Q2 Let $E(t)$ be the population of Elves

Let $D(t)$ be the population of Dwarves

By Malthus model

$$\frac{dE}{dt} = (\underbrace{B_E}_{\text{birth rate}} - \underbrace{D_E}_{\text{death rate}})E - P_E D$$

move out proportional to population of $D$

$$\frac{dD}{dt} = (B_D - D_D)D - P_D E$$

$$\therefore \begin{pmatrix} \frac{dE}{dt} \\ \frac{dD}{dt} \end{pmatrix} = \begin{pmatrix} B_E - D_E & -P_E \\ -P_D & B_D - D_D \end{pmatrix} \begin{pmatrix} E \\ D \end{pmatrix}$$

Note: $B_E > B_D$

$D_E < D_D \Rightarrow -D_D < -D_E$

$\Rightarrow B_E - D_E > B_D - D_D > 0$

r

Example

$$\begin{pmatrix} B_E - D_E & -P_E \\ -P_D & B_D - D_D \end{pmatrix} = \begin{pmatrix} 5 & -4 \\ -1 & 2 \end{pmatrix}$$

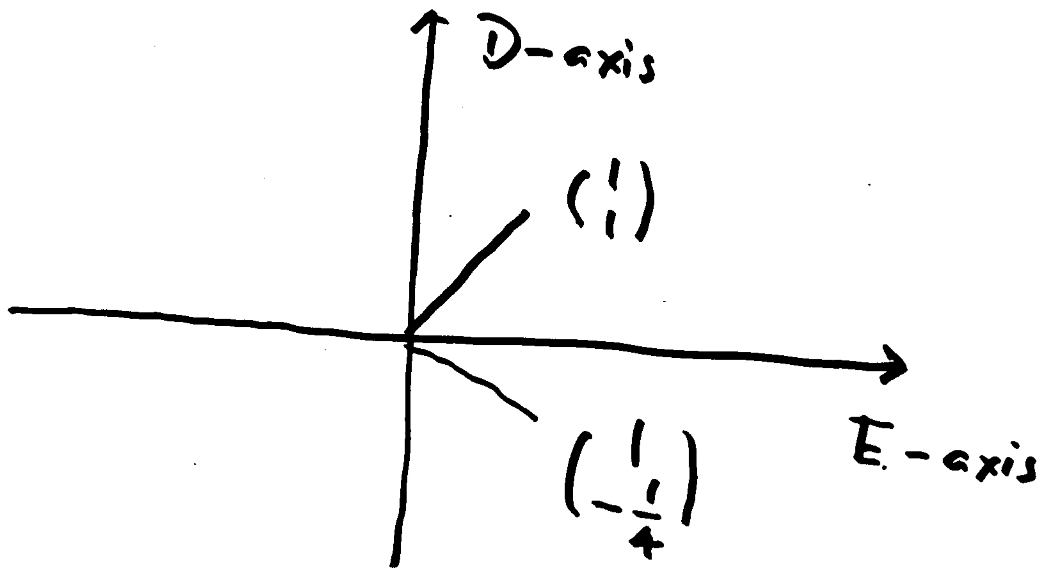
$$Tr = 7 \qquad \det = 6$$

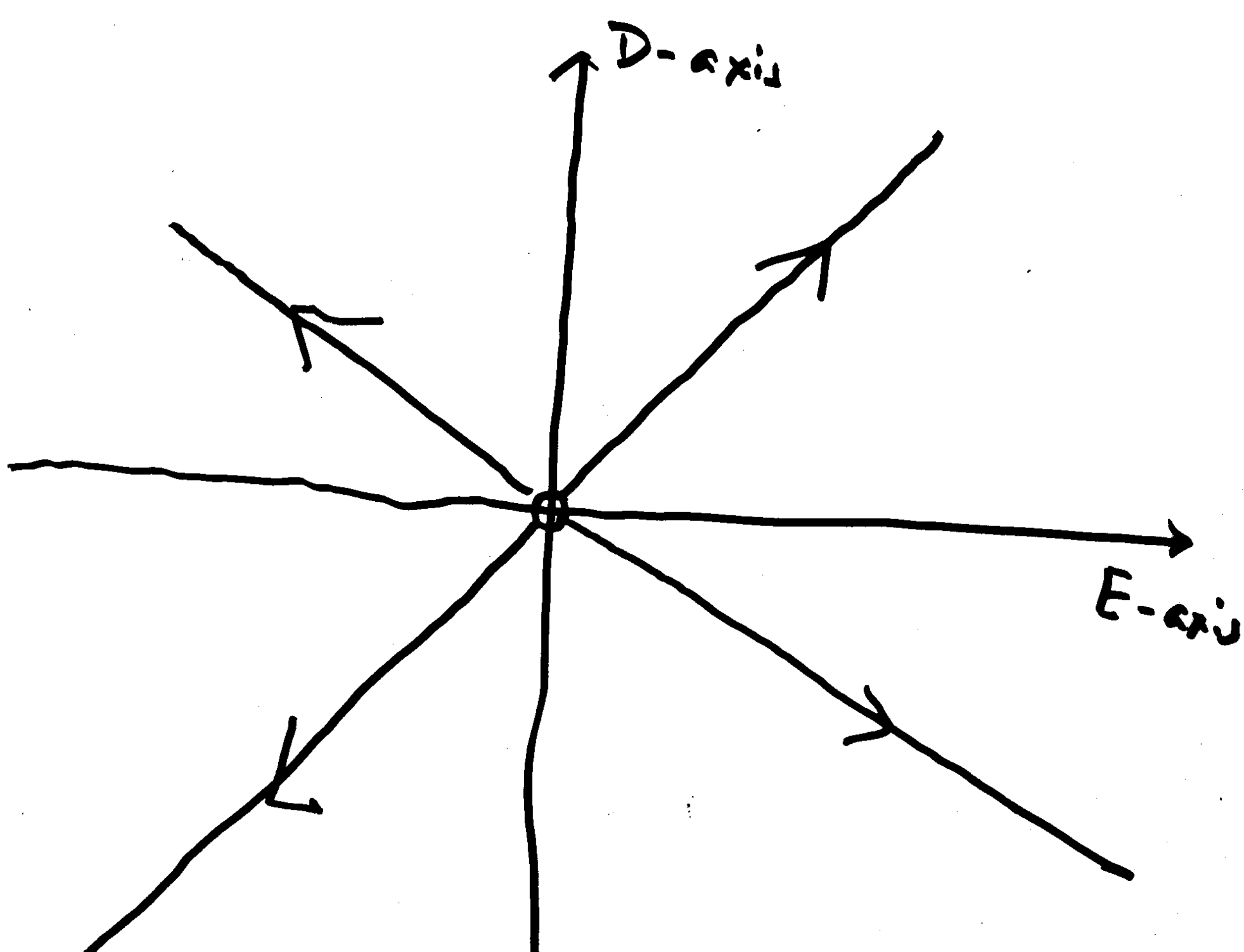
$$Tr^2 - 4\det = 25 > 0$$

Nodal source

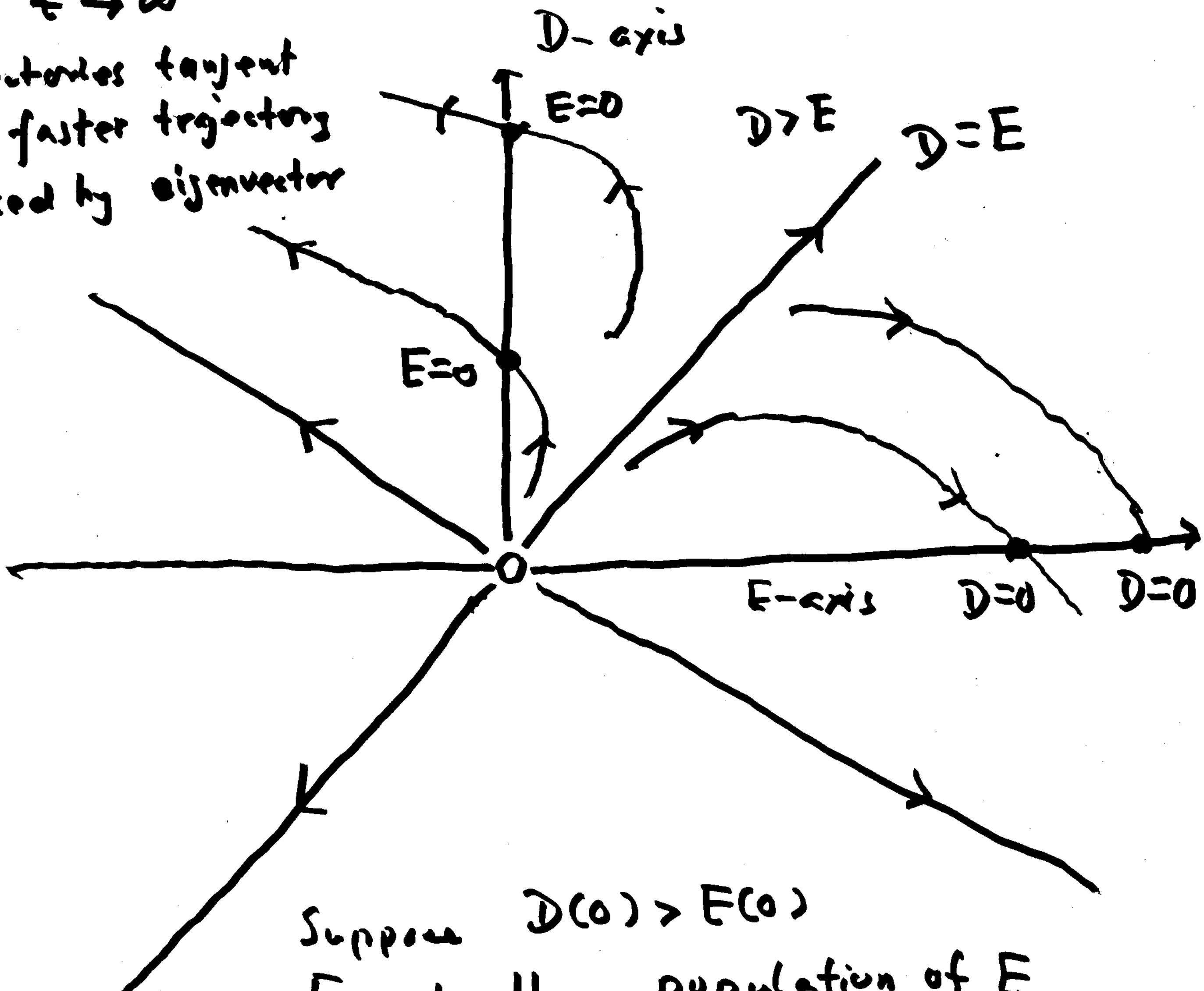
Eigenvalues $\lambda = 1$ $\lambda = 6$

Eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ -\frac{1}{4} \end{pmatrix}$



when $t \to \infty$
trajectories tangent
to faster trajectory
induced by eigenvector




Suppose $D(0) > E(0)$
Eventually population of E
$= 0$

Q 2

$0 < \lambda_1 < \lambda_2$

$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

claim: when $t \to \infty$, trajectories tangent to faster trajectory induced by eigenvector $\vec{v}$

<u>Proof.</u>

$$\begin{pmatrix} E(t) \\ D(t) \end{pmatrix} = c_1 \vec{u} e^{\lambda_1 t} + c_2 \vec{v} e^{\lambda_2 t}$$

$$= c_1 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} e^{\lambda_1 t} + c_2 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{\lambda_2 t}$$

$$\lim_{t \to \infty} \frac{D(t)}{E(t)} = \lim_{t \to \infty} \frac{c_1 u_2 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}}{c_1 u_1 e^{\lambda_1 t} + c_2 v_1 e^{\lambda_2 t}}$$

$$= \lim_{t \to \infty} \frac{c_1 u_2 e^{(\lambda_1 - \lambda_2) t} + c_2 v_2}{c_1 u_1 e^{(\lambda_1 - \lambda_2) t} + c_2 v_1}$$

$= \dfrac{v_2}{v_1}$ = gradient of eigenvector $\vec{v}$ with eigenvalue $\lambda_2$ $(\lambda_1 < \lambda_2)$

Q3

Rate of change of amount of uranium in each tank

= mass of uranium flow in /min

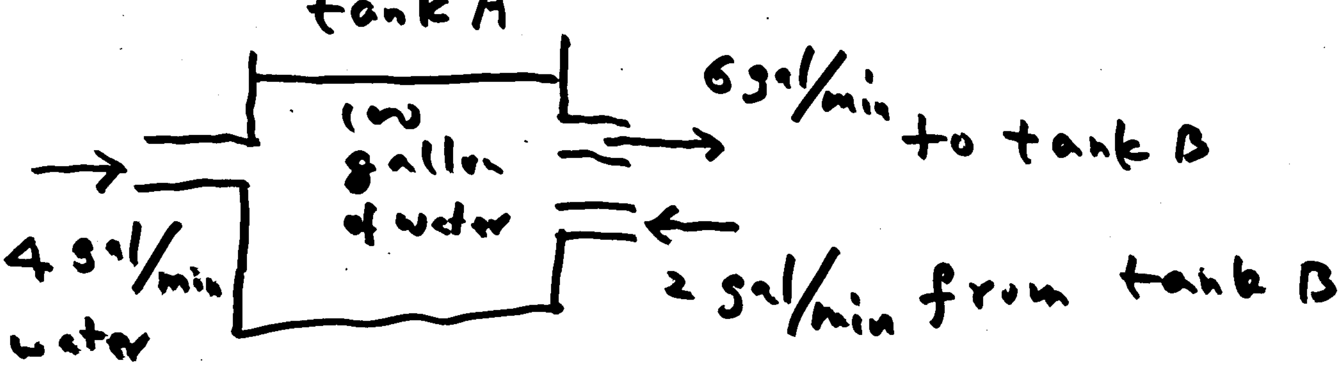
− mass of uranium flow out /min

$$= \frac{\text{mass of uranium flow in}}{\text{Volume of water}} \times \text{flow in rate} \left(\frac{\text{volume of water}}{\text{min}}\right)$$

$$- \frac{\text{mass of uranium flow out}}{\text{Volume of water}} \times \text{flow out rate}$$

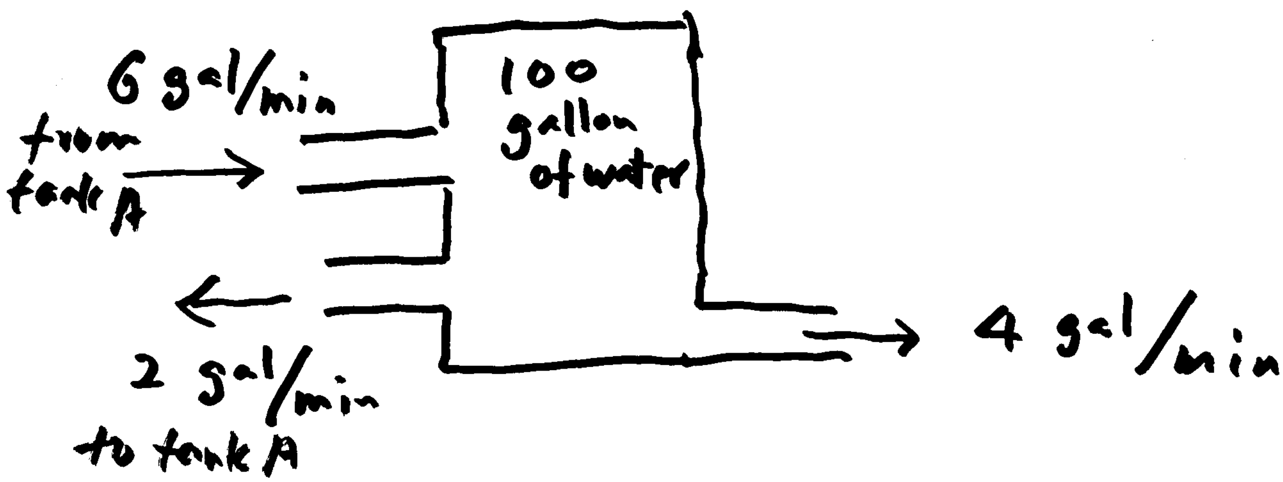
Let $x_A(t)$ be the mass of uranium in tank A at time $t$

Let $x_B(t)$ be the mass of uranium in tank B at time $t$




$$\frac{dx_A}{dt} = \frac{x_B}{100}\,2 - \frac{x_A}{100}\,6$$

tank B




$$\frac{d x_B}{dt} = \frac{x_A}{100} 6 - \frac{x_B}{100} 6$$

Write

$$\begin{bmatrix} \frac{d}{dt} x_A \\ \frac{d}{dt} x_B \end{bmatrix} = \frac{1}{100} \begin{bmatrix} -6 & 2 \\ 6 & -6 \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix}$$

Let $B = \frac{1}{100} \begin{bmatrix} -6 & 2 \\ 6 & -6 \end{bmatrix}$

eigenvalues of B

$$\lambda_1 = \frac{-6 + 2\sqrt{3}}{100} = -0.0253 < 0$$

$$\lambda_2 = \frac{-6 - 2\sqrt{3}}{100} = -0.0946 < 0$$

Corresponding eigenvectors

$$\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} \quad \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}$$

$$\therefore \begin{bmatrix} x_A \\ x_B \end{bmatrix} = c_1 e^{\lambda_1 t} \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} + c_2 e^{\lambda_2 t} \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}$$

Initial values $x_A(0) = 25$

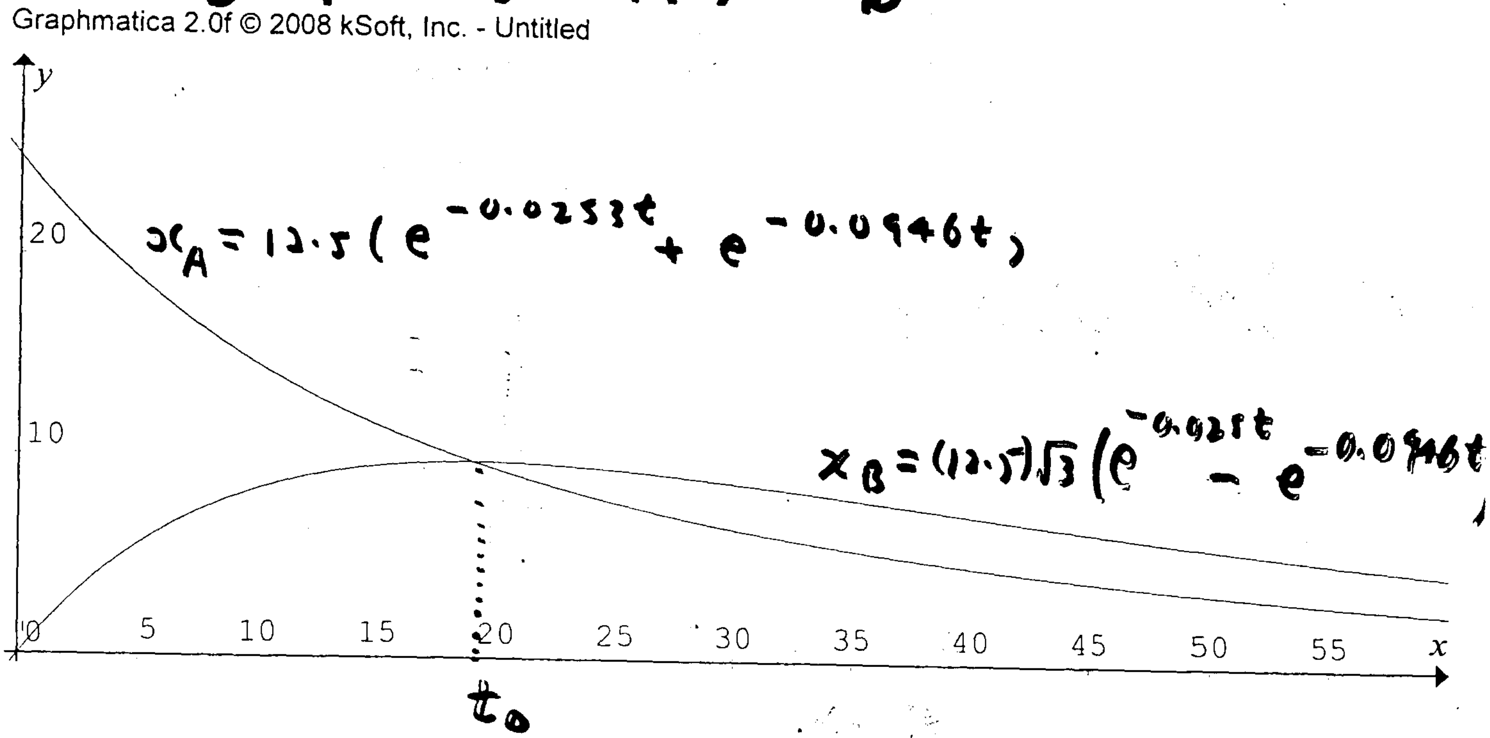
$x_B(0) = 0$

Hence $c_1 = 12.5$ $c_2 = 12.5$

$$\therefore x_A = 12.5\left(e^{\lambda_1 t} + e^{\lambda_2 t}\right)$$

$$x_B = (12.5)(\sqrt{3})\left(e^{\lambda_1 t} - e^{\lambda_2 t}\right)$$

2. graphs of $x_A$, $x_B$




when $t > t_0$

$x_B(t) > x_A(t)$

3. $x_B(t) > x_A(t)$ ?

Yes when $t > t_0$, see above graph

4. Two negative eigenvalues

O is Nodal sink