EE2023 Signals & Systems Quiz
Semester 1 AY2011/12
Date: 6 October 2011   Time Allowed: 1.5 hours

Q.1 Consider the periodic signal $x(t)$ given by the expression

$$x(t) = (2+2j)e^{-j3t} - 3je^{-j2t} + 5 + 3je^{j2t} + (2-2j)e^{j3t}$$

(a) What is the fundamental period and fundamental frequency of $x(t)$?

ANSWER:

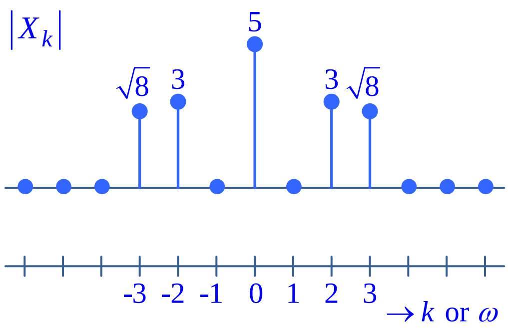
Fundamental frequency: $HCF\{2,3\} = 1\ rad/s\ \left(\text{or } \frac{1}{2\pi}\ Hz\right)$

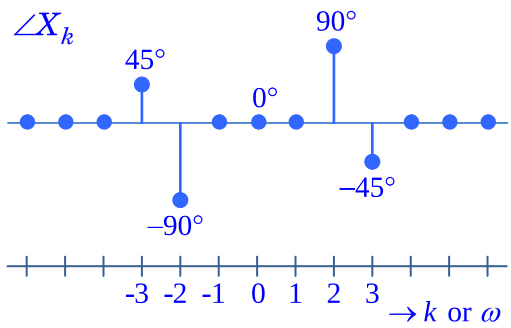
Fundamental period: $2\pi\ s$

(b) Sketch the amplitude and phase spectra of $x(t)$.

ANSWER:

$$x(t) = \sqrt{8}e^{j45^\circ}e^{-j3t} + 3e^{-j90^\circ}e^{-j2t} + 5 + 3e^{j90^\circ}e^{j2t} + \sqrt{8}e^{-j45^\circ}e^{j3t}$$





(c) Is $x(t)$ a real signal ? Justify your answer.

ANSWER:

$x(t)$ is a real signal because its magnitude and phase spectra are even and odd function of frequency, respectively. The Fourier series coefficients satisfy the condition that $X_{-k} = X_k^*$.

(d) What is the power of $x(t)$?

ANSWER:

Using Parseval's theorem:

Power of $x(t) = 2\times\left(\sqrt{8}\right)^2 + 2\times(3)^2 + 5^2 = 16 + 18 + 25 = 59$

Q.2 Derive the Fourier transform of the signal $x(t)$ shown in Figure Q2-1.

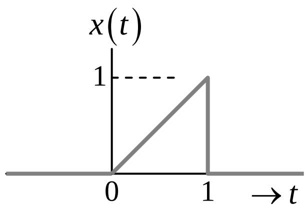


Figure Q2-1

ANSWER:

Method 1:

Let: $\left[y(t) = \text{rect}(t-0.5)\right] \rightleftarrows \left[Y(f) = \text{sinc}(f)e^{-j\pi f}\right]$

$\therefore x(t) = \int_{-\infty}^{t} y(\tau)d\tau - u(t-1)$

$$X(f) = \left[\frac{1}{j2\pi f}Y(f) + \frac{1}{2}\underbrace{Y(0)}_{1}\delta(f)\right] - \left[\frac{1}{2j\pi f} + \frac{1}{2}\delta(f)\right]e^{-j2\pi f}$$

$$= \left[\frac{1}{j2\pi f}\text{sinc}(f)e^{-j\pi f} + \frac{1}{2}\delta(f)\right] - \left[\frac{1}{2j\pi f} + \frac{1}{2}\delta(f)\right]e^{-j2\pi f}$$

$$= \frac{1}{j2\pi f}\text{sinc}(f)e^{-j\pi f} + \frac{1}{2}\delta(f) - \frac{1}{2j\pi f}e^{-j2\pi f} - \frac{1}{2}\delta(f)$$

$$= \frac{1}{j2\pi f}\text{sinc}(f)e^{-j\pi f} - \frac{1}{2j\pi f}e^{-j2\pi f}$$

Method 2:

Let: $\left[y(t) = \frac{dx(t)}{dt} = \text{rect}(t-0.5) - \delta(t-1)\right] \rightleftarrows \left[Y(f) = \text{sinc}(f)e^{-j\pi f} - e^{-j2\pi f}\right]$

$\therefore x(t) = \int_{-\infty}^{t} y(\tau)d\tau$

Since $Y(0) = 0$, we have

$$X(f) = \frac{1}{j2\pi f}Y(f) = \frac{1}{j2\pi f}\text{sinc}(f)e^{-j\pi f} - \frac{1}{j2\pi f}e^{-j2\pi f}$$

Derive the Fourier transform of the periodic signal $y(t)$ shown in Figure Q2-2.

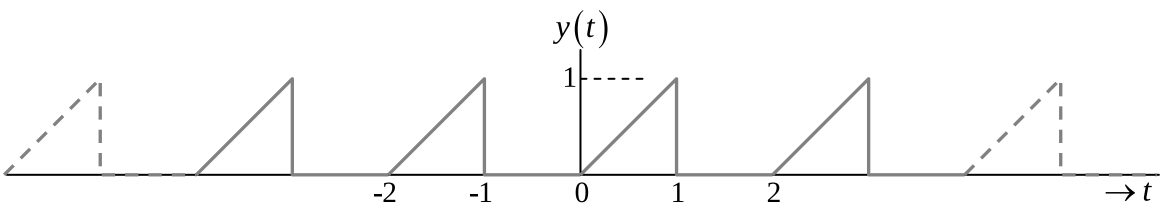


Figure Q2-2

ANSWER:

$$y(t) = \sum_{n=-\infty}^{\infty} x(t-2n) = x(t) * \sum_{n=-\infty}^{\infty} \delta(t-2n)$$

$$Y(f) = X(f)\cdot\frac{1}{2}\sum_{k=-\infty}^{\infty}\delta\left(f-\frac{k}{2}\right) = \left(\frac{1}{j2\pi f}\text{sinc}(f)e^{-j\pi f} - \frac{1}{j2\pi f}e^{-j2\pi f}\right)\cdot\frac{1}{2}\sum_{k=-\infty}^{\infty}\delta\left(f-\frac{k}{2}\right)$$

$$= \sum_{k=-\infty}^{\infty}\frac{1}{j2\pi k}\left(\text{sinc}\left(\frac{k}{2}\right)e^{-j0.5\pi k} - e^{-j\pi k}\right)\delta\left(f-\frac{k}{2}\right)$$

Q.3 A signal is modeled by $x(t) = 2\text{rect}\left(\frac{t}{2}\right) \ast \text{rect}\left(\frac{t}{2}\right)$ where $\ast$ denotes convolution.

Determine the spectrum, $X(f)$, of $x(t)$.

ANSWER:

Applying $\Im\left\{\text{rect}\left(\frac{t}{T}\right)\right\} = T\text{sinc}(fT)$ and the "Convolution in time-domain" property of the Fourier transform, we get

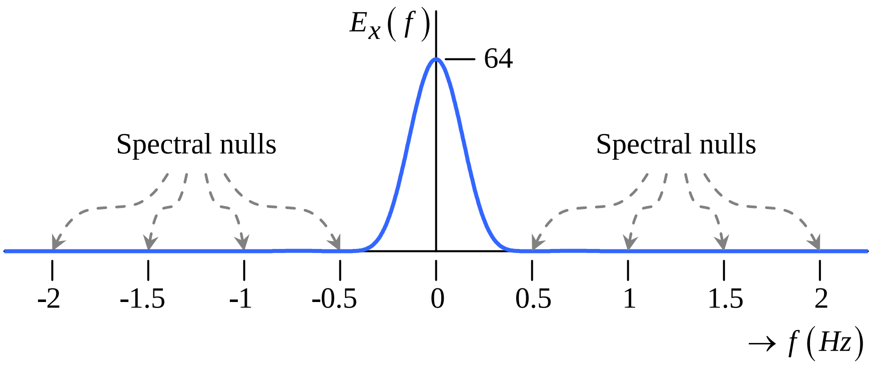
$$X(f) = 2\left(2\text{sinc}(2f)\cdot 2\text{sinc}(2f)\right) = 8\text{sinc}^2(2f).$$

Sketch and label the Energy Spectral Density (ESD) and Power Spectral Density (PSD) of $x(t)$ for frequencies between -2Hz and 2Hz.

ANSWER:

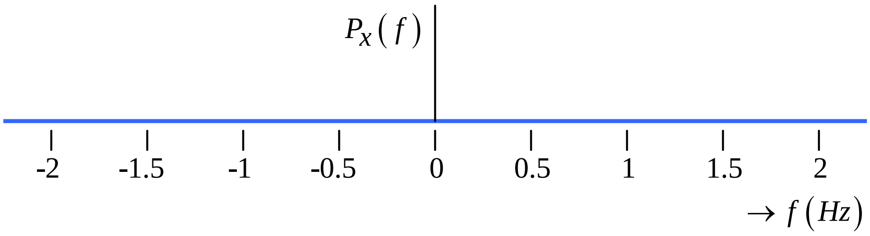
ENERGY SPECTRAL DENSITY (ESD):

$$E_x(f) = |X(f)|^2 = 64\text{sinc}^4(2f)$$



POWER SPECTRAL DENSITY (PSD):

Since the energy of $x(t)$ is finite, i.e. $0 < \int_{-\infty}^{\infty} E_x(f)\,df < \infty$, its average power must be equal to zero, or $\int_{-\infty}^{\infty} P_x(f)\,df = 0$. This together with the fact that $P_x(f) \geq 0$ implies that $P_x(f) = 0;\ \forall f$.



Q.4 Consider 2 signals, $x_1(t) = \sin 4\pi t$, $x_2(t) = 2\cos 8\pi t$. Suppose $y(t) = x_1(t)x_2(t)$.

Write down the Fourier Transforms, $X_1(f)$, $X_2(f)$ and $Y(f)$ where $X_1(f) \rightleftarrows x_1(t)$, $X_2(f) \rightleftarrows x_2(t)$ and $Y(f) \rightleftarrows y(t)$. Sketch their amplitude spectra.

ANSWER:

$$X_1(f) = \frac{1}{j2}\left[\delta(f-2) - \delta(f+2)\right]$$

$$= \begin{cases} 0.5\delta(f-2)e^{-90^\circ} \\ +0.5\delta(f+2)e^{90^\circ} \end{cases}$$

$$X_2(f) = \delta(f-4) + \delta(f+4)$$

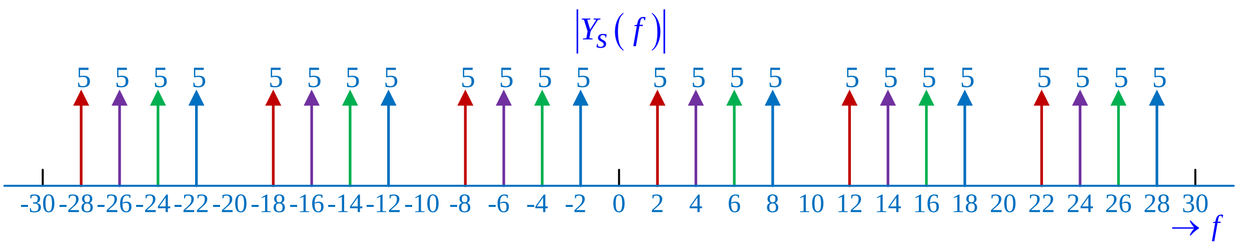
$$Y(f) = X_1(f) \ast X_2(f) = \frac{1}{j2}\left[\delta(f-2) - \delta(f+2)\right] \ast \left[\delta(f-4) + \delta(f+4)\right]$$

$$= \frac{1}{j2}\left[\delta(f-6) - \delta(f-2) + \delta(f+2) - \delta(f+6)\right]$$

$$= 0.5\delta(f-6)e^{-90^\circ} + 0.5\delta(f-2)e^{90^\circ} + 0.5\delta(f+2)e^{-90^\circ} + 0.5\delta(f+6)e^{90^\circ}$$

Assume that $y(t)$ is sampled with a sampling frequency of $f_s = 10$ Hz. Sketch the amplitude spectrum of the sampled signal.

ANSWER:



Can $y(t)$ be reconstructed completely from the sampled signal?

ANSWER:

$y(t)$ cannot be reconstructed completely using a lowpass because of alaising. Nyquist frequency is 12Hz and signal is sampled at 10Hz, which is below Nyquist frequency.

However, two very narrow band tuned filters with center frequencies 2Hz and 6Hz may be used to reconstruct $y(t)$.