NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2002-2003

Solutions to MA1505 MATHEMATICS I

November 2002 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided below.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **TEN (10)** questions and comprises **FORTY ONE (41)** printed pages.
3. Answer **ALL** questions. For each question, write your answer and your working in the space provided inside the booklet following that question.
4. The marks for each question are indicated at the beginning of the question.
5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Matriculation Number:

For official use only. Do not write below this line.

Question	1	2	3	4	5	6	7	8	9	10
Marks										

Question 7 (a) [5 marks]

Solve the differential equation

$$\frac{dy}{dx} - \left(1 + \frac{1}{x}\right) y = 2x + x^2, \qquad x > 0.$$

Answer.

Multiply by an integrating factor R, we have

$$R\frac{dy}{dx} - R\left(1+\frac{1}{x}\right)y = R(2x+x^2).$$

We choose R s.t.

$$R\frac{dy}{dx} - R\left(1+\frac{1}{x}\right)y = \frac{d}{dx}(Ry) = R\frac{dy}{dx} + \frac{dR}{dx}y$$

$$\therefore \quad \frac{dR}{dx} = -R\left(1+\frac{1}{x}\right)$$

$$\therefore \quad \frac{dR}{R} = -\left(1+\frac{1}{x}\right)dx$$

$$\Rightarrow \ln R = -x - \ln x \Rightarrow R = e^{-x-\ln x} = \frac{1}{x}e^{-x}$$

$$\therefore \quad \frac{d}{dx}\left(\frac{1}{x}e^{-x}y\right) = (2x+x^2)\frac{1}{x}e^{-x}$$

$$= 2e^{-x} + xe^{-x}$$

$$\therefore \quad \frac{1}{x}e^{-x}y = -2e^{-x} + \int xe^{-x}dx$$

$$= -2e^{-x} - xe^{-x} - e^{-x} + c$$

$$\therefore \quad \underline{\underline{y = -3x - x^2 + cxe^x}}$$

Question 7 (b) [5 marks]

By using the method of variation of parameters, or otherwise, solve the differential equation

$$y'' + y = \operatorname{cosec} x.$$

You may use the following formulae: $u' = \frac{-ry_2}{w}$, $v' = \frac{ry_1}{w}$, where $w = y_1y_2' - y_1'y_2$.

Answer.

First consider $z^2 + 1 = 0 \Rightarrow z = \pm i$

$\therefore$ $y_1 = \cos x$, $y_2 = \sin x$ are linearly independent solutions of $y'' + y = 0$.

Here $w = y_1 y_2' - y_1' y_2 = \cos^2 x + \sin^2 x = 1$

To find a particular solution of the given equation in the form $uy_1 + vy_2$, we have

$$u' = -\frac{\operatorname{cosec} x \sin x}{1} = -1 \Rightarrow u = -x$$

and $$v' = \frac{\operatorname{cosec} x \cos x}{1} = \frac{\cos x}{\sin x} \Rightarrow v = \ln|\sin x|$$

$\therefore$ The general solution of the given equation is

$$\underline{\underline{y = A\cos x + B\sin x - x\cos x + (\ln|\sin x|)\sin x}}$$

Question 8 (a) [5 marks]

A certain kind of insect has constant birth and death rates per capita. If the birth rate is known to be B% per month, and if the population doubles in a time T, find a formula for the death rate per capita. Also find a formula for the time, expressed in terms of T, needed for the population to triple.

Answer.

Let N denote the population at time t.

Let $D\%$ denote the death rate per month.

$$\therefore \frac{dN}{dt} = \{(B-D)/100\}N \Rightarrow N = Ce^{\frac{B-D}{100}t}$$

$$\because N(T) = 2N(0)$$

$$\therefore Ce^{\frac{B-D}{100}T} = 2C$$

$$\therefore \frac{B-D}{100}T = \ln 2$$

$$\therefore D = B - \frac{100\ln 2}{T}$$

Next $N(t) = 3N(0)$

$$\Rightarrow Ce^{\frac{B-D}{100}t} = 3C$$

$$\Rightarrow \frac{B-D}{100}t = \ln 3$$

$$\Rightarrow \frac{\ln 2}{T}t = \ln 3$$

$$\Rightarrow t = \frac{\ln 3}{\ln 2}T$$

Question 8 (b) [5 marks]

Solve the differential equation

$$y'' - 2y' + 5y = 0, \qquad y(0) = y'(0) = 1.$$

Answer.

Consider $z^2 - 2z + 5 = 0$

$$\therefore \quad z = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$$

$$\therefore \quad y = e^x (A\cos 2x + B\sin 2x)$$

$$\therefore \quad y' = e^x (A\cos 2x + B\sin 2x - 2A\sin 2x + 2B\cos 2x)$$

$$\therefore \quad y(0) = y'(0) = 1$$

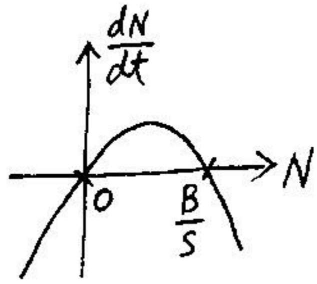
$$\Rightarrow \quad A = 1 \text{ and } A + 2B = 1$$

$$\Rightarrow \quad A = 1 \text{ and } B = 0.$$

Answer: $\underline{\underline{y = e^x \cos 2x}}$

Question 9 (a) [5 marks]

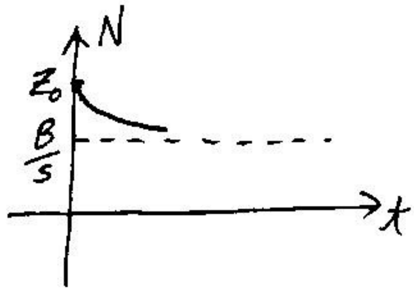
The population of zebras as a function of time in a certain national park, denoted by N, follows a logistic model given by $\frac{dN}{dt} = BN - SN^2$, where B and S are two positive constants. At time $t = 0$, the zebra population is Z_0, but in the long run the zebra population tends to Z_∞, with $Z_\infty < Z_0$. Find a formula for N in terms of Z_0, Z_∞, B and the time t.



Answer.

Since $\frac{dN}{dt} = BN - SN^2 = SN\left(\frac{B}{S} - N\right)$

$\therefore N \equiv \frac{B}{S}$ is the stable equilibrium solution.



$\because Z_0 > Z_\infty$

$\therefore Z_\infty = \frac{B}{S}$ and $N > Z_\infty \cdots ①$

Next $dt = \frac{dN}{BN - SN^2} = \left(\frac{1/B}{N} + \frac{S/B}{B - SN}\right)dN$

$\therefore t + C = \frac{1}{B}\ln N - \frac{1}{B}\ln|B - SN|$

$\therefore \frac{N}{|B - SN|} = ke^{Bt}$ where k = constant.

$N(0) = Z_0 \Rightarrow k = \frac{Z_0}{|B - SZ_0|} \Rightarrow \frac{N}{|B - SN|} = \frac{Z_0}{|B - SZ_0|}e^{Bt} \cdots ②$

$\because Z_0 > Z_\infty = \frac{B}{S}$ and $N > Z_\infty$ (by ①)

$\therefore ② \Rightarrow \frac{N}{SN - B} = \frac{Z_0}{SZ_0 - B}e^{Bt} \Rightarrow N = \dfrac{Z_\infty}{1 - \left(1 - \frac{Z_\infty}{Z_0}\right)e^{-Bt}}$

Question 9 (b) [5 marks]

Solve the differential equation

$$(1+y)\,y' + (\tan x)\,y^2 = 0, \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

Answer.

$$\because \quad (1+y)\frac{dy}{dx} = -(\tan x)\,y^2$$

$$\therefore \quad \left(\frac{1}{y^2} + \frac{1}{y}\right) dy = (-\tan x)\,dx$$

$$-\frac{1}{y} + \ln|y| = \ln(\cos x) + C$$

Question 10 (a) [5 marks]

Find the Laplace transform of the function

$$f(t) = \left(2 + 3e^t\right)^2.$$

You may use the following formulae: $L(1) = \frac{1}{s}$, $L(e^{at}) = \frac{1}{s-a}$.

Answer.

$$L(f) = L\left(4 + 12e^t + 9e^{2t}\right)$$

$$= 4L(1) + 12L(e^t) + 9L(e^{2t})$$

$$= \frac{4}{s} + \frac{12}{s-1} + \frac{9}{s-2}$$

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2003-2004

MA1505 MATHEMATICS I

November 2003 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided below.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **TEN (10)** questions and comprises **FORTY ONE (41)** printed pages.
3. Answer **ALL** questions. For each question, write your answer in the box and your working in the space provided inside the booklet following that question.
4. The marks for each question are indicated at the beginning of the question.
5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Matriculation Number:

For official use only. Do not write below this line.

Question	1	2	3	4	5	6	7	8	9	10
Marks										

Question 7 (a) [5 marks]

Find $f(x)$ which satisfies the differential equation

$$f'(x) + \frac{2}{x} f(x) = 8x, \qquad x > 0$$

and the initial condition $f(1) = 3$.

Answer 7(a)	$2x^2 + \frac{1}{x^2}$

(Show your working below and on the next page.)

$uf' + u\left(\frac{2}{x}\right)f = 8ux$

Let $uf' + u\frac{2}{x}f = \frac{d}{dx}(uf) = u'f + uf'$

$\therefore \quad u' = \frac{2u}{x}$

$\therefore \quad \frac{du}{u} = \frac{2\,dx}{x}$

$\therefore \ln u = 2\ln x = \ln x^2$

$\therefore u = x^2$

$\therefore \quad \frac{d}{dx}(x^2 f) = 8x^3$

$\therefore x^2 f = 2x^4 + C.$

$f(1) = 3 \Rightarrow 3 = 2 + C \Rightarrow C = 1$

$\therefore f(x) = \frac{2x^4+1}{x^2} = \underline{\underline{2x^2 + \frac{1}{x^2}}}$

Question 7 (b) [5 marks]

Solve the differential equation

$$y' = \frac{2y^4 + x^4}{xy^3}, \quad y(1) = 2.$$

Answer 7(b)	$y^4 = 17x^8 - x^4$

(Show your working below and on the next page.)

Let $y = ux$

$$\therefore u'x + u = \frac{2u^4x^4 + x^4}{u^3x^4} = \frac{2u^4+1}{u^3}$$

$$\therefore u'x = \frac{u^4+1}{u^3}$$

$$\therefore \frac{u^3}{u^4+1}du = \frac{dx}{x}$$

$$\therefore \frac{1}{4}\ln(u^4+1) = \ln x + C_1$$

$$\therefore u^4+1 = e^{4\ln x + C_2} = Cx^4$$

$$\therefore y^4 + x^4 = Cx^8$$

$y(1) = 2 \Rightarrow 16 + 1 = C \Rightarrow C = 17$

$$\therefore \underline{\underline{y^4 = 17x^8 - x^4}}$$

Question 8 (a) [5 marks]

Find the general solution of the differential equation

$$y'' - 2y' + 10y = 0.$$

Answer 8(a)	$y = C_1 e^x \cos 3x + C_2 e^x \sin 3x$

(Show your working below and on the next page.)

$$\lambda^2 - 2\lambda + 10 = 0$$

$$\Rightarrow \lambda = \frac{2 \pm \sqrt{4-40}}{2} = 1 \pm 3i$$

$$\therefore \; y = C_1 e^x \cos 3x + C_2 e^x \sin 3x$$

Question 8 (b) [5 marks]

Solve the differential equation

$$y'' - y = e^x, \quad y(0) = 2, \quad y'(0) = \frac{1}{2}.$$

Answer 8(b)	$y = e^x + e^{-x} + \frac{1}{2} x e^x$

(Show your working below and on the next page.)

$\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$

$\therefore$ The general solution of $y'' - y = 0$ is

$$y = c_1 e^x + c_2 e^{-x}.$$

Try $y = Axe^x$.

$\therefore$ $y' = Axe^x + Ae^x$

$y'' = Axe^x + 2Ae^x$

$\therefore$ $y'' - y = e^x \Rightarrow 2Ae^x = e^x \Rightarrow A = \frac{1}{2}$

$\therefore$ $y = c_1 e^x + c_2 e^{-x} + \frac{1}{2} x e^x$

$\therefore$ $y' = c_1 e^x - c_2 e^{-x} + \frac{1}{2} x e^x + \frac{1}{2} e^x$

$$\therefore \left.\begin{aligned} y(0) = 2 &\Rightarrow c_1 + c_2 = 2 \\ y'(0) = \tfrac{1}{2} &\Rightarrow c_1 - c_2 + \tfrac{1}{2} = \tfrac{1}{2} \end{aligned}\right\} \Rightarrow c_1 = c_2 = 1$$

$\therefore$ $y = e^x + e^{-x} + \frac{1}{2} x e^x$

Question 9 (a) [5 marks]

A tank initially holds 100 litres of a brine solution containing 20 kg of salt. Starting from time $t = 0$, fresh water is continuously poured into the tank at the rate of 5 litres/minute, while the well-stirred mixture leaves the tank continuously at the same rate. Find an expression for the amount (in kg) of salt in the tank at any time t.

Answer 9(a)	$20e^{-t/20}$

(Show your working below and on the next page.)

Let Q kg = amount of salt at time t.

$$\therefore \frac{dQ}{dt} = -\frac{5}{100}Q = -\frac{Q}{20}$$

$$\therefore \frac{dQ}{Q} = -\frac{1}{20}dt$$

$$\therefore \ln Q = -\frac{1}{20}t + C$$

$$\therefore Q = Ce^{-t/20}$$

At $t=0$, we have $Q=20$

$$\therefore 20 = C$$

$$\therefore \underline{\underline{Q = 20e^{-t/20}}}$$

Question 9 (b) [5 marks]

Find the inverse Laplace transform

$$L^{-1}\left\{\frac{1}{s\left(s^2+4\right)}\right\}.$$

Answer 9(b)	$\frac{1}{4} - \frac{1}{4}\cos 2t$

(Show your working below and on the next page.)

Let $\dfrac{1}{s(s^2+4)} = \dfrac{A}{s} + \dfrac{Bs+C}{s^2+4}$

$\therefore\ 1 = A(s^2+4) + Bs^2 + Cs = (A+B)s^2 + Cs + 4A$

$\therefore\ A = \frac{1}{4},\ B = -\frac{1}{4},\ C = 0$

$\therefore\ L^{-1}\left\{\dfrac{1}{s(s^2+4)}\right\} = L^{-1}\left\{\dfrac{1}{4}\cdot\dfrac{1}{s} - \dfrac{1}{4}\dfrac{s}{s^2+4}\right\}$

$= \frac{1}{4}L^{-1}\left(\frac{1}{s}\right) - \frac{1}{4}L^{-1}\left(\frac{s}{s^2+4}\right)$

$= \frac{1}{4} - \frac{1}{4}\cos 2t$

Question 10 (a) [5 marks]

Given that the Laplace transform of $\sqrt{t}$ is $L\left(\sqrt{t}\right) = \frac{1}{2}\sqrt{\pi}s^{-3/2}$. Find the Laplace transform of the function

$$f(t) = t^{3/2}.$$

Answer 10(a)	$\frac{3}{4}\sqrt{\pi}\, s^{-5/2}$

(Show your working below and on the next page.)

$$\int_0^t \sqrt{u}\,du = \frac{2}{3}u^{3/2}\Big|_0^t = \frac{2}{3}t^{3/2}$$

Using the formula $L\left(\int_0^t g(u)\,du\right) = \dfrac{L(g(t))}{s}$

with $g(u) = \sqrt{u}$, we have

$$L\left(\int_0^t \sqrt{u}\,du\right) = \frac{L(\sqrt{t})}{s} = \frac{1}{2}\sqrt{\pi}\, s^{-5/2}$$

$$\therefore\ L\left(\frac{2}{3}t^{3/2}\right) = \frac{1}{2}\sqrt{\pi}\, s^{-5/2}$$

$$\therefore\ L\left(t^{3/2}\right) = \frac{3}{4}\sqrt{\pi}\, s^{-5/2}$$

Question 10 (b) [5 marks]

Find the functions $x(t)$ and $y(t)$ which satisfy

$$\begin{cases} \frac{dx}{dt} = -2x + y \\ \frac{dy}{dt} = 2x - 3y \end{cases}$$

and the initial conditions $x(0) = 1$, $y(0) = 0$.

Answer 10(b)	$x = \frac{2}{3}e^{-t} + \frac{1}{3}e^{-4t}$ $y = \frac{2}{3}e^{-t} - \frac{2}{3}e^{-4t}$

(Show your working below and on the next page.)

Taking Laplace transform with $L(x) = X$, $L(y) = Y$, we have

$$\begin{cases} L(\frac{dx}{dt}) = -2X + Y \\ L(\frac{dy}{dt}) = 2X - 3Y \end{cases} \Rightarrow \begin{cases} sX - x(0) = -2X + Y \\ sY - y(0) = 2X - 3Y \end{cases}$$

$$\Rightarrow \begin{cases} (s+2)X - Y = 1 \\ -2X + (s+3)Y = 0 \end{cases}$$

$$\Rightarrow \begin{cases} X = \frac{s+3}{s^2+5s+4} = \frac{2/3}{s+1} + \frac{1/3}{s+4} \\ Y = \frac{2}{s^2+5s+4} = \frac{2/3}{s+1} + \frac{-2/3}{s+4} \end{cases}$$

$$\Rightarrow \begin{cases} x = \frac{2}{3}e^{-t} + \frac{1}{3}e^{-4t} \\ y = \frac{2}{3}e^{-t} - \frac{2}{3}e^{-4t} \end{cases}$$

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2004-2005

MA1505 MATHEMATICS I

November 2004 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided below.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **TEN (10)** questions and comprises **FORTY ONE (41)** printed pages.
3. Answer **ALL** questions. For each question, write your answer in the box and your working in the space provided inside the booklet following that question.
4. The marks for each question are indicated at the beginning of the question.
5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Matriculation Number:

For official use only. Do not write below this line.

Question	1	2	3	4	5	6	7	8	9	10
Marks										

Question 7 (a) [5 marks]

Solve the differential equation

$$x\frac{dy}{dx} - y = 2x^2 \sin 2x$$

with the initial condition $y = \pi$ when $x = \pi$.

Answer 7(a)	$y = x(2 - \cos 2x)$

(Show your working below and on the next page.)

$$u\frac{dy}{dx} - u\frac{y}{x} = 2ux \sin 2x$$

Let $u\frac{dy}{dx} - u\frac{y}{x} = \frac{d}{dx}(uy) = u\frac{dy}{dx} + \frac{du}{dx}y$

$\therefore \frac{du}{dx} = -\frac{u}{x} \Rightarrow \frac{du}{u} = -\frac{dx}{x} \Rightarrow \ln u = -\ln x$

$\therefore u = \frac{1}{x}$

$\therefore \frac{d}{dx}\left(\frac{1}{x}y\right) = 2\frac{1}{x}x \sin 2x = 2\sin 2x$

$\therefore \frac{1}{x}y = -\cos 2x + C$

$y(\pi) = \pi \Rightarrow 1 = -\cos 2\pi + C \Rightarrow C = 2$

$\therefore \underline{\underline{y = x(2 - \cos 2x)}}$

Question 7 (b) [5 marks]

Solve the differential equation

$$y' = \frac{x^2 + xy + y^2}{x^2}$$

with the initial condition $y = 0$ when $x = 1$.

Answer 7(b)	$y = x\tan(\ln\lvert x\rvert)$ $\Leftrightarrow \tan^{-1}\left(\frac{y}{x}\right) = \ln\lvert x\rvert$

(Show your working below and on the next page.)

Let $y = vx \Rightarrow y' = v'x + v$

$\therefore v'x + v = \dfrac{x^2 + vx^2 + v^2x^2}{x^2} = 1 + v + v^2$

$\therefore \dfrac{dv}{1+v^2} = \dfrac{dx}{x}$

$\therefore \tan^{-1} v = \ln\lvert x\rvert + C$

$\therefore \tan^{-1}\left(\frac{y}{x}\right) = \ln\lvert x\rvert + C$

$y(1) = 0 \Rightarrow C = 0$

$\therefore \underline{\underline{y = x\tan(\ln\lvert x\rvert)}}$

Question 8 (a) [5 marks]

Solve the differential equation

$$9y'' - 6y' + y = 0$$

with the initial conditions that $y = 1$ and $y' = 3$ when $x = 0$.

Answer 8(a)	$y = e^{\frac{1}{3}x} + \frac{8}{3} x e^{\frac{1}{3}x}$

(Show your working below and on the next page.)

$$9\lambda^2 - 6\lambda + 1 = 0$$

$$\Rightarrow (3\lambda - 1)^2 = 0$$

$\Rightarrow \lambda = \frac{1}{3}$ is a double root.

$$\therefore y = Ae^{\frac{1}{3}x} + Bxe^{\frac{1}{3}x}$$

$$y' = \frac{1}{3}Ae^{\frac{1}{3}x} + Be^{\frac{1}{3}x} + \frac{1}{3}Bxe^{\frac{1}{3}x}$$

$$y(0)=1,\ y'(0)=3 \Rightarrow A = 1,\ 3 = \frac{1}{3}A + B$$

$$\Rightarrow A = 1,\ B = \frac{8}{3}$$

$$\therefore \underline{\underline{y = e^{\frac{1}{3}x} + \frac{8}{3} x e^{\frac{1}{3}x}}}$$

Question 8 (b) [5 marks]

Solve the differential equation

$$y'' - 5y' + 6y = 18x^2.$$

Answer 8(b)	$y = Ae^{2x} + Be^{3x} + 3x^2 + 5x + \frac{19}{6}$

(Show your working below and on the next page.)

$$\lambda^2 - 5\lambda + 6 = 0 \Rightarrow \lambda = 2 \text{ or } 3.$$

Try $y = \tilde{A}x^2 + \tilde{B}x + \tilde{C}$

$y' = 2\tilde{A}x + \tilde{B}$

$y'' = 2\tilde{A}$

$\therefore\ 2\tilde{A} - 5(2\tilde{A}x + \tilde{B}) + 6(\tilde{A}x^2 + \tilde{B}x + \tilde{C}) = 18x^2$

$\therefore\ 6\tilde{A} = 18,\quad -10\tilde{A} + 6\tilde{B} = 0,\quad 2\tilde{A} - 5\tilde{B} + 6\tilde{C} = 0$

$\therefore\ \tilde{A} = 3,\quad \tilde{B} = 5,\quad \tilde{C} = \frac{19}{6}$

$\therefore\ y = Ae^{2x} + Be^{3x} + 3x^2 + 5x + \frac{19}{6}$

Question 9 (a) [5 marks]

Find the Laplace transform

$$L\,(t\cos 2t).$$

Answer 9(a)	$\dfrac{s^2-4}{(s^2+4)^2}$

(Show your working below and on the next page.)

Let $f(t) = t\cos 2t$, $\quad \therefore f(0) = 0$

$\therefore f'(t) = \cos 2t - 2t\sin 2t, \quad f'(0) = 1$

$f''(t) = -2\sin 2t - 2\sin 2t - 4t\cos 2t$

$\quad = -4\sin 2t - 4t\cos 2t$

$\because L(f'') = s^2 L(f) - sf(0) - f'(0)$

$\therefore -4L(\sin 2t) - 4L(t\cos 2t) = s^2 L(t\cos 2t) - 1$

$\therefore (s^2+4)L(t\cos 2t) = 1 - 4L(\sin 2t)$

$$= 1 - \frac{8}{s^2+4}$$

$$= \frac{s^2-4}{s^2+4}$$

$$\therefore L(t\cos 2t) = \frac{s^2-4}{(s^2+4)^2}$$

Question 9 (b) [5 marks]

Find the inverse Laplace transform

$$L^{-1}\left(\frac{2s^2-4}{(s-2)(s+1)(s-3)}\right).$$

Answer 9(b)	$-\frac{4}{3}e^{2t}-\frac{1}{6}e^{-t}+\frac{7}{2}e^{3t}$

(Show your working below and on the next page.)

Let $\dfrac{2s^2-4}{(s-2)(s+1)(s-3)} = \dfrac{A}{s-2} + \dfrac{B}{s+1} + \dfrac{C}{s-3}$

$\therefore\ 2s^2-4 = A(s+1)(s-3) + B(s-2)(s-3) + C(s-2)(s+1)$

$s=2 \Rightarrow 4 = -3A \Rightarrow A = -4/3$

$s=-1 \Rightarrow -2 = 12B \Rightarrow B = -1/6$

$s=3 \Rightarrow 14 = 4C \Rightarrow C = 7/2$

$$\therefore L^{-1}\left(\frac{2s^2-4}{(s-2)(s+1)(s-3)}\right) = L^{-1}\left(\frac{-4/3}{s-2} + \frac{-1/6}{s+1} + \frac{7/2}{s-3}\right)$$

$$= -\frac{4}{3}e^{2t} - \frac{1}{6}e^{-t} + \frac{7}{2}e^{3t}$$

Question 10 (a) [5 marks]

Solve the differential equation

$$\frac{d^2x}{dt^2} = 12\,(t-1)^2\,U\,(t-1)\ ,\ \ x'\,(0) = x\,(0) = 0$$

where $U\,(t-1) = \begin{cases} 0 & \text{if } t < 1 \\ 1 & \text{if } t > 1. \end{cases}$

Answer 10(a)	$x = (t-1)^4\,U(t-1)$

(Show your working below and on the next page.)

$$L\left(\frac{d^2x}{dt^2}\right) = L\{12(t-1)^2 U(t-1)\}$$

$$s^2\bar{X} - s\,x(0) - x'(0) = 12\,e^{-s}\left(\frac{2!}{s^3}\right), \text{ where } \bar{X} = L(x).$$

$$s^2\bar{X} = \frac{24}{s^3}e^{-s}$$

$$\bar{X} = \frac{24}{s^5}e^{-s}$$

$$x = L^{-1}\left(\frac{24}{s^5}e^{-s}\right) = \underline{\underline{(t-1)^4\,U(t-1)}}$$

Question 10 (b) [5 marks]

Find the functions $x(t)$ and $y(t)$ which satisfy

$$\begin{cases} \frac{dx}{dt} - y = \frac{t^2}{2} \\ x - \frac{dy}{dt} = 0 \end{cases}$$

and the initial conditions $x(0) = 0$, $y(0) = 1$.

Answer 10(b)	$x = e^t - e^{-t} - t$ $y = e^t + e^{-t} - \frac{t^2}{2} - 1$

(Show your working below and on the next page.)

Taking the Laplace transform, we have

$$\begin{cases} s\bar{X} - x(0) - \bar{Y} = \frac{1}{s^3} \\ \bar{X} - \{s\bar{Y} - y(0)\} = 0 \end{cases} \Rightarrow \begin{cases} s\bar{X} - \bar{Y} = \frac{1}{s^3} \\ \bar{X} - s\bar{Y} = -1 \end{cases}$$

$$\bar{X} = \frac{\begin{vmatrix} \frac{1}{s^3} & -1 \\ -1 & -s \end{vmatrix}}{\begin{vmatrix} s & -1 \\ 1 & -s \end{vmatrix}} = \frac{-\frac{1}{s^2} - 1}{-s^2 + 1} = \frac{s^2+1}{s^2(s^2-1)} = \frac{1}{s-1} - \frac{1}{s+1} - \frac{1}{s^2}$$

$$x = L^{-1}(\bar{X}) = \underline{\underline{e^t - e^{-t} - t}}$$

$$y = \frac{dx}{dt} - \frac{t^2}{2} = \underline{\underline{e^t + e^{-t} - 1 - \frac{t^2}{2}}}$$

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2005-2006

MA1505 MATHEMATICS I

November 2005 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided below.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **EIGHT (8)** questions and comprises **THIRTY THREE (33)** printed pages.
3. Answer **ALL** questions. For each question, write your answer in the box and your working in the space provided inside the booklet following that question.
4. The marks for each question are indicated at the beginning of the question.
5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Matriculation Number:

For official use only. Do not write below this line.

Question	1	2	3	4	5	6	7	8
Marks								

Question 5 (a) [5 marks]

The number of bacteria in a certain bacterium culture is 1000 at a certain initial time. Two hours after the initial time there are 1200 of them. Assuming constant birth and death rates per capita, how many bacteria will we have 6 hours after the initial time?

Answer 5(a)	1728

(Show your working below and on the next page.)

$$\frac{dN}{dt} = (B - D)N$$

$$\therefore \quad N = N_0 e^{(B-D)t}$$

$$t=0,\ N=1000 \Rightarrow N_0 = 1000$$

$$\therefore \quad N = 1000\, e^{(B-D)t}$$

$$t=2,\ N=1200 \Rightarrow 1200 = 1000\, e^{2(B-D)}$$

$$\Rightarrow B - D = \frac{1}{2} \ln \frac{12}{10}$$

$\therefore$ at $t=6$, we have

$$N = 1000\, e^{\left(\frac{1}{2}\ln\frac{12}{10}\right)6}$$

$$= 1000 \left(\frac{12}{10}\right)^3$$

$$= 12^3$$

$$= \underline{\underline{1728}}$$

Question 5 (b) [5 marks]

Solve the differential equation

$$\left(y^2 - x^2\right)\frac{dy}{dx} + 2xy = 0$$

with $x > 0$ and the initial condition $y = 1$ when $x = \sqrt{2}$.

Answer 5(b)	$x^2+y^2=3y$

(Show your working below and on the next page.)

Let $y = vx \Rightarrow \dfrac{dy}{dx} = \dfrac{dv}{dx}x + v$

$\therefore \dfrac{dv}{dx}x + v = \dfrac{2xy}{x^2-y^2} = \dfrac{2x^2v}{x^2-v^2x^2} = \dfrac{2v}{1-v^2}$

$\therefore \dfrac{dv}{dx}x = \dfrac{2v}{1-v^2} - v = \dfrac{v+v^3}{1-v^2} = \dfrac{v(1+v^2)}{1-v^2}$

$\therefore \dfrac{1-v^2}{v(1+v^2)}dv = \dfrac{dx}{x}$

$\therefore \left(\dfrac{1}{v} - \dfrac{2v}{1+v^2}\right)dv = \dfrac{dx}{x}$

$\therefore \ln|v| - \ln|1+v^2| = \ln x + \ln C_1$

$\therefore \dfrac{v}{1+v^2} = Cx \Rightarrow \dfrac{y/x}{1+(y/x)^2} = Cx \Rightarrow \dfrac{y}{x^2+y^2} = C$

$\therefore y = C(x^2+y^2)$

$x=\sqrt{2}, y=1 \Rightarrow \dfrac{1}{3} = C$

$\therefore \underline{\underline{x^2+y^2=3y}}$

Question 6 (a) [5 marks]

A water tank has a capacity of 120 litres. Initially the tank contains 90 litres of a salt solution with a concentration of 1 gram of salt per litre. A tap is then turned on and a salt solution with a concentration of 2 grams of salt per litre enters the tank at a rate of 4 litres per minute. At the same time when the tap is turned on, a drain is also turned on and the well-stirred mixture flows out of the tank at a rate of 3 litres per minute. How much salt in grams (round off your answer to the nearest integer) does the tank contain at the moment when it is full ?

Answer 6(a)	202 grams

(Show your working below and on the next page.)

Suppose the tank contains x grams of salt at t minutes pass initial time. Note that at this time, the tank contains $90+t(4-3) = 90+t$ litres of salt solution.

$$\therefore \frac{dx}{dt} = 8 - \frac{3x}{90+t} \Rightarrow \frac{dx}{dt} + \frac{3}{90+t}x = 8$$

$$\text{Integrating factor} = e^{\int \frac{3dt}{90+t}} = e^{3\ln(90+t)} = (90+t)^3$$

$$\therefore \frac{d}{dt}\{x(90+t)^3\} = 8(90+t)^3$$

$$\therefore x(90+t)^3 = 2(90+t)^4 + C$$

$$\therefore x = 2(90+t) + \frac{C}{(90+t)^3}$$

$t=0$, $x=90 \Rightarrow C = -(90)^4 \Rightarrow x = \dfrac{2(90+t)^4 - (90)^4}{(90+t)^3}$

The tank is full when $90+t=120 \Rightarrow t=30$

$$\Rightarrow x = \frac{2(120)^4-(90)^4}{(120)^3} = 202\tfrac{1}{32}$$

$$\approx \underline{\underline{202}}$$

Question 6 (b) [5 marks]

Solve the differential equation

$$y'' - y' - 2y = 0$$

with the initial conditions that $y = 1$ and $y' = 5$ when $x = 0$.

Answer 6(b)	$y = 2e^{2x} - e^{-x}$

(Show your working below and on the next page.)

$$\lambda^2 - \lambda - 2 = 0$$

$$\Rightarrow (\lambda - 2)(\lambda + 1) = 0 \Rightarrow \lambda = -1, 2.$$

$$\therefore \; y = Ae^{-x} + Be^{2x}$$

$$\therefore \; y' = -Ae^{-x} + 2Be^{2x}$$

$$y(0) = 1, \; y'(0) = 5 \Rightarrow \; A + B = 1$$

$$-A + 2B = 5$$

$$\Rightarrow B = 2, \; A = -1$$

$$\therefore \; \underline{\underline{y = -e^{-x} + 2e^{2x}}}$$

Question 7 (a) [5 marks]

Solve the differential equation

$$y'' - 5y' + 6y = 4e^{2x}$$

with the initial conditions that $y = 0$ and $y' = 1$ when $x = 0$.

Answer 7(a)	$y = 5e^{3x} - 5e^{2x} - 4xe^{2x}$

(Show your working below and on the next page.)

$\lambda^2 - 5\lambda + 6 = 0$

$(\lambda - 3)(\lambda - 2) = 0$

$\lambda = 2$ or 3.

Try $y = (Ax + B)e^{2x}$

$\therefore y' = Ae^{2x} + 2(Ax+B)e^{2x}$

$y'' = 4Ae^{2x} + 4(Ax+B)e^{2x}$

$y'' - 5y' + 6y = -Ae^{2x}$

$\therefore A = -4$

$\therefore y = c_1 e^{2x} + c_2 e^{3x} - 4xe^{2x}$

$y' = 2c_1 e^{2x} + 3c_2 e^{3x} - 4e^{2x} - 8xe^{2x}$

$y(0) = 0 \Rightarrow c_1 + c_2 = 0$

$y'(0) = 1 \Rightarrow 2c_1 + 3c_2 - 4 = 1$

$\therefore c_1 = -5, \ c_2 = 5$

$\therefore \underline{\underline{y = 5e^{3x} - 5e^{2x} - 4xe^{2x}}}$

Question 7 (b) [5 marks]

Solve the differential equation

$$y'' + y = \tan^2 x$$

with the initial conditions that $y = 1$ and $y' = 1$ when $x = 0$.

Answer 7(b)	$y = 3\cos x + \sin x - 2 + (\sin x)\ln\lvert\sec x + \tan x\rvert$

(Show your working below and on the next page.)

$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$

Let $y_1 = \cos x$, $y_2 = \sin x$

$\therefore W(y_1, y_2) = \begin{vmatrix} y_1 & y_1' \\ y_2 & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{vmatrix} = 1$

$u = -\int \frac{y_2 r}{W(y_1, y_2)} dx = -\int \frac{\sin^3 x}{\cos^2 x} dx = \int \frac{1-\cos^2 x}{\cos^2 x} d(\cos x)$

$= -\frac{1}{\cos x} - \cos x$

$v = \int \frac{y_1 r}{W(y_1, y_2)} dx = \int \frac{\sin^2 x}{\cos x} dx = \int \frac{1-\cos^2 x}{\cos x} dx$

$= \int (\sec x - \cos x)\, dx = \ln\lvert\sec x + \tan x\rvert - \sin x$

$\therefore y = C_1 \cos x + C_2 \sin x + u y_1 + v y_2$

$= C_1 \cos x + C_2 \sin x - 2 + (\sin x)\ln\lvert\sec x + \tan x\rvert$

$y(0) = 1 \Rightarrow C_1 = 3$

$y'(0) = 1 \Rightarrow C_2 = 1$

$\therefore \underline{\underline{y = 3\cos x + \sin x - 2 + (\sin x)\ln\lvert\sec x + \tan x\rvert}}$

Question 8 (a) [5 marks]

Find the Laplace transform

$$L\left\{(\sin t - \cos t)^2\right\}.$$

Answer 8(a)	$\dfrac{s^2-2s+4}{s(s^2+4)}$

(Show your working below and on the next page.)

$$L\{(\sin t - \cos t)^2\}$$

$$= L(\sin^2 t - 2\sin t\cos t + \cos^2 t)$$

$$= L(1 - \sin 2t)$$

$$= \frac{1}{s} - \frac{2}{s^2+2^2}$$

$$= \frac{1}{s} - \frac{2}{s^2+4} = \frac{s^2-2s+4}{s(s^2+4)}$$

Question 8 (b) [5 marks]

Find the inverse Laplace transform

$$L^{-1}\left(e^{-s}\frac{s+3}{s^2+4s+4}\right).$$

Answer 8(b)	$te^{-2(t-1)}U(t-1)$

(Show your working below and on the next page.)

Let $F(s) = \dfrac{s+3}{s^2+4s+4} = \dfrac{s+3}{(s+2)^2} = \dfrac{1}{s+2} + \dfrac{1}{(s+2)^2}$

Then
$$\begin{aligned} f(t) &= L^{-1}(F(s)) \\ &= e^{-2t} + e^{-2t}t \\ &= e^{-2t}(1+t) \end{aligned}$$

$$\begin{aligned} \therefore L^{-1}\left(e^{-s}\frac{s+3}{s^2+4s+4}\right) &= L^{-1}\left(e^{-s}F(s)\right) \\ &= f(t-1)U(t-1) \\ &= e^{-2(t-1)}t\,U(t-1) \\ &= te^{-2(t-1)}U(t-1) \end{aligned}$$

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2006-2007

MA1505 MATHEMATICS I

November 2006 Time allowed: 2 hours

Matriculation Number:

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided above.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **EIGHT (8)** questions and comprises **THIRTY THREE (33)** printed pages.
3. Answer **ALL** questions. For each question, write your answer in the box and your working in the space provided inside the booklet following that question.
4. The marks for each question are indicated at the beginning of the question.
5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

For official use only. Do not write below this line.

Question	1	2	3	4	5	6	7	8
Marks								

Question 8 (b) [5 marks]

Using the method of separation of variables, solve the partial differential equation

$$xu_x - yu_y = 0,$$

where $x > 0$ and $y > 0$.

Answer 8(b)	$u = k(xy)^c$

(Show your working below and on the next page.)

Let $u = XY$

$\therefore xX'Y - yXY' = 0 \Rightarrow xX'Y = yXY'$

$\Rightarrow x\dfrac{X'}{X} = y\dfrac{Y'}{Y} = c$

$\therefore \dfrac{X'}{X} = \dfrac{c}{x}$ and $\dfrac{Y'}{Y} = \dfrac{c}{y}$

$\therefore \ln|X| = c\ln|x| + a$ and $\ln|Y| = c\ln|y| + b$

$\therefore X = k_1 x^c$ and $Y = k_2 y^c$

$\therefore u = XY = k(xy)^c$

where k and c are constants.

Matriculation Number:

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2007-2008

MA1505 MATHEMATICS I

November 2007 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided above.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **EIGHT (8)** questions and comprises **THIRTY THREE (33)** printed pages.
3. Answer **ALL** questions. For each question, write your answer in the box and your working in the space provided inside the booklet following that question.
4. The marks for each question are indicated at the beginning of the question.
5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

For official use only. Do not write below this line.

Question	1	2	3	4	5	6	7	8
Marks								

Question 8 (b) [5 marks]

Find a solution of the form $u(x,y) = F(ax+y)$, where a is a constant and F is a differentiable single variable function, to the partial differential equation

$$u_x - 2u_y = 0,$$

that satisfies the condition $u(x,0) = \cos x$.

Answer 8(b)	$u(x,y) = \cos\frac{2x+y}{2}$

(Show your working below and on the next page.)

$$u_x = aF'(ax+y)$$

$$u_y = F'(ax+y)$$

$$u_x - 2u_y = 0 \Rightarrow aF'(ax+y) - 2F'(ax+y) = 0$$

$$\Rightarrow a = 2$$

$$\therefore u(x,y) = F(2x+y)$$

$$u(x,0) = \cos x \Rightarrow F(2x) = \cos x$$

$$\Rightarrow F(x) = \cos\frac{x}{2}$$

$$\therefore u(x,y) = F(2x+y) = \cos\frac{2x+y}{2}$$

$$\therefore u(x,y) = \cos\frac{2x+y}{2}$$

Matriculation Number:

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2008-2009

MA1505 MATHEMATICS I

November 2008 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided above.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **EIGHT (8)** questions and comprises **THIRTY THREE (33)** printed pages.
3. Answer **ALL** questions. For each question, write your answer in the box and your working in the space provided inside the booklet following that question.
4. The marks for each question are indicated at the beginning of the question.
5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

For official use only. Do not write below this line.

Question	1	2	3	4	5	6	7	8
Marks								

Question 8 (b) [5 marks]

Use the method of separation of variables to find $u(x, y)$ that satisfies the partial differential equation

$$u_x + u_y = (x - y)u,$$

given that $u(0, 0) = u(0, 2) = 1$.

Answer 8(b)	$u = e^{\frac{1}{2}x^2 - x - \frac{1}{2}y^2 + y}$

(Show your working below and on the next page.)

Let $u = X(x)Y(y)$

$\Rightarrow X'Y + XY' = (x-y)XY \Rightarrow \frac{X'}{X} + \frac{Y'}{Y} = x - y$

$\therefore \frac{X'}{X} - x = -\frac{Y'}{Y} - y = k$

$\therefore \frac{X'}{X} = x + k$ and $\frac{Y'}{Y} = -y - k$

$\ln|X| = \frac{1}{2}x^2 + kx + C_1$, $\ln|Y| = -\frac{1}{2}y^2 - ky + C_2$

$\therefore u = XY = Ce^{(\frac{1}{2}x^2 + kx - \frac{1}{2}y^2 - ky)}$

$u(0,0) = 1 \Rightarrow C = 1$

$u(0,2) = 1 \Rightarrow e^{-2-2k} = 1 \Rightarrow k = -1$

$\therefore u = e^{\frac{1}{2}x^2 - x - \frac{1}{2}y^2 + y}$

Matriculation Number:

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2009-2010

MA1505 MATHEMATICS I

November 2009 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided above.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **EIGHT (8)** questions and comprises **THIRTY THREE (33)** printed pages.
3. Answer **ALL** questions. For each question, write your answer in the box and your working in the space provided inside the booklet following that question.
4. The marks for each question are indicated at the beginning of the question.
5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

For official use only. Do not write below this line.

Question	1	2	3	4	5	6	7	8
Marks								

Question 2 (b) [5 marks]

Use the method of separation of variables to find $u(x,y)$ that satisfies the partial differential equation

$$2u_{xy} = [\sin(x+y) + \sin(x-y)]\,u\ ,$$

given that $u(0,0) = 1$ and $u(\pi,\pi) = e^2$.

Answer 2(b)	$u = e^{1-\cos x+\sin y}$

(Show your working below and on the next page.)

Let $u = XY$, $X = X(x)$, $Y = Y(y)$.

$\therefore\ 2X'Y' = (2\sin x\cos y)XY$

$\therefore\ \dfrac{X'}{X\sin x} = \dfrac{Y\cos y}{Y'} = k$

$\dfrac{X'}{X} = k\sin x \Rightarrow \ln|X| = -k\cos x \Rightarrow X = Ae^{-k\cos x}$

$\dfrac{Y'}{Y} = \dfrac{1}{k}\cos y \Rightarrow \ln|Y| = \dfrac{1}{k}\sin y \Rightarrow Y = Be^{\frac{1}{k}\sin y}$

$\therefore\ u = Ce^{-k\cos x + \frac{1}{k}\sin y}$

$\left.\begin{array}{l} u(0,0)=1 \Rightarrow 1 = Ce^{-k} \\ u(\pi,\pi)=e^2 \Rightarrow e^2 = Ce^{k} \end{array}\right\} \Rightarrow k=1,\ C=e$

$\therefore\ u = e^{1-\cos x+\sin y}$