3 – IMAGE TRANSFORMS (A)

A transform maps image data into a different mathematical space via a transformation equation. Many transforms are used in image processing, e.g., Fourier transform, cosine transform, Walsh-Hadamard transform, Haar transform, principal components transform, and wavelet transform.

The Fourier transform is the best known and the most widely used. It decomposes an image into its sinusoidal components, thus enabling easy examination and processing of certain frequencies of the image. It is used in algorithms for image analysis, filtering, compression and restoration, and 3D medical imaging.

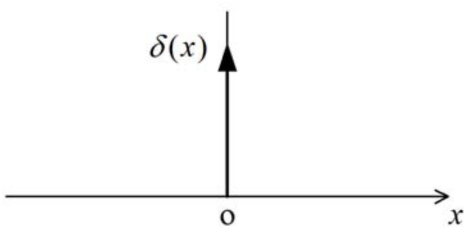
BASICS

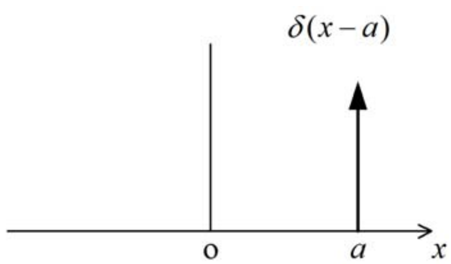
Delta Function

The 1D delta function (also called the Dirac function and the impulse function) is defined by the following two properties:

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases} \quad (1)$$

$$\int_{-\infty}^{+\infty} \delta(x)dx = 1 \quad (2)$$

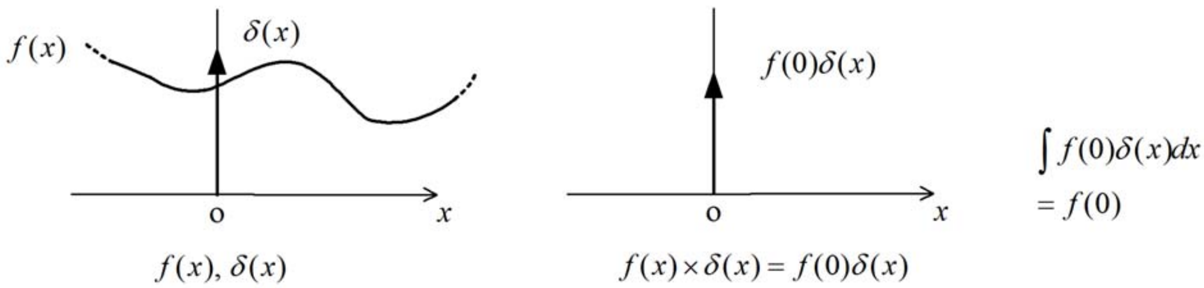




Useful properties:

$$f(x) \times \delta(x) = f(0)\delta(x) \tag{3}$$

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\delta(x)dx &= \int_{-\infty}^{\infty} f(0)\delta(x)dx \\ &= f(0)\int_{-\infty}^{\infty} \delta(x)dx \\ &= f(0) \end{aligned} \tag{4}$$

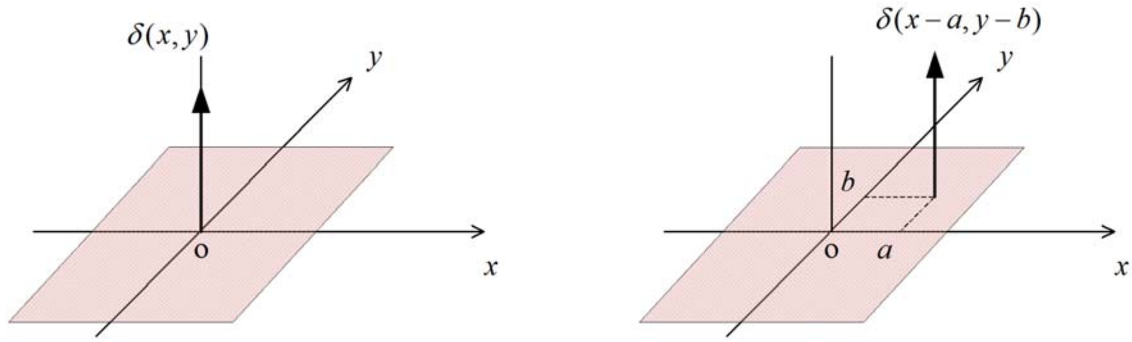


$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)) \tag{5}$$

The <u>2D delta function</u> $\delta(x, y)$ is defined by:

$$\delta(x,y) = \begin{cases} 0 & x, y \neq 0 \\ \infty & x, y = 0 \end{cases} \tag{6}$$

$$\iint_{-\infty}^{+\infty} \delta(x,y)dxdy = 1 \tag{7}$$



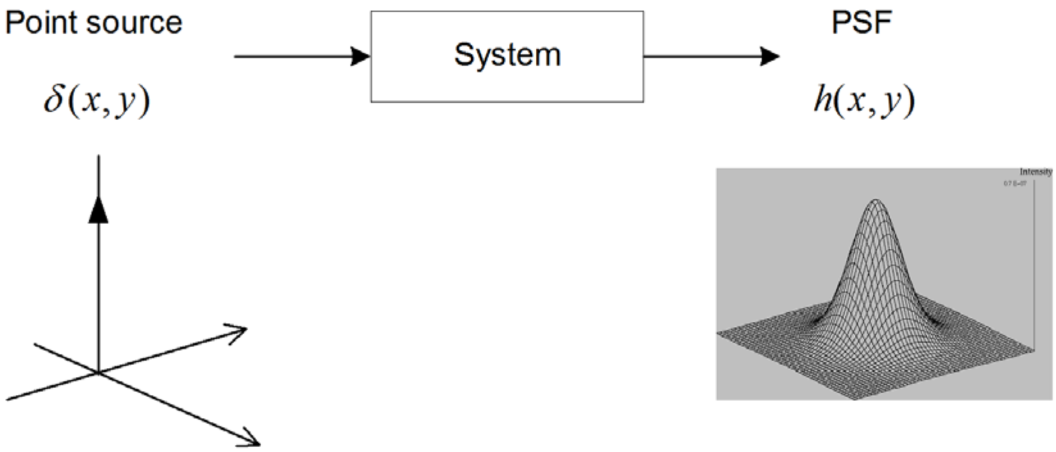
$$\begin{aligned}\iint_{-\infty}^{+\infty} f(x,y)\delta(x,y)dxdy &= f(0,0) && (8)\\ \iint_{-\infty}^{+\infty} f(x,y)\delta(x-a,y-b)dxdy &= f(a,b) && (9)\end{aligned}$$

System Response

The output of a system with $\delta(x, y)$ as the input function is termed the impulse response, i.e.,

$$h(x,y) \equiv \mathcal{O}\{\delta(x,y)\} \tag{10}$$

In optical systems, the impulse response is often called the point spread function (PSF).



The response of a system to an arbitrary input $f(x, y)$ is found by convolving $h(x, y)$ with $f(x, y)$:

$$\begin{aligned} g(x,y) &= h(x,y) \star f(x,y) \\ &= \int\!\!\int_{-\infty}^{+\infty} h(\alpha,\beta) f(x-\alpha, y-\beta) d\alpha\, d\beta \end{aligned} \tag{11}$$

It can easily be shown that

$$h(x,y) \star f(x,y) = f(x,y) \star h(x,y) \tag{12}$$

In the frequency domain, we have the system transfer function

$$H(u,v) \equiv \mathcal{F}\{h(x,y)\} \tag{13}$$

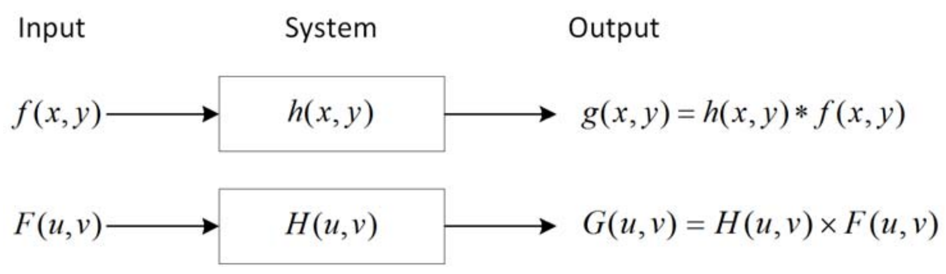
where $\mathcal{F}$ is the Fourier transform operator. In the case of optical systems, the transfer function is called the *optical transfer function* (OTF):

$$\text{OTF} = \mathcal{F}\{\text{PSF}\} \tag{14}$$

The input and output signals are related by

$$G(u,v) = H(u,v)F(u,v) \tag{15}$$

where $F(u,v) = \mathcal{F}\{f(x,y)\}$ and $G(u,v) = \mathcal{G}\{f(x,y)\}$.



THE FOURIER TRANSFORM

The 2D Fourier transform of the function $f(x, y)$ is defined as

$$F(u, v) = \int_{-\infty}^{+\infty}\int f(x, y) \exp[-j2\pi(ux + vy)]dxdy \tag{16}$$

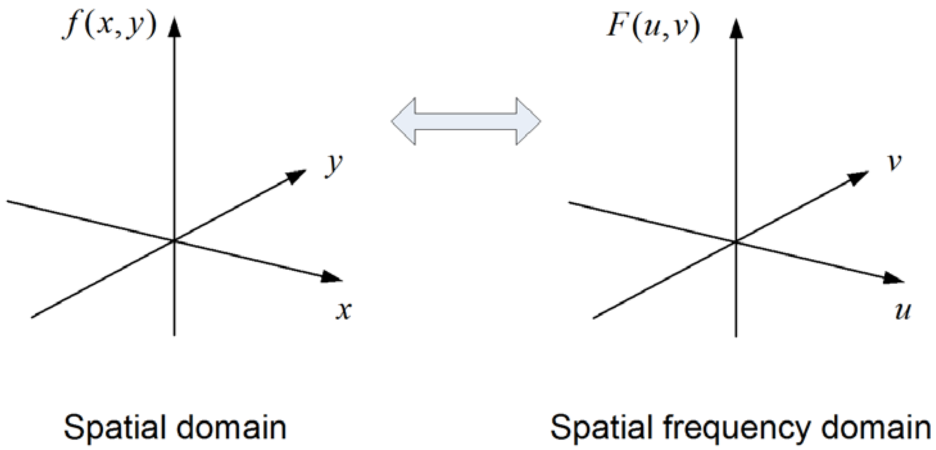
and its inverse is defined as

$$f(x, y) = \int_{-\infty}^{+\infty}\int F(u, v) \exp[j2\pi(ux + vy)]dudv \tag{17}$$

Note:

$$\exp[j2\pi(ux + vy)] = \cos[2\pi(ux + vy)] + j \sin[2\pi(ux + vy)] \tag{18}$$
$$\exp[-j2\pi(ux + vy)] = \cos[2\pi(ux + vy)] - j \sin[2\pi(ux + vy)] \tag{19}$$

$F(u, v)$ is called the frequency spectrum of $f(x, y)$. If x and y represent spatial coordinates, then u and v are the spatial frequencies (measured in cycles per unit distance) along the x and y axes, respectively.



The Fourier transform decomposes a signal $f(x, y)$ into complex exponentials. It provides information on the sinusoidal composition of a signal $f(x, y)$ at different spatial frequencies.

Example

Consider

$$f(x,y) = 5\sin(2\pi x) + 5 \quad (x \text{ in mm})$$

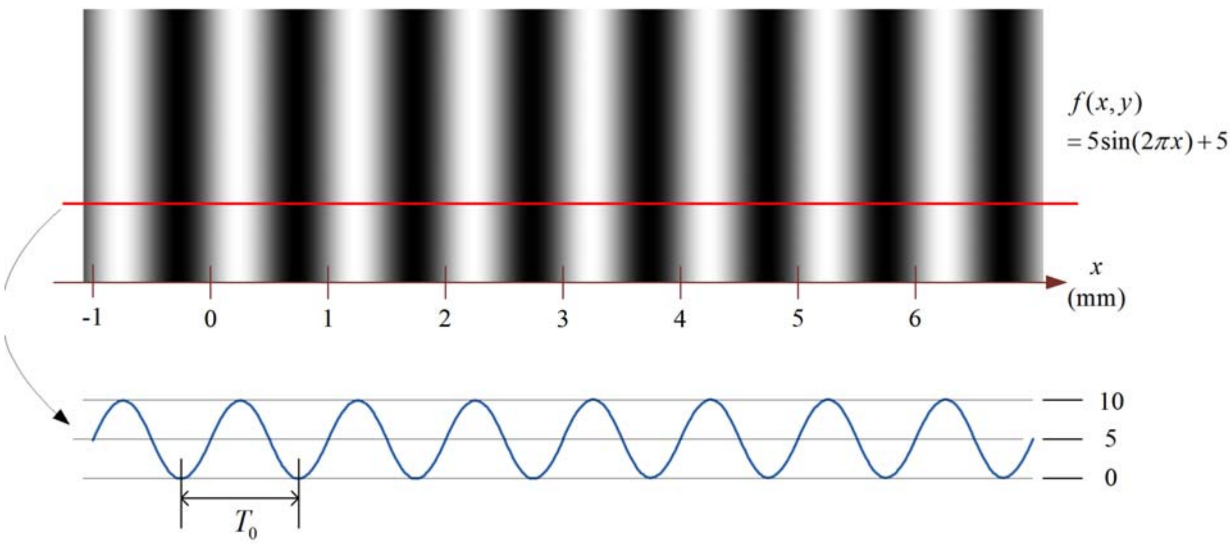
$$f(x,y) = 0 \ \Rightarrow x = \ldots, -0.25,\ 0.75,\ 1.75\ \ldots$$

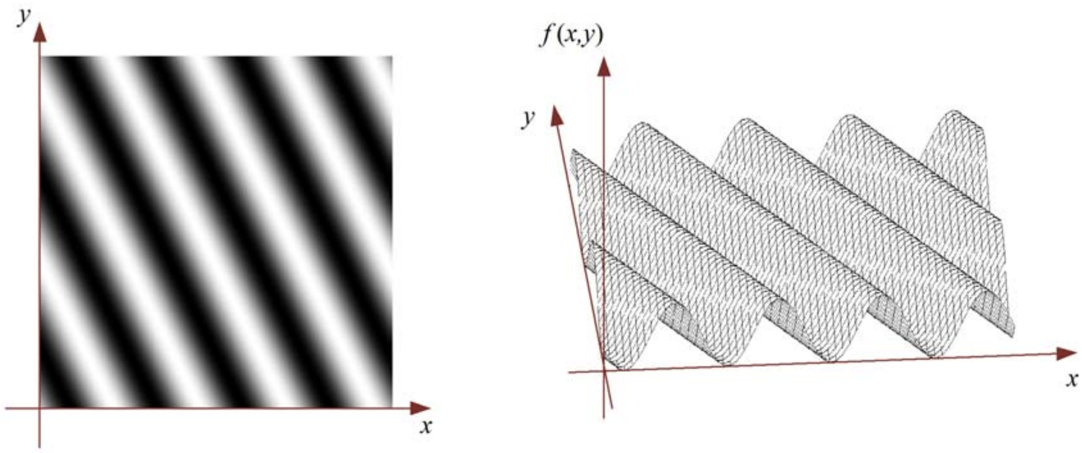
Hence, this sine wave has period

$$T_0 = 1 \text{ mm}$$

and spatial frequency

$$u_0 = \frac{1}{T_0} = 1 \text{ cycle/mm}$$





$f(x,y) = 0.5 + 0.5\sin(10x + 5y)$

The 2D Fourier transform can be computed in two separable steps:

$$
\begin{aligned}
F(u,v) &= \iint\limits_{-\infty}^{+\infty} f(x,y)\exp[-j2\pi(ux+vy)]dxdy \\
&= \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} f(x,y)\exp(-j2\pi ux)dx\right]\exp(-j2\pi vy)dy \qquad (20)\\
&= \int_{-\infty}^{\infty} F(u,y)\exp(-j2\pi vy)dy \qquad (21)
\end{aligned}
$$

where

$$F(u,y) = \int_{-\infty}^{\infty} f(x,y)\exp(-j2\pi ux)dx \qquad (22)$$

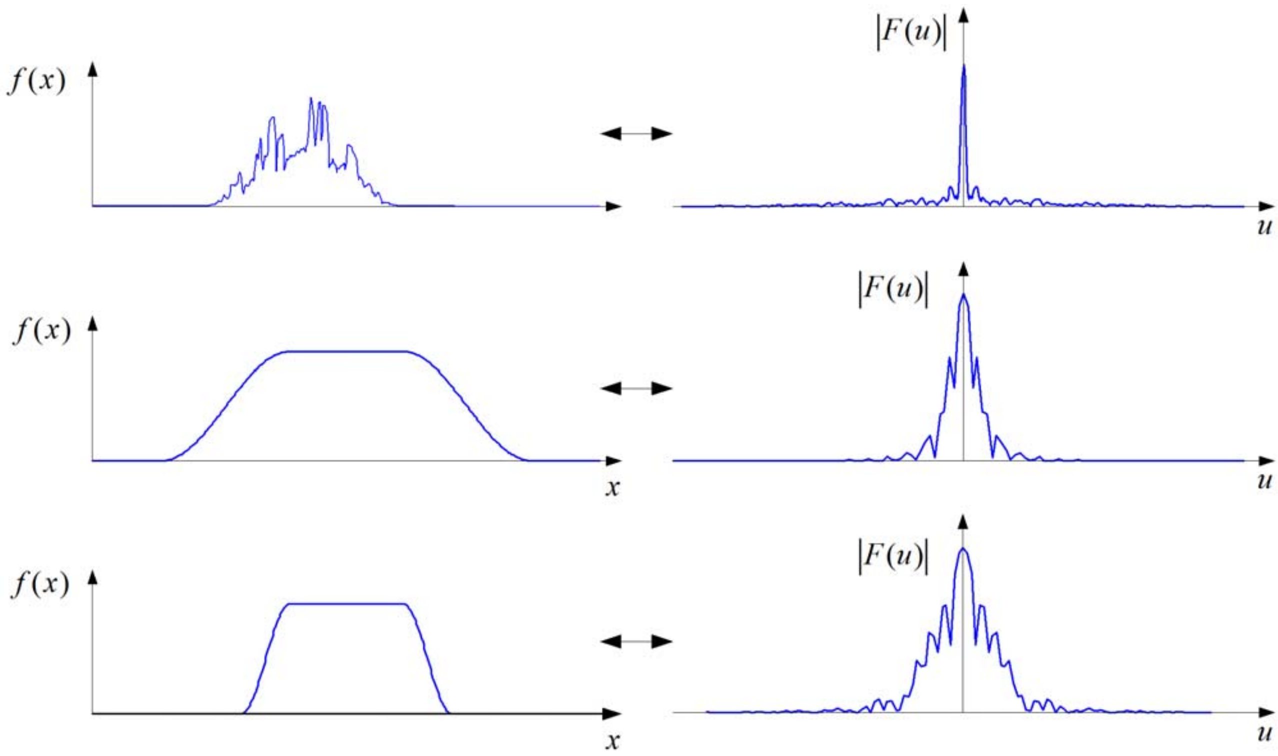
In general, the Fourier transform is complex:

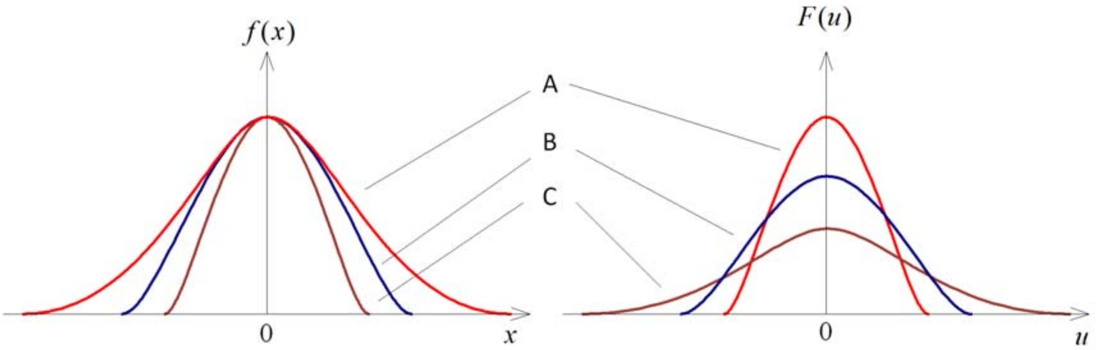
$$F(u,v) \equiv R(u,v) + jI(u,v) \equiv |F(u,v)|\exp[j\phi(u,v)]$$

with real part $R(u,v)$ and an imaginary part $I(u,v)$: We define

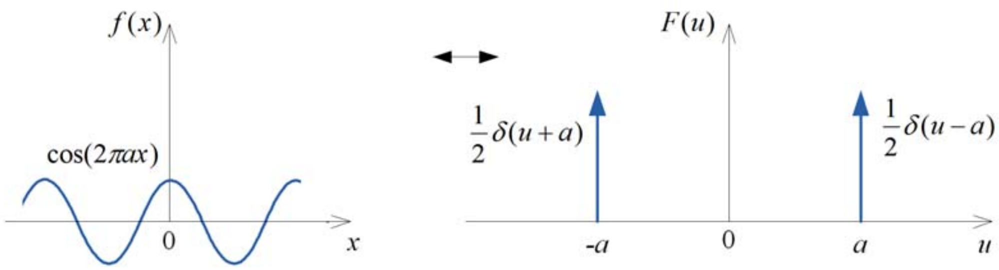
$$
\begin{array}{lll}
\text{Fourier spectrum:} & |F(u,v)| & = [R^2(u,v) + I^2(u,v)]^{1/2} \\
\text{Phase spectrum:} & \phi(u,v) & = \tan^{-1}[I(u,v)/R(u,v)] \\
\text{Power spectrum:} & P(u,v) & = |F(u,v)|^2 = R^2(u,v) + I^2(u,v)
\end{array}
$$

1D Fourier transform examples





$$\text{Gaussian function}: \exp(-ax^2) \leftrightarrow \sqrt{\frac{\pi}{a}}\exp(-\pi^2 u^2 / a)$$



2D Fourier transform examples

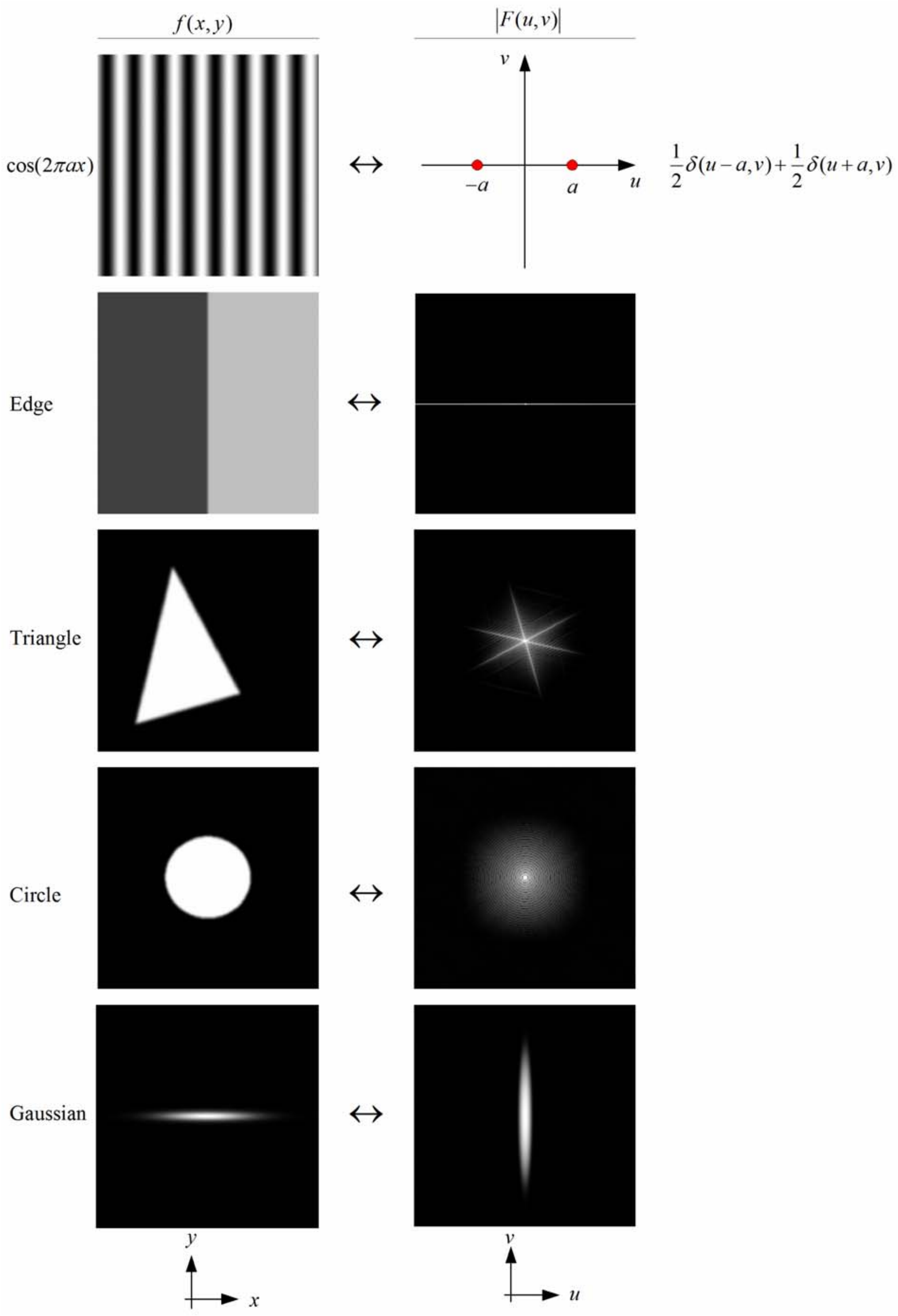
The images $f(x, y)$, are in column 1 and their respective Fourier spectra, $|F(u, v)|$, in column 2.

Note:

- The origin $(u, v) = (0, 0)$ corresponds to zero spatial frequency; it is located at the centre in the examples. $F(0, 0)$ is the DC term. Spatial frequency increases with distance from the origin.
- The spectral distribution of the image function $f(x, y)$ is given by $|F(u, v)|$, i.e., $|F(u, v)|$ shows how the spectral components of $f(x, y)$ vary as a function of spatial frequency.
- The Fourier spectrum $|F(u, v)|$ is symmetrical about the origin for a real image function, i.e.,

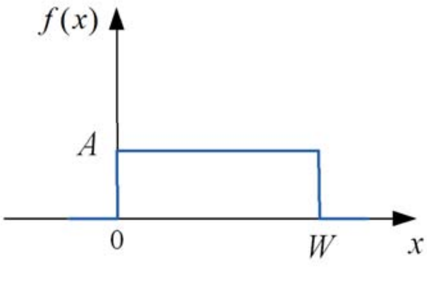
$$|F(u, v)| = |F(-u, -v)| \tag{23}$$

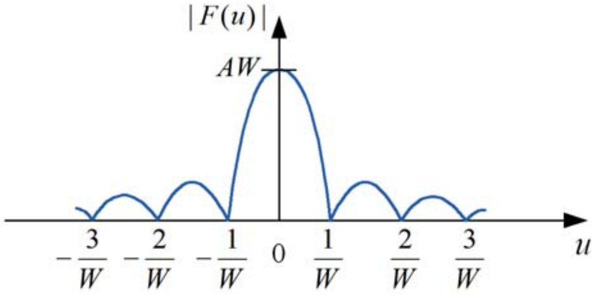
- $f(x, y)$: strong edge $\Rightarrow$
  $|F(u, v)|$: prominent lines with orientation perpendicular to the edge



$f(x,y)$
$|F(u,v)|$
$\cos(2\pi ax)$
$\leftrightarrow$
v
u
$-a$
a
$\frac{1}{2}\delta(u-a,v)+\frac{1}{2}\delta(u+a,v)$
Edge
Triangle
Circle
Gaussian
y
x
v
u

1D Example





The Fourier transform of $f(x)$ is

$$
\begin{aligned}
F(u) &= \int_{-\infty}^{\infty} f(x)\exp(-j2\pi ux)dx = A\int_0^W \exp(-j2\pi ux)dx \\
&= \frac{-A}{j2\pi u}[\exp(-j2\pi ux)]_0^W \\
&= \frac{-A}{j2\pi u}[\exp(-j2\pi uW) - 1] \\
&= \frac{-A}{j2\pi u}\exp(-j\pi uW)[\exp(-j\pi uW) - \exp(j\pi uW)] \\
&= \frac{-A}{j2\pi u}\exp(-j\pi uW)[-2j\sin(\pi uW)] \\
&= A\frac{\sin(\pi uW)}{\pi u}\exp(-j\pi uW) \\
&= AW\frac{\sin(\pi uW)}{\pi uW}\exp(-j\pi uW) \\
&= AW\text{sinc}(uW)\exp(-j\pi uW) \\
|F(u)| &= AW|\text{sinc}(uW)|
\end{aligned}
$$

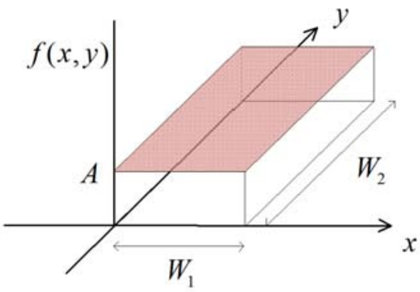
2D Example

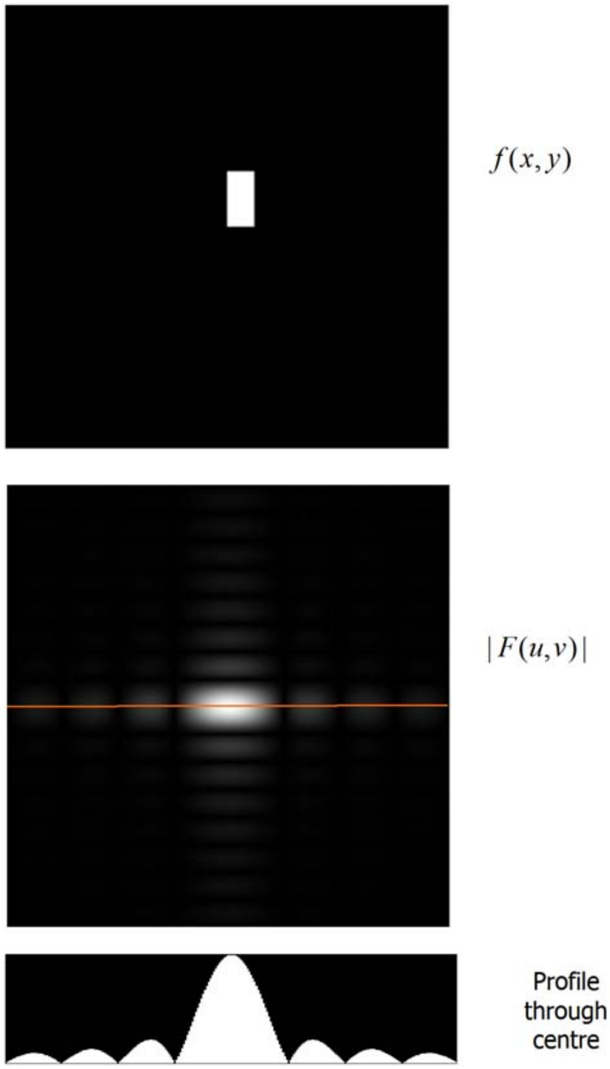
The Fourier transform of $f(x, y)$ is

$$
\begin{aligned}
F(u,v) &= \iint_{-\infty}^{+\infty} f(x,y)\exp[-j2\pi(ux+vy)]dx\,dy \\
&= A\int_0^{W_2}\int_0^{W_1} \exp[-j2\pi(ux+vy)]dx\,dy \\
&= A\int_0^{W_1} \exp(-j2\pi ux)dx \int_0^{W_2} \exp(-j2\pi vy)dy \\
&= A\left[\frac{\exp(-j2\pi ux)}{-j2\pi u}\right]_0^{W_1}\left[\frac{\exp(-j2\pi vy)}{-j2\pi v}\right]_0^{W_2} \\
&= AW_1W_2\mathrm{sinc}(uW_1)\mathrm{sinc}(vW_2)\exp[-j\pi(uW_1+vW_2)]
\end{aligned}
$$

The spectrum is

$$
|F(u,v)| = AW_1W_2|\mathrm{sinc}(uW_1)||\mathrm{sinc}(vW_2)|
$$





Properties of the 2D Continuous Fourier Transform

Linearity

The Fourier transform is a linear operator:

$$\mathcal{F}\{af_1(x,y) + bf_2(x,y)\} = aF_1(u,v) + bF_2(u,v) \tag{24}$$

where a and b are constants.

Conjugate symmetry

Conjugation property:

$$\mathcal{F}\{f^*(x,y)\} = F^*(-u,-v) \tag{25}$$

where * denotes the complex conjugation of a variable.

Proof (1D case)

$$\begin{aligned}
F(u) &= \int f(x)\exp(-j2\pi ux)dx \\
F^*(u) &= \int f^*(x)\exp(j2\pi ux)dx \\
F^*(-u) &= \int f^*(x)\exp(-j2\pi ux)dx \\
&= \mathcal{F}\{f^*(x)\}
\end{aligned}$$

i.e.,

$$f^*(x) \leftrightarrow F^*(-u) \tag{26}$$

We note that for real $f(x)$,

$$f(x) = f^*(x)$$

Taking the FT of both sides,

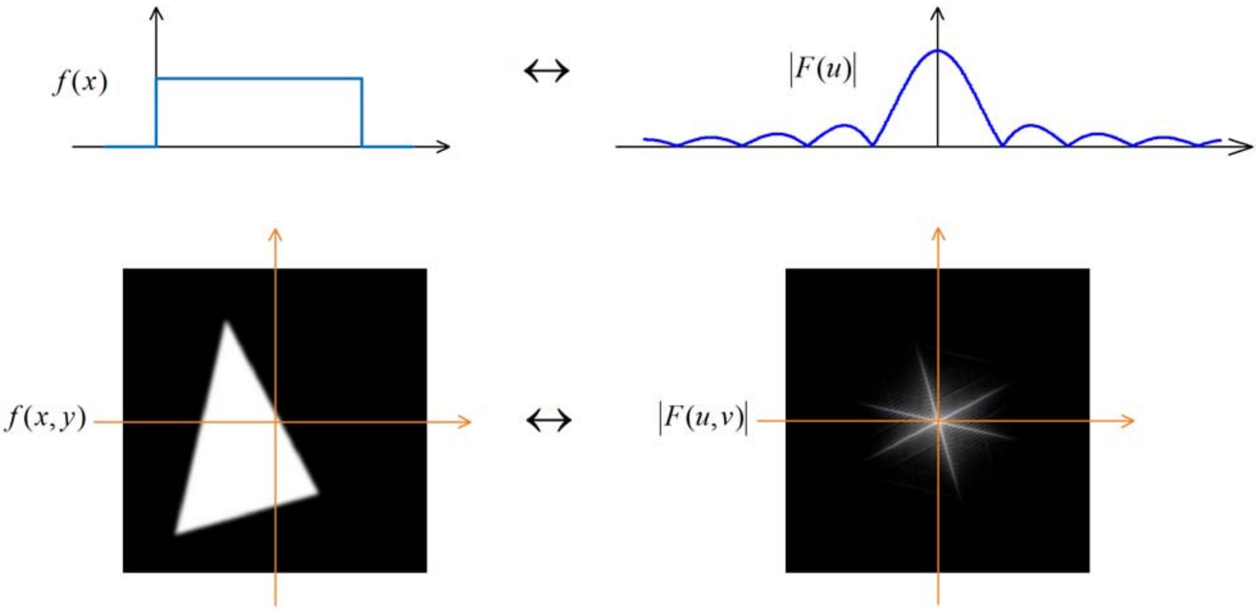
$$\begin{aligned} F(u) &= \mathcal{F}\{f^*(x)\} = F^*(-u) \\ |F(u)| &= |F^*(-u)| \\ &= |F(-u)| \end{aligned}$$

$\Rightarrow |F(u)|$ is an even function.

In the 2D case,

$$|F(u,v)| = |F(-u,-v)| \tag{27}$$

i.e., $|F(u,v)|$ is symmetrical about the origin.



Translation

A spatial shift (translation) in the input plane results in a phase shift in the output plane:

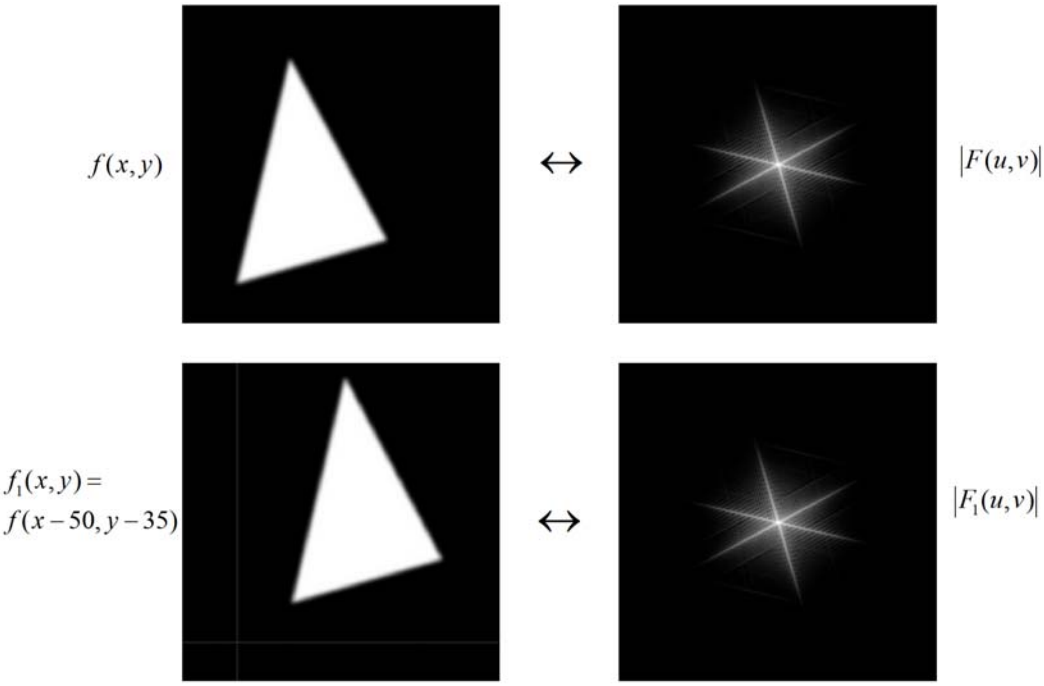
$$\mathcal{F}\{f(x-a, y-b)\} = F(u,v)\exp[-j2\pi(au+bv)] \tag{28}$$

For example,

$$\mathcal{F}\{f(x-25, y-50)\} = F(u,v)e^{-j50\pi u}e^{-j100\pi v}$$

From Eq. (28), we see that a spatial shift does not affect the magnitude of the transform:

$$\begin{aligned}|\mathcal{F}\{f(x-a, y-b)\}| &= |F(u,v)\exp[-j2\pi(au+bv)]| \\ &= |F(u,v)| \\ &= |\mathcal{F}\{f(x,y)\}|\end{aligned}$$



Proof (1D case)

$$\begin{aligned}\mathcal{F}\{f(x-a)\} &= \int f(x-a)\exp(-j2\pi ux)dx \\ &= \int f(\lambda)\exp(-j2\pi u(\lambda+a))dx \quad \text{by substituting } \lambda = x-a \\ &= \exp(-j2\pi au)\int f(\lambda)\exp(-j2\pi u\lambda)d\lambda \\ &= F(u)\exp(-j2\pi au)\end{aligned}$$

Scaling

A linear scaling of the spatial variables results in an inverse scaling of the spatial frequencies:

$$\mathcal{F}\{f(ax, by)\} = \frac{1}{|ab|} F(u/a, v/b) \tag{29}$$

For example,

$$\mathcal{F}\{f(2x, 2y)\} = \frac{1}{4} F(u/2, v/2)$$

Proof (1D case)

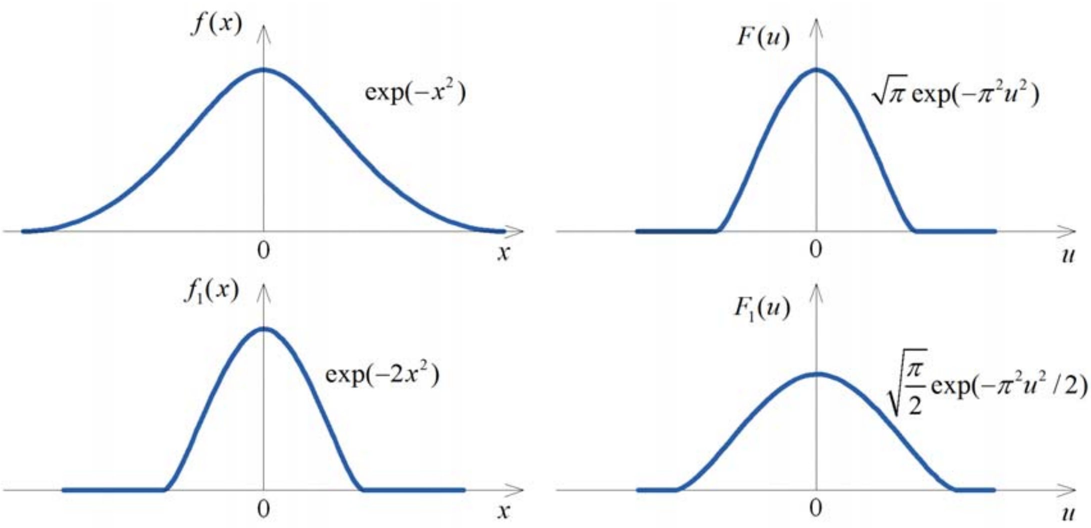
$$\begin{aligned} f(ax) &\leftrightarrow \int_{-\infty}^{\infty} f(ax)\exp(-j2\pi ux)dx \qquad a > 0 \\ &= \frac{1}{a}\int_{-\infty}^{\infty} f(\lambda)\exp(-j2\pi u\lambda/a)d\lambda \quad \text{by substituting } \lambda = ax \\ &= \frac{1}{a}F(u/a) \end{aligned}$$

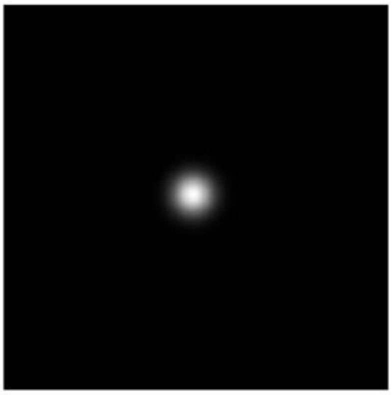
$$\begin{aligned} f(ax) &\leftrightarrow \int_{-\infty}^{\infty} f(ax)\exp(-j2\pi ux)dx \qquad a < 0 \\ &= -\frac{1}{a}\int_{-\infty}^{\infty} f(\lambda)\exp(-j2\pi u\lambda/a)d\lambda \quad \text{where } \lambda = ax \\ &= -\frac{1}{a}F(u/a) \end{aligned}$$

Hence

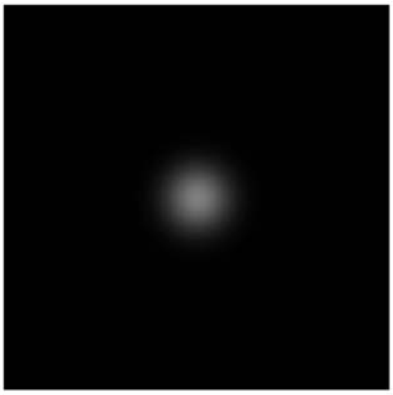
$$\mathcal{F}\{f(ax)\} = \frac{1}{|a|} F(u/a) \tag{30}$$

Examples

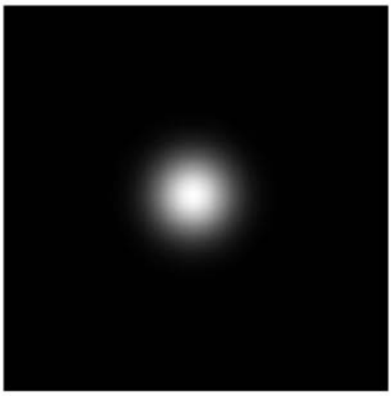




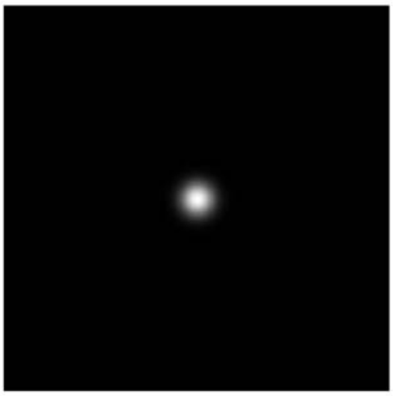
$f(x,y)$



$F(u,v)$



$f(0.5x,0.5y)$



$2F(2u,2v)$

Convolution

The 2D Fourier transform of two convolved functions is equal to the products of the transforms of the functions:

$$\mathcal{F}\{f(x,y) \star h(x,y)\} = F(u,v)H(u,v) \tag{31}$$

Conversely,

$$\mathcal{F}\{f(x,y)h(x,y)\} = F(u,v) \star H(u,v) \tag{32}$$

Rotation

If we introduce the polar coordinates

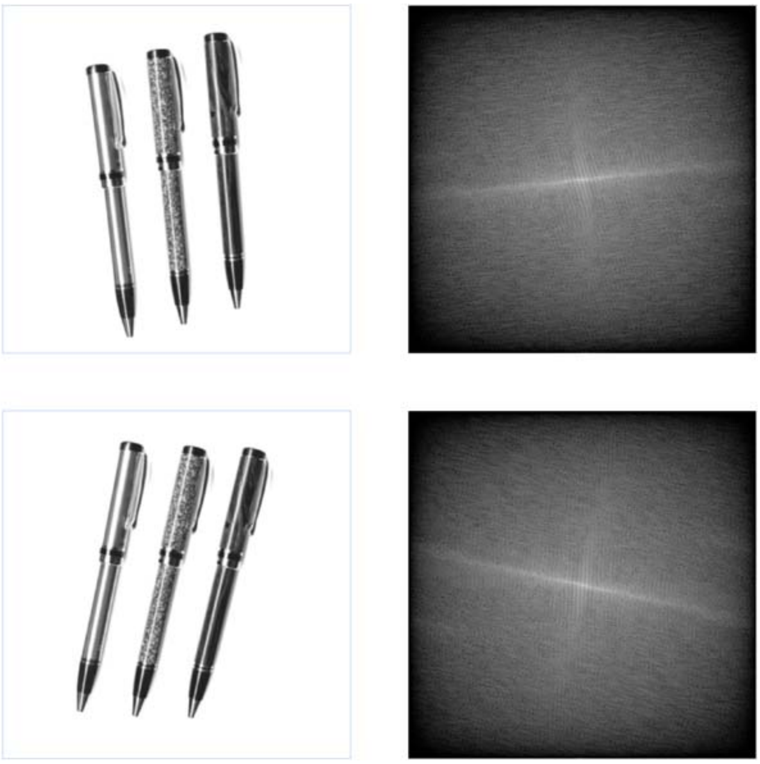
$$x = r\cos\theta \quad y = r\sin\theta \quad u = \omega\cos\phi \quad v = \omega\sin\phi$$

then $f(x,y)$ and $F(u,v)$ become $f(r,\theta)$ and $F(\omega,\phi)$, respectively. Direct substitution in the continuous Fourier transform pair yields

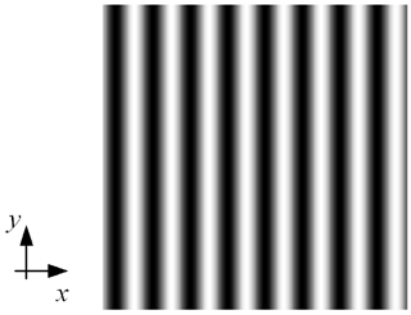
$$f(r,\theta+\theta_0) \Leftrightarrow F(\omega,\phi+\theta_0) \tag{33}$$

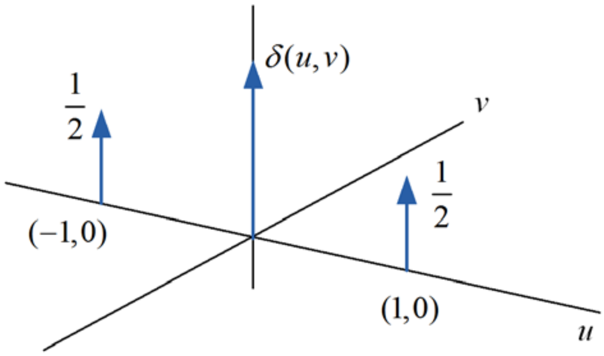
In other words, rotating $f(x,y)$ by an angle θ_0 rotates $F(u,v)$ by the same angle.

Example

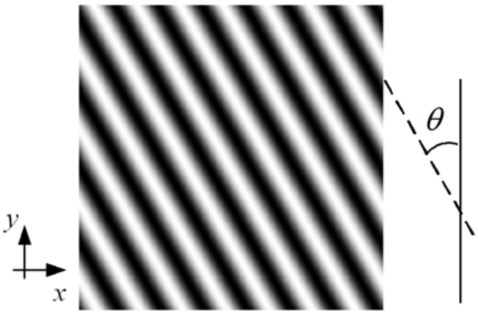


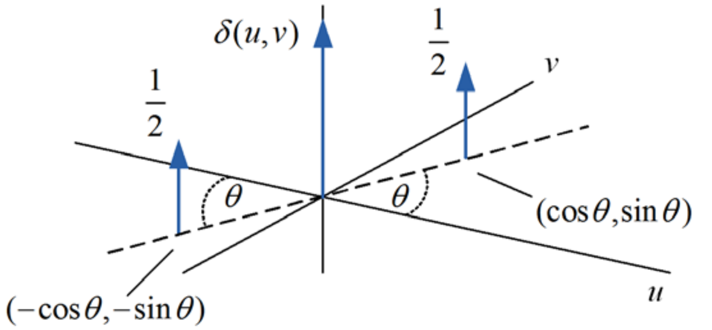
Example





$$1+\cos(2\pi x) \quad \leftrightarrow \quad \delta(u,v)+\frac{1}{2}\delta(u-1,v)+\frac{1}{2}\delta(u+1,v)$$





$$1+\cos[2\pi(x\cos\theta+y\sin\theta)] \quad \leftrightarrow \quad \begin{array}{l}\delta(u,v)+\\ \frac{1}{2}\delta(u+\cos\theta,v+\sin\theta)+\frac{1}{2}\delta(u-\cos\theta,v-\sin\theta)\end{array}$$