MA1506

Mathematics II

Chapter 1

1.1 Differential Equations

- A *differential equation* is an equation that contains one or more derivatives of a differentiable function.
- only ordinary d.e. , (partial d.e. only in chpt 8)

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + p(p+1)y = 0$$

PDE example: 1-D Heat Equation

$$a^2 \frac{\partial^2 w}{\partial x^2} = \frac{\partial w}{\partial t}$$

- Fourier series!

$$w(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 a^2 t} \sin nx$$

Linear Differential Equations

- A linear d.e. is of the following form:

a_k and ***F*** are functions of *x*.

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y^{(1)}(x) + a_0 y(x) = F$$

n-th Order

Linear Differential Equations

- A linear d.e. is of the following form:

$\boldsymbol{a_k}$ and $\boldsymbol{F}$ are functions of x.

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = F$$

n-th Order

Examples

$$y' = 5y$$

$$xy' - \sin x = 0$$

$$(y''')^2 + (y'')^5 - y' = e^x$$

non linear

3rd Order

General Solution

$$
\begin{aligned}
y' &= \cos x \\
\int y' dx &= \int \cos x \, dx \\
\Aboxed{y &= \sin x + C}
\end{aligned}
$$

General solution

In most cases, general solution of an ***n***-th order d.e. will have ***n*** arbitrary constants.

Particular Solution

When arbitrary constants are given specific values

$$
\left.\begin{aligned} y &= \sin x + 1 \\ y &= \sin x + 5 \end{aligned}\right\}
$$

Particular solutions of

$$
y' = \cos x
$$

Example 1

$$\frac{dy}{dx} = (1 + y^2)e^x$$

Integrate $\Rightarrow$ $$e^x dx = \frac{1}{1+y^2} dy$$

$$\begin{aligned} e^x &= \tan^{-1} y + c \\ \Rightarrow \tan^{-1} y &= e^x - c \\ \Rightarrow y &= \tan(e^x - c) \end{aligned}$$

Revision: Implicit Diff

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x$$

Let
$$y = \tan^{-1} x$$
$$x = \tan y$$

Diff wrt ***x***
$$1 = \sec^2 y \frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y}$$

$$\tan^{-1} x = y = \int \frac{1}{1+x^2}\,dx$$

1.2 Separable d.e. (first order)

usually nonlinear

$$M(x)dx = N(y)dy$$

$$\int M(x)dx = \int N(y)dy + c$$

Note: This is just a technique. See pg 26 of Farlow to see why it works

Example 2 : Radioactive decay

Experiments show that a radioactive substance decomposes at a rate proportional to the amount present. Starting with 2 mg at certain time, say *t=0*, what can be said about the amount available at a later time?

$$\frac{dx}{dt} \propto -x \quad \Rightarrow \quad \frac{dx}{dt} = -kx$$

Example 2

$$\frac{dx}{dt} = -kx$$

$$\int \frac{dx}{x} = \int -k dt$$

$$\ln|x| = -kt + c$$

$$\ln x = -kt + c$$

Initial condition

$$\ln 2 = 0 + c$$

$$x = e^{-kt+\ln 2} = 2e^{-kt}$$

Example 3

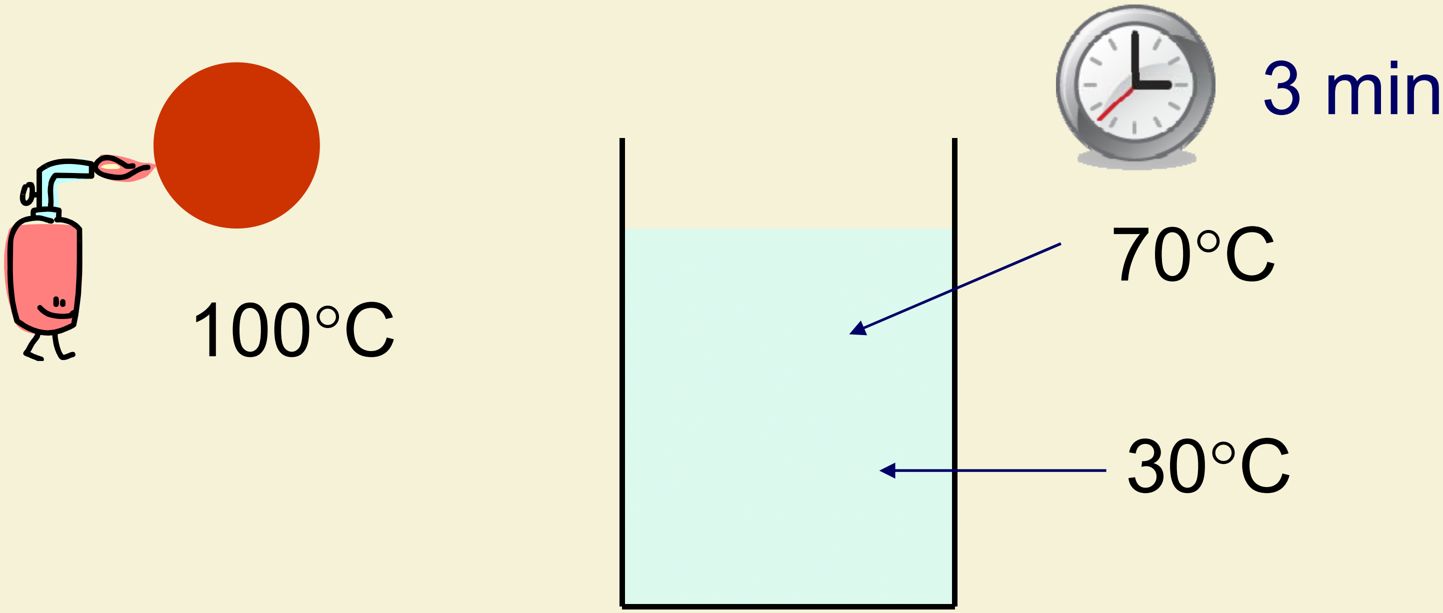
A copper ball is heated to 100°C. At t=0, it is placed in water which is maintained at 30°C. At the end of 3 mins, the temperature of the ball is reduced to 70°C. Find the time at which the temperature of the ball is 31°C.

<u>Physical information:</u>

Experiments show that the rate of change *dT/dt* of the temperature *T* of the ball wrt. *t* is proportional to the difference between *T* and the temp T_0 of the surrounding medium. Also, heat flows so rapidly in copper that at any time the temperature is practically the same at all points of the ball.

(Assume water is always 30°C)

Example 3



rate of change *dT/dt* of the temperature *T* of the ball wrt. *t* is proportional to the difference between *T* and the temp T_0 of the surrounding medium.

$$\frac{dT}{dt} = k(T - T_0)$$

Newton's Law of Cooling

Example 3 $\qquad \dfrac{dT}{dt} = k(T - T_0)$

$$\int \frac{dT}{T - T_o} = \int k\,dt$$

Initial condition

$\ln 70 = 0 + c$

$$\ln|T - T_0| = kt + c$$

$$\ln(T - T_0) = kt + c, \quad T \geq T_0$$

$$T - T_0 = e^{kt + \ln 70} = 70e^{kt}$$

$$\boxed{T = 30 + 70e^{kt}}$$

Example 3

$$\frac{dT}{dt} = k(T - T_0)$$

$$T = 30 + 70e^{kt}$$

2nd condition

$$70 - 30 = 70e^{3k} \Rightarrow k = \frac{1}{3}\ln\frac{4}{7}$$

Solve

$$31 \approx 30 + 70e^{-0.1865t_1}$$

$$t_1 \approx 22.8$$

Example 4

Suppose that a sky diver falls from rest towards the earth and the parachute opens at an instant $t=0$, when the sky diver's speed is $v(0)=10m/s$. Find the speed of the sky diver at any later time t.

[Physical assumptions and laws:

weight of the man + equipment = 712N,

air resistance = bv^2, where $b=30\ kg/m$.]

Example 4

$$m\frac{dv}{dt} = mg - bv^2$$

$$\frac{dv}{dt} = -\frac{b}{m}(v^2 - k^2), \qquad k^2 = \frac{mg}{b}$$

Terminating velocity

v > *k* then *v* decreases

v < *k* then *v* increases

Example 4

$$\frac{dv}{dt} = -\frac{b}{m}(v^2 - k^2), \quad k^2 = \frac{mg}{b}$$

$$\frac{1}{v^2 - k^2}\, dv = -\frac{b}{m}\, dt$$

$$\frac{1}{2k}\left(\frac{1}{v-k} - \frac{1}{v+k}\right) dv = -\frac{b}{m}\, dt.$$

$$\ln\left|\frac{v-k}{v+k}\right| = -\frac{2kb}{m} t + c_1,$$

Example 4

$$k^2 = \frac{mg}{b} = \frac{712}{30}$$

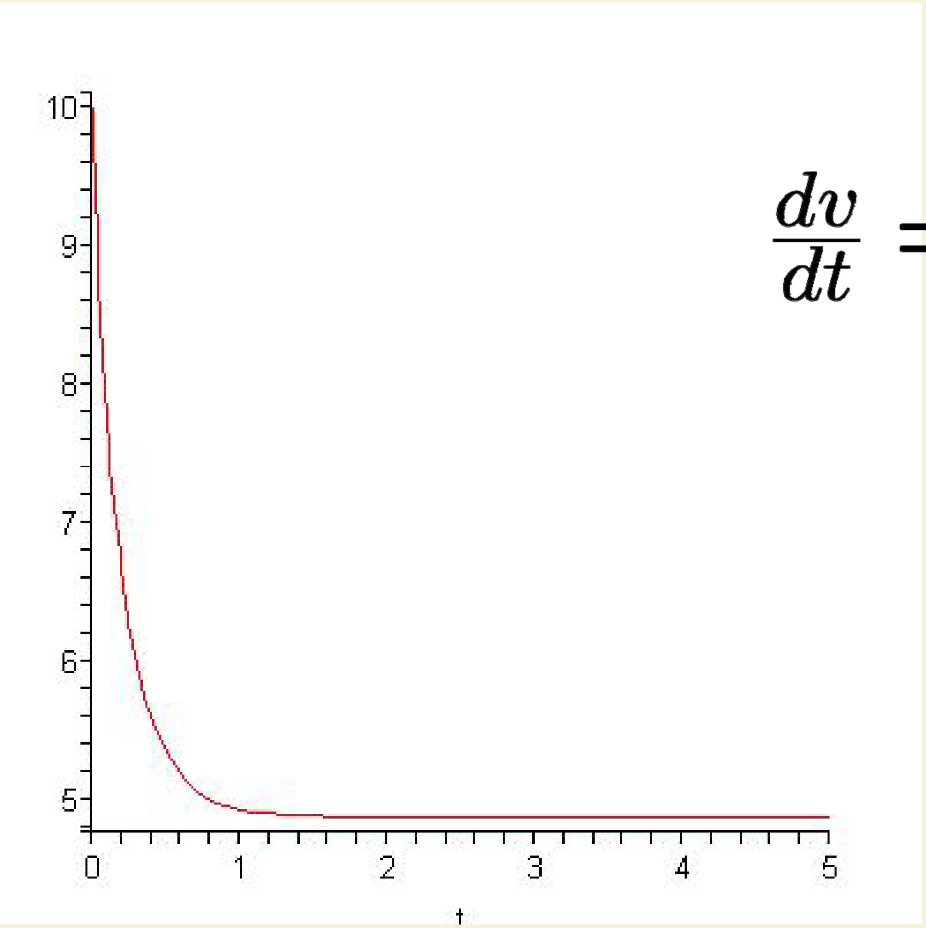
$$\ln\left|\frac{v-k}{v+k}\right| = -\frac{2kb}{m}t + c_1,$$

$$\frac{v-k}{v+k} = ce^{-pt},\ p = \frac{2kb}{m}, c = \pm e^{c_1}$$

How did I remove the absolute sign?

$$v = k\frac{1+ce^{-pt}}{1-ce^{-pt}} = 4.87\,\frac{1+0.345e^{-4.02t}}{1-0.345e^{-4.02t}}$$

$$v = k\frac{1+ce^{-pt}}{1-ce^{-pt}} = 4.87\,\frac{1+0.345e^{-4.02t}}{1-0.345e^{-4.02t}}$$



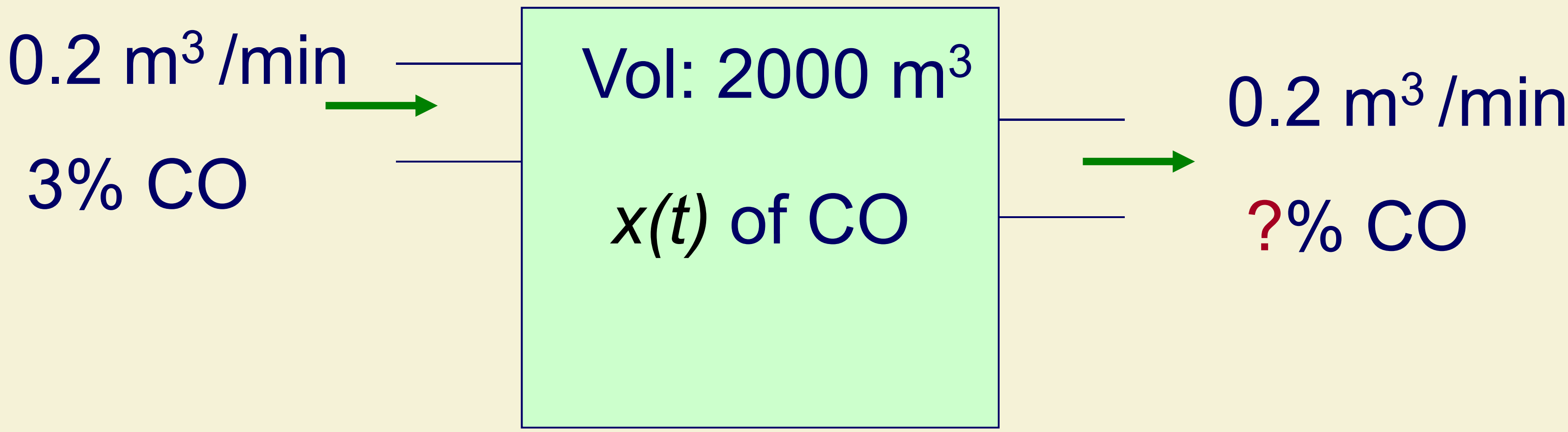
$$\frac{dv}{dt} = -\frac{b}{m}(v^2 - k^2)$$

Example 5

A conference room with volume $2000m^3$ contains air with 0.002% CO. At time, $t = 0$ the ventilation system starts blowing in air which contains 3% CO (by volume). If the ventilation system blows in (and extracts) air at a rate of $0.2m^3$/min, how long will it take for the air in the room to contain 0.015% CO?

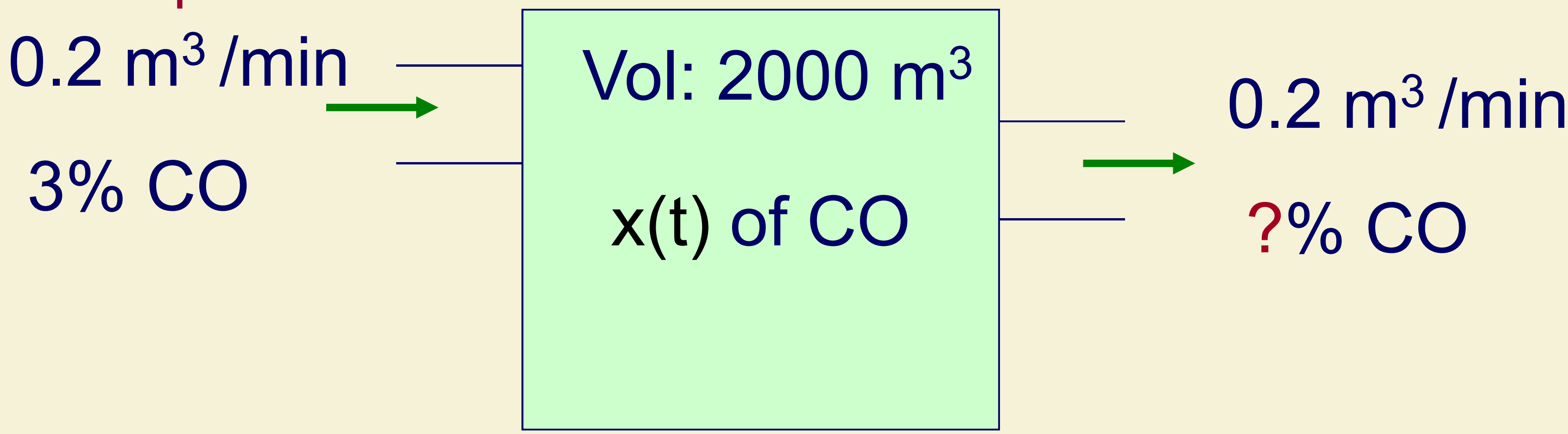
Example 5

Let $x(t)$ = vol of CO in the room at time t



$$
\begin{aligned}
\frac{dx}{dt} &= \text{inflow - outflow} \\
&= 0.03 \times 0.2 - \frac{x}{2000} \times 0.2 \\
&= 0.006 - 0.0001x
\end{aligned}
$$

Example 5



$$\frac{dx}{dt} = 0.006 - 0.0001x$$

$$\frac{dx}{60 - x} = 0.0001dt$$

$$-\ln|60 - x| = 0.0001t + c$$

$$x = 60 - ke^{-0.0001t}$$

Example 5

$$\boxed{x = 60 - ke^{-0.0001t}}$$

$$x(0) = 0.00002 \times 2000 = 0.04$$

$$\Rightarrow k = 59.96$$

0.015% CO means $x(t_1)$ = 0.00015 x 2000 = 0.3

$$0.3 = 60 - 59.96e^{-0.0001t_1}$$

$$t_1 = -10^4 \times \ln \frac{59.7}{59.96} \approx 43.5 \text{ min}$$

What happens when d.e. is not separable?

$$2xy\frac{dy}{dx} - y^2 + x^2 = 0$$

$$(2x - 4y + 5)y' + x - 2y + 3 = 0.$$

2 tricks:

- reduction to separable
- linear change of variables

Reduction to separable form

$$\boxed{y' = g\left(\frac{y}{x}\right)}$$

Set $\boxed{\frac{y}{x} = v} \Rightarrow y = vx$

$$\Rightarrow y' = v + xv'$$

$$\frac{dv}{g(v) - v} = \frac{dx}{x}$$

Example 6a : Reduction to separable form

$$2xy\frac{dy}{dx} - y^2 + x^2 = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y^2 - x^2}{2xy} = \frac{y}{2x} - \frac{x}{2y}$$

$$\boxed{\frac{y}{x} = v \Rightarrow y = vx \;\Rightarrow y' = v + xv'}$$

$$v + xv' = \frac{v}{2} - \frac{1}{2v}$$

$$\Rightarrow \quad \frac{2}{v + \frac{1}{v}} dv = -\frac{1}{x} dx$$

Example 6a : Reduction to separable form

$$\frac{2}{v+\frac{1}{v}}dv = \frac{2v}{v^2+1}dv = -\frac{1}{x}\,dx$$

$$\ln\left|v^2+1\right| = -\ln|x| + c$$

$$\frac{y^2}{x^2}+1 = \frac{c_1}{x}$$

$$y^2 + x^2 = c_1 x$$

Clarification on removing abs | |

$$\ln|v^2+1| = -\ln|x| + c$$

$x > 0$

$$\begin{aligned} \ln v^2+1 &= -\ln x + c \\ v^2+1 &= e^c\frac{1}{x} \end{aligned}$$

positive

$x < 0$

$$\begin{aligned} \ln v^2+1 &= -\ln(-x) + c \\ v^2+1 &= e^c\frac{1}{-x} \end{aligned}$$

negative

$$\frac{y^2}{x^2}+1 = \frac{c_1}{x}$$

Can be positive or negative

Example 6b : Reduction to separable form

$$y' = \frac{y}{x} + \frac{2x^3 \cos x^2}{y}$$

$$\Rightarrow v + xv' = v + \frac{2x^2 \cos x^2}{v}$$

Linear Change of Variable

A d.e. of the form

$$\boxed{y' = f(ax+by+c),}$$

where f is continuous and $b \neq 0$, can be solved by setting $u=ax+by+c.$

Example 7

$$(2x-4y+5)y'+x-2y+3=0.$$

Example 7: Linear Change

$$\boxed{(2x - 4y + 5)y' + x - 2y + 3 = 0.}$$

$$y' = \frac{-x+2y-3}{2x-4y+5}.$$

set $u = x - 2y \Rightarrow u' = 1 - 2y'$

$$\begin{aligned} \frac{1}{2}(1 - u') &= \frac{-u - 3}{2u + 5} \\ u' &= \frac{4u + 11}{2u + 5}. \end{aligned}$$

Example 7 $$u' = \frac{4u+11}{2u+5}.$$

$$\int \frac{\frac{1}{2}(4u+11-1)}{4u+11}du = \int 1dx$$

$$u - \frac{1}{4}\ln|4u+11| = 2x + c_1$$

$$4x + 8y + \ln|4x - 8y + 11| = c.$$

Separable d.e. (why it works)

$$\begin{aligned} M(x)dx &= N(y)dy \\ \int M(x)dx &= \int N(y)dy + c \end{aligned}$$

Note: we are actually diff wrt x on both sides and using chain rule. (See p. 26 of textbook)

Why this works?

$$\begin{aligned} M(x) &= N(y)\frac{dy}{dx} \\ \int M(x)dx &= \int N(y)\frac{dy}{dx}dx \end{aligned}$$

By chain rule / Implicit diff

$$\frac{d}{dx}f(y) = f'(y)\frac{dy}{dx}$$

1.3 Linear 1st Order d.e.

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Std form

linear in y

Integrating factor

$$R(x) = e^{\int^x P(s)ds}$$

$$R' = P(x) \times e^{\int^x P(s)ds} = PR$$

$$(Ry)' = RPy + Ry' = RQ$$

Example 8(i). $xy' - 3y = x^2, \quad x > 0$

Std form $$y' - \frac{3}{x}y = x$$

Integrating factor $$e^{-\int \frac{3}{s}ds} = e^{-3\ln x} = \frac{1}{x^3}$$

(Ry)'

RQ

$$\left(\frac{1}{x^3}y\right)' = \frac{1}{x^2}$$

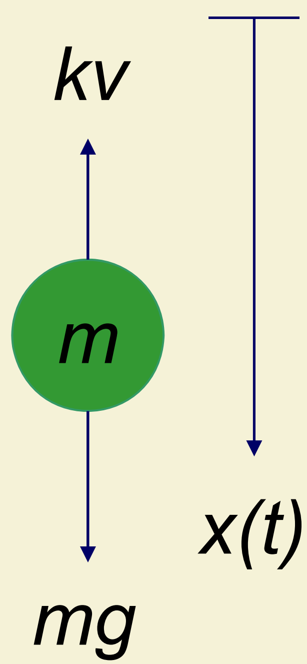
$$y = x^3\left(-\frac{1}{x} + c\right) = -x^2 + cx^3$$

Example 9

Consider an object of mass m dropped from rest in a medium that offers a resistance proportional to the magnitude of the instantaneous velocity of the object. The goal is to find the position *x(t)* and velocity *v(t)* at any time t.

2nd Law $$m\frac{dv}{dt} = mg - kv$$

$$v(0) = 0, x(0) = 0$$



Example 9 $$m\frac{dv}{dt} + kv = mg$$

Integrating factor $e^{\int \frac{k}{m} ds} = e^{\frac{kt}{m}}$

$$ve^{\frac{kt}{m}} = \int ge^{\frac{kt}{m}} = \frac{mg}{k}e^{\frac{kt}{m}} - C$$

$$\boxed{v(0) = 0}$$

$$\boxed{v = \frac{mg}{k} - Ce^{-\frac{kt}{m}} = \frac{mg}{k}(1 - e^{-\frac{kt}{m}})}$$

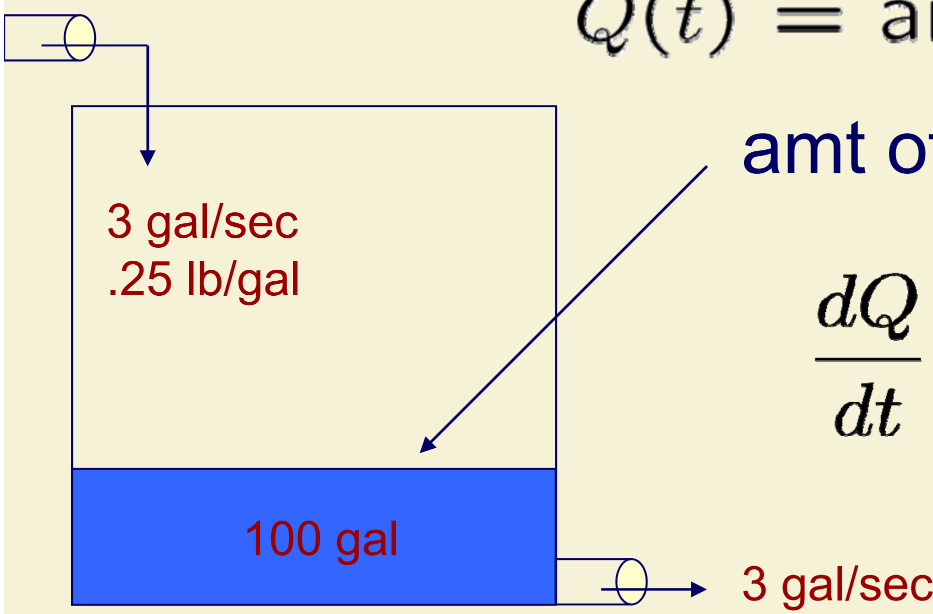
$$x(t) = \frac{mg}{k}(t + \frac{m}{k}e^{-\frac{kt}{m}}) + D$$

$$x(t) = \frac{mg}{k}t - \frac{m^2 g}{k^2}(1 - e^{-\frac{kt}{m}})$$

Example 10

At time *t* = *0* a tank contains 20 lbs of salt dissolved in 100 gal of water. Assume that water containing 0.25 lb of salt per gallon is entering the tank at a rate of 3 gal/min and the well stirred solution is leaving the tank at the same rate. Find the amount of salt at any time t.

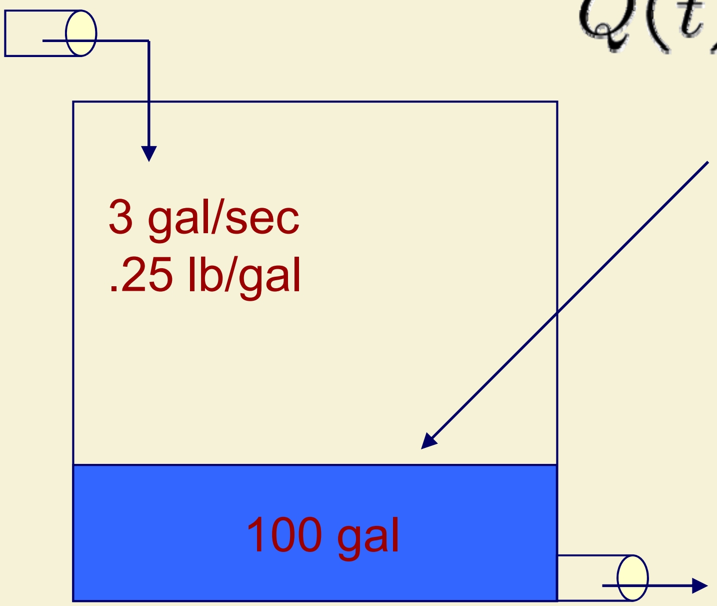
$$Q(t) = \text{amount of salt, } Q(0) = 20$$



amt of water = constant

$$\frac{dQ}{dt} = 3 \times 0.25 - \frac{3Q}{100}$$

Example 10



$Q(t) = \text{amount of salt, } Q(0) = 20$

amt of water = constant

$$\frac{dQ}{dt} = 3 \times 0.25 - \frac{3Q}{100}$$

$$\boxed{e^{\int \frac{3}{100}} = e^{\frac{3t}{100}}}$$

$$Qe^{\frac{3t}{100}} = \int 0.75 e^{\frac{3t}{100}} = 25 e^{\frac{3t}{100}} + C$$

$$\Rightarrow Q = 25 - 5e^{-\frac{3t}{100}}$$

$$\lim_{t \to \infty} Q(t) = 25$$

Example 11: radioactive decay redux

In Example 2, we saw that radioactive substances typically decay at a rate proportional to the amount present. Sometimes the product of a radioactive decay is itself a radioactive substance which in turn decays (at a different rate).

$\boxed{\text{U-234}} \rightarrow \boxed{\text{Th-230}} \rightarrow$

Half life: 245,000yrs 75,000yrs

Amt of time required for half of substance to decay

http://www.flickr.com/photos/eclogite/851474506/

Example 11: radioactive decay redux

U(t) : amt of Uranium, *T(t)* : amt of Thorium

$$\begin{aligned}\frac{dU}{dt} &= -k_U U,\\ \frac{dT}{dt} &= +k_U U - k_T T,\end{aligned}$$

constants

$U(0) = U_0, T(0) = 0$

Find the time *t*, given *T(t)/U(t)*.

Example 11

$$\frac{dU}{dt} = -k_U U,$$

Example 2

$$\Rightarrow U = U_0 e^{-k_U t}$$

Half life

$$U_0/2 = U_0 e^{-k_U \times 245000}$$

$$\boxed{\begin{aligned} \Rightarrow k_U &= \ln 2/245000 \approx 2.6 \times 10^{-6} \\ k_T &= \ln 2/75000 \approx 9.2 \times 10^{-6} \end{aligned}}$$

$$\frac{dT}{dt} = +k_U U - k_T T$$

$$\Rightarrow \quad \frac{dT}{dt} + k_T T = k_U U_0 e^{-k_U t}$$

Example 11

$$\frac{dT}{dt} + k_T T = k_U U_0 e^{-k_U t}$$

Int factor: $e^{k_T t}$

$$T e^{k_T t} = \int k_U U_0 e^{(k_T - k_U)t}$$

Since $T(0)=0$

$$\Rightarrow T(t) = \frac{k_U}{k_T - k_U} U_0 (e^{-k_U t} - e^{-k_T t})$$

$$U = U_0 e^{-k_U t}$$

is this positive?

$$\Rightarrow T/U = \frac{k_U}{k_T - k_U}\left(1 - e^{(k_U - k_T)t}\right)$$

Reduction to linear form: Bernoulli Equations

$$\boxed{y' + p(x)y = q(x)y^n}$$

linear only if

n=1 or *n=0*

Set $y^{1-n} = z \Rightarrow (1-n)y^{-n}y' = z'$

$$z' + (1-n)p(x)z = (1-n)q(x)$$

Example (ii): Bernoulli Equations

$$y' + y = x^2 y^2 \quad \Rightarrow \quad y^{-2} y' + y^{-1} = x^2$$

Set $\boxed{y^{1-2} = z \Rightarrow -y^{-2} y' = z'}$

$$z' - z = -x^2$$

Int fac: $e^{-\int 1} = e^{-x}$

$$e^{-x} z = \int -x^2 e^{-x} dx$$

http://integrals.wolfram.com

$$e^{-x} y^{-1} = e^{-x}(x^2 + 2x + 2) + A$$

Review: First Order d.e

- Separable $$M(x)dx = N(y)dy$$
- Linear $$\frac{dy}{dx} + P(x)y = Q(x)$$

Use integrating factor

What if neither applies?

Use some clever substitution

a) Reduction to separable, v = y/x

b) Linear change, u = ax+by +c

c) Bernoulli eq: $z = y^{1-n}$

Second Order DE

$$y'' = -y$$

Clever guess $y = \sin x$

$$y = 3\sin x + 4\cos x$$ also works!

Are there anymore solutions?

Second Order DE

$$y'' = y$$

Clever guess

$$y = e^x \qquad y = e^{-x}$$

Are there anymore solutions?

$$y = 123e^x + 456e^{-x}$$

An important class of hyperbolic functions:

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

http://www.sosmath.com/trig/hyper/hyper01/hyper01.html

1.4 2nd order linear d.e

$$y'' + p(x)y' + q(x)y = F(x)$$

Homogeneous if *F(x) = 0*

Examples

$$y'' + 4y = e^{-x}\sin x$$

nonlinear $x(y''y + (y')^2) + 2y'y = 0$

$$(1 - x^2)y'' - 2xy' + 6y = 0$$

Superposition principle (Homogeneous)

$$y'' + p(x)y' + q(x)y = 0$$

If y_1 and y_2 are solutions then

$c\,y_1 + d\,y_2$ is also a solution

constants

Reason: $$\frac{d}{dx}(f(x) + g(x)) = \frac{df}{dx} + \frac{dg}{dx}$$

Proof: Superposition principle

$$y'' + p(x)y' + q(x)y = 0$$

If y_1 and y_2 are solutions then so is $c\,y_1 + d\,y_2$

$$\begin{aligned} & (c_1y_1 + c_2y_2)'' + p(c_1y_1 + c_2y_2)' + q(c_1y_1 + c_2y_2) \\ = \; & c_1y_1'' + c_2y_2'' + pc_1y_1' + pc_2y_2' + qc_1y_1 + qc_2y_2 \\ = \; & c_1(y_1'' + py_1' + qy_1) + c_2(y_2'' + py_2' + qy_2) \\ = \; & c_1 \cdot 0 + c_2 \cdot 0 = 0. \end{aligned}$$

Caution

Does not hold for nonhomogeneous d.e.

$$y'' + y = 1$$

$y_1 = 1 + \cos x$ is a solution but not

$$y = 2y_1 = 2 + 2\cos x$$

Example 12

$$y'' - y = 0, \; \overbrace{y(0) = 5, \; y'(0) = 3}^{\text{Initial value problem}}$$

e^x, e^{-x} are solutions

$$\boxed{y = c_1 e^x + c_2 e^{-x}}$$

$$5 = c_1 + c_2$$
$$3 = c_1 - c_2$$

$$\Rightarrow \quad y = 4e^x + e^{-x}$$

Main Theorem for Hom. 2^{nd} order d.e

$$y'' + p(x)y' + q(x)y = 0$$

If y_1 and y_2 are linearly independent solutions then

General solution: $y = c_1\, y_1 + c_2\, y_2$

Particular solution: Fix some value of c_1 and c_2

Example

$$\boxed{y'' = -y}$$

$$\left.\begin{array}{l} y = \sin x \\ y = \cos x \end{array}\right\}$$ two linearly indep solutions

$$y = c_1 \sin x + c_2 \cos x$$ General solution

$$y = 3 \sin x + 4 \cos x$$ Particular solution

Hom. 2nd order d.e with constant coefficients

$$y'' + ay' + by = 0$$

Characteristic equation

$$\lambda^2 + a\lambda + b = 0$$

$$\boxed{\lambda_1, \lambda_2 = \frac{-a \pm \sqrt{a^2 - 4b}}{2}}$$

Case 1: two real roots $\quad y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$

Case 2: double root $\quad y = (c_1 + c_2 x) e^{-\frac{ax}{2}}$

Case 3: complex roots

$$y = e^{-\frac{a}{2}x}(c_1 \cos wx + c_2 \sin wx)$$

Ex 13: Case 1: two real roots $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$

$$y'' + y' - 2y = 0, \quad y(0) = 4, y'(0) = -5$$

$$\Downarrow$$

$$\lambda = \frac{-1 \pm \sqrt{1 - 4(-2)}}{2} = 1, -2$$

Gen sol: $y = c_1 e^{x} + c_2 e^{-2x}$

$$\begin{aligned} 4 &= c_1 + c_2 \\ -5 &= c_1 - 2c_2 \end{aligned} \quad \Rightarrow \quad y = e^{x} + 3e^{-2x}$$

Tip: Always verify your solution

$$y'' + y' - 2y = 0, \quad y(0) = 4, y'(0) = -5$$

Is my answer correct?

$$y = e^{x} + 3e^{-2x}$$

$$\Rightarrow y(0) = 1 + 3$$

$$\Rightarrow y' = e^{x} - 6e^{-2x} \qquad \Rightarrow y'(0) = 1 - 6$$

$$\Rightarrow y'' = e^{x} + 12e^{-2x}$$

Case 2: double roots $y = (c_1 + c_2 x)e^{-\frac{ax}{2}}$

$$\lambda_1, \lambda_2 = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$ = 0

Convince yourself

$$\boxed{y = xe^{-\frac{ax}{2}}}$$

satisfy $y'' + ay' + by = 0$

Ex 14: Case 2: double roots $y = (c_1 + c_2 x)e^{-\frac{ax}{2}}$

$$y'' - 4y' + 4y = 0, \quad y(0) = 3, y'(0) = 1$$

$$\Downarrow$$

$$\lambda = \frac{4 \pm \sqrt{16 - 4(4)}}{2} = 2$$

Gen sol: $y = (c_1 + c_2 x)e^{2x}$

$$\begin{aligned} 3 &= c_1 \\ 1 &= 2c_1 + c_2 \end{aligned} \Rightarrow y = (3 - 5x)e^{2x}$$

Case 3: Complex roots $\quad y'' + ay' + by = 0$

$$\lambda = \frac{-a \pm \sqrt{a^2 - 4b}}{2} \qquad < 0$$

$$\lambda = -\frac{a}{2} \pm iw \qquad w = \sqrt{b - \frac{a^2}{4}}$$

$$y = e^{-\frac{a}{2}x}(c_1 \cos wx + c_2 \sin wx)$$

Complex

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$$e^{u+i\theta} = e^u(\cos\theta + i\sin\theta)$$

Doing things the "complex" way $\quad y'' + ay' + by = 0$

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

also complex

$$\lambda = -\frac{a}{2} \pm iw$$

$$\begin{aligned} y \;&=\; (\alpha + i\beta)e^{-\frac{a}{2}x}(\cos wx + i\sin wx) \\ &\quad + (\gamma + i\delta)e^{-\frac{a}{2}x}(\cos wx - i\sin wx) \end{aligned}$$

Taking real part

$$y \;=\; e^{-\frac{a}{2}x}((\alpha + \gamma)\cos wx + (\delta - \beta)\sin wx)$$

Main Theorem: Hom. 2nd order d.e

$$y'' + p(x)y' + q(x)y = 0$$

If y_1 and y_2 are linearly independent solutions then every solution looks like

$$y = c_1\, y_1 + c_2\, y_2$$

Theorem : Precise Statement

Let $p(x)$ and $q(x)$ be continuous on the open interval (a,b) containing x_0 . For any real A and B, there exists a unique solution $y(x)$ defined on (a,b) to the IVP:

$$y'' + p(x)y' + q(x)y = 0$$

$$y(x_0) = A, y'(x_0) = B$$

Proof: Picard's theorem for 1st Order. Textbook Sect 1.5. Then use linear algebra to push it to 2nd Order

Ex 15: Case 3: complex

$$y = e^{-\frac{a}{2}x}(c_1 \cos wx + c_2 \sin wx)$$

$$y'' + 2y' + 5y = 0, \quad y(0) = 1, y'(0) = 5$$

$$\lambda = \frac{-2 \pm \sqrt{4 - 4(5)}}{2}$$

Gen sol: $y = e^{-x}(c_1 \cos 2x + c_2 \sin 2x)$

$$\begin{aligned} 1 &= c_1 \\ 5 &= -c_1 + 2c_2 \end{aligned}$$

$$\Rightarrow \quad y = e^{-x}(\cos 2x + 3 \sin 2x)$$

1.4.2 Nonhomogeneous d.e

$$y'' + p(x)y' + q(x)y = F(x)$$

nonhomogeneous if $F(x) \neq 0$

Different superposition principle

If y_1 , y_2 are solutions:

$$y_1'' + p(x)y_1' + q(x)y_1 = F(x)$$

$$y_2'' + p(x)y_2' + q(x)y_2 = F(x)$$

$$(y_1 - y_2)'' + p(y_1 - y_2)' + q(y_1 - y_2) = 0$$

1.4.2 Nonhomogeneous d.e

If y_1 , y_2 are solutions:

$$y_1'' + p(x)y_1' + q(x)y_1 = F(x)$$

$$y_2'' + p(x)y_2' + q(x)y_2 = F(x)$$

$$(y_1 - y_2)'' + p(y_1 - y_2)' + q(y_1 - y_2) = 0$$

Homogeneous solution!

1.4.2 Gen sol of nonhomogeneous d.e

$$y'' + p(x)y' + q(x)y = F(x)$$

General Solution

$$\boxed{y = y_h + y_p}$$

Solution to nonhom.

$$y_h'' + py_h' + qy_h = 0$$

$$y_p'' + py_p' + qy_p = F(x)$$

$$(y_h + y_p)'' + p(y_h + y_p)' + q(y_h + y_p) = F$$

Analogy: Odd / Even integers

- Even + Even = Even
- Odd + Even = Odd
- Odd – Odd = Even

$$
\begin{aligned}
y_{h_1} + y_{h_2} &= y_{h_3} \\
y_h + y_{p_1} &= y_{p_2} \\
y_{p_1} - y_{p_2} &= y_h
\end{aligned}
$$

Every odd integer = 1 + some even integer

= 999 + some even integer

All odd integers = 1 + All even integers

= 999 + All even integers

1.4.2 Gen sol of nonhomogeneous d.e

$$y'' + p(x)y' + q(x)y = F(x)$$

1) Find all homogeneous solution
2) Find one particular solution (to non-hom. d.e.)

General Solution

$$\boxed{y = y_h + y_p}$$

Particular, i.e. no arbitrary constants

1.4.2 Method of undetermined coeff

$$y'' + ay' + by = F(x)$$

Conditions: constants

- Polynomials
- Exponentials
- Sine/Cosine

Method: Make educated guesses!

Ex 16   Polynomial case

$$y'' - 4y' + y = x^2 + x + 2$$

Try polynomial of same degree

$$\begin{aligned} y &= Ax^2 + Bx + C \\ y' &= 2Ax + B \\ y'' &= 2A \end{aligned}$$

x^2	$A = 1,$
x	$B - 8A = 1 \Rightarrow B = 9,$
const	$2A - 4B + C = 2 \Rightarrow C = 36.$

Ex 18 Exponential case

$$y'' - 4y' + 2y = 2x^3 e^{2x}$$

Try exp of same type

$$\begin{aligned} y &= ue^{2x} \\ y' &= u'e^{2x} + 2ue^{2x} \\ y'' &= u''e^{2x} + 4u'e^{2x} + 4ue^{2x} \end{aligned}$$

$$u'' - 2u = 2x^3$$

Reduces to polynomial case

More complex numbers

Fact 1 $$e^{u+i\theta} = e^{u}(\cos\theta + i\sin\theta)$$

Fact 2 $$y = u + iv$$

$$y'' + ay' + by = h_1 + ih_2$$

$$\begin{aligned}(u'' + iv'') + a(u' + iv') + b(u + iv) &= h_1 + ih_2, \\ (u'' + au' + bu) + i(v'' + av' + bv) &= h_1 + ih_2\end{aligned}$$

Example $$\frac{1}{2+3i} = \frac{1}{2+3i} \times \frac{2-3i}{2-3i} = \frac{2}{13} + i\frac{-3}{13}$$

Ex 20 Sine/Cosine case

$$y'' + 4y = 16x \sin 2x$$

Solve
$$z'' + 4z = 16xe^{i2x}$$
$$= 16x(\cos 2x + i \sin 2x)$$

$$\begin{aligned} z &= ue^{i2x} \\ z' &= u'e^{i2x} + 2iue^{i2x} \\ z'' &= u''e^{i2x} + 4iu'e^{i2x} - 4ue^{i2x} \end{aligned}$$

$$u'' + 4iu' = 16x$$

Reduces to polynomial case

Ex 20 Sine/Cosine case

$$u'' + 4iu' = 16x$$

$$\begin{aligned} u &= Ax + B \\ u' &= A \\ u'' &= 0 \end{aligned} \quad \Rightarrow \quad 4iA = 16x$$

something is wrong

$$\begin{aligned} u &= Ax^2 + Bx + C \\ u' &= 2Ax + B \\ u'' &= 2A \end{aligned} \quad \Rightarrow \quad \begin{aligned} 2A + 4iB &= 0 \\ 8iA &= 16 \end{aligned}$$

$$\boxed{u = -2ix^2 + x}$$

Ex 20

$$y'' + 4y = 16x \sin 2x$$

$$z'' + 4z = 16xe^{i2x}$$

$$u = -2ix^2 + x$$

$$\begin{aligned} z &= (-2ix^2 + x)e^{i2x} \\ &= (-2ix^2 + x)(\cos 2x + i \sin 2x) \end{aligned}$$

$$y = \text{Im}(z) = x \sin 2x - 2x^2 \cos 2x.$$

Ex 20 Sine/Cosine case (Alternative)

$$y'' + 4y = 16x \sin 2x$$

Why?

Try $$y = (ax^2 + bx + c)\sin(2x) + (dx^2 + ex + f)\cos(2x)$$

LHS: $$2a\sin(2x) + 4(2ax + b)\cos(2x) + 2d\cos(2x) - 4(2dx + e)\sin(2x)$$

= RHS gives *d=-2, b=1, a=0, e=0*

$$y_p = x\sin(2x) - 2x^2\cos(2x)$$

Ex 20 : hom. solution coinciding with r.h.s

$$y'' + 4y = 16x \sin 2x$$

Hom Sol: $\lambda^2 = -4$

$$y_h = C_1 \cos 2x + C_2 \sin 2x$$

$$y_p = (ax + b) \cos 2x + (cx + d) \sin 2x$$

This guess is no good because *b* and *d* are redunant

Ex 21

$$y'' + 2y' + 5y = 16xe^{-x}\cos 2x$$

What’s the homogeneous solution?

Method of Variation of Parameters

$$y'' + \underbrace{p(x)}y' + \underbrace{q(x)}y = \underbrace{r(x)}$$

Conditions: 1) Continuous functions

2) Homogeneous solutions known

$$y_h = c_1 y_1 + c_2 y_2$$

Try $$y_p = u y_1 + v y_2$$

Method of Variation of Parameters

$$y'' + p(x)y' + q(x)y = r(x)$$

$$y_p = uy_1 + vy_2$$

$$y_p' = u'y_1 + uy_1' + v'y_2 + vy_2'$$

Set $\quad u'y_1 + v'y_2 = 0$

$$\begin{aligned} y_p' &= uy_1' + vy_2', \\ y_p'' &= u'y_1' + uy_1'' + v'y_2' + vy_2''. \end{aligned}$$

Method of Variation of Parameters

$$y'' + p(x)y' + q(x)y = r(x)$$

$$y_p = uy_1 + vy_2$$

$$u'y_1' + v'y_2' = r \qquad u'y_1 + v'y_2 = 0$$

$$u' = -\frac{y_2 r}{y_1 y_2' - y_1' y_2}, \quad v' = \frac{y_1 r}{y_1 y_2' - y_1' y_2}$$

Method of Variation of Parameters

Standard form $y'' + p(x)y' + q(x)y = r(x)$

$$y_p = uy_1 + vy_2$$

$$u = -\int \frac{y_2 r}{y_1 y_2' - y_1' y_2}\,dx,$$

$$v = \int \frac{y_1 r}{y_1 y_2' - y_1' y_2}\,dx.$$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Det of Wronskian

Example 23

$$y'' - y = e^{-x} \sin e^{-x} + \cos e^{-x}$$

$$y_h = c_1 e^{x} + c_2 e^{-x}$$

$$y_p = u e^{x} + v e^{-x}$$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix} = -2$$

$$\boxed{\text{Ex 23} \quad y'' - y = e^{-x} \sin e^{-x} + \cos e^{-x}}$$

$$y_p = ue^{x} + ve^{-x}$$

$$\begin{aligned} u &= \frac{1}{2} \int e^{-x} \left(e^{-x} \sin e^{-x} + \cos e^{-x}\right) dx \\ v &= -\frac{1}{2} \int e^{x} \left(e^{-x} \sin e^{-x} + \cos e^{-x}\right) dx \end{aligned}$$

$\boxed{\text{Ex 23} \quad y'' - y = e^{-x}\sin e^{-x} + \cos e^{-x}}$

$$\begin{aligned}
u &= \frac{1}{2}\int e^{-x}\left(e^{-x}\sin e^{-x} + \cos e^{-x}\right)dx \\
&= \frac{1}{2}\int w\sin w + \cos w dw \\
&= -\frac{1}{2}(2\sin e^{-x} - e^{-x}\cos e^{-x}).
\end{aligned}$$

$\boxed{w = e^{-x}}$

$$\text{Ex 23} \quad y'' - y = e^{-x}\sin e^{-x} + \cos e^{-x}$$

$$\begin{aligned} v &= -\frac{1}{2}\int e^{x}\left(e^{-x}\sin e^{-x} + \cos e^{-x}\right)dx \\ &= -\frac{1}{2}\int \frac{\sin w}{w} + \frac{\cos w}{w^2}dw \\ &= -\frac{1}{2}\frac{\cos w}{w} \\ &= -\frac{1}{2}e^{x}\cos e^{-x} \end{aligned}$$

$$w = e^{-x}$$

$$\boxed{y = c_1 e^{x} + c_2 e^{-x} - e^{x}\sin e^{-x}}$$

Summary: 2nd Order Linear D.E.

$$y'' + p(x)y' + q(x)y = F(x)$$

$$\begin{aligned} y &= y_h + y_p \\ &= \underbrace{c_1 y_1 + c_2 y_2}_{} + y_p \end{aligned}$$

If *p* and *q* are constants, use charac. equation

Method of undetermined coeff

Variation of parameters