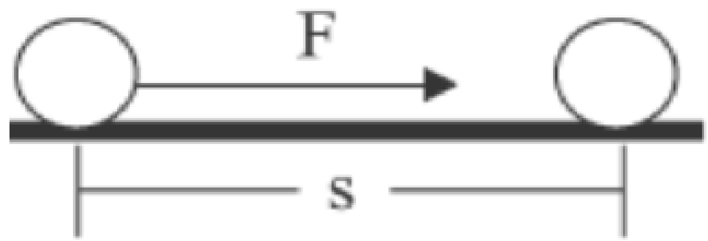
Chapter 9. Line Integrals

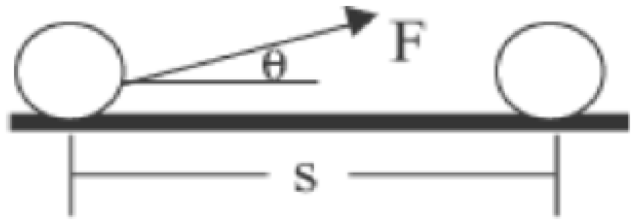
9.1.1 Work Done I



(i)

(i) Let $\mathbf{F}$ be a constant force acting on a particle in the displacement direction as shown in figure (i) above. Suppose the distance moved by the particle is s. The work done is given by

$$W = ||\mathbf{F}|| \times s.$$



(ii)

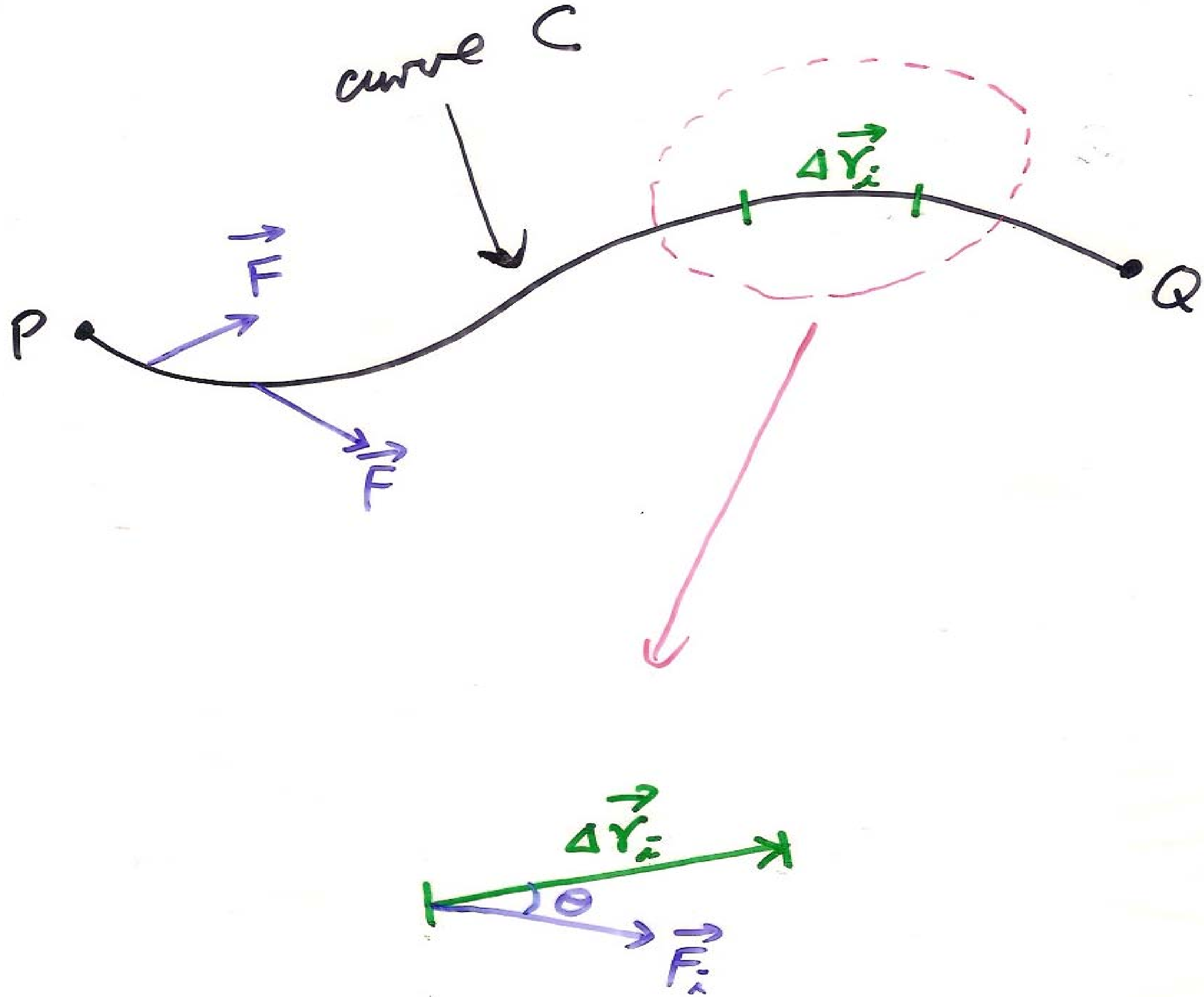
(ii) Let $\mathbf{F}$ be a constant force acting on a particle in the direction which form an angle θ against the displacement direction (see figure (ii) above). Suppose the distance moved by the particle is s. The work done is given by

$$W = ||\mathbf{F}|| \cos\theta \times s = (\mathbf{F}\cdot\mathbf{T}) \times s = \mathbf{F}\cdot s\mathbf{T}$$

where $\mathbf{T}$ is the unit vector in the displacement direction.

9.1.2 Work Done II

Let $\mathbf{F}(x, y, z)$ be a variable force acting on a particle which moves along the curve C with vector equation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ as shown in the figure below. Suppose the particle moves from point P to point Q. What is the work done?



curve C
P
Q
$\vec{F}$
$\vec{F}$
$\Delta\vec{r}_i$
$\Delta\vec{r}_i$
θ
$\vec{F}_i$

Workdone on $\Delta\vec{r}_i = W_i$

$$\approx \left(\|\vec{F}_i\| \cos\theta\right) \|\Delta\vec{r}_i\|$$

$$= \vec{F}_i \cdot \Delta\vec{r}_i$$

$W =$ Total workdone from P to $Q \approx \sum_{i=1}^{n} \vec{F}_i \cdot \Delta\vec{r}_i$

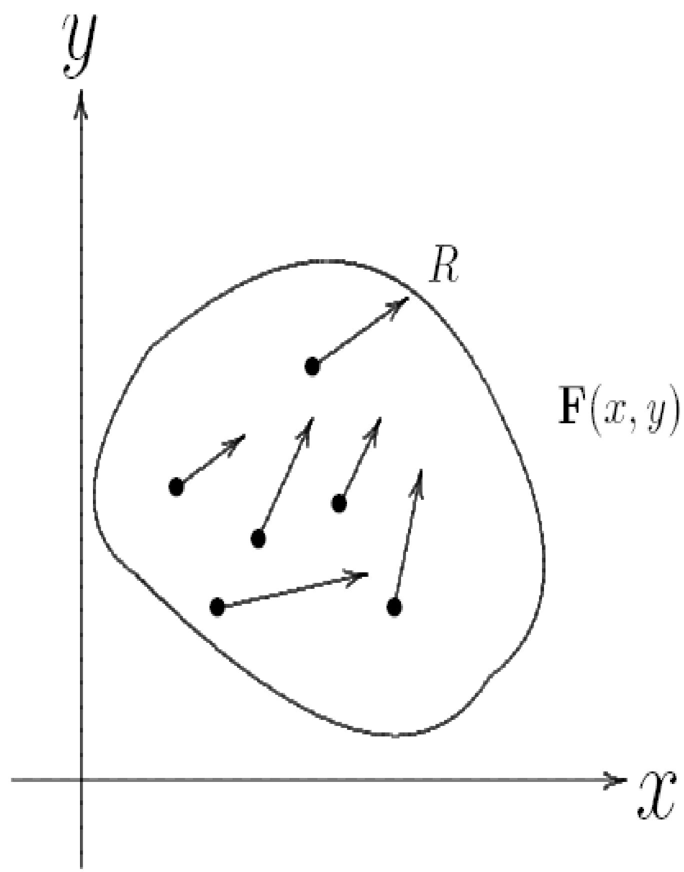
$n \to \infty \implies W = \int_C \vec{F} \cdot d\vec{r} =$ The line integral of $\vec{F}$ along C.

9.1.3 Vector Fields

The vector function $\mathbf{F}$ is called in general a vector field and the above integral is called the line integral of $\mathbf{F}$ along the curve C. We shall see in section 9.3.7 how to evaluate this type of integral.

9.2.1 Vector field (two variables)

Let R be a region in xy-plane. A **vector field** on R is a vector function $\mathbf{F}$ that assigns to each point (x, y) in R a two-dimensional vector $\mathbf{F}(x, y)$.



y
R
$\mathbf{F}(x, y)$
x

We may write $\mathbf{F}(x, y)$ in terms of its component functions. That is

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$$

or simply $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$.

9.2.2 Vector field (three variables)

Let D be a solid region in xyz-space. A **vector field** on D is a vector function $\mathbf{F}$ that assigns to each point (x, y, z) in D a three-dimensional vector $\mathbf{F}(x, y, z)$.

That is, $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$.

Example 9.2.3

$$\vec{F}(x,y) = (-y)\vec{i} + x\vec{j}$$

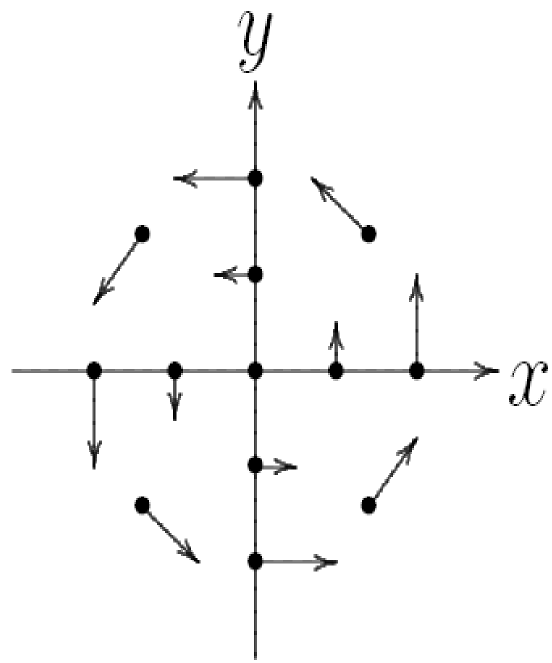
To show that $\vec{F} \perp$ to $\vec{r} = x\vec{i} + y\vec{j}$.

$$\because \vec{F}\cdot\vec{r} = (-y)x + x(y)$$

$$= -xy + xy$$

$$= 0$$

$$\therefore \vec{F} \perp \vec{r}.$$



$F(x,y) = (-y)\mathbf{i} + x\mathbf{j}$

9.2.4 Gradient fields

If $f(x, y)$ is a function of two variables, then

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$$

is a vector field in the xy-plane and it is called the **gradient (field)** of f.

Similarly, if $f(x, y, z)$ is a function of three variables, then

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}$$

is a vector field in the xyz-space and it is called the *gradient (field)* of f.

9.2.5 Example

The gradient field of $f(x, y) = xy^2 + x^3$ is

$$\nabla f(x, y) = (y^2 + 3x^2)\mathbf{i} + (2xy)\mathbf{j}.$$

9.2.6 Conservative fields

A vector field $\mathbf{F}$ is called a **conservative** vector field if it is the gradient of some (scalar) function. In other words, there is a function f such that $\mathbf{F} = \nabla f$. In this case, f is called a **potential** function for $\mathbf{F}$.

9.2.7 Example

By Example 9.2.5, $\mathbf{F}(x, y) = (y^2 + 3x^2)\mathbf{i} + (2xy)\mathbf{j}$ is conservative since it has a potential function $f(x, y) = xy^2 + x^3$.

9.2.8 Example

Let $\mathbf{F}(x, y) = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$. Find a potential function f for $\mathbf{F}$.

Solution: As $\nabla f = \mathbf{F}$, we have $f_x(x, y) = 3+2xy$. Integrating with respect to x, we get $f(x, y) = 3x + x^2y+g(y)$, where $g(y)$ is an integration constant, but it could be a function of y.

Thus $f_y(x, y) = x^2 + g'(y)$ so that $x^2 + g'(y) = x^2 - 3y^2$. That is, $g'(y) = -3y^2$.

Integrating $g'(y)$ with respect to y, we obtain $g(y) = -y^3 + K$, where K is a constant.

Consequently, $f(x, y) = 3x + x^2y - y^3 + K$.

9.2.9 Example

The gravitational field given by

$$\mathbf{G} = \left(\frac{-m_1 m_2 K x}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right) \mathbf{i} + \left(\frac{-m_1 m_2 K y}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right) \mathbf{j} + \left(\frac{-m_1 m_2 K z}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right) \mathbf{k}$$

is conservative because it is the gradient of the gravitational potential function

$$g(x, y, z) = \frac{m_1 m_2 K}{\sqrt{x^2 + y^2 + z^2}},$$

where K is the gravitational constant, m_1 and m_2 are the masses of two objects. Think of the mass m_1 at the origin that creates the field and g is the potential energy attained by the mass m_2 situated at (x, y, z).

9.2.10 Criteria of conservative fields

Throughout this chapter, we will assume the component functions of any vector field to have continuous partial derivatives, unless otherwise stated.

(a) Let $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ be a vector field on the xy-plane.

$$\text{If} \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \text{then } \mathbf{F} \text{ is conservative.}$$

$\vec{F} = P\vec{i} + Q\vec{j}$ is conservative

$\Rightarrow \vec{F} = \nabla f$ for some function f.

$\Rightarrow \vec{F} = f_x \vec{i} + f_y \vec{j}$

$$\Rightarrow \begin{cases} f_x = P \\ f_y = Q \end{cases}$$

$$\Rightarrow \begin{cases} f_{xy} = P_y \\ f_{yx} = Q_x \end{cases}$$

$\Rightarrow \dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ (if they are continuous)

(b) Let $\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i}+Q(x,y,z)\mathbf{j}+R(x,y,z)\mathbf{k}$

be a vector field on the xyz-space.

If $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}, \quad \dfrac{\partial P}{\partial z} = \dfrac{\partial R}{\partial x}, \quad \dfrac{\partial Q}{\partial z} = \dfrac{\partial R}{\partial y},$

then $\mathbf{F}$ is conservative.

The converse of (a) and (b) also hold.

9.2.11 Example

Consider the vector field

$$\mathbf{F}(x, y) = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}.$$

As $\dfrac{\partial(x^2 - 3y^2)}{\partial x} = 2x = \dfrac{\partial(3 + 2xy)}{\partial y}$,

$\mathbf{F}$ is conservative.

9.2.12 Example

Show that $\mathbf{F}(x, y, z) = xz\mathbf{i} + xyz\mathbf{j} - y^2\mathbf{k}$ is not conservative.

Solution: For example, $\dfrac{\partial(xyz)}{\partial x} = yz$ which is not equal to $\dfrac{\partial(xz)}{\partial y} = 0$.

So $\mathbf{F}$ is not conservative.

9.2.13 Exercise

Show that the vector field $\mathbf{F}(x, y, z) = x^2\mathbf{i}+y^2\mathbf{j}+z^2\mathbf{k}$ is conservative. Find a function f such that $\nabla f = \mathbf{F}$.

$$\vec{F} = x^2\,\vec{i} + y^2\,\vec{j} + z^2\,\vec{k}$$

$$\frac{\partial(x^2)}{\partial y} = 0 = \frac{\partial(y^2)}{\partial x}$$

$$\frac{\partial(x^2)}{\partial z} = 0 = \frac{\partial(z^2)}{\partial x}$$

$$\frac{\partial(y^2)}{\partial z} = 0 = \frac{\partial(z^2)}{\partial y}$$

$\therefore \vec{F}$ is conservative.

Let $\nabla f = \vec{F}$

$\therefore \dfrac{\partial f}{\partial x} = x^2$ (compare $\vec{i}$ component)

$\therefore f = \frac{1}{3}x^3 + g(y, z)$

Compare $\vec{j}$ component $\Longrightarrow g_y = y^2$

$\therefore g = \frac{1}{3}y^3 + h(z)$

$\therefore \ f = \frac{1}{3}x^3 + \frac{1}{3}y^3 + h(z)$

Compare $\vec{k}$ component

$\Rightarrow h' = z^2$

$\therefore \quad h = \frac{1}{3}z^3 + C$

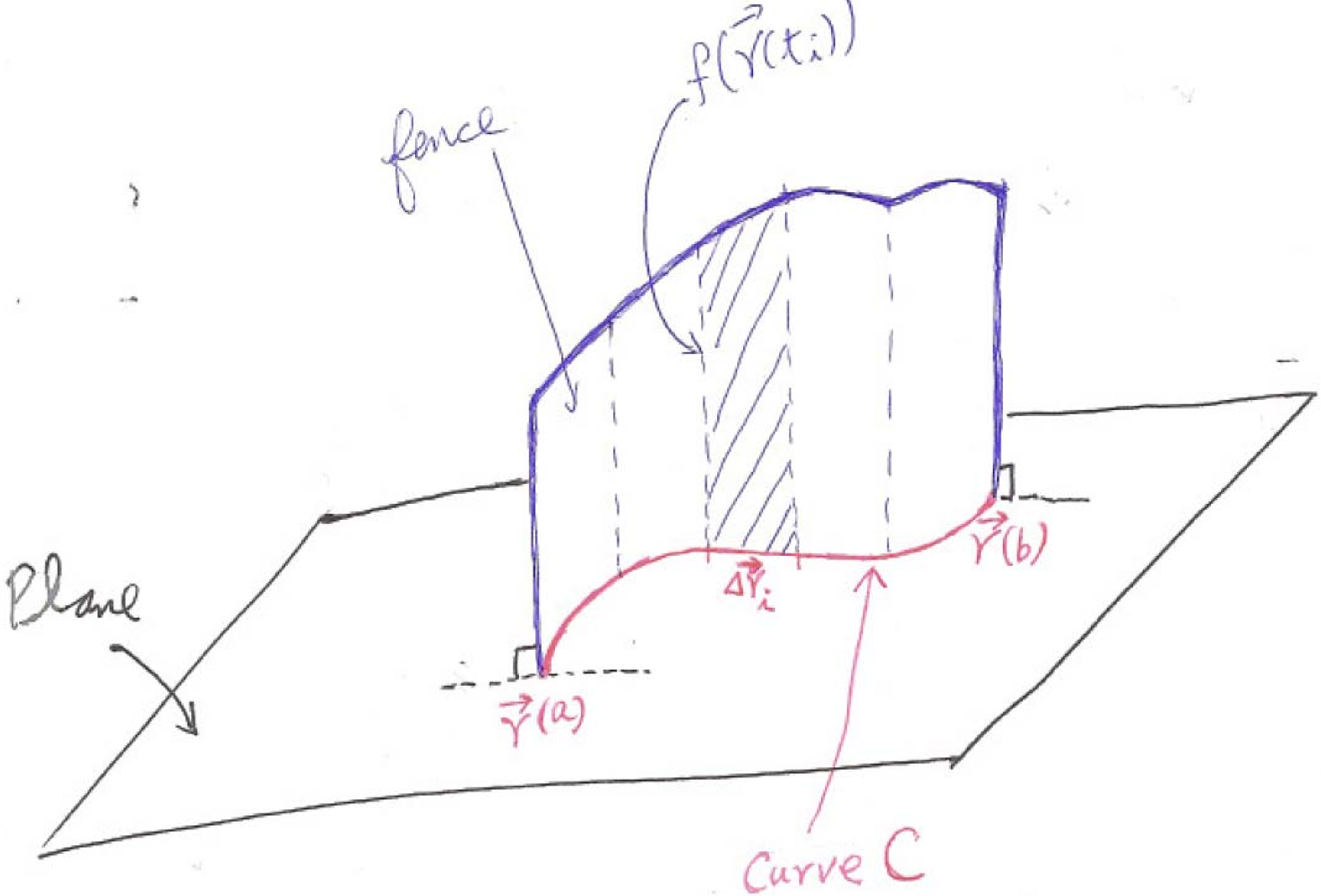
$$\therefore \quad \underline{\underline{f = \frac{1}{3}x^3 + \frac{1}{3}y^3 + \frac{1}{3}z^3 + C}}$$

9.3 Line Integrals

We have mentioned in section 9.1 that a line integral refers to an integration along a curve C. There are two types of line integrals. One is for vector fields and the other is for scalar functions.

9.3.1 Line integrals of scalar functions (Two variables)

Suppose we want to find the area of the following surface with the base, a plane curve C on the xy-plane and the top is described by a function $f(x, y)$.



$$\text{area of fence} \approx \sum_{i=1}^{n} f(\vec{r}(t_i))\,\|\Delta\vec{r}_i\|$$

$$n \to \infty \Rightarrow \text{area} = \int_C f(\vec{r}(t))\,\|d\vec{r}\|$$

$$d\vec{r} = \vec{r}\,'(t)\,dt \Rightarrow \|d\vec{r}\| = \|\vec{r}\,'(t)\|\,dt$$

$$\text{area} = \int_a^b f(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt$$

$$\int_C f(\vec{r}(t))\,\|d\vec{r}\|$$

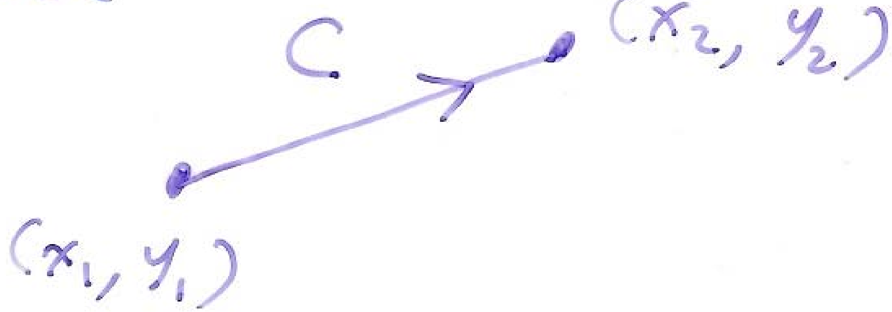
is also written as $\int_C f\,ds$

For a scalar function f:

$$\boxed{\int_C f\,ds = \int_a^b f(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt}$$

where C: $\vec{r}(t)$, $a \leq t \leq b$

Line segment



$$\vec{r}(t) = (x_1, y_1) + t\{(x_2, y_2) - (x_1, y_1)\}$$

$$0 \le t \le 1$$

$$= (x_1 + t(x_2 - x_1),\ y_1 + t(y_2 - y_1))$$

$$0 \le t \le 1$$

9.3.2 Example

Evaluate $\int_C (2y + x^2 y)\,ds$, where C is the upper half of the unit circle centered at the origin.

Solution: The vector function of C is given by $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$ with $0 \leq t \leq \pi$.

Thus $\|\mathbf{r}'(t)\| = \sqrt{\sin^2 t + \cos^2 t}$ and

$$\begin{aligned}
\int_C (2y + x^2 y)ds &= \int_0^{\pi} (2\sin t + \cos^2 t \sin t)\sqrt{\sin^2 t + \cos^2 t}dt \\
&= \int_0^{\pi} (2\sin t + \cos^2 t \sin t)\, dt \\
&= \left[-2\cos t - \frac{1}{3}\cos^3 t\right]_0^{\pi} \\
&= \frac{14}{3}
\end{aligned}$$

9.3.3 Line integrals of scalar functions (Three variables)

For line integral of a function $f(x, y, z)$ along a space curve C, we have the similar definitions:

$$\int_C f(x, y, z)\, ds =$$

$$\int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

9.3.4 Example

Evaluate $\int_C xy \sin z \, ds$, where C is the circular helix

$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$, $t \in [0, \pi/2]$.

Solution:

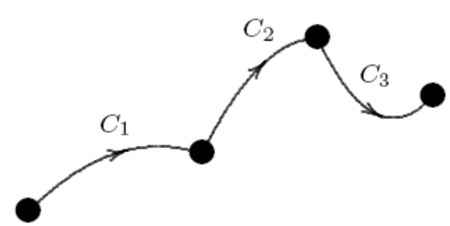
$$\begin{aligned}
&\int_C xy \sin z \, ds \\
=&\int_0^{\pi/2} (\cos t)(\sin t)(\sin t)\sqrt{\sin^2 t + \cos^2 t + 1}\, dt \\
=&\sqrt{2}\int_0^{\pi/2} \cos t \sin^2 t dt \\
=&\frac{\sqrt{2}}{3}\left[\sin^3 t\right]_0^{\pi/2} = \frac{\sqrt{2}}{3}
\end{aligned}$$

9.3.5 Piecewise smooth curves

We denote the union of a finite number of (smooth) curves $C_1, C_2, \cdots, C_n$ by

$$C = C_1 + C_2 + \cdots + C_n.$$

We say C is a *piecewise-smooth* curve.

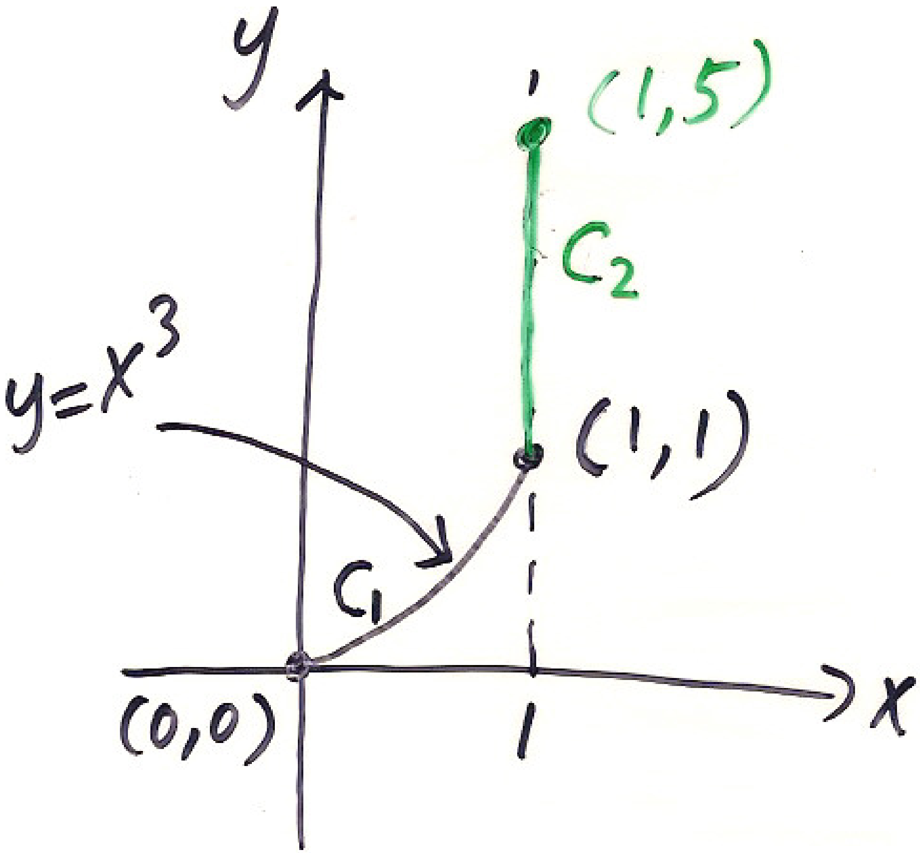


Then the line integral f along C is defined to be

$$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \cdots + \int_{C_n} f(x,y)\,ds.$$

9.3.6 Example

Evaluate $\displaystyle\int_C 9y\,ds$, where C consists of the arc C_1 of the cubic $y = x^3$ from $(0,0)$ to $(1,1)$ followed by the vertical line segment C_2 from $(1,1)$ to $(1,5)$.



y
(1,5)
C_2
$y=x^3$
(1,1)
C_1
(0,0)
1
x

$C_1: \vec{r}(t) = (t, t^3), \; 0 \leq t \leq 1$

$$\vec{r}\,'(t) = (1, 3t^2)$$

$$\|\vec{r}\,'(t)\| = \sqrt{1 + 9t^4}$$

$C_2: \vec{r}(t) = (1, t), \quad 1 \leq t \leq 5$

$$\vec{r}\,'(t) = (0, 1)$$

$$\|\vec{r}\,'(t)\| = \sqrt{0^2 + 1^2} = 1$$

$$\int_C 9y\,ds = \int_{C_1} 9y\,ds + \int_{C_2} 9y\,ds$$

$$= \int_0^1 9t^3\sqrt{1+9t^4}\,dt + \int_1^5 9t\,dt$$

$$= \frac{2}{3}\cdot\frac{1}{4}\left(1+9t^4\right)^{3/2}\Big|_0^1 + \frac{9}{2}t^2\Big|_1^5$$

$$= \frac{1}{6}\left\{10\sqrt{10}-1\right\} + 108$$

$$= \frac{1}{6}\left\{10\sqrt{10}+647\right\}$$

9.3.7 Line integrals of vector fields

Recall from first part, using workdone, we have

$$\text{Workdone} = \int_C \vec{F} \cdot d\vec{r}$$

$$\because \; d\vec{r} = \vec{r}\,'(t)\, dt,$$

we have

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = \int_a^b \{\vec{F} \cdot \vec{r}\,'(t)\}\, dt}$$

where $C : \vec{r}(t), \quad a \le t \le b.$

9.3.8 Example

Evaluate $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F}(x, y, z) = x\mathbf{i} + xy\mathbf{j} + xyz\mathbf{k}$$

and C is the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, $t \in [0, 2]$.

$$\vec{F} = x\vec{i} + xy\vec{j} + xyz\vec{k}$$

$$C: \quad \vec{r}(t) = t\vec{i} + t^2\vec{j} + t^3\vec{k}, \quad 0 \le t \le 2.$$

$$\vec{r}\,'(t) = \vec{i} + 2t\vec{j} + 3t^2\vec{k}$$

$$\vec{F}(\vec{r}(t)) = t\vec{i} + t(t^2)\vec{j} + t(t^2)(t^3)\vec{k}$$

$$= t\vec{i} + t^3\vec{j} + t^6\vec{k}$$

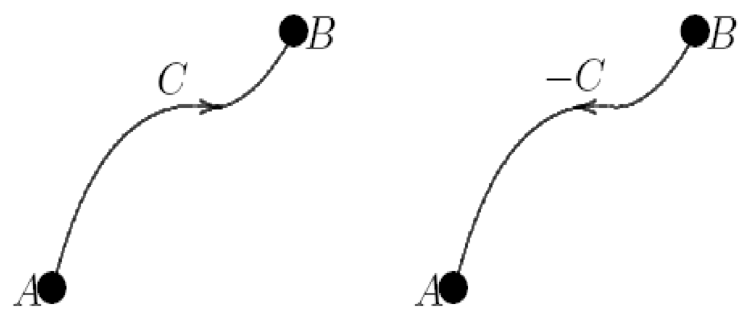
$$I = \int_0^2 \{ t + 2t(t^3) + 3t^2(t^6) \} \, dt$$

$$= \left[\frac{1}{2}t^2 + \frac{2}{5}t^5 + \frac{3}{9}t^9 \right]_0^2$$

$$= \frac{2782}{15}$$

9.3.9 Orientation of curves

The vector equation of a curve C determines an **orientation** (direction) of C. The same curve with the opposite orientation of C is denoted by $-C$.



We have

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_{C} \mathbf{F} \cdot d\mathbf{r}$$

as $\mathbf{r}'(t)$ changes sign in $-C$.

On the other hand, for line integral of scalar functions,

$$\int_{-C} f(x, y, z)\, ds = \int_{C} f(x, y, z)\, ds$$

since the arc length is always positive.

9.3.10 Line integrals in component form

Let $\vec{F} = P\vec{i} + Q\vec{j}$

$C: \vec{r} = x\vec{i} + y\vec{j}$

Then $d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy$$

Similarly, for three variable vector field

$\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$,

we can write the line integral as

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P dx + Q dy + R dz.$$

9.3.11 Example

Evaluate the line integral $\int_C y^2 dx + x dy$, where

(a) $C = C_1$ is the line segment from $(-5, -3)$ to $(0, 2)$,

(b) $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(0, 2)$.

$$I = \int_C y^2\,dx + x\,dy$$

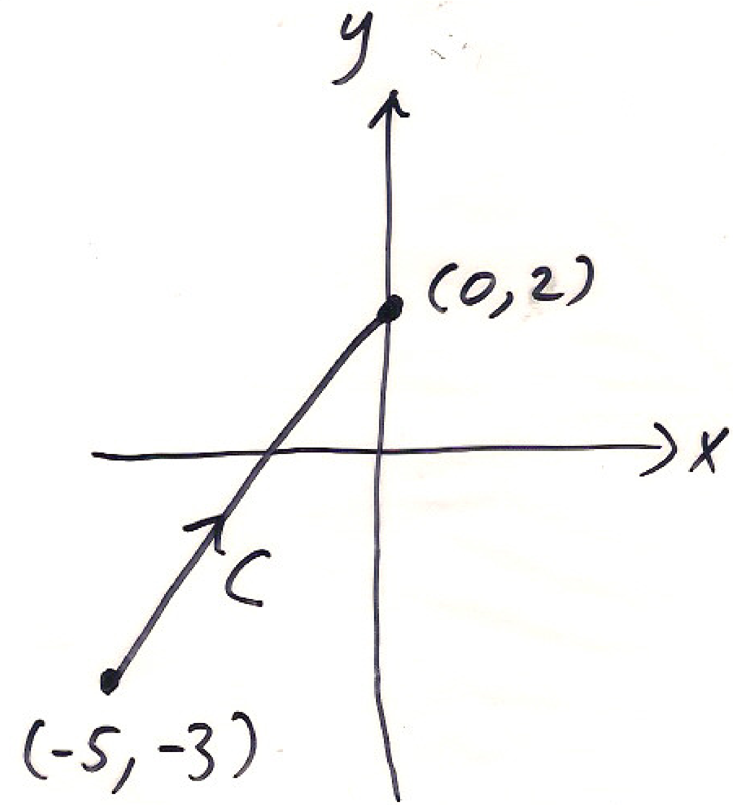
(a)

$C: \vec{r}(t) = (-5, -3)$

$+ t\{(0,2) - (-5,-3)\}$

$= (-5+5t, -3+5t)$

$0 \leq t \leq 1.$



$$I = \int_0^1 (-3+5t)^2 (5\,dt) + (-5+5t)(5\,dt)$$

$$= 5\int_0^1 (9-30t+25t^2-5+5t)\,dt$$

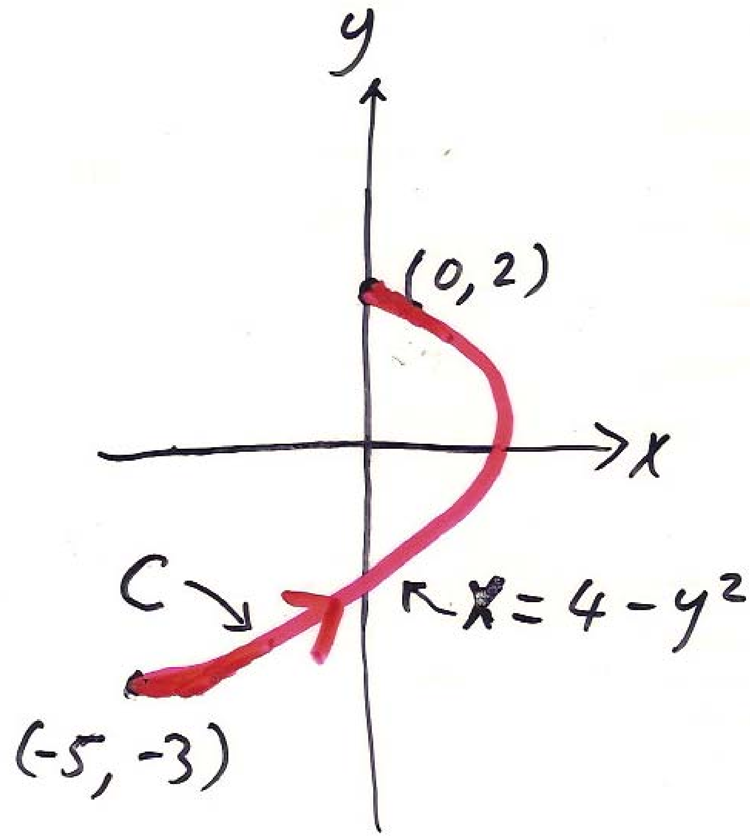
$$= 5\left[\frac{25}{3}t^3 - \frac{25}{2}t^2 + 4t\right]_0^1$$

$$= -\frac{5}{6}$$

(b)

$C: \vec{r}(t) = (4-t^2, t)$

$-3 \leq t \leq 2$



$$I = \int_{-3}^{2} t^2(-2t\,dt) + (4-t^2)\,dt$$

$$= \int_{-3}^{2} (-2t^3 + 4 - t^2)\,dt$$

$$= \left[-\frac{1}{2}t^4 + 4t - \frac{1}{3}t^3\right]_{-3}^{2}$$

$$= \frac{245}{6}$$

9.3.12 The fundamental theorem for line integrals

Suppose $\vec{F} = \nabla f = f_x \vec{i} + f_y \vec{j} + f_z \vec{k}$

Then $\int_C \vec{F} \cdot d\vec{r} = \int_C f_x \, dx + f_y \, dy + f_z \, dz$

$$= \int_a^b f_x \frac{dx}{dt} dt + f_y \frac{dy}{dt} dt + f_z \frac{dz}{dt} dt$$

$$= \int_a^b \left(f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt} \right) dt$$

$$= \int_a^b \left(\frac{df}{dt} \right) dt \quad \text{(By the Chain Rule)}$$

$$= f(\vec{r}(b)) - f(\vec{r}(a))$$

where $C : \vec{r}(t), \quad a \le t \le b.$

If f is a function of 2 or 3 variables whose gradient ∇f is continuous. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

9.3.13 Example

Find the work done by the (earth) gravitational field (see Example 9.2.9) in moving a particle of mass m from the point $(3, 4, 12)$ to the point $(1, 0, 0)$ along a curve C.

Solution:

$$W \equiv \int_C \mathbf{G} \cdot d\mathbf{r} = \int_C \nabla g \cdot d\mathbf{r} = g(1,0,0) - g(3,4,12).$$

Since the potential function $g(x, y, z) = \dfrac{mMK}{\sqrt{x^2 + y^2 + z^2}}$

where M is the mass of the earth and K the gravi-

tational constant, we have $W = 12mMK/13$.

9.3.14 Consequences of conservative fields

(I) If $\mathbf{F}$ is a conservative vector field, then $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ is *independent of path*,

i.e. $\displaystyle\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for any 2 paths C_1 and C_2 that have the same initial and terminal points.

(II) If $\mathbf{F}$ is a conservative vector field, then $\oint_{\ell} \mathbf{F} \cdot d\mathbf{r} = 0$ for any *closed* curve ℓ (i.e. a curve with terminal point coincides with its initial point).

Notation: If a curve ℓ is closed, we write the line integral as

$$\oint_{\ell} \mathbf{F} \cdot d\mathbf{r}.$$

9.3.15 Example

Let $\mathbf{F}(x,y) = (y^2+3x^2)\mathbf{i}+(2xy)\mathbf{j}$. Show that the line integral $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path and evaluate this integral over the curve C where C is

(i) given by $\mathbf{r}(t) = \cos t\,\mathbf{i} + e^t \sin t\,\mathbf{j}, \;\; t \in [0,\pi]$;

(ii) the unit circle.

Solution:

$$\frac{\partial}{\partial y}(y^2+3x^2) = 2y = \frac{\partial}{\partial x}(2xy)$$

$\therefore \vec{F}$ is conservative.

Let $\vec{F} = \nabla f = f_x \vec{i} + f_y \vec{j}$

$$f_x = y^2 + 3x^2 \Longrightarrow f = xy^2 + x^3 + g(y)$$

$$f_y = 2xy \Longrightarrow 2xy + g'(y) = 2xy$$

$$\Longrightarrow g'(y) = 0$$

$$\Longrightarrow g(y) = K \text{ a constant}$$

$$\therefore f = xy^2 + x^3 + K$$

(i) C joins $\vec{r}(0) = \vec{i}$ to $\vec{r}(\pi) = -\vec{i}$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = f(-\vec{i}) - f(\vec{i}) = (-1+K) - (1+K)$$

$$= \underline{\underline{-2}}$$

(ii) C is closed $\Rightarrow \oint_C \vec{F} \cdot d\vec{r} = \underline{\underline{0}}$

9.4 Green's Theorem

Let D be a bounded region in the xy-plane and ∂D the boundary of D. Suppose $P(x, y)$ and $Q(x, y)$ has continuous partial derivatives on D. Then

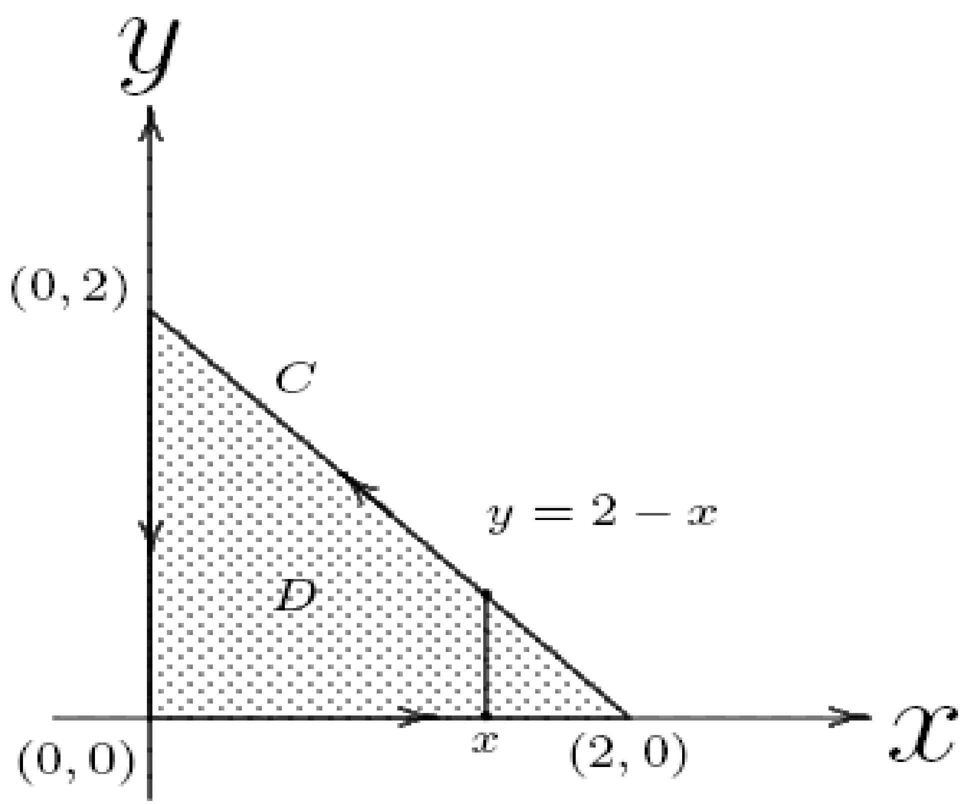
$$\oint_{\partial D} P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

The orientation of ∂D is such that, as one traverses along the boundary in this direction, the region D is always on the left hand side. We call this the **positive orientation** of the boundary.

9.4.1 Example

Evaluate $\oint_C 2xy\, dx + xy^2 dy$, where C is the triangular curve consisting of the line segments from $(0,0)$ to $(2,0)$, from $(2,0)$ to $(0,2)$ and from $(0,2)$ to $(0,0)$.

Solution:

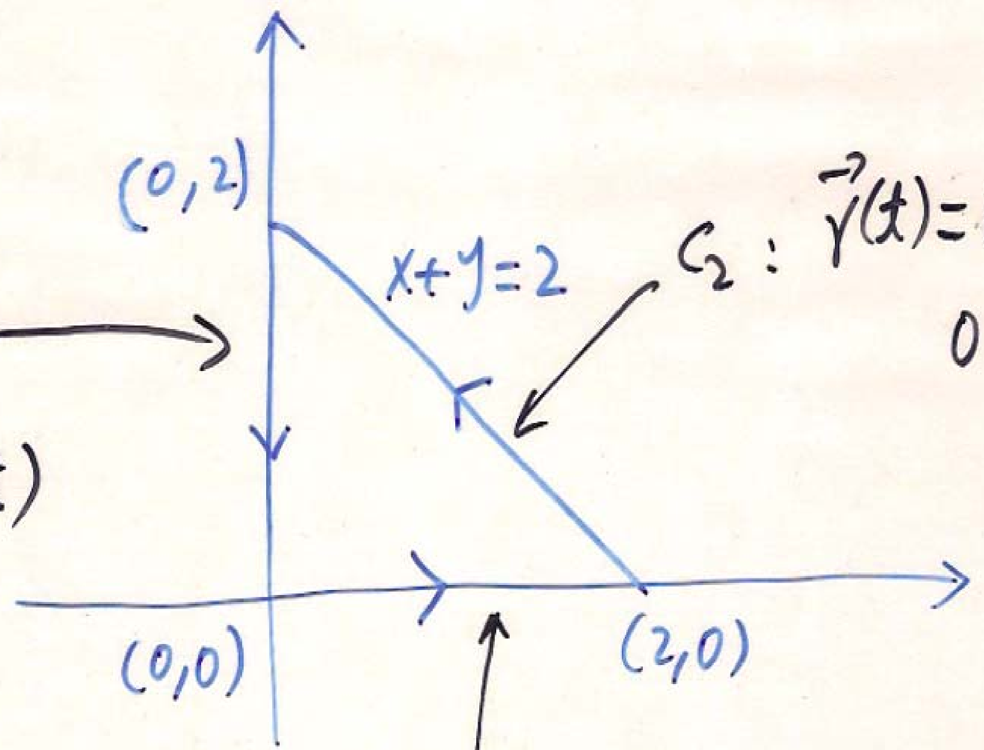


The region D is given by: $0 \leq y \leq 2-x,\ 0 \leq x \leq 2.$

By Green's Theorem,

$$\begin{aligned}
\oint_C 2xy\ dx + xy^2 dy &= \iint_D \left[\frac{\partial(xy^2)}{\partial x} - \frac{\partial 2xy}{\partial y}\right]\ dA \\
&= \iint_D (y^2 - 2x)\, dydx \\
&= \int_0^2 \int_0^{2-x} (y^2 - 2x)\, dydx \\
&= -\frac{4}{3}.
\end{aligned}$$

Check:



$-C_3$

$C_2 : \vec{r}(t) = (2-t, t)$

$0 \le t \le 2$

$C_3 : \vec{r}(t) = (0, t)$

$0 \le t \le 2$

$C_1 : \vec{r}(t) = (t, 0), \quad 0 \le t \le 2$

$$\oint_C 2xy\,dx + xy^2\,dy = \int_{C_1 + C_2 - C_3}$$

$$= \int_{C_1} + \int_{C_2} - \int_{C_3}$$

$$= \int_{C_2}$$

$$= \int_0^2 2(2-t)t\,(-dt) + (2-t)t^2\,dt$$

$$= \int_0^2 (-4t + 4t^2 - t^3)\,dt$$

$$= \left[-2t^2 + \frac{4}{3}t^3 - \frac{1}{4}t^4\right]_0^2$$

$$= \underline{\underline{-\frac{4}{3}}}$$

9.4.2 Example

Evaluate $\oint_C (4y - e^{x^2})dx + (9x + \sin(y^2 - 1))dy$, where

C is the circle $x^2 + y^2 = 4$.

Solution: C bounds the circular disk D of radius 2

and is given the positive orientation.

By Green's Theorem,

$$
\begin{aligned}
&\oint_C (4y - e^{x^2})dx + (9x + \sin y^2 - 1)dy \\
&= \iint_D \left[\frac{\partial(9x + \sin y^2 - 1)}{\partial x} - \frac{\partial(4y - e^{x^2})}{\partial y} \right] dA \\
&= \iint_D 5\, dA = 5 \iint_D dA \\
&= 5 \times (\text{area of } D) = 5(\pi 2^2) = 20\pi .
\end{aligned}
$$

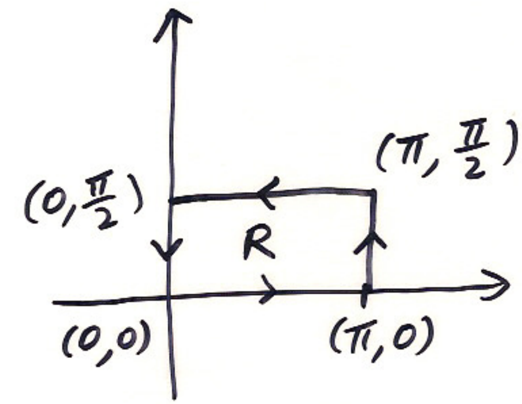
9.4.3 Exercise

Evaluate by Green's Theorem

$$\oint_C e^{-x} \sin y \, dx + e^{-x} \cos y \, dy$$

where C is the rectangle with vertices at $(0, 0)$, $(\pi, 0)$, $(\pi, \pi/2)$, $(0, \pi/2)$.

$$I = \int_C e^{-x}\sin y\,dx + e^{-x}\cos y\,dy$$



Using Green's Theorem,

$$I = \iint_R \left\{\frac{\partial}{\partial x}(e^{-x}\cos y) - \frac{\partial}{\partial y}(e^{-x}\sin y)\right\} dx\,dy$$

$$= \iint_R \left(-2e^{-x}\cos y\right) dx\,dy$$

$$= -2\int_0^{\frac{\pi}{2}}\int_0^{\pi} e^{-x}\cos y\,dx\,dy$$

$$= 2\int_0^{\frac{\pi}{2}}\left[e^{-x}\cos y\right]_{x=0}^{x=\pi}dy$$

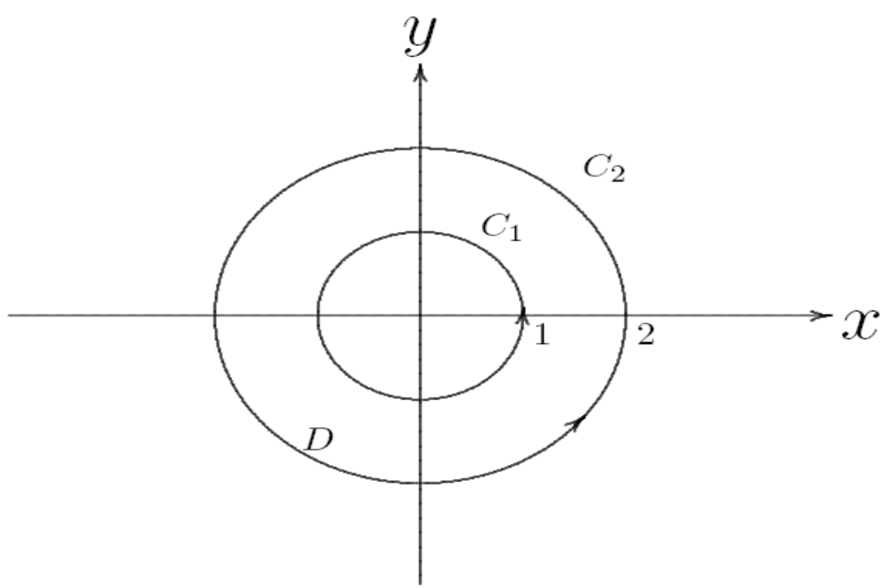
$$= 2(e^{-\pi}-1)\int_0^{\frac{\pi}{2}}\cos y\,dy$$

$$= 2(e^{-\pi}-1)\left[\sin y\right]_0^{\frac{\pi}{2}}$$

$$= \underline{\underline{2(e^{-\pi}-1)}}$$

9.4.4 Example

Let $\mathbf{F}(x,y) = y\mathbf{i} + y\mathbf{j}$ and D a region in xy-plane bounded by the two circles centered at the origin with radius 1 and 2.



Verify Green's Theorem.

Solution:

(i) Compute $\displaystyle\int_{\partial D} \mathbf{F} \cdot d\mathbf{r}$ directly:

The boundary of D is made up of two disjoint curves C_1 and C_2.

Now $C_1: \quad \mathbf{r}_1 = \cos t\mathbf{i} + \sin t\mathbf{j}$ and $C_2: \quad \mathbf{r}_2 = 2\cos t\mathbf{i} + 2\sin t\mathbf{j}$ with $t \in [0, 2\pi]$. Note that the equations give counterclockwise orientation to both curves.

However, to get positive orientation for the boundary of D, the outer boundary should traverse counterclockwise while the inner boundary should traverse clockwise.

Hence $\partial D = C_2 - C_1$.

$$\begin{aligned}
\int_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (\sin t\mathbf{i} + \sin t\mathbf{j}) \cdot (-\sin t\mathbf{i} + \cos t\mathbf{j})\, dt \\
&= \int_0^{2\pi} (-\sin^2 t + \sin t \cos t)\, dt \\
&= \int_0^{2\pi} \frac{1}{2}(\cos 2t - 1 + \sin 2t)\, dt \\
&= \frac{1}{2}\left[\frac{\sin 2t}{2} - t - \frac{\cos 2t}{2}\right]_0^{2\pi} = -\pi
\end{aligned}$$

Similarly, $\displaystyle\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = -4\pi$.

So $\displaystyle\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r} - \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = -3\pi$.

(ii) Using Green's Theorem, we have

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial y}{\partial x} - \frac{\partial y}{\partial y} \right) dA = \iint_D (-1)\, dA.$$

$$= - \text{area of } D$$

$$= - \{ \pi(2^2) - \pi(1^2) \}$$

$$= -3\pi$$