NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2003-2004

MA2214 Combinatorial Analysis

April 2004 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains a total of **FIVE (5)** questions and comprises **THREE (3)** printed pages.

2. Answer **ALL** questions.

3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Question 1 [20 marks]

(a) A factory canteen has 8 different food stalls. How many ways are there for 40 workers to queue randomly in front of these food stalls if

(i) there is no restriction?

(ii) there are at least 3 workers queuing in front of each food stall?

(b) Find the numbers of ways of distributing 20 different sweets to 6 children if

(i) there is no restriction;

(ii) one particular boy is given exactly 5 sweets and no one is empty-handed.

Question 2 [20 marks]

(a) Twenty boys and five girls are to be seated around a round table. Find the number of ways this can be done if

(i) there is no restriction;

(ii) all girls form a single block;

(iii) a particular girl G is adjacent to two particular boys B_1 and B_2;

(iv) any two girls are separated by at least two boys.

(b) Prove by a combinatorial argument that each of the following expressions is an integer for every positive integer n.

(i) $\dfrac{[(n+1)!]!}{[(n!)!]^{n+1}(n+1)!}$

(ii) $\dfrac{(7n+4)!}{(4n+3)!(3n+2)!}$

Question 3 [20 marks]

(a) Let $A = \{x \in \mathbb{Z} | 1 \leq x \leq 1000\}$. For each $m \in \{0, 1, 2, 3, 4\}$, let $E(m)$ denote the number of integers in the set A which are divisible by exactly m of the integers 4, 6, 10 and 25. Find $E(m)$ for each $m \in \{0, 1, 2, 3, 4\}$.

(b) Find the number of 9-digit integers formed by all of the nine numbers 1, 1, 1, 2, 2, 3, 3, 4, 4 if

(i) there is no restriction;

(ii) no two adjacent digits are identical.

Question 4 [20 marks]

(a) Given that $a_1 = 14, a_2 = 29, a_3 = 46$, and $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3} - 4n$ for any integer $n > 3$, find a_n in terms of n.

(b) For every positive integer n, let a_n denote the number of ways of paving a rectangular 2 metres by n metres floor using the 1 metre by 1 metre red tiles, 1 metre by 2 metres blue tiles and L-shaped green tiles (here the L-shaped tiles are 2 metres by 2 metres tiles with a corner 1 metre by 1 metre tile removed).

(i) Find a recurrence relation of a_n with the necessary initial conditions.

(ii) Find a_n in terms of n.

Question 5 [20 marks]

(a) For every positive integer n, let a_n denote the number of ways of distributing n distinct objects into 12 distinct boxes labelled from 1 to 12 such that the boxes 1, 2, 3 and 4 altogether contain an even number of objects, the boxes 5, 6, 7 and 8 altogether contain an odd number of objects, and the box 9 contains at least 3 objects.

(i) Write down the exponential generating function of a_n.

(ii) Find a_n in terms of n.

(b) A restaurant has 12 kinds of drink available. They are labelled from 1 to 12 by the owner of the restaurant. For every positive integer n, let a_n denote the number of all possible orders of n drinks received by a waiter of the restaurant such that, for each of these orders, the total number of drinks for types 1, 2, 3 and 4 altogether is even, the total number of drinks for types 5, 6, 7 and 8 altogether is odd, and the number of drinks for type 9 is at least 3.

(i) Write down the ordinary generating function of a_n.

(ii) Find a_n in terms of n.

END OF PAPER