NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2006-2007

MA1506 Mathematics 2

November 2006 — Time allowed : 2 hours

Matriculation Number:

INSTRUCTIONS TO CANDIDATES

1. **Write your matriculation number neatly in the space above.**
2. Do not insert loose papers into this booklet. This booklet will be collected at the end of the examination.
3. This examination paper contains a total of **FOURTEEN (14)** questions and comprises **TWENTY-EIGHT (28)** printed pages.
4. Answer **ALL** 14 questions. The marks for each question are indicated at the beginning of the question.
5. **Write your solution in the space below each question.**
6. Calculators may be used. However, you should lay out systematically the various steps in your calculations.

For official use only. Do not write in the boxes below.

Question	1	2	3	4	5	6	7
Marks							

Question	8	9	10	11	12	13	14
Marks							

Answer **all** *the questions.*

Question 1 [7 marks]

Find the extreme values of the function

$$f(x, y) = x + y,$$

subject to the constraint $x^2 + xy + 2y^2 = 14.$

(*More space for the solution to Question 1.*)

Question 2 [7 marks]

Let d be a *positive* constant. The plane $x + 2y + 2z = d$ is tangent to the surface

$$x^2 + 3y^2 + 6z^2 = 27.$$

Find the value of d.

(*More space for the solution to Question 2.*)

Question 3 [7 marks]

Find

$$\int_0^2 \int_{x^3}^8 x^2 \sin\left(y^2\right)\, dydx.$$

Question 4 [7 marks]

The region D in the first octant ($x \geq 0,\ y \geq 0, z \geq 0$) containing the point $(0,0,0)$ is bounded by the planes

$$2x + 2y + z \ = \ 4, \quad x + y \ = \ 1, \quad x = 0, \quad y = 0, \quad z = 0.$$

Find the volume of D.

(*More space for the solution to Question 4.*)

Question 5 [7 marks]

By changing to spherical coordinates, find the following iterated integral

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{1} \left(x^2 + y^2 + z^2\right) \, dz dy dx.$$

(*More space for the solution to Question 5.*)

Question 6 [7 marks]

Let $\mathbf{F}(x, y, z) \;=\; (2xz + \sin y)\mathbf{i} \;+\; (x\cos y)\mathbf{j} \;+\; (x^2 + \sin z)\mathbf{k}$.
Find a function $f(x, y, z)$ such that $\nabla f = \mathbf{F}$.

Hence, or otherwise, find the line integral $\displaystyle\int_C \mathbf{F} \bullet d\mathbf{r}$, where C is the curve described by

$$C: \quad \mathbf{r}(t) \;=\; (\cos t)\mathbf{i} \;+\; (\sin t)\mathbf{j} \;+\; t\mathbf{k}, \quad 0 \le t \le \pi.$$

(*More space for the solution to Question 6.*)

Question 7 [7 marks]

If the vector field $\mathbf{E}(x, y, z)$ represents an electric field, then Gauss's Law in electrostatics says that the net charge Q enclosed by a closed surface S is given by

$$Q = \epsilon_0 \int\int_S \mathbf{E} \bullet d\mathbf{S},$$

where S is given an orientation by the outward normal vector, and ϵ_0 is a certain constant called the permittivity of free space.

If the electric field is $\mathbf{E}(x, y, z) = y\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$, use Gauss's Law to find the net charge Q contained in the solid cylinder

$$0 \leq x^2 + y^2 \leq 4, \quad 0 \leq z \leq 5.$$

(*Leave your answer in terms of* ϵ_0.)

(*More space for the solution to Question 7.*)

Question 8 [8 marks]

The ellipsoid $2x^2 + y^2 + z^2 = 8$ is given an orientation by the outward normal vector. The plane $y + z = 0$ partitions the ellipsoid into two surfaces. If $\mathbf{F}(x, y, z) = z\mathbf{i} + x^2\mathbf{j} + y\mathbf{k}$, find

$$\int\int_S \text{curl } \mathbf{F} \bullet d\mathbf{S},$$

where S is the upper surface (containing the point $(0, 0, \sqrt{8})$).

(*More space for the solution to Question 8.*)

Question 9 [7 marks]

Use the method of separation of variables to find a solution $u(x, y)$ of the equation

$$u_x - u_y \;=\; 3(x^2 - y^2)u$$

satisfying $u(0, 0) = 2, \quad u(1, 0) = 2e^4.$

(*More space for the solution to Question 9.*)

Question 10 [7 marks]

Let R be a two-dimensional rectangular metal plate placed on the $xy-$plane such that the coordinates (x, y) of points in R satisfy

$$0 \leq x \leq 1, \quad 0 \leq y \leq 1.$$

The steady-state temperature $u(x, y)$ satisfies the Dirichlet boundary value problem:

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \quad \text{for} \quad 0 < x < 1, \quad 0 < y < 1 \\ u(0, y) &= 0, \quad u(1, y) = 2\sin(6\pi y) \quad \text{for} \quad 0 \leq y \leq 1 \\ u(x, 0) &= 0, \quad u(x, 1) = 4\sin(3\pi x)\cos(\pi x) \quad \text{for} \quad 0 \leq x \leq 1. \end{aligned}$$

Find $u(x, y)$.

(*More space for the solution to Question 10.*)

Question 11 [8 marks]

Use Laplace transforms to find the solution $w(x,t)$ of

$$w_x - 6x^2 w_t + w \;=\; x^2 e^{-x} \sin t, \quad w(x,0) = 0,$$

which is bounded for $x > 0, \quad t > 0$.

(*More space for the solution to Question 11.*)

Question 12 [7 marks]

Let

$$A = \begin{bmatrix} 1 & k & k \\ k & 1 & k \\ k & k & 1 \end{bmatrix},$$

where k is a constant.

(i) Find the value(s) of k for which A is <u>*not*</u> invertible.

(ii) If $k = -1$, find the inverse of A.

(*More space for the solution to Question 12.*)

Question 13 [7 marks]

Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 10 & -18 \\ 3 & -5 \end{bmatrix}.$$

Hence, or otherwise, solve the linear system

$$y_1' \;=\; 10y_1 - 18y_2, \quad y_2' \;=\; 3y_1 - 5y_2,$$

given the initial conditions

$$y_1(0) = 7, \quad y_2(0) = 3.$$

(*More space for the solution to Question 13.*)

Question 14 [7 marks]

Find a condition on the numbers p and q such that the following system of equations has *infinitely many solutions*:

$$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 2q \\ 3x & + & 4y & + & z & = & 6q \\ 2x & + & py & + & 2z & = & 4 \end{array}$$

(*More space for the solution to Question 14.*)

END OF PAPER