Chapter 6. SAMPLING DISTRIBUTIONS

March 8, 2011

Outline

Populations and Samples

Theories on Sampling Distribution of Mean

The Sampling Distribution of the Mean

when σ is known when σ is unknown

t, F and chi-square Distributions

1 Populations and Samples

- Population can be finite or infinite
  - Infinite: impossible to observe all its values
  - Finite: may be impractical or uneconomical to observe all its values.
- Example:
  - Infinite population: products from a production line; outcomes of flipping a coin;
  - Finite population: monthly income of Singaporean. products of a factory in one day. scores of students in NUS.

- Numerical descriptive measures of a population are called **parameters** e.g., p, μ, σ.

- The parameters are also are needed to calculate the probability and distribution, e.g. N(μ, σ^2), B(n, p), while the type of distribution is known!

Examples

- A pollster is sure that the responses to his "agree/disagree" question will follow a binomial distribution, but p, the proportion of those who "agree" in the population, is unknown.

- An agronomist believes that the yield per acre of a variety of wheat is approximately normally distributed, but the mean μ and the standard deviation σ of the yields are unknown.

2 Random Sample

- You must rely on the **sample** to learn about these parameters and study the properties of the population.
- To assure that a sample is representative of population, **random sample** is necessary.

Definition of Random Sample

A set of observations $X_1, X_2, ..., X_n$ constitute a random sample of size n from

- a finite population of size N, if its values are chosen so that each subset of n, the N elements of the population has the same probability of being selected.
- an infinite population with distribution f(x) if
  1. each X_i is a random variable whose distribution is given by f(x).
  2. These n random variables are independent.

Ways of getting Random Sample

- If the population is finite: serially number the elements of the population. Then, select a sample with the aid of a table of random digits or computer random number generator.
- Example:
  - If N = 500, n = 10.
  - Use three arbitrary selected columns of Table 7 (or number generated from computer, e.g. **sample(500, 10)** in R) to obtain 10 different three-digit numbers less than or equal to 500.
  - These 10 numbers are serial numbers of elements in the population that

we choose.

- What if N is very large or population size is infinite.
  - Use of random numbers becomes practically impossible.
  - Artificial or mechanical devices can be used to approach the randomness.
  - Example:
    - Selecting a sample from a production line: select one unit each half an hour.
    - Flip a coin: try to flip it such that neither side is intentionally favored.
- Care needs to be taken, when artificial or mechanical devices are used for selecting random samples: unconscious biases may be resulted!

3 Theories on Sampling Distribution of Mean

3.1 Estimator of μ: $\bar{X}$

- Now that a random sample of n observations, i.e. $X_1, X_2, ..., X_n$, has been taken.
- How do we estimate the population mean μ ? Answer: by $\bar{X}$.
  - X_1 is a random variable and so do $X_2, ..., X_n$.
  - $\bar{X}$ is a random variable as well.

Example of $\bar{X}$

- Question of interest: the average waiting time for buses of a student attending ST2334 every morning.

- Assume that the waiting time for each student every morning has the same uniform distribution with a $=$ 0, b $=$ 9.

- For the convenience of recording, only mins are reported. i.e. when I ask a students waiting time for the bus in a morning, the student answers 0min or 1min or 2mins, ...,

- Therefore, denote by X the waiting time for the bus in a morning. Then, P(X=i)=1/10, for each i=0,1,2,...,9.

- I requested 50 randomly selected students to report their waiting time for a bus each morning in 10 randomly selected mornings over the semester.

- For each student, 10 observations $X_1, X_2, ..., X_{10}$, are obtained. The sample mean $\bar{X}$ is then computed.

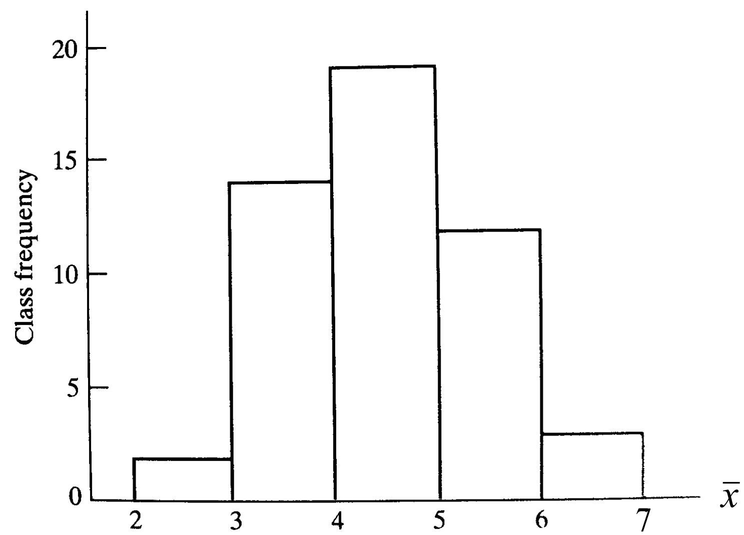
- Now that we obtain 50 sample means (50 students) as follows

4.4	3.2	5.0	3.5	4.1	4.4	3.6	6.5	5.3	4.4
3.1	5.3	3.8	4.3	3.3	5.0	4.9	4.8	3.1	5.3
3.0	3.0	4.6	5.8	4.6	4.0	3.7	5.2	3.7	3.8
5.3	5.5	4.8	6.4	4.9	6.5	3.5	4.5	4.9	5.3
3.6	2.7	4.0	5.0	2.6	4.2	4.4	5.6	4.7	4.3

- Group these means into a distribution

$\bar{X}$	[2.0, 3.0]	[3.0, 4.0]	[4.0, 5.0]	[5.0, 6.0]	[6.0, 7.0]
frequency	2	4	19	12	3

- Experimental Sampling Distribution of the Mean



3.2 Theoretical Sampling Distribution of $\bar{X}$

- **Theorem 6.1**: If a random sample of size n is taken from a population with mean μ and variance σ^2, then $\bar{X}$ is a random variable with mean μ and
  - For samples from an infinite population, its variance is

$$\frac{\sigma^2}{n}.$$

  - For samples from a finite population with size N, its variance is

$$\frac{\sigma^2}{n} \cdot \frac{N-n}{N-1},$$

where $\frac{N-n}{N-1}$ is called finite population correction factor

4 Sample Mean of Normally Distributed Population

- $\{X_1, X_2, ..., X_n\}$ is a random sample from $N(\mu, \sigma^2)$.
- Expectation of $\bar{X}$: μ.
- Variance of $\bar{X}$: σ^2/n.
- Thus, $\bar{X}$ has distribution: $N(\mu, \sigma^2)$.
- and

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

follows a standard normal distribution, i.e.

$$Z \sim N(0, 1)$$

Validity of sample mean

– The expectation of $\bar{X}$ is equal to the population mean μ.

– In "the long run"/"many times of estimation", $\bar{X}$ does not introduce any systematic bias as an estimator of μ.

– $\bar{X}$ can serve as a valid estimator of μ.

Accuracy of $\bar{X}$

- for infinite population, when n gets larger and larger, σ^2/n, the variance of $\bar{X}$, becomes smaller and smaller, that is, the accuracy of $\bar{X}$, as an estimator of μ keeps improving.

- for finite population, similar arguments apply, if the sample constitutes a substantial proportion of the population.

5 The population is not Normal: Law of Large Numbers (L.L.N.)

- If $X_1, X_2, ..., X_n$ are independent random variables with the same mean μ and variance (known or unknown), $\bar{X}$ gets more and more close to μ, as n gets bigger and bigger.

- **Theorem 6.2 (L.L.N.)**: If $X_1, X_2, ..., X_n$ are independent random variables with the same mean μ and variance σ^2 , when n is large,

$$\bar{X} \approx \mu.$$

6 Central Limit Theorem (C.L.T.)

- Next we work on the distribution of

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

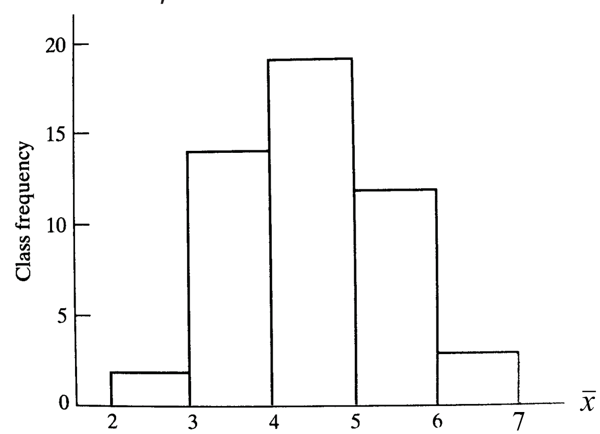
- We have known that Z is N(0,1), providing that $X_1, X_2, ..., X_n$ is a random sample from $N(\mu, \sigma^2)$

- **Theorem 6.3 (C.L.T.)**: If $\bar{X}$ is the mean of a random sample of size n taken from a population having mean μ and finite variance σ^2, then, if n is large,

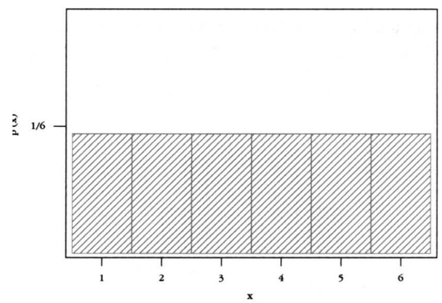
$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \approx N(0, 1)$$

- The C.L.T. states that, under rather general conditions, sums and means of random samples drawn from a population tend to possess an approximately normal distribution.

- **Example** Revisit Bus Waiting Time Example. Even though the population itself is a uniform distribution, the distribution of $\bar{X}$ is close to bell shaped.



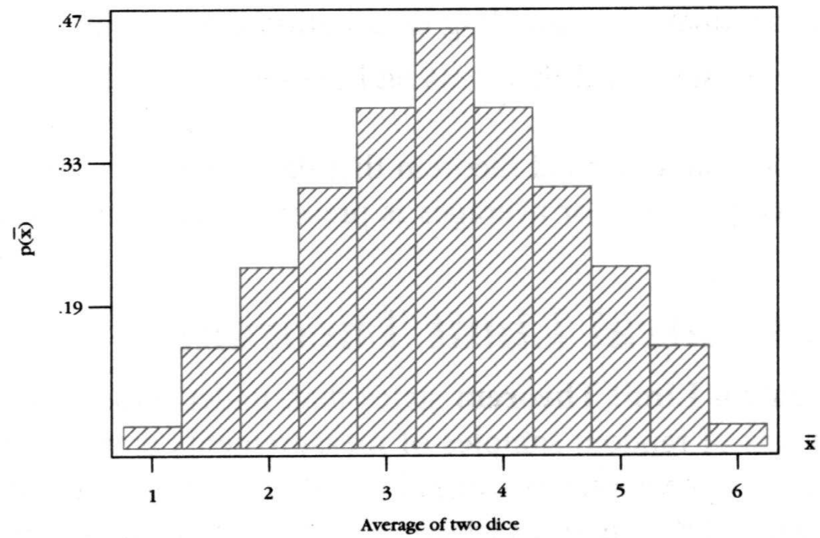
- **Example** The figure below shows the probability distribution of the number appearing on a single toss of a die.

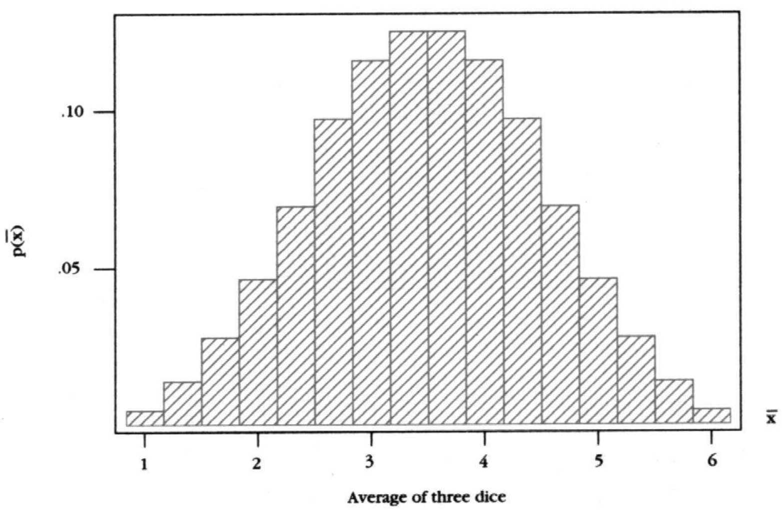


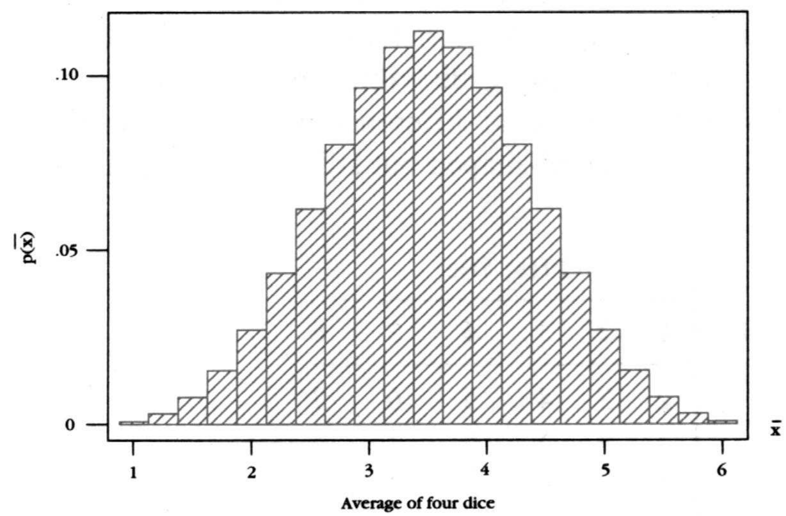
- Let us illustrate the sampling distributions of $\bar{X}$ for n $=$ 2, 3, 4, respectively.

- From the figures below, we can see that the distribution of $\bar{X}$ gets more and more close to normal, as n gets larger and larger.

- The convergence speed is fairly high: n$=$4, it is close enough to normal.


.47
.33
.19
p($\bar{x}$)
$\bar{x}$
1
2
3
4
5
6
Average of two dice


.10
.05
p($\bar{x}$)
$\bar{x}$
1
2
3
4
5
6
Average of three dice


.10
.05
0
p($\bar{x}$)
$\bar{x}$
1
2
3
4
5
6
Average of four dice

- Rule of Thumb
  - Rule of Thumb: In practice, the normal distribution provides an excellent approximation to the sampling distribution of the mean $\bar{X}$ if $n \geq 30$.
  - Note: if the random sample come from a normal population, $\bar{X}$ is normally distributed regardless of the value of n.

- **Example** If a 1-gallon can of paint covers on the average 513.3 square feet with a standard deviation of 31.5 square feet, what is the probability that the sample mean area covered by a sample of 40 of these 1-gallon cans will be anywhere from 510.0 to 520.0 square feet?

Note $\mu = 513.3$, $\sigma = 31.5$ and $n = 40$. Let $\bar{X}$ be the sample mean. We need to find

$$P(510 < \bar{X} < 520)$$

BY CLT, approximate $\bar{X}$ by $N(\mu, \sigma^2/n)$. We have

$$\begin{aligned} P(510 < \bar{X} < 520) &= P(\frac{510 - \mu}{\sigma/\sqrt{n}} < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < \frac{520 - \mu}{\sigma/\sqrt{n}}) \\ &= P(-0.066 < Z < 1.34) = 0.6553 \end{aligned}$$

7 The Sampling Distribution of the Mean

7.1 σ is Known, Data Normal

- $X_1, X_2, ..., X_n$ is a random sample from N(μ , σ^2).
- Expectation of $\bar{X}$: μ .
- Variance of $\bar{X}$: σ^2/n.
- $\bar{X}$ has distribution: N(μ , σ^2/n).
-

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

7.2 σ is Known, n is large (regardless of population distribution)

- $X_1, X_2, ..., X_n$ are random sample with the same μ and σ, σ is known, then, if n is large by CLT,

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \approx N(0, 1)$$

7.3 σ Unknown

- When σ is unknown, the above derivation is still correct, however, not applicable, i.e. $(\bar{X} - \mu)/(\sigma/\sqrt{n})$ contains the unknown quantity σ.

- Recall that we estimate σ by s , where

$$s = \sqrt{\sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{n-1}}$$

- The natural consideration is to replace σ by s, i.e. consider

$$t = \frac{\bar{X} - \mu}{s/\sqrt{n}}$$

- This is examined on two possibilities.

  – σ **Unknown, Data Normal, n small**

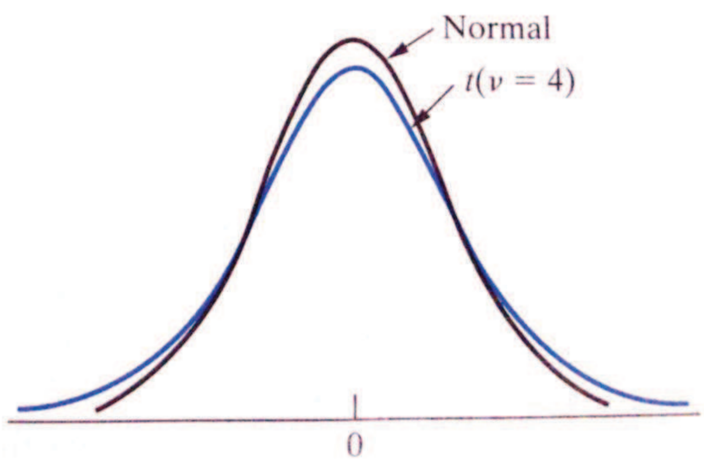
    * **Theorem 6.4**: If $\bar{X}$ is the mean of a random sample with mean μ and variance σ^2, then,

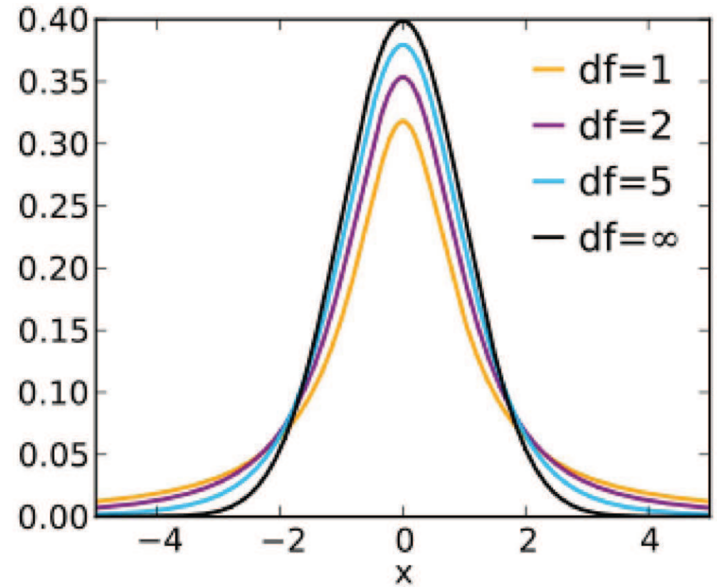
$$t = \frac{\bar{X} - \mu}{s/\sqrt{n}}$$

is a random variable having the t distribution with the parameter $\nu = n - 1$.

* The above theorem is true in cases of n large or small. However, it is mainly used in the case that n is small. The reason is because of the nature of t distribution, when $n \geq 30$, we can replace it by N(0, 1).

* **t-distribution** (or called student-distribution, denoted by $t(\nu)$) The shape of the p.d.f. of t-distribution is similar to that of standard normal distribution





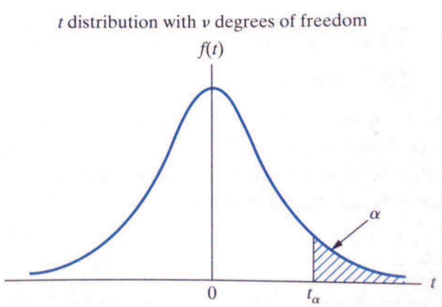
$*$ ν is called the degree of freedom (df). As $\nu \to \infty$, the distribution tends to N(0,1).

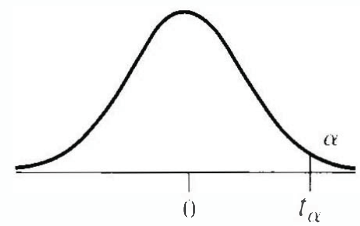
$*$ More about Properties of t Distribution

· The density function of t distribution is bell shaped, centered and being **symmetric** at 0.

· t distribution approaches N(0,1) as the parameter $\nu \to \infty$. when $n \geq 30$, we can replace it by N(0, 1).

· Table 4 contains selected values of t_α defined below for various values of ν, such that $P(t > t_\alpha) = \alpha$





Values of t_α

v	$\alpha = 0.10$	$\alpha = 0.05$	$\alpha = 0.025$	$\alpha = 0.01$	$\alpha = 0.00833$	$\alpha = 0.00625$	$\alpha = 0.005$	v
1	3.078	6.314	12.706	31.821	38.204	50.923	63.657	1
2	1.886	2.920	4.303	6.965	7.650	8.860	9.925	2
3	1.638	2.353	3.182	4.541	4.857	5.392	5.841	3
4	1.533	2.132	2.776	3.747	3.961	4.315	4.604	4
5	1.476	2.015	2.571	3.365	3.534	3.810	4.032	5
6	1.440	1.943	2.447	3.143	3.288	3.521	3.707	6
7	1.415	1.895	2.365	2.998	3.128	3.335	3.499	7
8	1.397	1.860	2.306	2.896	3.016	3.206	3.355	8
9	1.383	1.833	2.262	2.821	2.934	3.111	3.250	9
10	1.372	1.812	2.228	2.764	2.870	3.038	3.169	10
11	1.363	1.796	2.201	2.718	2.820	2.891	3.106	11
12	1.356	1.782	2.179	2.681	2.780	2.934	3.055	12
13	1.350	1.771	2.160	2.650	2.746	2.896	3.012	13
14	1.345	1.761	2.145	2.624	2.718	2.864	2.977	14
15	1.341	1.753	2.131	2.602	2.694	2.837	2.947	15

16	1.337	1.746	2.120	2.583	2.673	2.813	2.921	16
17	1.333	1.740	2.110	2.567	2.655	2.793	2.898	17
18	1.330	1.734	2.101	2.552	2.639	2.775	2.878	18
19	1.328	1.729	2.093	2.539	2.625	2.759	2.861	19
20	1.325	1.725	2.086	2.528	2.613	2.744	2.845	20
21	1.323	1.721	2.080	2.518	2.602	2.732	2.831	21
22	1.321	1.717	2.074	2.508	2.591	2.720	2.819	22
23	1.319	1.714	2.069	2.500	2.582	2.710	2.807	23
24	1.318	1.711	2.064	2.492	2.574	2.700	2.797	24
25	1.316	1.708	2.060	2.485	2.566	2.692	2.787	25
26	1.315	1.706	2.056	2.479	2.559	2.684	2.779	26
27	1.314	1.703	2.052	2.473	2.553	2.676	2.771	27
28	1.313	1.701	2.048	2.467	2.547	2.669	2.763	28
29	1.311	1.699	2.045	2.462	2.541	2.663	2.756	29
inf.	1.282	1.645	1.960	2.326	2.394	2.498	2.576	inf.

- **Example** The lecturer of ST2334 announced that the mean score of the midterm is 16 out of 30. A student doubts it, so he checked it by randomly choosing 5 classmates and asked them their scores of midterm. Suppose he was lucky and all randomly selected classmates told him their scores: 20, 19, 24, 22, 25.

  Question: should the student believe that the mean score of the midterm is 16?

  – Exam scores can usually be approximated by normal distribution. The student has n=5 sampled data

$$X_1 = 20, X_2 = 19, X_3 = 24, X_4 = 22, X_5 = 25.$$

– The question is $\mu = 16$? s is unknown.

– If $\mu \quad = 16$ were the case, $t = \frac{\bar{X}-\mu}{s/\sqrt{n}}$ should follow a t distribution with $\nu = 5 - 1 = 4.$

– Now that with the observed data $\mu \quad = 22$, s $=$ 2.55.

$$t = \frac{22 - 16}{2.55/\sqrt{5}} = 5.26$$

– Check Table 4, $t_{0.005} = 4.604$, indicating that only with chance 0.005, t should go beyond 4.604, providing that the lecturer is telling the true, i.e. $\mu \quad =$16.

– Should the student believe the mean score of the midterm is 16?

σ Unknown, n large

- $X_1, X_2, ..., X_n$ are random sample with the same μ , σ is unknown, then, if n is large by the CLT, we have

$$Z = \frac{\bar{X} - \mu}{s/\sqrt{n}} \approx N(0, 1)$$

- Compared with the situation that σ is known, n large, the only difference is that now σ is replaced with s.

8 The Sampling Distribution of the Variance

- For random $X_1, X_2, ..., X_n$, recall the sample variance is defined as

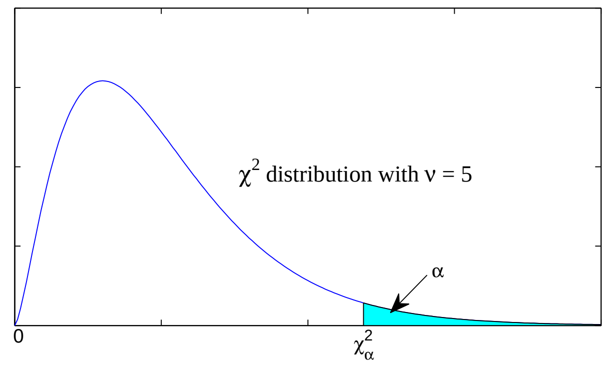
$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

- We also proved that $Es^2 = \sigma^2$

- **Theorem 6.5**. If the random sample of size n taken from a normal population having the variance σ^2, then

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2} = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{\sigma^2}$$

is a random variable having the **chi-square distribution, or χ^2−distribution** with the parameter $\nu = n - 1$.



– All chi-squares values are nonnegative.

– The chi-square distribution is a family of curves, each is determined by the degrees of freedom (ν).

– All Chi-Square Distributions are skewed to the right.

Values for χ^2_α

ν	$\alpha = 0.995$	$\alpha = 0.99$	$\alpha = 0.975$	$\alpha = 0.95$	$\alpha = 0.05$	$\alpha = 0.025$	$\alpha = 0.01$	$\alpha = 0.005$	ν
1	0.0000393	0.000157	0.000982	0.00393	3.841	5.024	6.635	7.879	1
2	0.0100	0.0201	0.0506	0.103	5.991	7.378	9.210	10.597	2
3	0.0717	0.115	0.216	0.352	7.815	9.348	11.345	12.838	3
4	0.207	0.297	0.484	0.711	9.488	11.143	13.277	14.860	4
5	0.412	0.554	0.831	1.145	11.070	12.833	15.086	16.750	5
6	0.676	0.872	1.237	1.635	12.592	14.449	16.812	18.548	6
7	0.989	1.239	1.690	2.167	14.067	16.013	18.475	20.278	7
8	1.344	1.646	2.180	2.733	15.507	17.535	20.090	21.955	8
9	1.735	2.088	2.700	3.325	16.919	19.023	21.666	23.589	9
10	2.156	2.558	3.247	3.940	18.307	20.483	23.209	25.188	10
11	2.603	3.053	3.816	4.575	19.675	21.920	24.725	26.757	11
12	3.074	3.571	4.404	5.226	21.026	23.337	26.217	28.300	12
13	3.565	4.107	5.009	5.892	22.362	24.736	27.688	29.819	13
14	4.075	4.660	5.629	6.571	23.685	26.119	29.141	31.319	14
15	4.601	5.229	6.262	7.261	24.996	27.488	30.578	32.801	15
16	5.142	5.812	6.908	7.962	26.296	28.845	32.000	34.267	16
17	5.697	6.408	7.564	8.672	27.587	30.191	33.409	35.718	17
18	6.265	7.015	8.231	9.390	28.869	31.526	34.805	37.156	18
19	6.844	7.633	8.907	10.117	30.144	32.852	36.191	38.582	19
20	7.434	8.260	9.591	10.851	31.410	34.170	37.566	39.997	20

- **Example** Suppose 6 random samples are drawn from N(μ, 4), define the sample variance

$$s^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{4}$$

Find c such that $P(s^2 > c) = 0.05$

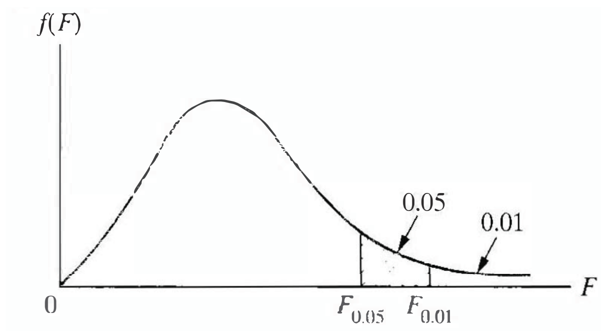
- **Theorem 6.6**. If s_1^2 and s_2^2 are the variances of independent random samples $X_1, ..., X_{n_1}$ and $X_1', ..., X_{n_2}'$ of size n_1 and n_2 respectively, taken from two normal populations having the same variance. then

$$F = \frac{s_1^2}{s_2^2} = \frac{\sum_{i=1}^{n_1}(X_i - \bar{X})^2/(n_1 - 1)}{\sum_{i=1}^{n_2}(X_i' - \bar{X}')^2/(n_2 - 1)}$$

is a random variable having the F distribution with the parameters $\nu_1 = n_1 - 1$ and and $\nu_2 = n_2 - 1$, where $\bar{X}' = (X_1' + ... + X_{n_2}')/n_2$.

- $F(\nu_1, \nu_2)$-distribution has two parameters, ν_1 and ν_2, are called the numerator and denominator degrees of freedom respectively.

- $F_{1-\alpha}(\nu_1, \nu_2) = 1/F_{\alpha}(\nu_2, \nu_1)$



Suppose $F_1 \sim F(\nu_1, \nu_2)$, c_1 such that $P(F_1 < c_1) = \alpha$ is $1/c_2$ with

$$P(F_2 > c_1) = \alpha$$

where $F_2 \sim F(\nu_2, \nu_1)$.

- **Example** If two independent random samples of size $n_1 = 7$ and $n_2 = 13$ are taken from a normal population. what is the probability that the variance of the first sample will be at least three times as large as that of the second sample?

  From Table 6 we find that $F_{0.05} =$ 3.00 for $\nu_1 = 6, \nu_2 = 12$, thus the desired probability is 0.05.

- **Example** Find the value of $F_{0.95}$ (corresponding to a left-hand tail probability of 0.05) for $\nu_1 = 10$ and $\nu_2 = 20$ degrees of freedom.

$$F_{0.95}(10, 20) = 1/F_{0.05}(20, 10) = 1/2.77 = 0.36$$

Values for $F_{0.01}(\nu_1, \nu_2)$

df2\df1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	22	24	26	28	30
3	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.35	27.23	27.13	27.05	26.98	26.92	26.87	26.83	26.79	26.75	26.72	26.69	26.64	26.60	26.56	26.53	26.50
4	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66	14.55	14.45	14.37	14.31	14.25	14.20	14.15	14.11	14.08	14.05	14.02	13.97	13.93	13.89	13.86	13.84
5	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16	10.05	9.96	9.89	9.82	9.77	9.72	9.68	9.64	9.61	9.58	9.55	9.51	9.47	9.43	9.40	9.38
6	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.79	7.72	7.66	7.61	7.56	7.52	7.48	7.45	7.42	7.40	7.35	7.31	7.28	7.25	7.23
7	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.54	6.47	6.41	6.36	6.31	6.28	6.24	6.21	6.18	6.16	6.11	6.07	6.04	6.02	5.99
8	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.73	5.67	5.61	5.56	5.52	5.48	5.44	5.41	5.38	5.36	5.32	5.28	5.25	5.22	5.20
9	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.18	5.11	5.05	5.01	4.96	4.92	4.89	4.86	4.83	4.81	4.77	4.73	4.70	4.67	4.65
10	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.77	4.71	4.65	4.60	4.56	4.52	4.49	4.46	4.43	4.41	4.36	4.33	4.30	4.27	4.25
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.46	4.40	4.34	4.29	4.25	4.21	4.18	4.15	4.12	4.10	4.06	4.02	3.99	3.96	3.94
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.22	4.16	4.10	4.05	4.01	3.97	3.94	3.91	3.88	3.86	3.82	3.78	3.75	3.72	3.70
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	4.02	3.96	3.91	3.86	3.82	3.78	3.75	3.72	3.69	3.66	3.62	3.59	3.56	3.53	3.51
14	8.86	6.51	5.56	5.04	4.70	4.46	4.28	4.14	4.03	3.94	3.86	3.80	3.75	3.70	3.66	3.62	3.59	3.56	3.53	3.51	3.46	3.43	3.40	3.37	3.35
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.73	3.67	3.61	3.56	3.52	3.49	3.45	3.42	3.40	3.37	3.33	3.29	3.26	3.24	3.21
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.62	3.55	3.50	3.45	3.41	3.37	3.34	3.31	3.28	3.26	3.22	3.18	3.15	3.12	3.10
17	8.40	6.11	5.19	4.67	4.34	4.10	3.93	3.79	3.68	3.59	3.52	3.46	3.40	3.35	3.31	3.27	3.24	3.21	3.19	3.16	3.12	3.08	3.05	3.03	3.00
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.43	3.37	3.32	3.27	3.23	3.19	3.16	3.13	3.10	3.08	3.03	3.00	2.97	2.94	2.92
19	8.19	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52	3.43	3.36	3.30	3.24	3.19	3.15	3.12	3.08	3.05	3.03	3.00	2.96	2.92	2.89	2.87	2.84
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.29	3.23	3.18	3.13	3.09	3.05	3.02	2.99	2.96	2.94	2.90	2.86	2.83	2.80	2.78
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26	3.18	3.12	3.07	3.02	2.98	2.94	2.91	2.88	2.85	2.83	2.78	2.75	2.72	2.69	2.67
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.09	3.03	2.98	2.93	2.89	2.85	2.82	2.79	2.76	2.74	2.70	2.66	2.63	2.60	2.58
26	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18	3.09	3.02	2.96	2.90	2.86	2.82	2.78	2.75	2.72	2.69	2.66	2.62	2.58	2.55	2.53	2.50
28	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12	3.03	2.96	2.90	2.84	2.79	2.75	2.72	2.68	2.65	2.63	2.60	2.56	2.52	2.49	2.46	2.44
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.91	2.84	2.79	2.74	2.70	2.66	2.63	2.60	2.57	2.55	2.51	2.47	2.44	2.41	2.39

Values for $F_{0.05}(\nu_1, \nu_2)$

df2\df1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	22	24	26	28	30
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.76	8.74	8.73	8.71	8.70	8.69	8.68	8.67	8.67	8.66	8.65	8.64	8.63	8.62	8.62
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.94	5.91	5.89	5.87	5.86	5.84	5.83	5.82	5.81	5.80	5.79	5.77	5.76	5.75	5.75
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.70	4.68	4.66	4.64	4.62	4.60	4.59	4.58	4.57	4.56	4.54	4.53	4.52	4.50	4.50
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.03	4.00	3.98	3.96	3.94	3.92	3.91	3.90	3.88	3.87	3.86	3.84	3.83	3.82	3.81
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.60	3.57	3.55	3.53	3.51	3.49	3.48	3.47	3.46	3.44	3.43	3.41	3.40	3.39	3.38
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.31	3.28	3.26	3.24	3.22	3.20	3.19	3.17	3.16	3.15	3.13	3.12	3.10	3.09	3.08
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.10	3.07	3.05	3.03	3.01	2.99	2.97	2.96	2.95	2.94	2.92	2.90	2.89	2.87	2.86
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.94	2.91	2.89	2.86	2.85	2.83	2.81	2.80	2.79	2.77	2.75	2.74	2.72	2.71	2.70
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.82	2.79	2.76	2.74	2.72	2.70	2.69	2.67	2.66	2.65	2.63	2.61	2.59	2.58	2.57
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.72	2.69	2.66	2.64	2.62	2.60	2.58	2.57	2.56	2.54	2.52	2.51	2.49	2.48	2.47
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.63	2.60	2.58	2.55	2.53	2.51	2.50	2.48	2.47	2.46	2.44	2.42	2.41	2.39	2.38
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.57	2.53	2.51	2.48	2.46	2.44	2.43	2.41	2.40	2.39	2.37	2.35	2.33	2.32	2.31
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.51	2.48	2.45	2.42	2.40	2.38	2.37	2.35	2.34	2.33	2.31	2.29	2.27	2.26	2.25
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.46	2.42	2.40	2.37	2.35	2.33	2.32	2.30	2.29	2.28	2.25	2.24	2.22	2.21	2.19
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.41	2.38	2.35	2.33	2.31	2.29	2.27	2.26	2.24	2.23	2.21	2.19	2.17	2.16	2.15
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.37	2.34	2.31	2.29	2.27	2.25	2.23	2.22	2.20	2.19	2.17	2.15	2.13	2.12	2.11
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.34	2.31	2.28	2.26	2.23	2.21	2.20	2.18	2.17	2.16	2.13	2.11	2.10	2.08	2.07
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.31	2.28	2.25	2.23	2.20	2.18	2.17	2.15	2.14	2.12	2.10	2.08	2.07	2.05	2.04
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.26	2.23	2.20	2.17	2.15	2.13	2.11	2.10	2.08	2.07	2.05	2.03	2.01	2.00	1.98
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.22	2.18	2.15	2.13	2.11	2.09	2.07	2.05	2.04	2.03	2.00	1.98	1.97	1.95	1.94
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.18	2.15	2.12	2.09	2.07	2.05	2.03	2.02	2.00	1.99	1.97	1.95	1.93	1.91	1.90
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.15	2.12	2.09	2.06	2.04	2.02	2.00	1.99	1.97	1.96	1.93	1.91	1.90	1.88	1.87
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.13	2.09	2.06	2.04	2.01	1.99	1.98	1.96	1.95	1.93	1.91	1.89	1.87	1.85	1.84