National University of Singapore
Department of Electrical & Computer Engineering

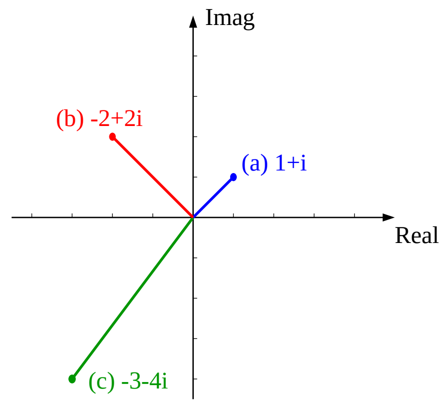
EE2023 Signals & Systems
Tutorial 8 Solutions

Section I

1. (a) $|1+j| = \sqrt{1^2+1^2} = \sqrt{2}$
   $\angle 1+j = \tan^{-1} 1 = 45^o$
   $\therefore 1+j = \sqrt{2}\angle 45^o = \sqrt{2}e^{j\frac{\pi}{4}}$

   (b) $|-2+2j| = \sqrt{2^2+2^2} = \sqrt{8}$
   $\angle -2+2j = 180^o - \tan^{-1} 1 = 135^o$
   $\therefore -2+2j = \sqrt{8}\angle 135^o = \sqrt{8}e^{j\frac{3\pi}{4}}$

   (c) $|-3-4j| = \sqrt{3^2+4^2} = 5$
   $\angle -3-4j = -(180^o - \tan^{-1}\frac{4}{3}) = -126.9^o$
   $\therefore -3-4j = 5\angle -126.9^o = 5e^{-2.215j}$



2. (a) As the output (transient + steady-state) signal is required, the expression should be found using Laplace Transform or calculus.
   Using the Laplace Transform method, the output of the first order system, $G(s)$, is

$$\begin{aligned} Y(s) &= \frac{2}{0.2s+1}\mathcal{L}\{\sin 3t\} \\ &= \frac{2}{0.2s+1}\,\frac{3}{s^2+9} \\ &= 0.88e^{-5t} - 0.88\cos 3t + 1.47\sin 3t \end{aligned}$$

   The output of the first order system at steady-state is

$$\begin{aligned} y_{ss}(t) &= \lim_{t\to\infty}[0.88e^{-5t} - 0.88\cos 3t + 1.47\sin 3t] \\ &= -0.88\cos 3t + 1.47\sin 3t \\ &= A\sin(3t+\phi) \end{aligned}$$

   where $A = \sqrt{0.88^2+1.47^2} = 1.71$ and $\phi = \tan^{-1}\dfrac{-0.88}{1.47} = -0.54$ rad
   The magnitude and phase of $G(s)|_{s=j\omega}$ is

$$\begin{aligned} |G(j\omega)|_{\omega=3} &= \left|\frac{2}{0.2\times 3j+1}\right| = 1.71 \\ \angle G(j\omega)|_{\omega=3} &= \angle\frac{2}{0.2\times 3j+1} = -0.54 \text{ rad} \end{aligned}$$

   Hence, it may be concluded that $\frac{A}{1} = |G(j\omega)|$ and $\phi = \angle G(j\omega)$ where $\omega = 3$ rad/s.

3. Since $\dfrac{\text{output amplitude}}{\text{input amplitude}} = |G(5j)| = 0.75$ and amplitude of the output signal is 4.5,

$$\text{Amplitude of input signal} = \frac{\text{output amplitude}}{|G(5j)|} = 6$$

Relationship between phase of input signal, $\angle G(j\omega)$ and phase of output signal is "Phase of input signal + $\angle G(j\omega)$ = Phase of output signal". Given that $\angle G(5j) = -69^o$,

$$\text{Phase of input signal} = -\angle G(5j) + \text{Phase of output signal} = 68^o - 38^o = 30^o = \frac{\pi}{6}$$

Hence, input signal is $6\ \sin(5t + \frac{\pi}{6}) = 6\ \cos(5t - 60^o)$" as $\cos(\omega t - 90^o) = \sin\ \omega t$.

4. (a) To solve the problem, first identify the high frequency asymptote, which is the right-most straight line segment. Then, the answer may be found using the definition of the gradient for a straight line i.e. $\frac{\text{vertical}}{\text{horizontal}}$. Note that the x-axis scale is non-linear (log scale), so the horizontal segment used to calculate the gradient must be multiples of one decade.

(b) First, it is necessary to understand how the Bode diagram of a complex transfer function is constructed using the Bode diagrams of simple factors derived in class and summarized on Page 6-14.

Suppose $G(s) = F_1(s)F_2(s)$, the product of two simple factors $F_1(s)$ and $F_2(s)$. The frequency response of $G(s)$

$$\begin{aligned}
20\log_{10}|G(j\omega)| &= 20\log_{10}|F_1(j\omega)F_2(j\omega)| \\
&= 20\log_{10}|F_1(j\omega)||F_2(j\omega)| \\
&= 20\log_{10}|F_1(j\omega)| + 20\log_{10}|F_2(j\omega)| \\
\angle G(j\omega) &= \angle F_1(j\omega)F_2(j\omega) \\
&= \angle F_1(j\omega) + \angle F_2(j\omega)
\end{aligned}$$

The equations shown above indicate that the Bode diagram of $G(s)$ is the sum of the Bode diagram of the individual factors. Hence, the task of constructing the Bode diagram of complex transfer functions using the simple factors reduces to adding the straight line asymptotes of the simple factors together. When two straight lines $y_1 = m_1x + c_1$ and $y_2 = m_2x + c_2$ are added together, the resulting straight line is

$$y_1 + y_2 = (m_1 + m_2)x + (c_1 + c_2)$$

The key result is the gradient of the new line is the sum of the slopes of the individual factors. Hence, the following conclusions can be drawn :

- A pole on the origin, or the factor $\frac{1}{s}$, contributes a straight line whose slope is -20 dB/decade for all frequencies to the magnitude plot. The phase contribution of $\frac{1}{s}$ is -90^o for all frequencies.

- A real pole, or the factor $\frac{1}{\tau s+1}$, causes the slope of the magnitude response to decrease by 20 dB/decade when frequency is larger than $\frac{1}{\tau}$. In the phase response, the factor $\frac{1}{\tau s+1}$ causes the phase response is decrease from 0^o to -90^o at the rate of -45^o/decade between the frequencies of $\frac{0.1}{\tau} = 0.1\omega_c$ and $\frac{10}{\tau} = 10\omega_c$.
- A real zero, or the factor $\tau s + 1$, causes the slope of the magnitude response to increase by 20 dB/decade when frequency is larger than the corner frequency $\omega_c = \frac{1}{\tau}$. In the phase response, the factor $\frac{1}{\tau s+1}$ causes the phase response to increase from 0^o to 90^o at the rate of 45^o/decade between the frequencies of $\frac{0.1}{\tau} = 0.1\omega_c$ and $\frac{10}{\tau} = 10\omega_c$.

In Figure 1, the slope of the low frequency asymptote (leftmost straight line segment) is -20 dB/decade so there is one integrator (pole on the origin). The slope becomes more gentle only after the corner frequency at 1 rad/s so $G(s)$ has one zero. After the other two corner frequencies, the slope becomes more negative so there are two real poles. Together with the pole on the origin, $G(s)$ has a total of 3 poles.

(c) Since the low frequency asymptote is $\frac{K}{(j\omega)^N}$, all points on the leftmost straight line segment must satisfy

$$20\log_{10} K - 20\log_{10}\omega$$

K can be derived by substituting any point on the low frequency asymptote into the equation. The most convenient point to use is $\omega = 0$ because $20\log_{10} K - 20\log_{10}\omega$ simplifies to $20\log_{10} K$ when $\omega = 1$.

Section II

1. Transfer function of the simplified suspension system,

$$\frac{X_o(s)}{X_i(s)} = \frac{bs+k}{s^2+bs+k}$$

Question states that the input signal due to the speed reducing strips on the road, $x_i(t)$, may be approximated by the following Fourier Series representation

$$x(t) = \frac{4}{\pi}\left[\sin\omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \ldots\right]$$

where $\omega = 1$ rad/s.
Since the input consists of 3 sinuosoidal waveforms ($\sin t$, $\sin 3t$ and $\sin 5t$) and system is linear, principle of superposition may be used to determine the solution i.e.

- Find the outputs when the inputs are the sinusoidal waveforms $\sin(\omega_1 t)$ when $\omega_1 = 1, 3, 5$ rad/s
- The output when the input is the periodic square wave is the sum of the output due to the 3 sinusoidal waveforms

Given that $m = 1$ kg, $k = 1\frac{\mathrm{N}}{\mathrm{m}}$ and $b = \sqrt{2}\frac{\mathrm{N}}{\mathrm{m/s}}$, the magnitude and phase of

$$G(j\omega_1) = \frac{j\sqrt{2}\omega_1 + 1}{(j\omega_1)^2 + j\sqrt{2}\omega_1 + 1}$$

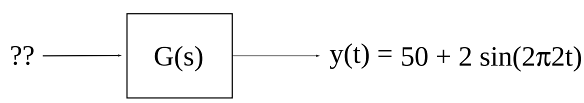
when $\omega_1 = 1$ rad/s, 3 rad/s and 5 rad/s are tabulated in the following table

ω_1 (rad/s)	$\lvert G(j\omega_1)\rvert$	$\angle G(j\omega_1)$ (rad)
1	1.2247	-0.6155
3	0.4814	-1.3147
5	0.2854	-1.4248

Hence, the steady-state output is

$$\begin{aligned} x_{o,ss}(t) &= \frac{4}{\pi}\Bigg[1.2247\sin(t - 0.6155) + \frac{0.4814}{3}\sin(3t - 1.3147) + \\ &\qquad \frac{0.2854}{5}\sin(5t - 1.4248) + \ldots\Bigg] \\ &= \frac{4}{\pi}\left[1.2247\sin(t - 0.6155) + 0.1605\sin(3t - 1.3147) + \right. \\ &\qquad \left. 0.005708\sin(5t - 1.4248) + \ldots\right] \end{aligned}$$

2. In this problem, the output and the transfer function is known and the task is to determine the input.



Question states that the plant (thermocouple and recorder) may be represented by a first order system with unity steady-state gain ($K = 1$), time constant (τ) of approximately 1 minute and no dead time i.e.

$$G(s) = \frac{K}{\tau s + 1} = \frac{1}{s + 1}$$

At steady state, the recorded temperature oscillates with a frequency of 2 cycles per minute between 52^{o}C and 48^{o}C i.e.

$$y(t) = 50 + 2\sin(2\pi f t) \qquad \text{where } f = 2 \text{ cycles per minute}$$

By the principle of superposition, the input will have two components because the output comprises of a dc component and a sinusoidal signal.

- The dc component of the output is 50. Since the transfer function has unity gain when $\omega = 0$ rad/s, the dc component of the input must also be 50.

- The frequency of the sinusoidal component is 2 cycles/min. The magnitude and phase of the plant, $G(s)$, at that frequency is

$$\begin{aligned}
G(j\omega) &= \frac{1}{2\pi fj+1} \\
&= \frac{1}{4\pi j+1} \\
|G(j\omega)|_{\omega=4\pi} &= \frac{1}{\sqrt{16\pi^2+1}} \\
&= 0.0793
\end{aligned}$$

It has been established that $|G(j\omega)|$ is the ratio of the output to the input. As the sinusoidal component in the ouput has an amplitude of 2, the amplitude of the input sinusoidal waveform is

$$\frac{2}{0.0793} = 25.2$$

Hence, the input temperature oscillates between $50-25.2 = 24.8^o$C and $50+25.2 = 75.2^o$C. Clearly, the results show that the recorder does not have sufficient bandwidth !

3. (a) Method is similar to Q4 in Section I. From the asymptotic magnitude response plot,

   - At 4 rad/s, slope changes from 0 to 20 dB/decade
     $\Rightarrow$ Presence of factor $\frac{1}{4}s+1$
   - At 10 rad/s, slope changes from 20 dB/decade to 0
     $\Rightarrow$ Presence of factor $\left(\frac{1}{10}s+1\right)^{-1}$
   - At 20 rad/s, slope changes from 0 to -40 dB/decade
     $\Rightarrow$ Presence of the factor $\left(\frac{1}{20}s+1\right)^{-2}$
   - Static gain $= 10^{\frac{13.9794}{20}} = 5$

   Hence, the transfer function is

$$\begin{aligned}
G(s) &= \frac{5\left(\frac{1}{4}s+1\right)}{\left(\frac{1}{10}s+1\right)\left(\frac{1}{20}s+1\right)^2} \\
G(s) &= \frac{5\times\frac{1}{4}(s+4)}{\frac{1}{10}(s+10)\frac{1}{20}(s+20)\frac{1}{20}(s+20)} \\
&= \frac{5000(s+4)}{(s+10)(s+20)(s+20)}
\end{aligned}$$

   $\therefore K = 5000$, $\alpha = 4$, $\beta = 10$, $\gamma = \lambda = 20$

   (b) For other plants to have the magnitude response in the question, it should have factors that only affects the phase response. One such factor is the transport delay as $|e^{-j\omega L}| = 1$ so $|G(j\omega)| = |G(j\omega)e^{-j\omega L}$. Another possibility is RHP poles/zeros because $|j\omega T+1| = |j\omega T-1|$.

Section III

1. Problem requires the steady-state values of signals when the input signal is a sinusoid so frequency response theorem is concept for formulating the solution. To use the frequency response theorem, a transfer function relating the input and output signal is needed so derive the necessary function

   - using Kirchoff current law
   - express the capacitor and inductor as impedences and applying the current division law.

2. Method is similar to Section II Q3. Parameters of the first order factor may be identified using the corner frequencies. The second order factor may be identified using the resonant peak and resonant frequency.