7. Linear Time-Invariant Systems

Linear time invariant (LTI) systems are generally modeled using differential equations in the time domain, or transfer functions in the Laplace transform domain. Transfer functions are defined by the ratio of the output (response) to the input (excitation) in the *s-domain* where s is the Laplace variable. This is illustrated in Fig.7-1.

$$\mathcal{L}\{x(t)\} = X(s) \longrightarrow \boxed{\boldsymbol{H(s)}} \longrightarrow Y(s) = \mathcal{L}\{y(t)\}$$

Fig.7-1 Transfer function: $H(s) = \dfrac{Y(s)}{X(s)} = \mathcal{L}\{h(t)\}$

The input-output relationship of a system can then be expressed in the *s-domain* as

$$Y(s) = H(s)X(s) \tag{7.1}$$

or in the *time-domain* (by taking inverse Laplace transform on both sides of (7.1)) as

$$y(t) = \underbrace{\mathcal{L}^{-1}\{H(s)X(s)\} = \overbrace{h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau}^{\text{convolution}}}_{\text{see Chapter 5, eqn.(5.10)}} \tag{7.2}$$

7.1 System Model

- Begin with the general time-domain differential equation model of a system (as defined in Chapter 4, eqn.(4.8))

$$\sum_{n=0}^{N} a_n \frac{d^n y(t)}{dt^n} = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m}. \tag{7.3}$$

- The transfer function $H(s)$ is obtained by taking the Laplace transform on both sides of (7.3), **assuming zero initial conditions**. This results in:

$$\Rightarrow \quad \sum_{n=0}^{N} a_n Y(s) s^n = \sum_{m=0}^{M} b_m X(s) s^m$$

$$\Rightarrow \quad H(s) = \frac{Y(s)}{X(s)} = \left(\sum_{m=0}^{M} b_m s^m \Big/ \sum_{n=0}^{N} a_n s^n \right) = \frac{b_M s^M + b_{M-1} s^{M-1} + \cdots + b_0}{a_N s^N + a_{N-1} s^{N-1} + \cdots + a_0}$$

$$\Rightarrow \quad H(s) = K \frac{\left(\frac{s}{z_1}+1\right)\left(\frac{s}{z_2}+1\right)\cdots\left(\frac{s}{z_M}+1\right)}{\left(\frac{s}{p_1}+1\right)\left(\frac{s}{p_2}+1\right)\cdots\left(\frac{s}{p_N}+1\right)}; \qquad K = \frac{b_0}{a_0} \tag{7.4a}$$

$$\Rightarrow \quad H(s) = K' \frac{(s+z_1)(s+z_2)\cdots(s+z_M)}{(s+p_1)(s+p_2)\cdots(s+p_N)}; \qquad K' = K \cdot \frac{p_1 p_2 \cdots p_N}{z_1 z_2 \cdots z_M} \tag{7.4b}$$

where $-p_n$ and $-z_m$ are the poles and zeros, respectively, of the system $H(s)$. Note, from (7.4), the different ways that a transfer function can be factorized.

- The general transfer function in (7.4) is said to be an N^{th}-order system with N poles and M zeros.
- The poles $(-p_n)$ and zeros $(-z_m)$ are roots obtained by, respectively, solving

$$a_N s^N + a_{N-1} s^{N-1} + \cdots + a_0 = 0 \tag{7.5}$$

and

$$b_M s^M + b_{M-1} s^{M-1} + \cdots + b_0 = 0\,. \tag{7.6}$$

- To express $H(s)$ in the forms of (7.4), we must have knowledge of $-p_n$ and $-z_m$. Otherwise, we will have to solve (7.5) and (7.6).

Example 7-1:

1^{st} - *order system*	:	$a_1 s + a_0 = 0$	➔	$-p_1 = -a_0/a_1$
2^{nd} - *order system*	:	$a_2 s^2 + a_1 s + a_0 = 0$	➔	$-p_1, -p_2 = \dfrac{-a_1 \pm \left(a_1^2 - 4a_2 a_0\right)^{0.5}}{2a_2}$
Higher order systems	:	*No exact algebraic formula to evaluate the roots. Roots are often found thru iterative algorithms such as the* ***Laguerre's method****.*		

- For practical systems, $N > M$. We say that the system has a ***pole-zero excess*** of $(N - M)$

7.2 System Stability (Role of Poles and Zeros)

To understand the role of poles and zeros, consider the output

$$Y(s) = H(s)X(s) = K \cdot \frac{p_1 p_2 \cdots p_N}{z_1 z_2 \cdots z_M} \frac{(s+z_1)(s+z_2)\cdots(s+z_M)}{(s+p_1)(s+p_2)\cdots(s+p_N)} X(s) \quad \cdots \quad \text{cf (6.4b)}$$

where $X(s)$ is the input. $X(s)$ may contain poles and zeros too. We distinguish the poles of $X(s)$ from the poles of $H(s)$ by referring to the former as ***input poles*** while the latter as ***system poles***.

Example 7-2 (Step Response of a General LTI System)

Suppose the input is a unit step signal $x(t) = u(t)$ which has a Laplace transform of $X(s) = 1/s$. Then

$$\begin{aligned} Y_{step}(s) &= K \cdot \frac{p_1 p_2 \cdots p_N}{z_1 z_2 \cdots z_M} \frac{(s+z_1)(s+z_2)\cdots(s+z_M)}{(s+p_1)(s+p_2)\cdots(s+p_N)} \cdot \frac{1}{s} \\ &= \frac{\alpha_1}{s+p_1} + \frac{\alpha_2}{s+p_2} + \cdots + \frac{\alpha_N}{s+p_N} + \frac{K}{s} \end{aligned} \tag{7.7}$$

where α_n are the constants obtained after partial factorization. Taking the inverse Laplace transform of $Y_{step}(s)$,

$$y_{step}(t) = \underbrace{\left[\alpha_1 e^{-p_1 t} + \alpha_2 e^{-p_2 t} + \cdots + \alpha_B e^{-p_N t}\right] u(t)}_{y_{tr}(t)} + \underbrace{K u(t)}_{y_{ss}(t)} = y_{tr}(t) + y_{ss}(t) \quad (7.8)$$

where

$y_{tr}(t)$ is the transient response which depends on the system poles $\left(-p_n;\ n = 1, 2, \cdots, N\right)$,

$y_{ss}(t)$ is the steady-state response which is constant.

Furthermore,

$$\lim_{t \to \infty} y_{tr}(t) = \begin{cases} 0 & \cdots \text{ if all the system poles have negative real part } \left(\forall n:\ \mathrm{Re}\left[-p_n\right] < 0\right) \\ \infty & \cdots \text{ if at least one system pole has positive real part } \left(\exists n:\ \mathrm{Re}\left[-p_n\right] > 0\right) \end{cases},$$

$$\lim_{t \to \infty} y_{ss}(t) = K.$$

From Example 7-2, we infer that, in general, the *system output* $y(t)$ *will be* ***bounded*** if:

- *Input* $x(t)$ *is bounded*
- *All system poles* $\left(-p_n;\ n = 1, 2, \cdots, N\right)$ *have negative real parts*

This leads us to the definition of system stability. Stability is defined in the context of ***bounded-input bounded-output* (BIBO)**. The different notions of stability are as follow:

- **BIBO stable:** if $\lim_{t\to\infty} y_{tr}(t) = 0$. The condition for this is that **all system poles must have negative real parts**.
- **Unstable:** if $\lim_{t\to\infty} y_{tr}(t) = \infty$. This occurs when **at least one system pole has positive real parts**.
- **Marginally: Stable** if $\lim_{t\to\infty} y_{tr}(t)$ has no fixed value or is non-zero. For example, $y_{tr}(t)$ could be sinusoidal and hence has no fixed steady-state value. This happens when the **system poles are on the imaginary axis, including the origin**.

The above provides a convenient way to check if a system is BIBO stable without having to calculate the output response $y(t)$ at all. System transfer functions are therefore very useful from this point of view. An overall view of the transient response in relation to the pole locations on the complex *s-plane* is shown in Fig.7-2.

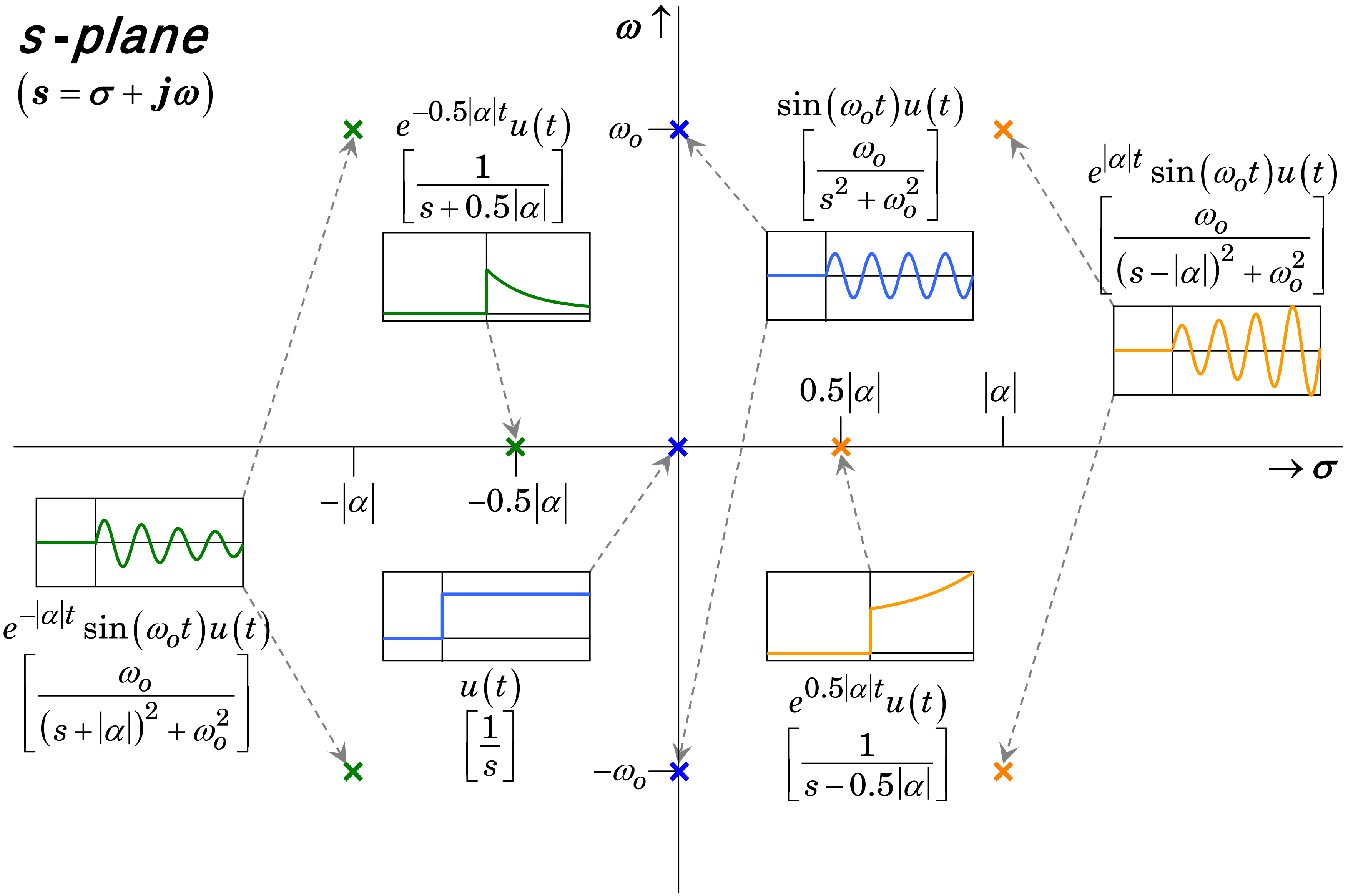


Fig.7-2: Responses of systems due to a unit impulse input

So what about the zeros?

- The zeros, $\left(z_m;\ m=1,2,\cdots,M\right)$, do not play any role in the stability of the system.
- The zeros only affect the values of the constants α_n during partial factorization, thus affecting only the transient response.

Example 7-3:

Consider the following second order system which has a real zero at $s=-1/\gamma$. Note that when $\gamma=0$, the zero is at infinity and has no effect on the system behavior at all.

System transfer function: $$H(s)=\frac{\gamma s+1}{40s^2+22s+1}=\frac{0.025(\gamma s+1)}{(s+0.5)(s+0.05)}$$

Input signal: $$x(t)=18u(t)\ \rightarrow\ X(s)=18/s$$

Output signal:
$$\begin{cases} Y(s)=\dfrac{0.025(\gamma s+1)}{(s+0.5)(s+0.05)}\cdot\dfrac{18}{s}=\dfrac{2-\gamma}{s+0.5}-\dfrac{20-\gamma}{s+0.05}+\dfrac{18}{s} \\ y(t)=\underbrace{\left[(2-\gamma)\exp(-0.5t)-(20-\gamma)\exp(-0.05t)\right]u(t)}_{y_{tr}(t)}+\underbrace{18u(t)}_{y_{ss}(t)} \end{cases}$$

The effects of the zero on the output for various values of γ are illustrated in Fig.7-3.

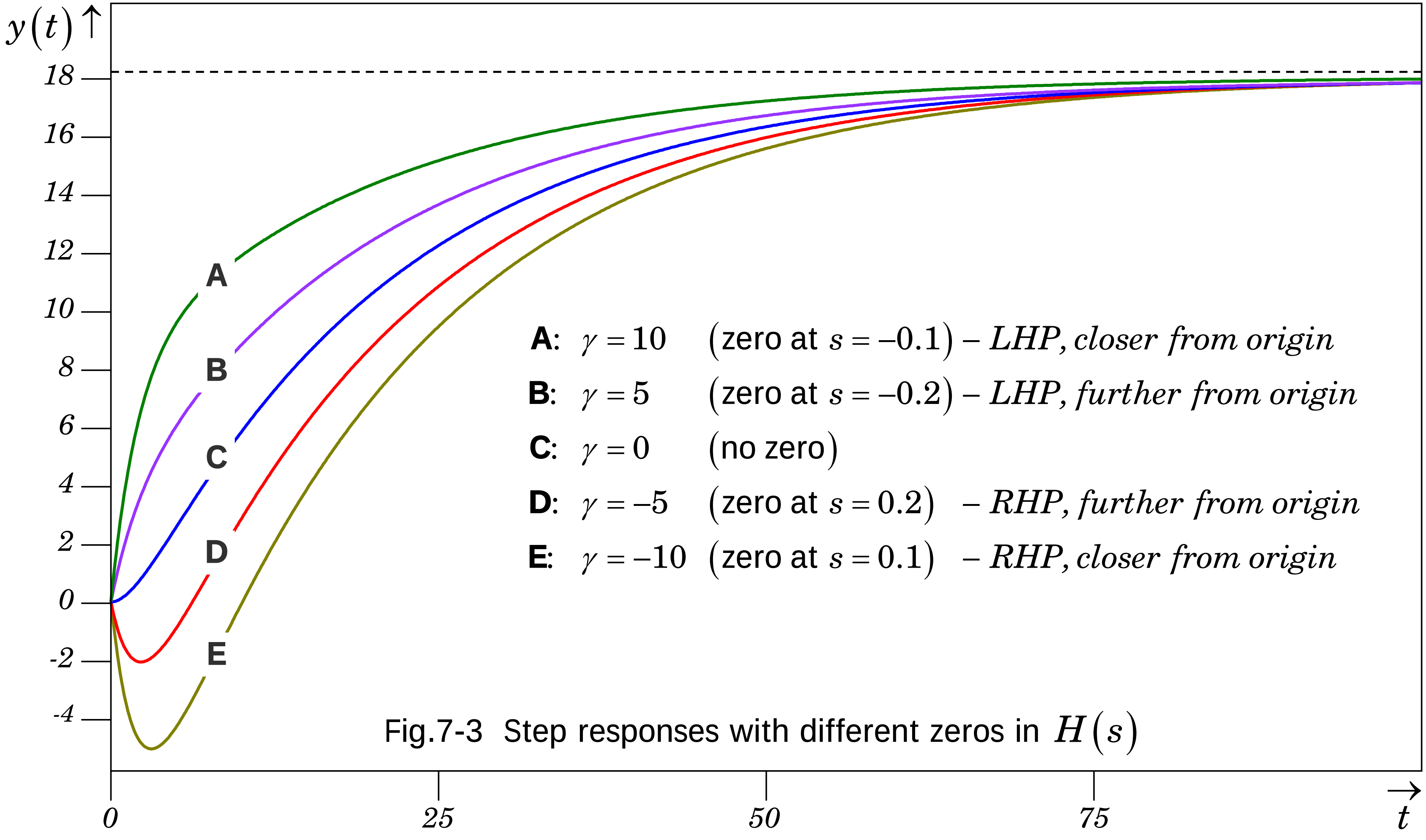


Fig.7-3 Step responses with different zeros in $H(s)$

Notice how the location of the zero affects $y(t)$.

7.2.1 DE and TF of First-order Systems

- Differential equation (DE) of a linear first-order system is generally written as

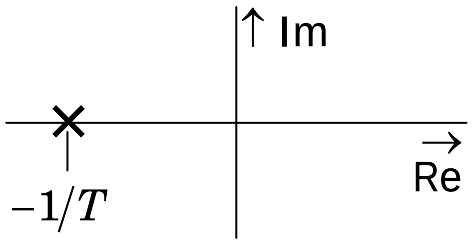
$$T\frac{dy(t)}{dt} + y(t) = Kx(t) \tag{7.9}$$

where $\begin{cases} x(t) & \text{: system input} \\ y(t) & \text{: system output} \\ K & \text{: DC (or Static) gain} \\ T & \text{: time-constant} \end{cases}$

- Transfer function (TF):

$$H(s) = \frac{Y(s)}{X(s)} = \frac{K}{Ts+1} \tag{7.10}$$

$$\text{Pole: } s_1 = -1/T$$



7.2.2 DE and TF of Second-order Systems

- Differential equation (DE) of a linear second-order system is generally written as

$$\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = K\omega_n^2 x(t) \qquad (7.11)$$

where
$$\begin{cases} x(t) & : \text{system input} \\ y(t) & : \text{system output} \\ \zeta & : \text{damping ratio} \\ \omega_n & : \text{undamped natural frequency } \left(\text{takes on particular meaning only when } 0 \le \zeta < 1\right) \\ K & : \text{DC (or Static) gain} \end{cases}$$

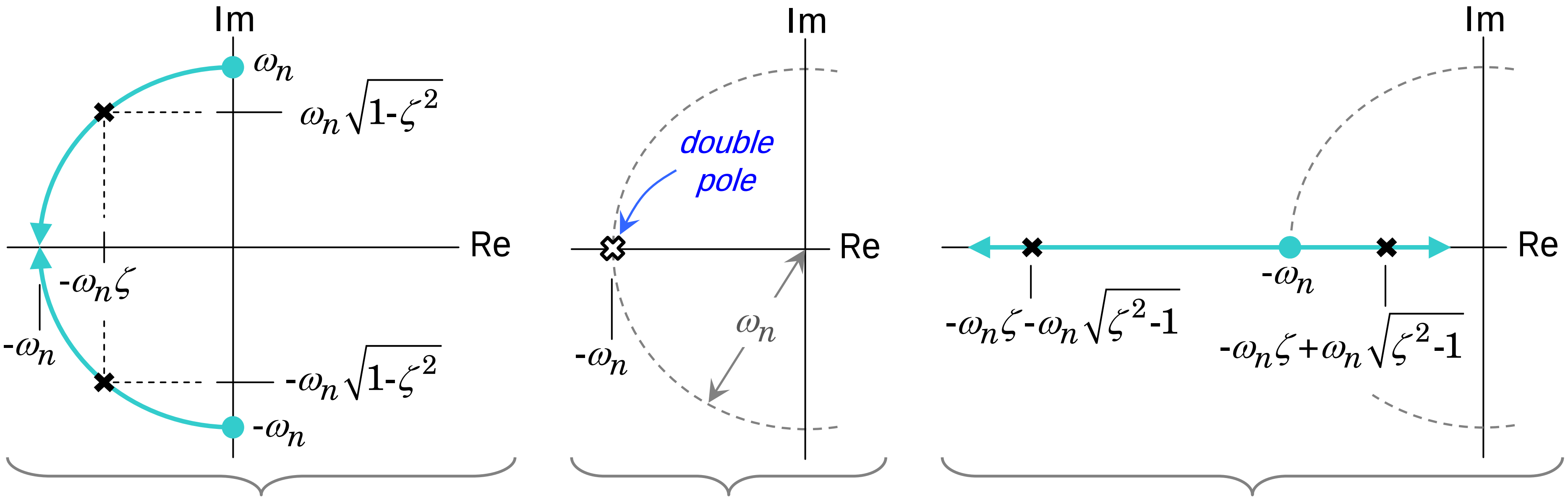
- Transfer function (TF):

$$H(s) = \frac{Y(s)}{X(s)} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\text{Poles: } s_{1,2} = -\omega_n\zeta \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$\begin{cases} \zeta = 0 & : \textit{imaginary poles} \cdots\cdots\cdots \left(\textit{Undamped}\right) \\ 0 \le \zeta < 1 & : \textit{complex poles} \cdots\cdots \left(\textit{Underdamped}\right) \\ \zeta = 1 & : \textit{double pole} \cdots\cdots \left(\textit{Critically damped}\right) \\ \zeta > 1 & : \textit{distinct real poles} \cdots \left(\textit{Overdamped}\right) \end{cases}$$

(7.12)



$0 \le \zeta < 1$
Underdamped

$\zeta = 1$
Critically damped

$\zeta > 1$
Overdamped

- *SPECIAL NOTATIONS for Underdamped Second-Order Systems*

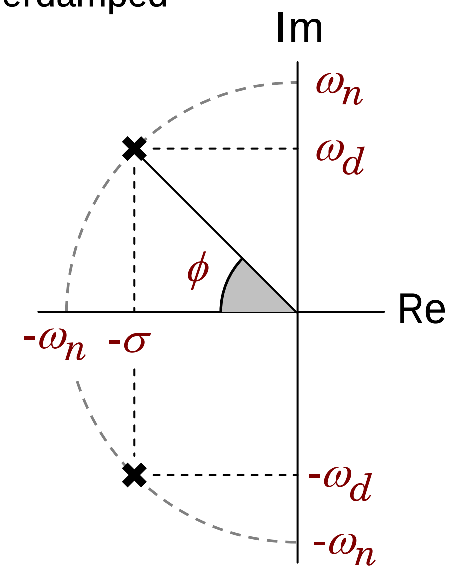
$$\sigma = \omega_n\zeta = \omega_n \cos(\phi)$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \omega_n \sin(\phi) \quad \cdots \text{ damped natural frequency}$$

$$H(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{K\omega_n^2}{(s+\sigma)^2 + \omega_d^2}$$

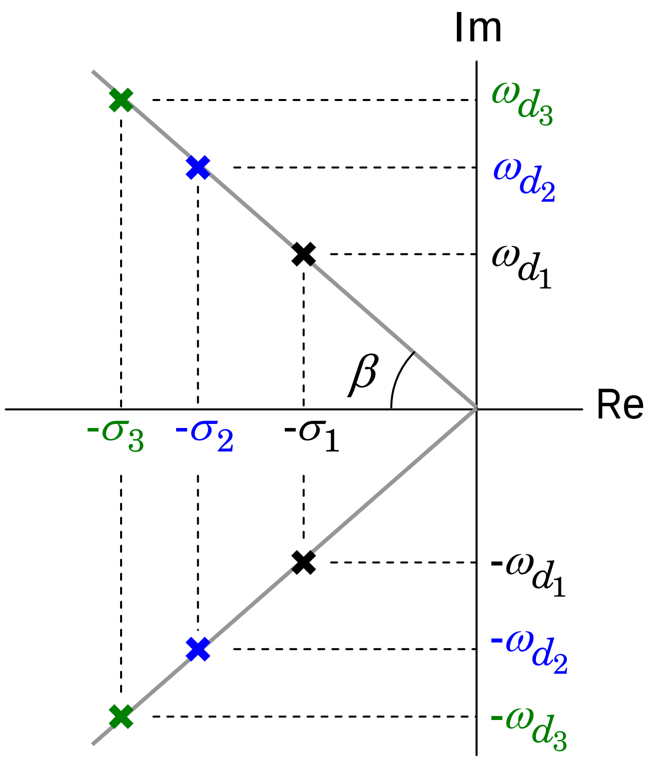
Poles located at $s = -\sigma \pm j\omega_d$

(◈)

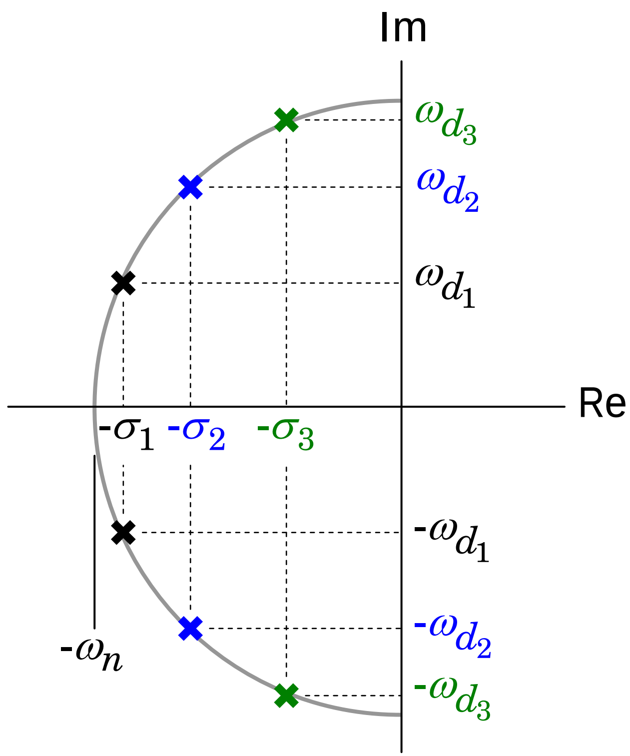


Note:

$$\left(\underbrace{\phi = \cos^{-1}(\zeta)}_{\text{Independent of } \omega_n}\right) \qquad \left(\underbrace{|s_{1,2}| = \omega_n}_{\text{Independent of } \zeta}\right)$$



Poles with same ζ are located on the same line with $\phi = \cos^{-1}(\zeta)$. Different poles have different σ and ω_d values.



Poles with the same ω_n are located on an arc with a radius of ω_n. Different poles on the same arc have different σ and ω_d values.

7.3 Response of LTI Systems to Unit Impulse, Unit Step and Sinusoids

- ***Impulse response*** – output response when the input is an impulse
- ***Step response*** – output response when the input is a step signal
- ***Sinusoidal response*** – output response when the input is a sinusoidal signal

Each of these responses is important to LTI systems because they define certain behaviours that can be generalized for such systems.

- ***Impulse response*** is related to the transfer function of the system.
- ***Step response*** is very commonly encountered in practice and they give information about some physical parameters in the LTI system
- ***Sinusoidal response*** is related to the frequency response of the system

7.3.1 Impulse Response

Suppose the system input is a unit impulse, i.e. $x(t)=\delta(t)$, and the corresponding system output is $y(t)=y_\delta(t)$.

$$\underset{\textit{Unit Impulse}}{\delta(t)} \longrightarrow \boxed{\boldsymbol{H}(\boldsymbol{s})} \longrightarrow \underset{\textit{Impulse Response}}{y_\delta(t)}$$

Then (7.1) becomes

$$Y_\delta(s)=H(s)\underbrace{\mathcal{L}\{\delta(t)\}}_{=1}=H(s). \tag{7.13}$$

and in the time-domain,

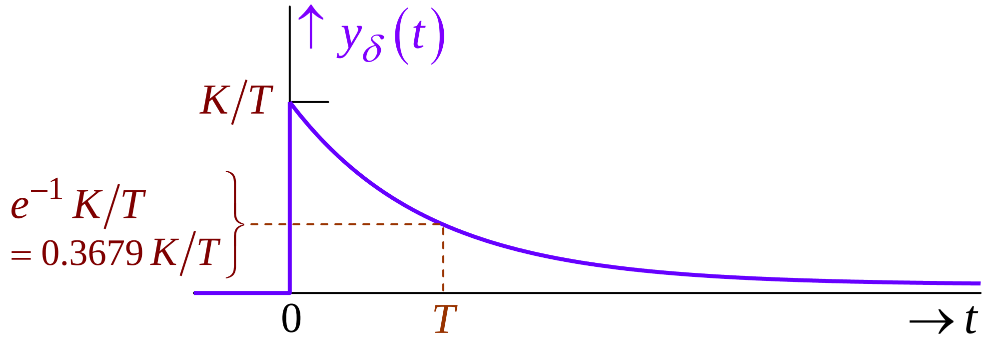
$$y_\delta(t)=\mathcal{L}^{-1}\{H(s)\}=h(t). \tag{7.14}$$

$\boldsymbol{y_\delta(t)=h(t)}$ is called the ***impulse response*** of the system. ***For a causal system, $\boldsymbol{y_\delta(t)}$ always takes the form of a right-sided function of time, i.e. $\boldsymbol{y_\delta(t)=0;\ t<0}$.***

- **First-order System:**

$$H(s) = \frac{K}{Ts+1}$$

$$y_\delta(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{K}{T}\exp\left(-\frac{t}{T}\right)u(t)$$



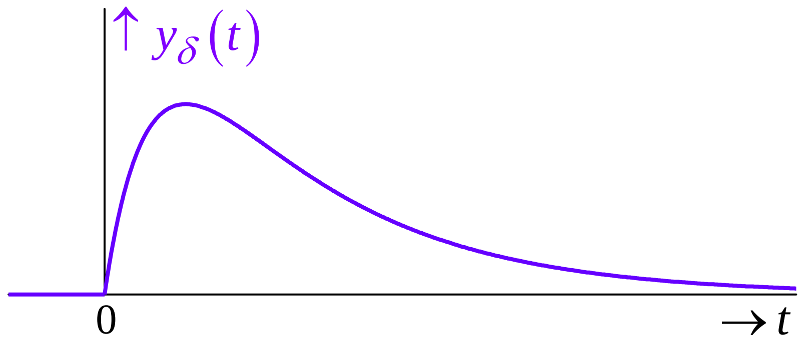
- **Second-order System:**

Overdamped $(\zeta > 1)$: Two distinct REAL poles

$$H(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{K\omega_n^2}{(s+a)(s+b)}$$

$$\frac{K\omega_n^2(b-a)^{-1}}{(s+a)} + \frac{K\omega_n^2(a-b)^{-1}}{(s+b)}$$

$$\cdots\cdots\cdots \begin{bmatrix} a = \omega_n\zeta + \omega_n\left(\zeta^2 - 1\right)^{1/2} \\ b = \omega_n\zeta - \omega_n\left(\zeta^2 - 1\right)^{1/2} \end{bmatrix} \quad (\bigstar)$$



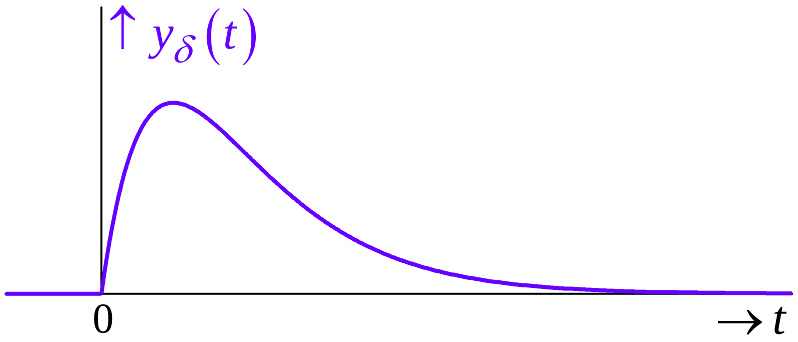
$$y_\delta(t) = \mathcal{L}^{-1}\{H(s)\} = \left[K_1\exp(-at) + K_2\exp(-bt)\right]u(t)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \left[K_1 = \frac{K\omega_n^2}{b-a} \qquad K_2 = \frac{K\omega_n^2}{a-b}\right] \quad (\bigstar)$$

Critically damped $(\zeta = 1)$: Repeated REAL poles

$$H(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{K\omega_n^2}{(s+\omega_n)^2}$$

$$y_\delta(t) = \mathcal{L}^{-1}\{H(s)\} = K\omega_n^2 t \exp(-\omega_n t)u(t)$$

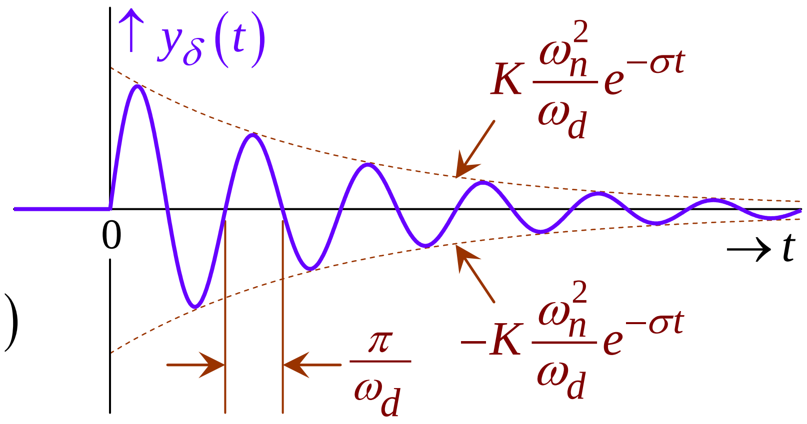


Underdamped $(0 < \zeta < 1)$: COMPLEX poles

$$H(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{K\omega_n^2}{(s+\sigma)^2 + \omega_d^2}$$

$$y_\delta(t) = \mathcal{L}^{-1}\{H(s)\} = K\frac{\omega_n^2}{\omega_d}\exp(-\sigma t)\sin(\omega_d t)u(t)$$

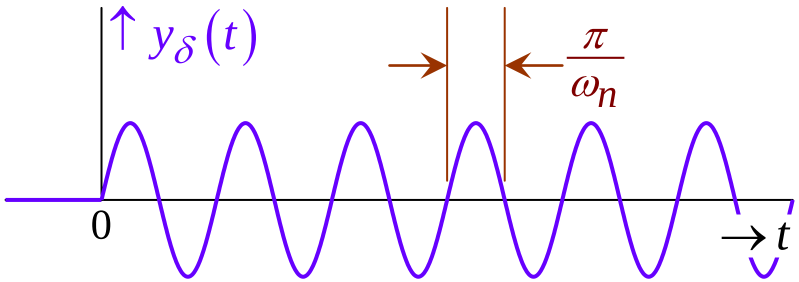
See (◈) - Pg.7-12 for definitions of σ *and* ω_d.



Undamped $(\zeta = 0)$: PURE IMAGINARY poles

$$H(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{K\omega_n^2}{s^2 + \omega_n^2}$$

$$y_\delta(t) = \mathcal{L}^{-1}\{H(s)\} = K\omega_n \sin(\omega_n t)u(t)$$



7.3.2 Step Response

Suppose the system input is a unit step, i.e. $x(t)=u(t)$, and the corresponding system output is $y(t)=y_{step}(t)$.

$$\underset{Unit\ Step}{u(t)} \longrightarrow \boxed{\boldsymbol{H(s)}} \longrightarrow \underset{Step\ Response}{y_{step}(t)}$$

Then (7.1) becomes

$$Y_{step}(s)=H(s)\mathcal{L}\{u(t)\}=H(s)\frac{1}{s}. \tag{7.15}$$

and in the time-domain, using Laplace transform *Property E* (see Chapter 5, Section 5.2),

$$y_{step}(t)=\mathcal{L}^{-1}\left\{H(s)\cdot\frac{1}{s}\right\}=\int_{0^-}^{t}h(\tau)d\tau\,. \tag{7.16}$$

$\boldsymbol{y_{step}(t)=\int_{0^-}^{t}h(\tau)d\tau}$ is called the ***step response*** of a causal system.

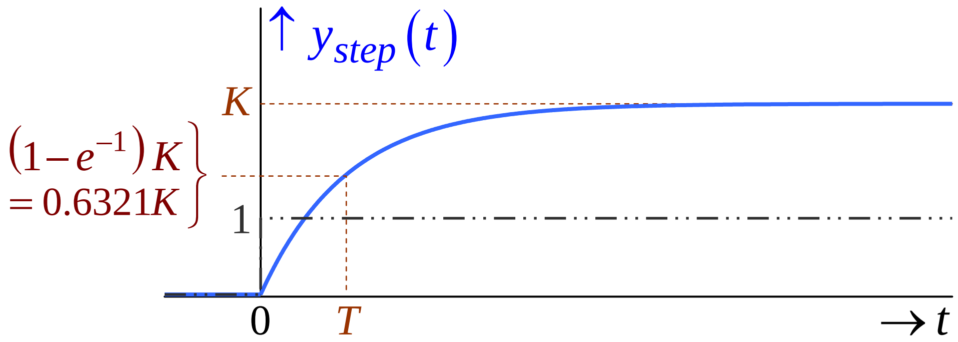
From (7.14) and (7.16), we have

$$\left[\boldsymbol{y_{step}(t)=\int_{0^-}^{t}y_\delta(\tau)d\tau}\right] \quad \text{and} \quad \left[\boldsymbol{y_\delta(\tau)=\frac{d}{dt}y_{step}(t)}\right] \tag{7.17}$$

- **First-order System:**

$$y_{step}(t) = \int_{0^-}^{t} \frac{K}{T}\exp\left(-\frac{\tau}{T}\right)d\tau$$

$$= K\left[1-\exp\left(-\frac{t}{T}\right)\right]u(t)$$



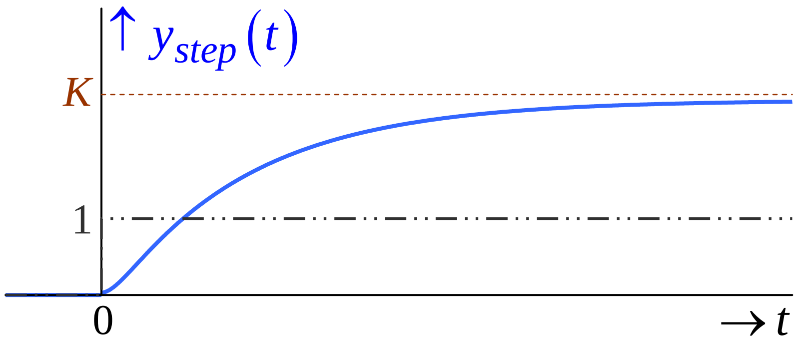
- **Second-order System:**

Overdamped $(\zeta > 1)$: Distinct REAL poles

$$y_{step}(t) = \int_{0^-}^{t} K_1\exp(-a\tau) + K_2\exp(-b\tau)\ d\tau$$

$$= \left[K - \frac{K_1}{a}\exp(-at) - \frac{K_2}{b}\exp(-bt)\right]u(t)$$

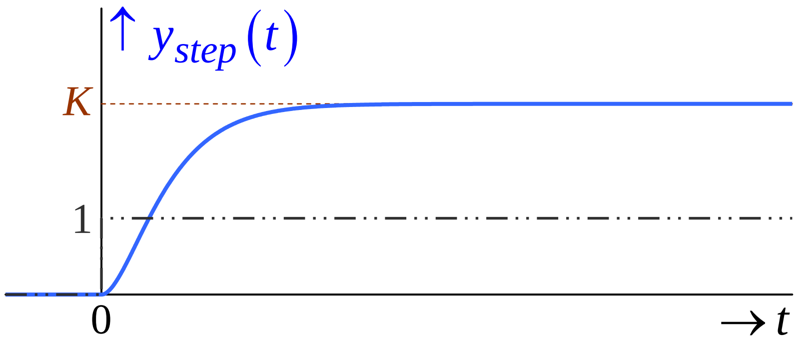
See (✱) - Pg.7-16 for definitions of K_1, K_2, a and b



Critically damped $(\zeta = 1)$: Repeated REAL poles

$$y_{step}(t) = \int_{0^-}^{t} K\omega_n^2\tau\exp(-\omega_n\tau)d\tau$$

$$= K\left[1-\exp(-\omega_n t) - \omega_n t\exp(-\omega_n t)\right]u(t)$$



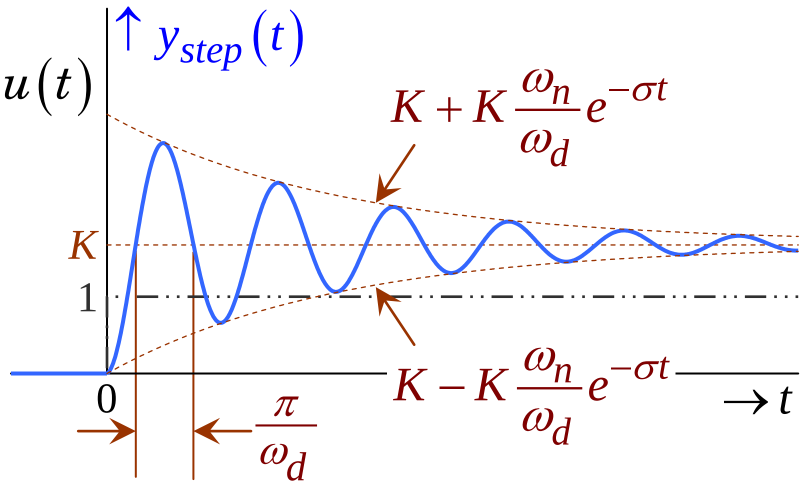
Underdamped $(\zeta < 1)$: COMPLEX poles

$$Y_{step}(s) = \frac{K\omega_n^2}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)} = \frac{K\left(\sigma^2 + \omega_d^2\right)}{s\left[(s+\sigma)^2 + \omega_d^2\right]} = \frac{K}{s} - \frac{K(s+\sigma)}{(s+\sigma)^2 + \omega_d^2} - \frac{K\sigma}{(s+\sigma)^2 + \omega_d^2}$$

$$y_{step}(t) = K\left[1 - e^{-\sigma t}\left[\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right]\right]u(t)$$

$$= K\left[1 - \frac{\omega_n}{\omega_d}e^{-\sigma t}\sin(\omega_d t + \phi)\right]u(t)$$

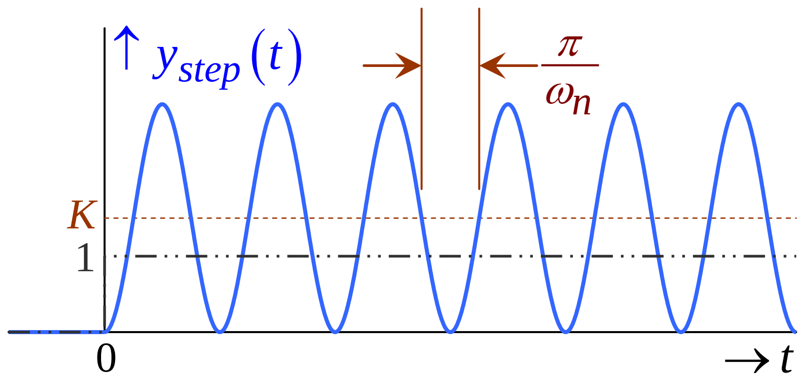
See (◈) - Pg.7-12 for definitions of σ, ω_d and ϕ.



Note: For this case, it is easier to obtain $y_{step}(t)$ from $\mathcal{L}^{-1}\left\{\frac{1}{s}H(s)\right\}$ than from $\int_{0^-}^{t} y_\delta(\tau)d\tau$.

Undamped $(\zeta = 0)$: PURE IMAGINARY poles

$$y_{step}(t) = \int_{0^-}^{t} K\omega_n \sin(\omega_n \tau)d\tau = K(1 - \cos\omega_n t)$$



7.3.3 Sinusoidal Response

What is the system output if the input is a sinusoid?

Intuitively, the output will also be a sinusoid as the solution to the system differential equations can be interpreted using two properties of sinusoidal signal:

- Each differentiation of a sinusoid results in another sinusoidal signal.
- Adding two sinusoids of the same frequency together results in another sinusoid with different amplitude and phase while preserving the common frequency of the signals.

Let the system transfer function be $H(s)$ and its input be

$$A\sin(\tilde{\omega}t)u(t) \rightleftarrows \frac{A\tilde{\omega}}{s^2+\tilde{\omega}^2}.$$

The corresponding output is thus given by

$$Y(s) = \frac{A\tilde{\omega}}{s^2+\tilde{\omega}^2}H(s).$$

Assume that $H(s)$ has stable poles at $s=-p_1,-p_2,\cdots\cdots$, then

$$Y(s) = A\left(\underbrace{\frac{A_1}{s+p_1}+\frac{A_2}{s+p_2}+\cdots}_{\text{System Poles}}+\underbrace{\frac{Cs+D}{s^2+\tilde{\omega}^2}}_{\text{Input Poles}}\right).$$

Hence,

$$y(t) = \underbrace{A\left[A_1 \exp(-p_1 t) + A_2 \exp(-p_2 t) + \cdots\cdots\cdots\right]}_{\text{Transient Response: } y_{tr}(t)} + \underbrace{A\left[C\cos(\tilde{\omega}t) + \frac{D}{\tilde{\omega}}\sin(\tilde{\omega}t)\right]}_{\text{Steady-state Response: } y_{ss}(t)}$$

- ***Transient Response*** : $\begin{cases} y_{tr}(t) = A\left[A_1 \exp(-p_1 t) + A_2 \exp(-p_2 t) + \cdots\cdots\cdots\right] \\ \text{Note: } \lim\limits_{t\to\infty} y_{tr}(t) = 0, \text{ i.e. the transient response decays to zero.} \end{cases}$

- ***Steady-state Response*** : $\begin{cases} y_{ss}(t) = A\left[C\cos\tilde{\omega}t + \dfrac{D}{\tilde{\omega}}\sin\tilde{\omega}t\right] = AM_{\tilde{\omega}}\sin\left(\tilde{\omega}t + \phi_{\tilde{\omega}}\right) \\ where \left(\underbrace{M_{\tilde{\omega}} = \left[C^2 + (D/\tilde{\omega})^2\right]^{0.5}}_{M_{-\tilde{\omega}} = M_{\tilde{\omega}}} \quad \underbrace{\phi_{\tilde{\omega}} = \tan^{-1}\dfrac{\tilde{\omega}C}{D}}_{\phi_{-\tilde{\omega}} = -\phi_{\tilde{\omega}}} \right) \end{cases}$ (7.18)

Clearly, the **steady-state** response is obtained by multiplying the amplitude of the input sinusoid by $M_{\tilde{\omega}}$ and adding $\phi_{\tilde{\omega}}$ to its phase. Hence, **dropping the tilde** ˜, the function

$$\boldsymbol{M_\omega \exp\left(j\phi_\omega\right)} \tag{7.19}$$

assumes the meaning of ***frequency response*** of the system, where $\boldsymbol{M_\omega}$ and $\boldsymbol{\phi_\omega}$ are, respectively, the magnitude and phase response of the system.

7.4 Frequency Response

- ***Frequency response*** is an intrinsic property of LTI systems as it characterizes how sinusoidal signals are altered in going through the system.

- We have alluded to the notion of frequency response of an LTI system in Chapters 2 when we discussed the convolution property of the Fourier transform.

- For a ***causal*** and ***stable*** system, we have shown in Chapter 5 Section 5.4 that the system ***frequency response*** is given by

$$H(j\omega) = H(s)\Big|_{s=j\omega}\ , \tag{7.20}$$

which is equivalent to the Fourier transform of the system impulse response $h(t)$.

- Like poles and zeros, ***frequency response*** is another important measure that is often used to quantify the behavior of an LTI system. The measure is in terms of the magnitude and phase response as defined by

$$H(j\omega) = \underbrace{\left|H(j\omega)\right|}_{\text{Magnitude Response}} \exp\Big[\, j\, \underbrace{\angle H(j\omega)}_{\text{Phase Response}}\, \Big] \tag{7.21}$$

- Comparing with (7.19) and (7.21), we have $\left(\boldsymbol{M_\omega} = \left|\boldsymbol{H}(\boldsymbol{j\omega})\right|,\ \boldsymbol{\phi_\omega} = \angle \boldsymbol{H}(\boldsymbol{j\omega})\right)$. The steady-state sinusoidal response given by (7.18) can then be summarized as follows.

NEW FACTS

$$A\cos(\tilde{\omega}t+\psi)u(t) \longrightarrow \boxed{H(j\omega)} \longrightarrow y_{ss}(t) = A\left|H(j\tilde{\omega})\right|\cos\left(\tilde{\omega}t+\psi+\angle H(j\tilde{\omega})\right)$$

$$A\sin(\tilde{\omega}t+\psi)u(t) \longrightarrow \boxed{H(j\omega)} \longrightarrow y_{ss}(t) = A\left|H(j\tilde{\omega})\right|\sin\left(\tilde{\omega}t+\psi+\angle H(j\tilde{\omega})\right)$$

$$\underbrace{Ae^{j(\tilde{\omega}t+\psi)}u(t)}_{\text{Sinusoid turned on at } t=0} \longrightarrow \boxed{H(j\omega)} \longrightarrow \underbrace{y_{ss}(t) = A\left|H(j\tilde{\omega})\right|e^{j\left(\tilde{\omega}t+\psi+\angle H(j\tilde{\omega})\right)}}_{\text{Steady-state Sinusoidal Response}}$$

OLD FACTS from 'Signals' Part

$$x(t) = A\cos\left(2\pi\tilde{f}t+\psi\right) \longrightarrow \boxed{H(f)} \longrightarrow y(t) = A\left|H(\tilde{f})\right|\cos\left(2\pi\tilde{f}t+\psi+\angle H(\tilde{f})\right)$$

$$x(t) = A\sin\left(2\pi\tilde{f}t+\psi\right) \longrightarrow \boxed{H(f)} \longrightarrow y(t) = A\left|H(\tilde{f})\right|\sin\left(2\pi\tilde{f}t+\psi+\angle H(\tilde{f})\right)$$

$$\underbrace{x(t) = Ae^{j\left(2\pi\tilde{f}t+\psi\right)}}_{\text{Sinusoid turned on at } t=-\infty} \longrightarrow \boxed{H(f)} \longrightarrow \underbrace{y(t) = A\left|H(\tilde{f})\right|e^{j\left(2\pi\tilde{f}t+\psi+\angle H(\tilde{f})\right)}}_{\mathfrak{I}^{-1}\{H(f)X(f)\}}$$

- In Chapter 2, we examined the frequency content of signals for frequency ranging from $-\infty$ to ∞. However, for LTI systems, it is customary to make use of Bode diagrams, which has a frequency axis extending only from 0 to ∞, to describe their responses to different frequencies.

7.4.1 Bode Diagrams

- Bode diagrams consist of two plots : ***magnitude vs frequency*** and ***phase vs frequency***. The former is commonly called ***Bode magnitude-plot*** and the latter ***Bode phase-plot***.

- Such plots are normally plotted on a semilogx graph paper which is a graph paper where the x-axis is not a linear frequency scale but instead, is a log frequency axis.

- Magnitude is generally plotted in dB units while the phase in degrees instead of radians.

  Absolute Magnitude : $\left|H(j\omega)\right|$ *Magnitude in dB :* $20\log_{10}\left|H(j\omega)\right|$

  Aside: The *Magnitude*, when expressed in dB, is frequently referred to as ***Gain*** since dB is a relative measure. For instance, if v_i and v_o are the input and output of a system, respectively, then we say that the system has a gain of $20\log_{10}\left(|v_o|/|v_i|\right)$ dB. Therefore, $20\log_{10}\left(|v_o|\right)$ is the system gain measured with respect to a ***unity input***, i.e. $v_i = 1$.

- Bode diagrams visualize the frequency response for only positive frequencies. This suffices for real systems since $H(j\omega)$ is conjugate symmetric.

- Bode diagrams are also approximated by straight lines, and the result is referred to as ***Bode straight-line plots***.

We shall illustrate the constructions of Bode diagrams of LTI systems by way of several examples in the following.

Example 7-4 :

Differentiator: $\boldsymbol{H(s) = Ks}$ ➔ $\boldsymbol{H(j\omega) = jK\omega}$

Magnitude response: $|H(j\omega)| = K\omega \equiv (20\log_{10} K + 20\log_{10}\omega)$ dB (7.22)

Phase response: $\angle H(j\omega) = \tan^{-1}(K\omega/0) = 90°$ (7.23)

From (7.22), we note that if $|H(j\omega)|$ *in dB is plotted against the* $\log_{10}\omega$, *then*

- *A straight line is obtained.*
- *The slope of this straight line is* $+20$dB *for every 10 times increase in frequency. We call this a* $+20$ dB/decade *slope.*

Bode plots of differentiators with different $\boldsymbol{K}$ ***values are shown in Fig.7-4.***

Fig.7-4(a)

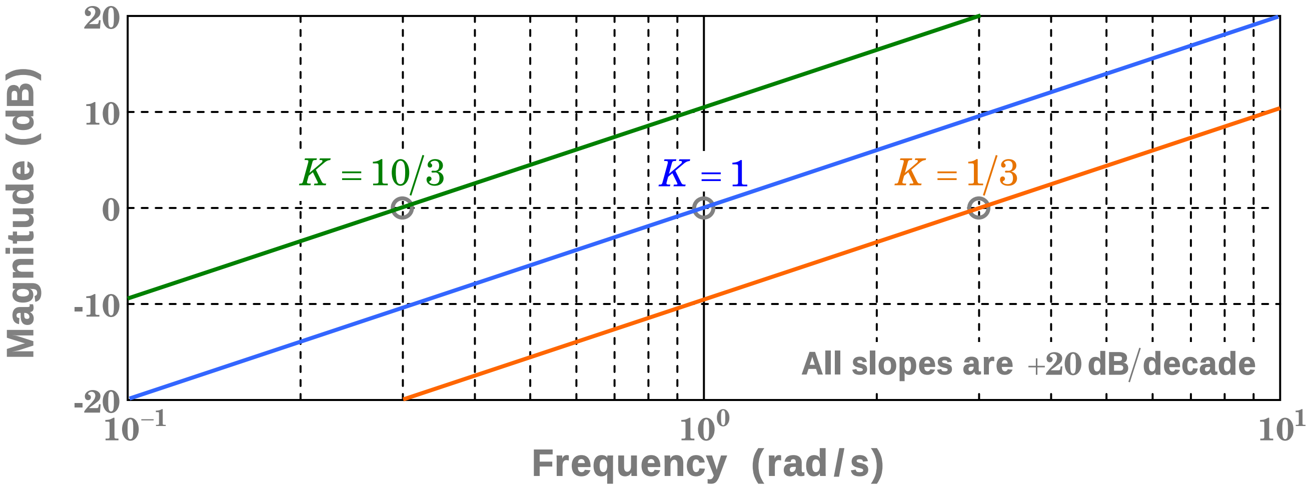
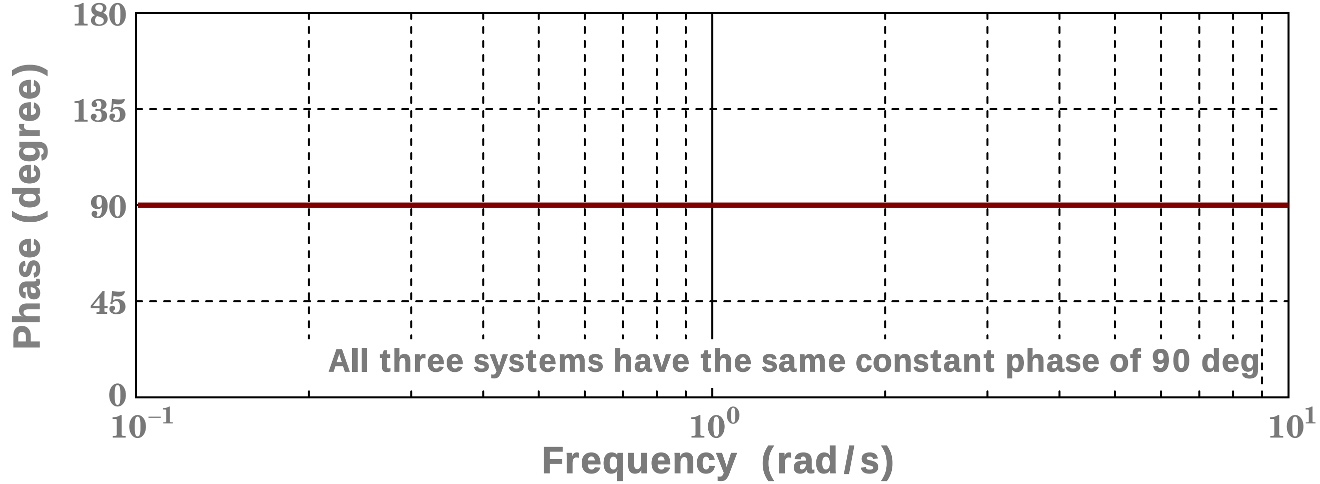


Fig.7-4(b)



The value of K *can be determined from the gain plot by locating the frequency* ω_o *where* $|H(j\omega_o)| = 1$ or 0dB. *Then* $K\omega_o = 1$ *and thus* $K = 1/\omega_o$.

For a ***cascade of*** N ***differentiators*** *where* $H(s) = Ks^N$, *the Bode diagram will have the following characteristics:*

- *Magnitude response is a straight line with slope of* $+20N$ dB/decade
- *Phase response is a constant at* $+90N°$ *for all frequencies.*
- *The value of* K *can be determined from the gain plot by locating the frequency* ω_o *where* $|H(j\omega_o)| = 1$ or 0dB. *Then* $K\omega_o^N = 1$ *and thus* $K = 1/\omega_o^N$.

Example 7-5

Integrator: $\boldsymbol{H(s) = K/s} \Rightarrow \boldsymbol{H(j\omega) = -j\,K/\omega}$

Magnitude response: $\left|H(j\omega)\right| = K/\omega \equiv \left(20\log_{10}K - 20\log_{10}\omega\right)\text{dB}$ (7.24)

Phase response: $\angle H(j\omega) = \tan^{-1}\left(-(K/\omega)/0\right) = -90°$ (7.25)

From (7.24), we note that if $\left|H(j\omega)\right|$ *in dB is plotted against the* $\log_{10}\omega$*, then*

- *A straight line is obtained.*
- *The slope of this straight line is* -20dB *for every 10 times increase in frequency. We call this a* $-20\,\text{dB/decade}$ *slope.*

Bode plots of integrators with different $\boldsymbol{K}$ ***values are shown in Fig.7-5.***

Fig.7-5(a)

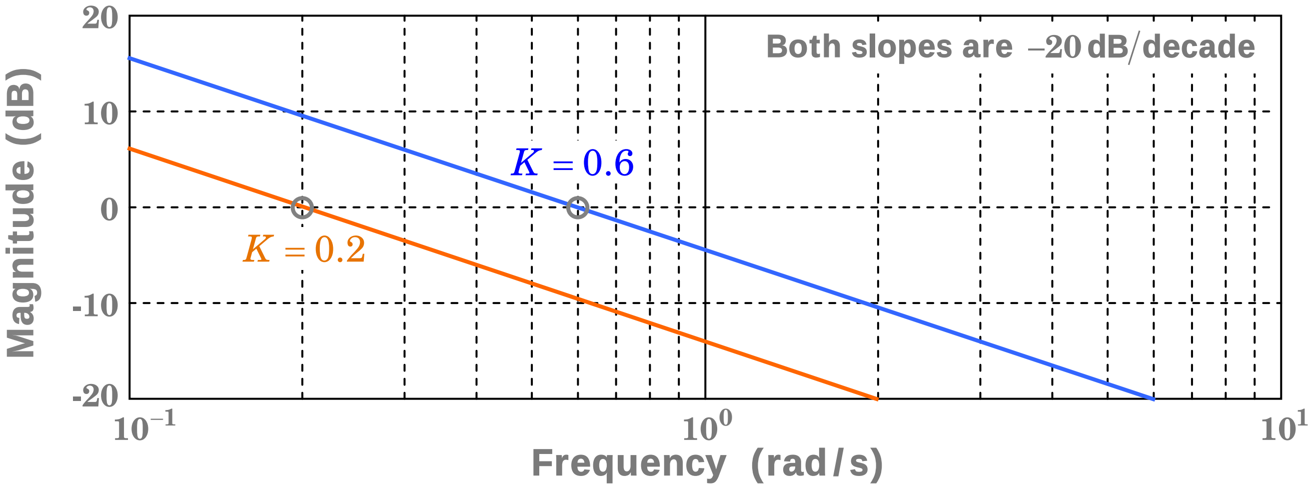
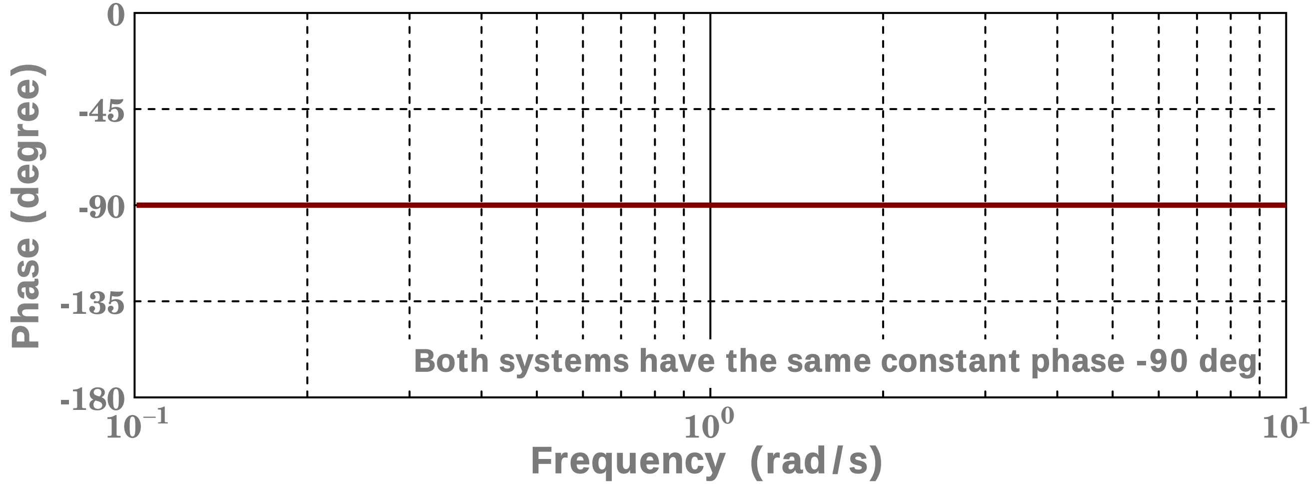


Fig.7-5(b)



The value of K can be determined from the gain plot by locating the frequency ω_o where $\left|H(j\omega_o)\right| = 1$ or 0dB. Then $K/\omega_o = 1$ and thus $K = \omega_o$.

For a cascade of N integrators where $H(s) = K/s^N$, the Bode diagram will have the following characteristics:

- *Magnitude response is a straight line with slope of* $-20N$ dB/decade
- *Phase response is a constant at $-90N°$ for all frequencies.*
- *The value of K can be determined from the gain plot by locating the frequency ω_o where $\left|H(j\omega_o)\right| = 1$ or 0dB. Then $K/\omega_o^N = 1$ and thus $K = \omega_o^N$.*

Example 7-6

First order system with one real pole: $\boldsymbol{H(s) = \frac{K}{sT+1}}$ ➔ $\boldsymbol{H(j\omega) = \frac{K}{j\omega T+1}}$

Pole: $s = -1/T$

Magnitude response: $$\left|H(j\omega)\right| = \frac{K}{\sqrt{\omega^2T^2+1}} \equiv \left(20\log_{10}K - 20\log_{10}\sqrt{\omega^2T^2+1}\right)\text{dB} \tag{7.26}$$

Phase response: $$\angle H(j\omega) = -\tan^{-1}(\omega T) \tag{7.27}$$

Case I: $\boldsymbol{\omega << 1/T}$

$$\left|H(j\omega)\right| \approx K \equiv 20\log_{10}K \text{ dB} \tag{7.28}$$

$$\angle H(j\omega) = -\tan^{-1}(\omega T) \to 0^\circ \tag{7.29}$$

Case II: $\boldsymbol{\omega = 1/T}$

$$\left|H(j\omega)\right| \approx K/\sqrt{2} \equiv \left(20\log_{10}K - 20\log_{10}\sqrt{2}\right) \text{ dB} = 20\log_{10}K - 3.01 \text{ dB} \tag{7.30}$$

$$\angle H(j\omega) = -\tan^{-1}(1) = -45^\circ \tag{7.31}$$

Case I: $\boldsymbol{\omega >> 1/T}$

$$\left|H(j\omega)\right| \approx K/(\omega T) \equiv \left(20\log_{10}K - 20\log_{10}\omega T\right) \text{ dB} \tag{7.32}$$

$$\angle H(j\omega) = -\tan^{-1}(\omega T) \to -90^\circ \tag{7.33}$$

Summary:

The magnitude response has two parts:

- *At low frequency, the magnitude is approximately constant and equal to the DC gain of* $20\log_{10} K$ *dB.* **(cf. (7.28))**

- *At high frequency, specifically after* $\omega = 1/T$*, the magnitude response look like that of an integrator with a slope of* -20 *dB/decade.* **(compare (7.32) with (7.24))**

- $\omega = 1/T$ *is called the corner frequency and at this frequency, the phase is* -45° *while the magnitude is* -3.01 *dB below the DC gain. The corner frequency corresponds to the pole at* $s = -1/T$*.* **(cf. (7.30))**

The phase response can be sketched as follows:

- *Low frequency approximation of* 0° *up to* $\omega = 0.1/T$*.* **(cf. (7.29))**

- *Between* $\omega = 0.1/T$ *and* $\omega = 10/T$*, draw a straight line through the* $(1/T, -45^\circ)$ *point.* **(cf. (7.31))**

- *High frequency approximation of* -90° *for* $\omega > 10/T$*.* **(cf. (7.33))**

Bode plots of $\boldsymbol{H(s) = \dfrac{K}{sT+1}}$ $($*with* $\boldsymbol{K = 2}$ *and* $\boldsymbol{T = 1}$$)$ ***are shown in Fig.7-6.***

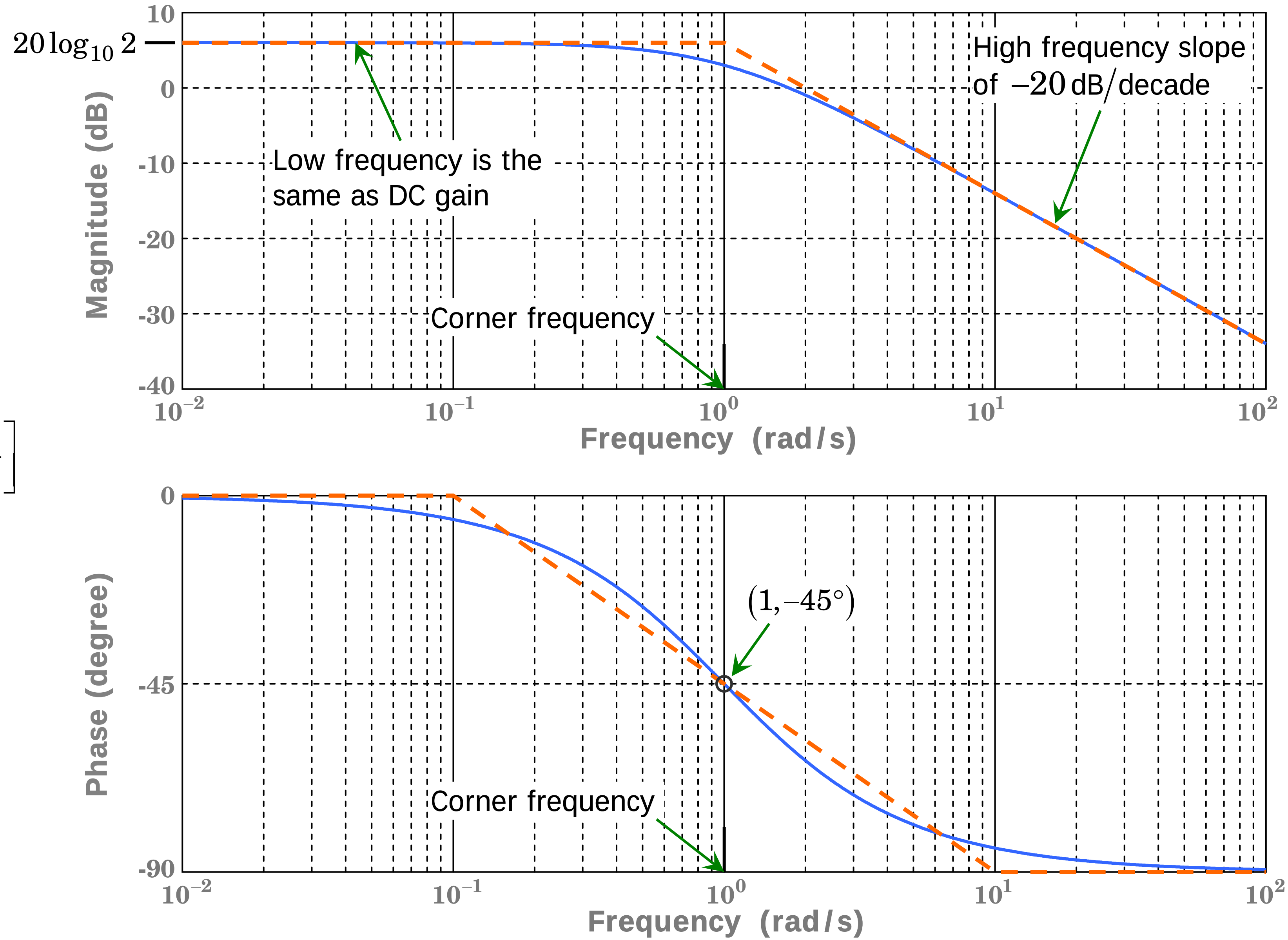


Fig.7-6

$$\left[H(s) = \frac{2}{s+1} \right]$$

Example 7-7:

Second order system with two real poles: $\begin{cases} \boldsymbol{H(s) = \dfrac{K}{(sT_1+1)(sT_2+1)}} & \textbf{or} \\ \boldsymbol{H(j\omega) = \dfrac{K}{(jT_1\omega+1)(jT_2\omega+1)}} \end{cases}$

Poles: $s=-1/T_1$ and $s=-1/T_2$ (*Assume* $T_1 > T_2$)

Magnitude response: $$|H(j\omega)| \equiv K_{dB} + \left|\frac{1}{jT_1\omega+1}\right|_{dB} + \left|\frac{1}{jT_2\omega+1}\right|_{dB} \tag{7.34}$$

Phase response: $$\angle H(j\omega) \equiv \angle\frac{1}{jT_1\omega+1} + \angle\frac{1}{jT_2\omega+1} \tag{7.35}$$

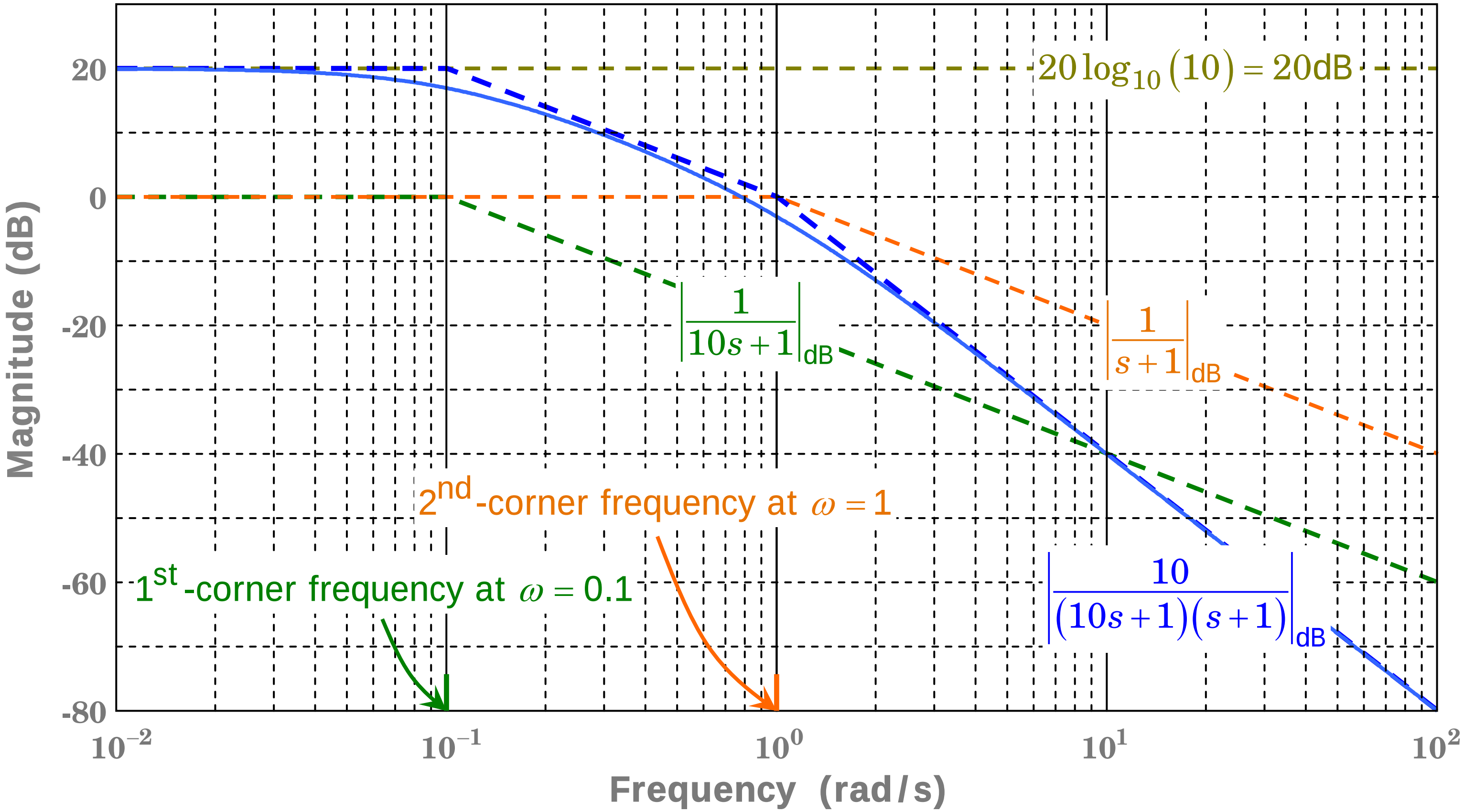
Summary of the Magnitude response: (See example in Fig.7-7(a))

- Low frequency magnitude is the same as DC gain ie $K_{dB} = 20\log_{10} K$ dB
- Since this is a second order overdamped system, there are two corner frequencies, one corresponding to each pole. Since $T_1 > T_2$, the first corer frequency is at $1/T_1$ and the second at $1/T_2$.

- There are 3 straight-line sections:
  - For $0 < \omega < 1/T_1$, straight horizontal line at K_{dB}.
  - For $1/T_1 < \omega < 1/T_2$, straight line with slope of -20 *dB/decade*.
  - For $\omega > 1/T_2$, straight line with slope of -40 *dB/decade*. This is because $H(s)$ behaves like a double integrator at high frequencies.

Fig.7-7(a)

$\left[\left|H(s)\right| : K = 10, \; T_1 = 10, \; T_2 = 1. \right]$

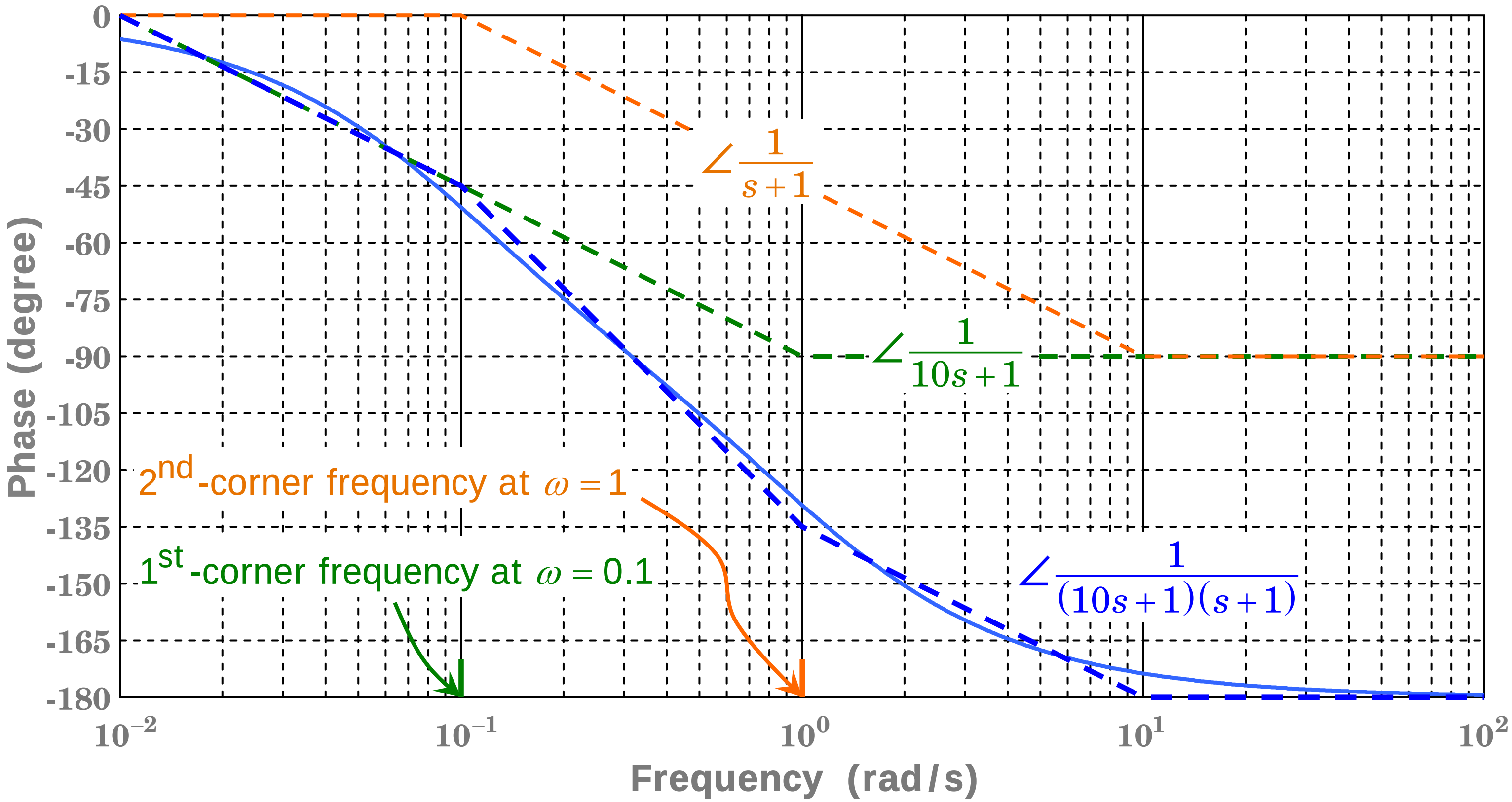


Summary of the Phase response: (See example in Fig.7-7(b))

- K factor does not introduce any phase.
- Each factor $\dfrac{1}{jT_1\omega+1}$ and $\dfrac{1}{jT_2\omega+1}$ contributes a phase response which is the same as the first order phase response in Example 7-6.
- Sum the two phase responses accordingly.

Fig.7-7(b)

$$\left[\begin{array}{l}\angle H(s): K=10, \\ T_1=10,\ T_2=1.\end{array}\right]$$



Example 7-8

Integrator cascaded with a first order system: $\begin{cases} \boldsymbol{H(s) = \dfrac{1}{s(sT+1)}} \quad \textbf{or} \\ \boldsymbol{H(j\omega) = \dfrac{1}{j\omega(jT\omega+1)}} \end{cases}$

The same method of constructing the magnitude and phase responses applies. The complete response is the sum of responses of $\dfrac{1}{s}$ and $\dfrac{1}{sT+1}$.

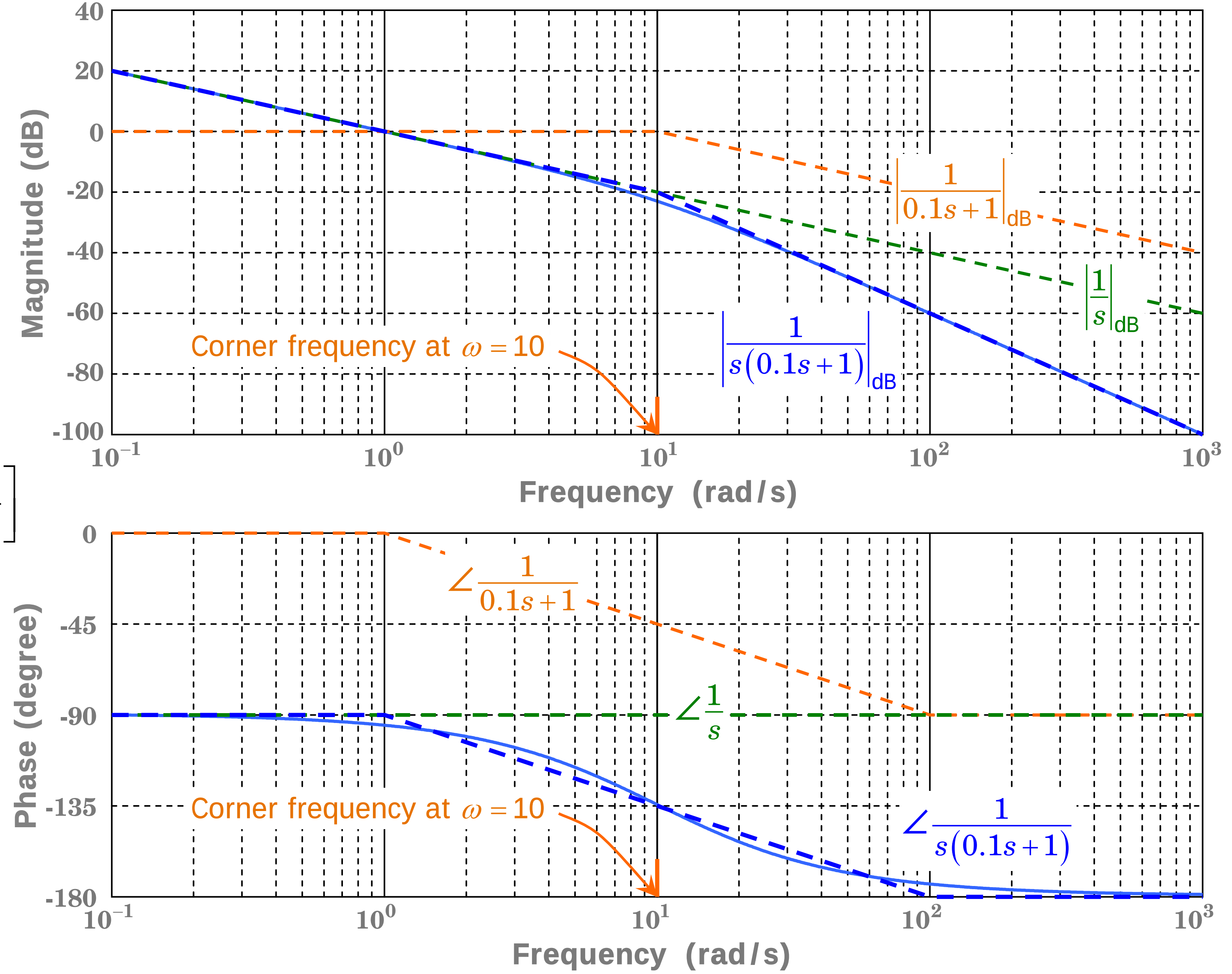
Magnitude response: $$\begin{cases} |H(j\omega)| \equiv \left|\dfrac{1}{j\omega}\right|_{dB} + \left|\dfrac{1}{jT\omega+1}\right|_{dB} \\ \qquad\qquad = -20\log_{10}\omega + \left|\dfrac{1}{jT\omega+1}\right|_{dB} \end{cases} \tag{7.36}$$

Phase response: $$\begin{cases} \angle H(j\omega) \equiv \angle\dfrac{1}{j\omega} + \angle\dfrac{1}{jT\omega+1} \\ \qquad\qquad = -90^\circ + \angle\dfrac{1}{jT\omega+1} \end{cases} \tag{7.37}$$

Bode plots of $\boldsymbol{H(s) = \dfrac{1}{s(sT+1)}}$ $(\textit{with } \boldsymbol{T = 0.1})$ ***are shown in Fig.7-8.***

Fig.7-8

$$\left[H(s) = \frac{1}{s(0.1s+1)} \right]$$



Example 7-9

Second order underdamped system: $\boldsymbol{H(s) = \dfrac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}}$ $\begin{cases} \textbf{Poles are complex} \\ \textbf{Damping ratio, } \zeta < 1 \end{cases}$

$H(s)$ can be rewritten in terms of the <u>normalized</u> frequency, $\tilde{\omega} = \dfrac{\omega}{\omega_n}$. Thus

$$H(s) = \frac{K}{\tilde{s}^2 + 2\zeta\tilde{s} + 1} \quad \text{or} \quad H(j\omega) = \frac{K}{\left(1 - \tilde{\omega}^2\right) + j2\tilde{\omega}\zeta} \tag{7.38}$$

where $\tilde{s} = \dfrac{s}{\omega_n} = j\tilde{\omega}$. Bode diagrams can be drawn in terms of the **normalized frequency $\tilde{\omega}$**.

Magnitude response: $$\left|H(j\omega)\right| \equiv 20\log_{10} K + 20\log_{10}\frac{1}{\sqrt{\left(1-\tilde{\omega}^2\right)^2 + 4\tilde{\omega}^2\zeta^2}} \text{ dB} \tag{7.39}$$

Phase response: $$\angle H(j\omega) = -\tan^{-1}\left(\frac{2\tilde{\omega}\zeta}{1-\tilde{\omega}^2}\right) \tag{7.40}$$

The approximate Bode straight-line magnitude plot is constructed based on the critically-damped case as follows:

- Corner frequency: $\tilde{\omega} = \mathbf{1}$ or $\omega = \omega_n$
- Slope change at the corner frequency: **-40 dB/decade**.

Bode plots of $\boldsymbol{H}(\boldsymbol{s}) = \dfrac{\boldsymbol{K}}{\tilde{\boldsymbol{s}}^2 + 2\zeta\tilde{\boldsymbol{s}} + 1}$ $\left(\text{with } \boldsymbol{K} = 10,\ \zeta = 0.2\right)$ *are shown in Fig.7-9*

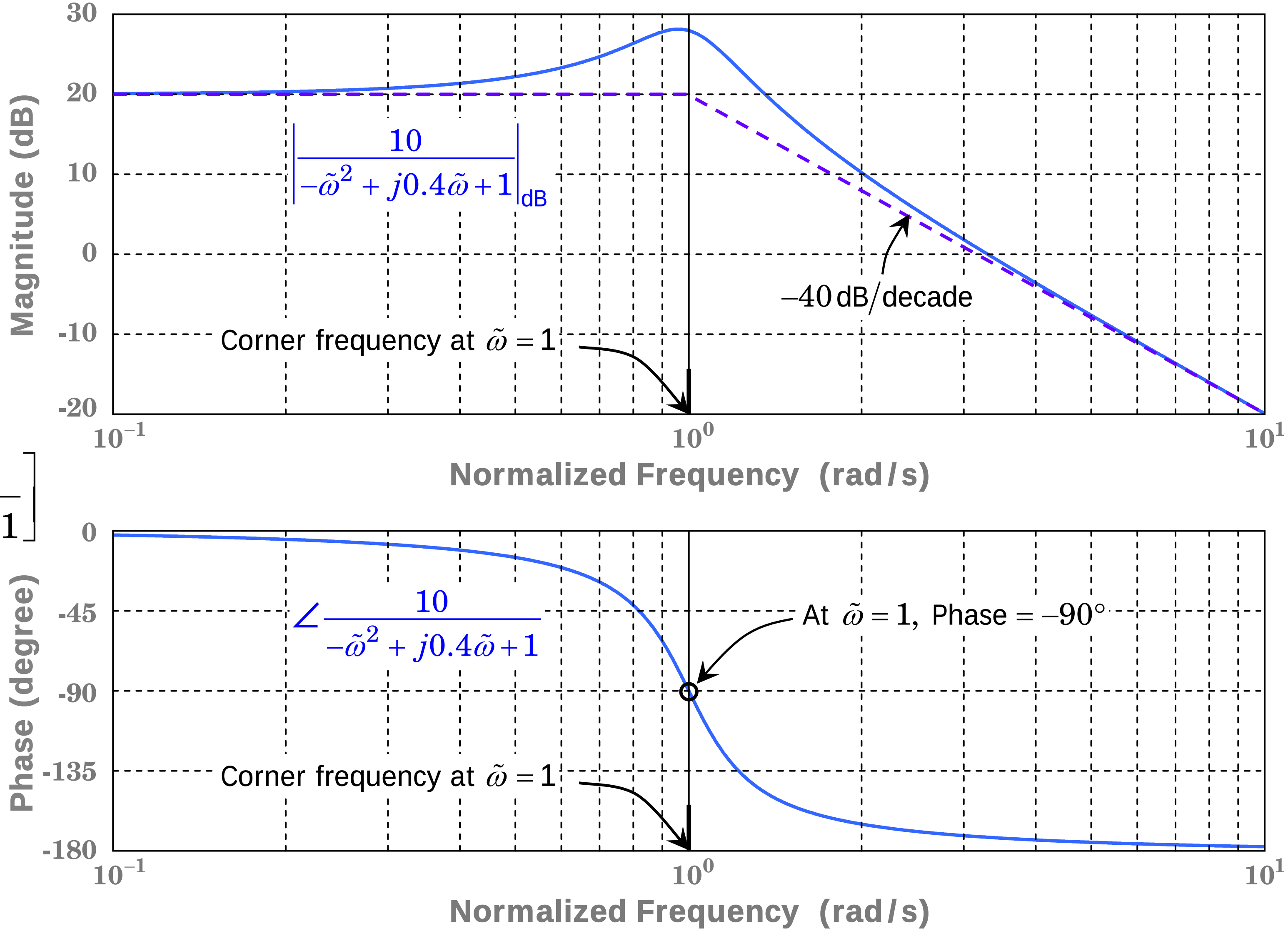


Fig.7-9

$$\left[H(s) = \frac{10}{\tilde{s}^2 + 0.4\tilde{s} + 1}\right]$$

Note that the magnitude response has a 'hump' or peak around $\tilde{\omega}=1$ or $\omega=\omega_n$. In general, the smaller the damping ratio, ζ, the larger is this peak. The family of Bode diagrams for different values of damping ratio is shown in Fig.7-10.

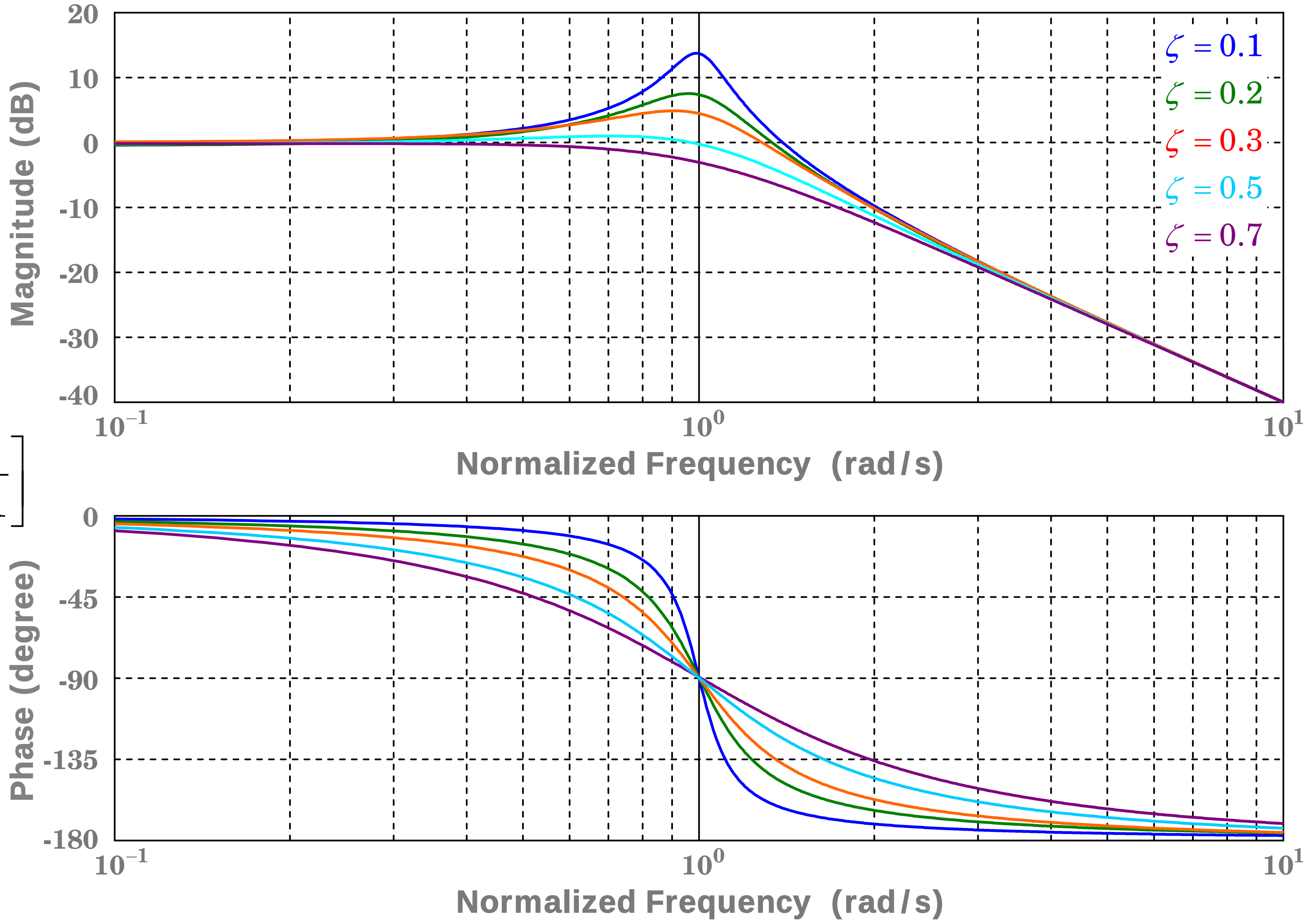


Fig.7-10

$$\left[H(s)=\frac{1}{\tilde{s}^2+2\zeta\tilde{s}+1}\right]$$

There is another concept related to underdamped second order systems when ζ is small. In particular, when $\zeta < \dfrac{1}{\sqrt{2}}$, a 'hump' or peak exists in the magnitude response and this is associated with the phenomenon of resonance.

Magnitude response (from (7.38)): $\left|H(j\omega)\right| = \dfrac{K}{\sqrt{\left(1-\tilde{\omega}^2\right)^2 + 4\zeta^2\tilde{\omega}^2}}$ where $\tilde{\omega} = \dfrac{\omega}{\omega_n}$

Maximizing $\left|H(j\omega)\right|$ is equivalent to minimizing $\left(1-\tilde{\omega}^2\right)^2 + 4\zeta^2\tilde{\omega}^2$. Therefore, setting

$$\frac{d}{d\tilde{\omega}}\left[\left(1-\tilde{\omega}^2\right)^2 + 4\zeta^2\tilde{\omega}^2\right]_{\tilde{\omega}=\tilde{\omega}_r} = 4\tilde{\omega}_r\left[\tilde{\omega}_r^2 - \left(1-2\zeta^2\right)\right] = 0$$

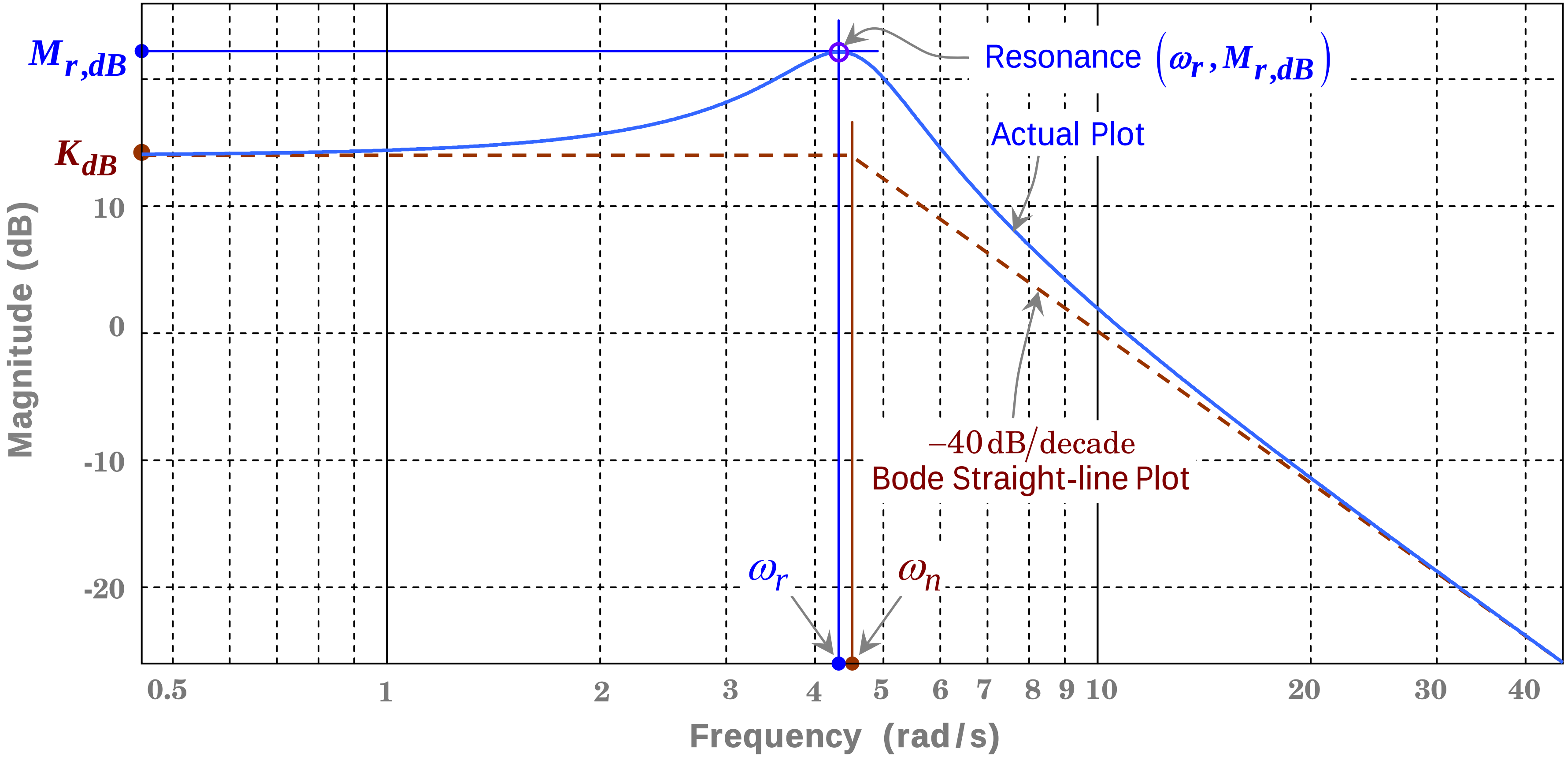
and solving, we get

$$\left.\begin{aligned} &\text{Resonance frequency} : \ \omega_r = \omega_n\sqrt{1-2\zeta^2} \\ &\text{Resonance peak} : \ M_r = \left|H(j\omega_r)\right| = \frac{K}{2\zeta\sqrt{1-\zeta^2}} \end{aligned}\right\} \begin{array}{c}\text{valid only}\\ \text{for}\\ \zeta < 1/\sqrt{2}\end{array} \qquad (7.41)$$

There is no resonance frequency or resonance peak when $\zeta \geq 1/\sqrt{2}$.

Note from (7.41) that $\omega_r \to \omega_n$ and $M_r \to \infty$ when $\zeta \to 0$.

The idea of resonance is one which allows receivers to be tuned to certain carrier frequencies.



7.5 Transportation Delay

- Transportation delay, t_d, is also called transport **lag** or **dead-time**.

- Transportation delay is a type of time delay that occurs in systems which require a finite time to move material or transmit signal from one point to another.

- In time domain, transportation delay gives rise to the following input$\left[x(t)\right]$-output$\left[y(t)\right]$ relationship:

$$y(t) = x(t - t_d) \tag{7.42}$$

where

$$\underbrace{\mathcal{L}\{y(t)\}}_{Y(s)} = \underbrace{\mathcal{L}\{x(t-t_d)\}}_{X(s)e^{-st_d}} \Rightarrow H(s) = \frac{Y(s)}{X(s)} = e^{-st_d}. \tag{7.43}$$

- Transfer function of the transport delay is non-rational.

- Time delays are a nuisance because it implies that the system is slow to react to any changes. This can create problems when there is a need for the system output to change quickly. For example, time delays in long distance calls that are routed via multiple satellite hops could be annoying.