EE2023 Signals & Systems Quiz
Semester 2 AY2011/12
Date: 8 March 2012 Time Allowed: 1.5 hours

Q1. Consider a periodic signal, $x(t)$, modelled by the following equation

$$x(t) = 2je^{-j3t} + (2+3j)e^{-j2t} + 5 + (2-3j)e^{j2t} - 2je^{j3t}$$

(a) What is the fundamental frequency of $x(t)$?

ANSWER:

Fundamental frequency = $HCF\{2,3\} = 1 \text{ rad}/s$

(b) By comparing $x(t)$ with the Fourier Series expansion equation, $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_o t}$, derive the magnitude, $|c_k|$, and phase, $\angle c_k$, of the Fourier Series coefficients when k = 0, 1, 2 and 3.

ANSWER:

$\omega_o = 1$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jkt} = \cdots + c_{-3}e^{-j3t} + c_{-2}e^{-j2t} + c_{-1}e^{-jt} + c_0 + c_1 e^{jt} + c_2 e^{j2t} + c_3 e^{j3t} + \cdots$$

Comparing coefficients:

$c_0 = 5 \rightarrow |c_0| = 5,\quad \angle c_0 = 0$

$c_1 = c_{-1} = 0 \rightarrow |c_1| = 0,\quad \angle c_1 = 0$

$c_2 = c_{-2}^{*} = 2-3j \rightarrow |c_2| = 13^{0.5} = 3.61\quad \angle c_2 = \tan^{-1}(-3/2) = -0.98 \text{ rad} = -56.31°$

$c_3 = c_{-3}^{*} = -2j \rightarrow |c_3| = 2,\quad \angle c_0 = -0.5\pi \text{ rad} = -90°$

(c) An alternative method for evaluating the Fourier Series coefficients, c_k, of $x(t)$ is

$$c_k = \frac{1}{T}\int_0^T x(t)e^{-j2\pi kt/T}dt$$

What is the value of T?

ANSWER:

$$\omega_o = 1 \rightarrow f_o = \frac{\omega_o}{2\pi} = \frac{1}{2\pi} \rightarrow T = \frac{1}{f_o} = 2\pi \text{ sec}$$

(d) Suppose the Fourier Series coefficients for the signal $y(t) = 4\sin(3t)$ is determined using the equation $c_k = \frac{1}{T}\int_0^T y(t)e^{-j2\pi kt/T}dt$, where T is the value determined in part (c). Can the resulting Fourier Series coefficients be used to correctly synthesize $y(t)$ via Fourier Series expansion ? Justify your answer.

ANSWER:

YES.

Period of $y(t)$ is $\frac{2\pi}{3}(\text{sec})$. $T = 2\pi(\text{sec})$ is an integer multiple of $\frac{2\pi}{3}(\text{sec})$.

Q2. Figure Q2 shows an exponentially decaying function $x(t)$ which is expressed as:

$$x(t) = \exp(-a|t|)$$

where $a > 0$.

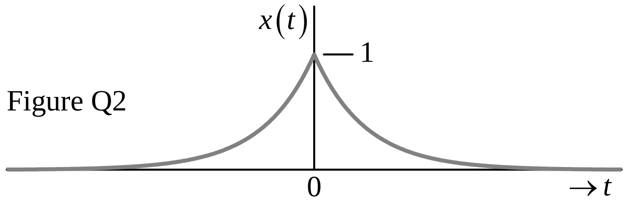


Figure Q2

(a) Determine the Fourier transform, $X(f)$, of the signal $x(t)$.

ANSWER:

$$X(f) = \int_{-\infty}^{\infty} x(t)\exp(-j2\pi ft)\,dt$$
$$= \int_{-\infty}^{0} \exp(at)\exp(-j2\pi ft)\,dt + \int_{0}^{\infty} \exp(-at)\exp(-j2\pi ft)\,dt$$
$$= \int_{-\infty}^{0} \exp[-(j2\pi f - a)t]\,dt + \int_{0}^{\infty} \exp[-(j2\pi f + a)t]\,dt$$
$$= \left.\frac{\exp[-(j2\pi f - a)t]}{-(j2\pi f - a)}\right|_{-\infty}^{0} + \left.\frac{\exp[-(j2\pi f + a)t]}{-(j2\pi f + a)}\right|_{0}^{\infty}$$
$$= \left[\frac{1}{(-j2\pi f + a)}\right] + \left[\frac{1}{(j2\pi f + a)}\right] = \frac{2a}{a^2 + 4\pi^2 f^2}$$

(b) Using the replication property of the Dirac-δ function, the periodic signal $x_p(t)$ can be obtained as:

$$x_p(t) = x(t) \ast \sum_{k=-\infty}^{\infty} \delta(t - kT_p)$$

where T_p is the period, and $\ast$ denotes convolution. Derive the Fourier transform, $X_p(f)$, of the periodic signal $x_p(t)$ based on this approach.

ANSWER:

$$X_p(f) = \Im\{x(t)\} \cdot \Im\left\{\sum_{k=-\infty}^{\infty} \delta(t - kT_p)\right\}$$
$$= X(f) \cdot \frac{1}{T_p} \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T_p}\right) = \frac{1}{T_p} \sum_{k=-\infty}^{\infty} X\left(\frac{k}{T_p}\right) \cdot \delta\left(f - \frac{k}{T_p}\right)$$
$$= \frac{1}{T_p} \sum_{k=-\infty}^{\infty} \frac{2a}{a^2 + 4\pi^2 (k/T_p)^2} \cdot \delta\left(f - \frac{k}{T_p}\right)$$

Q3. Consider an energy signal $x(t)$. Let $X(f)$, E and B denote its *spectrum*, *energy* and *bandwidth*, respectively. With $x(t)$, we form another signal $y(t) = -0.5x(t-5)$.

(a) Express the spectrum of $y(t)$ in terms of $X(f)$.

ANSWER:

Spectrum of $y(t)$: $\underbrace{Y(f) = -0.5X(f)\exp(-j10\pi f)}_{\text{using time-shifting property of FT}}$

(b) Express the energy of $y(t)$ in terms of E.

ANSWER:

Energy of $x(t)$: $E = \int_{-\infty}^{\infty} |X(f)|^2 \, df$

Energy of $y(t)$: $\underbrace{\int_{-\infty}^{\infty} |Y(f)|^2 \, df = 0.25\int_{-\infty}^{\infty} |X(f)|^2 \, df}_{\text{from part } (a)} = 0.25E$

(c) Express the bandwidth of $y(t)$ in terms of B.

ANSWER:

Bandwidth of $y(t)$: = B (amplitude-scaling and time-shifting do not affect bandwidth)

Q4. Consider a signal $x(t)$ (with Fourier Transform $X(f)$) whose amplitude spectrum is shown in Figure Q4 below.

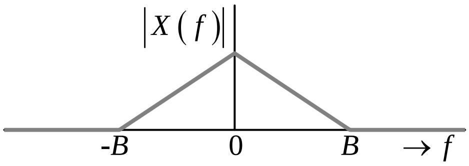


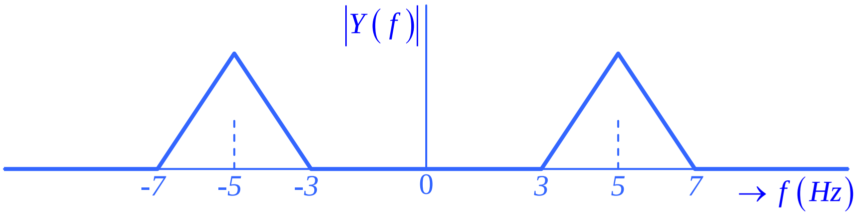
Figure Q4: Amplitude spectrum of $x(t)$

Consider also the signal $y(t) = x(t)\cos(10\pi t)$.

(a) If the bandwidth of $x(t)$ is $B = 2\ Hz$, sketch the amplitude spectrum of $y(t)$. Label clearly the frequency axis of the amplitude spectrum.

ANSWER:

Spectrum of $y(t)$: $Y(f) = X(f) * \frac{1}{2}\left[\delta(f+5) + \delta(f-5)\right] = \frac{1}{2}\left[X(f+5) + X(f-5)\right]$



(b) If $y(t)$ is sampled at a sampling frequency of 15 Hz, write down the expression for the sampled signal of $y(t)$ in terms of the comb function.

ANSWER:

Sampled $y(t)$: $y_S(t) = y(t) \cdot \sum_{n=-\infty}^{\infty} \delta\left(t - \frac{n}{15}\right) = x(t)\cos(10\pi t) \cdot \sum_{n=-\infty}^{\infty} \delta\left(t - \frac{n}{15}\right)$

(c) Sketch the amplitude spectrum of the sampled signal of $y(t)$.

ANSWER: