EE3206/EE3206E ICVIP

Tutorial Set B – Solutions

Question 1

Part (a)

The FT of $\delta(x, y)$ is

$$\begin{aligned} \mathcal{F}\{\delta(x,y)\} &= \iint\limits_{-\infty}^{+\infty} \delta(x,y) \exp[-j2\pi(ux+vy)]\, dx\, dy \quad \text{(by definition)} \\ &= \exp[-j2\pi(ux+vy)]\big|_{x,y=0} \\ &= 1 \end{aligned}$$

Part (b)

The inverse FT of $\delta(u, v)$ is

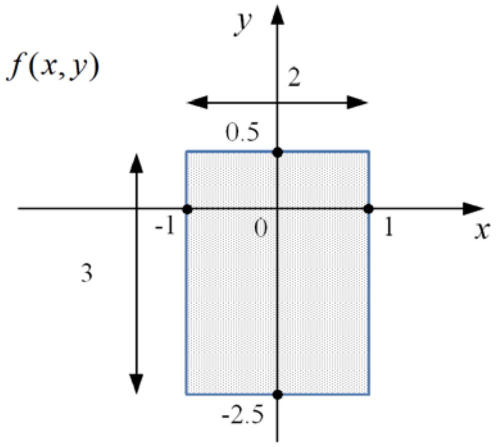
$$\begin{aligned} \mathcal{F}^{-1}\{\delta(u,v)\} &= \iint\limits_{-\infty}^{+\infty} \delta(u,v) \exp[j2\pi(ux+vy)]\, du\, dv \quad \text{(by definition)} \\ &= \exp[j2\pi(ux+vy)]\big|_{u,v=0} \\ &= 1 \end{aligned}$$

Taking the FT of both sides,

$$\delta(u, v) = \mathcal{F}\{1\}$$

i.e., the FT of $f(x, y) = 1$ is $\delta(u, v)$.

Question 2



The image function $f(x,y)$ can be regarded as the sum of two component functions $f_1(x,y)$ and $f_2(x,y)$:

$$f(x,y) = f_1(x,y) + f_2(x,y)$$

where

$$\begin{aligned} f_1(x,y) &= 10 \quad \text{for all} \quad x, y \\ f_2(x,y) &= \begin{cases} 40 & -1 \le x \le +1, \ -2.5 \le y \le +0.5 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

<u>*Method I*</u> - (not recommended)

$$\begin{aligned} F(u,v) &= \mathcal{F}\{f(x,y)\} \\ &= \int_{-\infty}^{+\infty}\!\!\int 10 \exp[-j2\pi(ux+vy)]dxdy \\ &\quad + \int_{-2.5}^{0.5}\int_{-1}^{1} 40 \exp[-j2\pi(ux+vy)]dxdy \\ &= \ldots \\ \text{Fourier spectrum} &= |F(u,v)| \end{aligned}$$

Method II

Use $$|F(u,v)| = |\mathcal{F}\{f(x,y)\}| = |\mathcal{F}\{f(x,y-1)\}|$$

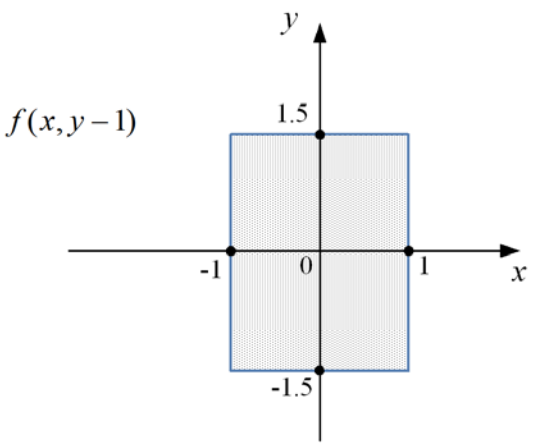
We have
$$f(x,y) = f_1(x,y) + f_2(x,y)$$

Hence,

$$\begin{aligned}
f(x,y-1) &= f_1(x,y) + f_2(x,y-1) \\
\mathcal{F}\{f(x,y-1)\} &= \iint_{-\infty}^{+\infty} 10\exp[-j2\pi(ux+vy)]\,dxdy \\
&\qquad + 40\int_{-1.5}^{+1.5}\int_{-1}^{1}\exp[-j2\pi(ux+vy)]\,dxdy \\
&= 10\delta(u,v) + 40\left[\frac{\exp(-j2\pi ux)}{-j2\pi u}\right]_{-1}^{+1}\left[\frac{\exp(-j2\pi vy)}{-j2\pi v}\right]_{-1.5}^{+1.5} \\
&= 10\delta(u,v) + 40[2\mathrm{sinc}(2u)]\,[3\mathrm{sinc}(3v)] \\
&= 10\delta(u,v) + 240\mathrm{sinc}(2u)\mathrm{sinc}(3v)
\end{aligned}$$

The Fourier spectrum of $f(x,y)$ is

$$\begin{aligned}
|F(u,v)| &= |\mathcal{F}\{f(x,y)\}| \\
&= |\mathcal{F}\{f(x,y-1)\}| \\
&= 10\delta(u,v) + 240|\mathrm{sinc}(2u)||\mathrm{sinc}(3v)|
\end{aligned}$$



Question 3

$$\begin{aligned}
\mathcal{F}^{-1}\{\delta(u-a, v-b)\} &= \int\int \delta(u-a,v-b)\exp[j2\pi(ux+vy)]dudv \\
&= \exp[j2\pi(ux+vy)]|_{u=a,v=b} \\
&= \exp[j2\pi(ax+by)]
\end{aligned}$$

Similarly,

$$\mathcal{F}^{-1}\{\delta(u+a, v+b)\} = \exp[-j2\pi(ax+by)]$$

Hence,

$$\begin{aligned}
\mathcal{F}^{-1}\{\delta(u-a, v-b) + \delta(u+a, v+b)\} &= \exp[j2\pi(ax+by)] + \exp[-j2\pi(ax+by)] \\
&= 2\cos 2\pi(ax+by)
\end{aligned}$$

Thus,

$$\cos 2\pi(ax+by) \leftrightarrow \frac{1}{2}\delta(u-a,v-b) + \frac{1}{2}\delta(u+a,v+b) \qquad (1)$$

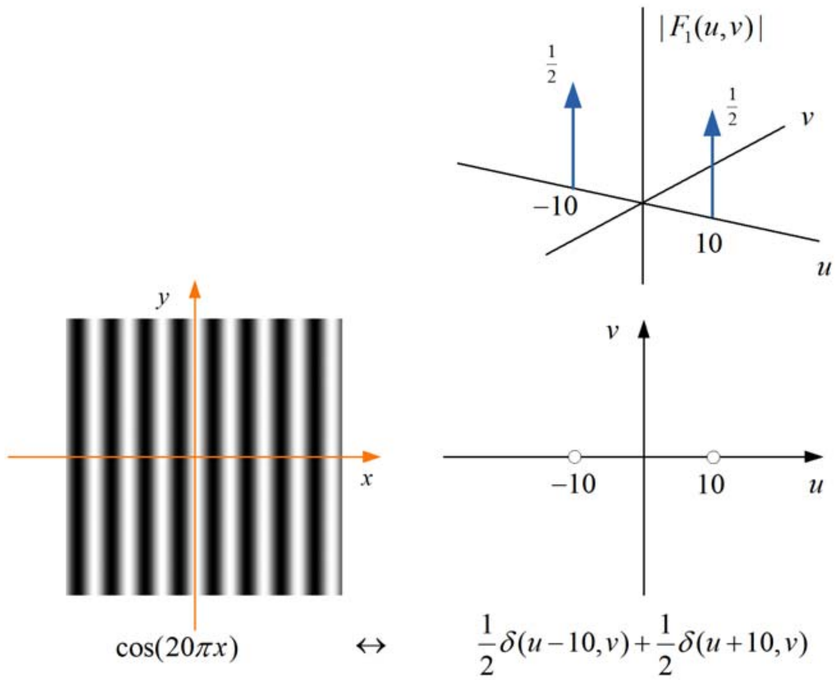
Part (a)

From Eq. (1), we let $a = 10$ and $b = 0$:

$$f_1(x,y) = \cos(20\pi x) \leftrightarrow \frac{1}{2}\delta(u-10,v) + \frac{1}{2}\delta(u+10,v)$$

Hence, the Fourier spectrum of $f_1(x,y)$ is

$$|F_1(u,v)| = \frac{1}{2}\delta(u-10,v) + \frac{1}{2}\delta(u+10,v)$$



Part (b)

We first obtain the Fourier transform of $\sin 2\pi(ax+by)$. Similar to Eq. (1),

$$\begin{aligned}\mathcal{F}^{-1}\{\delta(u-a,v-b)-\delta(u+a,v+b)\} &= \exp[j2\pi(ax+by)]-\exp[-j2\pi(ax+by)] \\ &= 2j\sin 2\pi(ax+by)\end{aligned}$$

Hence,

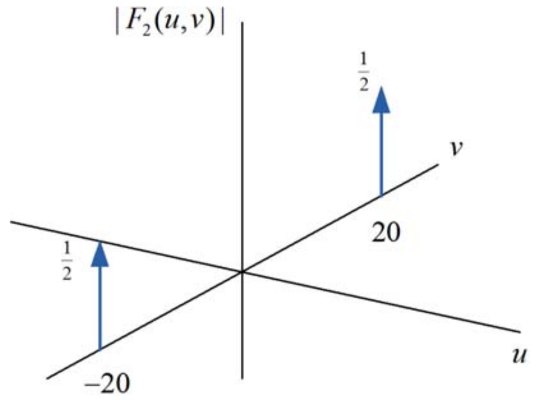
$$\begin{aligned}\sin 2\pi(ax+by) &\leftrightarrow \frac{1}{2j}\delta(u-a,v-b)-\frac{1}{2j}\delta(u+a,v+b) && (2)\\ &= -\frac{j}{2}\delta(u-a,v-b)+\frac{j}{2}\delta(u+a,v+b) && (3)\end{aligned}$$

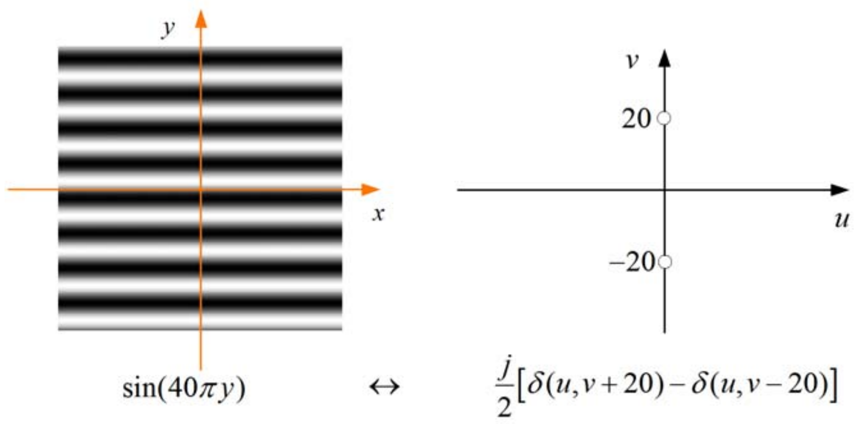
From Eq. (3), we let $a=0$ and $b=20$:

$$f_2(x,y)=\sin(40\pi y) \leftrightarrow \frac{j}{2}\delta(u,v+20)-\frac{j}{2}\delta(u,v-20)$$

Hence, the Fourier spectrum of $f_2(x,y)$ is

$$|F_2(u,v)|=\frac{1}{2}\delta(u,v+20)+\frac{1}{2}\delta(u,v-20)$$





Part (c)

$$
\begin{aligned}
f_3(x, y) &= \sin(30x + 40y) \\
&= \sin 2\pi \left(\frac{15}{\pi} x + \frac{20}{\pi} y \right) \\
F_3(u, v) &= \frac{j}{2} \delta \left(u + \frac{15}{\pi}, v + \frac{20}{\pi} \right) - \frac{j}{2} \delta \left(u - \frac{15}{\pi}, v - \frac{20}{\pi} \right)
\end{aligned}
$$

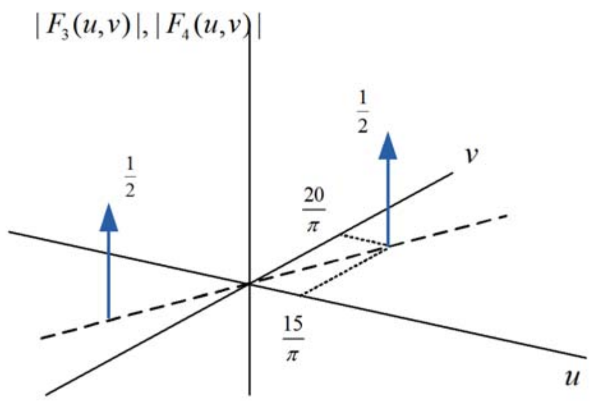
Hence, the Fourier spectrum of $f_3(x, y)$ is

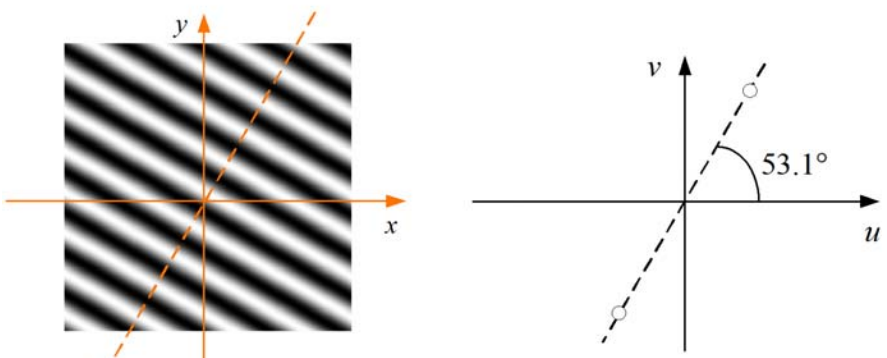
$$
|F_3(u, v)| = \frac{1}{2} \delta \left(u + \frac{15}{\pi}, v + \frac{20}{\pi} \right) + \frac{1}{2} \delta \left(u - \frac{15}{\pi}, v - \frac{20}{\pi} \right) \tag{4}
$$

Part (d)

$f_4(x, y) = \sin(30x + 40y + 30)$ is a translated version of $f_3(x, y)$. Hence

$$
|F_4(u, v)| = |F_3(u, v)|
$$





$\sin(30x + 40y) \quad \leftrightarrow \quad \frac{j}{2}\left[\delta\left(u + \frac{15}{\pi}, v + \frac{20}{\pi} \right) - \delta\left(u - \frac{15}{\pi}, v - \frac{20}{\pi} \right) \right]$

Question 4

Consider the transform $F(u)$ shifted by u_o, i.e.,

$$F_t(u) = F(u - u_0)$$

The inverse DFT of $F_t(u)$ is

$$\begin{aligned} f_t(x) &= \mathcal{F}^{-1}\{F(u - u_0)\} \\ &= f(x)\exp(j2\pi u_0 x/N) \end{aligned}$$

from the translation property of the DFT (Ch. 3B).

We wish to shift the transform $F(u)$ to the right by $N/2$:

$$F(u) \rightarrow F(u - N/2)$$

i.e., $u_0 = N/2$. Shifting $F(u)$ by u_0 is obtained by multiplying $f(x)$ by $\exp(j2\pi u_0 x/N)$. In this case, the exponential term is

$$\begin{aligned} \exp(j2\pi u_0 x/N) &= \exp\left(j2\pi \frac{N}{2}\frac{x}{N}\right) \\ &= \exp(j\pi x) \\ &= (e^{j\pi})^x \\ &= (-1)^x \end{aligned}$$

Thus,

$$f_t(x) = f(x)(-1)^x$$

and

$$\mathcal{F}\{f(x)(-1)^x\} = F(u - N/2)$$

$$\begin{aligned} F(u) &= \frac{1}{N}\sum_{x=0}^{N-1} f(x)\, e^{-j2\pi ux/N} \\ F(u - u_0) &= \frac{1}{N}\sum_{x=0}^{N-1} f(x)\, e^{-j2\pi (u-u_0)x/N} \\ &= \frac{1}{N}\sum_{x=0}^{N-1} \{f(x)\, e^{j2\pi u_0 x/N}\}\, e^{-j2\pi ux/N} \\ &= \mathcal{F}\{f(x)\, e^{j2\pi u_0 x/N}\} \end{aligned}$$

Question 5

Given

$$f(x) = 1, 1, 1, 1, 0, 0, 0, 0$$

Part (a)

$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\exp(-j2\pi ux/N) = \frac{1}{8}\sum_{x=0}^{3}\exp(-j2\pi ux/8)$$

$$\begin{aligned}
F(0) &= \frac{1}{8}\sum_{x=0}^{3}\exp(0) = 0.5 \\
F(1) &= \frac{1}{8}\sum_{x=0}^{3}\exp(-j2\pi(1)x/8) \\
&= \frac{1}{8}\exp(0) + \frac{1}{8}\exp(-j2\pi/8) + \frac{1}{8}\exp(-j2\pi\times 2/8) \\
&\qquad + \frac{1}{8}\exp(-j2\pi\times 3/8) \\
&= 0.327\angle -1.18
\end{aligned}$$

$$\begin{aligned}
F(2) &= \frac{1}{8}\sum_{x=0}^{3}\exp(-j2\pi(2)x/8) \\
&= \frac{1}{8}\exp(0) + \frac{1}{8}\exp(-j2\pi\times 2/8) + \frac{1}{8}\exp(-j2\pi\times 4/8) \\
&\qquad + \frac{1}{8}\exp(-j2\pi\times 6/8) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
F(3) &= \frac{1}{8}\sum_{x=0}^{3}\exp(-j2\pi(3)x/8) \\
&= \frac{1}{8}\exp(0) + \frac{1}{8}\exp(-j2\pi\times 3/8) + \frac{1}{8}\exp(-j2\pi\times 6/8) \\
&\qquad + \frac{1}{8}\exp(-j2\pi\times 9/8) \\
&= 0.135\angle -0.393
\end{aligned}$$

$$
\begin{aligned}
F(4) &= \frac{1}{8}\sum_{x=0}^{3} \exp(-j2\pi(4)x/8) \\
&= \frac{1}{8}\exp(0) + \frac{1}{8}\exp(-j2\pi \times 4/8) + \frac{1}{8}\exp(-j2\pi \times 8/8) \\
&\qquad + \frac{1}{8}\exp(-j2\pi \times 12/8) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
F(5) &= F^*(-5) \quad \text{(conjugate symmetry)} \\
&= F^*(8-5) \quad \text{(periodicity)} \\
&= F^*(3) \\
&= 0.135\angle + 0.393
\end{aligned}
$$

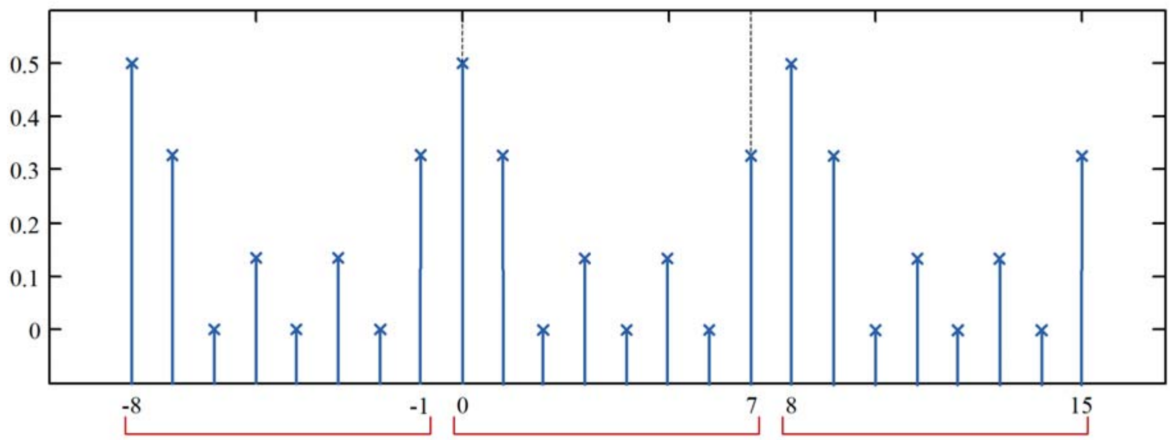
$$
\begin{aligned}
F(6) &= F^*(2) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
F(7) &= F^*(1) \\
&= 0.327\angle 1.18
\end{aligned}
$$

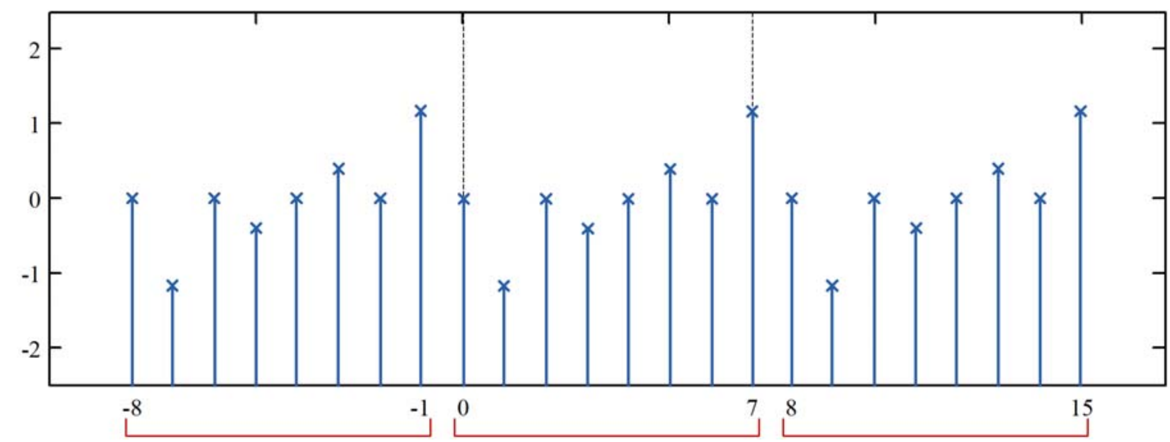
Part (b)

$$
\begin{aligned}
f(x) &= 1, \quad 1, \quad 1, \quad 1, \quad 0, \quad 0, \quad 0, \quad 0 \\
F(u) &= 0.5 \quad 0.327\angle -1.18, \quad 0, \quad 0.135\angle -0.393 \\
&\quad\; 0, \quad 0.135\angle +0.393, \quad 0, \quad 0.327\angle 1.18
\end{aligned}
$$

Magnitude of F(u)

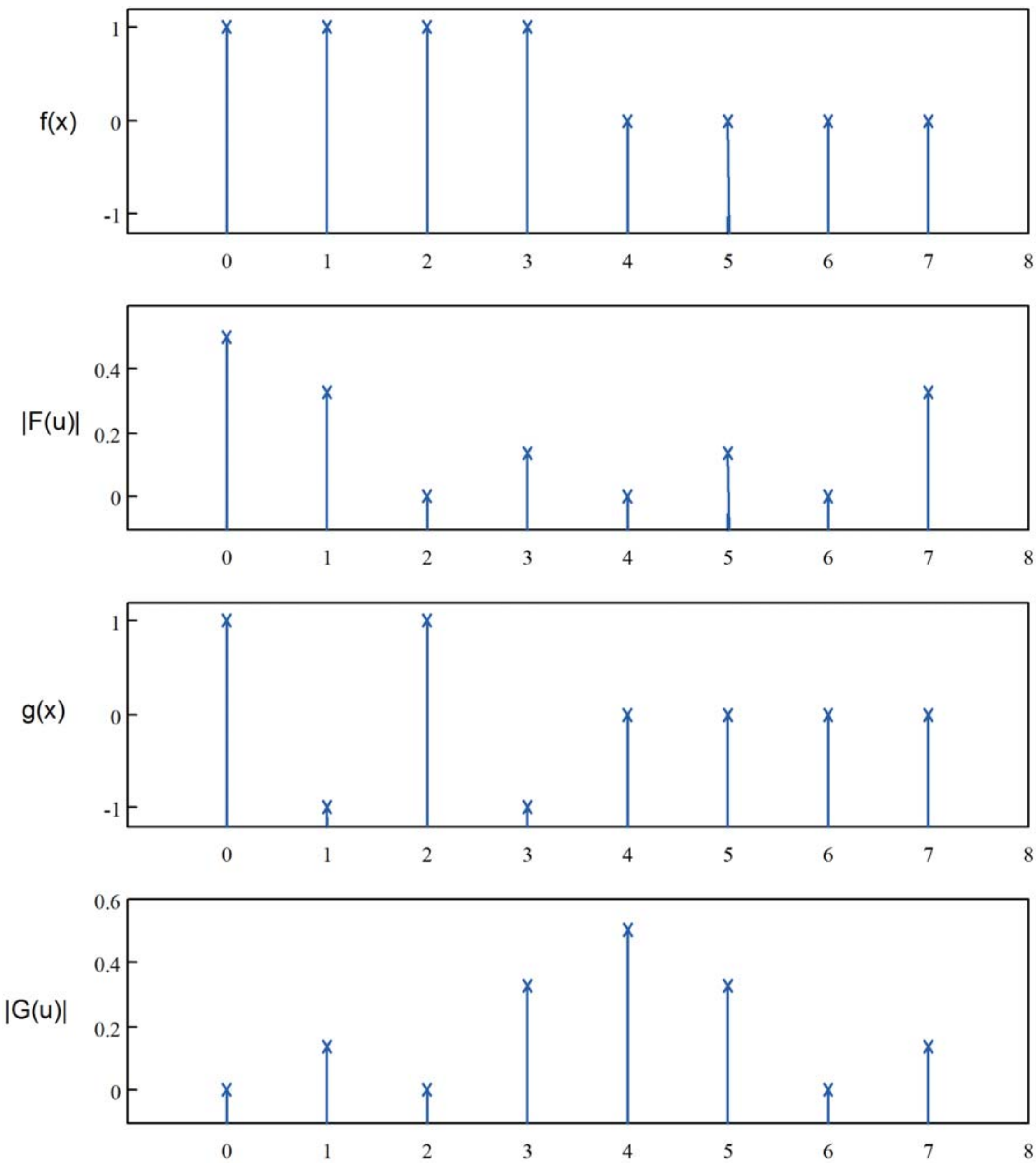


Phase of F(u)



Part (c)

$$\begin{aligned} g(x) &= (-1)^x f(x) = +1, \quad -1, \quad +1, \quad -1 \quad 0, \quad 0, \quad ,0, \quad 0 \\ G(u) &= 0, \quad 0.135\angle 0.393, \quad 0, \quad 0.327\angle 1.18, \\ & \quad 0.5, \quad 0.327\angle -1.18, \quad 0, \quad 0.135\angle -0.393 \end{aligned}$$



Question 6

Part (a)

$$\begin{aligned}
f_1(x) &= 1, 1, 1, 1 \\
F_1(u) &= \frac{1}{N}\sum_{x=0}^{N-1} f_1(x)\exp(-j2\pi ux/N) = \frac{1}{4}\sum_{x=0}^{3}\exp(-j2\pi ux/4) \\
F_1(0) &= \frac{1}{4}\sum_{x=0}^{3}\exp(0) = \frac{1}{4}\times 4 = 1 \\
F_1(1) &= \frac{1}{4}\sum_{x=0}^{3}\exp(-j2\pi(1)x/4) = \frac{1}{4}\times 0 = 0 \\
F_1(2) &= \frac{1}{4}\sum_{x=0}^{3}\exp(-j2\pi(2)x/4) = \frac{1}{4}\times 0 = 0 \\
F_1(3) &= \frac{1}{4}\sum_{x=0}^{3}\exp(-j2\pi(3)x/4) = \frac{1}{4}\times 0 = 0
\end{aligned}$$

Hence,

$$F_1(u) = 1, 0, 0, 0$$

Part(b)

$$\begin{aligned}
f_2(x) &= 1, 0, 0, 0 \\
F_2(u) &= \frac{1}{N}\sum_{x=0}^{N-1} f_2(x)\exp(-j2\pi ux/N) \\
&= \frac{1}{4}\sum_{x=0}^{0} 1 \\
&= \frac{1}{4}
\end{aligned}$$

$$F_2(0) = \frac{1}{4}, \qquad F_2(1) = \frac{1}{4}, \qquad F_2(2) = \frac{1}{4}, \qquad F_2(3) = \frac{1}{4}$$

Hence,

$$F(u) = \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$$

Part (c)

From the periodicity property, we note that

$$f_3(x) \;=\; f_2(x-1)$$

Hence,

$$\begin{aligned}
F_3(u) &= F_2(u)\exp(-j2\pi ua/4) \quad \text{(translation property)}\\
&= F_2(u)\exp(-j\pi u/2)
\end{aligned}$$

$$\begin{aligned}
F_3(0) &= F_2(0)\exp(0) = F_2(0)\\
F_3(1) &= F_2(1)\exp(-j\pi/2) = -jF_2(1)\\
F_3(2) &= F_2(2)\exp(-j\pi) = -F_2(2)\\
F_3(3) &= F_2(3)\exp(-j3\pi/2) = jF_2(3)
\end{aligned}$$

$$F_3(u) = \frac{1}{4}, -\frac{j}{4}, -\frac{1}{4}, \frac{j}{4}$$

Part (d)

$$f_4(x,y) = \begin{bmatrix} 1 & 1 & 1 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$F_4(x,v) = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$F_4(u,v) = \frac{1}{4}\begin{bmatrix} 1 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Part (e)

$$f_5(x,y) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$F_5(x,v) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$F_5(u,v) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that $F_5(u,v) = \delta(u,v)$.