Power Series

Power Series about $x = 0$

A ***power series*** about $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

where $c_0,\ c_1,\ \cdots,\ c_n,\ \cdots$ are constants while x is a variable.

A power series can be regarded as a function of x where it converges.

Power Series about $x = 0$ - Example

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

$a = 1$ and $r = x$

$$\frac{a}{1-r} = \frac{1}{1-x}$$

The geometric series

$$\boxed{a + ar + ar^2 + \cdots + ar^{n-1} + \cdots}$$

converges to the sum

$$\boxed{\frac{a}{1-r} \qquad \text{if } |r| < 1}$$

and

it diverges if $|r| \geq 1$.

This power series about $x = 0$ converges to $\dfrac{1}{1-x}$

when $|x| < 1$.

We state this as

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots, \quad -1 < x < 1.$$

Binomial expansion

Power Series about $x = a$

More generally, a ***power series*** about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

The number a is called the centre of the power series.

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Take $a = 0$

where $c_0,\ c_1,\ \cdots,\ c_n,\ \cdots$ are constants while x is a variable.

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The number a is called the centre of the power series.

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots, \quad -1 < x < 1.$$

Put $x = 2$

Left hand side $= \dfrac{1}{1-x}$

$$= \frac{1}{1-2}$$

$$= -1$$

Right hand side

$$= 1 + x + x^2 + \cdots + x^n + \cdots$$

$$= 1 + 2 + 4 + 8 + \ldots$$

$$> 0$$

Left hand side and Right hand side are not consistent!!

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots, \quad -1 < x < 1.$$

Put $x = -3$

Left hand side $= \dfrac{1}{1-x}$

$$= \frac{1}{1-(-3)}$$

$$= \frac{1}{4}$$

Right hand side

$= 1 + x + x^2 + \cdots + x^n + \cdots$

$= 1 - 3 + 9 - 27 + \ldots$

(integer)

Left hand side and Right hand side are not consistent!!

Problem

Given a ***power series*** about $x = a$,

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

we want to know for what values of x the power series is convergent.

The number a is called the centre of the power series.

We are interested in finding out

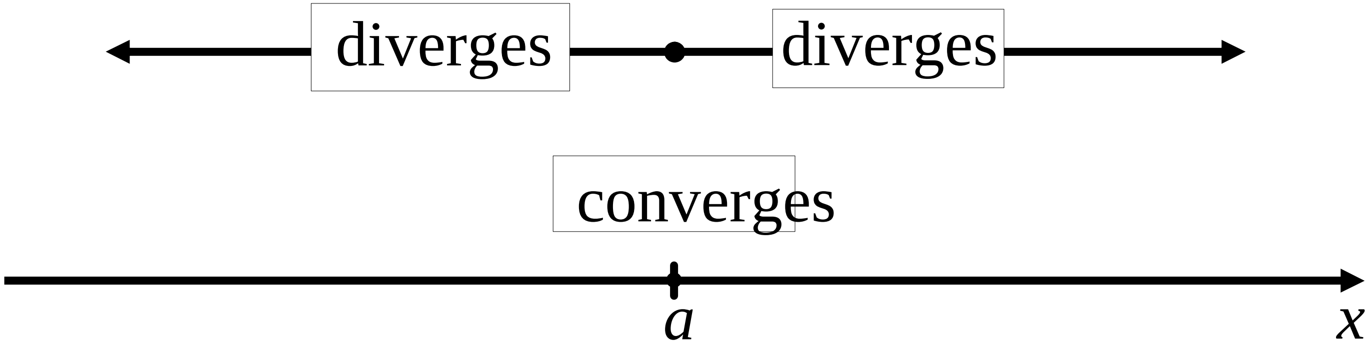
(1) interval of convergence ($x = a$ is the centre of the interval)

(2) radius of convergence R

Convergence of Power Series

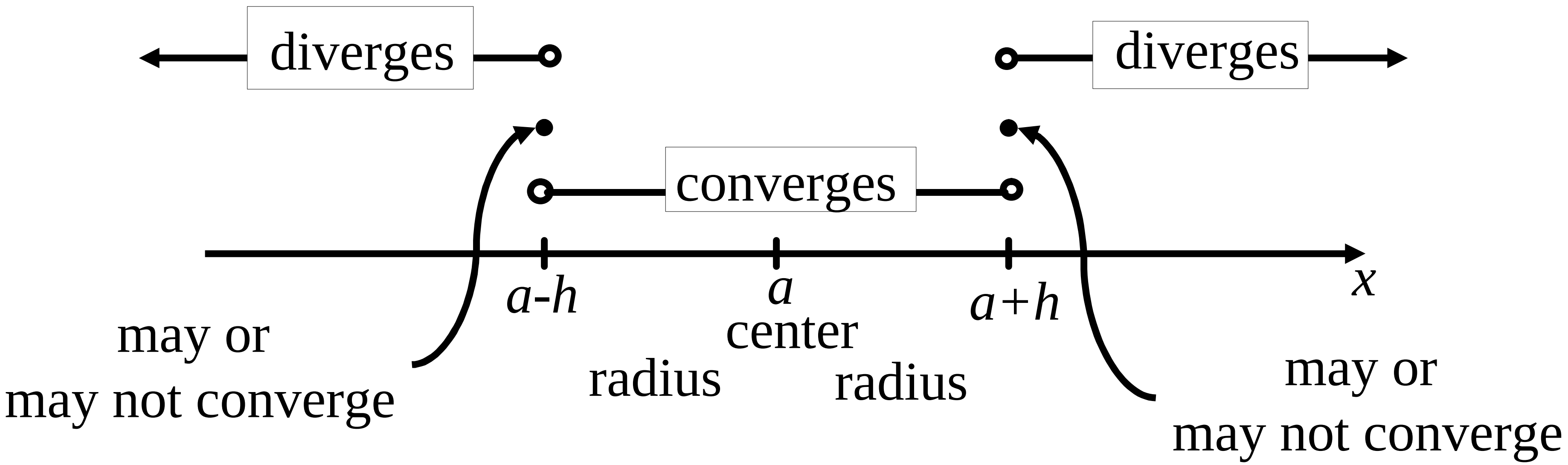
It can be shown that a power series always behaves in exactly one of the following 3 ways.

Case (i): Converges only at $x = a$ and diverges elsewhere.



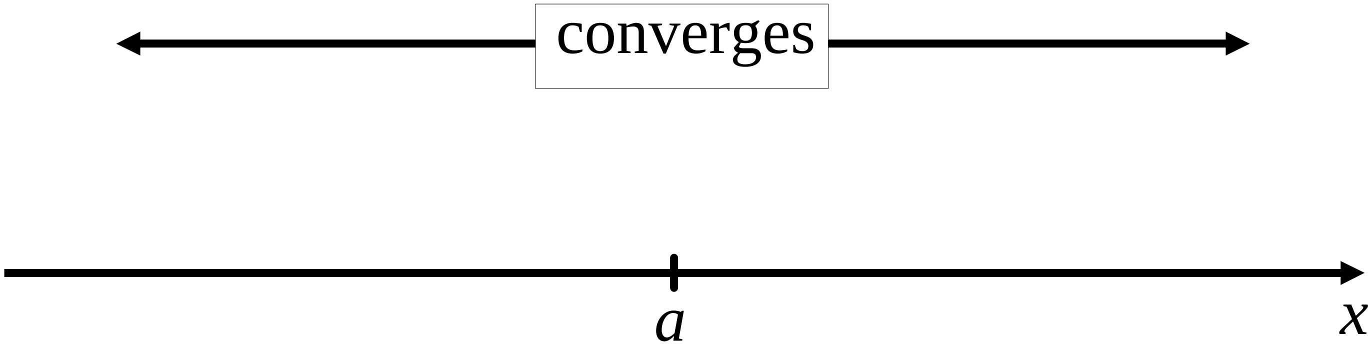
Convergence of Power Series

Case (ii): Converges for all x in the interval $(a-h, a+h)$ but diverges for $x < a-h$ and $x > a+h$.



Convergence of Power Series

Case (iii) : Converges for all values of x.



Radius of Convergence

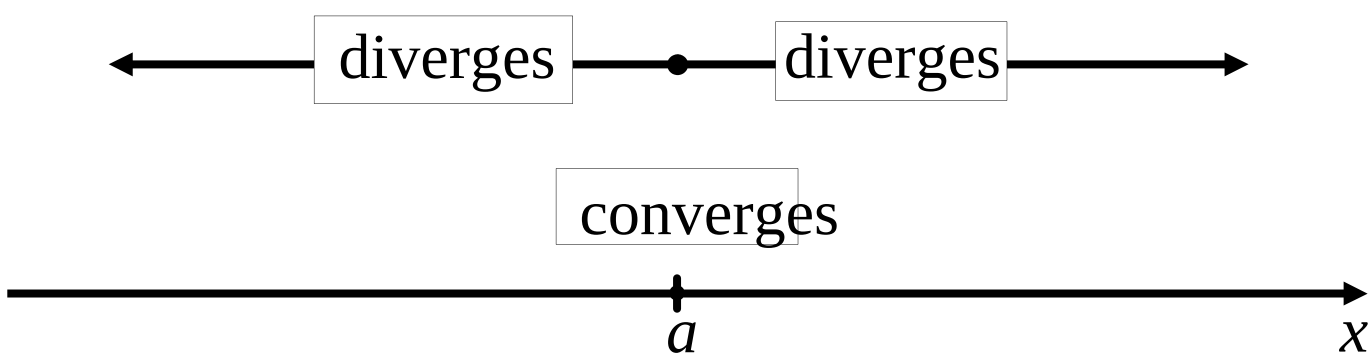
With reference to the cases for convergence of power series, the radius of convergence is as follows:

- Case (i): $R = 0$
- Case (ii): $R = h$
- Case (iii): $R = \infty$

Convergence of Power Series

It can be shown that a power series always behaves in exactly one of the following 3 ways.

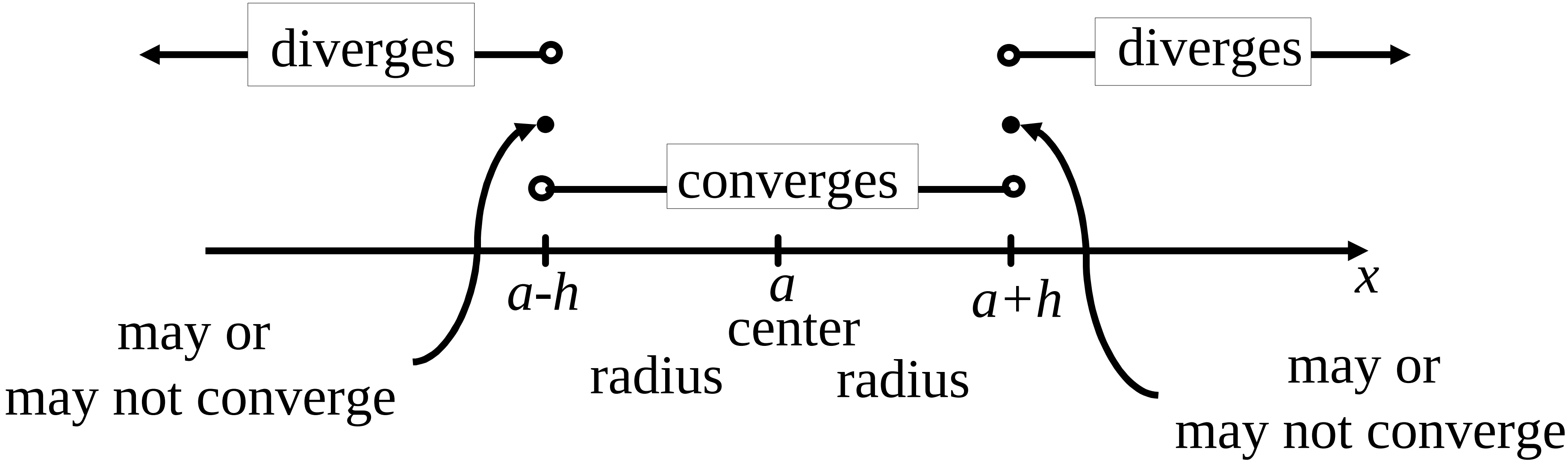
Case (i): Converges only at $x = a$ and diverges elsewhere.



Radius of convergence $R = 0$

Convergence of Power Series

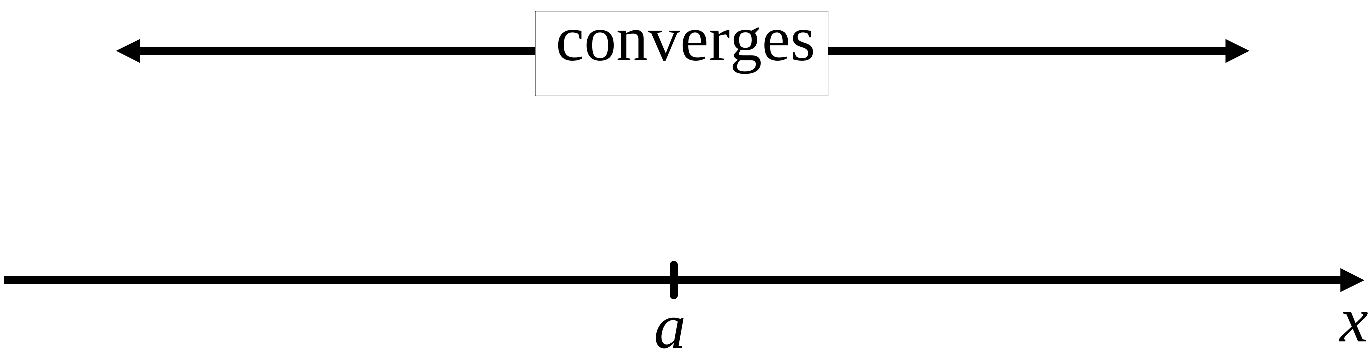
Case (ii): Converges for all x in the interval $(a-h, a+h)$ but diverges for $x < a-h$ and $x > a+h$.



$\text{Radius of convergence} = h$

Convergence of Power Series

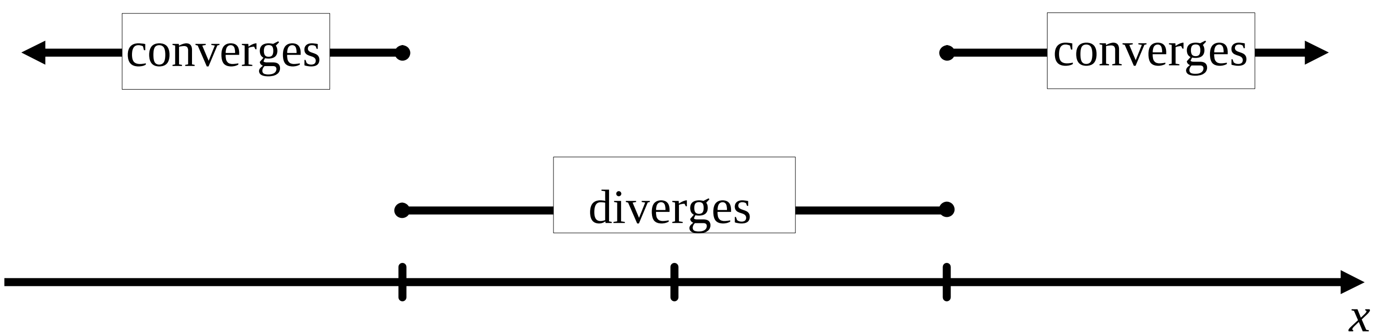
Case (iii) : Converges for all values of x.



Radius of convergence $= \infty$

Convergence of Power Series - Note

A power series cannot be convergent for two or more disjoint intervals.



The above cannot happen !!!

We are interested in finding out

(1) interval of convergence ($x = a$ is the centre of the interval)

(2) radius of convergence R

Question

How to find interval of convergence and radius of convergence ???

Use Ratio test

Let $\sum a_n$ be a series, and let

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = r.$$

(1) the series converges if $r < 1$.

(2) the series diverges if $r > 1$.

(3) no conclusion if $r = 1$.

Radius of Convergence - Example

(i) Find the radius of convergence of the power series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots$$

$$u_n=(-1)^{n-1}\frac{x^n}{n}$$

$$|u_n|=\left|(-1)^{n-1}\frac{x^n}{n}\right|$$

$$=\left|(-1)^{n-1}\right|\frac{|x|^n}{n}$$

$$=\frac{|x|^n}{n},\ \text{ since } \left|(-1)^{n-1}\right|=1$$

Replace n by $n+1$

$$|u_{n+1}|=\frac{|x|^{n+1}}{n+1}$$

(i) Find the radius of convergence of the power series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots$$

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(1) the series converges if $r<1$.

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(3) no conclusion if $r=1$.

$$|u_n|=\frac{|x|^n}{n}$$

$$|u_{n+1}|=\frac{|x|^{n+1}}{n+1}$$

Applying the ratio test,

$$\left|\frac{u_{n+1}}{u_n}\right|=\frac{|x|^{n+1}}{n+1}\times\frac{n}{|x|^n}=\frac{n}{n+1}|x|$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\frac{n}{n+1}|x|$$

$$=\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}|x|$$

$$=|x|$$

Thus, the series converges for $|x|<1$,

i.e., $-1<x<1$.

(i) Find the radius of convergence of the power series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

Let $\sum a_n$ be a series, and let

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = r.$$

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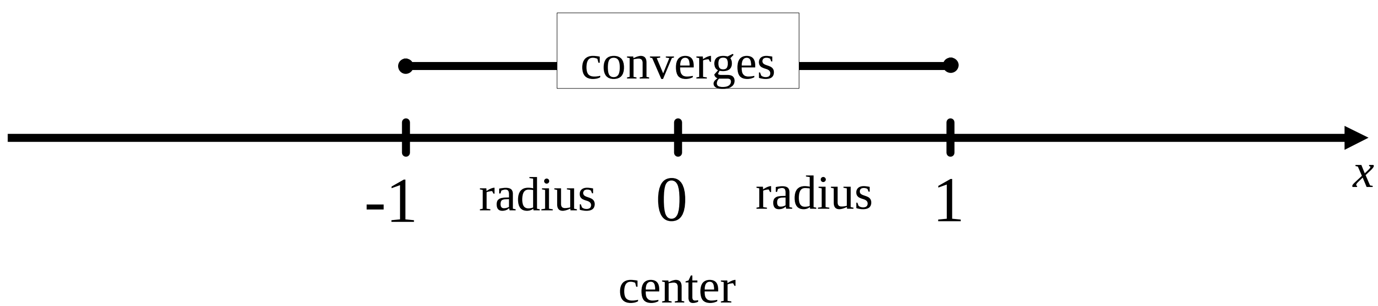
$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\frac{n}{n+1}|x|$$

$$= \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}|x|$$

$$= |x|$$

Thus, the series converges for $|x| < 1$,

i.e., $-1 < x < 1$.



Its center is at $a = 0$.

Radius of convergence $R = 1$.

Radius of Convergence - Example

Let $\sum a_n$ be a series, and let

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = r.$$

(1) the series converges if $r<1$.

(2) the series diverges if $r>1$.

(3) no conclusion if $r=1$.

(ii) Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$|u_n| = \frac{|x|^n}{n!}$$

$$|u_{n+1}| = \frac{|x|^{n+1}}{(n+1)!}$$

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right|$$

$$= \frac{1}{n+1}|x|$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\frac{|x|}{n+1}$$

$$= 0$$

Since $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = 0 < 1$ for all values of x

the series converges for all x.

Radius of convergence $R = \infty$.

Radius of Convergence - Example

(iii) Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} n!\,x^n = 1 + x + 2!\,x^2 + 3!\,x^3 + \cdots$$

Let $\sum a_n$ be a series, and let

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = r.$$

(1) the series converges if $r < 1$.

(2) the series diverges if $r > 1$.

(3) no conclusion if $r = 1$.

$$|u_n| = n!\,|x|^n \qquad |u_{n+1}| = (n+1)!\,|x|^{n+1}$$

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(n+1)!\,x^{n+1}}{n!\,x^n}\right| = (n+1)\,|x|$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}(n+1)\,|x| = \begin{cases} 0 & \text{if } x = 0 \\ \infty & \text{if } x \neq 0 \end{cases}$$

The series converges only at $x = 0$.

Radius of convergence $R = 0$.