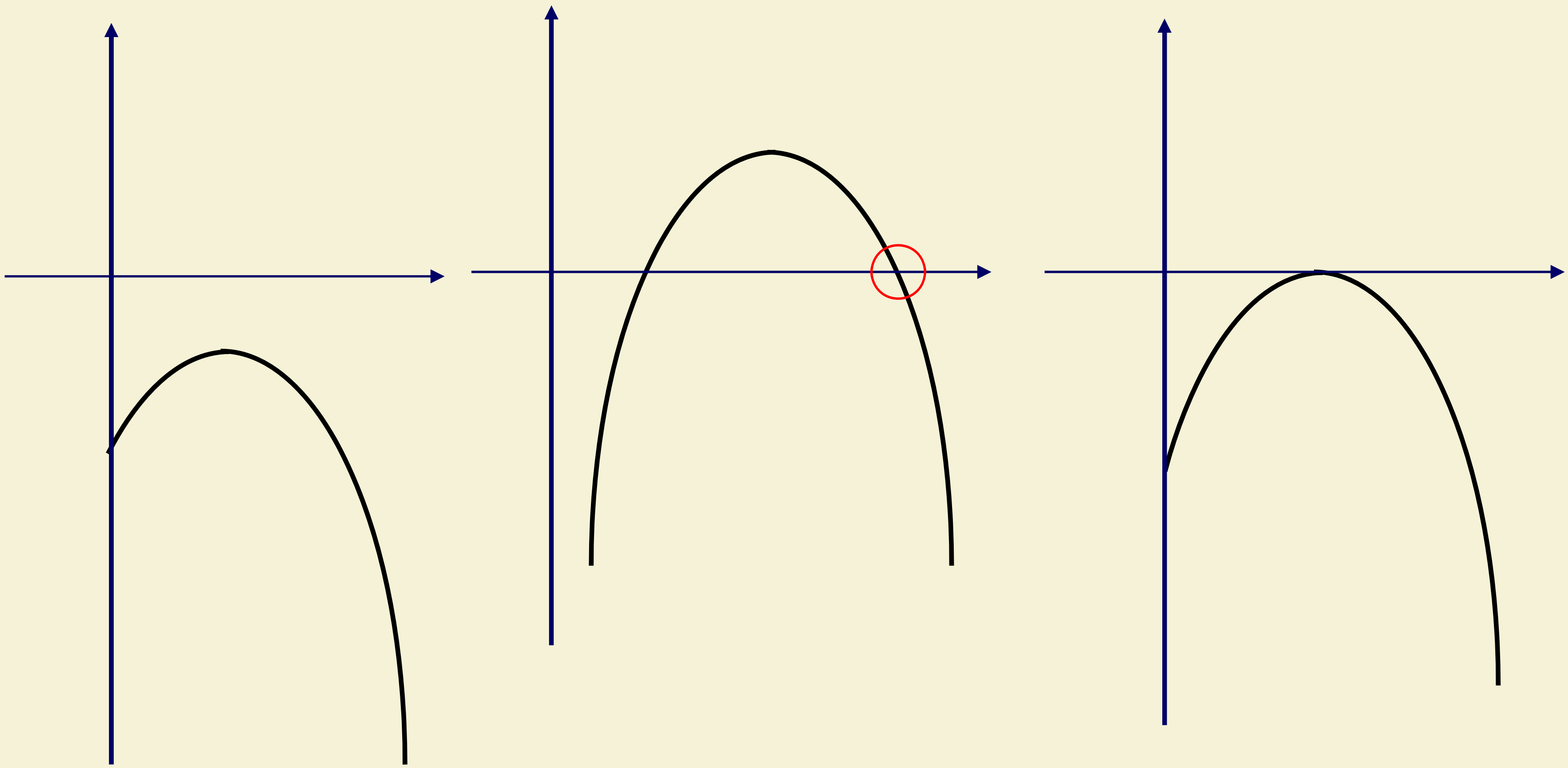
$$E = \frac{B^2}{4s}$$

unstable $\frac{B}{2s}$



3.5 Stable Equilibrium

$$E < \frac{B^2}{4s}$$

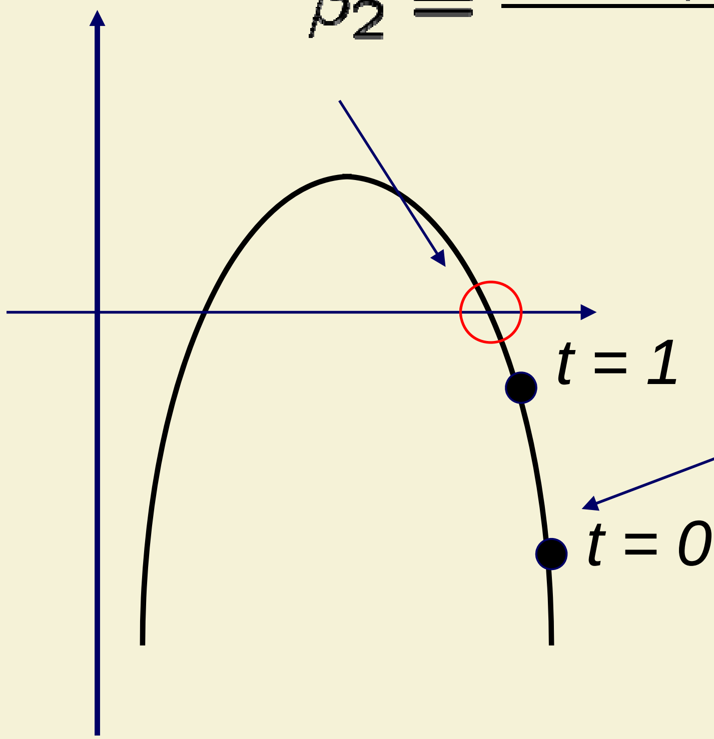


3.5 Remark

Assume you started from no fishing.

$\hat{N} = \frac{B}{s}$ carrying capacity

$$\beta_2 = \frac{B + \sqrt{B^2 - 4Es}}{2s} < \frac{B + \sqrt{B^2 - 0}}{2s} = \frac{B}{s}$$



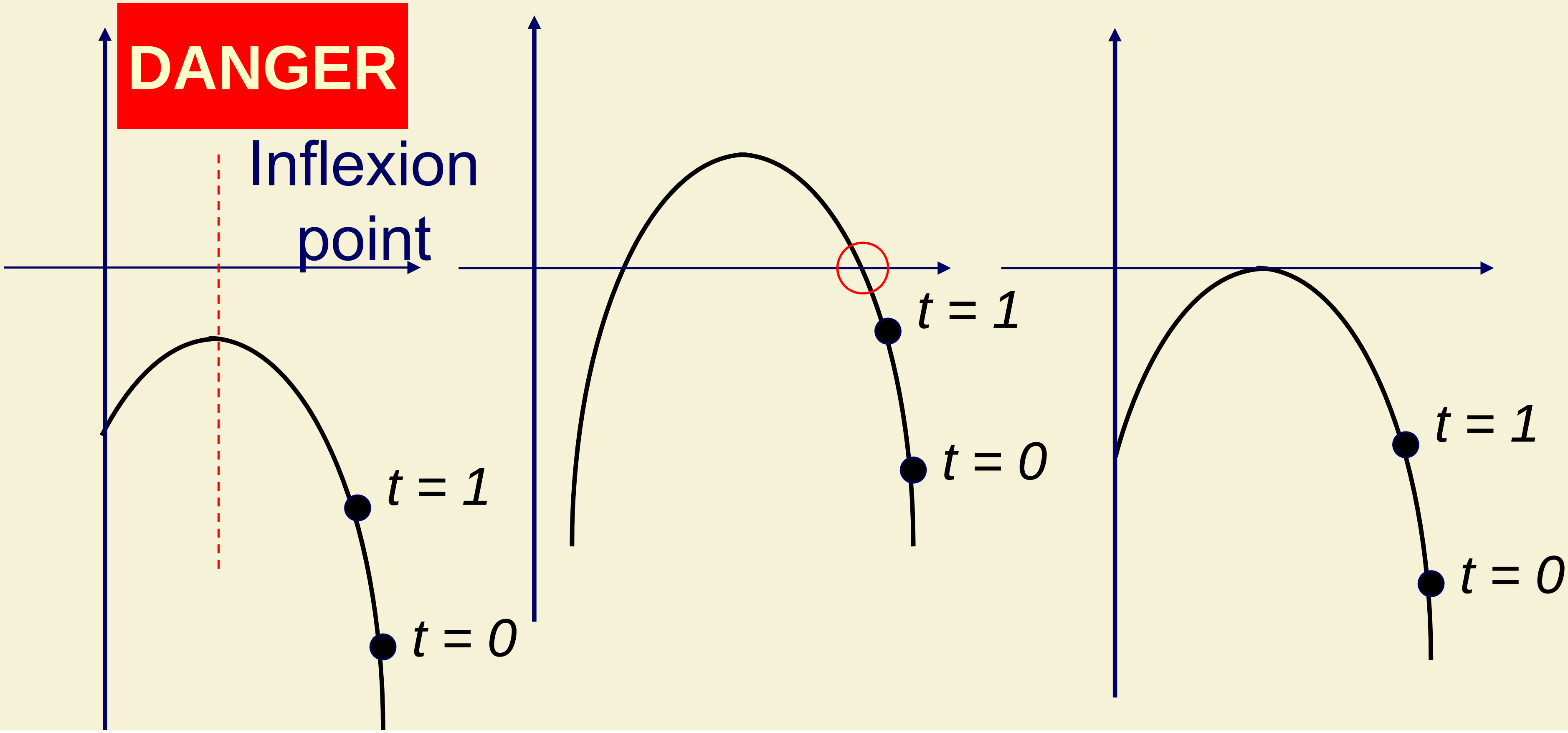
In order to not overfish

Rate $E < \frac{B^2}{4s}$

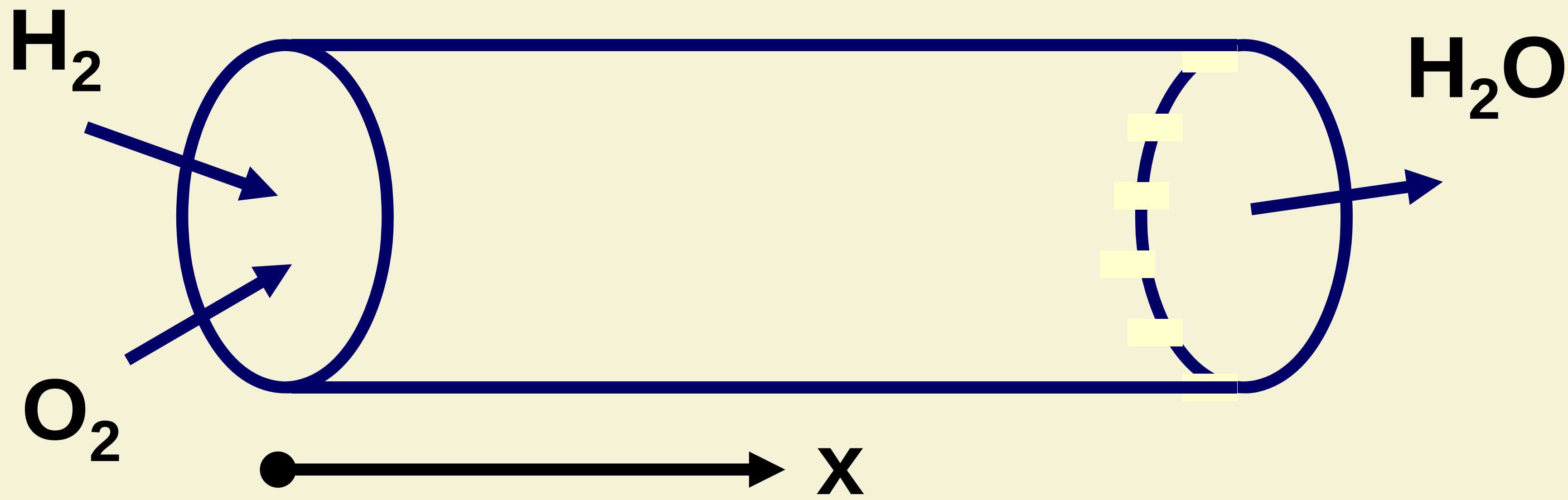
3.5 Remark

How to tell which situation are we in?

In all cases $\frac{dN}{dt}$ even when you overfish!



3.6 Plug Flow Reactor



Assumption: Oxygen is cheap

What is the concentration of Hydrogen?

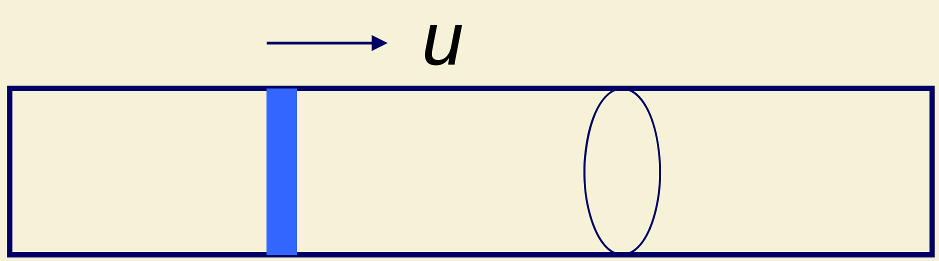
3.6 Plug Flow Reactor

What is the concentration of Hydrogen?

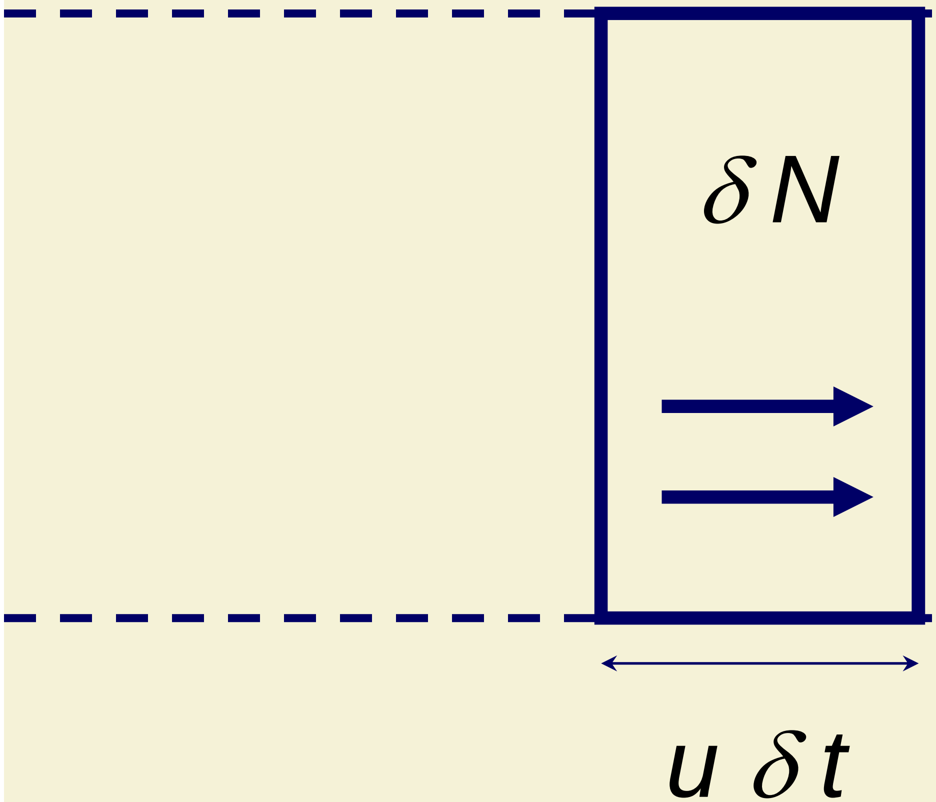
as a function of position in PFR

Assumptions:

1. Reagents flow at constant speed u
2. Uniform cross section area, A
3. No mixing upstream or downstream
4. Temp constant



3.6 PFR: Counting H_2



A = area of base

Concentration (1/m^3)

$$\delta N = C_{H_2}(x) \times A u \delta t$$

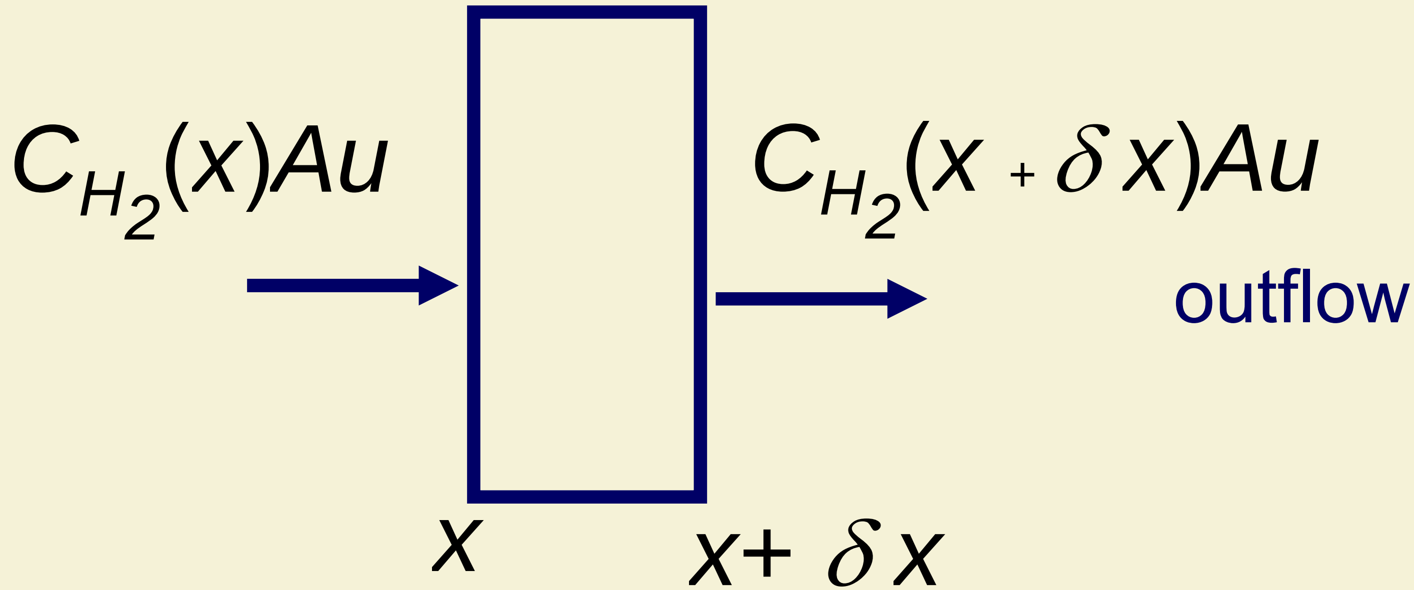
3.6 Plug Flow Reactor

$$\delta N = C_{H_2}(x) \times Au\delta t$$

$$\frac{dN}{dt} = \lim_{\delta t \to 0} \frac{\delta N}{\delta t} = C_{H_2}(x)Au$$

3.6 Plug Flow Reactor

$$\frac{dN}{dt} = C_{H_2}(x)Au$$



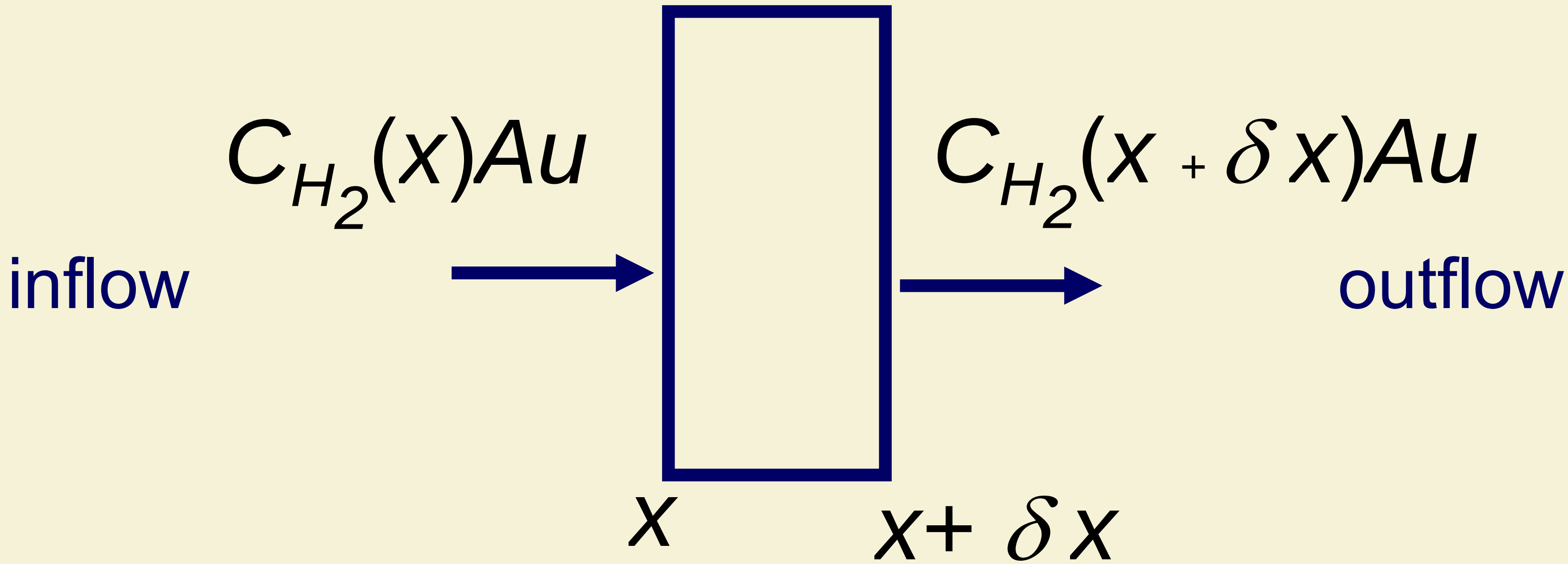
Hydrogen molecules destroyed at rate:

$$-2r\,A\,\delta x$$

r = rate of chemical reaction per unit volume (1/sm^3)

3.6 Plug Flow Reactor

$$\frac{dN}{dt} = C_{H_2}(x)Au$$



$$C_{H_2}(x)Au - 2rA\delta x = C_{H_2}(x+\delta x)Au$$

$2H_2 + O_2 = 2H_2O$

3.6 Plug Flow Reactor

$$\boxed{C_{H_2}(x)Au - C_{H_2}(x+\delta x)Au - 2rA\delta x = 0}$$

change in concentration

$$\delta C_{H_2} Au = -2rA\delta x$$

$$u\frac{dC_{H_2}}{dx} = u\lim_{\delta x \to 0}\frac{\delta C_{H_2}}{\delta x} = \underbrace{-2r}$$

Depends on … temp, conc of H_2 …

3.6 Plug Flow Reactor

Assume $r = kC_{H_2}(x)$

unit of *k* (1/s)

$$u\frac{dC_{H_2}}{dx} = -2r = -2kC_{H_2}$$

$$\frac{dC_{H_2}}{C_{H_2}} = -\frac{2k}{u}dx$$

dimensionless

$$C_{H_2} = C_{H_2}(0)e^{\frac{-2kx}{u}}$$

x/u = Time

Let ***X*** = length of PFR, ***T*** = total time, *u= X/T*

3.6 Plug Flow Reactor

$$C_{H_2}(exit) = C_{H_2}(entrance)e^{-2kT}$$

- PFRs are efficient
- Our model did not consider temperature

Digression : Chemical Reactions

Assume $r = kC_{H_2}(x)$

- First order reaction $\frac{dX}{dt} = kX$

(CH$_3$)$_3$CCl + NaOH → (CH$_3$)$_3$COH+NaCL

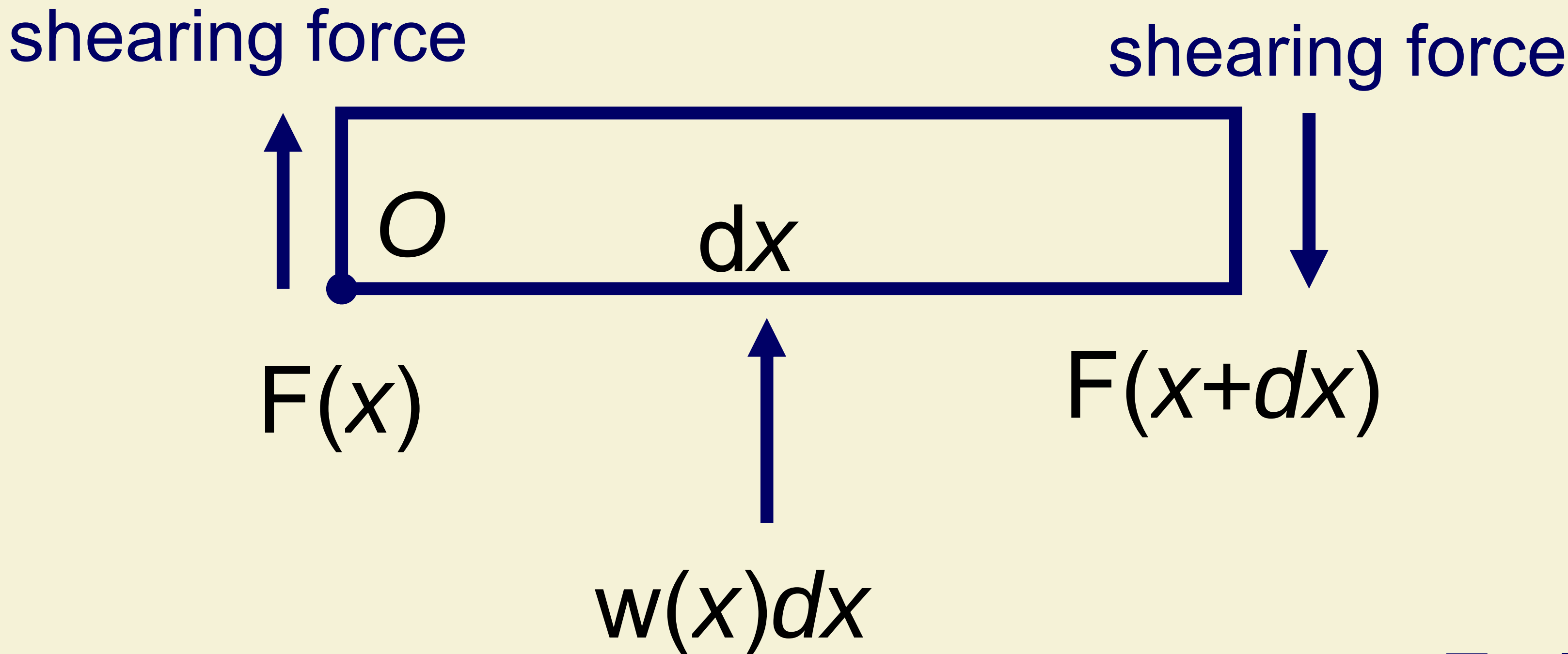
t-butyl chloride into t-butyl alcohol

- Second order reaction $\frac{dX}{dt} = k(\alpha - X)(\beta - X)$

CH$_3$CL + NaOH → CH$_3$ OH +NaCL

methyl chloride into methyl alcohol

3.7 Cantilevered Beams



Taylor

w(x) = Load = force per unit length

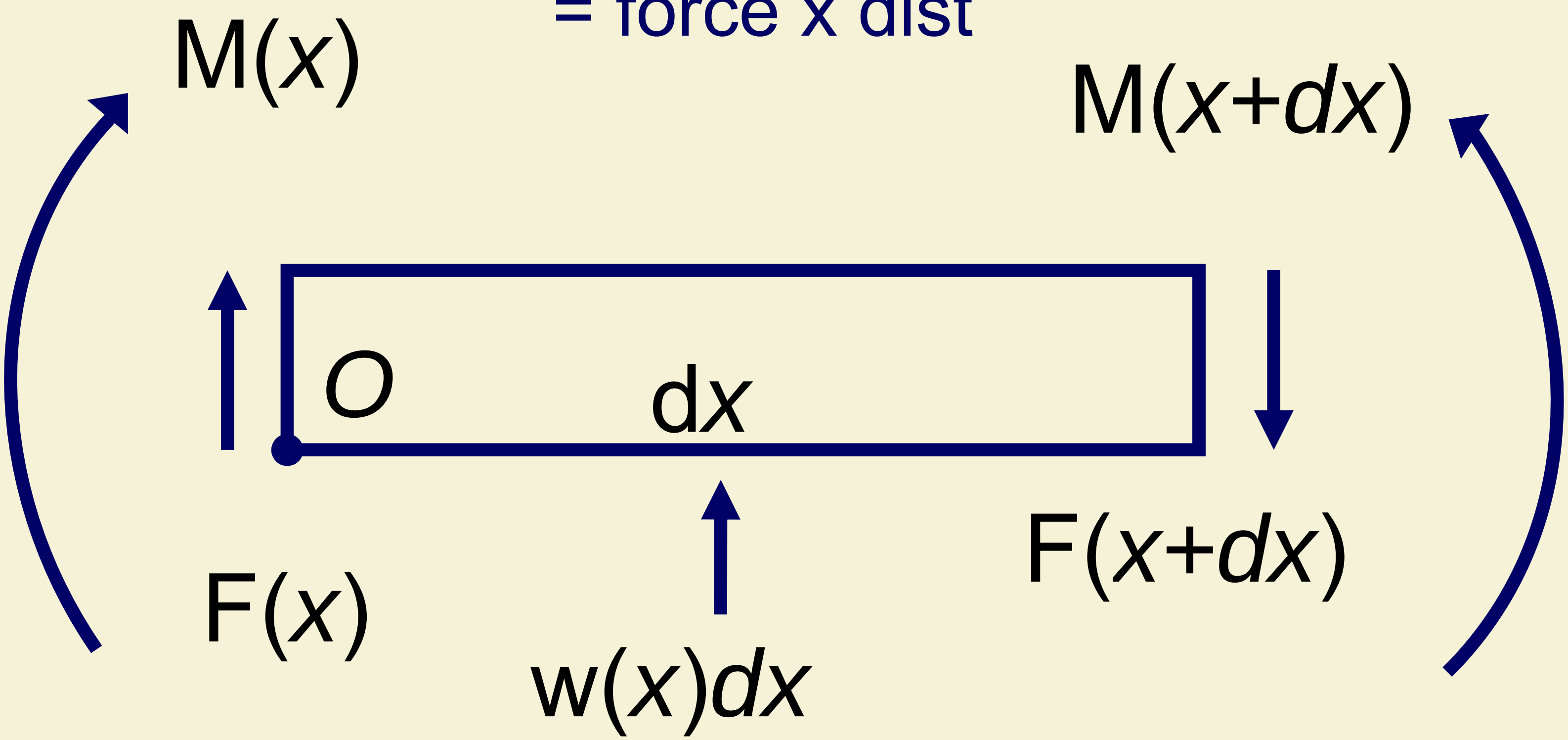
$$F(x) + w(x)dx = F(x + dx) = F(x) + \frac{dF}{dx}dx$$

$$\boxed{\frac{dF}{dx} = w(x)}$$

3.7 Cantilevered Beams

Torque (moment) = rotational force

= force x dist



$$M(x) + F(x+dx)dx = M(x+dx) + (w(x)dx)\frac{dx}{2}$$

3.7 Cantilevered Beams

$$M(x)+F(x+dx)dx=M(x+dx)+(w(x)dx)\frac{dx}{2}$$

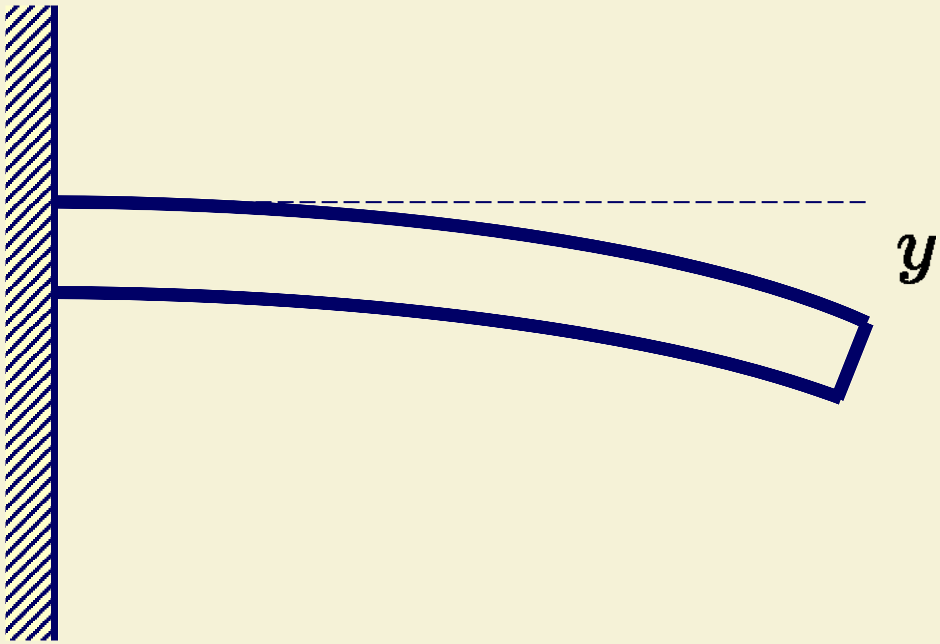
$$M(x)+F(x)dx+\frac{dF}{dx}(dx)^2$$

small

$$=M(x)+\frac{dM}{dx}dx+\frac{1}{2}w(x)(dx)^2$$

$$\frac{dM}{dx}=F \Rightarrow \boxed{\frac{d^2M}{dx^2}=\frac{dF}{dx}=w(x)}$$

3.7 Cantilevered Beams



- Deflection

Depends on

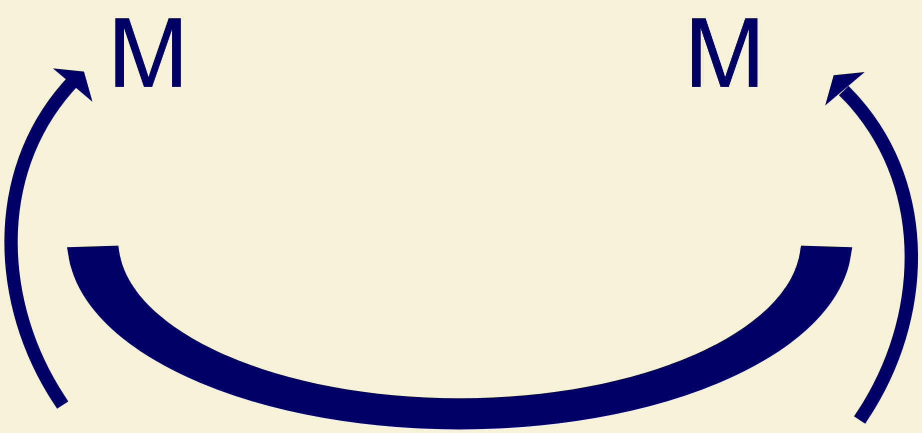
- Stiffness

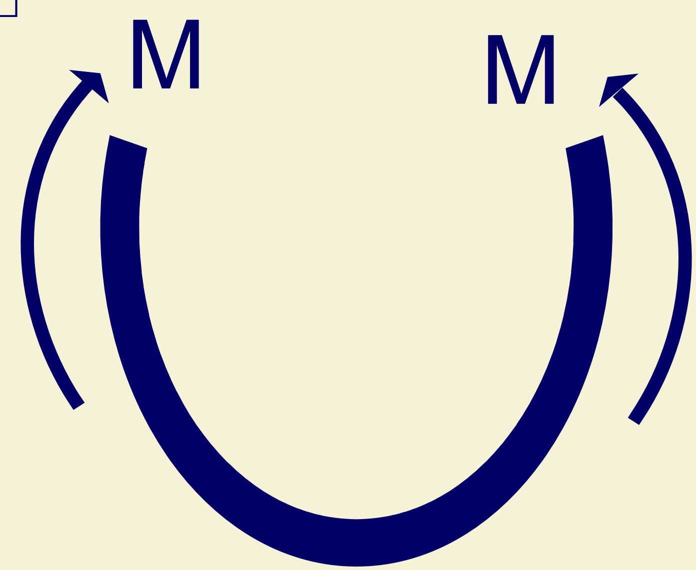
Stiffness depends on

- Material : Young's modulus, ***E***
- Shape of cross section , ***I***

3.7 Cantilevered Beams

Stiffness $$\boxed{EI = M/\frac{d^2y}{dx^2}}$$





$M/\frac{d^2y}{dx^2}$ large

$M/\frac{d^2y}{dx^2}$ small

3.7 Cantilevered Beams

$$EI = M / \frac{d^2 y}{dx^2}$$

$$\frac{d^2 y}{dx^2} = \frac{M}{EI}$$

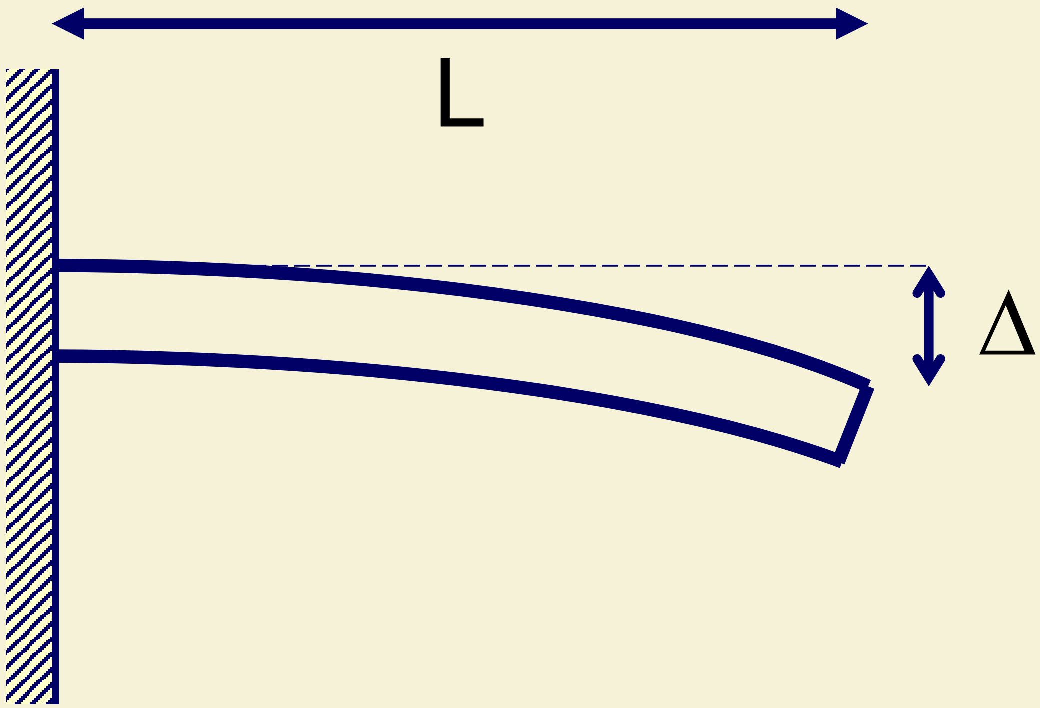
$$\Rightarrow \quad \frac{d^4 y}{dx^4} = \frac{1}{EI} \frac{d^2 M}{dx^2} = \frac{w(x)}{EI}$$

$$\boxed{\frac{d^4 y}{dx^4} = \frac{w(x)}{EI}}$$

http://en.wikipedia.org/wiki/Euler-Bernoulli_beam_equation

Find max deflection

$$\frac{d^4y}{dx^4} = \frac{w(x)}{EI}$$



up is positive

Assume: uniform mass

$$w(x) = -\alpha$$

Find max deflection

$$\frac{d^4y}{dx^4} = \frac{w(x)}{EI}$$

$$\frac{d^4y}{dx^4} = \frac{-\alpha}{EI} \Rightarrow \frac{d^3y}{dx^3} = -\frac{\alpha x}{EI} + A$$

Recall $$\frac{d^3y}{dx^3} = \frac{1}{EI}\frac{dM}{dx} = \frac{F(x)}{EI}$$

$$F(L) = 0 = EI\frac{d^3y}{dx^3}(L) = EI\left(-\frac{\alpha L}{EI} + A\right)$$

No shearing force at L.

$$\Rightarrow A = \frac{\alpha L}{EI}$$

Find max deflection

$$\frac{d^3y}{dx^3} = -\frac{\alpha x}{EI} + \frac{\alpha L}{EI} \Rightarrow \frac{d^2y}{dx^2} = -\frac{\alpha x^2}{2EI} + \frac{\alpha Lx}{EI} + B$$

Recall $\frac{d^2y}{dx^2} = \frac{M}{EI}$

$$M(L) = 0 = EI\frac{d^2y}{dx^2}(L)$$

No bending moment at L

$$= EI\left(-\frac{\alpha L^2}{2EI} + \frac{\alpha L^2}{EI} + B\right)$$

$$\Rightarrow \quad B = \frac{\alpha L^2}{2EI} - \frac{\alpha L^2}{EI} = -\frac{\alpha L^2}{2EI}$$

Find max deflection $$\frac{d^4y}{dx^4} = \frac{w(x)}{EI}$$

$$\frac{d^2y}{dx^2} = -\frac{\alpha x^2}{2EI} + \frac{\alpha Lx}{EI} - \frac{\alpha L^2}{2EI}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{\alpha x^3}{6EI} + \frac{\alpha Lx^2}{2EI} - \frac{\alpha L^2 x}{2EI} + C$$

$$\frac{dy}{dx}(0) = 0 \quad \Rightarrow \quad C=0$$

No curvature at x=0

$$\Rightarrow \quad y = -\frac{\alpha x^4}{24EI} + \frac{\alpha Lx^3}{6EI} - \frac{\alpha L^2x^2}{4EI} + D$$

Cantilever deflection formula

$$y = -\frac{\alpha x^4}{24EI} + \frac{\alpha L x^3}{6EI} - \frac{\alpha L^2 x^2}{4EI} + D$$

No deflection at x=0 ➡ ***D=0***

$$y = \frac{\alpha L^4}{2EI}\left(-\frac{1}{12}\left(\frac{x}{L}\right)^4 + \frac{1}{3}\left(\frac{x}{L}\right)^3 - \frac{1}{2}\left(\frac{x}{L}\right)^2\right)$$

$$\Delta = y(L) = \frac{\alpha L^4}{2EI}\left(-\frac{1}{12} + \frac{1}{3} - \frac{1}{2}\right)$$

$$= -\frac{\alpha L^4}{8EI}$$