NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2002-2003

MA2214 Combinatorial Analysis

April/May 2003 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains a total of **FIVE (5)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions.
3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Attempt **ALL** *questions. Each question carries 20 marks.*

Question 1 [20 marks]

(a) Let $S = \{1, 2, 3, \cdots, n+1\}$ where $n \geq 3$, and let

$$T = \{(a, b, c, d) \in S^4 \mid a, b, c < d\}.$$

(i) By counting $|T|$ in two different ways, show that

$$\sum_{r=1}^{n} r^3 = \binom{n+1}{2} + 6\binom{n+1}{3} + 6\binom{n+1}{4}.$$

(ii) Show further that the above result is indeed true for all integers $n \geq 1$.

(b) Prove by a combinatorial method that for each $n \in \mathbb{N}$, the following expressions are integers.

(i) $\dfrac{(n^2)!}{(n!)^{n+1}}$;

(ii) $\dfrac{(5n)!}{(4n+1)!\,n!}$.

Question 2 [20 marks]

(a) Assuming that all fruits of the same kind are identical, how many ways are there to give away 20 apples, 30 bananas and 40 oranges to 4 children if

(i) there is no restriction;

(ii) every child must have at least one of each kind of fruit;

(iii) every child must have at least one fruit.

(b) Let $S = \{(x_1, x_2, x_3, x_4) \in \mathbb{N}^4 \mid x_1 \geq 1,\, x_2 \geq x_1 + 3,\, x_3 \geq x_2,\, x_3 + 5 \leq x_4 \leq 30\}$. Find $|S|$.

Question 3 [20 marks]

Each of 8 boys attends a school gathering with both of his parents. To play a game these 24 people are to be divided into 8 groups of 3 each such that each group comprises a boy, a male parent and a female parent. How many ways can this be done if

(i) there is no restriction;

(ii) no boy is with both of his parents in his group;

(iii) no boy is with either of his parents in his group;

(iv) no boy is with either of his parents in his group and at least one female parent is not with her husband in her group.

Question 4 [20 marks]

(a) What is the probability that a roll of 6 distinct dice yields a sum of 16?

(b) For a non-negative integer r, let a_r be the number of integer solutions to the inequality

$$x_1 + x_2 + x_3 + x_4 \leq r,$$

with $x_1 \geq 1$, $x_2, x_3, x_4 \geq 3$. Find the ordinary generating function of a_r.

(c) By using (b) or otherwise, find the number of ways to select a set of 4 integers from 1 to 30 so that any two of the integers selected differ by at least 3. Justify your answer.

(d) Find the number of possible ways to assign a group of 10 people to 4 (different) committees such that each committee consists of an odd number of people.

Question 5 [20 marks]

(a) Solve the recurrence equation $a_n = a_{n-2} + 4n + 4$, given that $a_0 = -6$, $a_1 = 9$.

(b) Three sequences (a_n), (b_n) and (c_n) satisfy the following recurrence relations:

$$\begin{aligned} a_{n+1} &= \frac{1}{3}(b_n + c_n - a_n), \\ b_{n+1} &= \frac{1}{3}(c_n + a_n - b_n) + \frac{1}{3}, \\ c_{n+1} &= \frac{1}{3}(a_n + b_n - c_n) + \frac{2}{3}, \end{aligned}$$

with $a_0 = 1$, $b_0 = 0$ and $c_0 = 1$. Find a recurrence relation for the sequence $d_n = a_n + b_n + c_n$, and determine d_n for all $n \geq 0$.

(c) Find the ordinary generating function of the sequence $(u_n)_{n\geq 0}$ defined by

$$u_n - 3u_{n-1} - 4u_{n-2} = 0, \quad u_0 = 0,\ u_1 = 1.$$

Hence deduce a formula for u_n.

END OF PAPER