John Vince

Geometry for Computer Graphics

Formulae, Examples & Proofs

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John Vince MTech, PhD, CEng, FBCS
National Centre for Computer Animation
Bournemouth University, UK

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Dedication

This book is dedicated to my family, Annie, Samantha, Anthony, Megan, and Monty, who have not seen much of me over the past two years.

Preface

Anyone who has written programs for computer graphics, CAD, scientific visualization, computer games, virtual reality or computer animation will know that mathematics is extremely useful. Topics such as transformations, matrix algebra, vector algebra, curves and surfaces are at the heart of any application program in these areas, but the one topic that is really central is geometry, which is the theme of this book.

I recall many times when writing computer animation programs my own limited knowledge of geometry. I remember once having to create a 3D lattice of dodecahedrons as the basis for a cell growth model. At the time, I couldn't find a book on the subject and had to compute Platonic solid dihedral angles and vertex coordinates from scratch. The Internet had not been invented and I was left to my own devices to solve the problem. As it happened, I did solve it, and my new found knowledge of Platonic objects has never waned.

Fortunately, I no longer have to write computer programs, but many other people still do, and the need for geometry has not gone away. In fact, as computer performance has increased, it has become possible to solve amazingly complex three-dimensional geometric problems in real time.

The reason for writing this book is threefold: to begin with, I wanted to coordinate a wealth of geometry that is spread across all sorts of math books and the Internet; second, I wanted to illustrate how a formula was used in practice; third, I wanted to provide simple proofs for these formulas.

Personally, whenever I see an equation I want to know its origin. For example, why is the volume of a tetrahedron one-sixth of a set of vertices? Where does the 'one-sixth' come from? Take another example: why is the volume of a sphere four-thirds, π, radius cubed? Where does the 'four-thirds' come from? Why isn't it 'five-sixths'? This may be a personal problem I have about the origins of formulas but I do find that my understanding of a subject is increased when I understand its origins.

Quaternions are another example. There is still some mystique about what they are and how they work. I can think of no better way of understanding quaternions than to read about Sir William Rowan Hamilton and discover how he stumbled across his now famous non-commutative algebra.

I am the first to admit that I am not a mathematician, and this book is not intended to be read by mathematicians. A mathematician would have approached the subject with a greater logical rigour and employed formal structures that are relevant to the world of mathematics, but of little interest to a programmer wanting to find a formula for a parametric line equation intersecting a spherical surface.

For example, hyperplanes are a very powerful mathematical instrument for analyzing complex geometric scenarios, but this is not very relevant to a programmer who simply wants to know the line of intersection between two planes. Consequently, I have avoided the mathematical hierarchies used by mathematicians to compress their language into the smallest number of symbols. This is why I have avoided statements such as $H^n = R^{n-1} \times \{x_n | x_n \geq 0\ (x_n \in \mathbf{R})\}$, but included formulas such as: $A = \pi r^2$!

When I started this book I had no idea of its final structure. I asked colleagues if they had books on geometry that I could borrow. The first book I came across was *Mathematics Encyclopedia* edited by Max Shapiro. There I found a source of definitions that gave some initial breadth to the subject. I then discovered that I had in my own library *The VNR Concise Encyclopedia of Mathematics* edited by Gellert, Gottwald, Hellwich, Hästner and Kûstner. This book helped me understand some of the strategies used by mathematicians to resolve some standard geometric problems.

Then I discovered one of Springer's 'yellow' math books: *Handbook of Mathematics and Computational Science* by John Harris and Horst Stocker. Further 'yellow' books emerged from Springer: *Geometry I* by Marcel Berger, *Geometry: Plane and Fancy* by David Singer, and *Geometry: Our Cultural Heritage* by Audun Holme.

One of my favourite math books is *Mathematics: From the Birth of Numbers* by Jan Gullberg. It is a work of art, and Gullberg's clarity of writing inspired me to make my own explanations as precise and informative as possible.

It was only when I was half-way through my manuscript that I came across one of my favourite books *A Programmer's Geometry* by Adrian Bowyer and John Woodwark. When I opened it I realized that this is what my own book was about – a description of the geometric conditions that arise when lines, planes and spheres are brought into contact.

Early in my career I had met Adrian and John when they were at the University of Bath and they had showed me their ray casting programs and animations. Geometry was obviously an important part of their work. However, although their book covers a wide range of topics, it does not show the origins of their equations, and I spent many weeks devising compact proofs to substantiate their results. Nevertheless, their book has had a great impact on this book and I openly acknowledge their influence.

My personal library of math books is not extensive but reasonable. But there were many occasions when I had to resort to the Internet and do a Google search on topics such as 'Heron's formula', 'quaternions', 'Platonic objects', 'plane equations', etc. Such searches produced volumes of data but frequently the information I wanted was just not there. So over the past two years I have had no choice but to sit down and work out a solution for myself.

The book's scope was a problem – where should it start, and where should it end? I decided that I would begin with some important concepts of Euclidean plane geometry. For example, recognizing similar or congruent triangles is a very powerful problem-solving technique and provided some solid foundations for the rest of the book. Where to end was much more difficult. Some reviewers of early manuscripts suggested that I should embrace the geometric aspects of rendering, radiosity, physics, clipping, NURBS, and virtually the rest of computer graphics. I declined this advice as it would have changed the flavor of the book, which is primarily about geometry. Perhaps, I should not have included Bézier curves and patches, but I was tempted to include them as they developed the ideas of parametric formulas to control geometry.

Mathematicians have still not agreed upon a common notation for their mathematical instruments, which has made my life extremely frustrating in preparing this book. For example, some math books refer to vectors as $\vec{a}$ whilst others employ $\mathbf{a}$. The magnitude of a vector is expressed as $|\vec{a}|$ by one community and $||\mathbf{a}||$ by another. The scalar product is sometimes written as $\vec{a} \cdot \vec{b}$ or $\mathbf{a} \cdot \mathbf{b}$ and so on.

Some mathematicians use $\arctan \alpha$ in preference to $\tan^{-1} \alpha$ as the superscript is thought to be confusing. Even plane equations have two groups of followers: those that use $ax + by + cz + d = 0$ and others who prefer $ax + by + cz = d$. The difference may seem minor but one has to be very careful when applying the formulas involving these equations. But perhaps the biggest problem of all is the use of matrices as they can be used in two transposed modes. In the end, I selected what I thought was a logical notation and trust that the reader will find the usage consistent.

The book is designed to be used in three ways: the first section provides the reader with list of formulas across a wide range of geometric topics and hopefully will reveal a useful solution when referenced. Where relevant, I have provided alternative formulas for different mathematical representations. For example, a 2D line equation can be expressed in its general form or parametrically, which gives rise to two different solutions to a problem. I have also shown how a formula is simplified if a line equation is normalized or a normal vector has a unit length.

The second section places all the equations in some sort of context. For example, how to compute the angle between two planes; how to compute the area of an irregular polygon; or how to generate a parametric sinusoidal curve. I anticipate that this section will be useful to students who are discovering some of these topics for the first time.

The third section is the heart of the book and hopefully will be useful to lecturers teaching the geometric aspects of computer graphics. Students will also find this section instructive for two reasons: first it will show the origins of the formula; and second, it will illustrate different strategies for solving problems. I learnt a lot deriving these proofs. I discovered how important it was to create a scenario where the scalar product could be introduced, as this frequently removed an unwanted variable and secured the value of a parameter (often λ) which determined the final result. Similarly, the cosine rule was very useful as an opening problem-solving strategy.

Some proofs took days to produce. There were occasions when I after several hours work I had proved that $1 = 1$! There were occasions when a solution seemed impossible, but then after scanning several books I discovered a trick such as completing the square, or making a point on a line perpendicular to the origin.

This project has taught me many lessons: the first is that mathematics is nothing more than a game played according to a set of rules that keeps on growing. When the rules don't fit, they are changed to accommodate some new mathematical instrument. Vectors and quaternions are two such examples. Another lesson is that to become good at solving mathematical problems one requires a knowledge of the 'tricks of the trade' used by mathematicians. Alas, such tricks often demand knowledge of mathematics that is only taught to mathematicians.

I would like to acknowledge the advice given by my colleague Prof. Jian Zhang who offered constructive suggestions whilst preparing the manuscript. Also I would like to thank Rebecca Mowatt who provided vital editorial support throughout the entire project.

Finally, I thank the authors of all the books listed in the bibliography, as they made the book possible, and last, but not least, a very big thank you to Robert Gray who typeset the book.

John Vince
Ringwood

Contents

1 Geometry

Let no one enter who does not know geometry.

Inscription on Plato's door, probably at the Academy of Athens (*c.* 429–347 BC).

This section contains formulas often required in computer graphics and is organised into 19 groups:

Most of these formulas are developed in the section on Proofs and placed in context in the section on Examples.

Undefined results

The reader will probably be aware that the simplest of formulas must be treated with great care. For example $x = a/b$ appears rather innocent, but is undefined when $b = 0$. Similarly, $s = \sqrt{t}$ will only generate a real value when $t \geqslant 0$. Therefore similar care must be exercised when using vectors and quaternions. For example, if a vector is accidentally set null, e.g. $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ where $a = b = c = 0$ then $\mathbf{n} \bullet \mathbf{n} = 0$. This in itself is not a problem, but if this dot product is in the denominator of a formula, then the result is undefined and will terminate a computer program unless this condition is detected prior to the division.

Determinants

Some formulas in this section are expressed in determinant form simply because they provide a neat and compact notation. However, a determinant can be zero, therefore its value must be determined if it is used as a denominator in a formula.

Vectors

Formulas involving vectors can often be simplified if they are unit vectors. For example, the angle α between two vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ is given by

$$\alpha = \cos^{-1}\left(\frac{\mathbf{n}_1 \bullet \mathbf{n}_2}{\|\mathbf{n}_1\| \cdot \|\mathbf{n}_2\|}\right)$$

but if $\|\mathbf{n}_1\| = \|\mathbf{n}_2\| = 1$ $\qquad \alpha = \cos^{-1}(\mathbf{n}_1 \bullet \mathbf{n}_2)$

which saves unnecessary computation.

Matrices

Matrix transformations are another source of error when developing computer programs. Unfortunately, two systems are still in use and create untold havoc when a matrix is copied from a book or technical paper without knowing the source of the transform. For example, this text employs column vectors:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

where

$$x' = ax + by$$
$$y' = cx + dy$$

However, when using row vectors we have

$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where
$$x' = ax + cy$$
$$y' = bx + dy$$

which does not produce the same result!

The second example can be made identical to the first by transposing the matrix:

$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \cdot \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

where
$$x' = ax + by$$
$$y' = cx + dy$$

which is what the reader will have to do if they discover such a matrix. For example, a rotation matrix using row vectors is

$$\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

but when transposed creates the more familiar column vector form

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

Readers not familiar with matrices should appreciate that matrix multiplication is not commutative, i.e. in general $T_A \times T_B \neq T_B \times T_A$. This is easily seen using a simple example:

Given
$$T_A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad T_B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

then
$$T_A \times T_B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

whereas
$$T_B \times T_A = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \times \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ae + cf & be + df \\ ag + ch & bg + dh \end{bmatrix}$$

It is obvious that they do not produce the same result.

Efficiency

The formulas listed in this section are not selected on the basis of speed. Such strategies are beyond the scope of this book and the reader should investigate how these formulas have been developed by authors and researchers to improve their efficiency.

1.1 Lines, angles and trigonometry

1.1.1 Points and straight lines

The building blocks of Euclidian geometry are the *point* and the *straight line*. A point indicates position in space and has no size or magnitude. A moving point describes a line, which has length but no width. From these two concepts evolve the following axioms:

1. *Only one straight line can be drawn between two points.*
2. *Two straight lines intersect in one point only.*
3. *Two straight lines cannot enclose a space.*

As soon as we introduce two or more lines, the idea of a plane surface emerges. Such a surface can be tested as follows:

A straight line joining two points on a plane surface will also reside on that surface.

From these simple definitions explode the subject of two-dimensional Euclidian geometry.

Parallel lines

Parallel lines remain a constant distance apart and reside on a common surface.

1.1.2 Angles

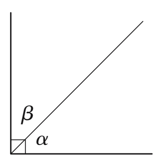
An angle is formed when two straight lines meet at a point. An angle is a spatial quantity and measures the rotational offset between the two lines when rotated about their common point or vertex. By definition, anti-clockwise angles are positive and clockwise angles are negative. Furthermore, by definition, one revolution equals 360° or 2π radians.

Acute, obtuse, right and straight angles

$0° <$ acute angle $<$ (right angle $= 90°$) $<$ obtuse angle $<$ (straight angle $= 180°$)

Complementary angles

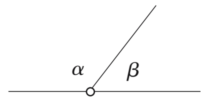
Complementary angles sum to 90°.



$\alpha + \beta = 90°$

Supplementary angles

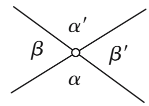
Supplementary angles sum to 180°.



$\alpha + \beta = 180°$

Vertical angles

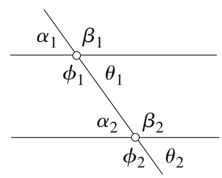
Two pairs of vertical angles are created by two intersecting straight lines.



$\alpha = \alpha'$ and $\beta = \beta'$

Interior, exterior, corresponding and opposite angles

Interior, exterior, corresponding and opposite angles arise when a straight line intersects a pair of parallel lines.



Interior angles $\alpha_2 \quad \beta_2 \quad \theta_1 \quad \phi_1$

Alternate interior angles

$$\theta_1 = \alpha_2$$
$$\phi_1 = \beta_2$$

Corresponding angles

$$\alpha_1 = \alpha_2$$
$$\beta_1 = \beta_2$$
$$\phi_1 = \phi_2$$
$$\theta_1 = \theta_2$$

Opposite angles

$$\alpha_1 = \theta_1$$
$$\phi_1 = \beta_1$$
$$\alpha_2 = \theta_2$$
$$\phi_2 = \beta_2$$

Exterior angles $\alpha_1 \quad \beta_1 \quad \theta_2 \quad \phi_2$

Alternate exterior angles

$$\alpha_1 = \theta_2$$
$$\beta_1 = \phi_2$$

1.1.3 Trigonometry

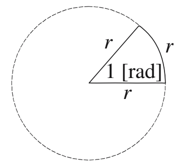
Angular measurement

By definition

Right angle	90°	$\frac{\pi}{2}$ [radians]
Straight angle	180°	π [radians]
One revolution	360°	2π [radians]

Radians

An angle of one radian subtends an arc length r with a circle of radius r.

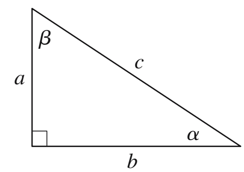


Transcendental trigonometric functions

$$\sin\alpha = \frac{a}{c} \qquad \cos\alpha = \frac{b}{c} \qquad \tan\alpha = \frac{a}{b}$$

$$\csc\alpha = \frac{1}{\sin\alpha} = \frac{c}{a} \qquad \sec\alpha = \frac{1}{\cos\alpha} = \frac{c}{b} \qquad \cot\alpha = \frac{1}{\tan\alpha} = \frac{b}{a}$$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} \qquad \cot\alpha = \frac{\cos\alpha}{\sin\alpha}$$



Useful trigonometric values

α	0°	30°	36°	45°	54°	60°	90°
$\sin\alpha$	0	$\frac{1}{2}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{1+\sqrt{5}}{4}$	$\frac{\sqrt{3}}{2}$	1
$\cos\alpha$	1	$\frac{\sqrt{3}}{2}$	$\frac{1+\sqrt{5}}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{1}{2}$	0
$\tan\alpha$	0	$\frac{\sqrt{3}}{3}$	$\sqrt{5-2\sqrt{5}}$	1	$\sqrt{\frac{5+2\sqrt{5}}{5}}$	$\sqrt{3}$	∞

α	0°	30°	45°	60°	90°
$\sin^2\alpha$	$\frac{0}{4}$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$
$\cos^2\alpha$	$\frac{4}{4}$	$\frac{3}{4}$	$\frac{2}{4}$	$\frac{1}{4}$	$\frac{0}{4}$

Cofunction identities

$$\sin\alpha = \cos\left(\frac{\pi}{2} - \alpha\right) = \cos\beta \qquad \cos\alpha = \sin\left(\frac{\pi}{2} - \alpha\right) = \sin\beta \qquad \tan\alpha = \cot\left(\frac{\pi}{2} - \alpha\right) = \cot\beta$$

$$\csc\alpha = \sec\left(\frac{\pi}{2} - \alpha\right) = \sec\beta \qquad \sec\alpha = \csc\left(\frac{\pi}{2} - \alpha\right) = \csc\beta \qquad \cot\alpha = \tan\left(\frac{\pi}{2} - \alpha\right) = \tan\beta$$

Even–odd identities

$$\sin(-\alpha) = -\sin\alpha \qquad \cos(-\alpha) = \cos\alpha \qquad \tan(-\alpha) = -\tan\alpha$$

$$\csc(-\alpha) = -\csc\alpha \qquad \sec(-\alpha) = \sec\alpha \qquad \cot(-\alpha) = -\cot\alpha$$

Pythagorean identities

$$\sin^2\alpha + \cos^2\alpha = 1 \qquad 1 + \tan^2\alpha = \sec^2\alpha \qquad 1 + \cot^2\alpha = \csc^2\alpha$$

Compound angle identities

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad \cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \qquad \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

Double-angle identities

$$\sin 2\alpha = 2\sin\alpha\cos\alpha \qquad \cos 2\alpha = 1 - 2\sin^2\alpha \qquad \tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha}$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha \qquad \cot 2\alpha = \frac{\cot^2\alpha - 1}{2\cot\alpha}$$

Multiple-angle identities

$$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha \qquad \cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$$

$$\tan 3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha} \qquad \cot 3\alpha = \frac{\cot^3\alpha - 3\cot\alpha}{3\cot^2\alpha - 1}$$

$$\sin 4\alpha = 4\sin\alpha\cos\alpha - 8\sin^3\alpha\cos\alpha \qquad \cos 4\alpha = 8\cos^4\alpha - 8\cos^2\alpha + 1$$

$$\tan 4\alpha = \frac{4\tan\alpha - 4\tan^3\alpha}{1 - 6\tan^2\alpha + \tan^4\alpha} \qquad \cot 4\alpha = \frac{\cot^4\alpha - 6\cot^2\alpha + 1}{4\cot^3\alpha - 4\cot\alpha}$$

$$\sin 5\alpha = 16\sin^5\alpha - 20\sin^3\alpha + 5\sin\alpha \qquad \cos 5\alpha = 16\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha$$

$$\tan 5\alpha = \frac{5\tan\alpha - 10\tan^3\alpha + \tan^5\alpha}{1 - 10\tan^2\alpha + 5\tan^4\alpha} \qquad \cot 5\alpha = \frac{\cot^5\alpha - 10\cot^3\alpha + 5\cot\alpha}{5\cot^4\alpha - 10\cot^2\alpha + 1}$$

Functions of the half-angle

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}} \qquad \cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

$$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} \qquad \cot\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}$$

Functions converting to the half-angle tangent form

$$\sin\alpha = \frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} \qquad \cos\alpha = \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} \qquad \tan\alpha = \frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}$$

$$\csc\alpha = \frac{1+\tan^2\frac{\alpha}{2}}{2\tan\frac{\alpha}{2}} \qquad \sec\alpha = \frac{1+\tan^2\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}} \qquad \cot\alpha = \frac{1-\tan^2\frac{\alpha}{2}}{2\tan^2\frac{\alpha}{2}}$$

Relationships between sums of functions

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \qquad \sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \qquad \cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\tan\alpha + \tan\beta = \frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta} \qquad \tan\alpha - \tan\beta = \frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta}$$

$$\cot\alpha + \cot\beta = \frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta} \qquad \cot\alpha - \cot\beta = -\frac{\sin(\alpha-\beta)}{\sin\alpha\sin\beta}$$

Inverse trigonometric functions

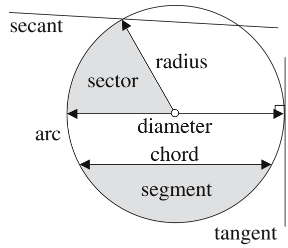
$$\sin(\sin^{-1}x) = x \qquad \cos(\cos^{-1}x) = x \qquad \tan(\tan^{-1}x) = x$$

$$\sin^{-1}(-x) = -\sin^{-1}x \qquad \cos^{-1}(-x) = \pi - \cos^{-1}x \qquad \tan^{-1}(-x) = -\tan^{-1}x$$

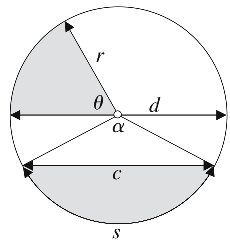
1.2 Circles

1.2.1 Properties of circles

A circle is the locus of all points in a plane equidistant from a center point.



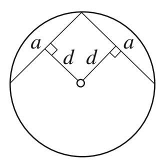
Circle



Area of a circle	$\pi r^2 = \frac{1}{4}\pi d^2$
Perimeter	$2\pi r = \pi d$
Length of arc	$s = \dfrac{\alpha^\circ}{360^\circ}\pi d$ or $s = r\alpha^{[rad]}$
Area of sector	$\dfrac{\theta^\circ}{360^\circ}\pi r^2$ or $\dfrac{r^2}{2}\alpha^{[rad]}$
Area of segment	$r^2\left(\dfrac{\alpha^\circ}{360^\circ}\pi - \dfrac{\sin\alpha}{2}\right)$ or $\dfrac{r^2}{2}\left(\alpha^{[rad]} - \sin\alpha^{[rad]}\right)$
Length of chord	$c = 2r\sin\dfrac{\alpha}{2}$

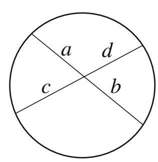
Chords

A chord is a straight line joining two points on the circumference of a circle. The rotational symmetry of a circle ensures that chords of equal lengths are equidistant from the center and vice versa.



The chord theorem

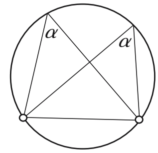
If two chords intersect, then the product of the intercepts on one chord equals the product of the intercepts on the other.



$ab = cd$

Peripheral angles subtended by a chord

Peripheral angles subtended by a common chord are equal.

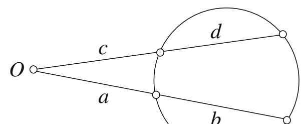


Secants

A secant of a circle is a straight line that intersects the circle's circumference in two points.

The secant theorem

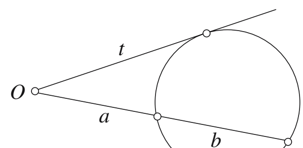
If two secants intersect at O outside a circle, then the product of the intercepts between O and the circle on one is equal to the product of the two intercepts on the other.



$a(a+b) = c(c+d)$

The secant–tangent theorem

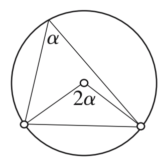
If two secants intersect at O outside a circle, and one of them is tangent to the circle, then the length of the intercept on the tangent between O and the point of contact is the geometric mean of the lengths of the intercepts of the other secant.



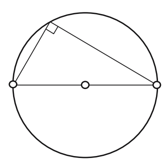
$t^2 = a(a+b)$

Arcs

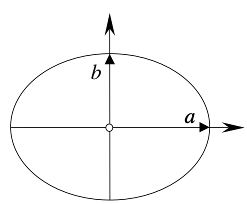
The central angle subtended by an arc is twice the angle on the circle.



When the central angle is 180° the angle at the periphery is 90°; and the arc is half the circumference.



1.2.2 Ellipses



Area on an ellipse

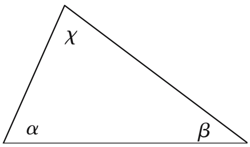
Area of an ellipse $\quad A = 4\pi ab$

1.3 Triangles

1.3.1 Types of triangle

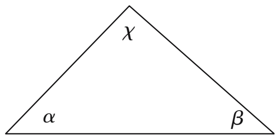
Acute-angled triangle

α, β and χ are acute.



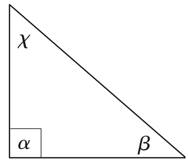
Obtuse-angled triangle

One angle (χ) is obtuse.



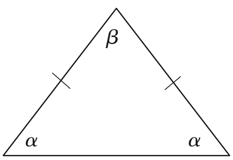
Right-angled triangle

One angle (α) equals 90°.



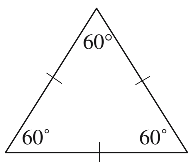
Isosceles triangle

Two equal sides and two equal base angles.



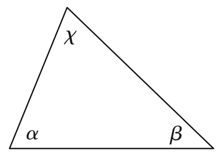
Equilateral triangle

All sides are equal and all angles equal 60°.



Scalene triangle

All sides are unequal and all angles unequal.



1.3.2 Similar triangles

Two triangles are similar ($\sim$) if corresponding angles are equal, and corresponding sides share a common ratio.

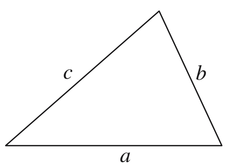
Conditions for similarity

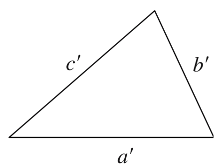
First triangle

Second triangle

Three corresponding sides are in the same ratio.

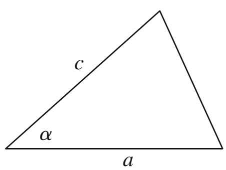
$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$

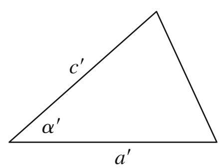




Two corresponding sides are in the same ratio, and the included angles are equal.

$$\frac{a}{a'} = \frac{c}{c'}$$
$$\alpha = \alpha'$$





Two corresponding angles are equal.

$\alpha = \alpha'$
$\beta = \beta'$

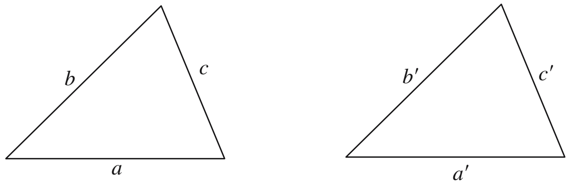
1.3.3 Congruent triangles

Two triangles are congruent (identical $\cong$) if corresponding sides and angles are equal.

Conditions for congruency **First triangle** **Second triangle**

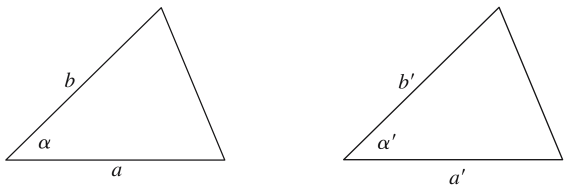
Three sides are equal.

$a = a'$
$b = b'$
$c = c'$



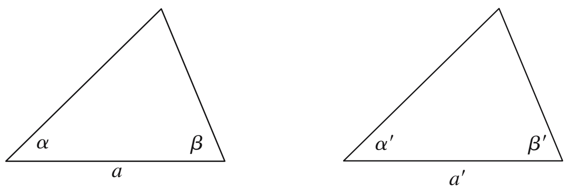
Two sides and the included angle are equal.

$a = a'$
$b = b'$
$\alpha = \alpha'$



One side and the adjoining angles are equal.

$a = a'$
$\alpha = \alpha'$
$\beta = \beta'$

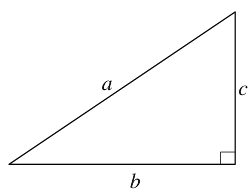


1.3.4 Theorem of Pythagoras

Pythagorean formula

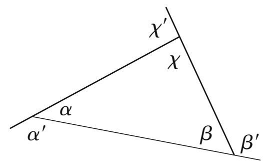
In a right-angled triangle, the square of the hypotenuse equals the sum of the squares of the other two sides.

$a^2 = b^2 + c^2$



1.3.5 Internal and external angles

Internal and external angles

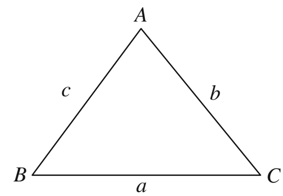


$\alpha + \beta + \chi = 180°$ [internal angles]

$\alpha' + \beta' + \chi' = 360°$ [external angles]

1.3.6 Sine, cosine and tangent rules

Sine rule



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Cosine rule

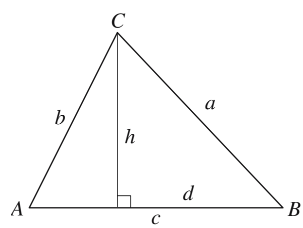
$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

Tangent rule

$$\frac{a+b}{a-b} = \frac{\tan\left(\frac{A+B}{2}\right)}{\tan\left(\frac{A-B}{2}\right)} \qquad \frac{b+c}{b-c} = \frac{\tan\left(\frac{B+C}{2}\right)}{\tan\left(\frac{B-C}{2}\right)} \qquad \frac{c+a}{c-a} = \frac{\tan\left(\frac{C+A}{2}\right)}{\tan\left(\frac{C-A}{2}\right)}$$

1.3.7 Area of a triangle

Normal formula



$$Area = \tfrac{1}{2} base \times height = \tfrac{1}{2} ch$$

$$Area = \tfrac{1}{2} bc \sin A$$

Heron's formula

$$Area = \sqrt{s(s-a)(s-b)(s-c)} \quad \text{where} \quad s = \tfrac{1}{2}(a+b+c)$$

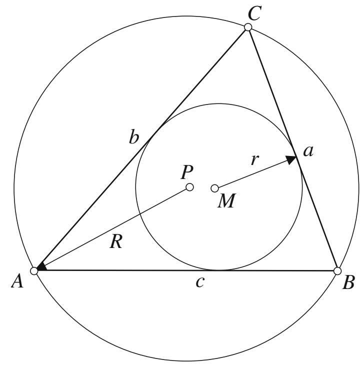
Determinant formula

$$Area = \frac{1}{2}\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}$$

Note: If the vertices are anti-clockwise, *Area* is +ve, else −ve.

1.3.8 Inscribed and circumscribed circles

General triangle



$A(x_A, y_A), B(x_B, y_B), C(x_C, y_C)$ are the vertices of a triangle with sides a, b, c.

r is the radius of the inscribed circle.

R is the radius of the circumscribed circle.

$M(x_M, y_M)$ is the center of the inscribed circle.

$P(x_P, y_P)$ is the center of the circumscribed circle.

$s = \frac{1}{2}(a+b+c)$ and $y_{AC} = y_C - y_A$ etc.

$$r = \frac{\text{Area } \triangle ABC}{s} \qquad R = \frac{abc}{4 \times \text{Area } \triangle ABC}$$

$$x_M = \frac{ax_A + bx_B + cx_C}{2s} \qquad y_M = \frac{ay_A + by_B + cy_C}{2s}$$

$$x_P = x_A + \frac{1}{2}\frac{\begin{vmatrix} y_{AC} & b^2 \\ y_{AB} & c^2 \end{vmatrix}}{\begin{vmatrix} x_{AB} & y_{AB} \\ x_{AC} & y_{AC} \end{vmatrix}} \qquad y_P = y_A + \frac{1}{2}\frac{\begin{vmatrix} b^2 & x_{AC} \\ c^2 & x_{AB} \end{vmatrix}}{\begin{vmatrix} x_{AB} & y_{AB} \\ x_{AC} & y_{AC} \end{vmatrix}}$$

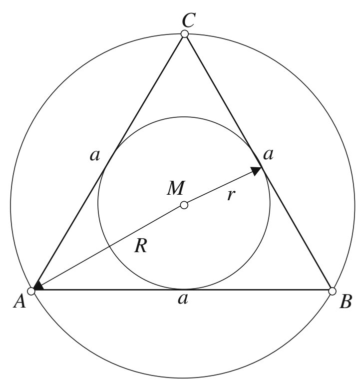
or

$$x_P = x_A + \frac{\begin{vmatrix} y_{AC} & b^2 \\ y_{AB} & c^2 \end{vmatrix}}{4 \times \text{Area } \triangle ABC} \qquad y_P = y_A + \frac{\begin{vmatrix} b^2 & x_{AC} \\ c^2 & x_{AB} \end{vmatrix}}{4 \times \text{Area } \triangle ABC}$$

or

$$x_P = x_A + \frac{R}{abc}\begin{vmatrix} y_{AC} & b^2 \\ y_{AB} & c^2 \end{vmatrix} \qquad y_P = y_A + \frac{R}{abc}\begin{vmatrix} b^2 & x_{AC} \\ c^2 & x_{AB} \end{vmatrix}$$

Equilateral triangle



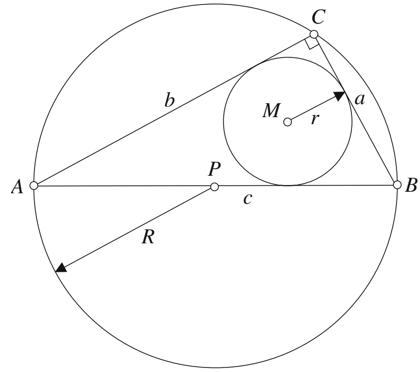
$$r = \tfrac{1}{6}a\sqrt{3}$$

$$R = \tfrac{1}{3}a\sqrt{3}$$

$$x_M = \tfrac{1}{3}(x_A + x_B + x_C)$$

$$y_M = \tfrac{1}{3}(y_A + y_B + y_C)$$

Right-angled triangle



$$r = \frac{2ab}{s}$$

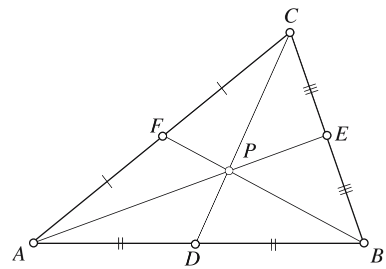
$$R = \tfrac{1}{2}\,hypotenuse$$

1.3.9 Centroid of a triangle

The medians of a triangle are concurrent, and intersect at its centroid two-thirds along a median connecting a vertex to the mid-point of the opposite side. The centroid is also the center of gravity of the triangle.

General triangle

AE, BF and CD are medians and P is the centroid.



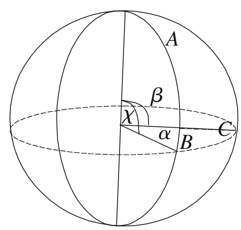
$$AP = \tfrac{2}{3}AE$$

$$BP = \tfrac{2}{3}BF$$

$$CP = \tfrac{2}{3}CD$$

1.3.10 Spherical trigonometry

Trigonometric rules



Sine rule $\quad \dfrac{\sin A}{\sin\alpha} = \dfrac{\sin B}{\sin\beta} = \dfrac{\sin C}{\sin\chi}$

Cosine rule $\quad \cos\alpha = \cos\beta\cos\chi + \sin\beta\sin\chi\cos A$

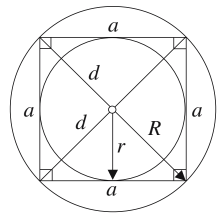
$$\cos A = -\cos B\cos C + \sin B\sin C\cos\alpha$$

Area of a spherical triangle

$$Area = \pi r^2 \frac{E}{180} \quad \text{where} \quad E = A + B + C - 180°$$

1.4 Quadrilaterals

Square



Diagonal $d = a\sqrt{2}$

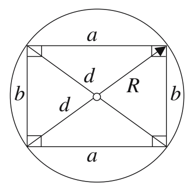
Area $A = a^2 = \frac{1}{2}d^2$

Inradius $r = \frac{1}{2}a$

Circumradius $R = \dfrac{a}{\sqrt{2}}$

Symmetry properties: A square has equal sides and equal diagonals, which bisect each other and the interior angles, and they intersect at right angles. The interior angles are right angles.

Rectangle



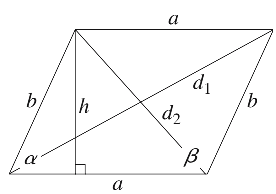
Diagonal $d = \sqrt{a^2 + b^2}$

Area $A = ab$

Circumradius $R = \frac{1}{2}d$

Symmetry properties: A rectangle has equal diagonals, which bisect each other, and the interior angles are right angles.

Parallelogram (rhomboid)



Diagonals $d_1 = \sqrt{a^2 + b^2 - 2ab\cos\beta}$

$d_2 = \sqrt{a^2 + b^2 - 2ab\cos\alpha}$

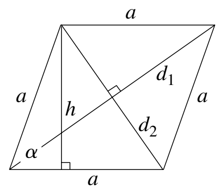
$d_1^2 + d_2^2 = 2(a^2 + b^2)$

Altitude $h = b\sin\alpha$

Area $A = ah = ab\sin\alpha$

Symmetry properties: A parallelogram has two pairs of parallel sides with equal opposite sides. Adjacent interior angles are supplementary and opposite interior angles are equal. The diagonals bisect each other.

Rhombus



Diagonals $d_1 = 2a\cos\dfrac{\alpha}{2}$

$$d_2 = 2a\sin\frac{\alpha}{2}$$

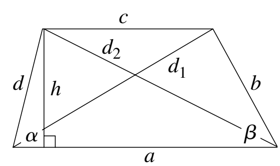
$$d_1^2 + d_2^2 = 4a^2$$

Altitude $h = a\sin\alpha$

Area $A = ah = a^2\sin\alpha = \frac{1}{2}d_1 d_2$

Symmetry properties: A rhombus has two pairs of parallel, equal sides. Adjacent interior angles are supplementary and opposite interior angles are equal. The diagonals bisect each other and the interior angles, and intersect at right angles.

Trapezium



Diagonals $d_1 = \sqrt{a^2 + b^2 - 2ab\cos\beta}$

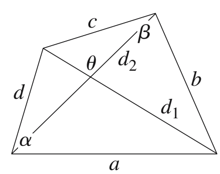
$$d_2 = \sqrt{a^2 + d^2 - 2ad\cos\alpha}$$

Altitude $h = d\sin\alpha = b\sin\beta$

Area $A = \frac{1}{2}(a + c)h$

Symmetry properties: A trapezium has one pair of parallel sides.

General quadrilateral



Area $A = \frac{1}{2}d_1 d_2\sin\theta$

$$A = \frac{1}{4}(b^2 + d^2 - a^2 - c^2)\tan\theta \quad \text{for } \theta < 90^\circ$$

$$A = \frac{1}{4}\sqrt{4d_1^2 d_2^2 - (b^2 + d^2 - a^2 - c^2)^2}$$

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\varepsilon}$$

where $s = \frac{1}{2}(a + b + c + d)$ and $\varepsilon = \frac{1}{2}(\alpha + \beta)$

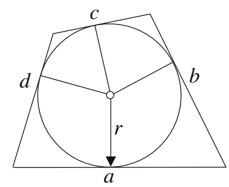
Symmetry properties: A general quadrilateral has all sides of different lengths and no sides parallel. The sum of interior angles = 360°, and the sum of exterior angles = 360°.

Tangent quadrilateral

Area $A = \frac{1}{2}r(a + b + c + d)$

$A = sr$

where $s = \frac{1}{2}(a + b + c + d)$



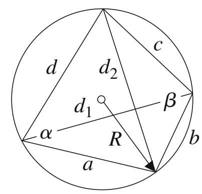
Symmetry properties: A tangent quadrilateral must have an inscribed circle.

Cyclic quadrilateral

Diagonals $d_1 = \sqrt{\dfrac{(ab + cd)(ac + bd)}{ad + bc}}$

$d_2 = \sqrt{\dfrac{(ac + bd)(ad + bc)}{ab + cd}}$

$d_1 d_2 = ac + bd$



Area $A = \sqrt{(s - a)(s - b)(s - c)(s - d)}$

where $s = \frac{1}{2}(a + b + c + d)$

Circumscribed radius $R = \frac{1}{4}\sqrt{\dfrac{(ac + bd)(ad + bc)(ab + cd)}{(s - a)(s - b)(s - c)(s - d)}}$

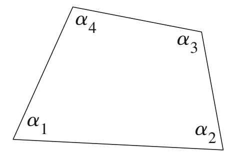
Symmetry properties: A cyclic quadrilateral must have a circumscribed circle. Opposite interior angles are supplementary (sum to 180°).

1.5 Polygons

1.5.1 Internal and external angles of a polygon

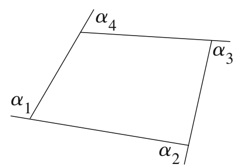
The internal angles of an n-gon sum to $(n - 2) \times 180°$.
Quadrilateral ($n = 4$)

$$\sum_{i=1}^{4} \alpha_i = 360°$$



The external angles of an n-gon sum to 360°.
Quadrilateral ($n = 4$)

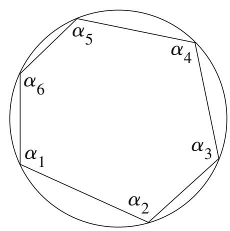
$$\sum_{i=1}^{4} \alpha_i = 360°$$



1.5.2 Alternate internal angles of a cyclic polygon

The alternate internal angles of a cyclic n-gon sum to $(n - 2) \times 90°$ [$n \geqslant 4$ and is even].
Cyclic hexagon ($n = 6$)

$$\alpha_1 + \alpha_3 + \alpha_5 = \alpha_2 + \alpha_4 + \alpha_6 = 360°$$



1.5.3 Area of a regular polygon

Area of a polygon using the number of edges

$$Area = \frac{1}{4} ns^2 \cot\left(\frac{\pi}{n}\right)$$

or
$$Area = nr^2 \sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right)$$

or
$$Area = \frac{1}{2} nr^2 \sin\left(\frac{2\pi}{n}\right)$$

where n = number of sides
s = length of side
r = radius of circumscribed circle

Area of a polygon using Cartesian coordinates

$$Area = \frac{1}{2}\sum_{i=0}^{n-1}(x_i y_{i+1(\mathrm{mod}\ n)} - y_i x_{i+1(\mathrm{mod}\ n)})$$

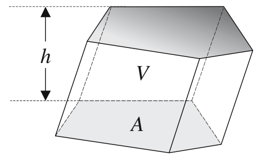
where the n vertices (x, y) are defined in counter-clockwise sequence.

1.6 Three-dimensional objects

1.6.1 Prisms

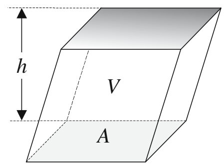
General prism

$$V = Ah$$



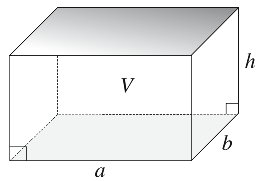
Parallelepiped

$$V = Ah$$



Rectangular parallelepiped

$$S = 2(ab + ah + bh)$$
$$V = abh$$



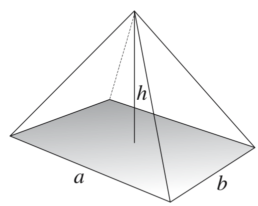
1.6.2 Pyramids

Rectangular pyramid

$$S = ab + \tfrac{1}{2}\left(a\sqrt{4h^2 + b^2} + b\sqrt{4h^2 + a^2}\right)$$

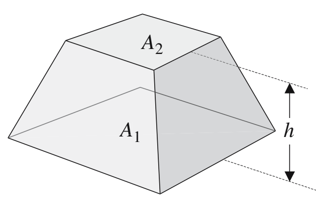
when $a = b$ $\qquad S = a^2 + a\sqrt{4h^2 + a^2}$

$$V = \tfrac{1}{3}abh$$

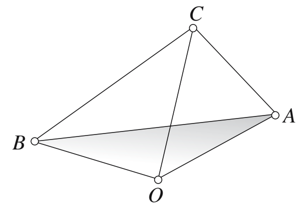


Volume of a frustum

$$V = \tfrac{1}{3}h\left(A_1 + A_2 + \sqrt{A_1A_2}\right)$$



Tetrahedron

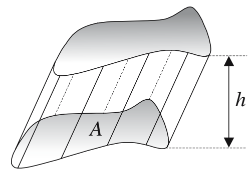


$$V = \frac{1}{6}\begin{vmatrix} x_a & y_a & z_a \\ x_b & y_b & z_b \\ x_c & y_c & z_c \end{vmatrix}$$

O is the origin.

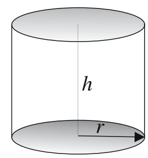
1.6.3 Cylinders

Irregular cylinder



$$V = Ah$$

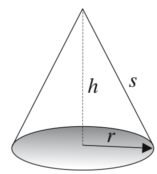
Cylinder



$$S = 2\pi r(r + h)$$
$$V = \pi r^2 h$$

1.6.4 Cones

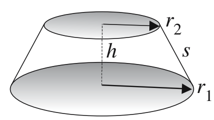
Right circular cone



Lateral surface area $A_L = \pi rs$

$$S = \pi r(r + s)$$
$$V = \tfrac{1}{3}\pi r^2 h$$

Right circular conical frustum



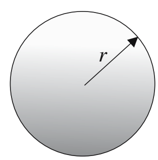
Lateral surface area $S_L = \pi s(r_1 + r_2)$

$$S = \pi(r_1^2 + r_2^2 + s(r_1 + r_2))$$

$$V = \tfrac{1}{3}\pi h(r_1^2 + r_2^2 + r_1 r_2)$$

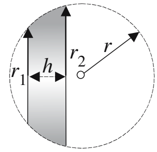
1.6.5 Spheres

Sphere



$$S = 4\pi r^2$$
$$V = \tfrac{4}{3}\pi r^3$$

Spherical segment



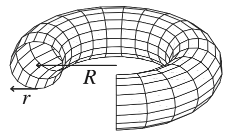
$$S = 2\pi rh$$

$$V = \tfrac{1}{6}\pi h(3r_1^2 + 3r_2^2 + h^2)$$

when $r_1 = 0$ $$V = \tfrac{1}{6}\pi h(3r_2^2 + h^2)$$

1.6.6 Tori

Circular torus

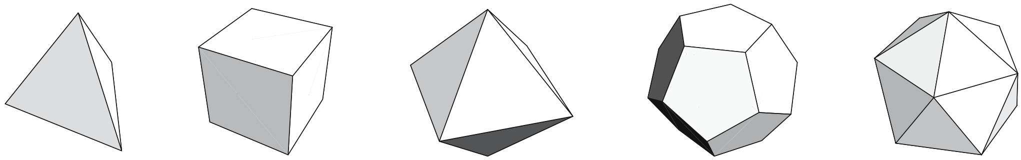


$$S = 4\pi^2 rR$$
$$V = 2\pi^2 r^2 R$$

1.6.7 Platonic solids

There are five Platonic solids: tetrahedron, cube (hexahedron), octahedron, dodecahedron and icosahedron.

Tetrahedron **Cube** **Octahedron** **Dodecahedron** **Icosahedron**



Each object is constructed from a common regular polygon and the inherent symmetry ensures that every vertex lies on a circumsphere of radius R_c. A second inner-sphere of radius R_{in} touches the mid-point of each face, whilst a third mid-sphere of radius R_{int} touches the mid-point of each edge. These radii, together with the surface area A, volume V and the dihedral angle between any neighboring pair of faces $\triangle$ can be expressed in terms of the parameters p, q, f and s, where

$p =$ the number of edges in a face
$q =$ the number of edges associated with a vertex
$f =$ the number of faces
$s =$ the edge length

The following steps show the formulas used for calculating R_{in}, R_{int}, R_c, Δ, A and V.

Ratio of in-sphere radius R_{in} to edge length s

$$\frac{R_{in}}{s} = \frac{1}{2}\frac{\cot\dfrac{\pi}{p}\cos\dfrac{\pi}{q}}{\sqrt{\sin^2\dfrac{\pi}{q} - \cos^2\dfrac{\pi}{p}}}$$

Ratio of mid-sphere radius R_{int} to edge length s

$$\frac{R_{int}}{s} = \frac{1}{2}\frac{\cos\frac{\pi}{p}}{\sqrt{\sin^2\frac{\pi}{q} - \cos^2\frac{\pi}{p}}}$$

Ratio of circumsphere radius R_c to edge length s

$$\frac{R_c}{s} = \frac{1}{2}\frac{\sin\frac{\pi}{q}}{\sqrt{\sin^2\frac{\pi}{q} - \cos^2\frac{\pi}{p}}}$$

Dihedral angle Δ

$$\Delta = 2\sin^{-1}\left(\frac{\cos\frac{\pi}{q}}{\sin\frac{\pi}{p}}\right)$$

Ratio of the surface area A to edge length s

$$\frac{A}{s^2} = fp\cot\frac{\pi}{p}$$

Ratio of the volume V to edge length s

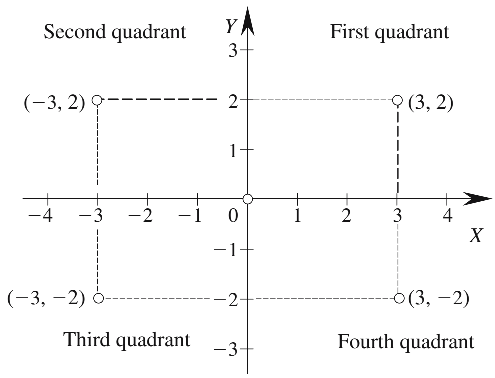
$$\frac{V}{s^3} = \frac{1}{3}AR_{in}$$

Characteristic	Tetrahedron	Cube	Octahedron	Dodecahedron	Icosahedron
Vertices	4	8	6	20	12
Edges	6	12	12	30	30
Faces	4	6	8	12	20
Edges/face (p)	3	4	3	5	3
Edges/vertex (q)	3	3	4	3	5
Dihedral angle (Δ)	70.52878°	90.0°	109.47122°	116.56505°	138.18969°
Area (A)	$s^3\sqrt{3}$	$6s^2$	$2s^2\sqrt{3}$	$3s^2\sqrt{25+10\sqrt{5}}$	$5s^2\sqrt{3}$
Volume (V)	$\frac{s^3\sqrt{2}}{12}$	s^3	$\frac{s^3\sqrt{2}}{3}$	$\frac{s^3(15+7\sqrt{5})}{4}$	$\frac{5s^3}{12}(3+\sqrt{5})$
R_{in}/s	$\frac{\sqrt{6}}{12}$ $= 0.204124$	$\frac{1}{2}$ $= 0.5$	$\frac{\sqrt{6}}{6}$ $= 0.408248$	$\frac{1}{20}\sqrt{250+110\sqrt{5}}$ $= 1.113516$	$\frac{1}{12}\sqrt{42+18\sqrt{5}}$ $= 0.755761$
R_{int}/s	$\frac{\sqrt{2}}{4}$ $= 0.353554$	$\frac{\sqrt{2}}{2}$ $= 0.707107$	$\frac{1}{2}$ $= 0.5$	$\frac{1}{4}\sqrt{14+6\sqrt{5}}$ $= 1.309017$	$\frac{1}{4}\sqrt{6+2\sqrt{5}}$ $= 0.89017$
R_c/s	$\frac{\sqrt{6}}{4}$ $= 0.612372$	$\frac{\sqrt{3}}{2}$ $= 0.866025$	$\frac{\sqrt{2}}{2}$ $= 0.707107$	$\frac{1}{4}\sqrt{18+6\sqrt{5}}$ $= 1.401259$	$\frac{1}{4}\sqrt{10+2\sqrt{5}}$ $= 0.951057$
A/s^2	$\sqrt{3}$ $= 1.732051$	6	$2\sqrt{3}$ $= 3.464102$	$3\sqrt{25+10\sqrt{5}}$ $= 20.645728$	$5\sqrt{3}$ $= 8.660254$
V/s^3	$\frac{\sqrt{2}}{12}$ $= 0.117851$	1	$\frac{\sqrt{2}}{3}$ $= 0.471405$	$\frac{1}{4}(15+7\sqrt{5})$ $= 7.663119$	$\frac{5}{12}(3+\sqrt{5})$ $= 2.181695$

1.7 Coordinate systems

1.7.1 Cartesian coordinates in $\mathbb{R}^2$

The Cartesian coordinates of a point in $\mathbb{R}^2$ are given by the ordered pair (x, y).



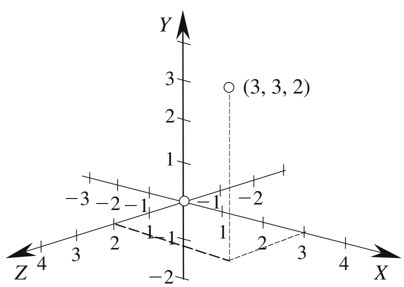
Distance in $\mathbb{R}^2$

Given two points (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$, the distance between them is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

1.7.2 Cartesian coordinates in $\mathbb{R}^3$

The Cartesian coordinates of a point in $\mathbb{R}^3$ are given by the ordered triple (x, y, z). The system illustrated is right handed with the z-axis coming towards the viewer. A left-handed axial system has the z-axis directed away from the viewer.



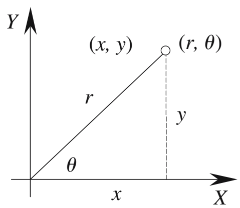
Distance in $\mathbb{R}^3$

Given two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in $\mathbb{R}^3$, the distance between them is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

1.7.3 Polar coordinates

The polar coordinates of a point (x, y) in $\mathbb{R}^2$ are given by the ordered pair (r, θ)



where $x = r\cos\theta$
$y = r\sin\theta$

and

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \quad \text{(1st and 4th quadrants only)}$$

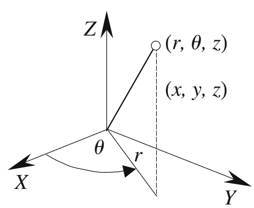
Distance in $\mathbb{R}^2$

Given two points (r_1, θ_1) and (r_2, θ_2) in $\mathbb{R}^2$, the distance between them is given by

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1)}$$

1.7.4 Cylindrical coordinates

The cylindrical coordinates of a point (x, y, z) in $\mathbb{R}^3$ are given by the ordered triple (r, θ, z)

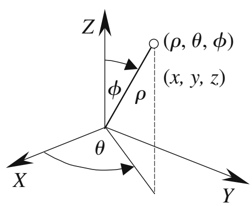


$$
\begin{aligned}
\text{where} \quad x &= r\cos\theta \\
y &= r\sin\theta \\
z &= z
\end{aligned}
$$

$$
\begin{aligned}
\text{and} \quad r &= \sqrt{x^2 + y^2} \\
\theta &= \tan^{-1}\left(\frac{y}{x}\right) \quad \text{(1st and 4th quadrants only)} \\
z &= z
\end{aligned}
$$

1.7.5 Spherical coordinates

The spherical coordinates of a point (x, y, z) in $\mathbb{R}^3$ are given by the ordered triple (ρ, θ, ϕ)



$$
\begin{aligned}
\text{where} \quad x &= \rho\sin\phi\cos\theta \\
y &= \rho\sin\phi\sin\theta \\
z &= \rho\cos\phi
\end{aligned}
$$

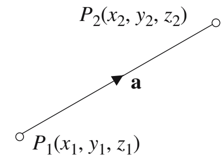
$$
\begin{aligned}
\text{and} \quad \rho &= \sqrt{x^2 + y^2 + z^2} \\
\theta &= \tan^{-1}\left(\frac{y}{x}\right) \quad \text{(1st and 4th quadrants only)} \\
\phi &= \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)
\end{aligned}
$$

Note: The z-axis is normally taken as the vertical axis.

1.8 Vectors

To simplify this summary all vectors have been described as 3D vectors, although where appropriate, the rules equally apply to 2D vectors.

1.8.1 Vector between two points



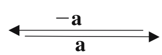
Given $P_1(x_1, y_1, z_1)$ and $P_2\ (x_2, y_2, z_2)$. **a** is a vector from P_1 to P_2.

$$\overrightarrow{P_1P_2} = \mathbf{a} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix} = \begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix}$$

1.8.2 Scaling a vector

$$s\mathbf{a} = \begin{bmatrix} sx_a \\ sy_a \\ sz_a \end{bmatrix}$$

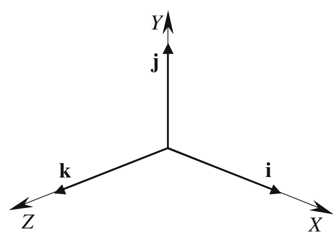
1.8.3 Reversing a vector



$$\mathbf{a} = \begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix} \qquad -\mathbf{a} = \begin{bmatrix} -x_a \\ -y_a \\ -z_a \end{bmatrix}$$

1.8.4 Unit Cartesian vectors

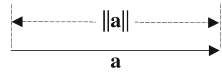
$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



1.8.5 Algebraic notation for a vector

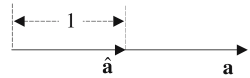
$$\mathbf{a} = x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k}$$

1.8.6 Magnitude of a vector



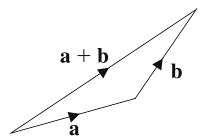
$$\|\mathbf{a}\| = \sqrt{x_a^2 + y_a^2 + z_a^2}$$

1.8.7 Normalizing a vector to a unit length



$$\hat{\mathbf{a}} = \frac{x_a}{\|\mathbf{a}\|}\mathbf{i} + \frac{y_a}{\|\mathbf{a}\|}\mathbf{j} + \frac{z_a}{\|\mathbf{a}\|}\mathbf{k}$$

1.8.8 Vector addition/subtraction



$$\mathbf{a} = \begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} \qquad \mathbf{a} \pm \mathbf{b} = \begin{bmatrix} x_a \pm x_b \\ y_a \pm y_b \\ z_a \pm z_b \end{bmatrix}$$

Commutative law of addition $\quad \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

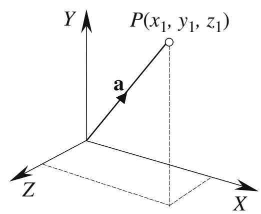
Associative law of addition $\quad (\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$

1.8.9 Compound scalar multiplication

Distributive law of multiplication

$$\begin{aligned} r(s\mathbf{a}) &= (rs)\mathbf{a} \\ r(\mathbf{a} + \mathbf{b}) &= r\mathbf{a} + r\mathbf{b} \quad \text{and} \\ (r + s)\mathbf{a} &= r\mathbf{a} + s\mathbf{a} \end{aligned}$$

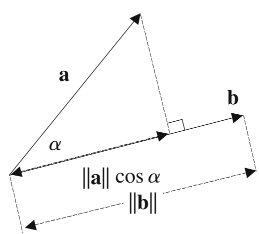
1.8.10 Position vector



Point $P_1(x_1, y_1, z_1)$ has a position vector **a**

$$\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$$

1.8.11 Scalar (dot) product



$$\mathbf{a} \bullet \mathbf{b} = x_a x_b + y_a y_b + z_a z_b = \|\mathbf{a}\| \cdot \|\mathbf{b}\| \cos\alpha$$

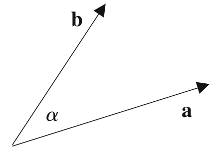
$$\mathbf{a} \bullet \mathbf{a} = \|\mathbf{a}\|^2$$

If **a** is a unit vector $\mathbf{a} \bullet \mathbf{a} = 1$

$$\mathbf{a} \bullet \mathbf{b} = 0 \quad \Leftrightarrow \quad \mathbf{a} \perp \mathbf{b}$$

Commutative law of multiplication	$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
Distributive law of multiplication	$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
	$(r\mathbf{a}) \cdot (s\mathbf{b}) = rs(\mathbf{a} \cdot \mathbf{b})$

1.8.12 Angle between two vectors

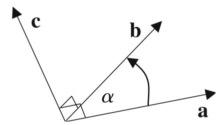


$$\mathbf{a} = x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k}$$
$$\mathbf{b} = x_b\mathbf{i} + y_b\mathbf{j} + z_b\mathbf{k}$$
$$\alpha = \cos^{-1}\left(\frac{x_a x_b + y_a y_b + z_a z_b}{\|\mathbf{a}\| \cdot \|\mathbf{b}\|}\right)$$

When **a** and **b** are unit vectors

$$\alpha = \cos^{-1}(x_a x_b + y_a y_b + z_a z_b)$$

1.8.13 Vector (cross) product



$$\mathbf{a} \times \mathbf{b} = \mathbf{c}$$

where $\|\mathbf{c}\| = \|\mathbf{a}\| \cdot \|\mathbf{b}\| \sin\alpha$

a, **b**, **c** form a right-handed system

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} y_a & z_a \\ y_b & z_b \end{vmatrix}\mathbf{i} + \begin{vmatrix} z_a & x_a \\ z_b & x_b \end{vmatrix}\mathbf{j} + \begin{vmatrix} x_a & y_a \\ x_b & y_b \end{vmatrix}\mathbf{k}$$

or

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_a & y_a & z_a \\ x_b & y_b & z_b \end{vmatrix}$$

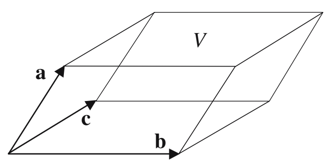
$$\mathbf{a} \times \mathbf{a} = 0$$

1.8.14 The commutative law does not hold: $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$

Distributive law $\quad (r\mathbf{a}) \times (s\mathbf{b}) = rs(\mathbf{a} \times \mathbf{b})$

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$
$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i} \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j} \qquad \mathbf{j} \times \mathbf{i} = -\mathbf{k}$$
$$\mathbf{k} \times \mathbf{j} = -\mathbf{i} \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

1.8.15 Scalar triple product



$$[\mathbf{a},\ \mathbf{b},\ \mathbf{c}] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} x_a & y_a & z_a \\ x_b & y_b & z_b \\ x_c & y_c & z_c \end{vmatrix}$$

Volume $\quad V = [\mathbf{a},\ \mathbf{b},\ \mathbf{c}]$

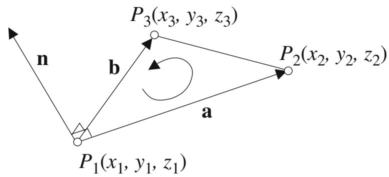
$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{b}, \mathbf{c}, \mathbf{a}] = [\mathbf{c}, \mathbf{a}, \mathbf{b}] = -[\mathbf{c}, \mathbf{b}, \mathbf{a}] = -[\mathbf{b}, \mathbf{a}, \mathbf{c}] = -[\mathbf{a}, \mathbf{c}, \mathbf{b}]$$

$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = 0 \Leftrightarrow \mathbf{a}, \mathbf{b}, \mathbf{c}$ are coplanar $(>0 \Leftrightarrow \mathbf{a}, \mathbf{b}, \mathbf{c})$ are right-handed.

1.8.16 Vector triple product

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \qquad (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$$

1.8.17 Vector normal to a triangle



Given three points P_1, P_2, P_3 defined in counter-clockwise sequence, $\mathbf{n}$ is the normal vector:

$$\mathbf{a} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} x_3 - x_1 \\ y_3 - y_1 \\ z_3 - z_1 \end{bmatrix} \qquad \mathbf{n} = \mathbf{a} \times \mathbf{b}$$

1.8.18 Area of a triangle

Given three points P_1, P_2, P_3, area A is:

$$A = \tfrac{1}{2} \|\mathbf{a} \times \mathbf{b}\| \qquad \mathbf{a} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} x_3 - x_1 \\ y_3 - y_1 \\ z_3 - z_1 \end{bmatrix}$$

1.9 Quaternions

1.9.1 Definition of a quaternion

A quaternion is a four-tuple formed by a scalar and a vector:

$$\mathbf{q} = [s, \mathbf{v}]$$

where s is a scalar and $\mathbf{v}$ is a vector.

Algebraically $\mathbf{q} = [s + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}]$

where s, x, y and z are all scalars.

1.9.2 Equal quaternions

Given $\mathbf{q}_1 = [s_1 + x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}]$

and $\mathbf{q}_2 = [s_2 + x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}]$

$\mathbf{q}_1 = \mathbf{q}_2$ if $s_1 = s_2 \quad x_1 = x_2 \quad y_1 = y_2 \quad z_1 = z_2$

1.9.3 Quaternion addition and subtraction

Given $\mathbf{q}_1 = [s_1 + x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}]$

and $\mathbf{q}_2 = [s_2 + x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}]$

$$\mathbf{q}_1 \pm \mathbf{q}_2 = [(s_1 \pm s_2) + (x_1 \pm x_2)\mathbf{i} + (y_1 \pm y_2)\mathbf{j} + (z_1 \pm z_2)\mathbf{k}]$$

1.9.4 Quaternion multiplication

Given $\mathbf{q}_1 = [s_1 + x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}]$

and $\mathbf{q}_2 = [s_2 + x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}]$

Hamilton's rules $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1 = \mathbf{ijk}$

$$\mathbf{ij} = \mathbf{k} \qquad \mathbf{jk} = \mathbf{i} \qquad \mathbf{ki} = \mathbf{j}$$

$$\mathbf{ji} = -\mathbf{k} \qquad \mathbf{kj} = -\mathbf{i} \qquad \mathbf{ik} = -\mathbf{j}$$

and summarized as

$$\begin{array}{cc} & \begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \end{array} \\ \begin{array}{c} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{array} & \begin{pmatrix} -1 & \mathbf{k} & -\mathbf{j} \\ -\mathbf{k} & -1 & \mathbf{i} \\ \mathbf{j} & -\mathbf{i} & -1 \end{pmatrix} \end{array}$$

$$\mathbf{q}_1\mathbf{q}_2 = [(s_1s_2 - x_1x_2 - y_1y_2 - z_1z_2) + (s_1x_2 + s_2x_1 + y_1z_2 - y_2z_1)\mathbf{i} + (s_1y_2 + s_2y_1 + z_1x_2 - z_2x_1)\mathbf{j} + (s_1z_2 + s_2z_1 + x_1y_2 - x_2y_1)\mathbf{k}]$$

which can be rewritten using the scalar and vector product notation

$$\mathbf{q}_1\mathbf{q}_2 = [(s_1s_2 - \mathbf{v}_1 \bullet \mathbf{v}_2), s_1\mathbf{v}_2 + s_2\mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2]$$

Note that quaternion multiplication is non-commutative.

1.9.5 Magnitude of a quaternion

Given $\mathbf{q} = [s + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}]$

$$\|\mathbf{q}\| = \sqrt{s^2 + x^2 + y^2 + z^2}$$

1.9.6 The inverse quaternion

Given $\mathbf{q} = [s + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}]$

then $$\mathbf{q}^{-1} = \frac{[s - x\mathbf{i} - y\mathbf{j} - z\mathbf{k}]}{\|\mathbf{q}\|^2}$$

and $$\mathbf{q}\mathbf{q}^{-1} = \mathbf{q}^{-1}\mathbf{q} = 1$$

1.9.7 Rotating a vector

A vector $\mathbf{p}$ is rotated to $\mathbf{p}'$ by a unit quaternion using:

$$\mathbf{p}' = \mathbf{q}\mathbf{p}\mathbf{q}^{-1}$$

where $$\mathbf{q} = \left[\cos\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta}{2}\right)\hat{\mathbf{V}}\right]$$

$\hat{\mathbf{V}}$ is the axis of rotation and θ the angle of rotation.

1.9.8 Quaternion as a matrix

Given $\mathbf{q} = [s, \mathbf{v}]$ where $s = \cos\left(\frac{\theta}{2}\right)$, $\mathbf{v} = \hat{\mathbf{n}}\sin\left(\frac{\theta}{2}\right)$

It is equivalent to the following matrix

$$\begin{bmatrix} s^2 + x^2 - y^2 - z^2 & 2(xy - sz) & 2(xz + sy) \\ 2(xy + sz) & s^2 + y^2 - x^2 - z^2 & 2(yz - sx) \\ 2(xz - sy) & 2(yz + sx) & s^2 + z^2 - x^2 - y^2 \end{bmatrix}$$

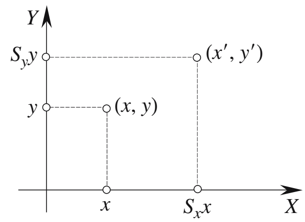
1.10 Transformations

The following transformations are divided into two groups: $\mathbb{R}^2$ and $\mathbb{R}^3$. The matrices are expressed in their homogeneous form, which ensures that they can be combined together. The reader should be aware that, in general, these transformations are not commutative, i.e. $T_1 \times T_2 \neq T_2 \times T_1$.

1.10.1 Scaling relative to the origin in $\mathbb{R}^2$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

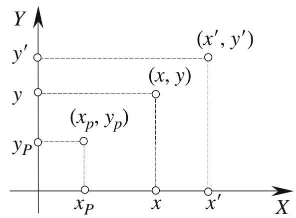
S_x = x-axis scaling factor
S_y = y-axis scaling factor



1.10.2 Scaling relative to a point in $\mathbb{R}^2$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & x_P(1-S_x) \\ 0 & S_y & y_P(1-S_y) \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

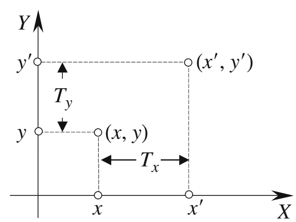
S_x = x-axis scaling factor
S_y = y-axis scaling factor
(x_p, y_p) = the reference point



1.10.3 Translation in $\mathbb{R}^2$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & T_x \\ 0 & 1 & T_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

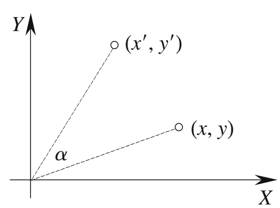
T_x = the x-axis translation
T_y = the y-axis translation



1.10.4 Rotation about the origin in $\mathbb{R}^2$

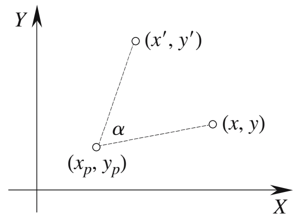
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

α = the angle of rotation



1.10.5 Rotation about a point in $\mathbb{R}^2$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha & x_p(1-\cos\alpha) + y_p \sin\alpha \\ \sin\alpha & \cos\alpha & y_p(1-\cos\alpha) - x_p \sin\alpha \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

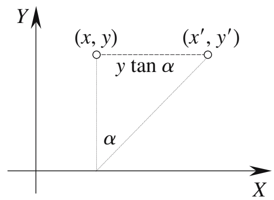


$\alpha =$ the angle of rotation
$(x_p, y_p) =$ the point of rotation

1.10.6 Shearing along the *x*-axis in $\mathbb{R}^2$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \tan\alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

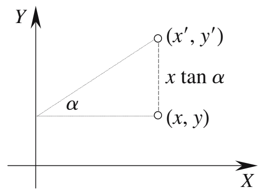
$\alpha =$ the shear angle



1.10.7 Shearing along the *y*-axis in $\mathbb{R}^2$

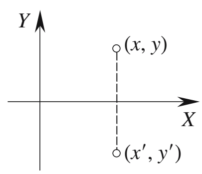
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \tan\alpha & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$\alpha =$ the shear angle



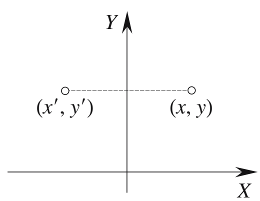
1.10.8 Reflection about the *x*-axis in $\mathbb{R}^2$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



1.10.9 Reflection about the *y*-axis in $\mathbb{R}^2$

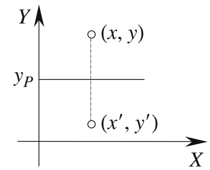
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



1.10.10 Reflection about a line parallel with the *x*-axis in $\mathbb{R}^2$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 2y_P \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

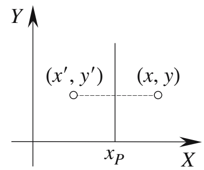
$y = y_P$ the axis of reflection



1.10.11 Reflection about a line parallel with the *y*-axis in $\mathbb{R}^2$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2x_P \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

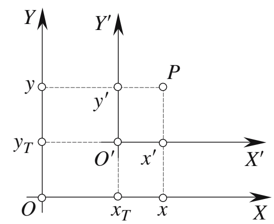
$x = x_P$ the axis of reflection



1.10.12 Translated change of axes in $\mathbb{R}^2$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -x_T \\ 0 & 1 & -y_T \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

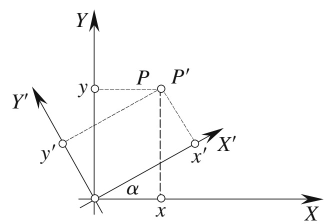
$(x_T, y_T) =$ the translation



1.10.13 Rotated change of axes in $\mathbb{R}^2$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$\alpha =$ the angle of rotation



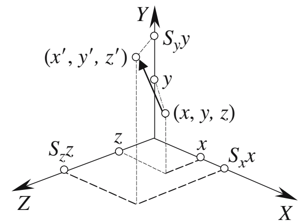
1.10.14 The identity matrix in $\mathbb{R}^2$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

1.10.15 Scaling relative to the origin in $\mathbb{R}^3$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

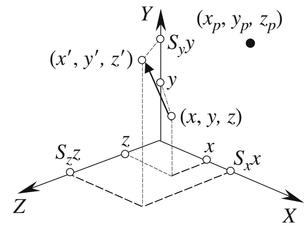
$S_x = x$-axis scaling factor
$S_y = y$-axis scaling factor
$S_z = z$-axis scaling factor



1.10.16 Scaling relative to a point in $\mathbb{R}^3$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 & x_p(1-S_x) \\ 0 & S_y & 0 & y_p(1-S_y) \\ 0 & 0 & S_z & z_p(1-S_z) \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

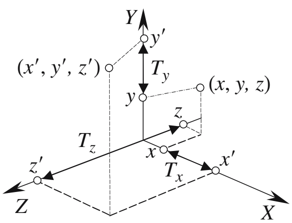
$S_x = x$-axis scaling factor
$S_y = y$-axis scaling factor
$S_z = z$-axis scaling factor
$(x_p, y_p, z_p) =$ the reference point



1.10.17 Translation in $\mathbb{R}^3$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

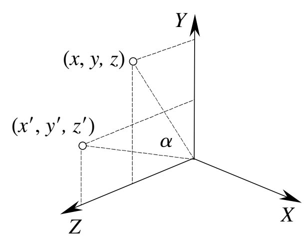
$(T_x, T_y, T_z) =$ the translation



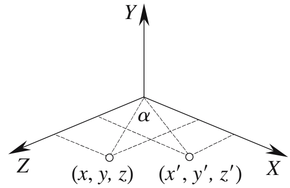
1.10.18 Rotation about the *x*-axis in $\mathbb{R}^3$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$\alpha =$ the angle of pitch about the x-axis



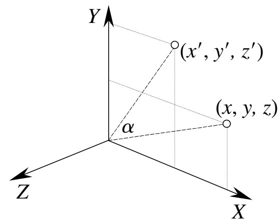
1.10.19 Rotation about the y-axis in $\mathbb{R}^3$



$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$\alpha =$ the angle of yaw about the y-axis

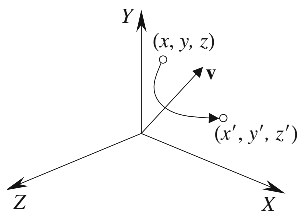
1.10.20 Rotation about the z-axis in $\mathbb{R}^3$



$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$\alpha =$ the angle of roll about the z-axis

1.10.21 Rotation about an arbitrary axis in $\mathbb{R}^3$



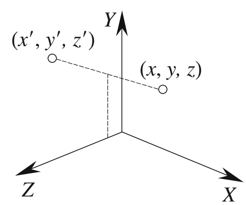
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} a^2K + \cos\alpha & abK - c\sin\alpha & acK + b\sin\alpha & 0 \\ abK + c\sin\alpha & b^2K + \cos\alpha & bcK - a\sin\alpha & 0 \\ acK - b\sin\alpha & bcK + a\sin\alpha & c^2K + \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$K = 1 - \cos\alpha$

axis $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and $||\mathbf{v}|| = 1$

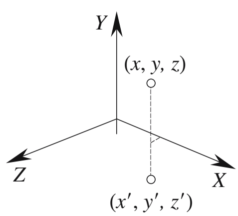
$\alpha =$ the angle of rotation about $\mathbf{v}$

1.10.22 Reflection about the yz-plane in $\mathbb{R}^3$



$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

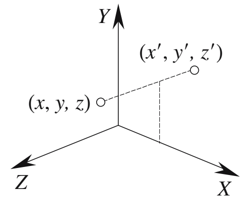
1.10.23 Reflection about the zx-plane in $\mathbb{R}^3$



$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

1.10.24 Reflection about the *xy*-plane in $\mathbb{R}^3$

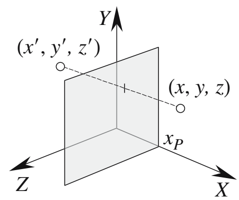
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



1.10.25 Reflection about a plane parallel with the *yz*-plane in $\mathbb{R}^3$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 2x_P \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

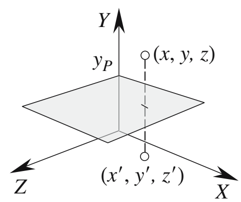
$x = x_p$ the position of the *yz*-plane



1.10.26 Reflection about a plane parallel with the *zx*-plane in $\mathbb{R}^3$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2y_P \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

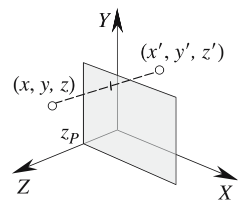
$y = y_p$ the position of the *zx*-plane



1.10.27 Reflection about a plane parallel with the *xy*-plane in $\mathbb{R}^3$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2z_P \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

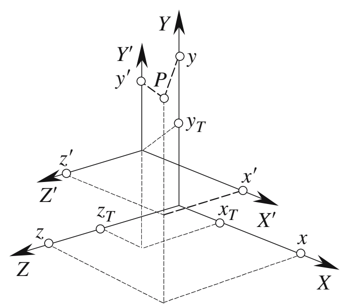
$z = z_p$ the position of the *xy*-plane



1.10.28 Translated change of axes in $\mathbb{R}^3$

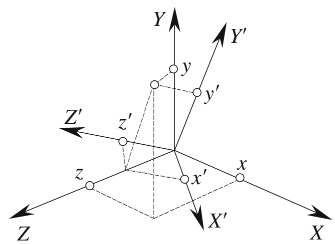
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -x_T \\ 0 & 1 & 0 & -y_T \\ 0 & 0 & 1 & -z_T \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$(x_T, y_T, z_T) =$ the translation



1.10.29 Rotated change of axes in $\mathbb{R}^3$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



r_{11}, r_{12}, r_{13} are the direction cosines of the secondary x-axis
r_{21}, r_{22}, r_{23} are the direction cosines of the secondary y-axis
r_{31}, r_{32}, r_{33} are the direction cosines of the secondary z-axis

1.10.30 The identity matrix in $\mathbb{R}^3$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

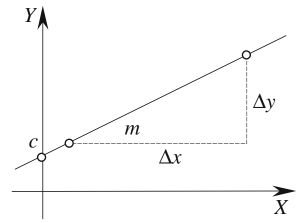
1.11 Two-dimensional straight lines

1.11.1 Normal form of the straight line equation

Given $\quad y = mx + c$

then $\quad m = \dfrac{\Delta y}{\Delta x}$ the slope of the line

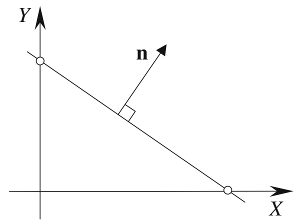
and $\quad c =$ the intercept with the y-axis



1.11.2 General form of the straight line equation

Given $\quad ax + by + c = 0$

then $\quad \mathbf{n} = a\mathbf{i} + b\mathbf{j}$



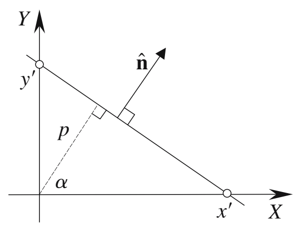
1.11.3 Hessian normal form of the straight line equation

Given $\quad x\cos\alpha + y\sin\alpha = p$

$|p|$ is the perpendicular distance from the origin to the line,

and $\quad x' = \dfrac{p}{\cos\alpha}$ and $y' = \dfrac{p}{\sin\alpha}$

unit vector $\quad \hat{\mathbf{n}} = \cos\alpha\mathbf{i} + \sin\alpha\mathbf{j}$



$ax + by + c = 0$ is converted into the Hessian normal form

by
$$\frac{ax}{\sqrt{a^2 + b^2}} + \frac{by}{\sqrt{a^2 + b^2}} + \frac{c}{\sqrt{a^2 + b^2}} = 0$$

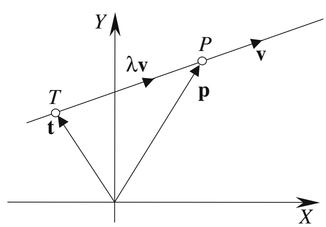
1.11.4 Parametric form of the straight line equation

Given $\quad \mathbf{p} = \mathbf{t} + \lambda\mathbf{v}$

where $\quad \mathbf{t} = x_T\mathbf{i} + y_T\mathbf{j}$

and $\quad \mathbf{v} = x_v\mathbf{i} + y_v\mathbf{j}$

$T(x_T, y_T)$ is a point on the line and λ is a scalar.



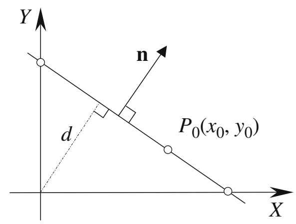
1.11.5 Cartesian form of the straight line equation

Given $ax + by = c$

then $c = d\,||\mathbf{n}|| = ax_0 + by_0$

where $P_0\,(x_0, y_0)$ is a point on the line.

The normalized form is $\dfrac{a}{||\mathbf{n}||}x + \dfrac{b}{||\mathbf{n}||}y = d$



1.11.6 Straight line equation from two points

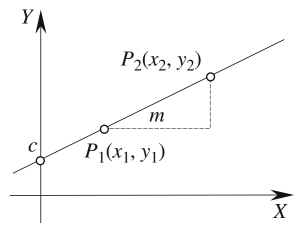
Normal form of the line equation

Given $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$

and $y = mx + c$

then $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

and $c = y_1 - x_1\left(\dfrac{y_2 - y_1}{x_2 - x_1}\right)$



General form of the line equation

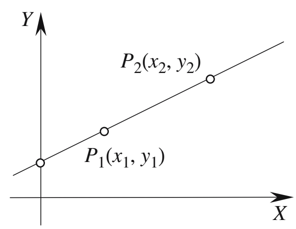
Given $P_1(x_1, y_1)$ and $P_2\,(x_2, y_2)$

and $Ax + By + C = 0$

then $A = y_2 - y_1$

$B = x_1 - x_2$

$C = -(x_1y_2 - x_2y_1)$



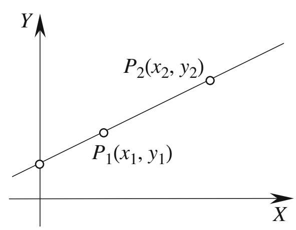
Cartesian form of the line equation

Given $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$

and $ax + by = c$

then $a = y_2 - y_1 \quad b = x_1 - x_2 \quad c = x_1y_2 - x_2y_1$

or $\begin{vmatrix} 1 & y_1 \\ 1 & y_2 \end{vmatrix} x + \begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix} y = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$



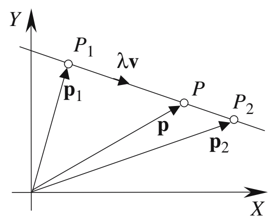
Parametric form of the line equation

Given $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$

and $\mathbf{p} = \mathbf{p}_1 + \lambda\mathbf{v}$

and $\mathbf{v} = \mathbf{p}_2 - \mathbf{p}_1$

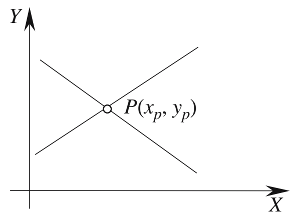
P is between P_1 and P_2 for $\lambda \in [0, 1]$.



1.11.7 Point of intersection of two straight lines

General form of the line equation

Given
$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$



then
$$\frac{x_P}{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}} = \frac{y_P}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}} = \frac{-1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Intersect at
$$x_p = \frac{c_2b_1 - c_1b_2}{a_1b_2 - a_2b_1} \qquad y_P = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}$$

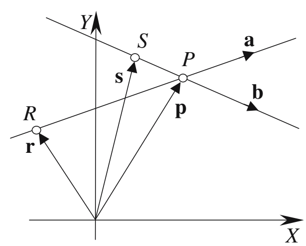
The lines are parallel if $a_1b_2 - a_2b_1 = 0$

Parametric form of the line equation

Given $\mathbf{p} = \mathbf{r} + \lambda\mathbf{a}$ $\quad \mathbf{q} = \mathbf{s} + \varepsilon\mathbf{b}$

where $\mathbf{r} = x_R\mathbf{i} + y_R\mathbf{j}$ $\quad \mathbf{s} = x_S\mathbf{i} + y_S\mathbf{j}$

and $\mathbf{a} = x_a\mathbf{i} + y_a\mathbf{j}$ $\quad \mathbf{b} = x_b\mathbf{i} + y_b\mathbf{j}$



then
$$\lambda = \frac{x_b(y_S - y_R) - y_b(x_S - x_R)}{x_by_a - x_ay_b}$$

and
$$\varepsilon = \frac{x_a(y_S - y_R) - y_a(x_S - x_R)}{x_by_a - x_ay_b}$$

Point of intersection $x_P = x_R + \lambda x_a$ $\quad y_p = y_R + \lambda y_a$

or $x_P = x_S + \varepsilon x_b$ $\quad y_P = y_S + \varepsilon y_b$

The lines are parallel if $x_by_a - x_ay_b = 0$

1.11.8 Angle between two straight lines

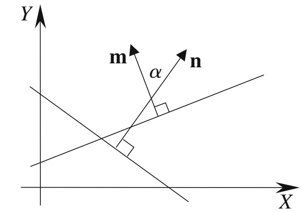
General form of the line equation

Given $a_1x + b_1y + c_1 = 0 \qquad a_2x + b_2y + c_2 = 0$

where $\mathbf{n} = a_1\mathbf{i} + b_1\mathbf{j} \qquad \mathbf{m} = a_2\mathbf{i} + b_2\mathbf{j}$

angle $$\alpha = \cos^{-1}\left(\frac{\mathbf{n}\bullet\mathbf{m}}{\|\mathbf{n}\|\cdot\|\mathbf{m}\|}\right)$$

If $\|\mathbf{n}\| = \|\mathbf{m}\| = 1 \qquad \alpha = \cos^{-1}(\mathbf{n}\bullet\mathbf{m})$

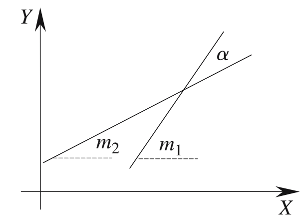


Normal form of the line equation

Given $y = m_1x + c_1 \qquad y = m_2x + c_2$

angle $$\alpha = \cos^{-1}\left(\frac{1+m_1m_2}{\sqrt{1+m_1^2}\sqrt{1+m_2^2}}\right)$$

or $$\alpha = \tan^{-1}\left(\frac{m_1-m_2}{1+m_1m_2}\right)$$



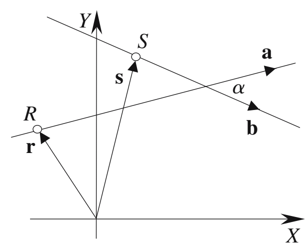
If the lines are perpendicular $m_1m_2 = -1$.

Parametric form of the line equation

Given $\mathbf{p} = \mathbf{r} + \lambda\mathbf{a} \qquad \mathbf{q} = \mathbf{s} + \varepsilon\mathbf{b}$

angle $$\alpha = \cos^{-1}\left(\frac{\mathbf{a}\bullet\mathbf{b}}{\|\mathbf{a}\|\cdot\|\mathbf{b}\|}\right)$$

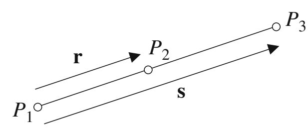
If $\|\mathbf{a}\| = \|\mathbf{b}\| = 1 \qquad \alpha = \cos^{-1}(\mathbf{a}\bullet\mathbf{b})$



1.11.9 Three points lie on a straight line

Given $P_1(x_1, y_1), \quad P_2(x_2, y_2) \quad \text{and} \quad P_3(x_3, y_3)$

and $\mathbf{r} = \overrightarrow{P_1P_2} \quad \text{and} \quad \mathbf{s} = \overrightarrow{P_1P_3}$



The three points lie on a straight line when $\mathbf{s} = \lambda\mathbf{r}$.

1.11.10 Parallel and perpendicular straight lines

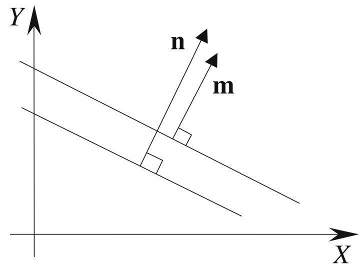
General form of the line equation

Given $a_1x + b_1y + c_1 = 0 \qquad a_2x + b_2y + c_2 = 0$

where $\mathbf{n} = a_1\mathbf{i} + b_1\mathbf{j} \qquad \mathbf{m} = a_2\mathbf{i} + b_2\mathbf{j}$

The lines are parallel if $\mathbf{n} = \lambda\mathbf{m}$.

The lines are mutually perpendicular if $\mathbf{n} \cdot \mathbf{m} = 0$.

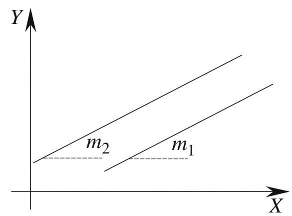


Normal form of the line equation

Given $y = m_1x + c_1 \qquad y = m_2x + c_2$

The lines are parallel if $m_1 = m_2$.

The lines are mutually perpendicular if $m_1m_2 = -1$.

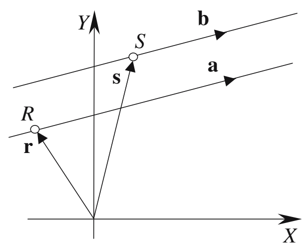


Parametric form of the line equation

Given $\mathbf{p} = \mathbf{r} + \lambda\mathbf{a} \qquad \mathbf{q} = \mathbf{s} + \varepsilon\mathbf{b}$

The lines are parallel if $\mathbf{a} = k\mathbf{b}$.

The lines are mutually perpendicular if $\mathbf{a} \cdot \mathbf{b} = 0$.



1.11.11 Position and distance of a point on a line perpendicular to the origin

General form of the line equation

Given $ax + by + c = 0$

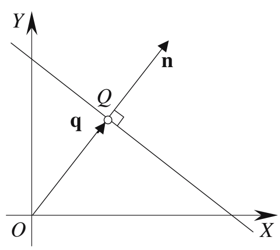
where $\mathbf{n} = a\mathbf{i} + b\mathbf{j}$

$\mathbf{q} = \lambda\mathbf{n}$

where $\lambda = \dfrac{-c}{\mathbf{n} \cdot \mathbf{n}}$

If $\|\mathbf{n}\| = 1 \qquad \lambda = -c$

Distance $OQ = \|\mathbf{q}\|$



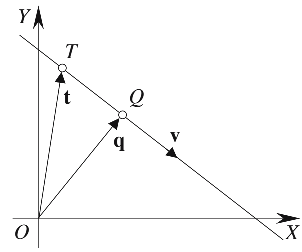
Parametric form of the line equation

Given $\mathbf{q} = \mathbf{t} + \lambda\mathbf{v}$

where $\lambda = \dfrac{-\mathbf{v}\cdot\mathbf{t}}{\mathbf{v}\cdot\mathbf{v}}$

If $\|\mathbf{v}\| = 1$ $\lambda = -\mathbf{v}\cdot\mathbf{t}$

Distance $OQ = \|\mathbf{q}\|$



1.11.12 Position and distance of the nearest point on a line to a point

General form of the line equation

Given $ax + by + c = 0$

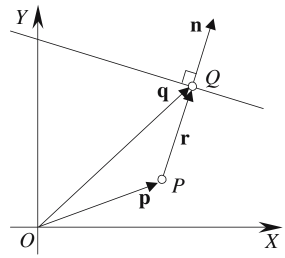
where $\mathbf{n} = a\mathbf{i} + b\mathbf{j}$

$\mathbf{q} = \mathbf{p} + \lambda\mathbf{n}$

where $\lambda = -\dfrac{\mathbf{n}\cdot\mathbf{p} + c}{\mathbf{n}\cdot\mathbf{n}}$

If $\|\mathbf{n}\| = 1$ $\lambda = -\mathbf{n}\cdot\mathbf{p} + c$

Distance $PQ = \|\lambda\mathbf{n}\|$



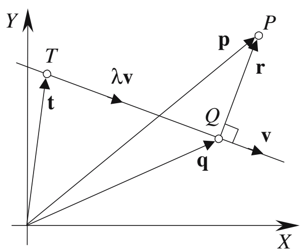
Parametric form of the line equation

Given $\mathbf{q} = \mathbf{t} + \lambda\mathbf{v}$

where $\lambda = \dfrac{\mathbf{v}\cdot(\mathbf{p} - \mathbf{t})}{\mathbf{v}\cdot\mathbf{v}}$

If $\|\mathbf{v}\| = 1$ $\lambda = \mathbf{v}\cdot(\mathbf{p} - \mathbf{t})$

Distance $PQ = \|\mathbf{p} - \mathbf{t} - \lambda\mathbf{v}\|$



1.11.13 Position of a point reflected in a line

General form of the line equation

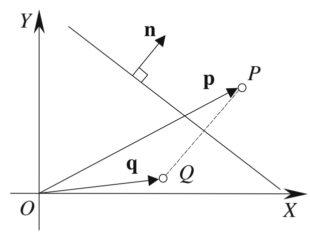
Given $ax + by + c = 0$

where $\mathbf{n} = a\mathbf{i} + b\mathbf{j}$

Q is P's reflection in the line $\mathbf{q} = \mathbf{p} - \lambda\mathbf{n}$

$$\lambda = \frac{2(\mathbf{n}\cdot\mathbf{p} + c)}{\mathbf{n}\cdot\mathbf{n}}$$

If $\|\mathbf{n}\| = 1$ $\lambda = 2(\mathbf{n}\cdot\mathbf{p} + c)$



Parametric form of the line equation

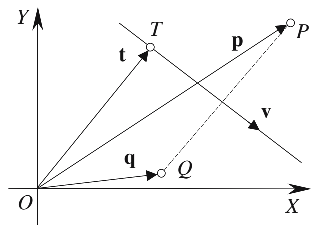
Given $\mathbf{s} = \mathbf{t} + \lambda\mathbf{v}$

Q is P's reflection in the line

$$\mathbf{q} = 2\mathbf{t} + \varepsilon\mathbf{v} - \mathbf{p}$$

where $$\varepsilon = \frac{2\mathbf{v} \bullet (\mathbf{p} - \mathbf{t})}{\mathbf{v} \bullet \mathbf{v}}$$

If $\|\mathbf{v}\| = 1$ $\quad \varepsilon = 2\mathbf{v} \bullet (\mathbf{p} - \mathbf{t})$

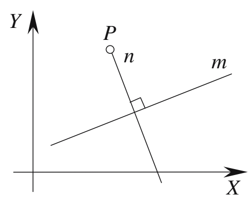


1.11.14 Normal to a line through a point

General form of the line equation

Given line m $\quad ax + by + c = 0$ and a point $P(x_p, y_p)$

Line n is $\quad -bx + ay + bx_p - ay_p = 0$



Parametric form of the line equation

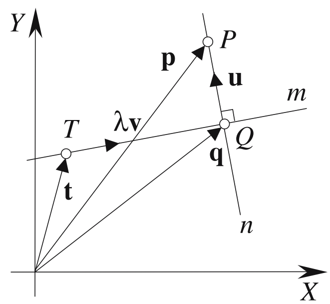
Given line m $\quad \mathbf{q} = \mathbf{t} + \lambda\mathbf{v}$ and a point P

$$\mathbf{u} = \mathbf{p} - (\mathbf{t} + \lambda\mathbf{v})$$

where $$\lambda = \frac{\mathbf{v} \bullet (\mathbf{p} - \mathbf{t})}{\mathbf{v} \bullet \mathbf{v}}$$

If $\| \mathbf{v} \| = 1$ $\quad \lambda = \mathbf{v} \bullet (\mathbf{p} - \mathbf{t})$

Line n is $\quad \mathbf{n} = \mathbf{p} + \varepsilon\mathbf{u}$ $\quad$ where ε is a scalar.



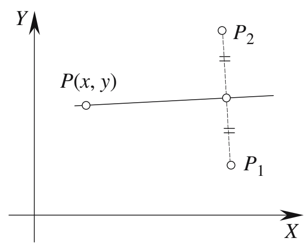
1.11.15 Line equidistant from two points

General form of the line equation

Given $\quad P_1(x_1, y_1)$ $\quad$ and $\quad P_2(x_2, y_2)$

The line equation is

$$(x_2 - x_1)x + (y_2 - y_1)y - \tfrac{1}{2}(x_2^2 - x_1^2 + y_2^2 - y_1^2) = 0$$



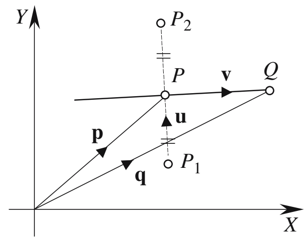
Parametric form of the line equation

Given $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$

$$\mathbf{q} = \mathbf{p} + \lambda\mathbf{v}$$

$$\mathbf{p} = \tfrac{1}{2}(\mathbf{p}_1 + \mathbf{p}_2)$$

$$\mathbf{v} = -(y_2 - y_1)\,\mathbf{i} + (x_2 - x_1)\,\mathbf{j}$$



1.11.16 Two-dimensional line segment

Line segment

$P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ define a line segment and $\mathbf{p}_1$ and $\mathbf{p}_2$ are their respective position vectors.

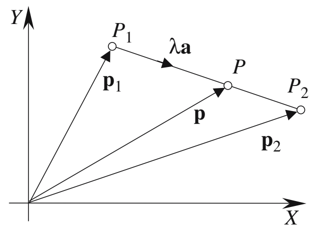
Therefore $\mathbf{p} = \mathbf{p}_1 + \lambda\mathbf{a}$

where $\mathbf{a} = \mathbf{p}_2 - \mathbf{p}_1$

therefore $x_P = x_1 + \lambda(x_2 - x_1)$

$y_P = y_1 + \lambda(y_2 - y_1)$

P is between P_1 and P_2 for $\lambda \in [0, 1]$.



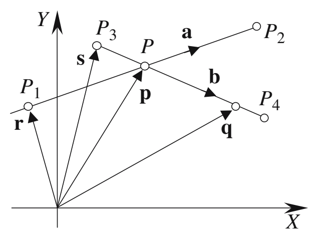
Intersection of two line segments

Given $\mathbf{p} = \mathbf{r} + \lambda\mathbf{a}$ and $\mathbf{q} = \mathbf{s} + \varepsilon\mathbf{b}$

where $\mathbf{a} = x_a\mathbf{i} + y_a\mathbf{j}$ and $\mathbf{b} = x_b\mathbf{i} + y_b\mathbf{j}$

then
$$\varepsilon = \frac{x_a(y_3 - y_1) - y_a(x_3 - x_1)}{x_b y_a - x_a y_b}$$

and
$$\lambda = \frac{x_b(y_3 - y_1) - y_b(x_3 - x_1)}{x_b y_a - x_a y_b}$$



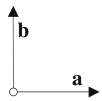
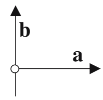
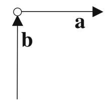
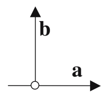
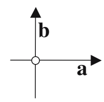
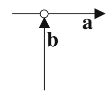
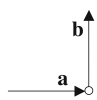
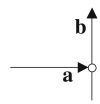
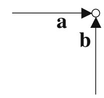
If $0 \leqslant \lambda \leqslant 1$ and $0 \leqslant \varepsilon \leqslant 1$ the lines intersect or touch one another. A possible point of intersection is given by

$$x_P = x_1 + \lambda x_a \qquad y_P = y_1 + \lambda y_a$$

or
$$x_P = x_3 + \varepsilon x_b \qquad y_P = y_3 + \varepsilon y_b$$

The line segments are parallel if $x_b y_a - x_a y_b = 0$.

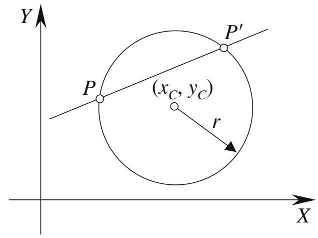
The table below illustrates the relative positions of the line segments for different values of λ and ε.

λ	ε		ε		ε	
0	0		$0 < \varepsilon < 1$		1	
$0 < \lambda < 1$	0		$0 < \varepsilon < 1$		1	
1	0		$0 < \varepsilon < 1$		1	

1.12 Lines and circles

1.12.1 Line intersecting a circle

General form of the line equation



Given a line $ax + by + c = 0$ where $a^2 + b^2 = 1$ and a circle radius r with center (x_C, y_C).

The potential intersection coordinates are given by

$$x = x_C - ac_T \pm \sqrt{c_T^2(a^2 - 1) + b^2 r^2}$$

$$y = y_C - bc_T \pm \sqrt{c_T^2(b^2 - 1) + a^2 r^2}$$

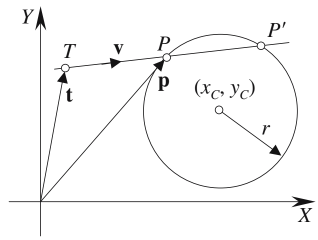
where $\quad c_T = ax_C + by_C + c$

Miss $\quad c_T^2(b^2 - 1) + a^2r^2 < 0$

Touch $\quad c_T^2(b^2 - 1) + a^2r^2 = 0$

Intersect $\quad c_T^2(b^2 - 1) + a^2r^2 > 0$

Parametric form of the line equation



Given a line $\mathbf{p} = \mathbf{t} + \lambda\mathbf{v}$ where $\|\mathbf{v}\| = 1$ and a circle radius r with center (x_C, y_C) with position vector $\mathbf{c} = x_C\mathbf{i} + y_C\mathbf{j}$.

The potential intersection coordinates are given by

$$x_P = x_T + \lambda x_v$$
$$y_P = y_T + \lambda y_v$$

where $\quad \lambda = \mathbf{s}\cdot\mathbf{v} \pm \sqrt{(\mathbf{s}\cdot\mathbf{v})^2 - \|\mathbf{s}\|^2 + r^2}$

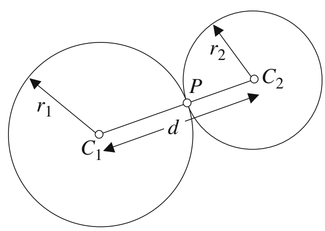
$\quad \mathbf{s} = \mathbf{c} - \mathbf{t}$

Miss $\quad (\mathbf{s}\cdot\mathbf{v})^2 - \|\mathbf{s}\|^2 + r^2 < 0$

Touch $\quad (\mathbf{s}\cdot\mathbf{v})^2 - \|\mathbf{s}\|^2 + r^2 = 0$

Intersect $\quad (\mathbf{s}\cdot\mathbf{v})^2 - \|\mathbf{s}\|^2 + r^2 > 0$

1.12.2 Touching and intersecting circles



Given two circles with radii r_1 and r_2 centered at C_1 (x_{C1}, y_{C1}) and C_2 (x_{C2}, y_{C2}) respectively.

Touch $\quad d = r_1 + r_2$

Touch point $$x_P = x_{C1} + \frac{r_1}{d}(x_{C2} - x_{C1})$$

$$y_P = y_{C1} + \frac{r_1}{d}(y_{C2} - y_{C1})$$

Separate $d > r_1 + r_2$

Intersect $r_1 + r_2 > d > |r_1 - r_2|$

Point(s) of intersection

$$x_{P1} = x_{C1} + \lambda x_d \mp \varepsilon y_d$$
$$y_{P1} = y_{C1} + \lambda y_d \pm \varepsilon x_d$$

where $$\lambda = \frac{r_1^2 - r_2^2 + d^2}{2d^2}$$

and $$\varepsilon = \left| \sqrt{\frac{r_1^2}{d^2} - \lambda^2} \right|$$

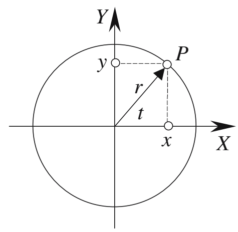
1.13 Second degree curves

1.13.1 Circle

General equation

Center origin $x^2 + y^2 = r^2$

Center (x_c, y_c) $(x - x_c)^2 + (y - y_c)^2 = r^2$



Parametric equation

Center origin
$$\left.\begin{array}{l} x = r\cos t \\ y = r\sin t \end{array}\right\} \quad 0 \le t \le 2\pi$$

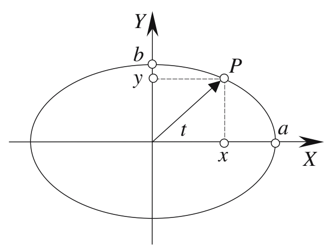
Center (x_c, y_c)
$$\left.\begin{array}{l} x = x_c + r\cos t \\ y = y_c + r\sin t \end{array}\right\} \quad 0 \le t \le 2\pi$$

1.13.2 Ellipse

General equation

Center origin
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Center (x_c, y_c)
$$\frac{(x - x_c)^2}{a^2} + \frac{(y - y_c)^2}{b^2} = 1$$



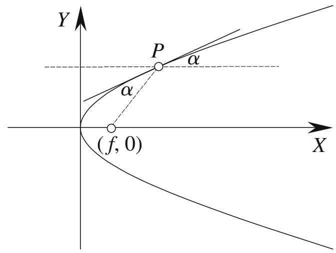
Parametric equation

Center origin
$$\left.\begin{array}{l} x = a\cos t \\ y = b\sin t \end{array}\right\} \quad 0 \le t \le 2\pi$$

Center (x_c, y_c)
$$\left.\begin{array}{l} x = x_c + a\cos t \\ y = y_c + b\sin t \end{array}\right\} \quad 0 \le t \le 2\pi$$

1.13.3 Parabola

General equation



Vertex origin $y^2 = 4fx$

Vertex (x_c, y_c) $(y - y_c)^2 = 4f(x - x_c)$

where f is the focus.

Reversing the axes $x^2 = 4fy$

Parametric equation

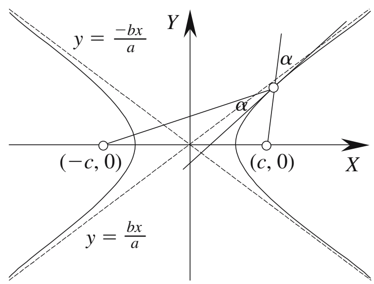
Vertex origin
$$x = t^2$$
$$y = 2\sqrt{f}\,t$$

Vertex (x_c, y_c)
$$x = x_c + t^2$$
$$y = y_c + 2t$$

Reversing the axes
$$x = 2\sqrt{f}\,t$$
$$y = t^2$$

1.13.4 Hyperbola

General equation



Centered at the origin, with the transverse axis coincident with the x-axis.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

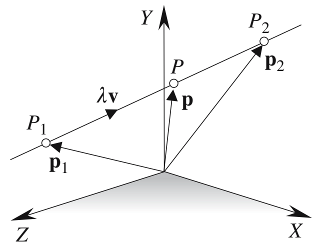
Foci at $(\pm c, 0)$ $c = \sqrt{a^2 + b^2}$

Parametric equation

Center origin
$$x = a \sec t$$
$$y = b \tan t$$

1.14 Three-dimensional straight lines

1.14.1 Straight line equation from two points



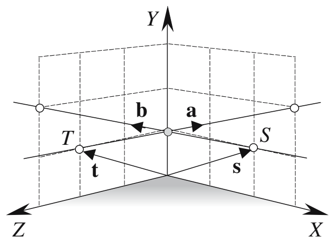
Given P_1 and P_2

$$\mathbf{v} = \mathbf{p}_2 - \mathbf{p}_1$$
$$\mathbf{p} = \mathbf{p}_1 + \lambda\mathbf{v}$$

P is between P_1 and P_2 for $\lambda \in [0, 1]$.

If $\|\mathbf{v}\| = 1$, λ corresponds to the linear distance along $\mathbf{v}$.

1.14.2 Intersection of two straight lines



Given $\quad \mathbf{p} = \mathbf{t} + \lambda\mathbf{a} \quad$ and $\quad \mathbf{q} = \mathbf{s} + \varepsilon\mathbf{b}$

where $\quad \mathbf{t} = x_t\mathbf{i} + y_t\mathbf{j} + z_t\mathbf{k} \quad$ and $\quad \mathbf{s} = x_s\mathbf{i} + y_s\mathbf{j} + z_s\mathbf{k}$

and $\quad \mathbf{a} = x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k} \quad$ and $\quad \mathbf{b} = x_b\mathbf{i} + y_b\mathbf{j} + z_b\mathbf{k}$

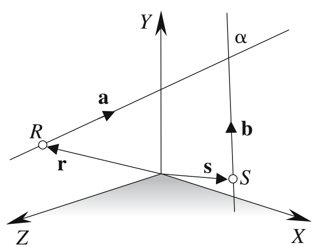
If $\mathbf{a} \times \mathbf{b} = 0$ the lines are parallel and do not intersect.

If $(\mathbf{t} - \mathbf{s}) \cdot (\mathbf{a} \times \mathbf{b}) \neq 0$ the lines do not intersect.

Solve
$$\begin{aligned} \lambda x_a - \varepsilon x_b &= x_s - x_t \\ \lambda y_a - \varepsilon y_b &= y_s - y_t \\ \lambda z_a - \varepsilon z_b &= z_s - z_t \end{aligned}$$

for values of λ and ε.

1.14.3 The angle between two straight lines



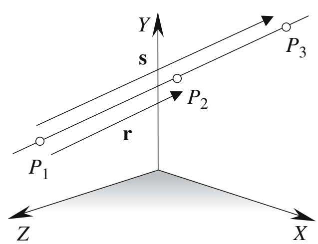
Given $\quad \mathbf{p} = \mathbf{r} + \lambda\mathbf{a}$

and $\quad \mathbf{q} = \mathbf{s} + \varepsilon\mathbf{b}$

Angle $\quad \alpha = \cos^{-1}\left(\dfrac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \cdot \|\mathbf{b}\|}\right)$

If $\|\mathbf{a}\| = \|\mathbf{b}\| = 1 \qquad \alpha = \cos^{-1}(\mathbf{a} \cdot \mathbf{b})$

1.14.4 Three points lie on a straight line



Given three points P_1, P_2, P_3.

Let $\qquad \mathbf{r} = \overrightarrow{P_1P_2} \quad$ and $\quad \mathbf{s} = \overrightarrow{P_1P_3}$

The points lie on a straight line when $\mathbf{s} = \lambda\mathbf{r}$ where λ is a scalar.

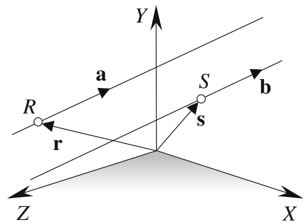
1.14.5 Parallel and perpendicular straight lines

Given $\mathbf{p} = \mathbf{r} + \mu\mathbf{a}$

and $\mathbf{q} = \mathbf{s} + \varepsilon\mathbf{b}$

The lines are parallel if $\mathbf{a} = \lambda\mathbf{b}$ where λ is a scalar.

The lines are perpendicular if $\mathbf{a} \cdot \mathbf{b} = 0$.



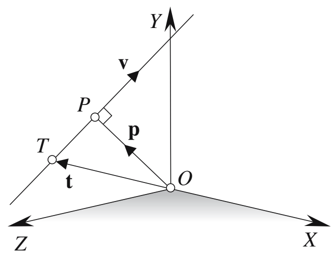
1.14.6 Position and distance of a point on a line perpendicular to the origin

Given $\mathbf{p} = \mathbf{t} + \lambda\mathbf{v}$

where $\lambda = \dfrac{-\mathbf{v} \cdot \mathbf{t}}{\mathbf{v} \cdot \mathbf{v}}$

If $\|\mathbf{v}\| = 1$ $\lambda = -\mathbf{v} \cdot \mathbf{t}$

Distance $OP = \|\mathbf{p}\|$



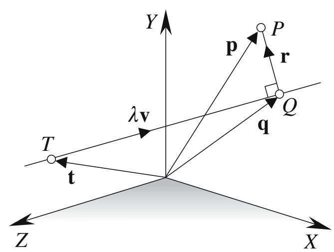
1.14.7 Position and distance of the nearest point on a line to a point

Given $\mathbf{q} = \mathbf{t} + \lambda\mathbf{v}$

where $\lambda = \dfrac{\mathbf{v} \cdot (\mathbf{p} - \mathbf{t})}{\mathbf{v} \cdot \mathbf{v}}$

If $\|\mathbf{v}\| = 1$ $\lambda = \mathbf{v} \cdot (\mathbf{p} - \mathbf{t})$

Distance $PQ = \|\mathbf{p} - \mathbf{t} - \lambda\mathbf{v}\|$

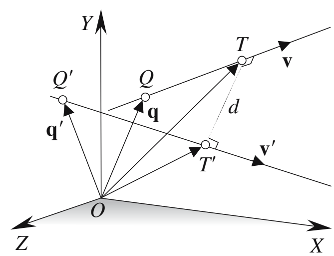


1.14.8 Shortest distance between two skew lines

Given $\mathbf{p} = \mathbf{q} + t\mathbf{v}$

and $\mathbf{p}' = \mathbf{q}' + \tau\mathbf{v}'$

Shortest distance $d = \dfrac{|(\mathbf{q} - \mathbf{q}') \cdot (\mathbf{v} \times \mathbf{v}')|}{\|\mathbf{v} \times \mathbf{v}'\|}$



1.14.9 Position of a point reflected in a line

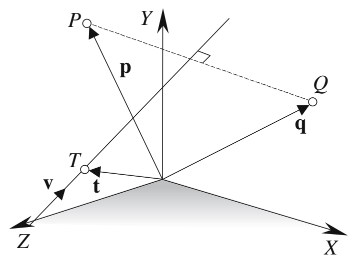
Given $\mathbf{s} = \mathbf{t} + \lambda\mathbf{v}$

and a point P with reflection Q

$$\mathbf{q} = 2\mathbf{t} + \varepsilon\mathbf{v} - \mathbf{p}$$

where $$\varepsilon = \frac{2\mathbf{v} \cdot (\mathbf{p} - \mathbf{t})}{\mathbf{v} \cdot \mathbf{v}}$$

If $\|\mathbf{v}\| = 1$ $\quad \varepsilon = 2\mathbf{v} \cdot (\mathbf{p} - \mathbf{t})$



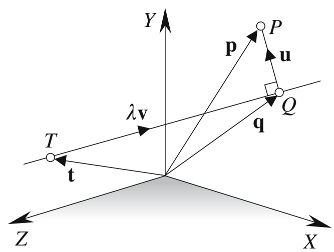
1.14.10 Normal to a line through a point

Given $\mathbf{q} = \mathbf{t} + \lambda\mathbf{v}$

the normal is $\mathbf{u} = \mathbf{p} - (\mathbf{t} + \lambda\mathbf{v})$

where $$\lambda = \frac{\mathbf{v} \cdot (\mathbf{p} - \mathbf{t})}{\mathbf{v} \cdot \mathbf{v}}$$

If $\|\mathbf{v}\| = 1$ $\quad \lambda = \mathbf{v} \cdot (\mathbf{p} - \mathbf{t})$

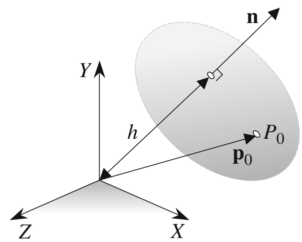


1.15 Planes

1.15.1 Cartesian form of the plane equation

Given $ax + by + cz = d$

where $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$



If P_0 is on the plane, and h is the perpendicular distance from the origin to the plane

$$d = \mathbf{n} \cdot \mathbf{p}_0 = h\|\mathbf{n}\|$$

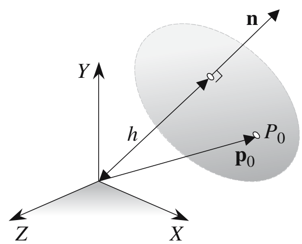
The normalized form is $Ax + By + Cz = D$

where $A = \dfrac{a}{\|\mathbf{n}\|} \qquad B = \dfrac{b}{\|\mathbf{n}\|} \qquad C = \dfrac{c}{\|\mathbf{n}\|} \qquad D = h$

1.15.2 General form of the plane equation

Given $Ax + By + Cz + D = 0$

where $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$



Its relationship to the Cartesian form is as follows:

$A = a \qquad B = b \qquad C = c \qquad D = -\mathbf{n} \cdot \mathbf{p}_0 = -d$

1.15.3 Hessian normal form of the plane equation

Given $Ax + By + Cz + D = 0$

The Hessian normal form is $n_1x + n_2y + n_3z + p = 0$

where

$$n_1 = \frac{A}{\sqrt{A^2 + B^2 + C^2}} \qquad n_2 = \frac{B}{\sqrt{A^2 + B^2 + C^2}}$$

$$n_3 = \frac{C}{\sqrt{A^2 + B^2 + C^2}} \qquad p = \frac{D}{\sqrt{A^2 + B^2 + C^2}}$$

In vector form: $P(x, y, z)$ is a point on the plane with position vector $\mathbf{p}$

then $\mathbf{p} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

and $\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$

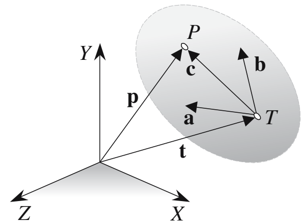
therefore $\mathbf{n} \cdot \mathbf{p} = -p$

1.15.4 Parametric form of the plane equation

Given $\mathbf{p} = \mathbf{t} + \lambda\mathbf{a} + \varepsilon\mathbf{b}$

$T(x_T, y_T, z_T)$ is on the plane with position vector **t**.
a and **b** are two unique vectors parallel to the plane
a point on the plane is given by

$$x_P = x_T + \lambda x_a + \varepsilon x_b$$
$$y_P = y_T + \lambda y_a + \varepsilon y_b$$
$$z_P = z_T + \lambda z_a + \varepsilon z_b$$



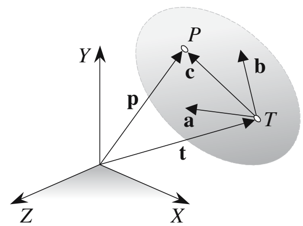
1.15.5 Converting from the parametric form to the general form

Given $\mathbf{p} = \mathbf{t} + \lambda\mathbf{a} + \varepsilon\mathbf{b}$

where $\|\mathbf{a}\| = \|\mathbf{b}\| = 1$

and $$\lambda = \frac{(\mathbf{a}\bullet\mathbf{b})(\mathbf{b}\bullet\mathbf{t}) - \mathbf{a}\bullet\mathbf{t}}{1-(\mathbf{a}\bullet\mathbf{b})^2}$$

and $$\varepsilon = \frac{(\mathbf{a}\bullet\mathbf{b})(\mathrm{a}\bullet\mathbf{t}) - \mathrm{b}\bullet\mathbf{t}}{1-(\mathbf{a}\bullet\mathbf{b})^2}$$



The normal vector is $\mathbf{p} = x_P\mathbf{i} + y_P\mathbf{j} + z_P\mathbf{k}$

$\|\mathbf{p}\|$ is the perpendicular distance from the plane to the origin

therefore $Ax + By + Cz + D = 0$

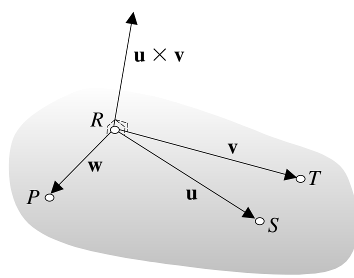
where $A = \dfrac{x_P}{\|\mathbf{p}\|} \quad B = \dfrac{y_P}{\|\mathbf{p}\|} \quad C = \dfrac{z_P}{\|\mathbf{p}\|} \quad D = -\|\mathbf{p}\|$

1.15.6 Plane equation from three points

Given R, S, T and $P(x, y, z)$ are on a plane

then $ax + by + cz + d = 0$

where
$$a = \begin{vmatrix} y_S - y_R & z_S - z_R \\ y_T - y_R & z_T - z_R \end{vmatrix}$$
$$b = \begin{vmatrix} z_S - z_R & x_S - x_R \\ z_T - z_R & x_T - x_R \end{vmatrix}$$
$$c = \begin{vmatrix} x_S - x_R & y_S - y_R \\ x_T - x_R & y_T - y_R \end{vmatrix}$$
$$d = -(ax_R + by_R + cz_R)$$



or
$$a = \begin{vmatrix} 1 & y_R & z_R \\ 1 & y_S & z_S \\ 1 & y_T & z_T \end{vmatrix} \qquad b = \begin{vmatrix} x_R & 1 & z_R \\ x_S & 1 & z_S \\ x_T & 1 & z_T \end{vmatrix}$$
$$c = \begin{vmatrix} x_R & y_R & 1 \\ x_S & y_S & 1 \\ x_T & y_T & 1 \end{vmatrix} \qquad d = -(ax_R + by_R + cz_R)$$

1.15.7 Plane through a point and normal to a line

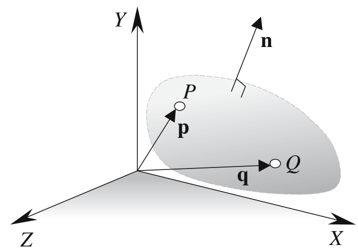
Given $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

$Q(x_Q, y_Q, z_Q)$ is on the plane with position vector $\mathbf{q}$

$P(x, y, z)$ is any point on the plane with position vector $\mathbf{p}$

then $\mathbf{n} \cdot (\mathbf{p} - \mathbf{q}) = 0$

or $ax + by + cz - (ax_Q + by_Q + cz_Q) = 0$



1.15.8 Plane through two points and parallel to a line

Given $M(x_M, y_M, z_M)$ and $N(x_N, y_N, z_N)$

and the line $\mathbf{p} = \mathbf{r} + \lambda\mathbf{a}$

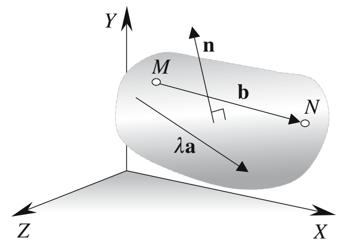
where $\mathbf{a} = x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k}$

then $\mathbf{b} = (x_N - x_M)\mathbf{i} + (y_N - y_M)\mathbf{j} + (z_N - z_M)\mathbf{k}$

and $\mathbf{a} \times \mathbf{b} = \mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

where $a = \begin{vmatrix} y_a & z_a \\ y_b & z_b \end{vmatrix} \quad b = \begin{vmatrix} z_a & x_a \\ z_b & x_b \end{vmatrix} \quad c = \begin{vmatrix} x_a & y_a \\ x_b & y_b \end{vmatrix}$

Plane equation is $ax + by + cz - (ax_M + by_M + cz_M) = 0$



1.15.9 Intersection of two planes

Given $a_1x + b_1y + c_1z + d_1 = 0$

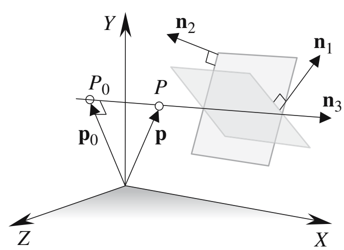
and $a_2x + b_2y + c_2z + d_2 = 0$

where $\mathbf{n}_1 = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$

and $\mathbf{n}_2 = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$

The line of intersection is $\mathbf{p} = \mathbf{p}_0 + \lambda\mathbf{n}_3$

where $\mathbf{n}_3 = \mathbf{n}_1 \times \mathbf{n}_2 = a_3\mathbf{i} + b_3\mathbf{j} + c_3\mathbf{k}$



and $$x_0 = \frac{d_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} - d_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}{DET}$$

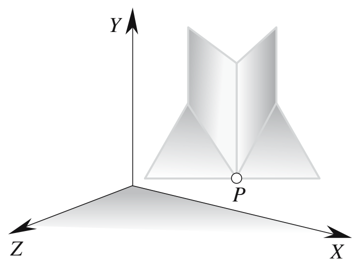
$$y_0 = \frac{d_2\begin{vmatrix} a_3 & c_3 \\ a_1 & c_1 \end{vmatrix} - d_1\begin{vmatrix} a_3 & c_3 \\ a_2 & c_2 \end{vmatrix}}{DET}$$

$$z_0 = \frac{d_2\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} - d_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}{DET}$$

and $$DET = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

If $DET = 0$ the line and plane are parallel.

1.15.10 Intersection of three planes



Given $$a_1x + b_1y + c_1z + d_1 = 0$$
$$a_2x + b_2y + c_2z + d_2 = 0$$
and $$a_3x + b_3y + c_3z + d_3 = 0$$

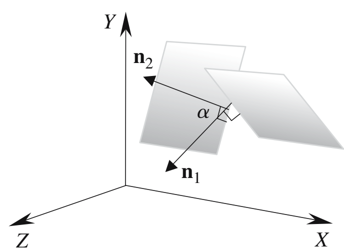
$P(x, y, z)$ is the point of intersection

where $$x = -\frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{DET} \quad y = -\frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{DET}$$

$$z = -\frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{DET} \quad DET = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

If $DET = 0$, two of the planes, at least, are parallel.

1.15.11 Angle between two planes

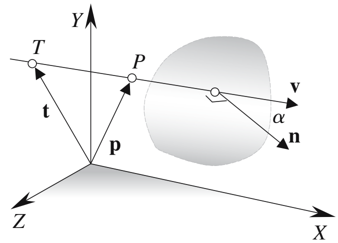


Given $$ax_1 + by_1 + cz_1 + d_1 = 0$$
and $$ax_2 + by_2 + cz_2 + d_2 = 0$$
where $$\mathbf{n}_1 = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$$
and $$\mathbf{n}_2 = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$$

$$\alpha = \cos^{-1}\left(\frac{\mathbf{n}_1 \bullet \mathbf{n}_2}{\|\mathbf{n}_1\| \cdot \|\mathbf{n}_2\|}\right)$$

If $\|\mathbf{n}_1\| = \|\mathbf{n}_2\| = 1 \quad \alpha = \cos^{-1}(\mathbf{n}_1 \bullet \mathbf{n}_2)$

1.15.12 Angle between a line and a plane



Given $\quad ax + by + cz + d = 0$

where $\quad \mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

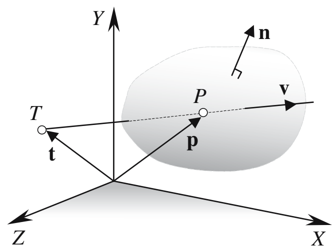
and the line equation is $\mathbf{p} = \mathbf{t} + \lambda\mathbf{v}$

then
$$\alpha = \cos^{-1}\left(\frac{\mathbf{n} \bullet \mathbf{v}}{\|\mathbf{n}\| \cdot \|\mathbf{n}\|}\right)$$

If $\|\mathbf{n}\| = \|\mathbf{v}\| = 1 \quad \alpha = \cos^{-1}(\mathbf{n} \bullet \mathbf{v})$

When the line is parallel with the plane $\mathbf{n} \bullet \mathbf{v} = 0$

1.15.13 Intersection of a line and a plane



Given $\quad ax + by + cz + d = 0$

where $\quad \mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

and line $\quad \mathbf{p} = \mathbf{t} + \lambda\mathbf{v}$

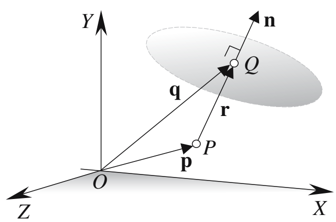
for the intersection point P

$$\lambda = \frac{-(\mathbf{n} \bullet \mathbf{t} + d)}{\mathbf{n} \bullet \mathbf{v}}$$

If $\|\mathbf{n}\| = \|\mathbf{v}\| = 1 \quad \lambda = -(\mathbf{n} \bullet \mathbf{t} + d)$

If $\mathbf{n} \bullet \mathbf{v} = 0$ the line and plane are parallel.

1.15.14 Position and distance of the nearest point on a plane to a point



Given $\quad ax + by + cz + d = 0$

where $\quad \mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

and Q is the nearest point on the plane to P

Position vector $\mathbf{q} = \mathbf{p} + \lambda\mathbf{n}$

Distance $\quad PQ = \|\lambda\mathbf{n}\|$

where
$$\lambda = \frac{-(\mathbf{n} \bullet \mathbf{p} + d)}{\mathbf{n} \bullet \mathbf{n}}$$

If $\|\mathbf{n}\| = 1 \quad \lambda = -(\mathbf{n} \bullet \mathbf{p} + d)$

1.15.15 Reflection of a point in a plane

Given $ax + by + cz + d = 0$

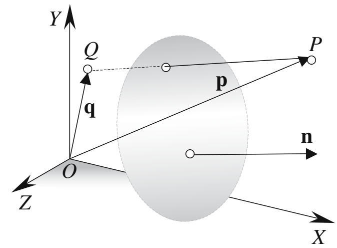
where $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

and Q is P's reflection.

Position vector $\mathbf{q} = \mathbf{p} + \lambda\mathbf{n}$

where $$\lambda = \frac{-2(\mathbf{n} \cdot \mathbf{p} + d)}{\mathbf{n} \cdot \mathbf{n}}$$

If $\|\mathbf{n}\| = 1$ $\quad \lambda = -2(\mathbf{n} \cdot \mathbf{p} + d)$



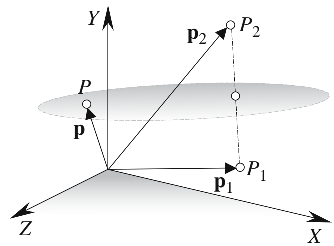
1.15.16 Plane equidistant from two points

Given $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$

where $P(x, y, z)$ is any point on the plane.

Plane equation is $(\mathbf{p}_2 - \mathbf{p}_1) \cdot (\mathbf{p} - \frac{1}{2}(\mathbf{p}_2 + \mathbf{p}_1)) = 0$

or $$(x_2 - x_1)x + (y_2 - y_1)y + (z_2 - z_1)z - \frac{1}{2}(x_2^2 - x_1^2 + y_2^2 - y_1^2 + z_2^2 - z_1^2) = 0$$



1.15.17 Reflected ray on a surface

Given $\mathbf{n}$ the surface normal vector

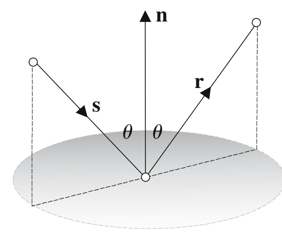
$\mathbf{s}$ the incident ray

$\mathbf{r}$ the reflected ray

then $\mathbf{r} = \mathbf{s} + \lambda\mathbf{n}$

where $$\lambda = \frac{-2\mathbf{n} \cdot \mathbf{s}}{\mathbf{n} \cdot \mathbf{n}}$$

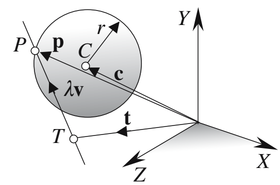
If $\|\mathbf{n}\| = 1$ $\quad \lambda = -2\mathbf{n} \cdot \mathbf{s}$



1.16 Lines, planes and spheres

1.16.1 Line intersecting a sphere

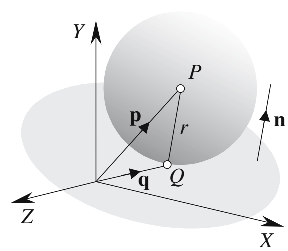
Given a sphere with radius r centered at C with position vector $\mathbf{c}$



and a line	$\mathbf{p} = \mathbf{t} + \lambda\mathbf{v}$
where	$\lVert\mathbf{v}\rVert = 1$
Position vector	$\mathbf{p} = \mathbf{t} + \lambda\mathbf{v}$
where	$\lambda = \mathbf{s}\cdot\mathbf{v} \pm \sqrt{(\mathbf{s}\cdot\mathbf{v})^2 - \lVert\mathbf{s}\rVert^2 + r^2}$
and	$\mathbf{s} = \mathbf{c} - \mathbf{t}$
Miss	$(\mathbf{s}\cdot\mathbf{v})^2 - \lVert\mathbf{s}\rVert^2 + r^2 < 0$
Touch	$(\mathbf{s}\cdot\mathbf{v})^2 - \lVert\mathbf{s}\rVert^2 + r^2 = 0$
Intersect	$(\mathbf{s}\cdot\mathbf{v})^2 - \lVert\mathbf{s}\rVert^2 + r^2 > 0$

1.16.2 Sphere touching a plane

Given a sphere with radius r centered at P

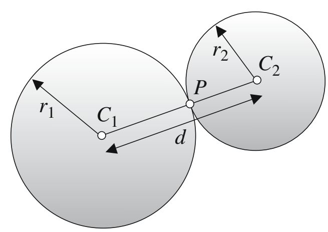


and a plane	$ax + by + cz + d = 0$
where	$\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$
then	$\mathbf{q} = \mathbf{p} + \lambda\mathbf{n}$
where	$\lambda = -\dfrac{\mathbf{n}\cdot\mathbf{p} + d}{\mathbf{n}\cdot\mathbf{n}}$
If $\lVert\mathbf{n}\rVert = 1$	$\lambda = -(\mathbf{n}\cdot\mathbf{p} + d)$

they touch at Q when $\lVert\lambda\mathbf{n}\rVert = r$

1.16.3 Touching spheres

Given two spheres: radius r_1, center C_1 (x_{C1}, y_{C1}, z_{C1}) and radius r_2, center C_2 (x_{C2}, y_{C2}, z_{C2})



$$d = \sqrt{(x_{C2} - x_{C1})^2 + (y_{C2} - y_{C1})^2 + (z_{C2} - z_{C1})^2}$$

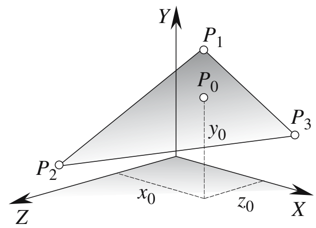
Intersect	$r_1 + r_2 > d > \lvert r_1 - r_2\rvert$
Separate	$d > r_1 + r_2$

Touch	$d = r_1 + r_2$
Touch point	$x_P = x_{C1} + \frac{r_1}{d}(x_{C2} - x_{C1})$
	$y_P = y_{C1} + \frac{r_1}{d}(y_{C2} - y_{C1})$
	$z_P = z_{C1} + \frac{r_1}{d}(z_{C2} - z_{C1})$

1.17 Three-dimensional triangles

1.17.1 Point inside a triangle

Given the vertices $P_1(x_1, y_1, z_1,), P_2(x_2, y_2, z_2)$ and $P_3(x_3, y_3, z_3)$ using barycentric coordinates we can write



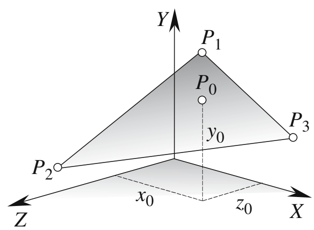
$$x_0 = \varepsilon x_1 + \lambda x_2 + \beta x_3$$
$$y_0 = \varepsilon y_1 + \lambda y_2 + \beta y_3$$
$$z_0 = \varepsilon z_1 + \lambda z_2 + \beta z_3$$

where $\quad \varepsilon + \lambda + \beta = 1$

P_0 is within the boundary of the triangle if $\varepsilon + \lambda + \beta = 1$ and $(\varepsilon, \lambda, \beta) \in [0, 1]$.

1.17.2 Unknown coordinate value inside a triangle

Given the vertices P_1, P_2, P_3 and a point $P_0(x_0, y_0, z_0)$ where only two of the coordinates are known, the third coordinate can be determined within the boundary of the triangle using barycentric coordinates. For example, if x_0 and z_0 are known we can find y_0 using barycentric coordinates:



$$y_0 = \varepsilon y_1 + \lambda y_2 + \beta y_3$$

where

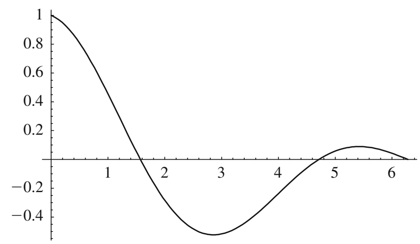
$$\frac{\varepsilon}{\begin{vmatrix} x_0 & z_0 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix}} = \frac{\lambda}{\begin{vmatrix} x_0 & z_0 & 1 \\ x_3 & z_3 & 1 \\ x_1 & z_1 & 1 \end{vmatrix}} = \frac{1}{\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix}}$$

P_0 is within the boundary of the triangle if $\varepsilon + \lambda + \beta = 1$ and $(\varepsilon, \lambda, \beta) \in [0, 1]$.

1.18 Parametric curves and patches

1.18.1 Parametric curve in $\mathbb{R}^2$

A parametric curve in $\mathbb{R}^2$ has two functions sharing a common parameter, with each function having independent control over the x and y-coordinates.



$$\left.\begin{aligned} x &= f(t) \\ y &= g(t) \end{aligned}\right\} \quad t \in [t_{\min},\ t_{\max}]$$

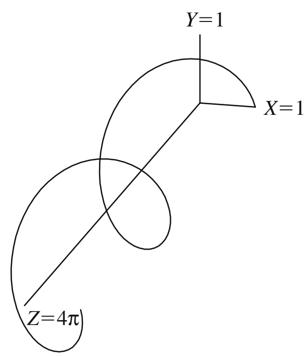
e.g.

$$t_{\max} = 2\pi$$

$$\left.\begin{aligned} a &= 1 - \frac{t}{t_{\max}} \\ x &= t \\ y &= a\cos t \end{aligned}\right\} \quad t \in [0,\ t_{\max}]$$

1.18.2 Parametric curve in $\mathbb{R}^3$

A parametric curve in $\mathbb{R}^3$ has three functions sharing a common parameter, with each function having independent control over the x, y and z-coordinates.



$$\left.\begin{aligned} x &= f(t) \\ y &= g(t) \\ z &= h(t) \end{aligned}\right\} \quad t \in [t_{\min},\ t_{\max}]$$

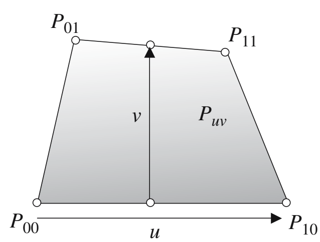
e.g.

$$\left.\begin{aligned} x &= \cos t \\ y &= \sin t \\ z &= t \end{aligned}\right\} \quad t \in [0,\ 4\pi]$$

1.18.3 Planar patch

Given $P_{00}, P_{10}, P_{11}, P_{01}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ that form a patch

$$P_{uv} = (1-v)[(1-u)P_{00} + uP_{10}] + v[(1-u)P_{01} + uP_{11}]$$



where $(u, v) \in [0, 1]$.

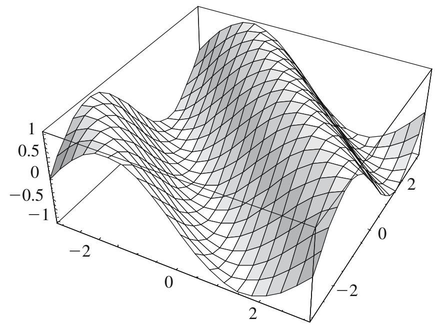
In matrix form

$$P_{uv} = [u \quad 1]\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} v \\ 1 \end{bmatrix}$$

1.18.4 Modulated surface

A function can be represented as a modulated surface by making the function's value modulate one of the Cartesian coordinates of the surface.

e.g. $\left.\begin{aligned} y &= f(x,z) \\ y &= \sin(x+z) \end{aligned}\right\} \quad (x,z) \in [-\pi, \pi]$

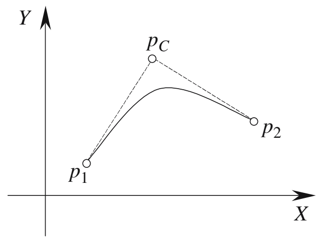


1.18.5 Quadratic Bézier curve

Given two points (x_1, y_1) and (x_2, y_2) and a control point (x_C, y_C) a quadratic Bézier curve has the form:

$$\mathbf{p}(t) = \mathbf{p}_1(1-t)^2 + \mathbf{p}_C 2t(1-t) + \mathbf{p}_2 t^2$$

or $$\mathbf{p}(t) = [t^2 \quad t \quad 1]\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_C \\ \mathbf{p}_2 \end{bmatrix}$$

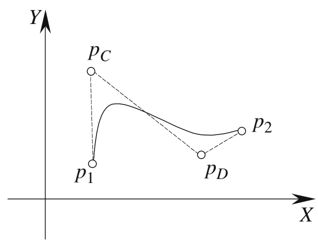


1.18.6 Cubic Bézier curve

Given two points (x_1, y_1) and (x_2, y_2) and two control points (x_C, y_C) and (x_D, y_D) a cubic Bézier curve has the form:

$$\mathbf{p}(t) = \mathbf{p}_1(1-t)^3 + \mathbf{p}_C 3t(1-t)^2 + \mathbf{p}_D 3t^2(1-t) + \mathbf{p}_2 t^3$$

or $$\mathbf{p}(t) = [t^3 \quad t^2 \quad t \quad 1]\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_C \\ \mathbf{p}_D \\ \mathbf{p}_2 \end{bmatrix}$$



1.18.7 Quadratic Bézier patch

Definition $$\mathbf{p}(u,v) = \sum_{i=0}^{2}\sum_{j=0}^{2} B_{i,2}(u) B_{j,2}(v) \mathbf{p}_{i,j}$$

where $\quad B_{i,2}(u) = \binom{2}{i} u^i (1-u)^{2-i} \quad \text{and} \quad B_{j,2}(v) = \binom{2}{j} v^j (1-v)^{2-j}$

as a matrix
$$\mathbf{p}(u,v) = [\,(1-u)^2 \quad 2u(1-u) \quad u^2\,] \begin{bmatrix} \mathbf{p}_{00} & \mathbf{p}_{01} & \mathbf{p}_{02} \\ \mathbf{p}_{10} & \mathbf{p}_{11} & \mathbf{p}_{12} \\ \mathbf{p}_{20} & \mathbf{p}_{21} & \mathbf{p}_{22} \end{bmatrix} \begin{bmatrix} (1-v)^2 \\ 2v(1-v) \\ v^2 \end{bmatrix}$$

or
$$\mathbf{p}(u,v) = [\,u^2 \quad u \quad 1\,] \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_{00} & \mathbf{p}_{01} & \mathbf{p}_{02} \\ \mathbf{p}_{10} & \mathbf{p}_{11} & \mathbf{p}_{12} \\ \mathbf{p}_{20} & \mathbf{p}_{21} & \mathbf{p}_{22} \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v^2 \\ v \\ 1 \end{bmatrix}$$

1.18.8 Cubic Bézier patch

Definition
$$\mathbf{p}(u,v) = \sum_{i=0}^{3} \sum_{j=0}^{3} B_{i,3}(u) B_{j,3}(v) \mathbf{p}_{i,j}$$

where $\quad B_{i,3}(u) = \binom{3}{i} u^i (1-u)^{3-i} \quad \text{and} \quad B_{j,2}(v) = \binom{2}{j} v^j (1-v)^{3-j}$

as a matrix

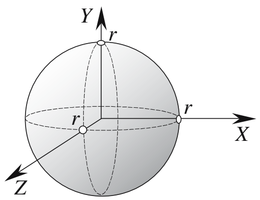
$$\mathbf{p}(u,v) = [(1-u)^3 \quad 3u(1-u)^2 \quad 3u^2(1-u) \quad u^3] \begin{bmatrix} \mathbf{p}_{00} & \mathbf{p}_{01} & \mathbf{p}_{02} & \mathbf{p}_{03} \\ \mathbf{p}_{10} & \mathbf{p}_{11} & \mathbf{p}_{12} & \mathbf{p}_{13} \\ \mathbf{p}_{20} & \mathbf{p}_{21} & \mathbf{p}_{22} & \mathbf{p}_{23} \\ \mathbf{p}_{30} & \mathbf{p}_{31} & \mathbf{p}_{32} & \mathbf{p}_{33} \end{bmatrix} \begin{bmatrix} (1-v)^3 \\ 3v(1-v)^2 \\ 3v^2(1-v) \\ v^3 \end{bmatrix}$$

or

$$\mathbf{p}(u,v) = [u^3 \quad u^2 \quad u \quad 1] \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_{00} & \mathbf{p}_{01} & \mathbf{p}_{02} & \mathbf{p}_{03} \\ \mathbf{p}_{10} & \mathbf{p}_{11} & \mathbf{p}_{12} & \mathbf{p}_{13} \\ \mathbf{p}_{20} & \mathbf{p}_{21} & \mathbf{p}_{22} & \mathbf{p}_{23} \\ \mathbf{p}_{30} & \mathbf{p}_{31} & \mathbf{p}_{32} & \mathbf{p}_{33} \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v^3 \\ v^2 \\ v \\ 1 \end{bmatrix}$$

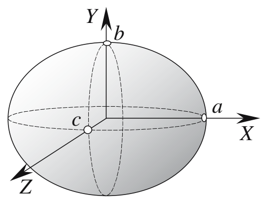
1.19 Second degree surfaces in standard form

Sphere



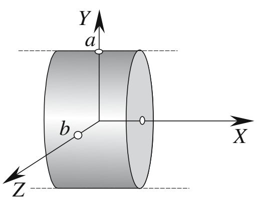
$$x^2 + y^2 + z^2 = r^2$$

Ellipsoid



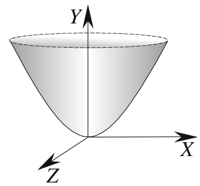
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Elliptic cylinder



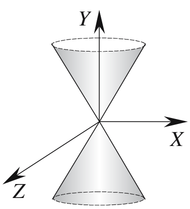
$$\frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$$

Elliptic paraboloid



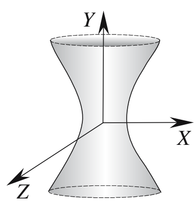
$$\frac{x^2}{a^2} + \frac{z^2}{b^2} = y$$

Elliptic cone



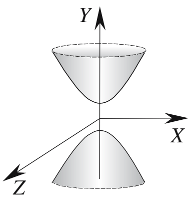
$$\frac{x^2}{a^2} + \frac{z^2}{b^2} - \frac{y^2}{c^2} = 0$$

Elliptic hyperboloid of one sheet

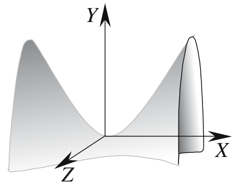


$$\frac{x^2}{a^2} + \frac{z^2}{b^2} - \frac{y^2}{c^2} = 1$$

Elliptic hyperboloid of two sheets



$$\frac{x^2}{a^2} + \frac{z^2}{b^2} - \frac{y^2}{c^2} = -1$$



$$y = \frac{x^2}{b^2} - \frac{z^2}{a^2}$$

2 Examples

Example is the school of mankind, and they will learn at no other.
Edmund Burke (1729–1797)

This section, like the previous section, is organised into 19 groups:

The following examples illustrate how geometric formulas are used in practice. Hopefully, the reader will see the advantages of using unit vectors, and the difference between using parametric equations and the general form of line equations and plane equations. There is no one strategy that overall is superior to another – much will depend upon the context.

Vectors

Vector notation provides a very compact way of expressing the solution to a geometric problem. For example, the formula for calculating the intersection of a line and plane is given by

$$\mathbf{p} = \mathbf{t} + \lambda\mathbf{v}$$

where $$\lambda = \frac{-(\mathbf{n} \bullet \mathbf{t} + d)}{\mathbf{n} \bullet \mathbf{v}}$$

The position vector **p** identifies a point *P* where the line intersects the plane. Therefore, the coordinates of *P* are given by

$$x_p = x_t + \lambda x_v$$

$$y_p = y_t + \lambda y_v$$

$$z_p = z_t + \lambda z_v$$

This sort of 'coordinate unpacking' is used throughout the examples in this section.

2.1 Trigonometry

Examples of cofunction identities

$$\sin\alpha = \cos\left(\frac{\pi}{2} - \alpha\right) = \cos\beta \qquad \sin 30° = \cos 60° = 0.5$$

$$\tan\alpha = \cot\left(\frac{\pi}{2} - \alpha\right) = \cot\beta \qquad \tan 45° = \frac{1}{\tan 45°} = 1$$

$$\csc\alpha = \sec\left(\frac{\pi}{2} - \alpha\right) = \sec\beta \qquad \frac{1}{\sin 30°} = \frac{1}{\cos 60°} = 2$$

Examples of even–odd identities

$$\sin(-\alpha) = -\sin\alpha \qquad \sin(-30°) = -\sin 30° = -0.5$$

$$\cos(-\alpha) = \cos\alpha \qquad \cos(-60°) = \cos 60° = 0.5$$

$$\tan(-\alpha) = -\tan\alpha \qquad \tan(-45°) = -\tan 45° = -1$$

Examples of Pythagorean identities

$$\sin^2\alpha + \cos^2\alpha = 1 \qquad \sin^2 30° + \cos^2 30° = \tfrac{1}{4} + \tfrac{3}{4} = 1$$

$$1 + \tan^2\alpha = \sec^2\alpha \qquad 1 + \tan^2 45° = \frac{1}{\cos^2 45°} = 2$$

$$1 + \cot^2\alpha = \csc^2\alpha \qquad 1 + \cot^2 45° = \frac{1}{\sin^2 45°} = 2$$

Examples of compound angle identities

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \sin(10° + 20°) = \sin 10°\cos 20° + \cos 10°\sin 20° = 0.5$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad \cos(10° + 50°) = \cos 10°\cos 50° - \sin 10°\sin 50° = 0.5$$

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \qquad \tan(20° + 25°) = \frac{\tan 20° + \tan 25°}{1 - \tan 20°\tan 25°} = 1$$

Examples of double-angle identities

$\sin 2\alpha = 2\sin\alpha\cos\alpha$ $\qquad$ $\sin 30^\circ = 2\sin 15^\circ\cos 15^\circ = 0.5$

$\cos 2\alpha = 1 - 2\sin^2\alpha$ $\qquad$ $\cos 60^\circ = 1 - 2\sin^2 30^\circ = 0.5$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$ $\qquad$ $\cos 60^\circ = \cos^2 30^\circ - \sin^2 30^\circ = 0.5$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\beta} \qquad \tan 45^\circ = \frac{2\tan 22.5^\circ}{1-\tan^2 22.5^\circ} = 1$$

Examples of multiple-angle identities

$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$ $\qquad$ $\sin 30^\circ = 3\sin 10^\circ - 4\sin^3 10^\circ = 0.5$

$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$ $\qquad$ $\cos 60^\circ = 4\cos^3 20^\circ - 3\cos 20^\circ = 0.5$

$$\tan 3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1-3\tan^2\alpha} \qquad \tan 45^\circ = \frac{3\tan 15^\circ - \tan^3 15^\circ}{1-3\tan^2 15^\circ} = 1$$

$\sin 4\alpha = 4\sin\alpha\cos\alpha - 8\sin^3\alpha\cos\alpha$ $\qquad$ $\sin 30^\circ = 4\sin 7.5^\circ\cos 7.5^\circ - 8\sin^3 7.5^\circ\cos 7.5^\circ = 0.5$

$\cos 4\alpha = 8\cos^4\alpha - 8\cos^2\alpha + 1$ $\qquad$ $\cos 60^\circ = 8\cos^4 15^\circ - 8\cos^2 15^\circ + 1 = 0.5$

$$\tan 4\alpha = \frac{4\tan\alpha - 4\tan^3\alpha}{1-6\tan^2\alpha + \tan^4\alpha} \qquad \tan 60^\circ = \frac{4\tan 15^\circ - 4\tan^3 15^\circ}{1-6\tan^2 15^\circ + \tan^4 15^\circ} = 1.732051$$

$\sin 5\alpha = 16\sin^5\alpha - 20\sin^3\alpha + 5\sin\alpha$ $\qquad$ $\sin 30^\circ = 16\sin^5 6^\circ - 20\sin^3 6^\circ + 5\sin 6^\circ = 0.5$

$\cos 5\alpha = 16\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha$ $\qquad$ $\cos 60^\circ = 16\cos^5 12^\circ - 20\cos^3 12^\circ + 5\cos 12^\circ = 0.5$

$$\tan 5\alpha = \frac{5\tan\alpha - 10\tan^3\alpha + \tan^5\alpha}{1-10\tan^2\alpha + 5\tan^4\alpha} \qquad \tan 45^\circ = \frac{5\tan 9^\circ - 10\tan^3 9^\circ + \tan^5 9^\circ}{1-10\tan^2 9^\circ + 5\tan^4 9^\circ} = 1$$

Functions of the half-angle

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}} \qquad \sin 30^\circ = \pm\sqrt{\frac{1-\cos 60^\circ}{2}} = \pm 0.5$$

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}} \qquad \cos 60^\circ = \pm\sqrt{\frac{1+\cos 120^\circ}{2}} = \pm 0.5$$

$$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} \qquad \tan 45^\circ = \pm\sqrt{\frac{1-\cos 90^\circ}{1+\cos 90^\circ}} = \pm 1$$

Functions converting to the half-angle tangent form

$$\sin\alpha = \frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} \qquad \sin 30° = \frac{2\tan 15°}{1+\tan^2 15°} = 0.5$$

$$\cos\alpha = \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} \qquad \cos 60° = \frac{1-\tan^2 30°}{1+\tan^2 30°} = 0.5$$

$$\tan\alpha = \frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}} \qquad \tan 45° = \frac{2\tan 22.5°}{1-\tan^2 22.5°} = 1$$

Relationships between sums of functions

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \qquad \sin 30° + \sin 30° = 2\sin 30°\cos 0° = 1$$

$$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \qquad \sin 60° - \sin 30° = 2\cos 45°\sin 15° = 0.366$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \qquad \cos 60° + \cos 60° = 2\cos 60°\cos 0° = 1$$

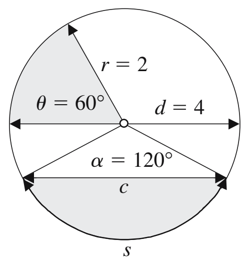
$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \qquad \cos 60° - \cos 30° = -2\sin 45°\sin 15° = -0.366$$

$$\tan\alpha + \tan\beta = \frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta} \qquad \tan 45° + \tan 45° = \frac{\sin 90°}{\cos 45°\cos 45°} = 2$$

$$\tan\alpha - \tan\beta = \frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta} \qquad \tan 60° - \tan 45° = \frac{\sin 15°}{\cos 60°\cos 45°} = 0.732$$

2.2 Circles

Example: Properties of circles



Circle

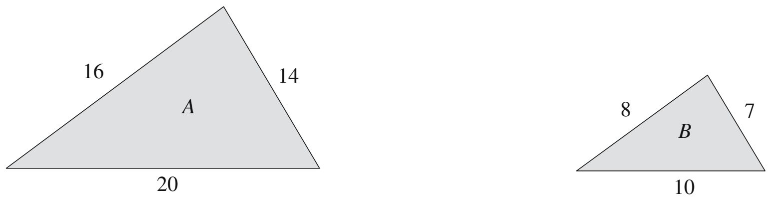
Area of circle	$A = \pi r^2$	$A = \pi 2^2 = 12.57$
Perimeter	$C = \pi d$	$C = \pi 4 = 12.57$
Length of arc	$s = \dfrac{\alpha°}{360°}\pi d$	$s = \dfrac{120°}{360°}\pi 4 = 4.19$
Area of sector	$\dfrac{\theta°}{360°}\pi r^2$	$\dfrac{60°}{360°}\pi 4 = 2.09$
Area of segment	$\dfrac{r^2}{2}\left(\alpha^{[rad]} - \sin\alpha^{[rad]}\right)$	$\dfrac{4}{2}\left(\dfrac{2}{3}\pi - \dfrac{\sqrt{3}}{2}\right) = 2.46$
Length of chord	$c = 2r\sin\dfrac{\alpha}{2}$	$c = 4\sin 60° = 3.46$

2.3 Triangles

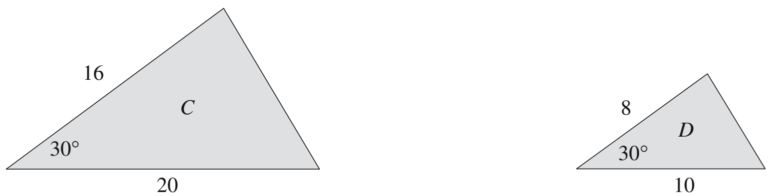
2.3.1 Checking for similar triangles

Triangles A and B are similar because three corresponding sides are in the same ratio:

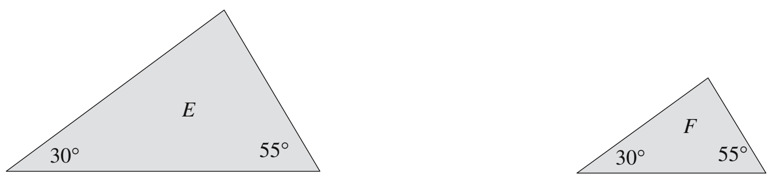
$$\frac{20}{10} = \frac{16}{8} = \frac{14}{7} = 2$$



Triangles C and D are similar because two corresponding sides are in the same ratio, and the included angles are equal: $\frac{20}{10} = \frac{16}{8} = 2$ and the included angles equal 30°.



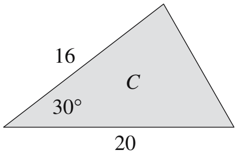
Triangles E and F are similar because two corresponding angles are equal.

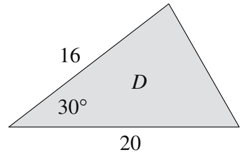


2.3.2 Checking for congruent triangles

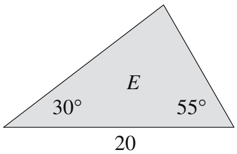
Triangles A and B are congruent because three corresponding sides are equal.

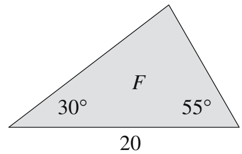
Triangles C and D are congruent because two corresponding sides are equal, and the included angles are equal.





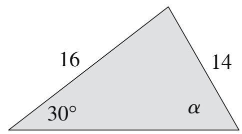
Triangles E and F are congruent because one side and the adjoining angles are equal.





2.3.3 Solving the angles and sides of a triangle

Use the sine rule to find angle α.

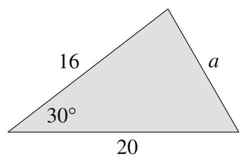


$$\frac{16}{\sin\alpha} = \frac{14}{\sin 30°}$$

$$\sin\alpha = \frac{16}{14}\sin 30°$$

$$\alpha = \sin^{-1}\left(\frac{16}{14}\sin 30°\right) = 34.85°$$

Use the cosine rule to find side a.

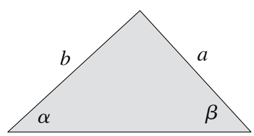


$$a^2 = 20^2 + 16^2 - 2 \times 20 \times 16 \cos 30°$$

$$a^2 = 400 + 256 - 720 \cos 30°$$

$$a = 5.7$$

Use the tangent rule to find side b.



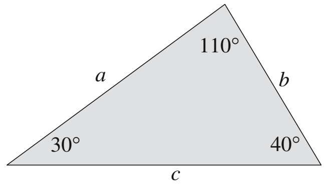
$$\frac{a+b}{a-b} = \frac{\tan\left(\dfrac{\alpha+\beta}{2}\right)}{\tan\left(\dfrac{\alpha-\beta}{2}\right)}$$

$a = 3 \qquad \alpha = 36.87° \qquad \beta = 53.13°$

$$\frac{3+b}{3-b} = \frac{\tan 45°}{\tan(-8.13°)} = \frac{1}{-0.14285} = -7$$

$$3 + b = -7(3 - b) \quad \therefore b = 4$$

Given $a - b$ use Mollweide's rule to find side c.



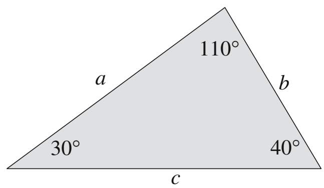
$$\frac{a-b}{c} = \frac{\sin\left(\dfrac{\alpha-\beta}{2}\right)}{\cos\left(\dfrac{\gamma}{2}\right)}$$

$$a - b = 2 \qquad \alpha = 40° \qquad \beta = 30° \qquad \gamma = 110°$$

$$\frac{2}{c} = \frac{\sin 5°}{\cos 55°} = 0.15195$$

$$c = 13.162$$

Given $a + b$ use Newton's rule to find side c.



$$\frac{a+b}{c} = \frac{\cos\left(\dfrac{\alpha-\beta}{2}\right)}{\sin\left(\dfrac{\gamma}{2}\right)}$$

$$a + \mathrm{b} = 16 \qquad \alpha = 40° \qquad \beta = 30° \qquad \gamma = 110°$$

$$\frac{16}{c} = \frac{\cos 5°}{\sin 55°} = 1.21613$$

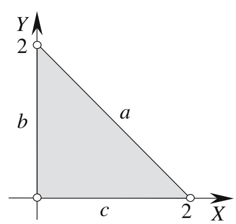
$$c = 13.15648$$

2.3.4 Calculating the area of a triangle

Use Heron's formula to calculate the area of a triangle.

$$a = \sqrt{8} \qquad b = 2 \qquad c = 2$$

Semiperimeter $s = \dfrac{\sqrt{8}+2+2}{2} = 2+\sqrt{2}$

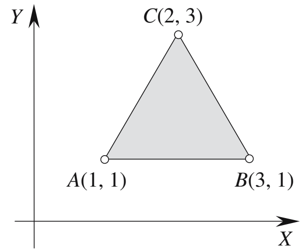


$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{(2+\sqrt{2})(2+\sqrt{2}-\sqrt{8})\sqrt{2}\sqrt{2}}$$

$$\text{Area} = 2$$

Use a determinant to calculate the area of a triangle.



$$\text{Area } \triangle ABC = \tfrac{1}{2}\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}$$

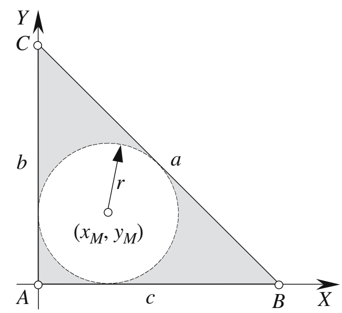
$$\begin{aligned} \text{Area} &= \tfrac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{vmatrix} \\ &= \tfrac{1}{2}(1+2+9-3-3-2) = 2 \end{aligned}$$

Reversing the vertex order:

$$\text{Area} = \tfrac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 3 & 1 & 1 \end{vmatrix} = \tfrac{1}{2}(3+3+2-1-2-9) = -2$$

2.3.5 The center and radius of the inscribed and circumscribed circles for a triangle

Calculate the center of the inscribed circle for triangle ABC.



$$a = \sqrt{8} \qquad b = 2 \qquad c = 2$$

$$A = (0,0) \qquad B = (2,0) \qquad C = (0,2)$$

$$x_M = \frac{ax_A + bx_B + cx_C}{a+b+c}$$

$$y_M = \frac{ay_A + by_B + cy_C}{a+b+c}$$

$$x_M = \frac{\sqrt{8}\times 0 + 2\times 2 + 2\times 0}{\sqrt{8}+2+2} = \frac{4}{4+\sqrt{8}}$$

$$y_M = \frac{\sqrt{8}\times 0 + 2\times 0 + 2\times 2}{\sqrt{8}+2+2} = \frac{4}{4+\sqrt{8}}$$

Position of the center

$$x_M = 0.5858 \qquad y_M = 0.5858$$

Calculate the radius of the inscribed circle for triangle ABC.

$$s = \frac{\sqrt{8}+2+2}{2} = \sqrt{2}+2$$

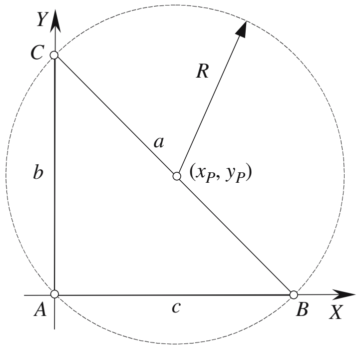
$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

$$r = \sqrt{\frac{(\sqrt{2}+2-\sqrt{8})(\sqrt{2})(\sqrt{2})}{\sqrt{2}+2}}$$

$$r = 2-\sqrt{2} = 0.5858$$

$$r = 0.5858 \qquad x_M = 0.5858 \qquad y_M = 0.5858$$

Calculate the radius of the circumscribed circle for triangle *ABC*.



$$a = \sqrt{8} \qquad b = 2 \qquad c = 2$$

$$A = (0,0) \qquad B = (2,0) \qquad C = (0,2)$$

$$R = \frac{abc}{4\times \text{Area } \triangle ABC} = \frac{\sqrt{8}\times 2\times 2}{4\times 2} = \sqrt{2}$$

Calculate the center of the circumscribed circle for triangle *ABC*.

$$x_P = x_A + \frac{R}{abc}\begin{vmatrix} y_{AC} & b^2 \\ y_{AB} & c^2 \end{vmatrix}$$

$$y_P = y_A + \frac{R}{abc}\begin{vmatrix} b^2 & x_{AC} \\ c^2 & x_{AB} \end{vmatrix}$$

$$x_P = \frac{\sqrt{2}}{\sqrt{8}\times 2\times 2}\begin{vmatrix} 2 & 4 \\ 0 & 4 \end{vmatrix} = 1$$

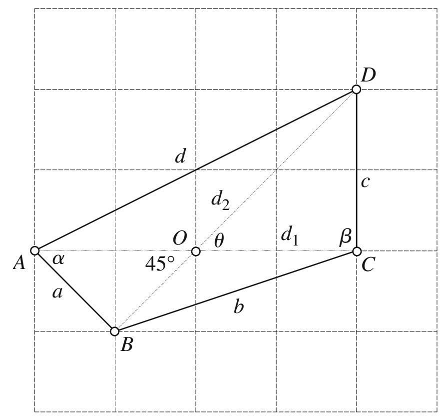
$$y_P = \frac{\sqrt{2}}{\sqrt{8}\times 2\times 2}\begin{vmatrix} 4 & 0 \\ 4 & 2 \end{vmatrix} = 1$$

$$R = \sqrt{2} \qquad x_p = 1 \qquad y_P = 1$$

2.4 Quadrilaterals

Example: Calculate the area of a quadrilateral.

$$a = \sqrt{2}$$
$$b = \sqrt{10}$$
$$c = 2$$
$$d = \sqrt{20}$$
$$AC = d_1 = 4$$
$$BD = d_2 = \sqrt{18}$$
$$s = \frac{a+b+c+d}{2} = 5.5243$$



By inspection

$$\triangle ABO = 1$$
$$\triangle BCO = 1$$
$$\triangle CDO = 2$$
$$\triangle DAO = 2$$

therefore Area $ABCD = 6$.

Here are four ways of computing the area:

$$\text{Area} = \frac{d_1 d_2}{2}\sin\theta = \frac{4\sqrt{18}}{2}\sin 45° = 6\sqrt{2}\,\frac{\sqrt{2}}{2} = 6$$

$$\text{Area} = \tfrac{1}{4}(b^2 + d^2 - a^2 - c^2)\tan\theta = \tfrac{1}{4}(10 + 20 - 2 - 4)\tan 45° = 6$$

$$\text{Area} = \tfrac{1}{4}\sqrt{4d_1^2 d_2^2 - (b^2 + d^2 - a^2 - c^2)^2}$$
$$= \tfrac{1}{4}\sqrt{4 \times 16 \times 18 - (10 + 20 - 2 - 4)^2} = 6$$

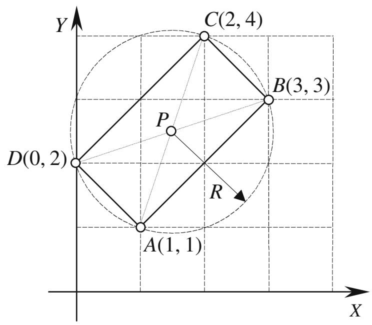
$$\text{Area} = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\varepsilon}$$

$$\varepsilon = \frac{\alpha + \beta}{2} = \frac{71.57° + 108.43°}{2} = 90°$$

$$\text{Area} = \sqrt{4.1101 \times 2.3620 \times 3.5243 \times 1.0522 - 40\cos^2 90°} = 6$$

It just so happens that the quadrilateral is a cyclic quadrilateral.

Example: Calculate the center and radius of the circumscribed circle for a rectangle.



$$P_A = (1,1) \qquad P_B = (3,3) \qquad P_C = (2,4) \qquad P_D = (0,2)$$

The center of the circumscribed circle is

$$x_P = \tfrac{1}{2}(x_A + x_C) \qquad y_P = \tfrac{1}{2}(y_A + y_C)$$

$$x_P = \tfrac{1}{2}(1+2) = 1.5 \qquad y_P = \tfrac{1}{2}(1+4) = 2.5$$

The radius of the circumscribed circle is

$$R = \tfrac{1}{2}\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (x_B - x_C)^2 + (y_B - y_C)^2}$$

$$R = \tfrac{1}{2}\sqrt{(3-1)^2 + (3-1)^2 + (3-2)^2 + (3-4)^2} = \tfrac{1}{2}\sqrt{10}$$

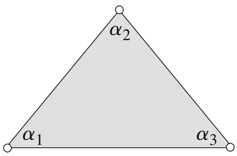
The circle has a radius of $\frac{1}{2}\sqrt{10}$ with a center at $(1.5, 2.5)$.

2.5 Polygons

Example: Determine the internal angles of a polygon

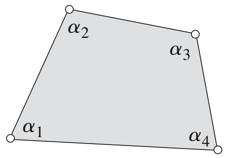
The internal angles of an n-sided polygon sum to $(n - 2) \times 180°$.

Triangle $(n = 3)$



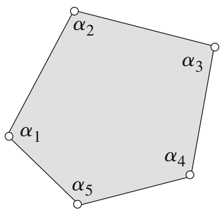
$$\sum_{i=1}^{3} \alpha_i = 180°$$

Quadrilateral $(n = 4)$



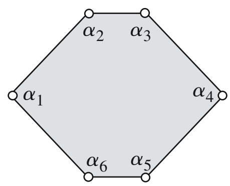
$$\sum_{i=1}^{4} \alpha_i = 360°$$

Pentagon $(n = 5)$



$$\sum_{i=1}^{5} \alpha_i = 540°$$

Hexagon $(n = 6)$

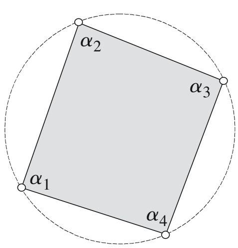


$$\sum_{i=1}^{6} \alpha_i = 720°$$

Example: Determine the alternate internal angles of a cyclic polygon

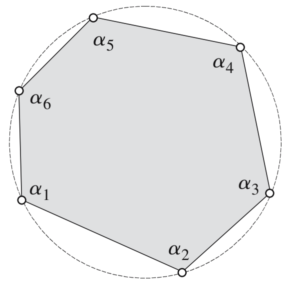
The alternate internal angles of an n-sided cyclic polygon sum to $(n - 2) \times 90°$ [$n \geqslant 4$ and is even].

Cyclic quadrilateral $(n = 4)$



$$\alpha_1 + \alpha_3 = \alpha_2 + \alpha_4 = 180°$$

Cyclic hexagon $(n = 6)$



$$\alpha_1 + \alpha_3 + \alpha_5 = \alpha_2 + \alpha_4 + \alpha_6 = 360°$$

Example: Calculate the area of regular polygon

$$\text{Area} = \tfrac{1}{4} n s^2 \cot \frac{\pi}{n}$$

where n = number of sides
s = length of side

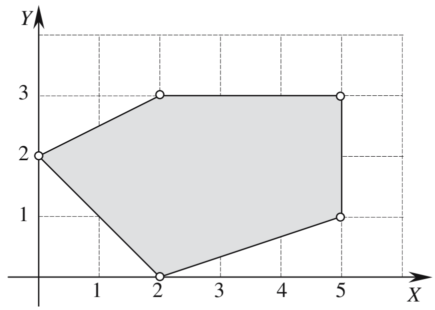
Let $s = 1$

n	Area
3	0.433
4	1
5	1.72
6	2.598
7	3.634
8	4.828

Example: Calculate the area of a polygon

The figure shows a polygon with the following vertices in counter-clockwise sequence

x	0	2	5	5	2
y	2	0	1	3	3



By inspection, the area is 10.5

The area of a polygon is given by

$$\text{Area} = \tfrac{1}{2} \sum_{i=0}^{n-1} (x_i y_{i+1(\text{mod } n)} - y_i x_{i+1(\text{mod } n)})$$

$$\text{Area} = \tfrac{1}{2}(0 \times 0 + 2 \times 1 + 5 \times 3 + 5 \times 3 + 2 \times 2 - 2 \times 2 - 0 \times 5 - 1 \times 5 - 3 \times 2 - 3 \times 0)$$

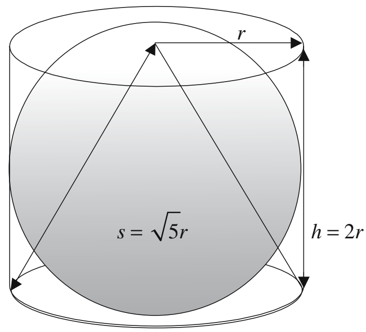
$$\text{Area} = \tfrac{1}{2}(36 - 15) = 10.5$$

2.6 Three-dimensional objects

2.6.1 Cone, cylinder and sphere

Example: Area and volume of a cone, cylinder and sphere

Area	$(h = 2r)\ (s = \sqrt{5}r)$	$(r = 1)$
Cone	$\pi r(r+s) = (1+\sqrt{5})\pi r^2$	$(1+\sqrt{5})\pi$
Sphere	$4\pi r^2$	4π
Cylinder	$2\pi r(r+h) = 6\pi r$	6π
Volume		
Cone	$\frac{1}{3}\pi r^2 h = \frac{2}{3}\pi r^3$	$\frac{2}{3}\pi$
Sphere	$\frac{4}{3}\pi r^3$	$\frac{4}{3}\pi$
Cylinder	$\pi r^2 h = 2\pi r^2$	2π



2.6.2 Conical frustum, spherical segment and torus

Example: Area and volume of a conical frustum, spherical segment and torus

Circular, conical frustum

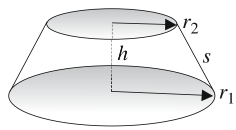
$$S = \pi(r_1^2 + r_2^2 + s(r_1 + r_2))$$

If $r_1 = 2 \quad r_2 = 1 \quad h = 1 \quad s = \sqrt{2}$

$$S = \pi(4 + 1 + \sqrt{2}(2+1)) = 29.03$$

$$V = \tfrac{1}{3}\pi h(r_1^2 + r_2^2 + r_1 r_2)$$

$$V = \tfrac{1}{3}\pi(4 + 1 + 2) = 7.33$$



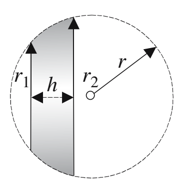
Spherical segment

$$S = 2\pi rh$$

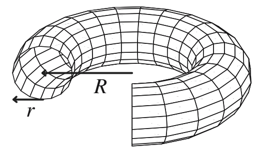
If $r = 1 \quad h = 1 \qquad S = 6.28$

$$V = \tfrac{1}{6}\pi h(3r_1^2 + 3r_2^2 + h^2)$$

If $r_1 = 0 \quad r_2 = 1 \quad h = 1 \qquad V = 2.09$
(half the volume)



Torus



$$S = 4\pi^2 rR$$

If $r = 1 \quad R = 1 \qquad S = 39.48$

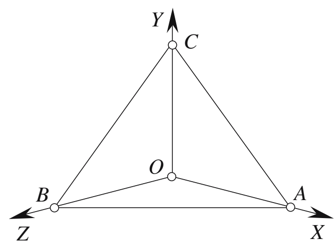
$$V = 2\pi^2 r^2 R$$

If $r = 1 \quad R = 1 \qquad V = 19.74$

2.6.3 Tetrahedron

Example: Volume of a tetrahedron

Tetrahedron



Let $A = (1,0,0) \quad B = (0,0,1) \quad C = (0,1,0)$

$$V = \frac{1}{6}\begin{vmatrix} x_a & y_a & z_a \\ x_b & y_b & z_b \\ x_c & y_c & z_c \end{vmatrix} = \frac{1}{6}\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \frac{1}{6}$$

Note: If the vertices are reversed the volume is negative.

$$V = \frac{1}{6}\begin{vmatrix} x_b & y_b & z_b \\ x_a & y_a & z_a \\ x_c & y_c & z_c \end{vmatrix} = \frac{1}{6}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -\frac{1}{6}$$

2.7 Coordinate systems

2.7.1 Cartesian coordinates in $\mathbb{R}^2$

Example: Distance in $\mathbb{R}^2$

Find the distance between the points (12, 16) and (9, 12).

Given $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

therefore $$d = \sqrt{(12-9)^2 + (16-12)^2} = \sqrt{9+16}$$
$$d = 5$$

2.7.2 Cartesian coordinates in $\mathbb{R}^3$

Example: Distance in $\mathbb{R}^3$

Find the distance between the points (12, 16, 22) and (9, 12, 20).

Given $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

therefore $$d = \sqrt{(12-9)^2 + (16-12)^2 + (22-20)^2}$$
$$= \sqrt{9+16+4} = \sqrt{29}$$
$$d = 5.39$$

2.7.3 Polar coordinates

Example: Conversion between Cartesian and polar coordinates

Find the polar coordinates (r, θ) for the points $(4, 3), (-4, 3), (-4, -3)$ and $(4, -3)$.

Given $$r = \sqrt{x^2 + y^2}$$

and $$\theta = \tan^{-1}\left(\frac{y}{x}\right) \quad \text{(1st and 4th quadrants only)}$$

For (4, 3) $$r = \sqrt{16+9} = 5$$

and $$\theta = \tan^{-1}\left(\frac{3}{4}\right) = 36.87°$$
$$(4, 3) \equiv (5, 36.87°)$$

For $(-4, 3)$ $$r = 5$$
and $$\theta = 180° - 36.87° = 143.13°$$
$$(-4, 3) \equiv (5, 143.13°)$$

For $(-4, -3)$ $\quad r = 5$

and $\quad \theta = 180° + 36.87° = 216.87°$

$(-4, -3) \equiv (5, 216.87°)$

For $(4, -3)$ $\quad r = 5$

and $\quad \theta = -36.87°$ or $323.13°$

$(4, -3) \equiv (5, 323.13°)$

Find the Cartesian coordinates (x, y) for the point $(5, 216.87°)$.

Given $\quad x = r\cos\theta$

and $\quad y = r\sin\theta$

For $(5, 216.87°)$ $\quad x = 5\cos 216.87° = -4$

and $\quad y = 5\sin 216.87° = -3$

$(5, 216.87°) \equiv (-4, -3)$

2.7.4 Cylindrical coordinates

Example: Conversion between Cartesian and cylindrical coordinates

Find the cylindrical coordinates (r, θ, z) for the points $(4, 3, 4)$, $(-4, 3, 4)$, $(-4, -3, 4)$ and $(4, -3, 4)$.

Given $\quad r = \sqrt{x^2 + y^2}$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \quad \text{(1st and 4th quadrants only)}$$

and $\quad z = z$

For $(4, 3, 4)$ $\quad r = \sqrt{16 + 9} = 5$

$$\theta = \tan^{-1}\left(\frac{3}{4}\right) = 36.87°$$

and $\quad z = 4$

$(4, 3, 4) \equiv (5, 36.87°, 4)$

For $(-4, 3, 4)$ $\quad r = 5$

$\theta = 180° - 36.87° = 143.13°$

and $\quad z = 4$

$(-4, 3, 4) \equiv (5, 143.13°, 4)$

For $(-4, -3, 4)$ $\quad r = 5$

$\theta = 180° + 36.87° = 216.87°$

and $\quad z = 4$

$(-4, -3, 4) \equiv (5, 216.87°, 4)$

For $(4, -3, 4)$ $\quad r = 5$

$\theta = -36.87°$ or $323.13°$

and $\quad z = 4$

$(4, -3, 4) \equiv (5, 216.87°, 4)$

Find the Cartesian coordinates (x, y, z) for the point $(5, 216.87°, 4)$.

Given $\quad x = r\cos\theta$

$y = r\sin\theta$

and $\quad z = z$

For $(5, 216.87°, 4)$ $\quad x = 5\cos 216.87° = -4$

$y = 5\sin 216.87° = -3$

$z = 4$

$(5, 216.87°, 4) \equiv (-4, -3, 4)$

2.7.5 Spherical coordinates

Example: Conversion between Cartesian and spherical coordinates

Find the spherical coordinates (ρ, θ, ϕ) for the points $(4, 3, 4), (-4, 3, 4), (-4, -3, 4)$ and $(4, -3, 4)$.

Given $$\rho = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \quad \text{(1st and 4th quadrants only)}$$

and $$\phi = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

For $(4, 3, 4)$ $$\rho = \sqrt{16 + 9 + 16} = \sqrt{41} = 6.403$$

$$\theta = \tan^{-1}\frac{3}{4} = 36.87°$$

and $$\phi = \cos^{-1}\frac{4}{6.403} = 51.34°$$

$(4, 3, 4) \equiv (6.403, 36.87°, 51.34°)$

For $(-4, 3, 4)$ $\quad \rho = 6.403$

$\theta = 180° - 36.87° = 143.13°$

and $\quad \phi = 51.34°$

$(-4, 3, 4) \equiv (6.403, 143.13°, 51.34°)$

For $(-4, -3, 4)$ $\quad \rho = 6.403$

$\theta = 180° + 36.87° = 216.87°$

and $\quad \phi = 51.34°$

$(-4, -3, 4) \equiv (6.403, 216.87°, 51.34°)$

For $(4, -3, 4)$ $\quad \rho = 6.403$

$$\theta = \tan^{-1}\left(\frac{-3}{4}\right) = -36.87° = 323.13°$$

and $\quad \phi = 51.34°$

$(4, -3, 4) \equiv (6.403, 323.13°, 51.34°)$

2.8 Vectors

2.8.1 Vector between two points

Given $P_1(1, 2, 3) \quad \text{and} \quad P_2(4, 6, 8)$

$$\overrightarrow{P_1P_2} = \mathbf{a} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

$$\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$$

2.8.2 Scaling a vector

Given $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$

scale by 3 $3\mathbf{a} = 9\mathbf{i} + 12\mathbf{j} + 15\mathbf{k}$

2.8.3 Reversing a vector

Given $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$

$$-\mathbf{a} = -3\mathbf{i} - 4\mathbf{j} - 5\mathbf{k}$$

2.8.4 Magnitude of a vector

Given $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$

$$\|\mathbf{a}\| = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50} = 7.071$$

2.8.5 Normalizing a vector to a unit length

Given $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$

$$\hat{\mathbf{a}} = \frac{3}{\sqrt{50}}\mathbf{i} + \frac{4}{\sqrt{50}}\mathbf{j} + \frac{5}{\sqrt{50}}\mathbf{k} = 0.424\mathbf{i} + 0.566\mathbf{j} + 0.707\mathbf{k}$$

check $\|\hat{\mathbf{a}}\| = \sqrt{\frac{9}{50} + \frac{16}{50} + \frac{25}{50}} = 1$

2.8.6 Vector addition/subtraction

Given $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k} \quad \text{and} \quad \mathbf{b} = 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$

$$\mathbf{a} + \mathbf{b} = 5\mathbf{i} + 8\mathbf{j} + 11\mathbf{k}$$

2.8.7 Position vector

Given a point $(3, 4, 5)$ its position vector is $3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$.

2.8.8 Scalar (dot) product

Given $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$

$$\mathbf{a} \bullet \mathbf{b} = 3 \times 2 + 4 \times 4 + 5 \times 6 = 52$$

2.8.9 Angle between two vectors

Given $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$

Let α be the angle between $\mathbf{a}$ and $\mathbf{b}$.

$$\|\mathbf{a}\| = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50} \quad \text{and} \quad \|\mathbf{b}\| = \sqrt{2^2 + 4^2 + 6^2} = \sqrt{56}$$

$$\alpha = \cos^{-1}\left(\frac{x_a x_b + y_a y_b + z_a z_b}{\|\mathbf{a}\| \cdot \|\mathbf{b}\|}\right)$$

$$\alpha = \cos^{-1}\left(\frac{3 \times 2 + 4 \times 4 + 5 \times 6}{\sqrt{50}\sqrt{56}}\right) = \cos^{-1}\left(\frac{52}{52.915}\right) = 10.67^\circ$$

2.8.10 Vector (cross) product

Given $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + \mathbf{j} + 8\mathbf{k}$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 5 \\ 1 & 1 & 8 \end{vmatrix} = 11\mathbf{i} - 19\mathbf{j} + \mathbf{k}$$

$11\mathbf{i} - 19\mathbf{j} + \mathbf{k}$ is orthogonal to $\mathbf{a}$ and $\mathbf{b}$.

Remember that $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$

Proof

$$\mathbf{b} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 8 \\ 3 & 2 & 5 \end{vmatrix} = -11\mathbf{i} + 19\mathbf{j} - \mathbf{k}$$

$-11\mathbf{i} + 19\mathbf{j} - \mathbf{k}$ is still orthogonal to $\mathbf{a}$ and $\mathbf{b}$ but is in the opposite direction to $11\mathbf{i} - 19\mathbf{j} + \mathbf{k}$.

2.8.11 Scalar triple product

Given $\mathbf{a} = 2\mathbf{j} + 2\mathbf{k} \qquad \mathbf{b} = 10\mathbf{k} \qquad \mathbf{c} = 5\mathbf{i}$

$$\mathbf{a}\bullet(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} x_a & y_a & z_a \\ x_b & y_b & z_b \\ x_c & y_c & z_c \end{vmatrix}$$

$$\text{Volume} = \mathbf{a}\bullet(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 0 & 2 & 2 \\ 0 & 0 & 10 \\ 5 & 0 & 0 \end{vmatrix} = 100$$

2.8.12 Vector normal to a triangle

Given $P_1(5,0,0) \qquad P_2(0,0,5) \qquad P_3(10,0,5)$

$$\mathbf{a} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} x_3 - x_1 \\ y_3 - y_1 \\ z_3 - z_1 \end{bmatrix}$$

$$\mathbf{a} = -5\mathbf{i} + 5\mathbf{k} \qquad \mathbf{b} = 5\mathbf{i} + 5\mathbf{k}$$

$$\mathbf{n} = \mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 & 0 & 5 \\ 5 & 0 & 5 \end{vmatrix} = 50\mathbf{j}$$

Surface normal $\mathbf{n} = 50\mathbf{j}$

2.8.13 Area of a triangle

Given $P_1(5,0,0) \qquad P_2(0,0,5) \qquad P_3(10,0,5)$

$$\mathbf{a} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} x_3 - x_1 \\ y_3 - y_1 \\ z_3 - z_1 \end{bmatrix}$$

$$\mathbf{a} = -5\mathbf{i} + 5\mathbf{k} \qquad \mathbf{b} = 5\mathbf{i} + 5\mathbf{k}$$

$$\text{Area} = \tfrac{1}{2}\,\|\mathbf{a}\times\mathbf{b}\| = \tfrac{1}{2}\left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 & 0 & 5 \\ 5 & 0 & 5 \end{vmatrix} \right\| = \tfrac{1}{2}\,\|50\mathbf{j}\|$$

Area $= 25$

2.9 Quaternions

2.9.1 Quaternion addition and subtraction

$$\mathbf{q}_1 \pm \mathbf{q}_2 = [(s_1 \pm s_2) + (x_1 \pm x_2)\mathbf{i} + (y_1 \pm y_2)\mathbf{j} + (z_1 \pm z_2)\mathbf{k}]$$

Given $\mathbf{q}_1 = [1 + 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}]$

and $\mathbf{q}_2 = [1 - \mathbf{i} + 2\mathbf{j} + 5\mathbf{k}]$

then $\mathbf{q}_1 + \mathbf{q}_2 = [2 + \mathbf{i} + 5\mathbf{j} + 9\mathbf{k}]$

2.9.2 Quaternion multiplication

$$\mathbf{q}_1\mathbf{q}_2 = [(s_1 s_2 - \mathbf{v}_1 \cdot \mathbf{v}_2), s_1\mathbf{v}_2 + s_2\mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2]$$

Given $\mathbf{q}_1 = [1 + \mathbf{i}]$

and $\mathbf{q}_2 = [1 + \mathbf{j}]$

then $\mathbf{q}_1\mathbf{q}_2 = [1 + \mathbf{i} + \mathbf{j} + \mathbf{k}]$

2.9.3 Magnitude of a quaternion

$$\|\mathbf{q}_1\| = \sqrt{s^2 + x^2 + y^2 + z^2}$$

Given $\mathbf{q}_1 = [1 + 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}]$

then $\|\mathbf{q}_1\| = \sqrt{1^2 + 2^2 + 3^2 + 4^2} = \sqrt{30}$

2.9.4 The inverse quaternion

$$\mathbf{q}_1^{-1} = \frac{[s - x\mathbf{i} - y\mathbf{j} - z\mathbf{k}]}{\|\mathbf{q}_1\|^2}$$

Given $\mathbf{q}_1 = [1 + 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}]$

then $\mathbf{q}_1^{-1} = \frac{1}{30}[1 - 2\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}] = [\frac{1}{30} - \frac{1}{15}\mathbf{i} - \frac{1}{10}\mathbf{j} - \frac{2}{15}\mathbf{k}]$

2.9.5 Rotating a vector

Rotate $\mathbf{p}$ using $\mathbf{p}' = \mathbf{q}\mathbf{p}\mathbf{q}^{-1}$ where $\mathbf{q} = [\cos(\frac{\theta}{2}), \sin(\frac{\theta}{2})\hat{\mathbf{v}}]$

Let $\mathbf{p}$ be the quaternion for $(1, 0, 0)$ i.e. $\mathbf{p} = [0 + \mathbf{i}]$

Let $\mathbf{q}$ be a unit quaternion aligned with the z-axis which rotates $\mathbf{p}$ 180°

i.e. $\quad \mathbf{q} = [\cos 90°, \sin 90°(\mathbf{k})] = [0 + \mathbf{k}]$

then $\quad \mathbf{q}^{-1} = [-\mathbf{k}]$

but $\quad \|\mathbf{q}\| = 1$

therefore $\quad \mathbf{p}' = [0 + \mathbf{k}] \cdot [0 + \mathbf{i}] \cdot [0 - \mathbf{k}] = [0 + \mathbf{j}] \cdot [0 - \mathbf{k}] = [0 - \mathbf{i}]$

$[0 - \mathbf{i}]$ points to the rotated point: $(-1, 0, 0)$, which is correct.

2.9.6 Quaternion as a matrix

$$R(\theta) = \begin{bmatrix} s^2 + x^2 - y^2 - z^2 & 2(xy - sz) & 2(xy + sz) \\ 2(xy + sz) & s^2 + y^2 - x^2 - z^2 & 2(yz - sx) \\ 2(xz - sy) & 2(yz + sx) & s^2 + z^2 - x^2 - y^2 \end{bmatrix}$$

Let's express the previous rotation quaternion as a matrix:

Given $\quad [0 + \mathbf{k}]$ then $s = 0, \quad x = 0, \quad y = 0, \quad z = 1$

therefore $\quad R(\theta) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

then $\quad \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

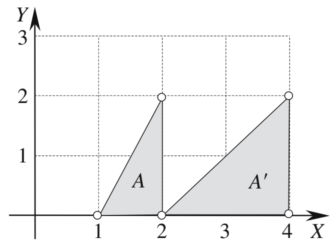
which confirms the previous result.

2.10 Transformations

In the following examples the coordinates of the original shape A are shown on the right-hand side of the transform enclosed in brackets, whilst the coordinates of the transformed shape A' are shown on the left-hand side.

2.10.1 Scaling relative to the origin in $\mathbb{R}^2$

Scale shape A by a factor of 2 in the x-direction and 1 in the y-direction relative to the origin.

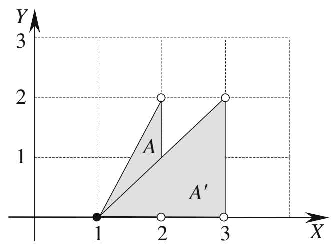


$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\underset{A'}{\begin{bmatrix} 2 & 4 & 4 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{\text{Transform}}{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.2 Scaling relative to a point in $\mathbb{R}^2$

Scale shape A by a factor of 2 in the x-direction and 1 in the y-direction relative to the point $(1, 0)$.

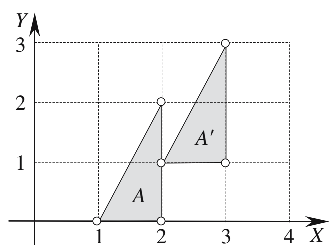


$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & x_P(1-S_x) \\ 0 & S_y & y_P(1-S_y) \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\underset{A'}{\begin{bmatrix} 1 & 3 & 3 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{\text{Transform}}{\begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.3 Translation in $\mathbb{R}^2$

Translate shape A by 1 in the x-direction and 1 in the y-direction.

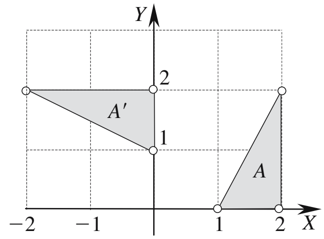


$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & T_x \\ 0 & 1 & T_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\underset{A'}{\begin{bmatrix} 2 & 3 & 3 \\ 1 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{\text{Transform}}{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.4 Rotation about the origin in $\mathbb{R}^2$

Rotate shape A 90° about the origin.

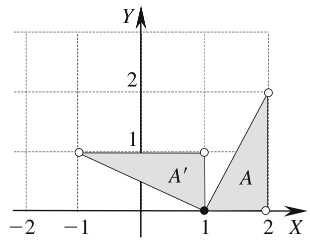


$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\underset{A'}{\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{\text{Transform}}{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.5 Rotation about a point in $\mathbb{R}^2$

Rotate shape A 90° about the point $(1, 0)$.

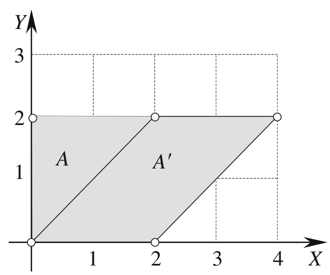


$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha & x_P(1-\cos\alpha) + y_P \sin\alpha \\ \sin\alpha & \cos\alpha & y_P(1-\cos\alpha) - x_P \sin\alpha \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\underset{A'}{\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{\text{Transform}}{\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.6 Shearing along the *x*-axis in $\mathbb{R}^2$

Shear shape A 45° along the x-axis.

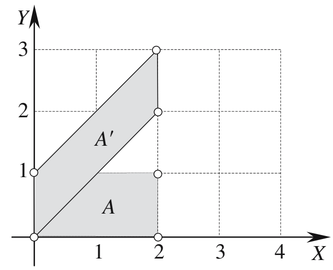


$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \tan\alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\underset{A'}{\begin{bmatrix} 0 & 2 & 4 & 2 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}} = \underset{\text{Transform}}{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}}$$

2.10.7 Shearing along the y-axis in $\mathbb{R}^2$

Shear shape A 45° along the y-axis.

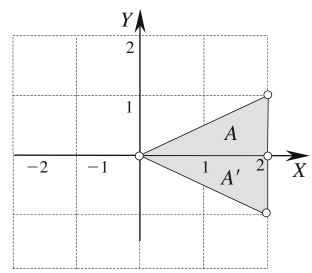


$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \tan\alpha & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\overset{A'}{\begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 2 & 3 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}} = \overset{\text{Transform}}{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \overset{A}{\begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}}$$

2.10.8 Reflection about the x-axis in $\mathbb{R}^2$

Reflect shape A about the x-axis.

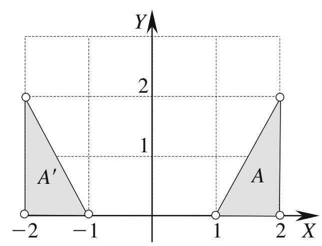


$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\overset{A'}{\begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}} = \overset{\text{Transform}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \overset{A}{\begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.9 Reflection about the y-axis in $\mathbb{R}^2$

Reflect shape A about the y-axis.



$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

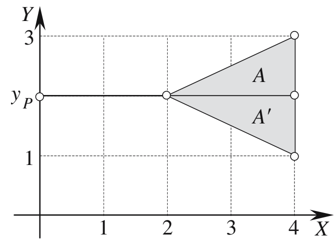
$$\overset{A'}{\begin{bmatrix} -1 & -2 & -2 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}} = \overset{\text{Transform}}{\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \overset{A}{\begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.10 Reflection about a line parallel with the *x*-axis in $\mathbb{R}^2$

Reflect shape A about the line $y_p = 2$.

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 2y_P \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\overset{A'}{\begin{bmatrix} 2 & 4 & 4 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}} = \overset{\text{Transform}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \overset{A}{\begin{bmatrix} 2 & 4 & 4 \\ 2 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}}$$

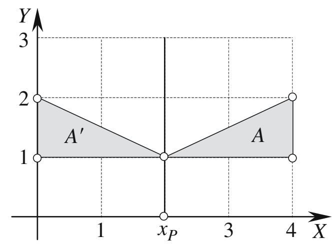


2.10.11 Reflection about a line parallel with the *y*-axis in $\mathbb{R}^2$

Reflect shape A about the line $x_p = 2$.

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2x_P \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\overset{A'}{\begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}} = \overset{\text{Transform}}{\begin{bmatrix} -1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \overset{A}{\begin{bmatrix} 2 & 4 & 4 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}}$$

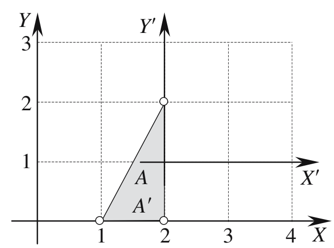


2.10.12 Translated change of axes in $\mathbb{R}^2$

The axes are subjected to a translation of $(2, 1)$.

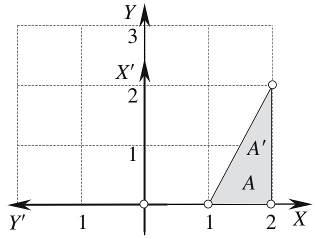
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -x_T \\ 0 & 1 & -y_T \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\overset{A'}{\begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}} = \overset{\text{Transform}}{\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}} \cdot \overset{A}{\begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}}$$



2.10.13 Rotated change of axes in $\mathbb{R}^2$

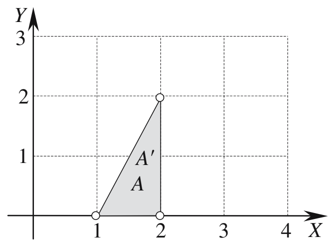
Rotate the axes 90°.



$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\underset{A'}{\begin{bmatrix} 0 & 0 & 2 \\ -1 & -2 & -2 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{\text{Transform}}{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.14 The identity matrix in $\mathbb{R}^2$

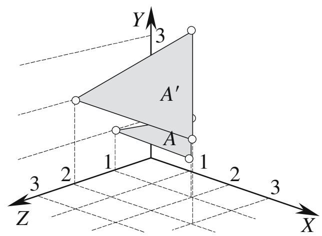


$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\underset{A'}{\begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{\text{Transform}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.15 Scaling relative to the origin in $\mathbb{R}^3$

Scale shape A 1.5 in the x-direction, 2 in the y-direction and 2 in the z-direction.

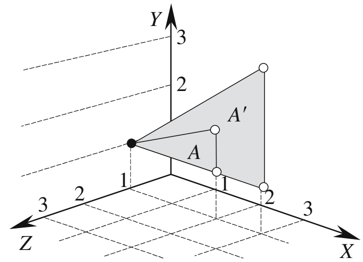


$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\underset{A'}{\begin{bmatrix} 0 & 3 & 3 \\ 2 & 2 & 4 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{\text{Transform}}{\begin{bmatrix} 1.5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 0 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.16 Scaling relative to a point in $\mathbb{R}^3$

Scale shape A 1.5 in the x-direction, 2 in the y-direction and 2 in the z-direction relative to the point $(0, 1, 1)$.

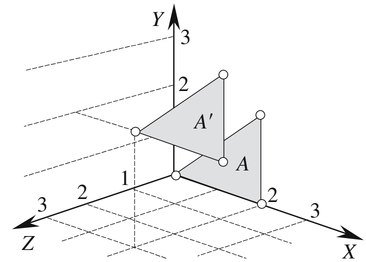


$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 & x_P(1-S_x) \\ 0 & S_y & 0 & y_P(1-S_y) \\ 0 & 0 & S_z & z_P(1-S_z) \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\underset{A'}{\begin{bmatrix} 0 & 3 & 3 \\ 1 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{\text{Transform}}{\begin{bmatrix} 1.5 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 0 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.17 Translation in $\mathbb{R}^3$

Translate shape A by $(2, 2, 3)$.

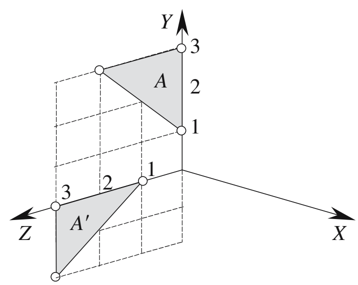


$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\underset{A'}{\begin{bmatrix} 2 & 4 & 4 \\ 2 & 2 & 4 \\ 3 & 3 & 3 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{\text{Transform}}{\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.18 Rotation about the x-axis in $\mathbb{R}^3$

Rotate shape A about the x-axis 90°.

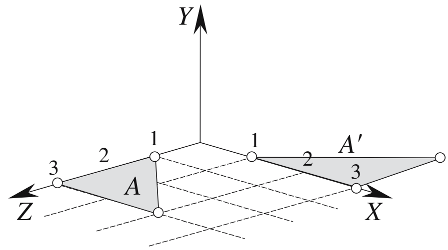


$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\underset{A'}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 1 & 3 & 3 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{\text{Transform}}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 3 & 3 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.19 Rotation about the y-axis in $\mathbb{R}^3$

Rotate shape A about the y-axis 90°.

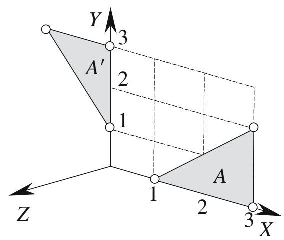


$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\overset{A'}{\begin{bmatrix} 1 & 3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix}} = \overset{\text{Transform}}{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \cdot \overset{A}{\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 1 & 3 & 3 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.20 Rotation about the z-axis in $\mathbb{R}^3$

Rotate shape A about the z-axis 90°.



$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\overset{A'}{\begin{bmatrix} 0 & 0 & -2 \\ 1 & 3 & 3 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}} = \overset{\text{Transform}}{\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \cdot \overset{A}{\begin{bmatrix} 1 & 3 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.21 Rotation about an arbitrary axis in $\mathbb{R}^3$

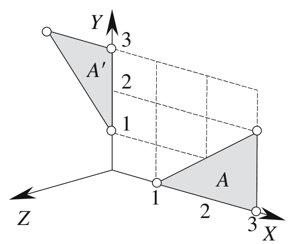
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} a^2K + \cos\alpha & abK - c\sin\alpha & acK + b\sin\alpha & 0 \\ abK + c\sin\alpha & b^2K + \cos\alpha & bcK - a\sin\alpha & 0 \\ acK - b\sin\alpha & bcK + a\sin\alpha & c^2K + \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$K = 1 - \cos\alpha$

Axis $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and $\|\mathbf{v}\| = 1$

Given $\mathbf{v} = \mathbf{k}$ and $\alpha = 90°$

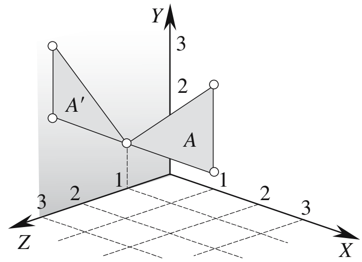
then $K = 1$



$$\overset{A'}{\begin{bmatrix} 0 & 0 & -2 \\ 1 & 3 & 3 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}} = \overset{\text{Transform}}{\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \cdot \overset{A}{\begin{bmatrix} 1 & 3 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.22 Reflection about the *yz*-plane in $\mathbb{R}^3$

Reflect shape A in the yz-plane.

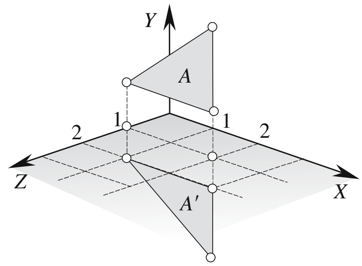


$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\underset{A'}{\begin{bmatrix} 0 & -2 & -2 \\ 1 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{\text{Transform}}{\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 0 & 2 & 2 \\ 1 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.23 Reflection about the *zx*-plane in $\mathbb{R}^3$

Reflect shape A in the zx-plane.

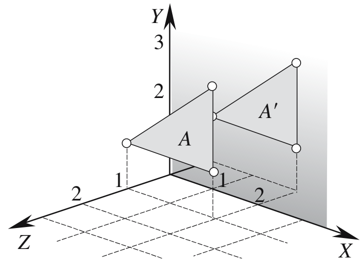


$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\underset{A'}{\begin{bmatrix} 0 & 2 & 2 \\ -1 & -1 & -3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{\text{Transform}}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 0 & 2 & 2 \\ 1 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.24 Reflection about the *xy*-plane in $\mathbb{R}^3$

Reflect shape A in the xy-plane.

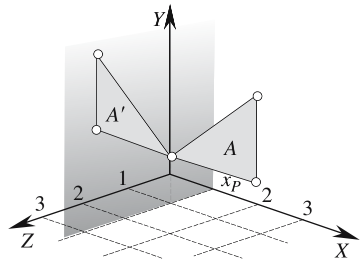


$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\underset{A'}{\begin{bmatrix} 0 & 2 & 2 \\ 1 & 1 & 3 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{\text{Transform}}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 0 & 2 & 2 \\ 1 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.25 Reflection about a plane parallel with the *yz*-plane in $\mathbb{R}^3$

Reflect shape A in the yz-plane $x_p = 1$.

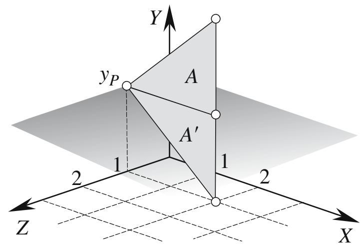


$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 2x_P \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\underset{A'}{\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{\text{Transform}}{\begin{bmatrix} -1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 1 & 3 & 3 \\ 1 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.26 Reflection about a plane parallel with the *zx*-plane in $\mathbb{R}^3$

Reflect shape A in the zx-plane $y_p = 2$.

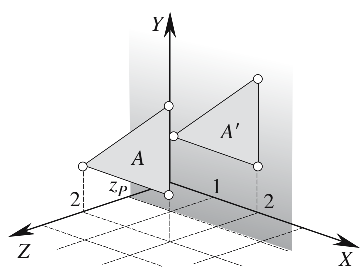


$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2y_P \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\underset{A'}{\begin{bmatrix} 0 & 2 & 2 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{\text{Transform}}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 0 & 2 & 2 \\ 2 & 2 & 4 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.27 Reflection about a plane parallel with the *xy*-plane in $\mathbb{R}^3$

Reflect shape A in the xy-plane $z_p = 1$.

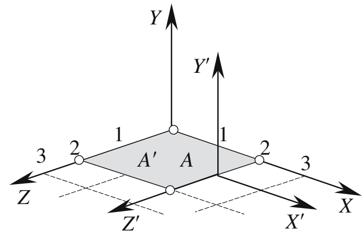


$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2z_P \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\underset{A'}{\begin{bmatrix} 0 & 2 & 2 \\ 1 & 1 & 3 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{\text{Transform}}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 0 & 2 & 2 \\ 1 & 1 & 3 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.28 Translated axes in $\mathbb{R}^3$

The axes are subjected to a translation of $(2, 0, 1)$.

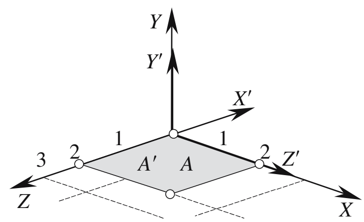


$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -x_T \\ 0 & 1 & 0 & -y_T \\ 0 & 0 & 1 & -z_T \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\underset{A'}{\begin{bmatrix} -2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}} = \underset{\text{Transform}}{\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}}$$

2.10.29 Rotated axes in $\mathbb{R}^3$

The axes are subjected to a rotation as illustrated.

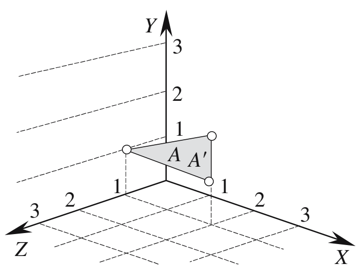


$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\underset{A'}{\begin{bmatrix} 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}} = \underset{\text{Transform}}{\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}}$$

2.10.30 The identity matrix in $\mathbb{R}^3$

$$\underset{A'}{\begin{bmatrix} 0 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{\text{Transform}}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 0 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}$$



2.11 Two-dimensional straight lines

2.11.1 Convert the normal form of the line equation to its general form and the Hessian normal form

Given the normal form of the line equation

$$y = -\frac{3}{4}x + \frac{5}{4}$$

The general form of the line equation is obtained by rearranging the equation to

$$3x + 4y - 5 = 0$$

The Hessian normal form is obtained be dividing throughout by the magnitude of the line's normal vector:

$$\frac{3x + 4y - 5}{\sqrt{3^2 + 4^2}} = 0$$

$$\frac{3}{5}x + \frac{4}{5}y - 1 = 0$$

The line intersects the x-axis at $x = 1\frac{2}{3}$ and the y-axis at $y = 1\frac{1}{4}$. The unit normal vector to the line $\hat{\mathbf{n}} = 0.6\mathbf{i} + 0.8\mathbf{j}$ and the perpendicular from the origin to the line is 1.

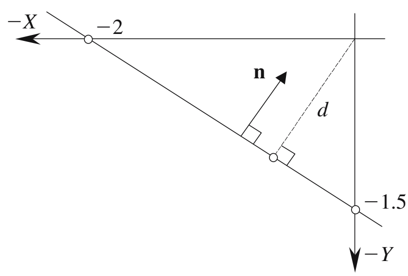
2.11.2 Derive the unit normal vector and perpendicular from the origin to the line for the line equation $3x + 4y + 6 = 0$

The normal vector is $\mathbf{n} = 3\mathbf{i} + 4\mathbf{j}$

The unit normal vector is $\hat{\mathbf{n}} = \frac{1}{\sqrt{3^2 + 4^2}}(3\mathbf{i} + 4\mathbf{j})$

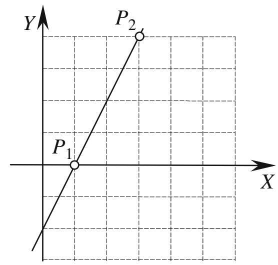
$$= 0.6\mathbf{i} + 0.8\mathbf{j}$$

The distance is $d = \frac{|c|}{\sqrt{3^2 + 4^2}} = \frac{6}{5} = 1.2$



2.11.3 Derive the straight-line equation from two points

Normal form of the line equation



Given $\quad P_1(x_1, y_1) \quad \text{and} \quad P_2(x_2, y_2)$

and $\quad y = mx + c$

then $$m = \frac{y_2 - y_1}{x_2 - x_1}$$

and $$c = y_1 - x_1\left(\frac{y_2 - y_1}{x_2 - x_1}\right)$$

If the two points are $P_1(1, 0)$ and $P_2(3, 4)$

then $$y = \left(\frac{4-0}{3-1}\right)x + 0 - 1\left(\frac{4-0}{3-1}\right)$$

and $\quad y = 2x - 2$

General form of the line equation

Given $\quad P_1(x_1, y_1) \quad \text{and} \quad P_2(x_2, y_2)$

and $\quad Ax + By + C = 0$

then $\quad A = y_2 - y_1 \qquad B = x_1 - x_2 \qquad C = -(x_1 y_2 - x_2 y_1)$

If the two points are $P_1(1, 0)$ and $P_2(3, 4)$

then $\quad (4 - 0)x + (1 - 3)y - (1 \times 4 - 3 \times 0) = 0$

and $\quad 4x - 2y - 4 = 0$

or $\quad 2x - y - 2 = 0$

Determinant form of the line equation

Given $$\begin{vmatrix} 1 & y_1 \\ 1 & y_2 \end{vmatrix} x + \begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix} y = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$

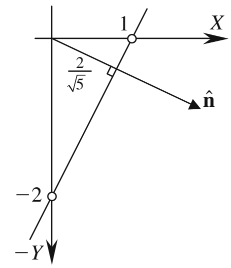
If the two points are $P_1(1, 0)$ and $P_2(3, 4)$

then $$\begin{vmatrix} 1 & 0 \\ 1 & 4 \end{vmatrix} x + \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} y = \begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix}$$

and $\quad 4x - 2y - 4 = 0$

or $\quad 2x - y - 2 = 0$

Hessian normal form of the line equation



Given $4x - 2y - 4 = 0$

The normalizing factor is $\dfrac{1}{\sqrt{a^2+b^2}} = \dfrac{1}{\sqrt{16+4}} = \dfrac{1}{\sqrt{20}}$

then $\dfrac{4}{\sqrt{20}}x - \dfrac{2}{\sqrt{20}}y - \dfrac{4}{\sqrt{20}} = 0$

and $\dfrac{2}{\sqrt{5}}x - \dfrac{1}{\sqrt{5}}y - \dfrac{2}{\sqrt{5}} = 0$

The normal unit vector to the line is $\hat{\mathbf{n}} = \dfrac{1}{\sqrt{5}}(2\mathbf{i} - \mathbf{j})$

The perpendicular from the origin to the line $= \dfrac{2}{\sqrt{5}}$

Parametric form of the line equation

Given $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$

and $\mathbf{p} = \mathbf{p}_1 + \lambda\mathbf{v}$

and $\mathbf{v} = \mathbf{p}_2 - \mathbf{p}_1$

If the two points are $P_1(1, 0)$ and $P_2(3, 4)$

$\mathbf{v} = 2\mathbf{i} + 4\mathbf{j}$

Therefore $x = 1 + 2\lambda$

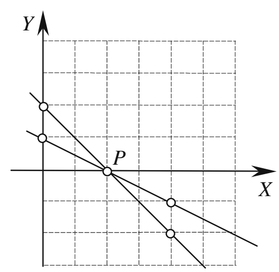
and $y = 4\lambda$

For example, when $\lambda = 0$ $x = 1$ $y = 0$

and when $\lambda = -0.5$ $x = 0$ $y = -2$

2.11.4 Point of intersection of two straight lines

General form of the line equation



Given $a_1x + b_1y + c_1 = 0$

and $a_2x + b_2y + c_2 = 0$

They intersect at $x_P = \dfrac{c_2b_1 - c_1b_2}{a_1b_2 - a_2b_1}$ $y_P = \dfrac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}$

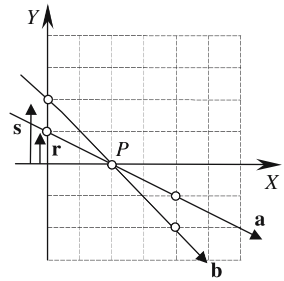
Let the straight lines be $2x + 2y - 4 = 0$ and $2x + 4y - 4 = 0$

Therefore $x_P = \dfrac{-4 \times 2 + 4 \times 4}{2 \times 4 - 2 \times 2} = \dfrac{8}{4} = 2$

and $y_P = \dfrac{2 \times -4 - 2 \times -4}{2 \times 4 - 2 \times 2} = \dfrac{0}{4} = 0$

The point of intersection is (2, 0) as confirmed by the diagram.

Parametric form of the line equation



Given $\mathbf{p} = \mathbf{r} + \lambda\mathbf{a}$ $\quad \mathbf{q} = \mathbf{s} + \varepsilon\mathbf{b}$

where $\mathbf{r} = x_R\mathbf{i} + y_R\mathbf{j}$ $\quad \mathbf{s} = x_S\mathbf{i} + y_S\mathbf{j}$

and $\mathbf{a} = x_a\mathbf{i} + y_a\mathbf{j}$ $\quad \mathbf{b} = x_b\mathbf{i} + y_b\mathbf{j}$

then $\lambda = \dfrac{x_b(y_S - y_R) - y_b(x_S - x_R)}{x_b y_a - x_a y_b}$

Point of intersection $x_P = x_R + \lambda x_a$ $\quad y_P = y_R + \lambda y_a$

Given $\mathbf{r} = \mathbf{j}$ $\quad \mathbf{a} = 2\mathbf{i} - \mathbf{j}$ $\quad \mathbf{s} = 2\mathbf{j}$ $\quad \mathbf{b} = 2\mathbf{i} - 2\mathbf{j}$

$$\lambda = \frac{2(2-1) + 2(0-0)}{2 \times (-1) - 2 \times (-2)} = \frac{2}{2} = 1$$

$$x_P = 0 + 2 = 2 \qquad y_P = 1 + 1 \times (-1) = 0$$

The point of intersection is (2, 0) as confirmed by the diagram.

2.11.5 Calculate the angle between two straight lines

General form of the line equation

Given $a_1x + b_1y + c_1 = 0$ $\quad a_2x + b_2y + c_2 = 0$

where $\mathbf{n} = a_1\mathbf{i} + b_1\mathbf{j}$ $\quad \mathbf{m} = a_2\mathbf{i} + b_2\mathbf{j}$

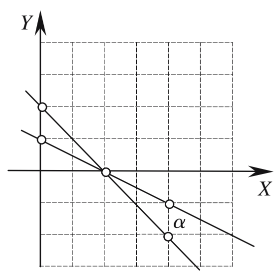
Angle $\alpha = \cos^{-1}\left(\dfrac{\mathbf{n} \bullet \mathbf{m}}{\|\mathbf{n}\| \cdot \|\mathbf{m}\|}\right)$

Let the line equations be $2x + 2y - 4 = 0$

and $2x + 4y - 4 = 0$

Therefore $\alpha = \cos^{-1}\left(\dfrac{2 \times 2 + 2 \times 4}{\sqrt{2^2 + 2^2}\sqrt{2^2 + 4^2}}\right)$

$= 18.435°$



Normal form of the line equation

Given $y = m_1 x + c_1 \qquad y = m_2 x + c_2$

Angle $\alpha = \cos^{-1}\left(\dfrac{1 + m_1 m_2}{\sqrt{1+m_1^2}\sqrt{1+m_2^2}}\right)$

Let the line equations be $y = -x + 2$ and $y = -\dfrac{x}{2} + 1$

where $m_1 = -1 \qquad m_2 = -\frac{1}{2}$

Therefore $\alpha = \cos^{-1}\left(\dfrac{1 + (-1)(-\frac{1}{2})}{\sqrt{1+(-1)^2}\sqrt{1+(-\frac{1}{2})^2}}\right) = 18.435°$

Parametric form of the line equation

Given $\mathbf{p} = \mathbf{r} + \lambda\mathbf{a} \qquad \mathbf{q} = \mathbf{s} + \varepsilon\mathbf{b}$

Angle $\alpha = \cos^{-1}\left(\dfrac{\mathbf{a} \bullet \mathbf{b}}{\|\mathbf{a}\| \cdot \|\mathbf{b}\|}\right)$

Let the line equations be $\mathbf{p} = \mathbf{r} + \lambda\mathbf{a}$ and $\mathbf{q} = \mathbf{s} + \varepsilon\mathbf{b}$

where $\mathbf{r} = \mathbf{j} \qquad \mathbf{a} = 2\mathbf{i} - \mathbf{j} \qquad \mathbf{s} = -2\mathbf{j} \qquad \mathbf{b} = 2\mathbf{i} - 2\mathbf{j}$

Therefore $\alpha = \cos^{-1}\left(\dfrac{2 \times 2 + (-1)(-2)}{\sqrt{5}\sqrt{8}}\right) = 18.435°$

2.11.6 Test if three points lie on a straight line

Given $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$

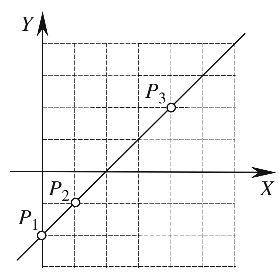
and $\mathbf{r} = \overrightarrow{P_1P_2}$ and $\mathbf{s} = \overrightarrow{P_1P_3}$

The three points lie on a straight line when $\mathbf{s} = \lambda\mathbf{r}$.

Let the points be $P_1(0, -2) \qquad P_2(1, -1) \qquad P_3(4, 2)$

Therefore $\mathbf{r} = \mathbf{i} + \mathbf{j}$ and $\mathbf{s} = 4\mathbf{i} + 4\mathbf{j}$

and $\mathbf{s} = 4\mathbf{r}$



Therefore the points lie on a straight line as confirmed by the diagram.

2.11.7 Test for parallel and perpendicular lines

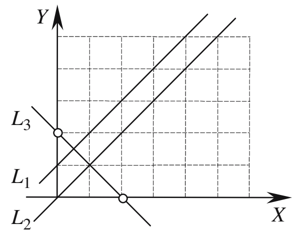
General form of the line equation

Given $a_1x + b_1y + c_1 = 0$ $a_2x + b_2y + c_2 = 0$

where $\mathbf{n} = a_1\mathbf{i} + b_1\mathbf{j}$ $\mathbf{m} = a_2\mathbf{i} + b_2\mathbf{j}$

The lines are parallel if $\mathbf{n} = \lambda\mathbf{m}$.

The lines are mutually perpendicular if $\mathbf{n} \cdot \mathbf{m} = 0$.



Given three lines $L_1: x - y + 1 = 0$

$L_2: x - y = 0$

$L_3: x + y - 2 = 0$

L_1 and L_2 are parallel because the normal vectors to the lines are

$$\mathbf{n}_1 = \mathbf{i} - \mathbf{j} \quad \text{and} \quad \mathbf{n}_2 = \mathbf{i} - \mathbf{j}$$

and $\mathbf{n}_1 = \lambda\mathbf{n}_2 \quad (\lambda = 1)$

L_1 and L_2 are perpendicular because

$$\mathbf{n} \cdot \mathbf{m} = 0 \qquad 1 \times 1 + (-1) \times 1 = 0$$

Normal form of the line equation

Given $y = m_1x + c_1$ $y = m_2x + c_2$

The lines are parallel if $m_1 = m_2$.

The lines are mutually perpendicular if $m_1m_2 = -1$

Given three lines $L_1 : y = x + 1$

$L_2 : y = x$

$L_3 : y = -x + 2$

L_1 and L_2 are parallel because

$$m_1 = m_2 = 1$$

L_1 and L_3 are perpendicular because

$$m_1m_3 = -1 \qquad 1 \times (-1) = -1$$

Parametric form of the line equation

Given $\mathbf{p} = \mathbf{r} + \lambda\mathbf{a}$ $\mathbf{q} = \mathbf{s} + \varepsilon\mathbf{b}$

where $\mathbf{a} = x_a\mathbf{i} + y_a\mathbf{j}$ $\mathbf{b} = x_b\mathbf{i} + y_b\mathbf{j}$

The lines are parallel if $\mathbf{a} = k\mathbf{b}$.

The lines are mutually perpendicular if $\mathbf{a} \cdot \mathbf{b} = 0$.

Given three lines $\mathbf{p} = \mathbf{r} + \lambda\mathbf{a}$ $\mathbf{q} = \mathbf{s} + \varepsilon\mathbf{b}$ $\mathbf{u} = \mathbf{t} + \beta\mathbf{c}$

where $L_1 : \mathbf{a} = \mathbf{i} + \mathbf{j}$
and $L_2 : \mathbf{b} = \mathbf{i} + \mathbf{j}$
and $L_3 : \mathbf{c} = \mathbf{i} - \mathbf{j}$

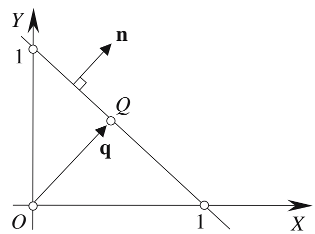
L_1 and L_2 are parallel because $\mathbf{a} = \mathbf{b}$

L_1 and L_3 are perpendicular because

$$x_a x_c + y_a y_c = 0 \qquad 1 \times 1 + 1 \times (-1) = 0$$

2.11.8 Find the position and distance of the nearest point on a line to the origin

General form of the line equation



Given $ax + by + c = 0$

where $\mathbf{n} = a\mathbf{i} + b\mathbf{j}$

$\mathbf{q} = \lambda \mathbf{n}$

where $\lambda = \dfrac{-c}{\mathbf{n} \bullet \mathbf{n}}$

Distance $OQ = \|\mathbf{q}\| = \|\lambda \mathbf{n}\|$

Given the line equation $x + y - 1 = 0$

where $a = 1 \qquad b = 1 \qquad c = -1$

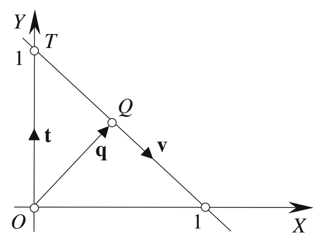
Therefore $\lambda = \frac{1}{2}$

and $x_Q = \lambda x_n = \frac{1}{2} \qquad y_Q = \lambda y_n = \frac{1}{2}$

The nearest point is $Q\left(\frac{1}{2}, \frac{1}{2}\right)$

Distance $OQ = \frac{1}{2}\|\mathbf{n}\| = \frac{1}{2}\sqrt{2} = 0.7071$

Parametric form of the line equation



Given $\mathbf{q} = \mathbf{t} + \lambda \mathbf{v}$

where $\lambda = \dfrac{-\mathbf{v} \bullet \mathbf{t}}{\mathbf{v} \bullet \mathbf{v}}$

Distance $OQ = \|\mathbf{q}\|$

Given the direction vectors $\mathbf{t} = \mathbf{j} \qquad \mathbf{v} = \mathbf{i} - \mathbf{j}$

$$\lambda = \tfrac{1}{2}$$

$$x_Q = x_T + \lambda x_v = 0 + \tfrac{1}{2} \times 1 = \tfrac{1}{2}$$

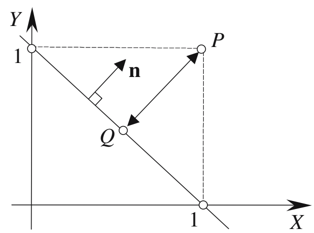
$$y_Q = y_T + \lambda y_v = 1 + \tfrac{1}{2} \times (-1) = \tfrac{1}{2}$$

The nearest point is $Q\left(\tfrac{1}{2}, \tfrac{1}{2}\right)$

Distance $OQ = \|\mathbf{t} + \lambda\mathbf{v}\| = \|\tfrac{1}{2}\mathbf{i} + \tfrac{1}{2}\mathbf{j}\| = 0.7071$

2.11.9 Find the position and distance of the nearest point on a line to a point

General form of the line equation



Given $ax + by + c = 0$

where $\mathbf{n} = a\mathbf{i} + b\mathbf{j}$

$$\mathbf{q} = \mathbf{p} + \lambda\mathbf{n}$$

where $\lambda = -\dfrac{\mathbf{n} \bullet \mathbf{p} + c}{\mathbf{n} \bullet \mathbf{n}}$

Distance $PQ = \|\lambda\mathbf{n}\|$

Given $P(1, 1)$ and $x + y - 1 = 0$

then $a = 1 \qquad b = 1 \qquad c = -1$

$$\lambda = -\frac{2-1}{2} = -\tfrac{1}{2}$$

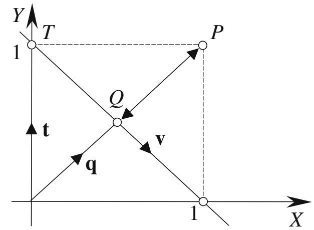
Therefore $x_Q = x_P + \lambda x_n = 1 - \tfrac{1}{2} \times 1 = \tfrac{1}{2}$

$$y_Q = y_P + \lambda y_n = 1 - \tfrac{1}{2} \times 1 = \tfrac{1}{2}$$

The nearest point is $Q\left(\tfrac{1}{2}, \tfrac{1}{2}\right)$

Distance $PQ = \|\lambda\mathbf{n}\| = \tfrac{1}{2}\|\mathbf{i} + \mathbf{j}\| = 0.7071$

Parametric form of the line equation



Given $\mathbf{q} = \mathbf{t} + \lambda\mathbf{v}$

where $\lambda = \dfrac{\mathbf{v} \bullet (\mathbf{p} - \mathbf{t})}{\mathbf{v} \bullet \mathbf{v}}$

Distance $PQ = \|\mathbf{p} - \mathbf{t} - \lambda\mathbf{v}\|$

Given the direction vectors $\mathbf{t} = \mathbf{j}$ and $\mathbf{v} = \mathbf{i} - \mathbf{j}$

and $\mathbf{p} = \mathbf{i} + \mathbf{j}$

$$\lambda = \tfrac{1}{2}$$

$$x_Q = x_T + \lambda x_v = 0 + \tfrac{1}{2} = \tfrac{1}{2}$$

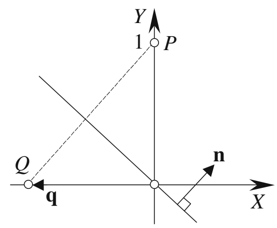
$$y_Q = y_T + \lambda y_v = 1 - \tfrac{1}{2} = \tfrac{1}{2}$$

The nearest point is $Q\left(\tfrac{1}{2}, \tfrac{1}{2}\right)$

Distance $PQ = \|\mathbf{p} - \mathbf{t} - \lambda\mathbf{v}\| = \|\tfrac{1}{2}\mathbf{i} + \tfrac{1}{2}\mathbf{j}\| = 0.7071$

2.11.10 Find the reflection of a point in a line passing through the origin

General form of the line equation



Given $ax + by + c = 0$

where $\mathbf{n} = a\mathbf{i} + b\mathbf{j}$

$$\mathbf{q} = \mathbf{p} - \lambda\mathbf{n}$$

$$\lambda = \frac{2(\mathbf{n} \bullet \mathbf{p} + c)}{\mathbf{n} \bullet \mathbf{n}}$$

Given the line equation $x + y = 0$

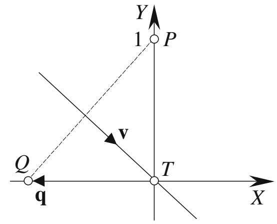
where $a = 1 \quad b = 1 \quad c = 0 \quad P(1, 1)$

$$\lambda = \frac{2 \times 1}{2} = 1$$

Therefore
$$x_Q = x_P - \lambda x_n = 0 - 1 \times 1 = -1$$
$$y_Q = y_P - \lambda y_n = 1 - 1 \times 1 = 0$$

The reflection point is $Q(-1, 0)$

Parametric form of the line equation



Given $\mathbf{s} = \mathbf{t} + \lambda\mathbf{v}$

$\mathbf{q} = 2\mathbf{t} + \varepsilon\mathbf{v} - \mathbf{p}$

where $$\varepsilon = \frac{2\mathbf{v} \bullet (\mathbf{p} - \mathbf{t})}{\mathbf{v} \bullet \mathbf{v}}$$

Given $x_P = 0 \quad y_P = 1 \quad \mathbf{t} = 0 \quad \mathbf{v} = \mathbf{i} - \mathbf{j}$

$$\varepsilon = \frac{2 \times (-1)}{2} = -1$$

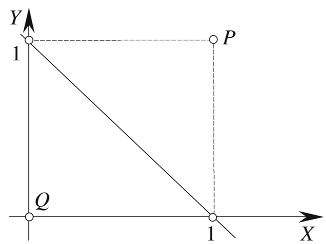
Therefore $x_Q = 2x_T + \varepsilon x_v - x_P = 2 \times 0 - 1 \times 1 - 0 = -1$

$y_Q = 2y_T + \varepsilon y_v - y_P = 2 \times 0 - 1 \times (-1) - 1 = 0$

The reflection point is $Q(-1, 0)$

2.11.11 Find the reflection of a point in a line

General form of the line equation



Given $ax + by + c = 0$

where $\mathbf{n} = a\mathbf{i} + b\mathbf{j}$

$\mathbf{q} = \mathbf{p} - \lambda\mathbf{n}$

$$\lambda = \frac{2(\mathbf{n} \bullet \mathbf{p} + c)}{\mathbf{n} \bullet \mathbf{n}}$$

Given the line equation $x + y - 1 = 0$

where $a = 1 \quad b = 1 \quad c = -1 \quad x_P = 1 \quad y_P = 1$

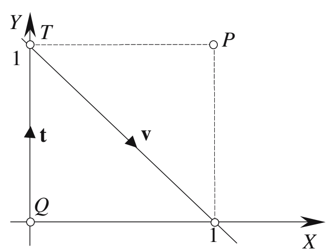
$$\lambda = \frac{2 \times (2 - 1)}{2} = 1$$

Therefore $x_Q = x_P - \lambda x_n = 1 - 1 \times 1 = 0$

$y_Q = y_P - \lambda y_n = 1 - 1 \times 1 = 0$

The reflection point is $Q(0, 0)$

Parametric form of the line equation



Given $\mathbf{s} = \mathbf{t} + \lambda\mathbf{v}$

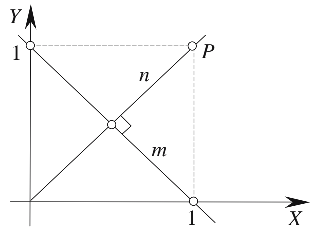
$\mathbf{q} = 2\mathbf{t} + \varepsilon\mathbf{v} - \mathbf{p}$

where $$\varepsilon = \frac{2\mathbf{v} \bullet (\mathbf{p} - \mathbf{t})}{\mathbf{v} \bullet \mathbf{v}}$$

Given	$x_P = 1 \quad y_P = 1 \quad \mathbf{t} = \mathbf{j} \quad \mathbf{v} = \mathbf{i} - \mathbf{j}$
	$\varepsilon = \dfrac{2 \times 1}{2} = 1$
Therefore	$x_Q = 2x_T + \varepsilon x_v - x_P = 2 \times 0 + 1 \times 1 - 1 = 0$
	$y_Q = 2y_T + \varepsilon y_v - y_P = 2 \times 1 + 1 \times (-1) - 1 = 0$
The reflection point is	$Q(0, 0)$

2.11.12 Find the normal to a line through a point

General form of the line equation

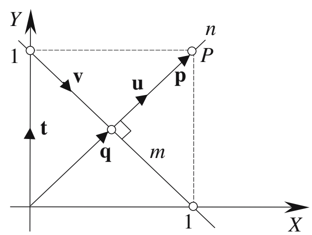


If line m is $\quad ax + by + c = 0$

and line n is perpendicular to m passing through the point $P(x_P, y_P)$

The line equation for n is	$-bx + ay + bx_P - ay_P = 0$
Given m is	$x + y - 1 = 0$
then	$a = 1 \quad b = 1 \quad x_P = 1 \quad y_P = 1$
Line n is	$-x + y = 0$

Parametric form of the line equation

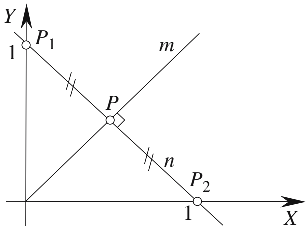


Given line m	$\mathbf{q} = \mathbf{t} + \lambda\mathbf{v}$ and a point P
	$\mathbf{u} = \mathbf{p} - (\mathbf{t} + \lambda\mathbf{v})$
where	$\lambda = \dfrac{\mathbf{v} \bullet (\mathbf{p} - \mathbf{t})}{\mathbf{v} \bullet \mathbf{v}}$
line n is	$\mathbf{p} + \varepsilon\mathbf{u}$ where ε is a scalar.
Given	$\mathbf{v} = \mathbf{i} - \mathbf{j} \quad \mathbf{p} = \mathbf{i} + \mathbf{j} \quad \mathbf{t} = \mathbf{j}$
	$\lambda = \dfrac{(\mathbf{i} - \mathbf{j}) \bullet \mathbf{i}}{(\mathbf{i} - \mathbf{j}) \bullet (\mathbf{i} - \mathbf{j})} = \dfrac{1}{2}$
	$\mathbf{u} = (\mathbf{i} + \mathbf{j}) - (\mathbf{j} + \frac{1}{2}(\mathbf{i} - \mathbf{j})) = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$
Line n is	$\mathbf{n} = (\mathbf{i} + \mathbf{j}) + \varepsilon(\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}) = (1 + \frac{1}{2}\varepsilon)\mathbf{i} + (1 + \frac{1}{2}\varepsilon)\mathbf{j}$

where ε is a scalar, which is equivalent to $-x + y = 0$.

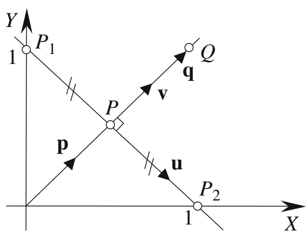
2.11.13 Find the line equidistant from two points

General form of the line equation



Given	$ax + by + c = 0$
Line n is	$x + y - 1 = 0$
Line m is given by	$(x_2 - x_1)\,x + (y_2 - y_1)\,y - \frac{1}{2}(x_2^2 - x_1^2 + y_2^2 - y_1^2) = 0$
with	$P_1(0,1)$ and $P_2(1,0)$
Line m is	$(1-0)\,x + (0-1)\,y - \frac{1}{2}(1-0+0-1) = 0$
	$x - y = 0$

Parametric form of the line equation

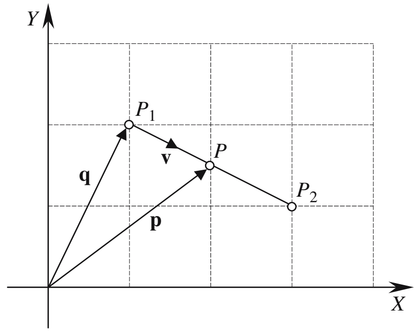


Given	$\mathbf{q} = \mathbf{p} + \lambda\mathbf{v}$
where	$\mathbf{q} = (\frac{1}{2}(x_1 + x_2) - \lambda(y_2 - y_1))\mathbf{i} + (\frac{1}{2}(y_1 + y_2) + \lambda(x_2 - x_1)\mathbf{j}$
with	$P_1(0,1)$ and $P_2(1,0)$
Therefore	$\mathbf{q} = (\frac{1}{2}(0+1) - \lambda(0-1))\mathbf{i} + (\frac{1}{2}(1+0) + \lambda(1-0)\mathbf{j}$
	$\mathbf{q} = (\frac{1}{2} + \lambda)\mathbf{i} + (\frac{1}{2} + \lambda)\mathbf{j}$

e.g. when $\lambda = 0$ we have $P\left(\frac{1}{2}, \frac{1}{2}\right)$ and when $\lambda = \frac{1}{2}$ the point is $Q(1,1)$

This is equivalent to $y = x$ or $-x + y = 0$

2.11.14 Creating the parametric line equation for a line segment



P_1 (x_1, y_1) and P_2 (x_2, y_2) delimit the line segment and the parametric line equation is given by

$$\mathbf{p} = \mathbf{q} + \lambda\,\mathbf{v}$$

where $\qquad \mathbf{q} = x_1\mathbf{i} + y_1\mathbf{j} \quad \text{and} \quad \mathbf{v} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j}$

Therefore

$$x_P = x_1 + \lambda(x_2 - x_1)$$
$$y_P = y_1 + \lambda(y_2 - y_1)$$

Given P_1 $(1, 2)$ and $P_2(3, 1)$. P is between P_1 and P_2 for $\lambda \in [0, 1]$

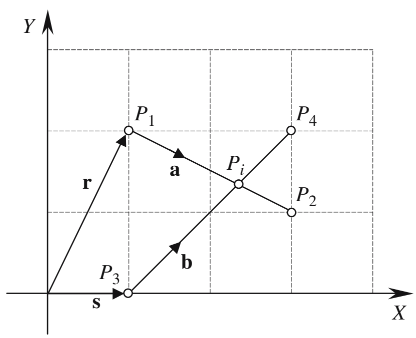
i.e.

$$x_P = 1 + \lambda(3 - 1) = 1 + 2\lambda$$
$$y_P = 2 + \lambda(1 - 2) = 2 - \lambda$$

For example, when $\lambda = 0.5$

$$x_{\frac{1}{2}} = 2 \quad \text{and} \quad y_{\frac{1}{2}} = 1.5$$

2.11.15 Intersecting two line segments



Given two line segments with equations $\mathbf{r} + \lambda\mathbf{a}$ and $\mathbf{s} + \varepsilon\mathbf{b}$

where $\qquad \mathbf{a} = x_a\mathbf{i} + y_a\mathbf{j} \quad \text{and} \quad \mathbf{b} = x_b\mathbf{i} + y_b\mathbf{j}$

The point of intersection is $x_i = x_r + \lambda x_a \quad y_i = y_r + \lambda y_a$

where
$$\lambda = \frac{x_b(y_3 - y_1) - y_b(x_3 - x_1)}{x_b y_a - x_a y_b}$$

Let the two line segments be $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_3P_4}$ with $P_1(1, 2), P_2(3, 1), P_3(1, 0), P_4(3, 1)$

Therefore $\quad \mathbf{r} = \mathbf{i} + 2\mathbf{j} \quad$ and $\quad \mathbf{a} = 2\mathbf{i} - \mathbf{j}$

$\mathbf{s} = \mathbf{i} \quad$ and $\quad \mathrm{b} = 2\mathbf{i} + 2\mathbf{j}$

Therefore
$$\lambda = \frac{2(0-2) - 2(1-1)}{2 \times (-1) - 2 \times 2} = \tfrac{2}{3}$$

As $0 < \lambda < 1$ there is a point of intersection

$$x_i = 1 + \tfrac{2}{3} \times 2 = 2\tfrac{1}{3}$$

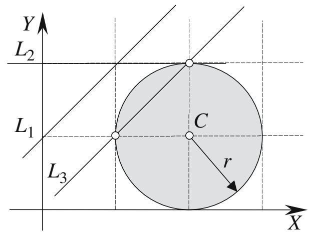
$$y_i = 2 + \tfrac{2}{3}(-1) = 1\tfrac{1}{3}$$

The point of intersection is $(2\frac{1}{3}, 1\frac{1}{3})$, which is correct.

2.12 Lines and circles

2.12.1 Line intersecting a circle

General form of the line equation



The diagram shows a circle radius $r = 1$ centered at $C(x_C, y_C) = (2, 1)$ and three lines: L_1, L_2 and L_3 that miss, touch and intersect the circle respectively.

The line equation is $\quad ax + by + c = 0$

Point(s) of intersection
$$x = x_C - ac_T \pm \sqrt{c_T^2(a^2 - 1) + b^2r^2} \tag{1}$$

$$y = y_C - bc_T \pm \sqrt{c_T^2(b^2 - 1) + a^2r^2}$$

where $\quad c_T = ax_C + by_C + c$

Miss condition

Line L_1 is
$$-x + y - 1 = 0 \tag{2}$$

L_1 normalized is
$$-\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y - \frac{1}{\sqrt{2}} = 0$$

where
$$a = -\frac{1}{\sqrt{2}} \qquad b = \frac{1}{\sqrt{2}} \qquad c = -\frac{1}{\sqrt{2}}$$

then
$$c_T = -\frac{2}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}$$

$$c_T^2(b^2 - 1) + a^2r^2 = 2\left(\tfrac{1}{2} - 1\right) + \tfrac{1}{2} = -\tfrac{1}{2}$$

The negative discriminant confirms the non-intersection.

Touch condition

Line L_2 is $\qquad y - 2 = 0$ (which is already normalized) $\qquad$ (3)

therefore $\qquad a = 0 \qquad b = 1 \qquad c = -2$

and $\qquad c_T = 0 + 1 - 2 = -1$

$$c_T^2(b^2 - 1) + a^2r^2 = 1(1 - 1) = 0$$

The zero discriminant confirms the touch condition:

using (1) $\qquad x = 2$

and (2) $\qquad y = 2$

Therefore the touching point is $(2, 2)$ which is correct.

Intersect condition

Line L_3 is $\qquad x - y = 0 \qquad$ (4)

L_3 normalized is $\qquad \frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y = 0$

where $\qquad a = \frac{1}{\sqrt{2}} \qquad b = -\frac{1}{\sqrt{2}} \qquad c = 0$

$$c_T = \frac{2}{\sqrt{2}} - \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$c_T^2(b^2 - 1) + a^2r^2 = \tfrac{1}{4}$$

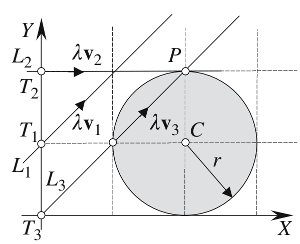
The positive discriminant confirms the intersect condition.

Using (1) $\qquad x = 2 - \frac{1}{2} \pm \sqrt{\frac{1}{4}} = 2$ and 1

and (4) $\qquad y = 2$ and 1

The intersection points are $(2, 2)$ and $(1, 1)$ which are correct.

Parametric form of the line equation



The diagram shows a circle radius $r = 1$ centered at $C(x_C, y_C) = (2, 1)$ and three lines: L_1, L_2 and L_3 that miss, touch, and intersect the circle respectively.

The lines are $\quad \mathbf{p}_1 = \mathbf{t}_1 + \lambda \mathbf{v}_1 \quad \mathbf{p}_2 = \mathbf{t}_2 + \lambda \mathbf{v}_2 \quad \mathbf{p}_3 = \mathbf{t}_3 + \lambda \mathbf{v}_3$

where $\quad \mathbf{t}_1 = \mathbf{j} \quad \mathbf{v}_1 = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$

$\mathbf{t}_2 = 2\mathbf{j} \quad \mathbf{v}_2 = \mathbf{i}$

$\mathbf{t}_3 = 0 \quad \mathbf{v}_3 = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$

and $\quad \mathbf{c} = 2\mathbf{i} + \mathbf{j}$

Let us substitute the lines into the following equations:

Point(s) of intersection $\quad x_p = x_T + \lambda x_v$

$y_P = y_T + \lambda y_v$

where $\quad \lambda = \mathbf{s} \bullet \mathbf{v} \pm \sqrt{(\mathbf{s} \bullet \mathbf{v})^2 - \|\mathbf{s}\|^2 + r^2}$

$\mathbf{s} = \mathbf{c} - \mathbf{t}$

L_1: $\quad \mathbf{s} = 2\mathbf{i}$

$(\mathbf{s} \bullet \mathbf{v})^2 - \|\mathbf{s}\|^2 + r^2 = 2 - 4 + 1 = -1$

The negative discriminant confirms a miss condition.

L_2: $\quad \mathbf{s} = 2\mathbf{i} - \mathbf{j}$

$(\mathbf{s} \bullet \mathbf{v})^2 - \|\mathbf{s}\|^2 + r^2 = 4 - 5 + 1 = 0$

The zero discriminant confirms a touch condition.

Therefore $\quad \lambda = 2$

The touch point is $\quad x_P = 2 \quad y_P = 2 \quad$ which is correct.

L_3: $\quad \mathbf{s} = 2\mathbf{i} + \mathbf{j}$

$(\mathbf{s} \bullet \mathbf{v})^2 - \|\mathbf{s}\|^2 + r^2 = 4.5 - 5 + 1 = \frac{1}{2}$

The positive discriminant confirms an intersect condition.

Therefore $\quad \lambda = \frac{3}{\sqrt{2}} \pm \frac{1}{\sqrt{2}} = 2\sqrt{2} \quad \text{and} \quad \sqrt{2}$

The intersection points are

$$\lambda = 2\sqrt{2} \qquad x_p = 0 + 2\sqrt{2}\,\frac{1}{\sqrt{2}} = 2$$

$$y_P = 0 + 2\sqrt{2}\frac{1}{\sqrt{2}} = 2$$

$\lambda = \sqrt{2}$

$$x_P = 0 + \sqrt{2}\frac{1}{\sqrt{2}} = 1$$

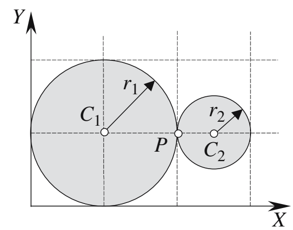
$$y_P = 0 + \sqrt{2}\frac{1}{\sqrt{2}} = 1$$

The intersection points are (1, 1) and (2, 2) which are correct.

2.12.2 Touching and intersecting circles

Touching circles

The diagram shows two circles touching one another at a point P(x_P, y_P).



One circle with radius $r_1 = 1$ is centered at $C_1(1, 1)$, the other with radius $r_2 = 0.5$ is centered at C_2 (2.5, 1).

Given $$d = \sqrt{(x_{C2} - x_{C1})^2 + (y_{C2} - y_{C1})^2}$$

The touch condition is $$d = r_1 + r_2$$

The touch point is $$x_P = x_{C1} + \frac{r_1}{d}(x_{C2} - x_{C1}) \quad \text{and} \quad y_P = y_{C1} + \frac{r_1}{d}(y_{C2} - y_{C1})$$

then $$d = \sqrt{(2.5 - 1)^2 + (1 - 1)^2} = 1.5$$

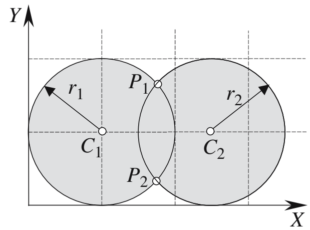
The touch condition is satisfied.

$$x_P = 1 + \frac{1}{1.5}(2.5 - 1) = 2$$

$$y_P = 1 + \frac{1}{1.5}(1 - 1) = 1$$

Therefore the touch point is $P(2, 1)$ which is correct.

Intersecting circles



The diagram shows two circles intersecting one another at points $P_1(x_{P1}, y_{P1})$ and $P_2(x_{P2}, y_{P2})$.

One circle with radius $r_1 = 1$ is centered at $C_1(1, 1)$, the other with radius $r_2 = 1$ is centered at $C_2(2.5, 1)$.

The intersect condition $\quad d < r_1 + r_2$

The points of intersection are

$$x_{P1} = x_{C1} + \lambda x_d - \varepsilon y_d$$

$$y_{P1} = y_{C1} + \lambda y_d + \varepsilon x_d$$

$$x_{P2} = x_{C1} + \lambda x_d + \varepsilon y_d$$

$$y_{P2} = y_{C1} + \lambda y_d - \varepsilon x_d$$

where

$$\lambda = \frac{r_1^2 - r_2^2 + d^2}{2d^2}$$

and

$$\varepsilon = \left|\sqrt{\frac{r_1^2}{d^2} - \lambda^2}\right|$$

$$d = 1$$

Therefore the intersect condition is satisfied.

$$\lambda = \frac{1 - 1 + 2.25}{2 \times 2.25} = \frac{1}{2}$$

$$\varepsilon = \left|\sqrt{\frac{4}{9} - \frac{1}{4}}\right| = \frac{\sqrt{7}}{6}$$

therefore

$$x_{P1} = 1 + \frac{1}{2}\frac{3}{2} = 1\tfrac{3}{4} \qquad y_{P1} = \frac{\sqrt{7}}{6}\frac{3}{2} = \tfrac{\sqrt{7}}{4}$$

and

$$x_{P2} = 1 + \frac{1}{2}\frac{3}{2} = 1\tfrac{3}{4} \qquad y_{P2} = -\frac{\sqrt{7}}{6}\frac{3}{2} = -\tfrac{\sqrt{7}}{4}$$

The intersection points are $\left(1\tfrac{3}{4}, \tfrac{\sqrt{7}}{4}\right)$ and $\left(1\tfrac{3}{4}, -\tfrac{\sqrt{7}}{4}\right)$ which are correct.

2.13 Second degree curves

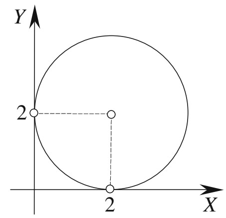
2.13.1 Circle

General equation

Center (x_C, y_C) $\quad (x - x_C)^2 + (y - y_C)^2 = r^2$

Given a radius $r = 2$ and center $(2, 2)$

then $\quad (x - 2)^2 + (y - 2)^2 = 4$



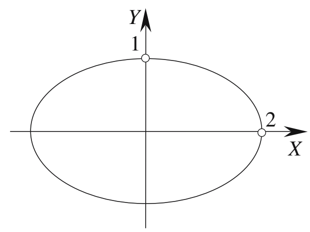
2.13.2 Ellipse

General equation

Center origin $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

with $a = 2, b = 1$

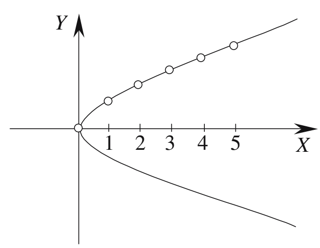
then $\quad \dfrac{x^2}{4} + y^2 = 1$



2.13.3 Parabola

Parametric equation

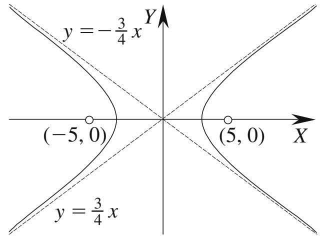
Vertex origin $\quad \left.\begin{aligned} x &= t^2 \\ y &= 2t \end{aligned}\right\} \quad t \in [-5, 5]$



t	0	± 1	± 2	± 3	± 4	± 5
x	0	1	4	9	16	25
y	0	± 2	± 4	± 6	± 8	± 10

2.13.4 Hyperbola

General equation



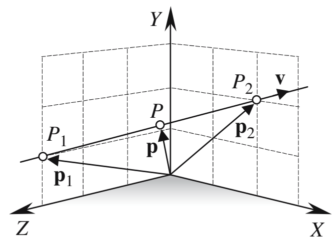
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

Foci at $(\pm c, 0)$ $\qquad c=\sqrt{a^2+b^2}$

then $\qquad \dfrac{x^2}{16}-\dfrac{y^2}{9}=1 \qquad$ with $\quad c=5$

2.14 Three-dimensional straight lines

2.14.1 Derive the straight-line equation from two points



Given P_1 and P_2 $\quad \mathbf{v} = \mathbf{p}_2 - \mathbf{p}_1$

$\mathbf{p} = \mathbf{p}_1 + \lambda\mathbf{v}$

Given $\quad P_1(0, 1, 3)$ and $P_2(2, 2, 0)$

$P_1 = \mathbf{j} + 3\mathbf{k}$

$\mathbf{v} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$

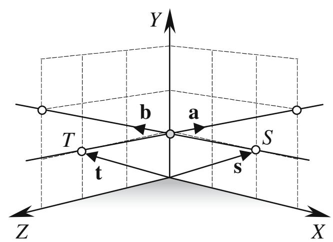
and $\quad \mathbf{p} = \mathbf{p}_1 + \lambda\mathbf{v}$

2.14.2 Intersection of two straight lines

Given two lines $\quad \mathbf{p} = \mathbf{t} + \lambda\mathbf{a}$ and $\mathbf{q} = \mathbf{s} + \varepsilon\mathbf{b}$

where $\quad \mathbf{t} = x_t\mathbf{i} + y_t\mathbf{j} + z_t\mathbf{k}$ and $\mathbf{s} = x_s\mathbf{i} + y_s\mathbf{j} + z_s\mathbf{k}$

$\mathbf{a} = x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k}$ and $\mathbf{b} = x_b\mathbf{i} + y_b\mathbf{j} + z_b\mathbf{k}$



Step 1: If $\mathbf{a} \times \mathbf{b} = 0$ the lines are parallel and do not intersect.

Step 2: If $(\mathbf{t} - \mathbf{s}) \cdot (\mathbf{a} \times \mathbf{b}) \neq 0$ the lines do not touch.

Step 3: Solving

$$\lambda x_a - \varepsilon x_b = x_s - x_t$$
$$\lambda y_a - \varepsilon y_b = y_s - y_t$$
$$\lambda z_a - \varepsilon z_b = z_s - z_t$$

provides values for λ and ε which, when substituted in the original line equations, reveal the intersection point.

Given $\quad \mathbf{t} = \mathbf{j} + 2\mathbf{k}$ and $\mathbf{s} = 2\mathbf{i} + \mathbf{j}$

$\mathbf{a} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = -2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$

Step 1: Prove that the lines are not parallel.
Although it is obvious that **a** and **b** are not parallel, let's prove it by ensuring that $\mathbf{a} \times \mathbf{b} \neq 0$.

	i	**j**	**k**
a	3	1	−2
b	−2	1	3
a × **b**	5	5	5

Therefore the lines are not parallel.

Step 2: Prove that the lines are touching.
If $(\mathbf{t} - \mathbf{s}) \cdot (\mathbf{a} \times \mathbf{b}) = 0$ the lines touch.
Therefore $(2\mathbf{i} + 2\mathbf{k}) \cdot (5\mathbf{i} + 5\mathbf{j} + 5\mathbf{k}) = 0$ so the lines touch.

Step 3: Compute the intersection point.
Create the three equations:

$$3\lambda + 2\varepsilon = 2 \quad (1)$$
$$\lambda - \varepsilon = 0 \quad (2)$$
$$-2\lambda - 3\varepsilon = -2 \quad (3)$$

From (2) $\lambda = \varepsilon$

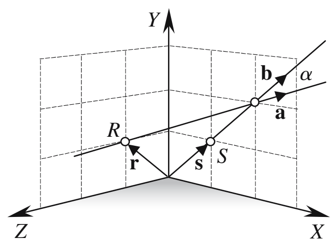
Substituting $\lambda = \varepsilon$ in (1) $\lambda = \frac{2}{5}$ and $\varepsilon = \frac{2}{5}$

Substitute λ and ε in the original line equations

$$\mathbf{p} = (\mathbf{j} + 2\mathbf{k}) + \tfrac{2}{5}(3\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = \tfrac{6}{5}\mathbf{i} + \tfrac{7}{5}\mathbf{j} + \tfrac{6}{5}\mathbf{k}$$

The intersection point is $\left(\frac{6}{5}, \frac{7}{5}, \frac{6}{5}\right)$

2.14.3 Calculate the angle between two straight lines



Given $\mathbf{p} = \mathbf{r} + \lambda\mathbf{a}$

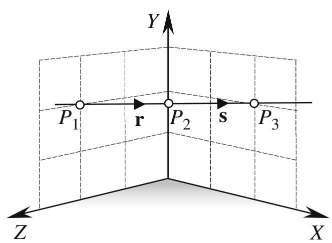
and $\mathbf{q} = \mathbf{s} + \varepsilon\mathbf{b}$

angle $\alpha = \cos^{-1}\left(\dfrac{\mathbf{a} \bullet \mathbf{b}}{\|\mathbf{a}\| \cdot \|\mathbf{b}\|}\right)$

Given $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + \mathbf{j}$

$$\alpha = \cos^{-1}\left(\frac{(2\mathbf{i} + \mathbf{j} - \mathbf{k}) \bullet (\mathbf{i} + \mathbf{j})}{\sqrt{6}\sqrt{2}}\right)$$
$$= \cos^{-1}\left(\frac{3}{\sqrt{12}}\right) = 30°$$

2.14.4 Test if three points lie on a straight line



Given three points P_1, P_2, P_3.

Let $\mathbf{r} = \overrightarrow{P_1P_2}$ and $\mathbf{s} = \overrightarrow{P_1P_3}$

The points lie on a straight line when $\mathbf{s} = \lambda\mathbf{r}$ where λ is a scalar.

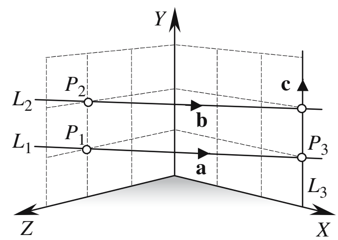
Given $P_1(0,2,2)$ $P_2(1,2,1)$ $P_3(2,2,0)$

therefore $\mathbf{r} = \mathbf{i} - \mathbf{k}$ and $\mathbf{s} = 2\mathbf{i} - 2\mathbf{k}$

and $\mathbf{s} = 2\mathbf{r}$

Therefore the points lie on a straight line.

2.14.5 Test for parallel and perpendicular straight lines



Given $\mathbf{p} = \mathbf{r} + \mu\mathbf{a}$
and $\mathbf{q} = \mathbf{s} + \varepsilon\mathbf{b}$

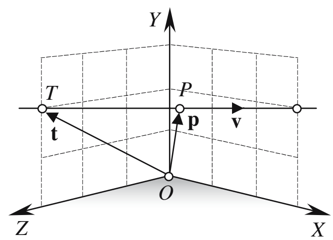
The lines are parallel if $\mathbf{a} = \lambda\mathbf{b}$ where λ is a scalar.
The lines are perpendicular if $\mathbf{a} \cdot \mathbf{b} = 0$.

Given three lines L_1: $\mathbf{p}_1 + \mu\mathbf{a}$
L_2: $\mathbf{p}_2 + \varepsilon\mathbf{b}$
L_3: $\mathbf{p}_3 + \lambda\mathbf{c}$

where $\mathbf{a} = 3\mathbf{i} - 2\mathbf{k}$
$\mathbf{b} = 3\mathbf{i} - 2\mathbf{k}$
$\mathbf{c} = \mathbf{j}$

L_1 and L_2 are parallel because $\mathbf{a} = \mathbf{b}$.
L_1 and L_3 are perpendicular because $\mathbf{a} \cdot \mathbf{c} = (3\mathbf{i} - 2\mathbf{k}) \cdot (\mathbf{j}) = 0$.

2.14.6 Find the position and distance of the nearest point on a line to the origin



Given $\mathbf{p} = \mathbf{t} + \lambda\mathbf{v}$

where $\lambda = \dfrac{-\mathbf{v} \cdot \mathbf{t}}{\mathbf{v} \cdot \mathbf{v}}$

Distance $OP = \|\mathbf{p}\|$

Given $\mathbf{t} = 2\mathbf{j} + 3\mathbf{k} \quad \mathbf{v} = 3\mathbf{i} - 3\mathbf{k}$

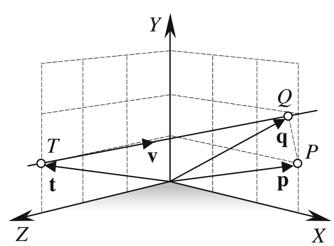
$$\lambda = \frac{-(3\mathbf{i} - 3\mathbf{k}) \cdot (2\mathbf{j} + 3\mathbf{k})}{(3\mathbf{i} - 3\mathbf{k}) \cdot (3\mathbf{i} - 3\mathbf{k})} = \frac{9}{18} = \tfrac{1}{2}$$

therefore

$$x_P = x_T + \lambda x_v = 0 + \tfrac{1}{2} \times 3 = 1\tfrac{1}{2}$$
$$y_P = y_T + \lambda y_v = 2 + \tfrac{1}{2} \times 0 = 2$$
$$z_P = z_T + \lambda z_v = 3 + \tfrac{1}{2} \times (-3) = 1\tfrac{1}{2}$$

Distance $OP = \|\mathbf{p}\| = \|\tfrac{3}{2}\mathbf{i} + 2\mathbf{j} + \tfrac{3}{2}\mathbf{k}\| = 2.92$

2.14.7 Find the position and distance of the nearest point on a line to a point



Given $\mathbf{q} = \mathbf{t} + \lambda\mathbf{v}$

where $\lambda = \dfrac{\mathbf{v} \cdot (\mathbf{p} - \mathbf{t})}{\mathbf{v} \cdot \mathbf{v}}$

Distance $PQ = \|\mathbf{p} - (\mathbf{t} + \lambda\mathbf{v})\|$

Given $\mathbf{t} = \mathbf{j} + 3\mathbf{k}$
$\mathbf{v} = 3\mathbf{i} + \mathbf{j} - 3\mathbf{k}$

and $\mathbf{p} = 3\mathbf{i} + \mathbf{j}$

$$\lambda = \frac{(3\mathbf{i}+\mathbf{j}-3\mathbf{k})\bullet(3\mathbf{i}-3\mathbf{k})}{(3\mathbf{i}+\mathbf{j}-3\mathbf{k})\bullet(3\mathbf{i}+\mathbf{j}-3\mathbf{k})} = \frac{18}{19}$$

then $x_Q = x_T + \lambda x_v = 0 + \frac{18}{19}\times 3 = 2.842$

$y_Q = y_T + \lambda y_v = 1 + \frac{18}{19}\times 1 = 1.947$

$z_Q = z_T + \lambda z_v = 3 + \frac{18}{19}\times(-3) = 0.1579$

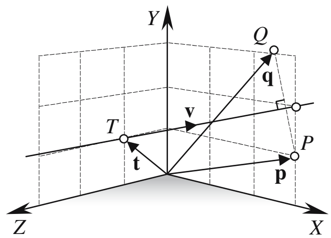
Distance $PQ = \|(3\mathbf{i}+\mathbf{j}) - ((\mathbf{j}+3\mathbf{k}) + \frac{18}{19}(3\mathbf{i}+\mathbf{j}-3\mathbf{k}))\| = 0.9733$

2.14.8 Find the reflection of a point in a line

Given $\mathbf{s} = \mathbf{t} + \lambda\mathbf{v}$ and a point P with reflection Q

$\mathbf{q} = 2\mathbf{t} + \varepsilon\mathbf{v} - \mathbf{p}$

where $\varepsilon = \dfrac{2\mathbf{v}\bullet(\mathbf{p}-\mathbf{t})}{\mathbf{v}\bullet\mathbf{v}}$



Given $\mathbf{t} = \mathbf{j} + \mathbf{k}$

$\mathbf{v} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$

and $\mathbf{p} = 3\mathbf{i} + \mathbf{j}$

$$\varepsilon = \frac{2(3\mathbf{i}+\mathbf{j}-\mathbf{k})\bullet(3\mathbf{i}-\mathbf{k})}{(3\mathbf{i}+\mathbf{j}-\mathbf{k})\bullet(3\mathbf{i}+\mathbf{j}-\mathbf{k})} = \frac{20}{11}$$

then $x_Q = 2x_T + \varepsilon x_v - x_P = 2\times 0 + \frac{20}{11}\times 3 - 3 = 2.4545$

$y_Q = 2y_T + \varepsilon y_v - y_P = 2\times 1 + \frac{20}{11}\times 1 - 1 = 2.8181$

$z_Q = 2z_T + \varepsilon z_v - z_P = 2\times 1 + \frac{20}{11}\times(-1) - 0 = 0.1818$

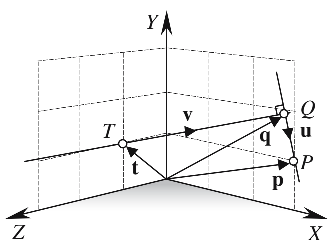
The reflection point is $Q(2.45, 2.82, 0.18)$

2.14.9 Find the normal to a line through a point

Given $\mathbf{q} = \mathbf{t} + \lambda\mathbf{v}$

the normal is $\mathbf{u} = \mathbf{p} - (\mathbf{t} + \lambda\mathbf{v})$

where $\lambda = \dfrac{\mathbf{v}\bullet(\mathbf{p}-\mathbf{t})}{\mathbf{v}\bullet\mathbf{v}}$



Given $\mathbf{t} = \mathbf{j} + \mathbf{k}$

$\mathbf{v} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$

and $\mathbf{p} = 3\mathbf{i} + \mathbf{j}$

therefore $$\lambda = \frac{(3\mathbf{i}+\mathbf{j}-\mathbf{k})\bullet(3\mathbf{i}-\mathbf{k})}{(3\mathbf{i}+\mathbf{j}-\mathbf{k})\bullet(3\mathbf{i}+\mathbf{j}-\mathbf{k})} = \frac{10}{11}$$

and $\quad x_u = x_P - (x_T + \lambda x_v) = 3 - (0 + \frac{10}{11} \times 3) = 0.2727$

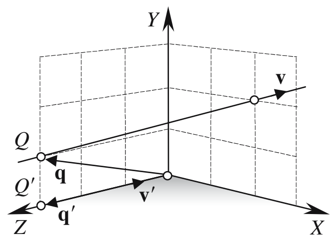
$$y_u = y_P - (y_T + \lambda y_v) = 1 - (1 + \tfrac{10}{11} \times 1) = -0.909$$

$$z_u = z_P - (z_T + \lambda z_v) = 0 - (1 + \tfrac{10}{11} \times (-1)) = -0.0909$$

therefore $\quad \mathbf{u} = 0.273\mathbf{i} - 0.909\mathbf{j} - 0.091\mathbf{k}$

The line equation for the normal is $\quad \mathbf{n} = \mathbf{p} + \varepsilon\mathbf{u}$

2.14.10 Find the shortest distance between two skew lines



Given $\quad \mathbf{p} = \mathbf{q} + t\mathbf{v}$

and $\quad \mathbf{p}' = \mathbf{q}' + \tau\mathbf{v}'$

Shortest distance $\quad d = \dfrac{\|(\mathbf{q} - \mathbf{q}') \bullet (\mathbf{v} \times \mathbf{v}')\|}{\|\mathbf{v} \times \mathbf{v}'\|}$

Given $\quad \mathbf{q} = \mathbf{j} + 3\mathbf{k}$

$\mathbf{q}' = 3\mathbf{k}$

$\mathbf{v} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$

$\mathbf{v}' = -\mathbf{k}$

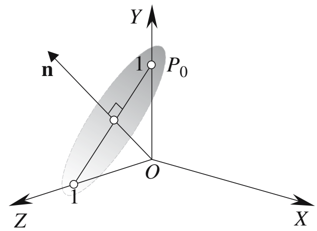
Calculate $\mathbf{v} \times \mathbf{v}'$

	i	**j**	**k**
$\mathbf{v}$	2	1	−3
$\mathbf{v}'$	0	0	−1
$\mathbf{v} \times \mathbf{v}'$	−1	2	0

$$d = \frac{\|\mathbf{j} \bullet (-\mathbf{i} + 2\mathbf{j})\|}{\|-\mathbf{i} + 2\mathbf{j}\|} = \frac{2}{\sqrt{5}} = 0.8944$$

2.15 Planes

2.15.1 Cartesian form of the plane equation



Given $ax + by + cz = d$

where the normal is $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

$\mathbf{p}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$

and $d = \mathbf{n} \cdot \mathbf{p}_0$

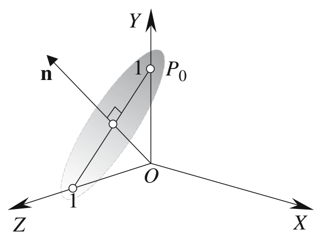
If the normal is $\mathbf{n} = \mathbf{j} + \mathbf{k}$

and the point $P_0(0, 1, 0)$

then $0x + 1y + 1z = 0 \times 0 + 1 \times 1 + 1 \times 0 = 1$

The plane equation is $y + z = 1$

2.15.2 General form of the plane equation



Given $ax + by + cz - (ax_0 + by_0 + cz_0) = 0$

where the normal is $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

and a point is $\mathbf{p}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$

If the normal is $\mathbf{n} = \mathbf{j} + \mathbf{k}$

and the point $P_0(0, 1, 0)$

then $0x + 1y + 1z - (0 \times 0 + 1 \times 1 + 1 \times 0) = 0$

The plane equation is $y + z - 1 = 0$

2.15.3 Hessian normal form of the plane equation

To convert the previous equation into Hessian normal form, rearrange the formula and divide throughout by $\|\mathbf{n}\|$.

Given $y + z - 1 = 0$

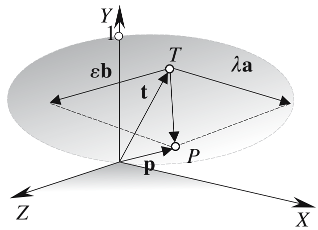
where the normal is $\mathbf{n} = \mathbf{j} + \mathbf{k}$

$\|\mathbf{n}\| = \sqrt{2}$

therefore $$\frac{1}{\sqrt{2}}y + \frac{1}{\sqrt{2}}z - \frac{1}{\sqrt{2}} = 0$$

or $$\tfrac{1}{2}\sqrt{2}y + \tfrac{1}{2}\sqrt{2}z - \tfrac{1}{2}\sqrt{2} = 0$$

2.15.4 Parametric form of the plane equation



Given vectors **a** and **b** that are parallel to the plane and point T is on the plane

where $$\mathbf{c} = \lambda\mathbf{a} + \varepsilon\mathbf{b}$$
and $$\mathbf{p} = \mathbf{t} + \mathbf{c}$$
then
$$x_P = x_T + \lambda x_a + \varepsilon x_b$$
$$y_P = y_T + \lambda y_a + \varepsilon y_b$$
$$z_P = z_T + \lambda z_a + \varepsilon z_b$$

The plane is parallel with the xz-plane and intersects the y-axis at $y = 1$. Let **a** and **b** be unit vectors parallel with the plane

i.e. $$\mathbf{a} = \mathbf{i} \qquad \mathbf{b} = \mathbf{k}$$

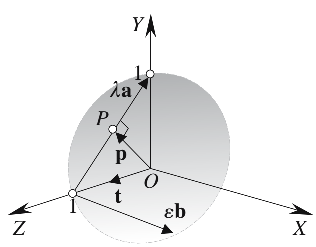
and $T(1, 1, 1)$ is a point on the plane

therefore $$\mathbf{t} = \mathbf{i} + \mathbf{j} + \mathbf{k}$$
and $$\mathbf{p} = \mathbf{t} + \lambda\mathbf{a} + \varepsilon\mathbf{b}$$

As **a** and **b** are unit vectors, λ and ε measure Euclidean distances. Therefore if $\lambda = 2$ and $\varepsilon = 1$

$$x_P = 1 + 2 \times 1 + 1 \times 0 = 3$$
$$y_P = 1 + 2 \times 0 + 1 \times 0 = 1$$
$$z_P = 1 + 2 \times 0 + 1 \times 1 = 2$$

2.15.5 Converting a plane equation from parametric form to general form



Given $$\mathbf{p} = \mathbf{t} + \lambda\mathbf{a} + \varepsilon\mathbf{b}$$

for P to be perpendicular to O

$$\lambda = \frac{(\mathbf{a} \bullet \mathbf{b})(\mathbf{b} \bullet \mathbf{t}) - (\mathbf{a} \bullet \mathbf{t})\|\mathbf{b}\|^2}{\|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a} \bullet \mathbf{b})^2}$$

and $$\varepsilon = \frac{(\mathbf{a} \bullet \mathbf{b})(\mathbf{a} \bullet \mathbf{t}) - (\mathbf{b} \bullet \mathbf{t})\|\mathbf{a}\|^2}{\|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a} \bullet \mathbf{b})^2}$$

then
$$\frac{x_P}{\|\mathbf{p}\|}x + \frac{y_P}{\|\mathbf{p}\|}y + \frac{z_P}{\|\mathbf{p}\|}z - \|\mathbf{p}\| = 0$$

We know in advance that the general equation of this plane is

$$\tfrac{1}{2}\sqrt{2}y + \tfrac{1}{2}\sqrt{2}z - \tfrac{1}{2}\sqrt{2} = 0$$

and intersects the y-axis and z-axis at $y = 1$ and $z = 1$ respectively.
The vectors for the parametric equation are

$$\mathbf{a} = \mathbf{j} - \mathbf{k}$$
$$\mathbf{b} = \mathbf{i}$$
$$\mathbf{t} = \mathbf{k}$$

therefore
$$\lambda = \frac{(0)(0) - (-1) \times 1}{2 \times 1 - (0)} = \frac{1}{2}$$

and
$$\varepsilon = \frac{(0)(-1) - (0) \times 2}{2 \times 1 - (0)} = 0$$

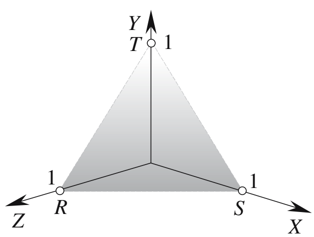
therefore
$$x_P = 0 + \tfrac{1}{2} \times 0 + 0 \times 1 = 0$$
$$y_P = 0 + \tfrac{1}{2} \times 1 + 0 \times 0 = \tfrac{1}{2}$$
$$z_P = 1 + \tfrac{1}{2}(-1) + 0 \times 0 = \tfrac{1}{2}$$
$$\|\mathbf{p}\| = \sqrt{0^2 + \tfrac{1}{2}^2 + \tfrac{1}{2}^2} = \tfrac{1}{2}\sqrt{2}$$

The plane equation is
$$0x + \frac{\tfrac{1}{2}}{\tfrac{1}{2}\sqrt{2}}y + \frac{\tfrac{1}{2}}{\tfrac{1}{2}\sqrt{2}}z - \tfrac{1}{2}\sqrt{2} = 0$$

and
$$\tfrac{1}{2}\sqrt{2}y + \tfrac{1}{2}\sqrt{2}z - \tfrac{1}{2}\sqrt{2} = 0$$

or
$$y + z - 1 = 0$$

2.15.6 Plane equation from three points



Given three points $R(x_R, y_R, z_R), S(x_S, y_S, z_S), T(x_T, y_T, z_T)$
the plane equation is $ax + by + cz + d = 0$
where

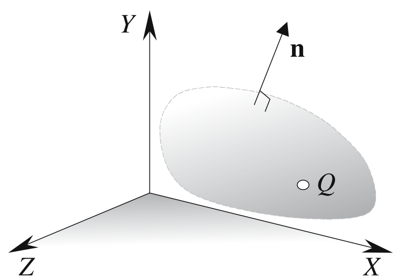
$$a = \begin{vmatrix} y_R & z_R & 1 \\ y_S & z_S & 1 \\ y_T & z_T & 1 \end{vmatrix} \qquad b = \begin{vmatrix} z_R & x_R & 1 \\ z_S & x_S & 1 \\ z_T & x_T & 1 \end{vmatrix} \qquad c = \begin{vmatrix} x_R & y_R & 1 \\ x_S & y_S & 1 \\ x_T & y_T & 1 \end{vmatrix} \qquad d = -(ax_R + by_R + cz_R)$$

If the three points are $R(0, 0, 1), S(1, 0, 0), T(0, 1, 0)$

$$a = \begin{vmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 1 \qquad b = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \qquad c = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1 \qquad d = -(1 \times 0 + 1 \times 0 + 1 \times 1) = -1$$

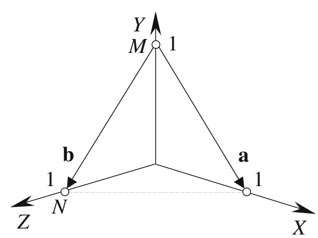
then the plane equation is $x + y + z - 1 = 0$

2.15.7 Plane through a point and normal to a line



Given $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and $Q(x_Q, y_Q, z_Q)$
the plane equation is $ax + by + cz - (ax_Q + by_Q + cz_Q) = 0$
If the line is $\mathbf{n} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $Q(0, 1, 0)$
the plane is $x + y + z - 1 = 0$

2.15.8 Plane through two points and parallel to a line



Given a line's direction vector **a** and two points $M(x_M, y_M, z_M)$ and $N(x_N, y_N, z_N)$
where $\mathbf{a} = x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k}$

and	$\mathbf{b} = (x_N - x_M)\mathbf{i} + (y_N - y_M)\mathbf{j} + (z_N - z_M)\mathbf{k}$
the plane equation is	$ax + by + cz - (ax_M + by_M + cz_M) = 0$
where	$a = y_a z_b - y_b z_a \qquad b = z_a x_b - z_b x_a \qquad c = x_a y_b - x_b y_a$
Given	$M = (0,1,0)$ and $N = (0,0,1)$
and	$\mathbf{a} = \mathbf{i} - \mathbf{j}$
therefore	$\mathbf{a} \times \mathbf{b} = \mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = -\mathbf{i} - \mathbf{j} - \mathbf{k}$
and	$-x - y - z - (0 - 1 + 0) = 0$
The plane equation is	$-x - y - z + 1 = 0$
or	$x + y + z - 1 = 0$

2.15.9 Intersection of two planes

Given two planes $\quad a_1x + b_1y + c_1z + d_1 = 0 \qquad a_2x + b_2y + c_2z + d_2 = 0$

where $\quad \mathbf{n}_1 = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k} \qquad \mathbf{n}_2 = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$

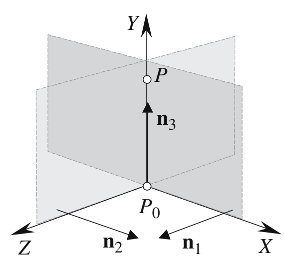
The direction vector of the intersection line is given by $\mathbf{n}_3 = \mathbf{n}_1 \times \mathbf{n}_2$
and the point P_0 on the intersection line is given by

$$DET = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad x_0 = \frac{d_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} - d_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}{DET}$$

$$y_0 = \frac{d_2\begin{vmatrix} a_3 & c_3 \\ a_1 & c_1 \end{vmatrix} - d_1\begin{vmatrix} a_3 & c_3 \\ a_2 & c_2 \end{vmatrix}}{DET} \qquad z_0 = \frac{d_2\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} - d_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}{DET}$$

Example 1

Let the two intersecting planes be the xy-plane and the xz-plane, which means that the line of intersection will be the y-axis.



The plane equations are $z = 0$ and $x = 0$

where $\quad \mathbf{n}_1 = \mathbf{k} \qquad \mathbf{n}_2 = \mathbf{i} \qquad d_1 = 0 \qquad d_2 = 0$

and
$$\mathbf{n}_3 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \mathbf{j}$$

Therefore $$DET = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 1 \qquad x_0 = \frac{0\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$$

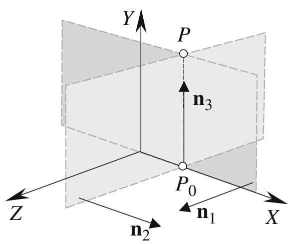
$$y_0 = \frac{0\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0 \qquad z_0 = \frac{0\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}}{1} = 0$$

therefore the line equation is $\mathbf{p} = \lambda \mathbf{n}_3$

where $\mathbf{n}_3 = \mathbf{j}$

Example 2

Let the two intersecting planes be the xy-plane and the plane $x = 1$, which means that the line of intersection will be parallel with the y-axis passing through the point $(1, 0, 0)$



The plane equations are $z = 0$ and $x - 1 = 0$

where $\mathbf{n}_1 = \mathbf{k} \qquad \mathbf{n}_2 = \mathbf{i} \qquad d_1 = 0 \qquad d_2 = -1$

and $$\mathbf{n}_3 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \mathbf{j}$$

and $$DET = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 1 \qquad x_0 = \frac{-1\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - 0\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}}{1} = 1$$

$$y_0 = \frac{-1\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0 \qquad z_0 = \frac{-1\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}}{1} = 0$$

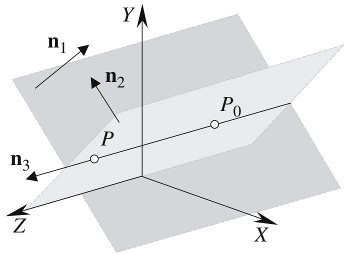
Therefore the line equation is $\mathbf{p} = \mathbf{p}_0 + \lambda \mathbf{n}_3$

where $\mathbf{p}_0 = \mathbf{i}$

and $\mathbf{n}_3 = \mathbf{j}$

Example 3

Let the two intersecting planes be $x + y - 1 = 0$ and $-x + y = 0$.



Therefore $\mathbf{n}_1 = \mathbf{i} + \mathbf{j}$ $\mathbf{n}_2 = -\mathbf{i} + \mathbf{j}$ $d_1 = -1$ $d_2 = 0$

and
$$\mathbf{n}_3 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ -1 & 1 & 0 \end{vmatrix} = 2\mathbf{k}$$

$$DET = \begin{vmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 4 \qquad x_0 = \frac{0\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix}}{4} = \frac{1}{2}$$

$$y_0 = \frac{0\begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 0 & 2 \\ -1 & 0 \end{vmatrix}}{4} = \frac{1}{2} \qquad z_0 = \frac{0\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + 1\begin{vmatrix} -1 & 1 \\ 0 & 0 \end{vmatrix}}{4} = 0$$

Therefore the line equation is $\mathbf{p} = \mathbf{p}_0 + \lambda \mathbf{n}_3$

where $\mathbf{p}_0 = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$

and $\mathbf{n}_3 = 2\mathbf{k}$

2.15.10 Intersection of three planes

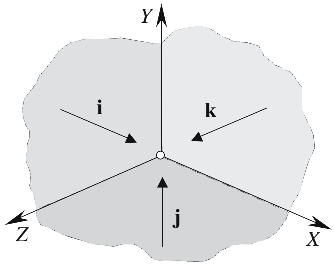
Given three planes
$$a_1x + b_1y + c_1z + d_1 = 0$$
$$a_2x + b_2y + c_2z + d_2 = 0$$
$$a_3x + b_3y + c_3z + d_3 = 0$$

the intersection point (x, y, z) is

$$x = -\frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{DET} \qquad y = -\frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{DET} \qquad z = -\frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{DET}$$

where
$$DET = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Example 1



Given the planes $x = 0 \quad y = 0 \quad z = 0$

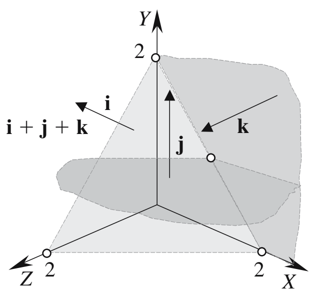
which are the three orthogonal planes intersecting at the origin.

$$DET = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$$x = -\begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0 \qquad y = -\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0 \qquad z = -\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

The intersection point is the origin, which is correct.

Example 2



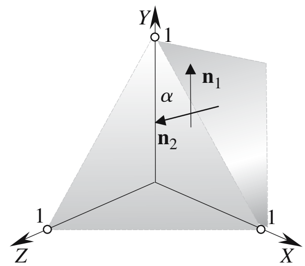
Given the planes $x + y + z - 2 = 0 \quad z = 0 \quad y - 1 = 0$

$$DET = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -1$$

$$x = -\frac{\begin{vmatrix} -2 & 1 & 1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{vmatrix}}{-1} = 1 \qquad y = -\frac{\begin{vmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix}}{-1} = 1 \qquad z = -\frac{\begin{vmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{vmatrix}}{-1} = 0$$

The intersection point is $(1, 1, 0)$ which is correct.

2.15.11 Angle between two planes



Given two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$

where $\mathbf{n}_1 = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{n}_2 = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$

the angle between the normals is $\alpha = \cos^{-1}\left(\dfrac{\mathbf{n}_1 \bullet \mathbf{n}_2}{\|\mathbf{n}_1\| \cdot \|\mathbf{n}_2\|}\right)$

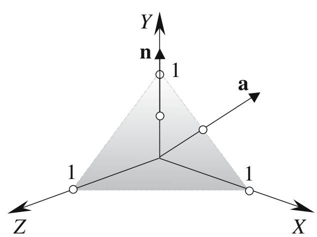
Given the planes $x + y + z - 1 = 0$ and $z = 0$

where $\mathbf{n}_1 = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{n}_2 = \mathbf{k}$

$$\|\mathbf{n}_1\| = \sqrt{3} \quad \text{and} \quad \|\mathbf{n}_2\| = 1$$

$$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) = 54.74°$$

2.15.12 Angle between a line and a plane



Given the plane $ax + by + cz + d = 0$

where $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

and the line $\mathbf{p} = \mathbf{r} + \lambda\mathbf{a}$

the angle between the line and the plane's normal is

$$\alpha = \cos^{-1}\left(\frac{\mathbf{n} \bullet \mathbf{a}}{\|\mathbf{n}\| \cdot \|\mathbf{a}\|}\right)$$

Given the plane $x + y + z - 1 = 0$

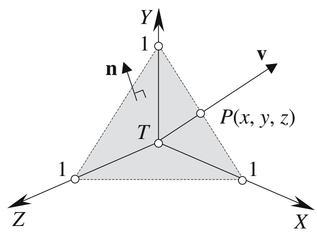
then $\mathbf{n} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

and $\mathbf{a} = \mathbf{i} + \mathbf{j}$

$$\|\mathbf{n}\| = \sqrt{3} \quad \text{and} \quad \|\mathbf{a}\| = \sqrt{2}$$

$$\alpha = \cos^{-1}\left(\frac{2}{\sqrt{6}}\right) = 35.26°$$

2.15.13 Intersection of a line and a plane



Given a plane	$ax + by + cz + d = 0$
where	$\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$
and a line	$\mathbf{p} = \mathbf{t} + \lambda\mathbf{v}$
for the intersection point P	$\lambda = \dfrac{-(\mathbf{n} \cdot \mathbf{t} + d)}{\mathbf{n} \cdot \mathbf{v}}$

Example 1

Given the plane	$x + y + z - 1 = 0$
and the line	$\mathbf{p} = \mathbf{t} + \lambda\mathbf{v}$
where	$\mathbf{t} = 0$
and	$\mathbf{v} = \mathbf{i} + \mathbf{j}$
then	$\lambda = \dfrac{-(1 \times 0 + 1 \times 0 + 1 \times 0 - 1)}{1 \times 1 + 1 \times 1 + 1 \times 0} = \dfrac{1}{2}$

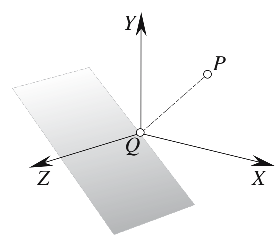
The point of intersection is $P\left(\frac{1}{2}, \frac{1}{2}, 0\right)$.

Example 2

With the same plane	$x + y + z - 1 = 0$
but	$\mathbf{t} = \mathbf{i} + \mathbf{j} + \mathbf{k}$
and	$\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$
	$\lambda = \dfrac{-(1 \times 1 + 1 \times 1 + 1 \times 1 - 1)}{1 \times 1 + 1 \times 1 + 1 \times 1} = -\dfrac{2}{3}$
	$\mathbf{p} = \mathbf{t} + \lambda\mathbf{v}$

The point of intersection is $P\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

2.15.14 Position and distance of the nearest point on a plane to a point



Given the plane	$ax + by + cz + d = 0$
where	$\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

and a point P with position vector $\mathbf{p}$.

The position vector of the nearest point Q is given by $\mathbf{q} = \mathbf{p} + \lambda\mathbf{n}$

where $$\lambda = \frac{-(\mathbf{n} \bullet \mathbf{p} + d)}{\mathbf{n} \bullet \mathbf{n}}$$

The distance PQ is $PQ = \|\lambda\mathbf{n}\|$

Given the plane $x + y = 0$

where $\mathbf{n} = \mathbf{i} + \mathbf{j}$

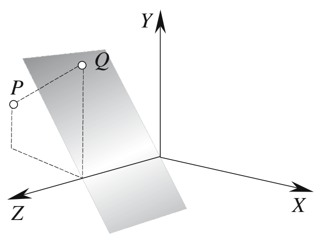
and a point $P(1, 1, 0)$ where $\mathbf{p} = \mathbf{i} + \mathbf{j}$

$$\lambda = \frac{-(2)}{2} = -1$$

The nearest point is $Q(0, 0, 0)$ the origin.

The distance is $PQ = \|-1(\mathbf{i} + \mathbf{j})\| = \sqrt{2}$

2.15.15 Reflection of a point in a plane



Given the plane $ax + by + cz + d = 0$

where $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

and P is a point with position vector $\mathbf{p}$

P's reflection Q is given by $\mathbf{q} = \mathbf{p} + \lambda\mathbf{n}$

where $$\lambda = \frac{-2(\mathbf{n} \bullet \mathbf{p} + d)}{\mathbf{n} \bullet \mathbf{n}}$$

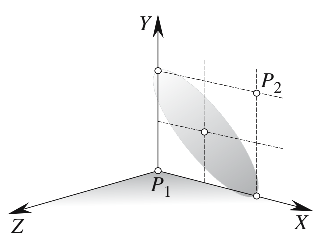
Given the plane $x + y = 0$ and $P(-1, 0, 1)$

$\mathbf{n} = \mathbf{i} + \mathbf{j}$

$$\lambda = \frac{-2(-1)}{2} = 1$$

The reflection point is $(0, 1, 1)$.

2.15.16 Plane equidistant from two points



Given two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ the plane equation is

$$(x_2 - x_1)x + (y_2 - y_1)y + (z_2 - z_1)z - \tfrac{1}{2}(x_2^2 - x_1^2 + y_2^2 - y_1^2 + z_2^2 - z_1^2) = 0$$

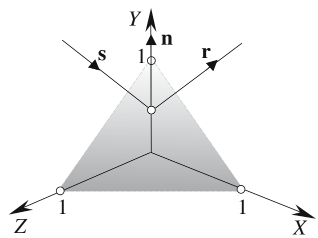
Given $P_1(0, 0, 0)$ and $P_2(2, 2, 0)$

the plane equation is $2x + 2y - \tfrac{1}{2}(4 + 4) = 0$

or $x + y - 2 = 0$

2.15.17 Reflected ray on a surface

Given the surface normal $\mathbf{n}$
the incident ray $\mathbf{s}$
the reflected ray $\mathbf{r}$



then $\mathbf{r} = \mathbf{s} + \lambda\mathbf{n}$

where $$\lambda = \frac{-2\mathbf{n} \bullet \mathbf{s}}{\mathbf{n} \bullet \mathbf{n}}$$

Given $\mathbf{n} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

and $$\mathbf{s} = \mathbf{i} - \frac{1}{4}\mathbf{j} - \frac{1}{4}\mathbf{k}$$

then $$\lambda = -\frac{1}{3}$$

and $$x_r = 1 - \frac{1}{3} = \frac{2}{3}$$

$$y_r = -\frac{1}{4} - \frac{1}{3} = -\frac{7}{12}$$

$$z_r = -\frac{1}{4} - \frac{1}{3} = -\frac{7}{12}$$

with $$\mathbf{r} = \frac{2}{3}\mathbf{i} - \frac{7}{12}\mathbf{j} - \frac{7}{12}\mathbf{k}$$

Let's check this vector out. Its magnitude should equal the magnitude of the incident vector **s**, and the reflection angle should equal the incident angle.

$$||\mathbf{s}|| = \sqrt{1^2 + \left(\frac{-1}{4}\right)^2 + \left(\frac{-1}{4}\right)^2} = \frac{\sqrt{18}}{4}$$

$$||\mathbf{r}|| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{-7}{12}\right)^2 + \left(\frac{-7}{12}\right)^2} = \frac{\sqrt{18}}{4}$$

The reflection angle equals $\theta = \cos^{-1}\left(\dfrac{\mathbf{n} \bullet \mathbf{r}}{\|\mathbf{n}\| \cdot \|\mathbf{r}\|}\right)$

The incident angle equals $\alpha = \cos^{-1}\left(\dfrac{\mathbf{n} \bullet -\mathbf{s}}{\|\mathbf{n}\| \cdot \|-\mathbf{s}\|}\right)$

For $\theta = \alpha$ $\dfrac{\mathbf{n} \bullet \mathbf{r}}{\|\mathbf{n}\| \cdot \|\mathbf{r}\|} = \dfrac{\mathbf{n} \bullet -\mathbf{s}}{\|\mathbf{n}\| \cdot \|-\mathbf{s}\|}$

but $\|\mathbf{s}\| = \|\mathbf{r}\|$

therefore $\mathbf{n} \bullet \mathbf{r} = \mathbf{n} \bullet -\mathbf{s}$

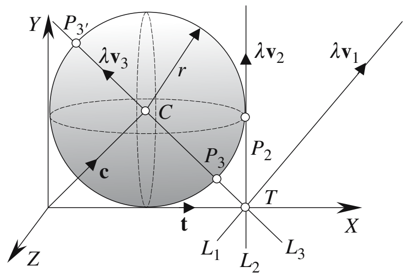
$$\mathbf{n} \bullet \mathbf{r} = (\mathbf{i} + \mathbf{j} + \mathbf{k}) \bullet \left(\frac{2}{3}\mathbf{i} - \frac{7}{12}\mathbf{j} - \frac{7}{12}\mathbf{k}\right) = -\frac{1}{2}$$

$$\mathbf{n} \bullet -\mathbf{s} = (\mathbf{i} + \mathbf{j} + \mathbf{k}) \bullet \left(-\mathbf{i} + \frac{1}{4}\mathbf{j} + \frac{1}{4}\mathbf{k}\right) = -\frac{1}{2}$$

which confirms that the angle of reflection equals the angle of incidence.

2.16 Lines, planes and spheres

2.16.1 Line intersecting a sphere



Given a sphere with radius r located at C with position vector $\mathbf{c}$

and a line equation $\quad \mathbf{p} = \mathbf{t} + \lambda\mathbf{v}$ where $\|\mathbf{v}\| = 1$

a touch, miss or intersect condition is determined by λ

where $$\lambda = \mathbf{s} \cdot \mathbf{v} \pm \sqrt{(\mathbf{s} \cdot \mathbf{v})^2 - \|\mathbf{s}\|^2 + r^2}$$

and $$\mathbf{s} = \mathbf{c} - \mathbf{t}$$

The diagram shows a sphere with radius $r = 1$ centered at C with position vector $\mathbf{c} = \mathbf{i} + \mathbf{j}$ and three lines L_1, L_2 and L_3 that miss, touch and intersect the sphere respectively.

The lines are of the form $\quad \mathbf{p} = \mathbf{t} + \lambda\,\mathbf{v}$

therefore $$\mathbf{p}_1 = \mathbf{t}_1 + \lambda\mathbf{v}_1 \qquad \mathbf{p}_2 = \mathbf{t}_2 + \lambda\mathbf{v}_2 \qquad \mathbf{p}_3 = \mathbf{t}_3 + \lambda\mathbf{v}_3$$

where $$\mathbf{t}_1 = 2\mathbf{i} \qquad \mathbf{v}_1 = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

$$\mathbf{t}_2 = 2\mathbf{i} \qquad \mathbf{v}_2 = \mathbf{j}$$

$$\mathbf{t}_3 = 2\mathbf{i} \qquad \mathbf{v}_3 = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

and $$\mathbf{c} = \mathbf{i} + \mathbf{j}$$

Let us substitute the lines in the original equations:

L_1: $$\mathbf{s} = -\mathbf{i} + \mathbf{j}$$
$$(\mathbf{s} \cdot \mathbf{v})^2 - \|\mathbf{s}\|^2 + r^2 = 0 - 2 + 1 = -1$$

The negative discriminant confirms a miss condition.

L_2: $$\mathbf{s} = -\mathbf{i} + \mathbf{j}$$
$$(\mathbf{s} \cdot \mathbf{v})^2 - \|\mathbf{s}\|^2 + r^2 = 1 - 2 + 1 = 0$$

The zero discriminant confirms a touch condition, therefore $\lambda = 1$.
The touch point is $P_2(2, 1, 0)$ which is correct.

L_3: $$\mathbf{s} = -\mathbf{i} + \mathbf{j}$$
$$(\mathbf{s} \cdot \mathbf{v})^2 - \|\mathbf{s}\|^2 + r^2 = 2 - 2 + 1 = 1$$

The positive discriminant confirms an intersect condition therefore $$\lambda = \frac{2}{\sqrt{2}} \pm 1 = 1 + \sqrt{2} \quad \text{or} \quad \sqrt{2} - 1$$

The intersection points are:

if $\lambda = 1 + \sqrt{2}$

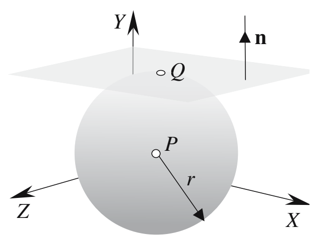
$$x_P = 2 + (1 + \sqrt{2})\left(-\frac{1}{\sqrt{2}}\right) = 1 - \frac{1}{\sqrt{2}}$$
$$y_P = 0 + (1 + \sqrt{2})\frac{1}{\sqrt{2}} = 1 + \frac{1}{\sqrt{2}}$$
$$z_P = 0$$

if $\lambda = \sqrt{2} - 1$

$$x_P = 1 + (\sqrt{2} - 1)\left(-\frac{1}{\sqrt{2}}\right) = 1 + \frac{1}{\sqrt{2}}$$
$$y_P = 0 + (\sqrt{2} - 1)\frac{1}{\sqrt{2}} = 1 - \frac{1}{\sqrt{2}}$$
$$z_P = 0$$

The intersection points are $P_{3'}\left(1 - \frac{1}{\sqrt{2}},\ 1 + \frac{1}{\sqrt{2}}, 0\right)$ and $P_3\left(1 + \frac{1}{\sqrt{2}},\ 1 - \frac{1}{\sqrt{2}}, 0\right)$ which are correct.

2.16.2 Sphere touching a plane



Given a plane $ax + by + cz + d = 0$

where $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

the nearest point Q on the plane to a point P is given by

$$\mathbf{q} = \mathbf{p} + \lambda\mathbf{n}$$

where $$\lambda = -\frac{\mathbf{n} \bullet \mathbf{p} + d}{\mathbf{n} \bullet \mathbf{n}}$$

The distance is given by $\|\lambda\mathbf{n}\|$

for a plane and a sphere $\|\lambda\mathbf{n}\| = r$

The diagram shows a sphere radius $r = 1$ centered at $P(1, 1, 1)$

The plane equation is $y - 2 = 0$

therefore $\mathbf{n} = \mathbf{j}$

and $\mathbf{p} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

therefore $\lambda = -(1 - 2) = 1$

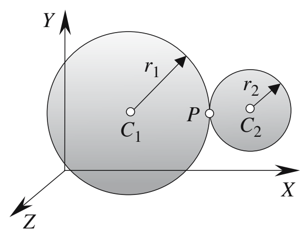
which equals the sphere's radius and therefore the sphere and the plane touch.

The touch point is

$$x_Q = 1 + 1 \times 0 = 1$$
$$y_Q = 1 + 1 \times 1 = 2$$
$$z_Q = 1 + 1 \times 0 = 1$$

therefore the touch point is $Q(1, 2, 1)$ which is correct.

2.16.3 Touching spheres



Given

$$d = \sqrt{(x_{C2} - x_{C1})^2 + (y_{C2} - y_{C1})^2 + (z_{C2} - z_{C1})^2}$$

the touch condition is

$$d = r_1 + r_2$$

the touch point is

$$x_P = x_{C1} + \frac{r_1}{d}(x_{C2} - x_{C1})$$
$$y_P = y_{C1} + \frac{r_1}{d}(y_{C2} - y_{C1})$$
$$z_P = z_{C1} + \frac{r_1}{d}(z_{C2} - z_{C1})$$

Given that one sphere with radius $r_1 = 1$ is centered at $C_1(1, 1, 1)$ and the other with radius $r_2 = 0.5$ is centered at $C_2(2.5, 1, 1)$

then

$$d = \sqrt{(2.5 - 1)^2 + (1 - 1)^2 + (1 - 1)^2} = 1.5$$

The touch condition is satisfied

and

$$x_P = 1 + \frac{1}{1.5}(2.5 - 1) = 2$$
$$y_P = 1 + \frac{1}{1.5}(1 - 1) = 1$$
$$z_P = 1 + \frac{1}{1.5}(1 - 1) = 1$$

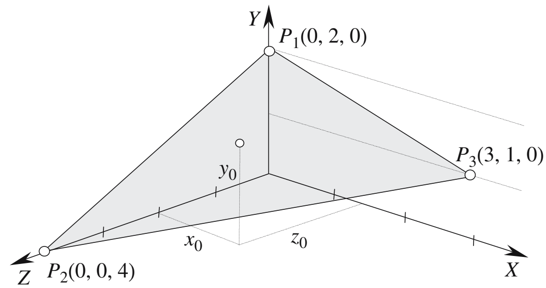
therefore the touch point is $P(2, 1, 1)$ which is correct.

2.17 Three-dimensional triangles

2.17.1 Coordinates of a point inside a triangle

To locate points inside and outside the triangle P_1, P_2, P_3 using barycentric coordinates.

For any point $P_0(x_0, y_0, z_0)$ we can state



$$x_0 = \varepsilon x_1 + \lambda x_2 + \beta x_3$$
$$y_0 = \varepsilon y_1 + \lambda y_2 + \beta y_3$$
$$z_0 = \varepsilon z_1 + \lambda z_2 + \beta z_3$$

where $\varepsilon + \lambda + \beta = 1$

The table below shows values of P_0 for various values of ε, λ and β. Let us check that the positions of P_0 reside on the plane of the triangle.

The vertices of the triangle are $P_1(0, 2, 0), P_2(0, 0, 4), P_3(3, 1, 0)$ therefore the Cartesian plane equation is

$ax + by + cz = d$ (see plane equation from three points)

where

$$a = \begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix} \qquad b = \begin{vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{vmatrix} \qquad c = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \qquad d = ax_1 + by_1 + cz_1$$

$$a = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 4 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 4 \qquad b = \begin{vmatrix} 0 & 0 & 1 \\ 4 & 0 & 1 \\ 0 & 3 & 1 \end{vmatrix} = 12 \qquad c = \begin{vmatrix} 0 & 2 & 1 \\ 0 & 0 & 1 \\ 3 & 1 & 1 \end{vmatrix} = 6 \qquad \begin{aligned} d &= 4 \times 0 + 12 \times 2 + 6 \times 0 \\ &= 24 \end{aligned}$$

therefore the plane equation is $4x + 12y + 6z = 24$

The table also confirms that the values of P_0 satisfy the plane equation.

ε	λ	β	x_0	y_0	z_0	$4x_0 + 12y_0 + 6z_0$
1	0	0	0	2	0	24
0	1	0	0	0	4	24
0	0	1	3	1	0	24
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$1\frac{1}{2}$	1	1	24
0	$\frac{1}{2}$	$\frac{1}{2}$	$1\frac{1}{2}$	$\frac{1}{2}$	2	24
$\frac{1}{2}$	$\frac{1}{2}$	0	0	1	2	24
$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1	1	$\frac{4}{3}$	24

2.17.2 Unknown coordinate value inside a triangle

The x and z-coordinates of a point P_0 are known and it is required to determine its y-coordinate inside the triangle P_1, P_2, P_3.

Using barycentric coordinates we have

$$y_0 = \varepsilon y_1 + \lambda y_2 + (1 - \varepsilon - \lambda) y_3$$

where
$$\frac{\varepsilon}{\begin{vmatrix} x_0 & z_0 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix}} = \frac{\lambda}{\begin{vmatrix} x_0 & z_0 & 1 \\ x_3 & z_3 & 1 \\ x_1 & z_1 & 1 \end{vmatrix}} = \frac{1}{\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix}}$$

For P_0 to be inside the triangle $(\varepsilon, \lambda) \in [0, 1]$.

If P_0 is positioned at P_1 i.e. $x_0 = z_0 = 0$, y_0 should be 2.

Therefore
$$\frac{\varepsilon}{\begin{vmatrix} 0 & 0 & 1 \\ 0 & 4 & 1 \\ 3 & 0 & 1 \end{vmatrix}} = \frac{\lambda}{\begin{vmatrix} 0 & 0 & 1 \\ 3 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 0 & 0 & 1 \\ 0 & 4 & 1 \\ 3 & 0 & 1 \end{vmatrix}}$$

and
$$\frac{\varepsilon}{-12} = \frac{\lambda}{0} = \frac{1}{-12}$$

which makes $\varepsilon = 1$ and $\lambda = 0$

therefore $y_0 = 1 \times 2 + 0 \times 0 + (1 - 1 - 0)1 = 2$ which is correct.

The table below shows the values of $\varepsilon, \lambda, 1 - \varepsilon - \lambda$ and y_0 for different values of x_0 and z_0.

Let us check that the interpolated values of P_0 reside on the plane of the triangle.

The vertices of the triangle are $P_1(0, 2, 0), P_2(0, 0, 4), P_3(3, 1, 0)$ therefore the Cartesian plane equation is $ax + by + cz = d$ (see plane equation from three points)

where

$$a = \begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix} \quad b = \begin{vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{vmatrix} \quad c = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad d = ax_1 + by_1 + cz_1$$

$$a = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 4 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 4 \quad b = \begin{vmatrix} 0 & 0 & 1 \\ 4 & 0 & 1 \\ 0 & 3 & 1 \end{vmatrix} = 12 \quad c = \begin{vmatrix} 0 & 2 & 1 \\ 0 & 0 & 1 \\ 3 & 1 & 1 \end{vmatrix} = 6 \quad \begin{aligned} d &= 4 \times 0 + 12 \times 2 + 6 \times 0 \\ &= 24 \end{aligned}$$

therefore the plane equation is $4x + 12y + 6z = 24$

x_0	y_0	z_0	ε	λ	$1-\varepsilon-\lambda$	$4x+12y+6z$
0	2	0	1	0	0	24
3	1	0	0	0	1	24
0	0	4	0	1	0	24
1	$\frac{2}{3}$	2	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$	24
2	$\frac{5}{6}$	1	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{2}{3}$	24
1	$\frac{7}{6}$	1	$\frac{5}{12}$	$\frac{1}{4}$	$\frac{1}{3}$	24

The table below also confirms that the above values of P_0 satisfy the plane equation. Let us test a point outside the triangle's boundary, e.g. $P_0(4,0,0)$

$$\frac{\varepsilon}{\begin{vmatrix} 4 & 0 & 1 \\ 0 & 4 & 1 \\ 3 & 0 & 1 \end{vmatrix}} = \frac{\lambda}{\begin{vmatrix} 4 & 0 & 1 \\ 3 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 0 & 0 & 1 \\ 0 & 4 & 1 \\ 3 & 0 & 1 \end{vmatrix}}$$

$$\frac{\varepsilon}{4} = \frac{\lambda}{0} = \frac{1}{-12}$$

therefore $\varepsilon = -\frac{1}{3}$

which confirms that P_0 is outside the triangle's boundary.

Similarly, for $P_0(0,0,5)$

$$\frac{\varepsilon}{\begin{vmatrix} 0 & 5 & 1 \\ 0 & 4 & 1 \\ 3 & 0 & 1 \end{vmatrix}} = \frac{\lambda}{\begin{vmatrix} 0 & 5 & 1 \\ 3 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 0 & 0 & 1 \\ 0 & 4 & 1 \\ 3 & 0 & 1 \end{vmatrix}}$$

$$\frac{\varepsilon}{3} = \frac{\lambda}{-15} = \frac{1}{-12}$$

therefore $\varepsilon = -\frac{1}{4}$ and $\lambda = 1\frac{1}{4}$

which confirms that P_0 is also outside the triangle's boundary.

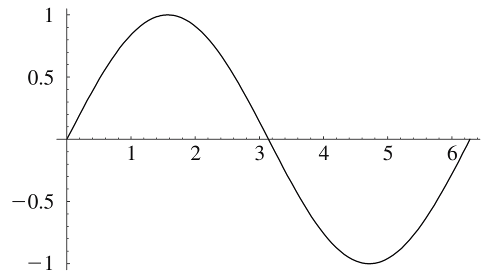
2.18 Parametric curves and patches

The following examples illustrate how various curves can be created by mixing together different parametric functions.

2.18.1 Parametric curves in $\mathbb{R}^2$

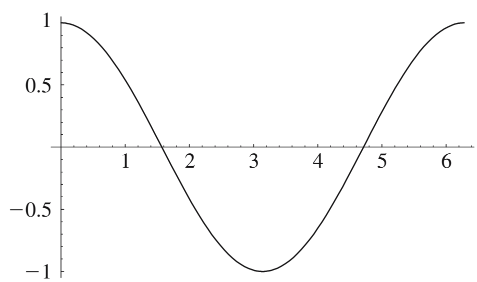
Sine curve

$$t_{\max} = 2\pi$$
$$a = 1$$
$$\left.\begin{aligned} x &= t \\ y &= a \sin t \end{aligned}\right\} \quad t \in [0,\ t_{\max}]$$



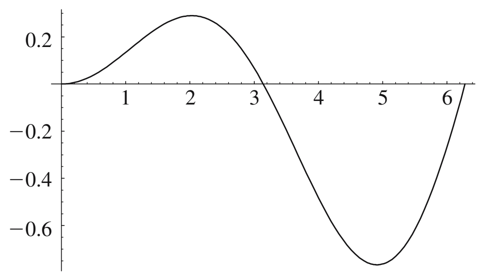
Cosine curve

$$t_{\max} = 2\pi$$
$$a = 1$$
$$\left.\begin{aligned} x &= t \\ y &= a \cos t \end{aligned}\right\} \quad t \in [0,\ t_{\max}]$$



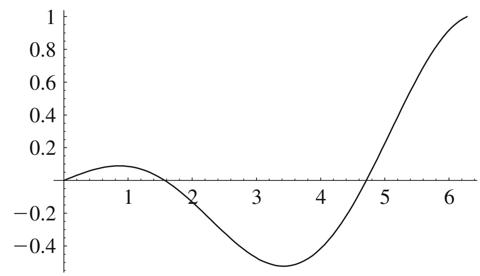
Sine curve with growing amplitude

$$t_{\max} = 2\pi$$
$$\left.\begin{aligned} a &= \frac{t}{t_{\max}} \\ x &= t \\ y &= a \sin t \end{aligned}\right\} \quad t \in [0,\ t_{\max}]$$



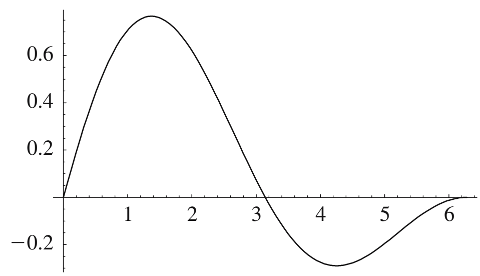
Cosine curve with growing amplitude

$$t_{\max} = 2\pi$$
$$\left.\begin{aligned} a &= \frac{t}{t_{\max}} \\ x &= t \\ y &= a\cos t \end{aligned}\right\} \quad t \in [0,\ t_{\max}]$$



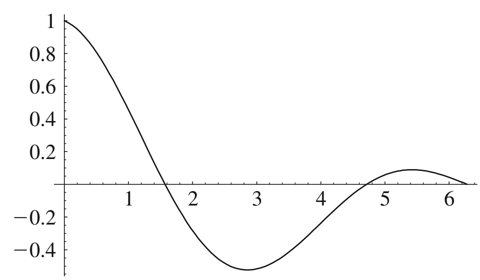
Sine curve with decaying amplitude

$$t_{\max} = 2\pi$$
$$\left.\begin{aligned} a &= 1 - \frac{t}{t_{\max}} \\ x &= t \\ y &= a\sin t \end{aligned}\right\} \quad t \in [0,\ t_{\max}]$$



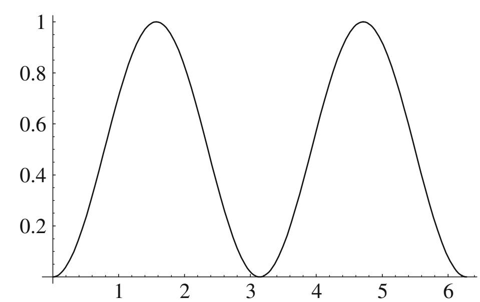
Cosine curve with decaying amplitude

$$t_{\max} = 2\pi$$
$$\left.\begin{aligned} a &= 1 - \frac{t}{t_{\max}} \\ x &= t \\ y &= a\cos t \end{aligned}\right\} \quad t \in [0,\ t_{\max}]$$



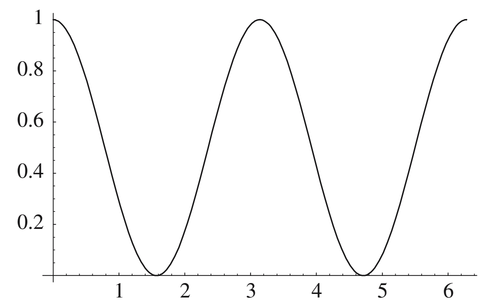
Sine-squared curve

$$t_{\max} = 2\pi$$
$$a = 1$$
$$\left.\begin{aligned} x &= t \\ y &= a\sin^2 t \end{aligned}\right\} \quad t \in [0,\ t_{\max}]$$



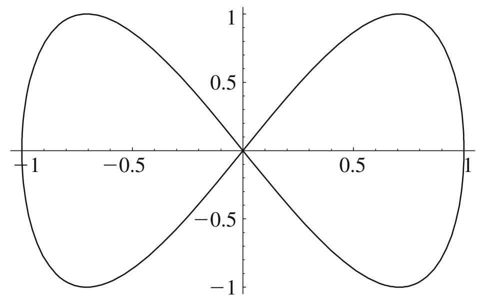
Cosine-squared curve

$$t_{max} = 2\pi$$
$$a = 1$$
$$\left.\begin{aligned} x &= t \\ y &= a\cos^2 t \end{aligned}\right\} \quad t \in [0,\ t_{max}]$$



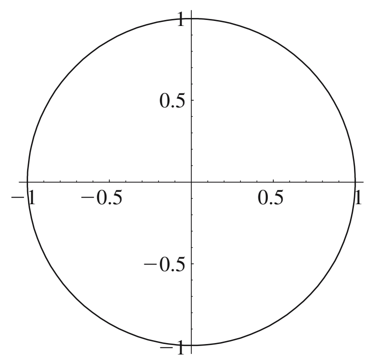
Lissajous curve

$$t_{max} = 2\pi$$
$$a = 1$$
$$\left.\begin{aligned} x &= a\sin t \\ y &= a\sin 2t \end{aligned}\right\} \quad t \in [0,\ t_{max}]$$



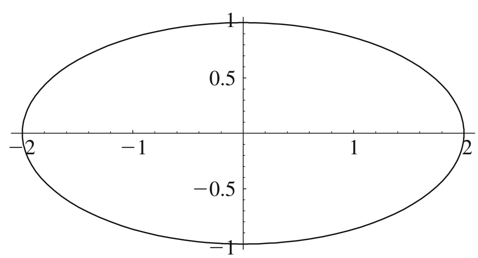
Circle

$$t_{max} = 2\pi$$
$$a = 1$$
$$\left.\begin{aligned} x &= a\cos t \\ y &= a\sin t \end{aligned}\right\} \quad t \in [0,\ t_{max}]$$



Ellipse

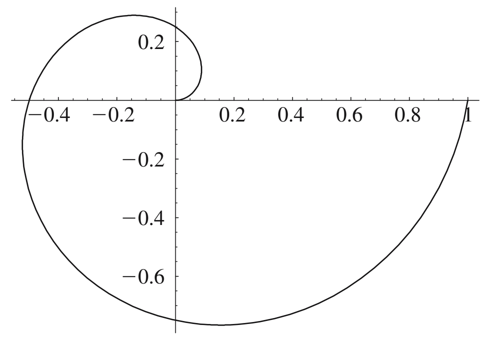
$$t_{max} = 2\pi$$
$$a = 2$$
$$b = 1$$
$$\left.\begin{aligned} x &= a\cos t \\ y &= b\sin t \end{aligned}\right\} \quad t \in [0,\ t_{max}]$$



Spiral

$$t_{\max} = 2\pi$$

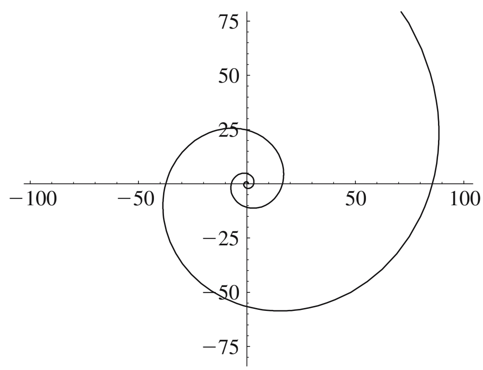
$$\left.\begin{aligned} r &= \frac{t}{t_{\max}} \\ x &= r\cos t \\ y &= r\sin t \end{aligned}\right\} \quad t \in [0,\ t_{\max}]$$



Logarithmic spiral

$$t_{\max} = 2\pi$$
$$a = 0.6$$
$$b = 3.8$$

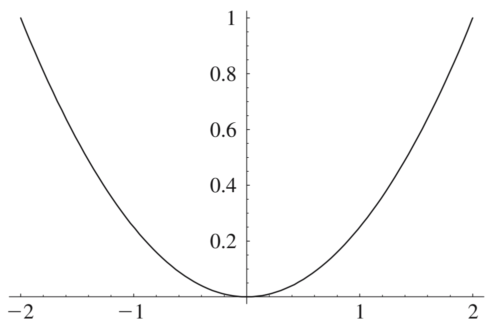
$$\left.\begin{aligned} x &= ae^{t}\cos bt \\ y &= ae^{t}\sin bt \end{aligned}\right\} \quad t \in [0,\ t_{\max}]$$



Parabola

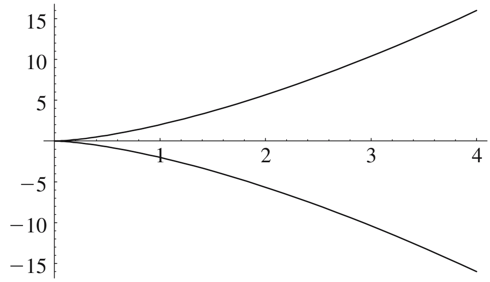
$$t_{\max} = 4$$
$$p = 2$$

$$\left.\begin{aligned} x &= t \\ y &= \frac{1}{2p}t^{2} \end{aligned}\right\} \quad t \in \left[-\frac{t_{\max}}{2},\ \frac{t_{\max}}{2}\right]$$



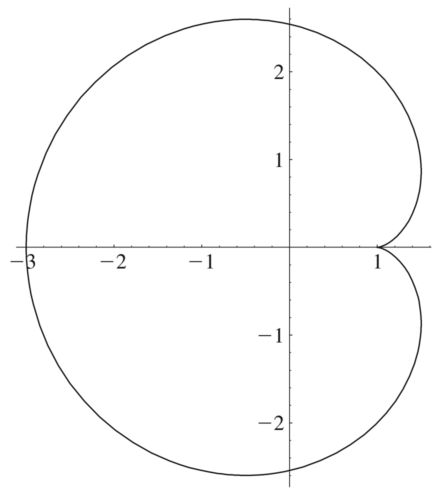
Neil's parabola

$$t_{max} = 4$$
$$a = 2$$
$$\left.\begin{aligned} x &= t^2 \\ y &= at^3 \end{aligned}\right\} \quad t \in \left[-\frac{t_{max}}{2},\ \frac{t_{max}}{2}\right]$$



Cardioid

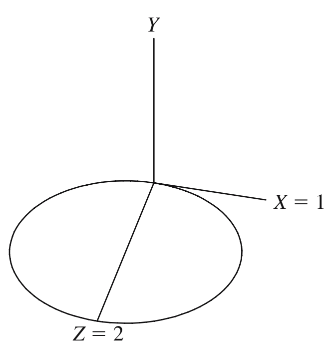
$$t_{max} = 2\pi$$
$$a = 1$$
$$\left.\begin{aligned} x &= a(2\cos t - \cos 2t) \\ y &= a(2\sin t - \sin 2t) \end{aligned}\right\} \quad t \in [0,\ t_{max}]$$



2.18.2 Parametric curves in $\mathbb{R}^3$

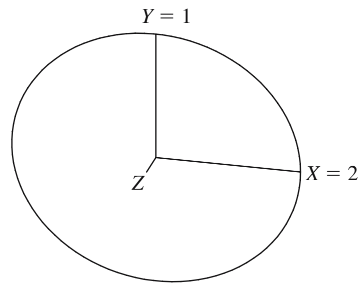
Circle

$$t_{max} = 2\pi$$
$$a = 1$$
$$\left.\begin{aligned} x &= a\cos t \\ y &= 0 \\ z &= a + a\sin t \end{aligned}\right\} \quad t \in [0,\ t_{max}]$$



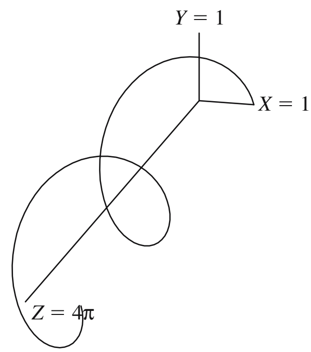
Ellipse

$$t_{\max} = 2\pi$$
$$a = 2$$
$$b = 1$$
$$\left.\begin{aligned} x &= a\cos t \\ y &= b\sin t \\ z &= 0 \end{aligned}\right\} \quad t \in [0,\ t_{\max}]$$



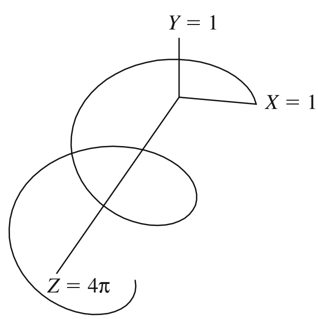
Spiral 1

$$t_{\max} = 4\pi$$
$$a = 1$$
$$\left.\begin{aligned} x &= a\cos t \\ y &= a\sin t \\ z &= t \end{aligned}\right\} \quad t \in [0,\ t_{\max}]$$



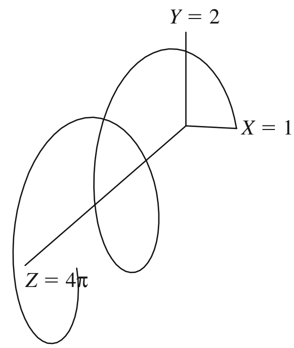
Spiral 2

$$t_{\max} = 4\pi$$
$$a = 2$$
$$b = 1$$
$$\left.\begin{aligned} x &= a\cos t \\ y &= b\sin t \\ z &= t \end{aligned}\right\} \quad t \in [0,\ t_{\max}]$$



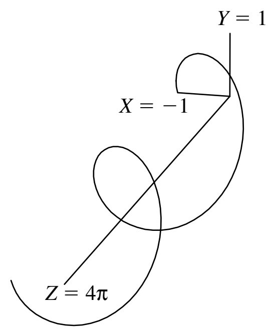
Spiral 3

$$t_{\max} = 4\pi$$
$$a = 1$$
$$b = 2$$
$$\left.\begin{aligned} x &= a\cos t \\ y &= b\sin t \\ z &= t \end{aligned}\right\} \quad t \in [0,\ t_{\max}]$$



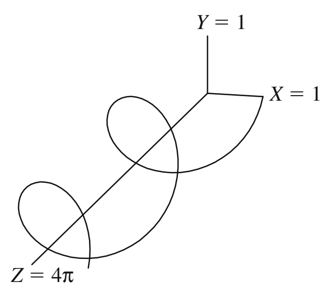
Spiral 4

$$t_{\max} = 4\pi$$
$$a = 1$$
$$\left.\begin{aligned} x &= -a\cos t \\ y &= a\sin t \\ z &= t \end{aligned}\right\} \quad t \in [0,\ t_{\max}]$$



Spiral 5

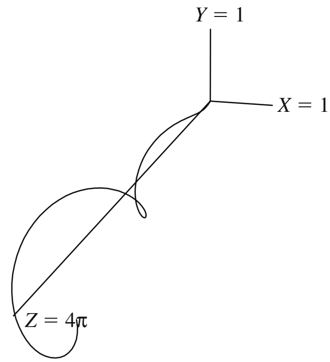
$$t_{\max} = 4\pi$$
$$a = 1$$
$$\left.\begin{aligned} x &= a\cos t \\ y &= -a\sin t \\ z &= t \end{aligned}\right\} \quad t \in [0,\ t_{\max}]$$



Spiral 6

$$t_{\max} = 4\pi$$

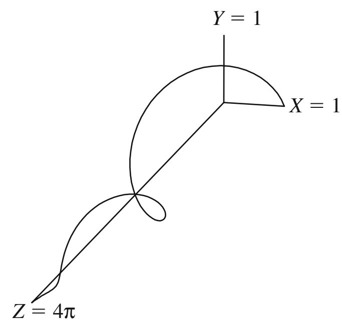
$$\left.\begin{aligned} r &= \frac{t}{t_{\max}} \\ x &= r\cos t \\ y &= r\sin t \\ z &= t \end{aligned}\right\} \quad t \in [0,\ t_{\max}]$$



Spiral 7

$$t_{\max} = 4\pi$$

$$\left.\begin{aligned} r &= 1 - \frac{t}{t_{\max}} \\ x &= r\cos t \\ y &= r\sin t \\ z &= t \end{aligned}\right\} \quad t \in [0,\ t_{\max}]$$

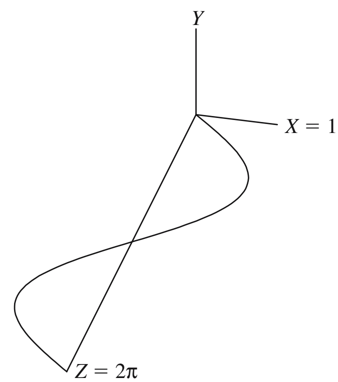


Sinusoid

$$t_{\max} = 2\pi$$

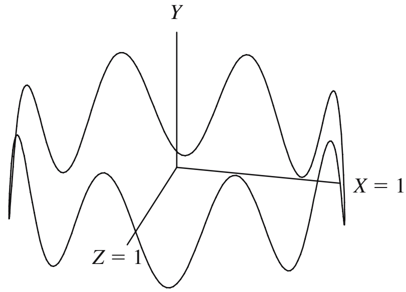
$$a = 1$$

$$\left.\begin{aligned} x &= a\sin t \\ y &= 0 \\ z &= t \end{aligned}\right\} \quad t \in [0,\ t_{\max}]$$



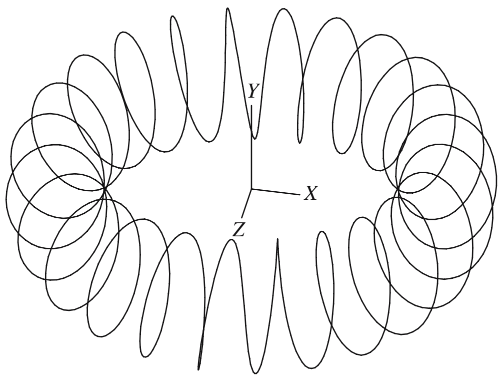
Sinusoidal ring

$$t_{max} = 2\pi$$
$$a = 1$$
$$b = 0.2$$
$$n = 8$$
$$\left.\begin{aligned} x &= a\cos t \\ y &= b\sin nt \\ z &= a\sin t \end{aligned}\right\} \quad t \in [0,\ t_{max}]$$



Coiled ring

$$t_{max} = 2\pi$$
$$R = 2 \quad \text{(major radius)}$$
$$r = 0.5 \quad \text{(minor radius)}$$
$$n = 24$$
$$\left.\begin{aligned} x &= (R + r\cos nt)\cos t \\ y &= r\sin nt \\ z &= -(R + r\cos nt)\sin t \end{aligned}\right\} \quad t \in [0,\ t_{max}]$$



2.18.3 Planar patch

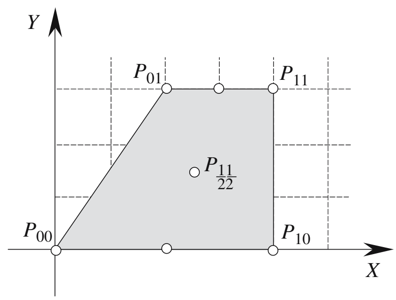
Given $P_{00}, P_{10}, P_{11}, P_{01}$ in $\mathbb{R}^2$

$$P_{uv} = \begin{bmatrix} u & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v \\ 1 \end{bmatrix}$$

Given $P_{00}(0,0), P_{01}(2,3), P_{11}(4,3), P_{10}(4,0)$

$$x_{\frac{1}{2}\frac{1}{2}} = \begin{bmatrix} \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} = 2\tfrac{1}{2}$$

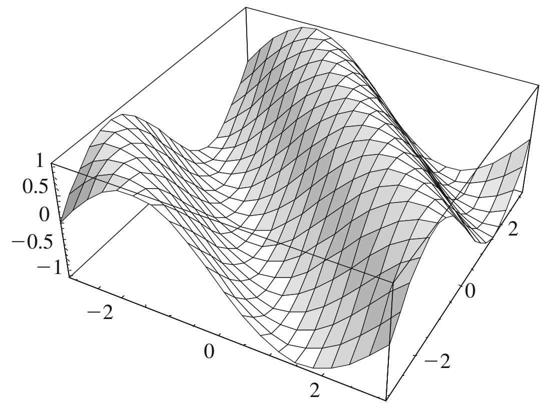
$$y_{\frac{1}{2}\frac{1}{2}} = \begin{bmatrix} \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} = 1\tfrac{1}{2}$$



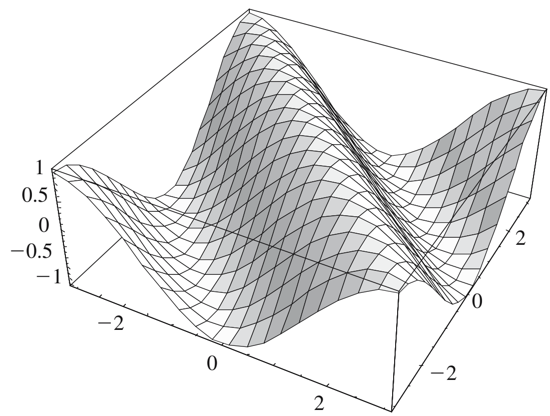
2.18.4 Parametric surfaces in $\mathbb{R}^3$

Modulated surface

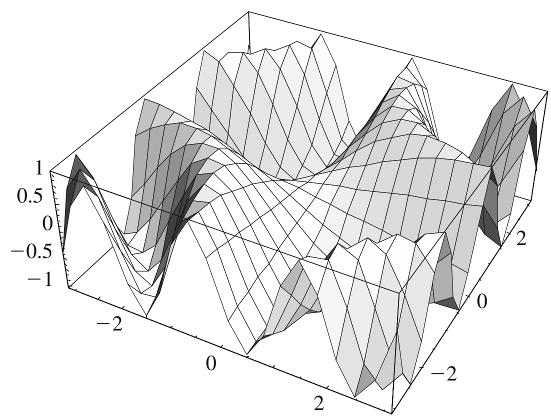
$y = \sin(x + z)$
$T = \pi$
$a = 1$
$y = a\sin(x + z)\}\quad (x, z) \in [-T,\ T]$



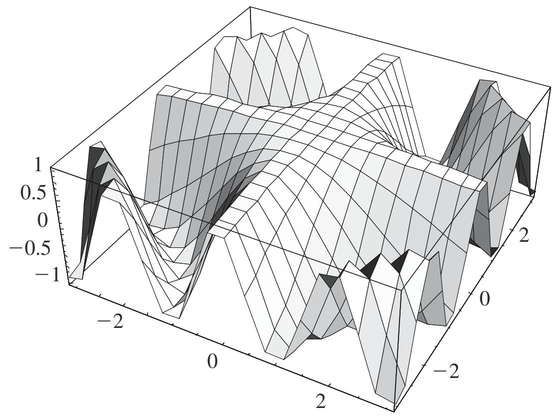
$y = \cos(x + z)$
$T = \pi$
$a = 1$
$y = a\cos(x + z)\}\quad (x, z) \in [-T,\ T]$



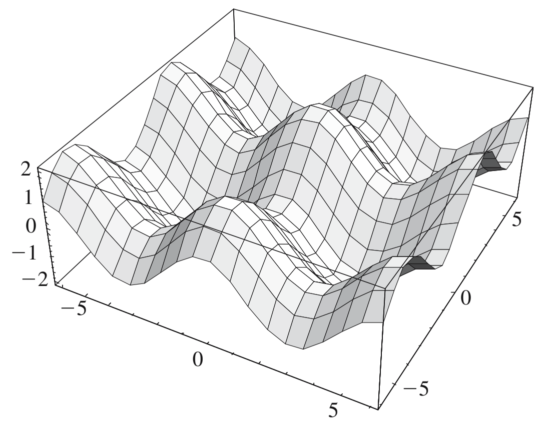
$y = \sin(xz)$
$T = \pi$
$a = 1$
$y = a\sin(xz)\}\quad (x, z) \in [-T,\ T]$



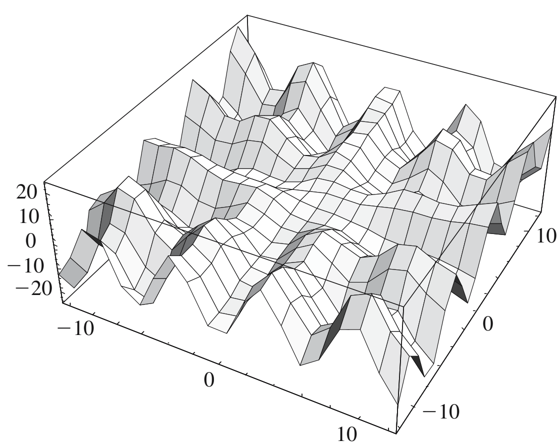
$y = \cos(xz)$
$T = \pi$
$a = 1$
$y = a\cos(xz)\}\quad (x, z) \in [-T,\ T]$



$y = \cos x + \sin z$
$T = 2\pi$
$a = 1$
$y = a\cos x + a\sin z\}\quad (x, z) \in [-T,\ T]$



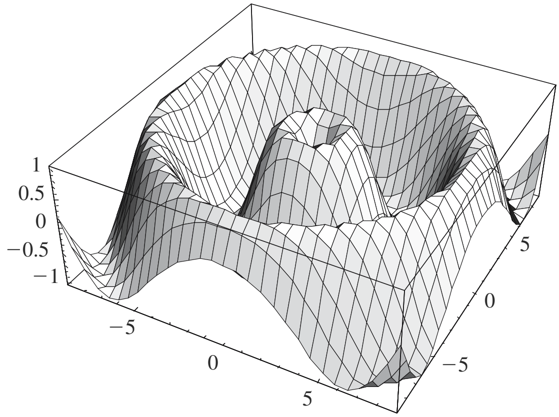
$y = z\cos x + x\sin z$
$T = 4\pi$
$y = z\cos x + x\sin z\}\quad (x, z) \in [-T,\ T]$



$\sin\left(\sqrt{x^2+y^2}\right)$

$T = 9$

$\sin\left(\sqrt{x^2+y^2}\right)\Big\}\quad (x, y) \in [-T, T]$



2.18.5 Quadratic Bézier curve

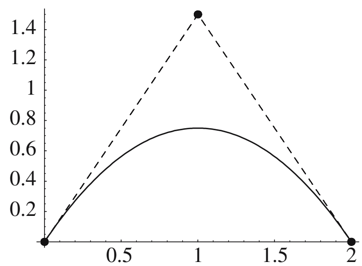
Quadratic Bézier curve in $\mathbb{R}^2$

A quadratic Bézier curve is given by

$\mathbf{p}(t) = (1-t)^2\,\mathbf{p}_1 + 2t(1-t)\,\mathbf{p}_C + t^2\mathbf{p}_2$

Given the points $P_1(0,0), P_C(1,1.5), P_2(2,0)$

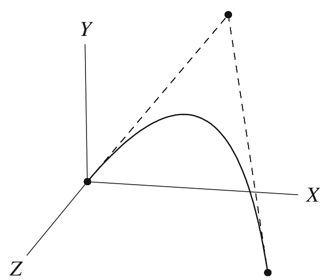
the quadratic Bézier curve is shown with its control points.



Quadratic Bézier curve in $\mathbb{R}^3$

Given the points $P_1(0,0,0), P_C(2,2.5,0), P_2(3,0,3)$

the quadratic Bézier curve is shown with its control points.



2.18.6 Cubic Bézier curve

Cubic Bézier curve in $\mathbb{R}^2$

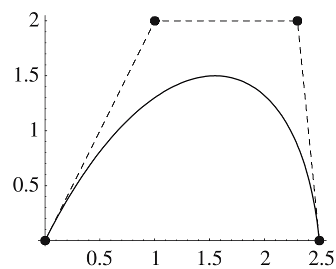
A cubic Bézier curve is given by

$\mathbf{p}(t) = (1-t)^3\,\mathbf{p}_1 + 3t(1-t)^2\,\mathbf{p}_{C1} + 3t^2\,(1-t)\,\mathbf{p}_{C2} + t^3\mathbf{p}_2$

Given the points

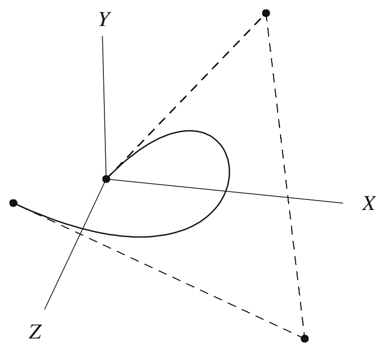
$P_1(0,0), P_{C1}(1,2), P_{C2}(2.3,2), P_2(2.5,0)$

the cubic Bézier curve is shown with its control points.



Cubic Bézier curve in $\mathbb{R}^3$

Given the points $P_1(0,0,0), P_{C1}(2,2.5,0), P_{C2}(3,0,3), P_2(0,2,4)$ the cubic Bézier curve is shown with its control points.



2.18.7 Quadratic Bézier patch

A quadratic surface patch is described by

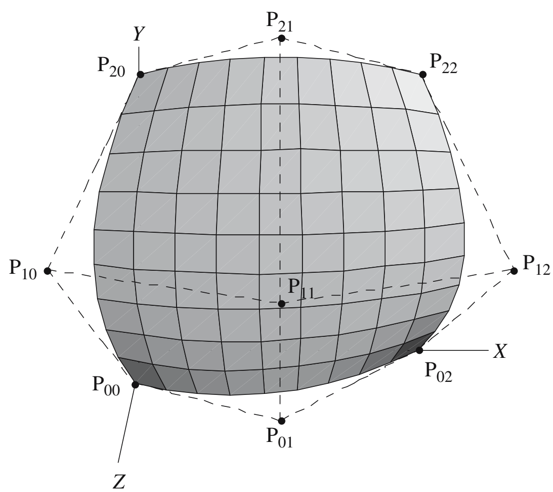
$$\mathbf{p}(u,v) = [(1-u)^2 \quad 2u(1-u) \quad u^2]\begin{bmatrix} \mathbf{p}_{00} & \mathbf{p}_{01} & \mathbf{p}_{02} \\ \mathbf{p}_{10} & \mathbf{p}_{11} & \mathbf{p}_{12} \\ \mathbf{p}_{20} & \mathbf{p}_{21} & \mathbf{p}_{22} \end{bmatrix}\begin{bmatrix} (1-v)^2 \\ 2v(1-v) \\ v^2 \end{bmatrix}$$

Given $\quad \mathbf{p}_{00} = (0,0,1) \quad \mathbf{p}_{01} = (1,0,2) \quad \mathbf{p}_{02}\,(2,0,0)$

$\mathbf{p}_{10} = (-0.5,\ 1,2) \quad \mathbf{p}_{11} = (1,\ 1,3) \quad \mathbf{p}_{12}(2\tfrac{1}{2},\ 1,2)$

$\mathbf{p}_{20} = (0,2\tfrac{1}{2},0) \quad \mathbf{p}_{21} = (1,2\tfrac{1}{2},2) \quad \mathbf{p}_{22}(2,2,0)$

The surface patch is shown in the diagram



2.18.8 Cubic Bézier patch

A cubic surface patch is described by

$$\mathbf{p}(u,v) = [(1-u)^3 \quad 3u(1-u)^2 \quad 3u^2(1-u) \quad u^3] \begin{bmatrix} \mathbf{p}_{00} & \mathbf{p}_{01} & \mathbf{p}_{02} & \mathbf{p}_{03} \\ \mathbf{p}_{10} & \mathbf{p}_{11} & \mathbf{p}_{12} & \mathbf{p}_{13} \\ \mathbf{p}_{20} & \mathbf{p}_{21} & \mathbf{p}_{22} & \mathbf{p}_{23} \\ \mathbf{p}_{30} & \mathbf{p}_{31} & \mathbf{p}_{32} & \mathbf{p}_{33} \end{bmatrix} \begin{bmatrix} (1-v)^3 \\ 3v(1-v)^2 \\ 3v^2(1-v) \\ v^3 \end{bmatrix}$$

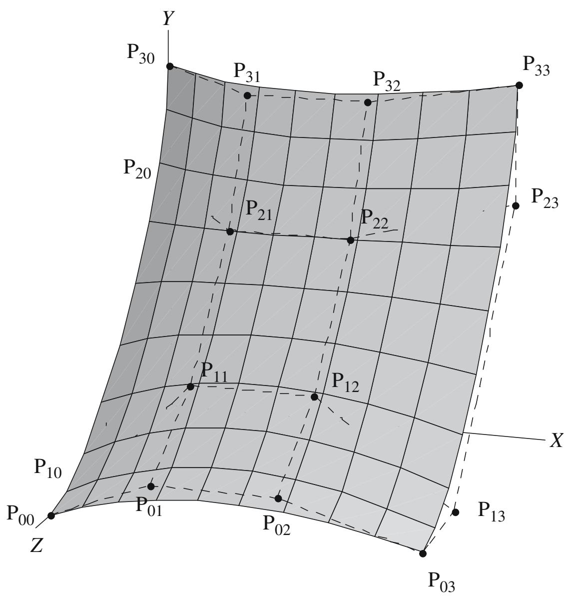
Given $\mathbf{p}_{00} = (0,0,3) \quad \mathbf{p}_{01} = (1, \frac{1}{2}, 3\frac{1}{2}) \quad \mathbf{p}_{02}(2, \frac{1}{2}, 3\frac{1}{2}) \quad \mathbf{p}_{03}(3,0,3)$

$\mathbf{p}_{10} = (0,0,2) \quad \mathbf{p}_{11} = (1,\ 1, 2\frac{1}{2}) \quad \mathbf{p}_{12}(2,\ 1, 2\frac{1}{2}) \quad \mathbf{p}_{13}(3,0,2)$

$\mathbf{p}_{20} = (0,2,0) \quad \mathbf{p}_{21} = (1,2,1\frac{1}{2}) \quad \mathbf{p}_{22}(2,2,1\frac{1}{2}) \quad \mathbf{p}_{23}(3,2,0)$

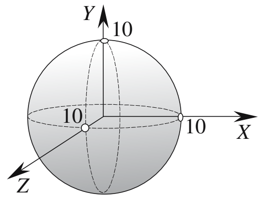
$\mathbf{p}_{30} = (0,3,0) \quad \mathbf{p}_{31} = (1,3,\ 1) \quad \mathbf{p}_{32}(2,3,\ 1) \quad \mathbf{p}_{33}(3,3,0)$

The surface patch is shown in the diagram



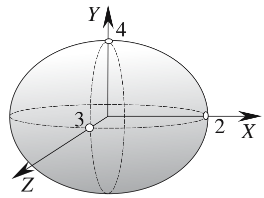
2.19 Second degree surfaces in standard form

Sphere



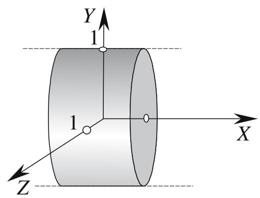
$$x^2 + y^2 + z^2 = 100$$

Ellipsoid



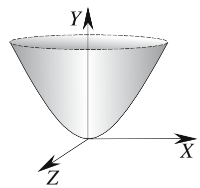
$$\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{9} = 1$$

Elliptic cylinder



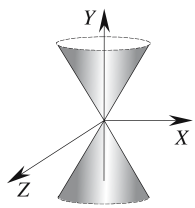
$$y^2 + z^2 = 1$$

Elliptic paraboloid



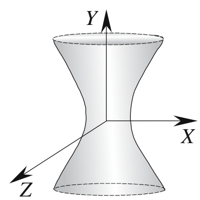
$$x^2 + z^2 = y$$

Elliptic cone



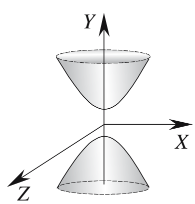
$$x^2 + z^2 = y^2$$

Elliptic hyperboloid of one sheet

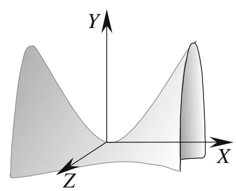


$$x^2 + z^2 = 1 + y^2$$

Elliptic hyperboloid of two sheets



$$x^2 + z^2 = y^2 - 1$$



$$y = x^2 - z^2$$

3 Proofs

We must never assume that which is incapable of proof.
G.H. Lewes (1817–1878)

This third section is divided into 18 groups:

3.1 Trigonometry
3.2 Circles
3.3 Triangles
3.4 Quadrilaterals
3.5 Polygons
3.6 Three-dimensional objects
3.7 Coordinate systems
3.8 Vectors
3.9 Quaternions
3.10 Transformations
3.11 Two-dimensional straight lines
3.12 Lines and circles
3.13 Second degree curves
3.14 Three-dimensional straight lines
3.15 Planes
3.16 Lines, planes and spheres
3.17 Three-dimensional triangles
3.18 Parametric curves and patches

Not everyone will be interested in why a formula has a particular form. For some, all that matters is that it provides the correct numerical result. However, students and academics may have other interests – they may be interested in the origins of the formula and the strategy used in its derivation.

Some formulas are extremely simple and are readily derived using the sine rule or cosine rule. Others are much more subtle and require techniques such as completing the square, recognizing ratios in virtual triangles, substituting trigonometric or vector formulas to simplify the current status of the formula.

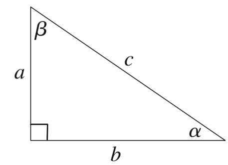
What is apparent from these proofs is that deriving a proof is not always obvious. Remember, that it took Sir William Rowan Hamilton over a decade to crack the non-commutative rules behind quaternions; yet today, any student can be taught the ideas behind vectors and quaternions in one or two hours. Therefore readers should not be surprised how easy it is to prove that $1 = 1$, even after working through several pages of complex algebra! Such dead ends are often due to working with statements that are linearly related in some way.

In many of the proofs involving vectors, a vector equation is derived which reflects a geometric condition. By itself, this equation is unable to reveal an answer, but by taking the scalar product of its terms with a suitable vector, the equation is simplified because the dot product of a critical pair of vectors is known to be zero. This is a very powerful problem-solving technique and should be remembered by the reader.

The following proofs are the heart of this book. They may not always reveal the most elegant route to the final result, and if the reader can discover a more elegant strategy, hopefully they will derive pleasure in the process, which is what mathematics should be about.

3.1 Trigonometry

3.1.1 Trigonometric functions and identities



By definition
$$\sin\alpha = \frac{a}{c}$$
$$\cos\alpha = \frac{b}{c}$$
$$\tan\alpha = \frac{a}{b} = \frac{ac}{cb} = \frac{\sin\alpha}{\cos\alpha}$$

similarly
$$\cot\alpha = \frac{b}{a} = \frac{bc}{ca} = \frac{\cos\alpha}{\sin\alpha}$$

$$\boxed{\tan\alpha = \frac{\sin\alpha}{\cos\alpha}} \quad \boxed{\cot\alpha = \frac{\cos\alpha}{\sin\alpha}}$$

3.1.2 Cofunction identities

$$\sin\alpha = \frac{a}{c} = \cos\beta$$
$$\cos\alpha = \frac{b}{c} = \sin\beta$$
$$\tan\alpha = \frac{a}{b} = \cot\beta$$
$$\csc\alpha = \frac{c}{a} = \sec\beta$$
$$\sec\alpha = \frac{c}{b} = \csc\beta$$
$$\cot\alpha = \frac{b}{a} = \tan\beta$$

3.1.3 Pythagorean identities

$$a^2 + b^2 = c^2 \tag{1}$$

Divide (1) by c^2
$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{c^2}{c^2} = 1$$

therefore
$$\sin^2\alpha + \cos^2\alpha = 1 \tag{2}$$

Divide (2) by $\cos^2\alpha$
$$\frac{\sin^2\alpha}{\cos^2\alpha} + \frac{\cos^2\alpha}{\cos^2\alpha} = \frac{1}{\cos^2\alpha}$$

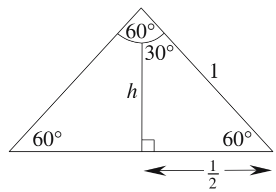
therefore
$$1 + \tan^2\alpha = \sec^2\alpha$$

Divide (2) by $\sin^2\alpha$ $$\frac{\sin^2\alpha}{\sin^2\alpha}+\frac{\cos^2\alpha}{\sin^2\alpha}=\frac{1}{\sin^2\alpha}=\csc^2\alpha$$

therefore $$1+\cot^2\alpha=\csc^2\alpha$$

$\sin^2\alpha+\cos^2\alpha=1$	$1+\tan^2\alpha=\sec^2\alpha$	$1+\cot^2\alpha=\csc^2\alpha$

3.1.4 Useful trigonometric values



sin 30° cos 30° tan 30°

Pythagoras $$h^2+\left(\tfrac{1}{2}\right)^2=1^2$$

$$h=\sqrt{1-\tfrac{1}{4}}=\tfrac{1}{2}\sqrt{3}$$

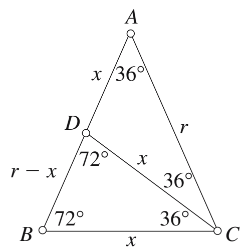
$$\sin 30°=\tfrac{1}{2}=\cos 60°$$

$$\cos 30°=\tfrac{1}{2}\sqrt{3}=\sin 60°$$

$$\tan 30°=\tfrac{1}{3}\sqrt{3}$$

and $$\tan 60°=\sqrt{3}$$

sin 36° cos 36° tan 36°



Given

$\triangle ABC$ is isosceles, therefore $AC=AB=r$

$\triangle BCD$ is isosceles, therefore $CD=x$

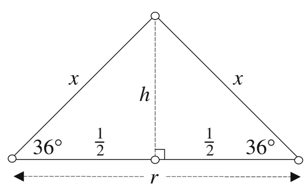
$\triangle DAC$ is isosceles, therefore $DA=x$ and $BD=r-x$

$\triangle CBD$ is similar to $\triangle ACB$, therefore

$$\frac{x}{r}=\frac{r-x}{x}$$

$$x^2+xr-r^2=0$$

which has roots $$x=\tfrac{1}{2}(-r\pm\sqrt{r^2+4})$$



Let $r=1$ $$x=\tfrac{1}{2}(-1\pm\sqrt{5})$$

$$\cos 36°=\frac{\frac{1}{2}}{x}=\frac{1}{-1+\sqrt{5}}$$

$$\cos 36°=\frac{1}{-1+\sqrt{5}}\frac{(-1-\sqrt{5})}{(-1-\sqrt{5})}=\frac{1+\sqrt{5}}{4}$$

$$\cos 36°=\frac{1+\sqrt{5}}{4}=\sin 54°$$

but $$\sin^2 36° + \cos^2 36° = 1$$

$$\sin 36° = \sqrt{1 - \cos^2 36°}$$

$$\sin 36° = \sqrt{1 - \left(\frac{1+\sqrt{5}}{4}\right)^2}$$

$$\sin 36° = \frac{\sqrt{10 - 2\sqrt{5}}}{4} = \cos 54°$$

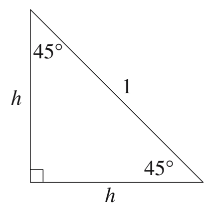
$$\tan 36° = \frac{\sin 36°}{\cos 36°} = \frac{\sqrt{10-2\sqrt{5}}}{1+\sqrt{5}}$$

$$\tan 36° = \sqrt{5 - 2\sqrt{5}}$$

$$\tan 54° = \frac{\sin 54°}{\cos 54°} = \frac{1+\sqrt{5}}{\sqrt{10-2\sqrt{5}}}$$

$$\tan 54° = \sqrt{\frac{5+2\sqrt{5}}{5}}$$

sin 45° cos 45° tan 45°



$$h^2 + h^2 = 1$$

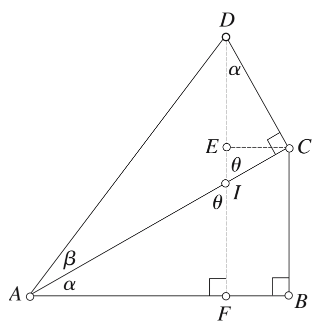
$$h = \sqrt{\tfrac{1}{2}} = \tfrac{1}{2}\sqrt{2}$$

$$\sin 45° = \frac{h}{1} = \tfrac{1}{2}\sqrt{2}$$

$$\cos 45° = \frac{h}{1} = \tfrac{1}{2}\sqrt{2}$$

$$\tan 45° = \frac{h}{h} = 1$$

3.1.5 Compound angle identities



$\triangle ABC$ and $\triangle ACD$ are right-angled triangles

DF is perpendicular to AB

EC is parallel to AB

therefore $BC = FE$ and $EC = FB$

$\angle AIF = \angle DIC \quad \therefore \angle IDC = \alpha$

sin($\boldsymbol{\alpha \pm \beta}$)

$$\frac{BC}{AC} = \frac{FE}{AC} = \sin\alpha \quad \therefore FE = AC\sin\alpha$$

$$\frac{ED}{DC} = \cos\alpha \qquad ED = DC\cos\alpha$$

$$\frac{AC}{AD} = \cos\beta \quad \text{and} \quad \frac{DC}{AD} = \sin\beta$$

$$\sin(\alpha+\beta) = \frac{FD}{AD} = \frac{FE}{AD} + \frac{ED}{AD} = \sin\alpha\frac{AC}{AD} + \cos\alpha\frac{DC}{AD}$$

$$\boxed{\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta}$$

If β is negative

$$\sin(\alpha-\beta) = \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta)$$

$$\boxed{\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta}$$

cos($\boldsymbol{\alpha \pm \beta}$)

$$\frac{AB}{AC} = \cos\alpha \qquad \therefore AB = \cos\alpha\, AC$$

$$\frac{EC}{DC} = \sin\alpha \qquad \therefore EC = \sin\alpha\, DC$$

$$\frac{AC}{AD} = \cos\beta \quad \text{and} \quad \frac{DC}{AD} = \sin\beta$$

$$\cos(\alpha+\beta) = \frac{AF}{AD} = \frac{AB}{AD} - \frac{EC}{AD} = \cos\alpha\frac{AC}{AD} - \sin\alpha\frac{DC}{AD}$$

$$\boxed{\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

If β is negative

$$\cos(\alpha-\beta) = \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta)$$

$$\boxed{\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta}$$

tan($\boldsymbol{\alpha \pm \beta}$)

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} \tag{1}$$

Divide (1) by $\cos\alpha\cos\beta$

$$\boxed{\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}}$$

If β is negative

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan(-\beta)}{1 + \tan\alpha\tan(-\beta)}$$

$$\boxed{\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}}$$

cot(α ± β) $$\cot(\alpha+\beta)=\frac{\cos(\alpha+\beta)}{\sin(\alpha+\beta)}=\frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\sin\alpha\cos\beta+\cos\alpha\sin\beta} \tag{2}$$

Divide (2) by $\sin\alpha\sin\beta$

$$\boxed{\cot(\alpha+\beta)=\frac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}}$$

If β is negative $$\cot(\alpha-\beta)=\frac{\cot\alpha\cot(-\beta)+1}{\cot\alpha-\cot(-\beta)}$$

$$\boxed{\cot(\alpha-\beta)=\frac{\cot\alpha\cot\beta+1}{\cot\alpha-\cot\beta}}$$

3.1.6 Double-angle identities

Substituting $\beta=\alpha$ in the compound angle identities produces

$$\boxed{\sin 2\alpha = 2\sin\alpha\cos\alpha}$$

$$\boxed{\cos 2\alpha = 1-2\sin^2\alpha}$$

but $\cos^2\alpha+\sin^2\alpha=1$ $\quad\therefore \cos 2\alpha=\cos^2\alpha-\sin^2\alpha$

$$\boxed{\tan 2\alpha=\frac{2\tan\alpha}{1-\tan^2\alpha}}$$

$$\boxed{\cot 2\alpha=\frac{\cot^2\alpha-1}{2\cot\alpha}}$$

3.1.7 Multiple-angle identities

Letting β equal multiples of α in the compound-angle identities produces

$$\boxed{\sin 3\alpha = 3\sin\alpha-4\sin^3\alpha}$$

$$\boxed{\cos 3\alpha = 4\cos^3\alpha-3\cos\alpha}$$

$$\boxed{\tan 3\alpha=\frac{3\tan\alpha-\tan^3\alpha}{1-3\tan^2\alpha}}$$

$$\boxed{\cot 3\alpha=\frac{\cot^3\alpha-3\cot\alpha}{3\cot^2\alpha-1}}$$

$$\boxed{\sin 4\alpha = 4\sin\alpha\cos\alpha-8\sin^3\alpha\cos\alpha}$$

$\cos 4\alpha = 8\cos^4\alpha - 8\cos^2\alpha + 1$
$\tan 4\alpha = \dfrac{4\tan\alpha - 4\tan^3\alpha}{1 - 6\tan^2\alpha + \tan^4\alpha}$
$\cot 4\alpha = \dfrac{\cot^4\alpha - 6\cot^2\alpha + 1}{4\cot^3\alpha - 4\cot\alpha}$
$\sin 5\alpha = 16\sin^5\alpha - 20\sin^3\alpha + 5\sin\alpha$
$\cos 5\alpha = 16\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha$
$\tan 5\alpha = \dfrac{5\tan\alpha - 10\tan^3\alpha + \tan^5\alpha}{1 - 10\tan^2\alpha + 5\tan^4\alpha}$
$\cot 5\alpha = \dfrac{\cot^5\alpha - 10\cot^3\alpha + 5\cot\alpha}{5\cot^4\alpha - 10\cot^2\alpha + 1}$

3.1.8 Functions of the half-angle

$\boldsymbol{\sin\dfrac{\alpha}{2}}$

Double-angle identity $\qquad \cos 2\alpha = 1 - 2\sin^2\alpha$

$$\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2}$$

$$\sin^2\frac{\alpha}{2} = \frac{1 - \cos\alpha}{2}$$

$$\boxed{\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos\alpha}{2}}}$$

$\boldsymbol{\cos\dfrac{\alpha}{2}}$

Double-angle identity $\qquad \cos 2\alpha = 1 - 2\sin^2\alpha$

$$\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2} = 1 - 2\left(1 - \cos^2\frac{\alpha}{2}\right) = 2\cos^2\frac{\alpha}{2} - 1$$

$$\boxed{\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos\alpha}{2}}}$$

$$\tan\frac{\alpha}{2}$$

$$\tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \frac{\pm\sqrt{\dfrac{1-\cos\alpha}{2}}}{\pm\sqrt{\dfrac{1+\cos\alpha}{2}}} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

$$\boxed{\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}}$$

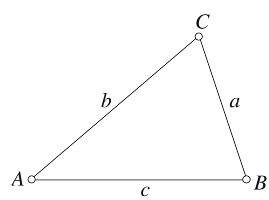
$$\cot\frac{\alpha}{2}$$

$$\cot\frac{\alpha}{2} = \frac{1}{\tan\frac{\alpha}{2}} = \pm\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}$$

$$\boxed{\cot\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}}$$

3.1.9 Functions of the half-angle using the perimeter of a triangle

Cosine rule $\quad a^2 = b^2 + c^2 - 2bc\cos A$



$$\cos A = \frac{a^2-b^2-c^2}{-2bc} = \frac{b^2+c^2-a^2}{2bc}$$

but $\quad \cos A = 1 - 2\sin^2\frac{A}{2}$

therefore $\quad 1 - 2\sin^2\frac{A}{2} = \frac{b^2+c^2-a^2}{2bc}$

$$2\sin^2\frac{A}{2} = 1 - \left(\frac{b^2+c^2-a^2}{2bc}\right) = \frac{2bc-(b^2+c^2-a^2)}{2bc}$$

$$2\sin^2\frac{A}{2} = \frac{a^2-(b-c)^2}{2bc}$$

$$\sin^2\frac{A}{2} = \frac{a^2-(b-c)^2}{4bc} = \frac{(a-b+c)(a+b-c)}{4bc}$$

Let $\quad 2s = a + b + c$

therefore $$\sin^2\frac{A}{2}=\frac{(2s-2b)(2s-2c)}{4bc}=\frac{(s-b)(s-c)}{bc}$$

$$\boxed{\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}}$$

Similarly $$\cos A=2\cos^2\frac{A}{2}-1=\frac{b^2+c^2-a^2}{2bc}$$

therefore $$2\cos^2\frac{A}{2}=1+\frac{b^2+c^2-a^2}{2bc}=\frac{(b+c)^2-a^2}{2bc}$$

$$2\cos^2\frac{A}{2}=\frac{(b+c-a)(b+c+a)}{2bc}$$

$$\cos^2\frac{A}{2}=\frac{2s(2s-2a)}{4bc}=\frac{s(s-a)}{bc}$$

$$\boxed{\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}}$$

For $\tan\frac{A}{2}$ divide $\sin\frac{A}{2}$ by $\cos\frac{A}{2}$

$$\boxed{\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}}$$

3.1.10 Functions converting to the half-angle tangent form

sin α

Double-angle identity $$\sin 2\alpha=2\sin\alpha\cos\alpha$$

$$\sin\alpha=2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}=\frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}=\frac{2\tan\frac{\alpha}{2}}{\sec^2\frac{\alpha}{2}}$$

$$\boxed{\sin\alpha=\frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}}$$

cos α

Double-angle identity $\cos 2\alpha = 1 - 2\sin^2\alpha$

$$\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2} = 1 - \frac{2\tan^2\frac{\alpha}{2}}{\sec^2\frac{\alpha}{2}} = \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$$

$$\boxed{\cos\alpha = \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}}$$

tan α

Double-angle identity $\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$

$$\boxed{\tan\alpha = \frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}}$$

Similarly

$$\boxed{\csc\alpha = \frac{1+\tan^2\frac{\alpha}{2}}{2\tan\frac{\alpha}{2}}}$$

$$\boxed{\sec\alpha = \frac{1+\tan^2\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}}$$

$$\boxed{\cot\alpha = \frac{1-\tan^2\frac{\alpha}{2}}{2\tan^2\frac{\alpha}{2}}}$$

3.1.11 Relationships between sums of functions

$\sin\alpha + \sin\beta$

$$\begin{aligned}
&\sin(\alpha+\beta)\cos(\alpha-\beta)\\
&\quad = (\sin\alpha\cos\beta + \cos\alpha\sin\beta)(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = \sin\alpha\cos\alpha\cos^2\beta\\
&\qquad + \sin\beta\cos\beta\sin^2\alpha + \sin\beta\cos\beta\cos^2\alpha + \sin\alpha\cos\alpha\sin^2\beta\\
&\quad = \sin\alpha\cos\alpha(\cos^2\beta + \sin^2\beta) + \sin\beta\cos\beta(\cos^2\alpha + \sin^2\alpha)
\end{aligned}$$

but $\cos^2\theta + \sin^2\theta = 1$

$$\therefore \sin(\alpha+\beta)\cos(\alpha-\beta) = \sin\alpha\cos\alpha + \sin\beta\cos\beta$$

$$\sin(\alpha+\beta)\cos(\alpha-\beta) = \tfrac{1}{2}\sin 2\alpha + \tfrac{1}{2}\sin 2\beta$$

$$\boxed{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) = \sin\alpha + \sin\beta}$$

$\sin\alpha - \sin\beta$

$$\begin{aligned}
&\sin(\alpha-\beta)\cos(\alpha+\beta)\\
&\quad = (\sin\alpha\cos\beta - \cos\alpha\sin\beta)(\cos\alpha\cos\beta - \sin\alpha\sin\beta)\\
&\quad = \sin\alpha\cos\alpha\cos^2\beta - \sin\beta\cos\beta\sin^2\alpha - \sin\beta\cos\beta\cos^2\alpha + \sin\alpha\cos\alpha\sin^2\beta\\
&\quad = \sin\alpha\cos\alpha(\cos^2\beta + \sin^2\beta) - \sin\beta\cos\beta(\cos^2\alpha + \sin^2\alpha)
\end{aligned}$$

but $\cos^2\theta + \sin^2\theta = 1$

$$\therefore \sin(\alpha-\beta)\cos(\alpha+\beta) = \sin\alpha\cos\alpha - \sin\beta\cos\beta$$

$$\sin(\alpha-\beta)\cos(\alpha+\beta) = \tfrac{1}{2}\sin 2\alpha - \tfrac{1}{2}\sin 2\beta$$

$$\boxed{2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right) = \sin\alpha - \sin\beta}$$

$\cos\alpha + \cos\beta$

$$\begin{aligned}
&\cos(\alpha+\beta)\cos(\alpha-\beta)\\
&\quad = (\cos\alpha\cos\beta - \sin\alpha\sin\beta)(\cos\alpha\cos\beta + \sin\alpha\sin\beta)\\
&\quad = \cos^2\alpha\cos^2\beta + \sin\alpha\cos\alpha\sin\beta\cos\beta - \sin\alpha\cos\alpha\sin\beta\cos\beta - \sin^2\alpha\sin^2\beta\\
&\quad = \cos^2\alpha\cos^2\beta - \sin^2\alpha\sin^2\beta
\end{aligned}$$

but $\sin^2\alpha + \cos^2\alpha = 1$

$$\therefore \sin^2\alpha = 1 - \cos^2\alpha$$

$$\begin{aligned}
\cos(\alpha+\beta)\cos(\alpha-\beta) &= \cos^2\alpha\cos^2\beta - \sin^2\beta(1-\cos^2\alpha)\\
&= \cos^2\alpha\cos^2\beta - \sin^2\beta + \sin^2\beta\cos^2\alpha\\
&= \cos^2\alpha(\cos^2\beta + \sin^2\beta) - \sin^2\beta\\
&= \cos^2\alpha - \sin^2\beta\\
&= 1 - \sin^2\alpha - \sin^2\beta
\end{aligned}$$

but $\cos 2\theta = 1 - 2\sin^2\theta$

$$\therefore -\sin^2\theta = \tfrac{1}{2}(\cos 2\theta - 1)$$

$$\begin{aligned}\cos(\alpha+\beta)\cos(\alpha-\beta) &= 1 + \tfrac{1}{2}(\cos 2\theta - 1) + \tfrac{1}{2}(\cos 2\beta - 1)\\ &= \tfrac{1}{2}\cos 2\alpha + \tfrac{1}{2}\cos 2\beta\end{aligned}$$

$$\boxed{2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) = \cos\alpha + \cos\beta}$$

$\cos\alpha - \cos\beta$

$$\begin{aligned}&\sin(\alpha+\beta)\sin(\alpha-\beta)\\ &\quad= (\sin\alpha\cos\beta + \cos\alpha\sin\beta)(\sin\alpha\cos\beta - \cos\alpha\sin\beta)\\ &\quad= \sin^2\alpha\cos^2\beta - \sin\alpha\cos\alpha\sin\beta\cos\beta + \sin\alpha\cos\alpha\sin\beta\cos\beta - \cos^2\alpha\sin^2\beta\\ &\quad= \sin^2\alpha\cos^2\beta + \cos^2\alpha\sin^2\beta\\ &\quad= -(\cos^2\alpha\sin^2\beta - \sin^2\alpha\cos^2\beta)\end{aligned}$$

but $\sin^2\alpha + \cos^2\alpha = 1$

$$\therefore \cos^2\alpha = 1 - \sin^2\alpha$$

$$\begin{aligned}&= -((1 - \sin^2\alpha)\sin^2\beta - \sin^2\alpha\cos^2\beta)\\ &= -(\sin^2\beta - \sin^2\alpha\sin^2\beta - \sin^2\alpha\cos^2\beta)\\ &= -(\sin^2\beta - \sin^2\alpha(\sin^2\beta + \cos^2\beta))\\ &= -(\sin^2\beta - \sin^2\alpha)\end{aligned}$$

but $\cos 2\theta = 1 - 2\sin^2\theta$

$$\therefore \sin^2\theta = \tfrac{1}{2}(1 - \cos 2\theta)$$

$$\sin(\alpha+\beta)\sin(\alpha-\beta) = -(\tfrac{1}{2}(1 - \cos 2\beta) - \tfrac{1}{2}(1 - \cos 2\alpha))$$

$$= -\tfrac{1}{2}(\cos 2\alpha - \cos 2\beta)$$

$$\boxed{-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) = \cos\alpha - \cos\beta}$$

$\tan\alpha + \tan\beta$

$$\frac{\sin(\alpha+\beta)}{\cos\alpha\ \cos\beta} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta}$$

$$\boxed{\frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta} = \tan\alpha + \tan\beta}$$

$\tan\alpha - \tan\beta$

$$\frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta} = \frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\alpha\cos\beta}$$

$$\boxed{\frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta} = \tan\alpha - \tan\beta}$$

$\cot\alpha + \cot\beta$

$$\frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\sin\alpha\sin\beta}$$

$$\boxed{\frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta} = \cot\alpha + \cot\beta}$$

$\cot\alpha - \cot\beta$

$$\frac{\sin(\alpha-\beta)}{\sin\alpha\sin\beta} = \frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\sin\alpha\sin\beta}$$

$$\frac{\sin(\alpha-\beta)}{\sin\alpha\sin\beta} = \cot\beta - \cot\alpha$$

$$\boxed{-\frac{\sin(\alpha-\beta)}{\sin\alpha\sin\beta} = \cot\alpha - \cot\beta}$$

3.1.12 Inverse trigonometric functions

$$\sin(\sin^{-1}x) = x$$
$$\cos(\cos^{-1}x) = x$$
$$\tan(\tan^{-1}x) = x$$
$$\sin^{-1}(-x) = -\sin^{-1}x$$
$$\cos^{-1}(-x) = \pi - \cos^{-1}x$$
$$\tan^{-1}(-x) = -\tan^{-1}x$$

Domain

$\sin^{-1}x \quad [-\frac{\pi}{2}, \frac{\pi}{2}]$

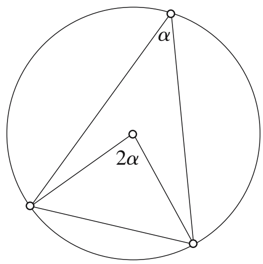
$\cos^{-1}x \quad [0, \pi]$

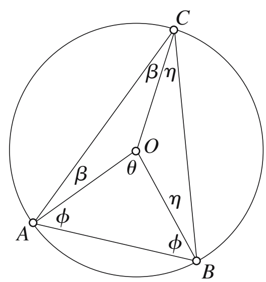
$\tan^{-1}x \quad]-\frac{\pi}{2}, \frac{\pi}{2}[$ (Open interval: extends to both limits but includes neither)

3.2 Circles

3.2.1 Proof: Angles subtended by the same arc

This theorem states that from an arc, the angle subtended at the center of a circle is twice that subtended at a point on the periphery.





Strategy: Construct the geometry with such a scenario and analyze the resulting triangles.

$\triangle$OAB, $\triangle$OBC, $\triangle$OCA are isosceles triangles (OA, OB, OC are radii)

$\triangle$**OAB**	$2\phi = 180° - \theta$	(1)
$\triangle$**ABC**	$2\phi + 2(\beta + \eta) = 180°$	
Let $\alpha = \beta + \eta$	$2\alpha = 180° - 2\phi$	(2)
Substituting (1) in (2)	$2\alpha = 180° - (180° - \theta)$	
therefore	$\boxed{2\alpha = \theta}$	

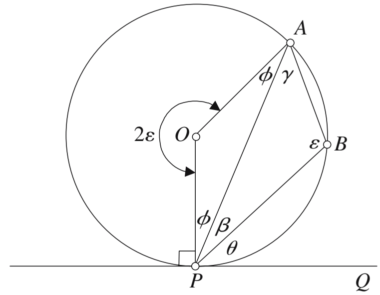
Corollary

1. Peripheral angles subtended by the same arc are equal.
2. When the arc is a semicircle the central angle equals 180°, which makes the peripheral angle 90° [Theorem of Thales].

3.2.2 Proof: Alternate segment theorem

The alternate segment theorem states that when a line PQ is tangent to a circle at P the alternate segment angles θ and γ are equal.

Strategy: Use the fact that the central angle subtended by an arc is twice the angle at the periphery.



Angle subtended by an arc reflex angle $\angle POA = 2\varepsilon$

$\angle POA = 360° - 2\varepsilon$

OP is a radius and tangent to PQ

Interior angles of $\triangle POA$	$360° - 2\varepsilon + 2\phi = 180°$	
therefore	$\varepsilon = 90° + \phi$	(1)
$\angle OPQ$ = right angle	$\beta + \theta + \phi = 90°$	(2)
Interior angles of $\triangle PAB$	$\beta + \gamma + \varepsilon = 180°$	(3)
Substituting (1) in (3)	$\beta + \gamma + \phi = 90°$	(4)
Comparing (2) and (4)	$\theta = \gamma$	

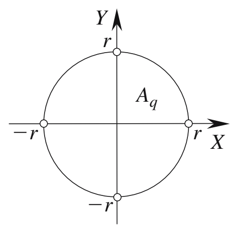
$$\boxed{\text{The alternate angles are equal } \theta = \gamma}$$

3.2.3 Proof: Area of a circle, sector and segment

Area of a circle

Strategy: Use integral calculus to find the area of a quadrant and multiply this by 4.

The equation of a circle is $x^2 + y^2 = r^2$ where r is the radius.



Equation of quadrant curve is given by $\quad y = \sqrt{r^2 - x^2}$

therefore $\quad A_q = \int_0^r \sqrt{r^2 - x^2}\,dx$

Let $\quad x = r\sin(\theta)$

therefore $\quad \sqrt{r^2 - x^2} = r\cos\theta$

and $\quad dx = r\cos\theta\,d\theta$

Establish new limits:

when $x = 0$ $\quad r\sin\theta = 0 \quad \therefore \theta = 0$

when $x = r$ $\quad r\sin\theta = r \quad \therefore \theta = \frac{\pi}{2}$

$$A_q = \int_0^{\frac{\pi}{2}} r\cos\theta\, r\cos\theta\,d\theta$$

$$= r^2\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$

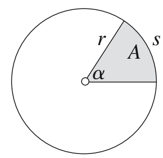
$$= \frac{r^2}{2}\int_0^{\frac{\pi}{2}} (1 + \cos 2\theta)d\theta$$

$$= \frac{r^2}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}} = \frac{r^2}{2}\left[\frac{\pi}{2}\right]$$

$$A_q = \frac{\pi r^2}{4}$$

$$\boxed{\text{Area of circle} = \pi r^2}$$

Area of a sector



Strategy: The area of a sector is found by using the sector's interior angle or arc length to create a fraction of the total area.

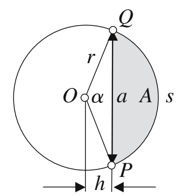
Area using arc angle [°] $$A = \frac{\alpha°}{360°}\pi r^2$$

Area using arc angle [*rad*] $$A = \frac{\alpha}{2\pi}\pi r^2 = \tfrac{1}{2}\alpha r^2$$

Area using arc length $$A = \frac{s}{2\pi r}\pi r^2 = \tfrac{1}{2}sr$$

$$\boxed{A = \frac{\alpha°}{360°}\pi r^2 \qquad A = \tfrac{1}{2}sr \qquad A = \tfrac{1}{2}\theta r^2}$$

Area of a segment



Strategy: Compute the area of the segment as a function of α by subtracting the area of triangle OPQ from the area of the sector.

Area of segment $$A = \frac{\alpha°}{360°}\pi r^2 - \frac{1}{2}ah \tag{1}$$

but $$h = r\cos\frac{\alpha}{2} \tag{2}$$

and $$a = 2r\sin\frac{\alpha}{2} \tag{3}$$

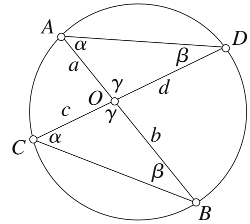
Substituting (2) and (3) in (1) $$A = \frac{\alpha°}{360°}\pi r^2 - r^2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}$$

$$A = r^2\left(\frac{\alpha°}{360°}\pi - \frac{\sin\alpha}{2}\right)$$

or using radians $$A = \frac{r^2}{2}(\alpha - \sin\alpha)$$

3.2.4 Proof: Chord theorem

Strategy: Create two triangles from the intersecting chords and prove that they are similar.



Let AB and CD be two chords intersecting at O

$\angle DAB = \angle BCD = \alpha$ (subtend equal arcs)

Similarly $\angle ADC = \angle ABC = \beta$ (subtend equal arcs)

$\angle AOD = \angle COB = \gamma$ (opposite angles)

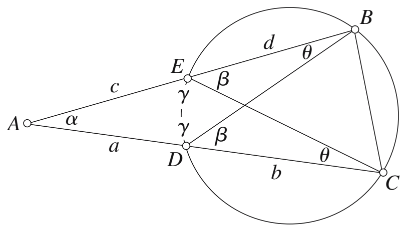
therefore $\triangle AOD$ and $\triangle COB$ are similar

Consequently $$\frac{a}{d} = \frac{c}{b}$$

and $ab = cd$

3.2.5 Proof: Secant theorem

The secant theorem states that if two secants intersect at O outside a circle, then the product of the intercepts between O and the circle on one is equal to the product of the two intercepts on the other.



Strategy: Create two triangles from the intersecting secants and prove that they are similar.

BC is a common chord

therefore $\angle CEB = \angle CDB = \beta$ (subtend equal arcs)

and $\angle AEC = \angle ADB = \gamma$ (complementary to β)

and $\angle EBD = \angle ECD = \theta$ (subtend equal arcs)

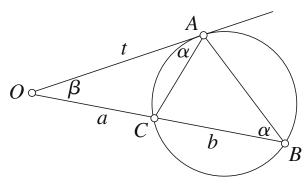
Therefore $\triangle$s ABD and ACE are similar

Therefore $$\frac{c}{a+b} = \frac{a}{c+d}$$

or $a(a+b) = c(c+d)$

3.2.6 Proof: Secant–tangent theorem

The secant–tangent theorem states that if two secants intersect at O outside a circle, and one of them is tangent to the circle, then the length of the intercept on the tangent between O and the point of contact is the geometric mean of the lengths of the intercepts of the other secant.



Strategy: Identify two similar triangles from the construction lines and form ratios of their sides.

Prove that $\triangle OAC$ and $\triangle OBA$ are similar

$\angle OAC = \angle OBA$ (alternate segment theorem)

Let $\angle AOB = \beta$ (common to both triangles)

therefore $\angle OCA = 180° - \alpha - \beta$

and $\angle OAB = 180° - \alpha - \beta$

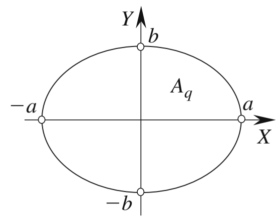
There are three common angles, therefore the triangles are similar

therefore $$\frac{t}{a+b} = \frac{a}{t}$$

and $$t^2 = a(a+b)$$

3.2.7 Proof: Area of an ellipse

Strategy: Use integral calculus to find the area of a quadrant and multiply this by 4.



The equation of an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where a and b are the radii.

Equation of the quadrant curve is given by $y = \frac{b}{a}\sqrt{a^2 - x^2}$

$$A_q = \frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$$ (area under curve between the limits 0 and a)

Let $x = a \sin\theta$

therefore $\sqrt{a^2 - x^2} = a\cos\theta$

and $dx = a\cos\theta\, d\theta$

Establish new limits:

when $x = 0$ $\quad a\sin\theta = 0 \quad \therefore \theta = 0$

when $x = a$ $\quad a\sin\theta = a \quad \therefore \theta = \frac{\pi}{2}$

$$A_q = \frac{b}{a}\int_0^{\frac{\pi}{2}} a\cos\theta\, a\cos\theta\,\mathrm{d}\theta$$

$$= ab\int_0^{\frac{\pi}{2}} \cos^2\theta\,\mathrm{d}\theta$$

$$= \frac{ab}{2}\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\mathrm{d}\theta$$

$$= \frac{ab}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}} = \frac{ab}{2}\left[\frac{\pi}{2}\right]$$

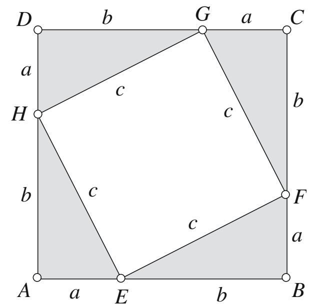
$$A_q = \frac{\pi ab}{4}$$

$$\boxed{\text{Area of ellipse} = \pi ab}$$

3.3 Triangles

3.3.1 Proof: Theorem of Pythagoras

Strategy 1: Place a rotated square inside a larger square and resolve the geometry.



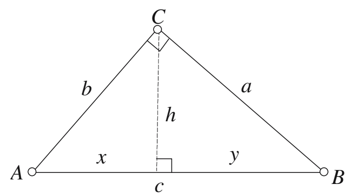
ABCD and *EFGH* are squares.
By symmetry, the diagram can be annotated as shown.
The area of $ABCD = (a + b)^2$ which must equal the area of the shaded triangles and the inner square *EFGH*.

$$(a+b)^2 = 4 \times \tfrac{1}{2}ab + c^2$$

$$a^2 + 2ab + b^2 = 2ab + c^2$$

Pythagorean theorem $\boxed{a^2 + b^2 = c^2}$

Strategy 2: Use the altitude in a right-angled triangle to resolve the geometry.



$\triangle ABC$ is a right-angled triangle, therefore

$$\frac{a}{c} = \frac{y}{a} \quad \text{and} \quad \frac{b}{c} = \frac{x}{b}$$

therefore $a^2 = cy$ and $b^2 = cx$

and $a^2 + b^2 = cx + cy = c(x + y)$

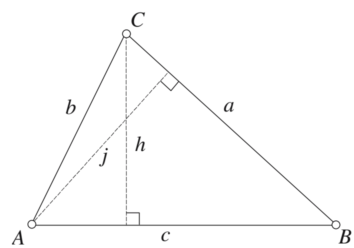
but $x + y = c$

therefore $\boxed{a^2 + b^2 = c^2}$

3.3.2 Proofs: Properties of triangles

Sine rule

Strategy: Drop a perpendicular to divide the triangle in two and then declare definitions of the sines of the two base angles.



$$\frac{h}{b} = \sin A \quad \text{and} \quad \frac{h}{a} = \sin B$$

$$b \sin A = a \sin B$$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

Similarly $\quad \dfrac{j}{b} = \sin C \quad$ and $\quad \dfrac{j}{c} = \sin B$

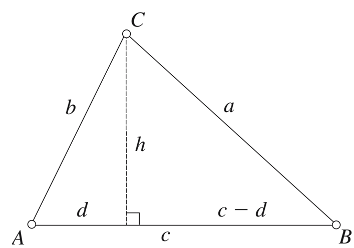
$$b \sin C = c \sin B$$

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

Sine rule $\quad \boxed{\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}}$

Cosine rule

Strategy: Drop a perpendicular to divide the triangle in two and apply the theorem of Pythagoras to both triangles.



Pythagoras $\quad h^2 = a^2 - (c-d)^2 \quad$ and $\quad h^2 = b^2 - d^2$

$$a^2 - c^2 + 2cd - d^2 = b^2 - d^2$$

$$a^2 = b^2 + c^2 - 2cd$$

but $\quad d = b \cos A$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Similarly for the other combinations.

Cosine rule

$$\boxed{\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos A \\ b^2 &= a^2 + c^2 - 2ac \cos B \\ c^2 &= a^2 + b^2 - 2ab \cos C \end{aligned}}$$

Tangent rule

Sine rule $\quad \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

therefore $\quad a = c\dfrac{\sin A}{\sin C} \quad$ and $\quad b = c\dfrac{\sin B}{\sin C}$

$$a + b = c\left(\frac{\sin A + \sin B}{\sin C}\right) \quad \text{and} \quad a - b = c\left(\frac{\sin A - \sin B}{\sin C}\right)$$

therefore $\quad \dfrac{a+b}{a-b} = \dfrac{\sin A + \sin B}{\sin A - \sin B} = \dfrac{2\sin\big((A+B)/2\big)\cos\big((A-B)/2\big)}{2\sin\big((A-B)/2\big)\cos\big((A+B)/2\big)}$

Tangent rule

$$\boxed{\begin{aligned} \frac{a+b}{a-b} &= \frac{\tan\left((A+B)/2\right)}{\tan\left((A-B)/2\right)} \\ \frac{b+c}{b-c} &= \frac{\tan\left((B+C)/2\right)}{\tan\left((B-C)/2\right)} \\ \frac{a+c}{a-c} &= \frac{\tan\left((A+C)/2\right)}{\tan\left((A-C)/2\right)} \end{aligned}}$$

Mollweide's formulas

Sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

therefore

$$b = a\frac{\sin B}{\sin A} \quad \text{and} \quad c = a\frac{\sin C}{\sin A}$$

and

$$b - c = a\left(\frac{\sin B - \sin C}{\sin A}\right)$$

$$\frac{b-c}{a} = \frac{\sin B - \sin C}{\sin A} = \frac{2\sin\left((B-C)/2\right)\cos\left((B+C)/2\right)}{2\sin\left(A/2\right)\cos\left(A/2\right)}$$

but for a triangle
therefore

$$\sin\left(\frac{A}{2}\right) = \cos\left(\frac{B+C}{2}\right)$$

Mollweide's rule

$$\boxed{\begin{aligned} \frac{b-c}{a} &= \frac{\sin\left((B-C)/2\right)}{\cos(A/2)} \\ \frac{c-a}{b} &= \frac{\sin\left((C-A)/2\right)}{\cos(B/2)} \\ \frac{a-b}{c} &= \frac{\sin\left((A-B)/2\right)}{\cos(C/2)} \end{aligned}}$$

Newton's rule

Furthermore $$b + c = a\left(\frac{\sin B + \sin C}{\sin A}\right)$$

therefore $$\frac{b+c}{a} = \frac{\sin B + \sin C}{\sin A} = \frac{2\sin\big((B+C)/2\big)\cos\big((B-C)/2\big)}{2\sin(A/2)\cos(A/2)}$$

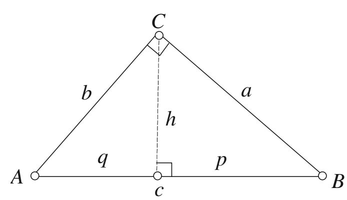
But for a triangle $$\cos\left(\frac{A}{2}\right) = \sin\left(\frac{B+C}{2}\right)$$

therefore

Newton's rule

$$\boxed{\begin{aligned} \frac{b+c}{a} &= \frac{\cos\big((B-C)/2\big)}{\sin(A/2)} \\ \frac{c+a}{b} &= \frac{\cos\big((C-A)/2\big)}{\sin(B/2)} \\ \frac{a+b}{c} &= \frac{\cos\big((A-B)/2\big)}{\sin(C/2)} \end{aligned}}$$

3.3.3 Proof: Altitude theorem



Strategy: Use the same technique used to prove the theorem of Pythagoras.

$\triangle ABC$ is a right-angled triangle.
The altitude h divides AB into lengths p and q

therefore $$\frac{a}{c} = \frac{p}{a} = \cos B$$

and $$p = \frac{a^2}{c} \qquad (1)$$

and $$\frac{b}{c} = \frac{q}{b} = \cos A$$

and $$q = \frac{b^2}{c} \qquad (2)$$

From (1) and (2) $$pq = \frac{a^2b^2}{c^2}$$

but $a = c \sin A$ and $b = c \sin B$

therefore $ab = c^2 \sin A \sin B$

and $$\frac{ab}{c} = c \sin A \sin B \tag{3}$$

but $$\sin A = \frac{h}{b} \quad \text{and} \quad \sin B = \frac{h}{a} \tag{4}$$

Substitute (4) in (3) $$\frac{ab}{c} = c\frac{h}{b}\frac{h}{a} = \frac{h^2c}{ab}$$

therefore $$h = \frac{ab}{c}$$

Altitude theorem $$\boxed{pq = h^2 = \frac{a^2b^2}{c^2}}$$

3.3.4 Proof: Area of a triangle

Basic formula

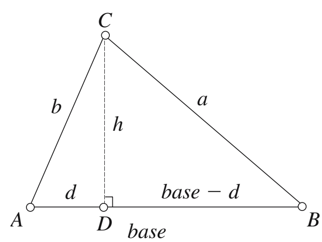
Strategy: Divide the triangle into two right-angled triangles, whose area is equal to half a rectangle.

$\triangle ADC$ area $= \frac{1}{2}dh$

$\triangle BCD$ area $= \frac{1}{2}(base - d)h$

$\triangle ABC$ area $= \frac{1}{2}(base - d)h + \frac{1}{2}dh$

$$\boxed{\text{area} = \tfrac{1}{2} base \cdot h}$$

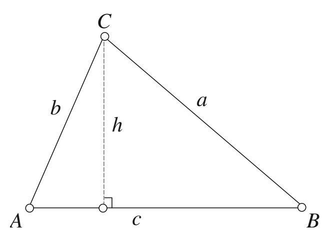


Angle formula

$\triangle ABC$ area $= \frac{1}{2}hc$

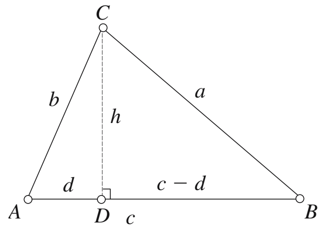
but $h = b \sin A$

$$\boxed{\text{area} = \tfrac{1}{2} bc \sin A}$$



Heron's formula

Strategy: Drop a perpendicular to divide the triangle in two, apply the theorem of Pythagoras to both triangles and resolve.



$\triangle ADC$ $$h^2 = a^2 - (c-d)^2 = b^2 - d^2$$

$$a^2 = b^2 + c^2 - 2cd$$

$$d = \frac{b^2 + c^2 - a^2}{2c}$$

$\triangle DBC$ $$h^2 = b^2 - \left(\frac{b^2 + c^2 - a^2}{2c}\right)^2$$

$$4c^2h^2 = 4b^2c^2 - (b^2 + c^2 - a^2)^2$$

$$4c^2h^2 = (2bc + (b^2 + c^2 - a^2))(2bc - (b^2 + c^2 - a^2))$$

$$4c^2h^2 = ((b+c)^2 - a^2)(a^2 - (b-c)^2)$$

$$4c^2h^2 = (a+b+c)(-a+b+c)(a-b+c)(a+b-c)$$

Let $$2s = a + b + c$$

therefore $$(-a+b+c) = 2(s-a)$$

$$(a-b+c) = 2(s-b)$$

$$(a+b-c) = 2(s-c)$$

therefore $$4c^2h^2 = 16s(s-a)(s-b)(s-c)$$

$$ch = 2\sqrt{s(s-a)(s-b)(s-c)}$$

but $$\text{area} = \tfrac{1}{2}ch$$

Heron's formula $$\boxed{\text{area} = \sqrt{s(s-a)(s-b)(s-c)}}$$

Alternatively:

Area of a triangle $$\text{area} = \tfrac{1}{2}bc\sin A$$

but $$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$$

$$\text{area} = \tfrac{1}{2}bc\,2\sin\frac{A}{2}\cos\frac{A}{2} = bc\sin\frac{A}{2}\cos\frac{A}{2}$$

but $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$

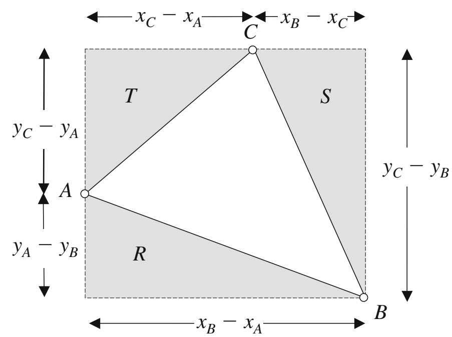
and $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$

therefore $\text{area} = bc\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-a)}{bc}}$

$$\boxed{\text{area} = \sqrt{s(s-a)(s-b)(s-c)}}$$

Area of a triangle using a determinant

Strategy: Show that the expansion of a determinant is equivalent to the area of an arbitrary triangle.



$\triangle ABC$ $\quad$ area $\triangle ABC$ = area of rectangle $- \triangle R - \triangle S - \triangle T$

$$\text{area} = (x_B - x_A)(y_C - y_B) - \tfrac{1}{2}(x_B - x_A)(y_A - y_B) - \tfrac{1}{2}(x_C - x_A)(y_C - y_A) - \tfrac{1}{2}(x_B - x_C)(y_C - y_B)$$

$$\text{area} = \tfrac{1}{2}(x_A y_B + x_B y_C + x_C y_A - x_A y_C - x_B y_A - x_C y_B)$$

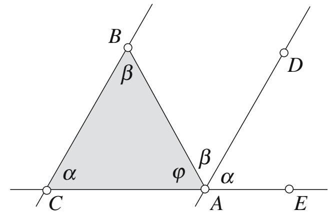
$$\boxed{\text{area} = \tfrac{1}{2}\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}}$$

[Note that the determinant produces a positive value for anti-clockwise vertices and a negative value for clockwise vertices, which means that it can also be used to identify the order of vertices.]

3.3.5 Proof: Internal and external angles of a triangle

Internal angles

Strategy: Exploit the geometric properties of parallel lines with the geometry of a triangle.



CAE is a straight line, and *AD* is parallel with *CB*.

$$E\hat{A}D = A\hat{C}B = \alpha$$

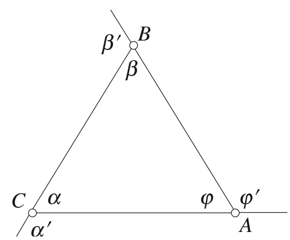
Alternate angles $D\hat{A}B = A\hat{B}C = \beta$

Let $B\hat{A}C = \varphi$

therefore $E\hat{A}C = \alpha + \beta + \varphi = 180°$

$\boxed{\text{The internal angles of a triangle sum to 180°.}}$

External angles



Internal angles of a triangle $\alpha + \beta + \varphi = 180°$

By definition $\alpha + \alpha' = \beta + \beta' = \varphi + \varphi' = 180°$

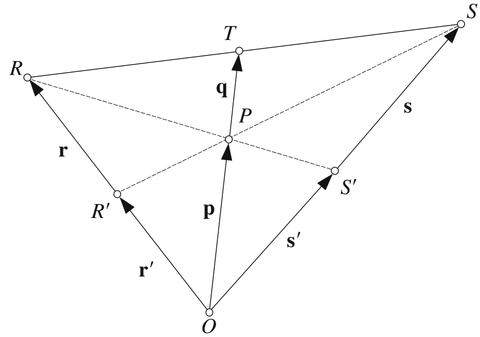
$$\alpha + \beta + \varphi + \alpha' + \beta' + \varphi' = 3 \times 180° = 540°$$

$$\alpha' + \beta' + \varphi' = 360°$$

$\boxed{\text{The external angles of a triangle sum to 360°.}}$

3.3.6 Proof: The medians of a triangle are concurrent at its centroid

$\triangle OSR$



Let R' and S' be the mid-points of *OR* and *OS* respectively.

Let *P* be the point of intersection of the medians RS' and SR'.

Let *T* be the point where the line through *O* and *P* meets *RS*.

Strategy: Prove that *OT* is a median of the triangle, i.e. *T* bisects *RS*.

Since R' and S' are mid-points of OR and OS respectively

$$\mathbf{r}' = \tfrac{1}{2}\mathbf{r}$$

and

$$\mathbf{s}' = \tfrac{1}{2}\mathbf{s}$$

$$\overrightarrow{RP} = \lambda(\overrightarrow{RS'}) \quad \text{for some } \lambda$$

Therefore

$$\mathbf{p} = \mathbf{r} + \overrightarrow{RP} \quad = \mathbf{r} + \lambda(\mathbf{s}' - \mathbf{r}) \quad = (1-\lambda)\,\mathbf{r} + \tfrac{1}{2}\lambda\mathbf{s}$$

$$\overrightarrow{SP} = \varepsilon\,(\overrightarrow{SR'}) \quad \text{for some } \varepsilon$$

Therefore

$$\mathbf{p} = \mathbf{s} + \overrightarrow{SP} = \mathbf{s} + \varepsilon(\mathbf{r}' - \mathbf{s}) = (1-\varepsilon)\mathbf{s} + \tfrac{1}{2}\varepsilon\mathbf{r}$$

$$\mathbf{p} = (1-\lambda)\mathbf{r} + \tfrac{1}{2}\lambda\mathbf{s} = (1-\varepsilon)\mathbf{s} + \tfrac{1}{2}\varepsilon\mathbf{r} \tag{1}$$

$$(1 - \lambda - \tfrac{1}{2}\varepsilon)\mathbf{r} = (1 - \varepsilon - \tfrac{1}{2}\lambda)\mathbf{s} \tag{2}$$

since $\mathbf{r}$ and $\mathbf{s}$ are not collinear (2) can only be true if

$$(1 - \lambda - \tfrac{1}{2}\varepsilon) = 0 = (1 - \varepsilon - \tfrac{1}{2}\lambda)$$

therefore

$$\lambda = \tfrac{2}{3} \quad \text{and} \quad \varepsilon = \tfrac{2}{3}$$

P is $\tfrac{2}{3}$ along RS' and $\tfrac{2}{3}$ along SR'.

We must now prove that $RT = \tfrac{1}{2}RS$ and $OP = \tfrac{2}{3}OT$

As $\mathbf{p}$ and $\mathbf{q}$ are collinear

$$\mathbf{q} = \mu\mathbf{p} \quad \text{for some } \mu$$

Using (1)

$$\mathbf{p} = \tfrac{1}{3}\mathbf{r} + \tfrac{1}{3}\mathbf{s}$$

$$\mathbf{q} = \tfrac{1}{3}\mu\mathbf{r} + \tfrac{1}{3}\mu\mathbf{s}$$

$\overrightarrow{RT}$ and $\overrightarrow{RS}$ are also collinear

$$\overrightarrow{RT} = \varphi\overrightarrow{RS} \quad \text{for some } \varphi$$

$$\mathbf{q} = \tfrac{1}{3}\mu\mathbf{r} + \tfrac{1}{3}\mu\mathbf{s} = \mathbf{r} + \overrightarrow{RT} = \mathbf{r} + \varphi(\mathbf{s} - \mathbf{r}) = (1-\varphi)\mathbf{r} + \varphi\mathbf{s}$$

Therefore

$$(\tfrac{1}{3}\mu - 1 + \varphi)\mathbf{r} = (\varphi - \tfrac{1}{3}\mu)\mathbf{s} \tag{3}$$

since **r** and **s** are not collinear (3) can only be true if

$$\left(\tfrac{1}{3}\mu - 1 + \varphi\right) = 0 = \left(\varphi - \tfrac{1}{3}\mu\right)$$

therefore $\qquad \varphi = \tfrac{1}{2} \quad \text{and} \quad \mu = \tfrac{3}{2}$

$$\overrightarrow{RT} = \tfrac{1}{2}\overrightarrow{RS} \quad \text{and} \quad \mathbf{p} = \tfrac{2}{3}\mathbf{q}$$

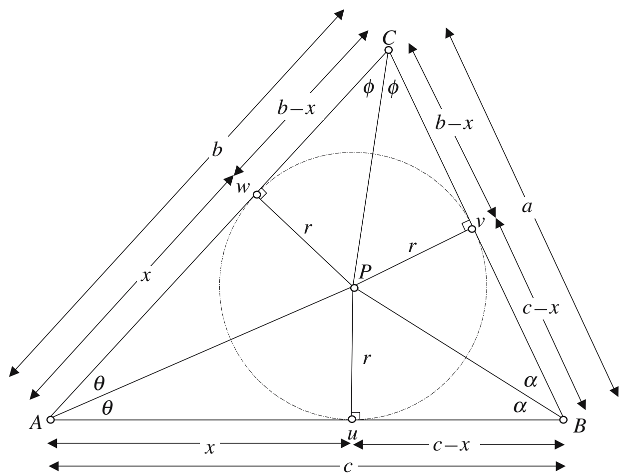
which confirms that

The three medians intersect at a point two-thirds along each median.

3.3.7 Proof: Radius and center of the inscribed circle for a triangle

Radius

Strategy: Create the geometry formed by the intersecting angle bisectors of a triangle and drop perpendiculars, each of which equals the radius of the inscribed circle. Apply Heron's area formula of a triangle to reveal the radius.



AP, BP and CP bisect angles A, B and C respectively.

r is the radius of the inscribed circle.

The tangency points are u, v and w.

Using congruent triangles, let $\qquad Au = x \qquad Aw = x$

$$uB = c - x \qquad vB = c - x$$

$$wC = b - x \qquad Cv = b - x$$

$\triangle ABC \qquad$ Perimeter $= a + b + c = 2x + 2(c - x) + 2(b - x)$

Semiperimeter $s = \frac{1}{2}(a+b+c) = x + c - x + b - x = b + c - x$

$$\text{area} = rx + r(c-x) + r(b-x)$$

$$\text{area} = r(x + c - x + b - x) = r\,(b + c - x) = rs$$

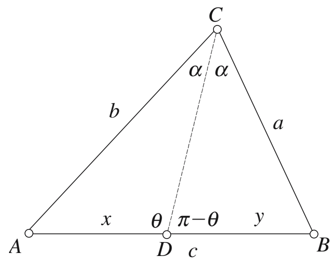
$$r = \frac{\text{area}\ \triangle ABC}{s}$$

but $\text{area} = \sqrt{s(s-a)(s-b)(s-c)}$

therefore $r = \dfrac{\sqrt{s(s-a)(s-b)(s-c)}}{s}$

$$\boxed{r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}}$$

Center



Strategy: A circle can be drawn inside a triangle such that it touches every side. The center of the circle is the unique point where the angle bisectors meet. The proof exploits a relationship between the sides of a triangle and the edge intersected by the angle bisector.

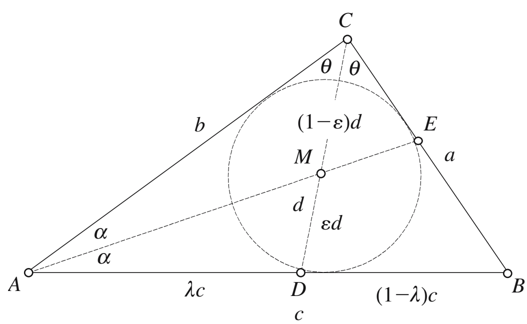
Let $BC = a$ $AC = b$ $AB = c$ $AD = x$ $DB = y$
DC bisects angle C and divides AB at D into lengths x and y.

Using the sine rule $\dfrac{x}{\sin\alpha} = \dfrac{b}{\sin\theta}$ $\quad \therefore \dfrac{x}{b} = \dfrac{\sin\alpha}{\sin\theta}$

$$\frac{y}{\sin\alpha} = \frac{a}{\sin(\pi-\theta)} \qquad \therefore \frac{y}{a} = \frac{\sin\alpha}{\sin(\pi-\theta)} = \frac{\sin\alpha}{\sin\theta}$$

$$\frac{x}{b} = \frac{y}{a} \qquad \therefore \frac{x}{y} = \frac{b}{a}$$

General triangle



$\triangle ABC$

Let $BC = a$ $CA = b$
$AB = c$ $DC = d$

AE bisects angle A and CD bisects angle C.

$M\ (x_M, y_M)$ is the center of the inscribed circle.

Let $0 \leqslant (\lambda, \varepsilon) \leqslant 1$

$$AD = \lambda c \qquad \therefore DB = (1-\lambda)c$$

$$\frac{\lambda c}{(1-\lambda)c} = \frac{b}{a} \qquad \therefore \lambda a = (1-\lambda)b$$

therefore

$$\lambda = \frac{b}{a+b} \qquad \therefore 1-\lambda = \frac{a}{a+b}$$

$\triangle$***ADC***

$$DM = \varepsilon d \qquad \therefore MC = (1-\varepsilon)d$$

$$\frac{\varepsilon d}{(1-\varepsilon)d} = \frac{\lambda c}{b} = \frac{bc}{b(a+b)} = \frac{c}{a+b}$$

$$\varepsilon(a+b) = (1-\varepsilon)c$$

$$\varepsilon = \frac{c}{a+b+c} \qquad \therefore 1-\varepsilon = \frac{a+b}{a+b+c}$$

$$x_D = \lambda x_B + (1-\lambda)x_A$$

$$x_D = \frac{b}{a+b}x_B + \frac{a}{a+b}x_A$$

$$x_M = \varepsilon x_C + (1-\varepsilon)x_D$$

$$x_M = \frac{c}{a+b+c}x_C + \frac{a+b}{a+b+c}\left(\frac{b}{a+b}x_B + \frac{a}{a+b}x_A\right)$$

$$x_M = \frac{ax_A + bx_B + cx_C}{a+b+c}$$

Similarly for y_M

$$y_M = \frac{ay_A + by_B + cy_C}{a+b+c}$$

Center

$$\boxed{x_M = \frac{ax_A + bx_B + cx_C}{a+b+c} \qquad y_M = \frac{ay_A + by_B + cy_C}{a+b+c}}$$

Equilateral triangle

For an equilateral triangle all sides are length a.

Center

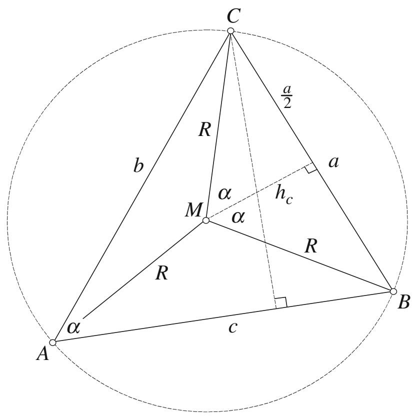
$$\boxed{x_M = \tfrac{1}{3}(x_A + x_B + x_C) \qquad y_M = \tfrac{1}{3}(y_A + y_B + y_C)}$$

3.3.8 Proof: Radius and center of the circumscribed circle for a triangle

Radius

Strategy: The circumcenter of a triangle is equidistant from its vertices, which enables its radius to be defined in terms of the triangle's area.

General triangle



Chord theorem $\qquad \angle BAC = \alpha \quad \text{and} \quad \angle BMC = 2\alpha$

$$\sin\alpha = \frac{a}{2R} \tag{1}$$

$\triangle ABC$

$$\text{area} = \tfrac{1}{2} c h_c \tag{2}$$

$$\frac{h_c}{b} = \sin\alpha \qquad \therefore h_c = b\sin\alpha \tag{3}$$

Substitute (3) in (2)

$$\text{area} = \frac{bc}{2}\sin\alpha \tag{4}$$

Substitute (1) in (4)

$$\text{area} = \frac{abc}{4R}$$

$$\boxed{R = \frac{abc}{4 \times \text{area} \triangle ABC}}$$

Equilateral triangle

If $\triangle ABC$ is an equilateral triangle with side a

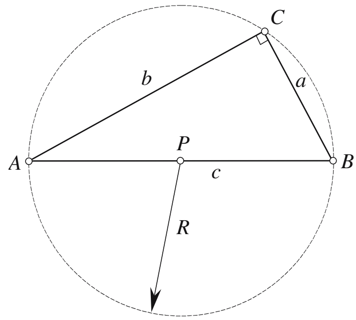
$$\text{area} = \frac{a^2\sqrt{3}}{4} \tag{5}$$

$$R = \frac{a^3}{4 \times \text{area } \triangle ABC} \tag{6}$$

$$\boxed{R = \frac{a\sqrt{3}}{3}}$$

Right-angled triangle

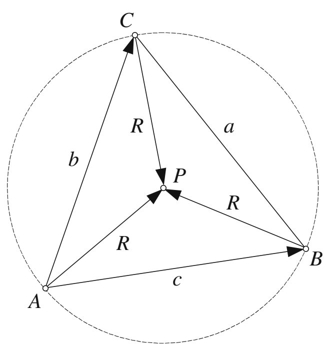
$$\boxed{R = \frac{\textit{hypotenuse}}{2}}$$



Center

Strategy: The center of the circumscribed circle is equidistant from the triangle's vertices. Locating this center is established by vector analysis.

General triangle



Let P be the center of the circumscribed circle of radius R.

Then $$AP = BP = CP = R$$

$$\overrightarrow{AB} = \overrightarrow{AP} - \overrightarrow{BP}$$

$$x_{BP} = x_{AP} - x_{AB} \tag{7}$$

$$y_{BP} = y_{AP} - y_{AB} \tag{8}$$

but $$\|\overrightarrow{BP}\| = \|\overrightarrow{AP}\| = \|\overrightarrow{CP}\| = R$$

$$x_{BP}^2 + y_{BP}^2 = x_{AP}^2 + y_{AP}^2 \tag{9}$$

Substituting (7) and (8) in (9)

$$(x_{AP} - x_{AB})^2 + (y_{AP} - y_{AB})^2 = x_{AP}^2 + y_{AP}^2$$

$$x_{AP}^2 - 2x_{AB}x_{AP} + x_{AB}^2 + y_{AP}^2 - 2y_{AP}y_{AB} + y_{AB}^2 = x_{AP}^2 + y_{AP}^2$$

$$x_{AB}^2 + y_{AB}^2 = 2x_{AB}x_{AP} + 2y_{AP}y_{AB}$$

$$c^2 = 2(x_{AB}\,x_{AP} + y_{AP}\,y_{AB}) \tag{10}$$

Similarly $$\overrightarrow{AC} = \overrightarrow{AP} - \overrightarrow{CP}$$

$$x_{CP} = x_{AP} - x_{AC} \tag{11}$$

$$y_{CP} = y_{AP} - y_{AC} \tag{12}$$

but $$\|\overrightarrow{CP}\| = \|\overrightarrow{AP}\| = R$$

$$x_{CP}^2 + y_{CP}^2 = x_{AP}^2 + y_{AP}^2 \tag{13}$$

Substitute (11) and (12) in (13)

$$(x_{AP} - x_{AC})^2 + (y_{AP} - y_{AC})^2 = x_{AP}^2 + y_{AP}^2$$

$$x_{AP}^2 - 2x_{AP}x_{AC} + x_{AC}^2 + y_{AP}^2 - 2y_{AP}y_{AC} + y_{AC}^2 = x_{AP}^2 + y_{AP}^2$$

$$x_{AC}^2 + y_{AC}^2 = 2x_{AP}x_{AC} + 2y_{AP}y_{AC}$$

$$b^2 = 2(x_{AP}x_{AC} + y_{AP}\,y_{AC}) \tag{14}$$

Combine (10) and (14) to reveal y_{AP}

$$-x_{AB}b^2 = -2(x_{AB}x_{AP}x_{AC} - x_{AB}\,y_{AP}\,y_{AC})$$

$$x_{AC}c^2 = 2(x_{AB}x_{AP}x_{AC} + x_{AC}y_{AP}y_{AB})$$

$$x_{AC}c^2 - x_{AB}b^2 = 2(x_{AC}y_{AP}y_{AB} - x_{AB}y_{AP}y_{AC})$$

$$x_{AC}c^2 - x_{AB}b^2 = 2y_{AP}\,(x_{AC}y_{AB} - x_{AB}y_{AC})$$

$$y_{AP} = \frac{x_{AC}c^2 - x_{AB}b^2}{2(x_{AC}y_{AB} - x_{AB}y_{AC})}$$

This can be represented in determinant form:

$$y_{AP} = \frac{1}{2}\frac{\begin{vmatrix} x_{AC} & b^2 \\ x_{AB} & c^2 \end{vmatrix}}{\begin{vmatrix} x_{AC} & y_{AC} \\ x_{AB} & y_{AB} \end{vmatrix}} \tag{15}$$

Combine (10) and (14) to reveal x_{AP}

$$-y_{AB}b^2 = -2(x_{AP}x_{AC}y_{AB} + y_{AP}y_{AC}y_{AB})$$

$$y_{AC}c^2 = 2(x_{AB}x_{AP}y_{AC} + y_{AP}y_{AB}y_{AC})$$

$$y_{AC}c^2 - y_{AB}b^2 = 2(x_{AB}x_{AP}y_{AC} - x_{AP}x_{AC}y_{AB})$$

$$y_{AC}c^2 - y_{AB}b^2 = 2x_{AP}(x_{AB}y_{AC} - x_{AC}y_{AB})$$

$$x_{AP} = \frac{y_{AC}c^2 - y_{AB}b^2}{2(x_{AB}y_{AC} - x_{AC}y_{AB})}$$

In determinant form

$$x_{AP} = \frac{1}{2}\frac{\begin{vmatrix} y_{AC} & b^2 \\ y_{AB} & c^2 \end{vmatrix}}{\begin{vmatrix} x_{AB} & y_{AB} \\ x_{AC} & y_{AC} \end{vmatrix}} \tag{16}$$

The coordinates of $P(x_P, y_P)$ are given by

$$x_P = x_A + \frac{1}{2}\frac{\begin{vmatrix} y_{AC} & b^2 \\ y_{AB} & c^2 \end{vmatrix}}{\begin{vmatrix} x_{AB} & y_{AB} \\ x_{AC} & y_{AC} \end{vmatrix}} \tag{17}$$

$$y_P = y_A + \frac{1}{2}\frac{\begin{vmatrix} x_{AC} & b^2 \\ x_{AB} & c^2 \end{vmatrix}}{\begin{vmatrix} x_{AC} & y_{AC} \\ x_{AB} & y_{AB} \end{vmatrix}} \tag{18}$$

(18) can be arranged to have the same denominator as (17):

$$\textbf{Center} \qquad \boxed{x_P = x_A + \frac{1}{2}\frac{\begin{vmatrix} y_{AC} & b^2 \\ y_{AB} & c^2 \end{vmatrix}}{\begin{vmatrix} x_{AB} & y_{AB} \\ x_{AC} & y_{AC} \end{vmatrix}} \qquad y_P = y_A + \frac{1}{2}\frac{\begin{vmatrix} b^2 & x_{AC} \\ c^2 & x_{AB} \end{vmatrix}}{\begin{vmatrix} x_{AB} & y_{AB} \\ x_{AC} & y_{AC} \end{vmatrix}}}$$

Further developments

If the area of the triangle is already known then we can show that the determinant

$\begin{vmatrix} x_{AB} & y_{AB} \\ x_{AC} & y_{AC} \end{vmatrix}$ is related to the area:

$$\begin{aligned}
\begin{vmatrix} x_{AB} & y_{AB} \\ x_{AC} & y_{AC} \end{vmatrix} &= \begin{vmatrix} (x_B - x_A) & (y_B - y_A) \\ (x_C - x_A) & (y_C - y_A) \end{vmatrix} \\
&= (x_B - x_A)(y_C - y_A) - (x_C - x_A)(y_B - y_A) \\
&= x_B y_C - x_B y_A - x_A y_C + x_A y_A - x_C y_B - x_C y_B + x_C y_B + x_C y_A \\
&= x_A y_B + x_B y_C + x_C y_A - x_B y_A - x_C y_B - x_A y_C \\
&= \begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix} = 2 \times \text{area } \triangle ABC
\end{aligned}$$

Therefore
$$x_P = x_A + \frac{1}{4}\frac{\begin{vmatrix} y_{AC} & b^2 \\ y_{AB} & c^2 \end{vmatrix}}{\text{area } \triangle ABC}$$

and
$$y_P = y_A + \frac{1}{4}\frac{\begin{vmatrix} b^2 & x_{AC} \\ c^2 & x_{AB} \end{vmatrix}}{\text{area } \triangle ABC}$$

Similarly, if the radius R of the circumscribed circle is known, we can exploit the relationship

$$\triangle ABC \qquad \text{area} = \frac{abc}{4R}$$

$$\textbf{Center} \qquad \boxed{x_P = x_A + \frac{R}{abc}\begin{vmatrix} y_{AC} & b^2 \\ y_{AB} & c^2 \end{vmatrix} \qquad y_P = y_A + \frac{R}{abc}\begin{vmatrix} b^2 & x_{AC} \\ c^2 & x_{AB} \end{vmatrix}}$$

Equilateral triangle

All sides equal a

$$\text{area}\,\triangle ABC = \frac{a^2\sqrt{3}}{4}$$

$$x_P = x_A + \frac{1}{4}\frac{\begin{vmatrix} y_{AC} & b^2 \\ y_{AB} & c^2 \end{vmatrix}}{\text{area}\,\triangle ABC}$$

$$x_P = x_A + \frac{\begin{vmatrix} y_{AC} & a^2 \\ y_{AB} & a^2 \end{vmatrix}}{a^2\sqrt{3}}$$

$$x_P = x_A + \frac{a^2 y_{AC} - a^2 y_{AB}}{a^2\sqrt{3}}$$

$$x_P = x_A + \frac{y_C - y_A - y_B + y_A}{\sqrt{3}}$$

Center

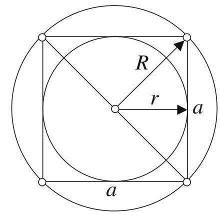
$$\boxed{x_P = x_A + \frac{\sqrt{3}}{3}(y_C - y_B) \qquad y_P = y_A + \frac{\sqrt{3}}{3}(x_B - x_C)}$$

3.4 Quadrilaterals

3.4.1 Proof: Properties of quadrilaterals

Quadrilaterals embrace the square, rectangle, parallelogram, rhombus, trapezium, general quadrilateral, tangent quadrilateral and cyclic quadrilateral. Proofs are given for some of the more useful formulas and we begin with the square.

Square



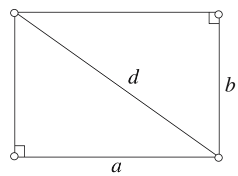
Diagonal $d = a\sqrt{2}$

Area $A = a^2 = \frac{1}{2}d^2$

Inradius $r = \dfrac{a}{2}$

Circumradius $R = \dfrac{a}{\sqrt{2}}$ (see the proof for a rectangle)

Rectangle

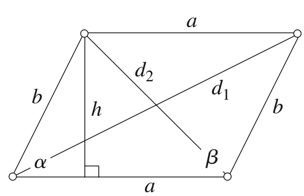


Diagonal $d = \sqrt{a^2 + b^2}$

Area $A = ab$

Circumradius $R = \dfrac{d}{2}$ (see the proof)

Parallelogram



Diagonals $d_1 = \sqrt{a^2 + b^2 - 2ab\cos\beta}$ (cosine rule)

and $d_2 = \sqrt{a^2 + b^2 - 2ab\cos\alpha}$

$$d_1^2 + d_2^2 = 2(a^2 + b^2) - 2ab(\cos\alpha + \cos\beta)$$

but $\alpha + \beta = 180°$

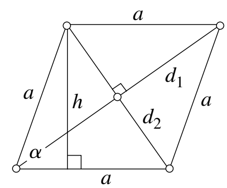
therefore $d_1^2 + d_2^2 = 2(a^2 + b^2)$ (parallelogram law)

Altitude $h = b\sin\alpha$

Area $A = ah$

Rhombus

A rhombus is a parallelogram with equal sides.



Diagonals $d_1 = 2a\cos\dfrac{\alpha}{2}$ and $d_2 = 2a\sin\dfrac{\alpha}{2}$

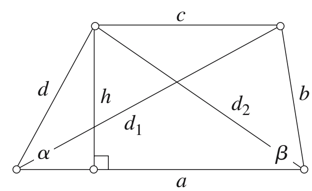
therefore $d_1^2 + d_2^2 = 4a^2$

Altitude $h = a\sin\alpha$

Area $A = ah = a^2\sin\alpha = \frac{1}{2}d_1d_2$

Trapezium

A trapezium has one pair of parallel sides.



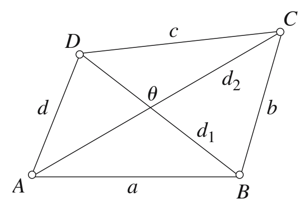
Diagonals $d_1 = \sqrt{a^2 + b^2 - 2ab\cos\beta}$ (cosine rule)

and $d_2 = \sqrt{a^2 + d^2 - 2ad\cos\alpha}$

Altitude $h = d\sin\alpha = b\sin\beta$

Area $A = \frac{1}{2}(a+c)h$

General quadrilateral

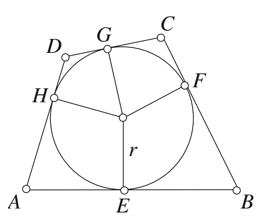


Area $A = \frac{1}{2}d_1d_2\sin\theta$ (see proof)

$A = \frac{1}{4}(b^2 + d^2 - a^2 - c^2)\tan\theta$ (see proof)

$A = \frac{1}{4}\sqrt{4d_1^2d_2^2 - (b^2 + d^2 - a^2 - c^2)}$ (see proof)

Tangent quadrilateral



$AB = a$
$BC = b$
$CD = c$
$DA = d$

Because the intercepts of two tangents from a single point to a circle are equal:

$$|AE| = |AH|,\ |EB| = |BF|,\ |FC| = |CG|,\ |GD| = |HD|$$

therefore $\quad |AE| + |EB| + |CG| + |GD| = |BF| + |FC| + |AH| + |HD|$

and $\quad a + c = b + d$

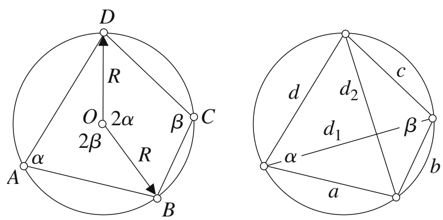
Area $\quad A = \frac{1}{2}ra + \frac{1}{2}rb + \frac{1}{2}rc + \frac{1}{2}rd = \frac{1}{2}r(a + b + c + d)$

Area $\quad A = sr$

where $\quad s = \frac{1}{2}(a + b + c + d)$

Cyclic quadrilateral

In a cyclic quadrilateral the sum of the opposite interior angles equals 180°, which enables the vertices to reside on the circumscribed circle.



The vertices *A, B, C, D* lie on the circumference of a circle, radius *R*.

Let $\quad \angle A = \alpha \quad$ and $\quad \angle C = \beta$

The chord theorem confirms $\quad \angle BOD = 2\angle BAD = 2\alpha \quad$ (the internal angle)

Similarly $\quad \angle BOD = 2\angle BCD = 2\beta \quad$ (the external angle)

but $\quad 2\alpha + 2\beta = 360°$

therefore $\quad \alpha + \beta = 180°$

For any quadrilateral $\quad A = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\varepsilon}$

where $\quad \varepsilon = \frac{1}{2}(\alpha + \beta) \quad$ and $\quad s = \frac{1}{2}(a + b + c + d)$

therefore $\quad A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$

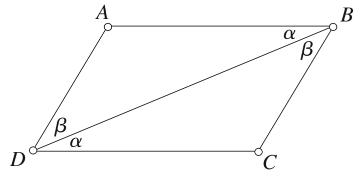
It can also be shown that $\quad R = \frac{1}{4}\sqrt{\dfrac{(ac+bd)(ad+bc)(ab+cd)}{(s-a)(s-b)(s-c)(s-d)}}$

and the diagonals are $\quad d_1 = \sqrt{\dfrac{(ab+cd)(ac+bd)}{ad+bc}} \quad$ and $\quad d_2 = \sqrt{\dfrac{(ac+bd)(ad+bc)}{ab+cd}}$

and $\quad d_1 d_2 = ac + bd$

3.4.2 Proof: The opposite sides and angles of a parallelogram are equal

Definition: A parallelogram is a quadrilateral in which both pairs of sides are parallel.
Strategy: Divide the parallelogram into two triangles and prove that they are congruent.



By definition	AB is parallel to DC
and	AD is parallel to BC
also	BD is a line intersecting all the lines
$\triangle$s ABD, CBD	
	$\angle ABD = \angle CDB = \alpha$ (alternate angles)
	$\angle ADB = \angle CBD = \beta$ (alternate angles)
	BD is common to both triangles
therefore	$\triangle$s ABD, CBD are congruent
which implies that	$AB = DC$ and $AD = BC$
i.e.	**the opposite sides of a parallelogram are equal**
and	$\angle ABC = \angle ADC = \alpha + \beta$
i.e.	**the opposite angles of a parallelogram are equal**

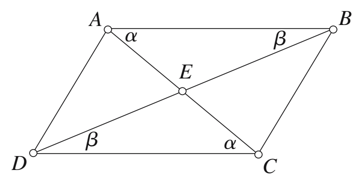
Since $\triangle$s ABD, CBD are congruent they have the same area and must bisect the parallelogram.

Corollary

1. If one angle of a parallelogram is a right angle, all the angles are right angles.
2. If two adjacent sides of a parallelogram are equal, all the sides are equal.

3.4.3 Proof: The diagonals of a parallelogram bisect each other

Strategy: Prove that triangles AEB and CED are congruent.



$\triangle$s AEB, CED

$AB = DC$ (opposite sides of a parallelogram are equal)

$\angle EAB = \angle ECD = \alpha$ (alternate angles)

$\angle EBA = \angle EDC = \beta$ (alternate angles)

therefore $\triangle$s AEB, CED are congruent

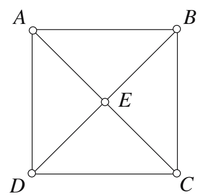
which implies that $AE = EC$ and $BE = ED$

i.e. **the diagonals of a parallelogram are bisected**

3.4.4 Proof: The diagonals of a square are equal, intersect at right angles and bisect the opposite angles

Definition: A square is a quadrilateral with both pairs of opposite sides parallel, one of its angles a right angle and two adjacent sides equal.

Strategy: Prove that triangles ADC and BCD are congruent.



$\triangle$s ADC, BCD

$AD = BC$ (opposite sides of a parallelogram)

DC is common to both triangles

$\angle ADC = \angle BCD$ (corollary: opposite sides of a parallelogram)

therefore $\triangle$s ADC, BCD are congruent

which implies $AC = BD$

i.e. **the diagonals of a square are equal**

$\triangle$s AED, CED

$AE = EC$ (diagonals bisect each other)

$AD = DC$ (sides of a square)

ED is common

therefore $\triangle$s AED, CED are congruent

which implies $\angle AED = \angle DEC$

These are right angles and the diagonals intersect at right angles.

Since $\triangle$s AED, CED are congruent

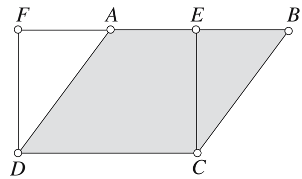
$\angle ADE = \angle CDE$

which implies $\angle ADC$ is bisected

i.e. **the diagonals of a square bisect opposite angles**

3.4.5 Proof: Area of a parallelogram

Strategy: Prove that $\triangle$s *BCE, ADF* are congruent.



ABCD is a parallelogram

CE and DF are equal and perpendicular to AB

$\triangle$s *BCE, ADF*

$\angle CBE = \angle DAF$ (corresponding angles)

$\angle DFA = \angle CEB$ (right angles)

$CB = DA$ (opposite sides of a parallelogram are equal)

therefore $\triangle$s *BCE, ADF* are congruent

Therefore quadrilateral $ADCE + \triangle BCE =$ quadrilateral $ADCE + \triangle ADF$

i.e. parallelogram $ABCD =$ rectangle $ECDF$

Therefore the area of a parallelogram is equal to the area of the rectangle with the same base and same height.

area of a parallelogram = base $\times$ height

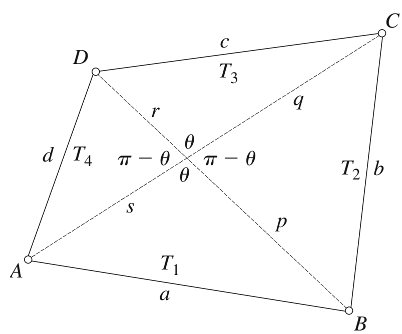
Corollary

Parallelograms having the same base and height share a common area.

3.4.6 Proof: Area of a quadrilateral

Using lengths of diagonals

Strategy: Divide the quadrilateral into four triangles and sum the individual areas.



Let $AC = d_1 = s + q$ and $BD = d_2 = r + p$
Area of ABCD = sum of the areas of triangles $\triangle T_1, \triangle T_2, \triangle T_3, \triangle T_4$

$$\text{area } \triangle T_1 = \tfrac{1}{2} sp \sin\theta$$

$$\text{area } \triangle T_2 = \tfrac{1}{2} pq \sin(\pi - \theta) = \tfrac{1}{2} pq \sin\theta$$

$$\text{area } \triangle T_3 = \tfrac{1}{2} qr \sin\theta$$

$$\text{area } \triangle T_4 = \tfrac{1}{2} rs \sin(\pi - \theta) = \tfrac{1}{2} rs \sin\theta$$

$$\text{area of } ABCD = \frac{1}{2}(sp + pq + qr + rs)\sin\theta$$

$$= \tfrac{1}{2}(p + r)(q + s)\sin\theta$$

$$\boxed{\text{Area of } ABCD = \tfrac{1}{2} d_1 d_2 \sin\theta} \qquad (1)$$

Using lengths of sides

Strategy: Apply the cosine rule to develop a relationship between the squares of the sides.

$$a^2 = s^2 + p^2 - 2ps\cos\theta$$
$$c^2 = r^2 + q^2 - 2rq\cos\theta$$
$$a^2 + c^2 = r^2 + s^2 + p^2 + q^2 - 2ps\cos\theta - 2rq\cos\theta$$
$$b^2 = p^2 + q^2 - 2pq\cos(\pi - \theta) = p^2 + q^2 + 2pq\cos\theta$$
$$d^2 = r^2 + s^2 - 2rs\cos(\pi - \theta) = r^2 + s^2 + 2rs\cos\theta$$
$$b^2 + d^2 = r^2 + s^2 + p^2 + q^2 + 2pq\cos\theta + 2rs\cos\theta$$
$$b^2 + d^2 - (a^2 + c^2) = 2pq\cos\theta + 2rs\cos\theta + 2ps\cos\theta + 2rq\cos\theta$$
$$b^2 + d^2 - a^2 - c^2 = 2(pq + rs + ps + rq)\cos\theta$$
$$b^2 + d^2 - a^2 - c^2 = 2(p + r)(q + s)\cos\theta$$
$$b^2 + d^2 - a^2 - c^2 = 2d_1 d_2 \cos\theta$$

$$d_1 d_2 = \frac{b^2 + d^2 - a^2 - c^2}{2\cos\theta} \qquad (2)$$

Substitute (2) in (1)

$$\text{area of } ABCD = \tfrac{1}{2}\left(\frac{b^2 + d^2 - a^2 - c^2}{2\cos\theta}\right)\sin\theta$$

$$\boxed{\text{Area of } ABCD = \tfrac{1}{4}(b^2 + d^2 - a^2 - c^2)\tan\theta} \qquad (3)$$

Using lengths of diagonals and sides

Strategy: Develop (2) by expressing the trigonometric function in terms of the diagonal lengths.

$$d_1 d_2 = \frac{b^2 + d^2 - a^2 - c^2}{2\cos\theta}$$

$$d_1^2 d_2^2 = \frac{(b^2 + d^2 - a^2 - c^2)^2}{4\cos^2\theta} = \frac{(b^2 + d^2 - a^2 - c^2)^2 \sec^2\theta}{4}$$

but $\quad 1 + \tan^2\theta = \sec^2\theta$

$$4d_1^2 d_2^2 = (b^2 + d^2 - a^2 - c^2)^2(1 + \tan^2\theta)$$

$$= (b^2 + d^2 - a^2 - c^2)^2 + (b^2 + d^2 - a^2 - c^2)^2 \tan^2\theta$$

$$4d_1^2 d_2^2 - (b^2 + d^2 - a^2 - c^2)^2 = (b^2 + d^2 - a^2 - c^2)^2 \tan^2\theta$$

$$\sqrt{4d_1^2 d_2^2 - (b^2 + d^2 - a^2 - c^2)^2} = (b^2 + d^2 - a^2 - c^2)\tan\theta$$

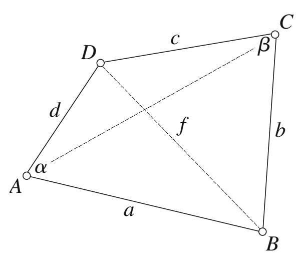
Using (3)

$$\sqrt{4d_1^2 d_2^2 - (b^2 + d^2 - a^2 - c^2)^2} = 4 \times \text{area of } ABCD$$

$$\boxed{\text{Area of } ABCD = \tfrac{1}{4}\sqrt{4d_1^2 d_2^2 - (b^2 + d^2 - a^2 - c^2)^2}}$$

3.4.7 Proof: Area of a general quadrilateral using Heron's formula

Strategy: Use the cosine rule to create an equation in the form of the difference of two squares.



Apply the cosine rule to $\triangle ABD$ and $\triangle BCD$

$$a^2 + d^2 - 2ad\cos\alpha = f^2 \tag{1}$$
$$b^2 + c^2 - 2bc\cos\beta = f^2 \tag{2}$$

Subtract (2) from (1)

$$a^2 + d^2 - b^2 - c^2 = 2(ad\cos\alpha - bc\cos\beta) \tag{3}$$

$$\text{area } \triangle ABD = \tfrac{1}{2}ad\sin\alpha \tag{4}$$

$$\text{area } \triangle BCD = \tfrac{1}{2}bc\sin\beta \tag{5}$$

Add (4) and (5)

$$\text{area } ABCD = A_q = \tfrac{1}{2}(ad\sin\alpha + bc\sin\beta) \tag{6}$$

$$(4A_q)^2 = 4(ad\sin\alpha + bc\sin\beta)^2 \tag{7}$$

Square (3)

$$(a^2 + d^2 - b^2 - c^2)^2 = 4(ad\cos\alpha - bc\cos\beta)^2 \tag{8}$$

Add (7) and (8)

$$16A_q^2 + (a^2 + d^2 - b^2 - c^2)^2 = 4(ad\sin\alpha + bc\sin\beta)^2 + 4(ad\cos\alpha - bc\cos\beta)^2$$

$$= 4(a^2d^2\sin^2\alpha + b^2c^2\sin^2\beta + 2abcd\sin\alpha\sin\beta$$
$$+ a^2d^2\cos^2\alpha + b^2c^2\cos^2\beta + 2abcd\cos\alpha\cos\beta)$$

$$= 4(a^2d^2 + b^2c^2 + 2abcd(\sin\alpha\sin\beta + \cos\alpha\cos\beta)) \tag{9}$$

Substitute $\cos(\alpha + \beta) = \cos 2\varepsilon = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ in (9)
(note the substitution $2\varepsilon = \alpha + \beta$)

$$= 4(a^2d^2 + b^2c^2 - 2abcd\cos 2\varepsilon)$$

$$16A_q^2 + (a^2 + d^2 - b^2 - c^2)^2 = 4(a^2d^2 + b^2c^2 - 2abcd\cos 2\varepsilon) \tag{10}$$

Substitute $\cos 2\varepsilon = 2\cos^2\varepsilon - 1$ in (10)

$$16A_q^2 + (a^2 + d^2 - b^2 - c^2)^2 = 4(a^2d^2 + b^2c^2 - 2abcd(2\cos^2\varepsilon - 1))$$

$$= 4(a^2d^2 + b^2c^2 - 4abcd\cos^2\varepsilon + 2abcd)$$

$$= 4((ad + bc)^2 - 4abcd\cos^2\varepsilon)$$

$$16A_q^2 = 4(ad + bc)^2 - (a^2 + d^2 - b^2 - c^2)^2 - 16abcd\cos^2\varepsilon$$

$$16A_q^2 = (2ad + 2bc)^2 - (a^2 + d^2 - b^2 - c^2)^2 - 16abcd\cos^2\varepsilon$$

Solve the difference of two squares

$$16A_q^2 = (2ad + 2bc + a^2 + d^2 - b^2 - c^2)(2ad + 2bc - a^2 - d^2 + b^2 + c^2) - 16abcd\cos^2\varepsilon$$

$$16A_q^2 = (a + b - c + d)(a - b + c + d)(a + b + c - d)(-a + b + c + d) - 16abcd\cos^2\varepsilon$$

Substitute $2s = a + b + c + d$

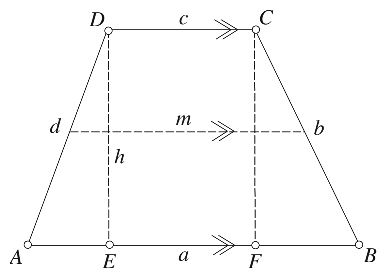
$$16A_q^2 = 16(s-c)(s-b)(s-c)(s-a) - 16abcd\cos^2\varepsilon$$

$$\boxed{A_q = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\varepsilon}}$$

For a cyclic quadrilateral $\alpha + \beta = 180°$ therefore $\varepsilon = 90°$ and $\cos 90° = 0$

$$\boxed{A_{cq} = \sqrt{(s-a)(s-b)(s-c)(s-d)}}$$

3.4.8 Proof: Area of a trapezoid



Area *ABCD*

$$\text{area} = \text{area } EFCD + \text{area } AED + \text{area } FBC$$

$$\text{area} = ch + \tfrac{1}{2}rh + \tfrac{1}{2}sh$$

$$\text{area} = h(c + \tfrac{1}{2}(r+s)) \qquad (1)$$

but $\quad a = c + r + s$

therefore $\quad r + s = a - c \qquad (2)$

Substitute (2) in (1) $\quad \text{area} = h(c + \tfrac{1}{2}(a-c))$

$$\text{area} = \tfrac{1}{2}h(a+c)$$

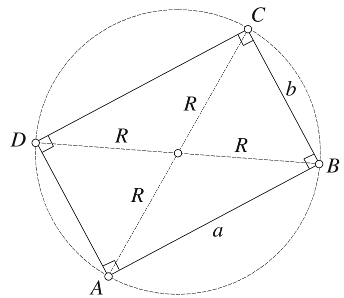
Let $\quad m = \tfrac{1}{2}(a+c)$

$$\boxed{\text{Area} = m \cdot h \quad \text{where } m = \tfrac{1}{2}(a+c)}$$

3.4.9 Proof: Radius and center of the circumscribed circle for a rectangle

To find the radius

Strategy: The circumcenter of a rectangle is located at the intersection of the rectangle's diagonals, which can be located using the Pythagorean theorem.



$$a^2 + b^2 = (2R)^2 = 4R^2$$

$$R = \tfrac{1}{2}\sqrt{a^2 + b^2}$$

or

$$R = \tfrac{1}{2}\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (x_B - x_C)^2 + (y_B - y_C)^2}$$

For a square $b = a$, therefore

$$\boxed{R = \tfrac{1}{2}\sqrt{2}a}$$

To find the center

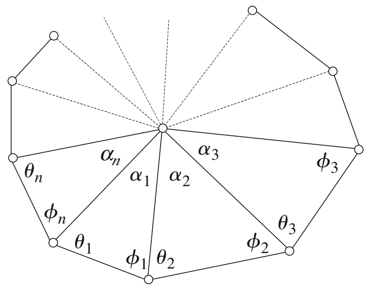
Strategy: Show that the rectangle's diagonals are diameters of the circumscribing circle. $\angle A$ and $\angle B$ are right angles, therefore AC and BD must be equal diameters of the circumscribing circle (Chord theorem). The point P must be the center of the circle. The coordinates of the center P are given by

$$\boxed{\begin{aligned} x_P &= \tfrac{1}{2}(x_A + x_C) \quad \text{or} \quad = \tfrac{1}{2}(x_B + x_D) \\ y_P &= \tfrac{1}{2}(y_A + y_C) \quad \text{or} \quad = \tfrac{1}{2}(y_B + y_D) \end{aligned}}$$

3.5 Polygons

3.5.1 Proof: The internal angles of a polygon

Strategy: Divide the polygon into triangles and analyze their internal triangles.



Let the number of sides to the polygon be n.

Internal angles of a triangle $\theta_i + \phi_i + \alpha_i = 180° \quad 1 \leqslant i \leqslant n$

For one revolution
$$\sum_{i=1}^{n} \alpha_i = 360° \tag{1}$$

Internal angles of n triangles
$$\sum_{i=1}^{n} (\theta_i + \phi_i + \alpha_i) = 180n$$

therefore
$$\sum_{i=1}^{n} (\theta_i + \phi_i) + \sum_{i=1}^{n} \alpha_i = 180n \tag{2}$$

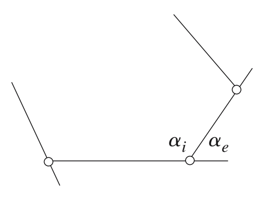
Substitute (1) in (2)
$$\sum_{i=1}^{n} (\theta_i + \phi_i) + 360° = 180n$$

therefore
$$\sum_{i=1}^{n} (\theta_i + \phi_i) = 180n - 360°$$

The internal angles of an n-sided polygon sum to $180(n - 2)°$.

3.5.2 Proof: The external angles of a polygon

Strategy: Exploit the relationship for the internal angles of a polygon for the external angles.



Let the number of sides to the polygon be n.

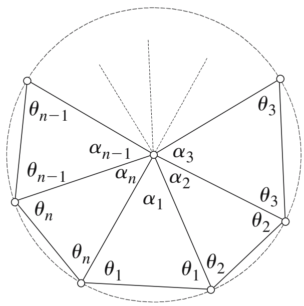
α_i is an internal angle, and α_e is the complementary external angle

therefore	$\alpha_i + \alpha_e = 180°$
With n such combinations	$n(\alpha_i + \alpha_e) = 180n$
and for n internal angles	$n\alpha_i = 180(n-2)$
therefore	$180(n-2) + n\alpha_e = 180n$
	$180n - 360° + n\alpha_e = 180n$
therefore	$n\alpha_e = 360°$

The external angles of an n-sided polygon sum to 360°.

3.5.3 Proof: Alternate internal angles of a cyclic polygon

Strategy: Divide the polygon into triangles and analyze their angles.



Let the number of sides to the polygon be n.
The internal angles of a triangle in the polygon

$$2\theta_i + \alpha_i = 180° \quad 1 \leqslant i \leqslant n$$

For one revolution
$$\sum_{i=1}^{n} \alpha_i = 360° \tag{1}$$

For n triangles
$$\sum_{i=1}^{n} (2\theta_i + \alpha_i) = 180n$$

$$\sum_{i=1}^{n} 2\theta_i + \sum_{i=1}^{n} \alpha_i = 180n \tag{2}$$

Substitute (1) in (2)
$$\sum_{i=1}^{n} 2\theta_i + 360° = 180n$$

$$\sum_{i=1}^{n} 2\theta_i = 180n - 360°$$

therefore $$\sum_{i=1}^{n} 2\theta_i = 90(n-2)$$

i.e. $$(\theta_1+\theta_2)+(\theta_3+\theta_4)+(\theta_5+\theta_6)+\cdots+(\theta_{n-1}+\theta_n)=90(n-2)$$

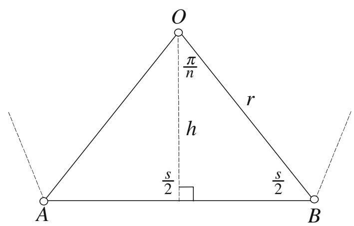
or $$(\theta_2+\theta_3)+(\theta_4+\theta_5)+(\theta_6+\theta_7)+\cdots+(\theta_n+\theta_1)=90(n-2)$$

[where $n \geqslant 4$ and is even]

$$\boxed{\text{The alternate internal angles sum to } 90(n-2)^\circ.}$$

3.5.4 Proof: Area of a regular polygon

Strategy: Given a regular polygon with n sides, side length s, and radius r of the circumscribed circle, its area is computed by dividing it into n isosceles triangles and summing their total area.



The isosceles triangle OAB is formed by an edge s and the center O of the polygon.

$$\frac{\frac{1}{2}s}{h} = \tan\left(\frac{\pi}{n}\right)$$

therefore $$h = \tfrac{1}{2}s\cot\left(\frac{\pi}{n}\right)$$

$$\text{area of } \triangle OAB = \tfrac{1}{2}sh = \tfrac{1}{4}s^2\cot\left(\frac{\pi}{n}\right)$$

$$\boxed{\text{Area} = \tfrac{1}{4}ns^2\cot\left(\frac{\pi}{n}\right)}$$

But $$\frac{\frac{1}{2}s}{r} = \sin\left(\frac{\pi}{n}\right)$$

therefore $$\tfrac{1}{2}s = r\sin\left(\frac{\pi}{n}\right)$$

$$\frac{h}{r} = \cos\left(\frac{\pi}{n}\right)$$

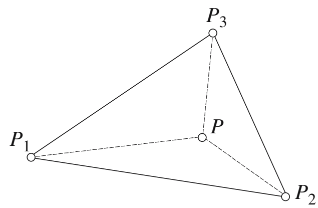
$$h = r\cos\left(\frac{\pi}{n}\right)$$

therefore area of $\triangle OAB = \frac{1}{2}sh = r^2 \sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right) = \frac{1}{2}r^2 \sin\left(\frac{2\pi}{n}\right)$

$$\boxed{\text{Area} = \frac{1}{2}nr^2 \sin\left(\frac{2\pi}{n}\right)}$$

3.5.5 Proof: Area of a polygon

Strategy: Divide the polygon (e.g. a triangle) into three arbitrary smaller triangles. Then derive the area of the polygon from the areas of the individual triangles.



Let P_1, P_2, P_3 be the counter-clockwise vertices of a triangle. Also, let $P(x, y)$ be an arbitrary point inside $\triangle P_1P_2P_3$.

The area of a triangle is $\text{area} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

therefore,

$$\text{area of } \triangle P_1P_2P_3 = \text{area of } \triangle P_1P_2P + \text{area of } \triangle P_2P_3P + \text{area of } \triangle P_3P_1P$$

$$\text{Area } A \text{ of } \triangle P_1P_2P_3 = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x & y & 1 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x & y & 1 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_3 & y_3 & 1 \\ x_1 & y_1 & 1 \\ x & y & 1 \end{vmatrix}$$

$$\begin{aligned} \text{area} = \tfrac{1}{2}(&x_1y_2 + xy_1 + x_2y - x_1y - x_2y_1 - xy_2 + x_2y_3 + xy_2 + x_3y \\ &-x_2y - x_3y_2 - xy_3 + x_3y_1 + xy_3 + x_1y - x_3y - x_1y_3 - xy_1) \end{aligned}$$

$$\text{area} = \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3) \qquad (1)$$

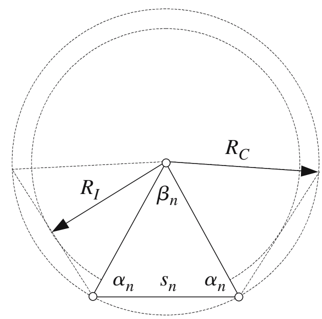
$$\boxed{\text{Area} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}$$

From (1) the area of a polygon with n sides is

$$\boxed{\text{Area} = \frac{1}{2}\sum_{i=0}^{n-1}(x_i y_{i+1(\text{mod } n)} - y_i x_{i+1(\text{mod } n)})}$$

3.5.6 Proof: Properties of regular polygons

Let n be the number of sides to the regular polygon, and s_n be the edge length.
R_I and R_C are the radii of the internal and outer circles respectively.



Apex angle is $$\beta_n = \frac{360°}{n}$$

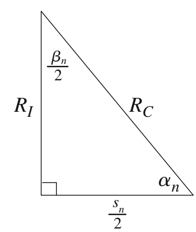
Let the base angle be α_n

The internal angles of a triangle $2\alpha_n + \beta_n = 180°$

$$2\alpha_n + \frac{360°}{n} = 180°$$

The base angle is $$\alpha_n = \left(1 - \frac{2}{n}\right)90°$$

Inradius R_I



$$\frac{s_n}{2R_I} = \tan\left(\frac{\beta_n}{2}\right) = \tan\left(\frac{\pi}{n}\right)$$

$$R_I = \frac{s_n}{2}\cot\left(\frac{\pi}{n}\right)$$

also $$\frac{R_I}{R_C} = \cos\left(\frac{\beta_n}{2}\right) = \cos\left(\frac{\pi}{n}\right)$$

$$\boxed{\text{The inradius } R_I = R_C \cos\left(\frac{\pi}{n}\right)}$$

Circumradius R_C

$$\frac{s_n}{2R_C} = \sin\left(\frac{\beta_n}{2}\right) = \sin\left(\frac{\pi}{n}\right)$$

$$\boxed{\text{The circumradius } R_C = \frac{s_n}{2\sin\left(\frac{\pi}{n}\right)}}$$

Area A_n

Calculate the area of one isosceles triangle in the regular polygon.

$$\frac{s_n}{2} R_I = \frac{s_n^2}{4} \cot\left(\frac{\pi}{n}\right)$$

$$\boxed{\text{Area of the polygon is } A_n = n \frac{s_n^2}{4} \cot\left(\frac{\pi}{n}\right)}$$

or

$$\boxed{A_n = \tfrac{1}{2} n s_n R_I}$$

or

$$\boxed{A_n = \tfrac{1}{2} n s_n R_C \cos\left(\tfrac{\pi}{n}\right)}$$

3.6 Three-dimensional objects

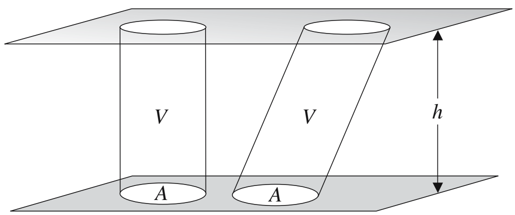
3.6.1 Proof: Volume of a prism

Strategy: The approximate volume of an object can be determined by cutting it into a large number of thin slices and summing their individual volumes. Integral calculus develops this idea by making the slices infinitesimally thin and securing a limiting value.

In general, one can write $$V = \int_a^b \underbrace{A(x)}_{\text{area of the cross-section}} \quad \underbrace{dx}_{\text{thickness of the slice}}$$

If the volume is considered as an infinite set of slices, it is unaffected by any linear or rotational offset applied to the slices, because any offset will not alter the individual volume of a slice. This is known as Cavalieri's theorem, after Bonaventura Cavalieri (1598–1647).

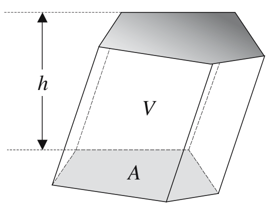
This implies that objects with the same cross-section and height possess equal volumes. For example, the following objects have equal volumes:



where volume $V = Ah$

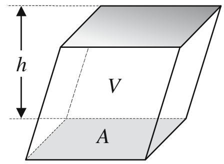
The volume of any prism obeys this formula.

General prism



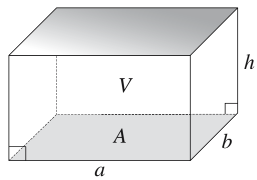
$$V = Ah$$

Parallelepiped



$V = Ah$

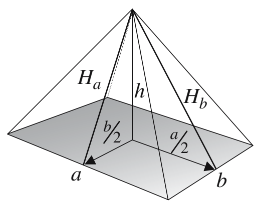
Rectangular parallelepiped



$A = ab$
$V = abh$

3.6.2 Proof: Surface area of a rectangular pyramid

Strategy: Divide the surface area into its component parts.



Slant heights $\quad H_a = \sqrt{h^2 + \tfrac{1}{4}b^2} \quad$ and $\quad H_b = \sqrt{h^2 + \tfrac{1}{4}a^2}$

Surface area $\quad A =$ area of base + area of 4 triangles

$$A = ab + (\tfrac{1}{2}aH_a + \tfrac{1}{2}aH_a + \tfrac{1}{2}bH_b + \tfrac{1}{2}bH_b)$$

$$A = ab + aH_a + bH_b$$

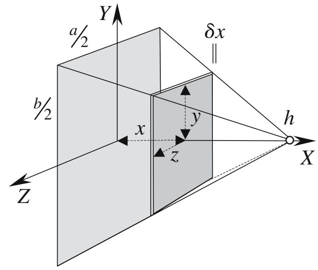
$$A = ab + a\sqrt{h^2 + \tfrac{1}{4}b^2} + b\sqrt{h^2 + \tfrac{1}{4}a^2}$$

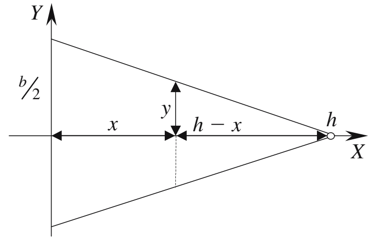
Surface area	$A = ab + \tfrac{1}{2}(a\sqrt{4h^2 + b^2} + b\sqrt{4h^2 + a^2})$
when $a = b$	$A = a^2 + a\sqrt{4h^2 + a^2}$

3.6.3 Proof: Volume of a rectangular pyramid

Strategy: Use integral calculus to find the volume of a pyramid by summing vertical cross-sections.

Let the dimensions of the pyramid be





base: $a \times b$ and height: h

Area of slice $A_s = 4yz$

Volume of slice $V_s = 4yz\,\delta x$

but $\dfrac{y}{h-x} = \dfrac{b/2}{h}$

therefore $y = \dfrac{b}{2h}(h-x)$

Similarly $\dfrac{z}{h-x} = \dfrac{a/2}{h}$

therefore $z = \dfrac{a}{2h}(h-x)$

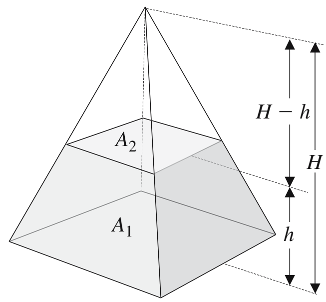
Volume of slice
$$V_s = \frac{ab}{h^2}(h-x)^2\,\delta x$$
$$= \frac{ab}{h^2}(h^2 - 2xh + x^2)\delta x$$

Volume of pyramid
$$V = \frac{ab}{h^2}\int_0^h (h^2 - 2xh + x^2)dx$$
$$= \frac{ab}{h^2}\left[h^2x - hx^2 + \frac{x^3}{3}\right]_0^h$$
$$= \frac{ab}{h^2}\left(h^3 - h^3 + \frac{h^3}{3}\right)$$
$$V = \tfrac{1}{3}abh$$

$$\boxed{\text{Volume of a pyramid } = \tfrac{1}{3}abh}$$

Note that the formula can be expressed as $V = \frac{1}{3}$ area of *base* × *height*

3.6.4 Volume of a rectangular pyramidal frustum



Volume of frustum = volume of whole pyramid − volume of top pyramid

$$\begin{aligned} V_F &= \tfrac{1}{3}A_1H - \tfrac{1}{3}A_2(H-h) \\ &= \tfrac{1}{3}H(A_1 - A_2) + \tfrac{1}{3}hA_2 \end{aligned} \tag{1}$$

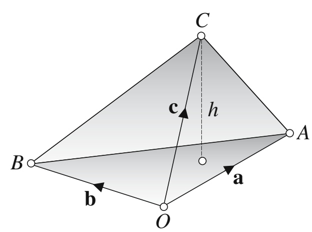
but
$$\sqrt{\frac{A_2}{A_1}} = \frac{H-h}{H}$$

therefore
$$H = \frac{h\sqrt{A_1}}{\sqrt{A_1} - \sqrt{A_2}} \tag{2}$$

Substitute (2) in (1)
$$\begin{aligned} V_F &= \tfrac{1}{3}h\frac{\sqrt{A_1}}{\sqrt{A_1} - \sqrt{A_2}}(A_1 - A_2) + \tfrac{1}{3}hA_2 \\ &= \tfrac{1}{3}h\left(\frac{\sqrt{A_1}}{\sqrt{A_1} - \sqrt{A_2}}(\sqrt{A_1} - \sqrt{A_2})(\sqrt{A_1} + \sqrt{A_2}) + A_2\right) \\ V_F &= \tfrac{1}{3}h(A_1 + A_2 + \sqrt{A_1A_2}) \end{aligned}$$

$$\boxed{\text{Volume of a frustum } = \tfrac{1}{3}h(A_1 + A_2 + \sqrt{A_1A_2})}$$

3.6.5 Proof: Volume of a triangular pyramid



Strategy: Use the volume of a pyramid to derive the volume of a triangular pyramid.

Volume of a pyramid is $\tfrac{1}{3}$ area of *base* × *height*

Area of base is $\tfrac{1}{2}\|\mathbf{a} \times \mathbf{b}\|$

Volume of pyramid is $\frac{1}{3} \cdot \frac{1}{2} \|\mathbf{a} \times \mathbf{b}\| h$

Volume of a parallelepiped is $\|\mathbf{a} \times \mathbf{b}\| \times h = \begin{vmatrix} x_a & y_a & z_a \\ x_b & y_b & z_b \\ x_c & y_c & z_c \end{vmatrix}$

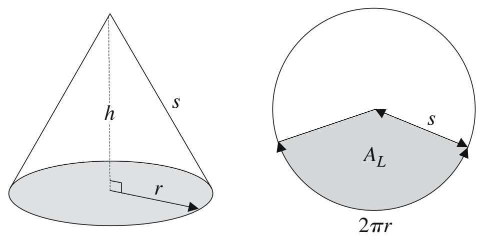
The volume of a pyramid is $\frac{1}{6} \begin{vmatrix} x_a & y_a & z_a \\ x_b & y_b & z_b \\ x_c & y_c & z_c \end{vmatrix}$

Note: The volume is positive if the vertices A, B, C appear clockwise from O, otherwise it is negative.

3.6.6 Proof: Surface area of a right cone

Strategy: Develop the lateral surface area of a right cone from the sector of a circle.

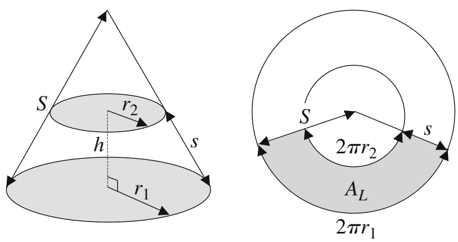
The sector marked A_L will form the lateral surface area of a right cone with radius r and slant height s.



$$\text{Area of sector } = \frac{2\pi r}{2\pi s}\pi r^2 = \pi r s$$

Lateral surface area is	$A_L = \pi r s$
Total surface area with base	$A = \pi r(r + s)$

3.6.7 Proof: Surface area of a right conical frustum



Lateral surface area of the frustum = lateral area of whole cone − lateral area of top cone

$$A_L = \pi r_1 S - \pi r_2(S - s)$$

$$A_L = \pi(S(r_1 - r_2) + s r_2) \tag{1}$$

but $$\frac{r_1}{r_2} = \frac{S}{S - s}$$

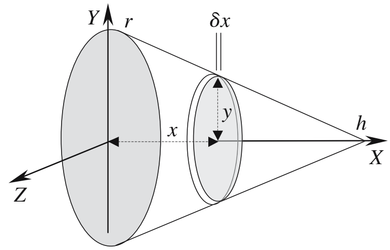
therefore $$S = \frac{r_1 s}{r_1 - r_2} \tag{2}$$

Substitute (2) in (1) $$A_L = \pi\left(\frac{r_1 s}{r_1 - r_2}(r_1 - r_2) + sr_2\right)$$

Lateral surface area	$A_L = \pi s(r_1 + r_2)$
Total surface area	$A = \pi(r_1{}^2 + r_2{}^2 + s(r_1 + r_2))$

3.6.8 Proof: Volume of a cone

Strategy: Use integral calculus to find the volume of a cone by summing vertical cross-sections.

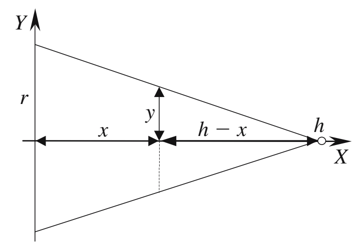


Cone with radius r and height h.

Area of disk $= \pi y^2$

Volume of disk $= \pi y^2 \delta x$

but $$\frac{y}{h - x} = \frac{r}{h}$$



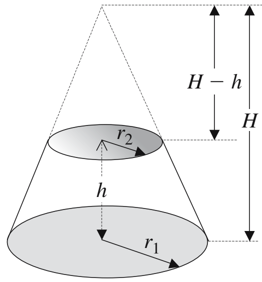
therefore $$y = \frac{r}{h}(h - x)$$

Volume of disk $$= \pi\left(\frac{r}{h}(h - x)\right)^2 \delta x$$

Volume of cone
$$\begin{aligned}
&= \int_0^h \frac{\pi r^2}{h^2}(h^2 - 2hx + x^2)dx \\
&= \frac{\pi r^2}{h^2}\int_0^h (h^2 - 2hx + x^2)dx \\
&= \frac{\pi r^2}{h^2}\left[h^2 x - hx^2 + \frac{x^3}{3}\right]_0^h \\
&= \frac{\pi r^2}{h^2}\left(h^3 - h^3 + \frac{h^3}{3}\right) \\
&= \frac{\pi r^2 h}{3}
\end{aligned}$$

$$\boxed{\text{Volume of a cone} = \tfrac{1}{3}\pi r^2 h}$$

3.6.9 Proof: Volume of a right conical frustum



Volume of frustum = volume of whole cone − volume of top cone

$$\begin{aligned} V_F &= \tfrac{1}{3}\pi r_1^2 H - \tfrac{1}{3}\pi r_2^2 (H - h) \\ &= \tfrac{1}{3}\pi H(r_1^2 - r_2^2) + \tfrac{1}{3}\pi r_2^2 h \end{aligned} \tag{1}$$

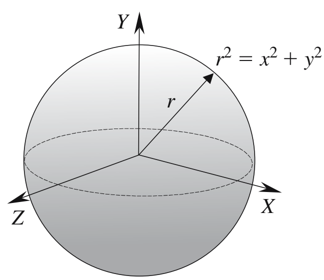
but
$$\frac{r_2}{r_1} = \frac{H - h}{H}$$

therefore
$$H = \frac{r_1}{r_1 - r_2} h \tag{2}$$

Substitute (2) in (1)
$$\begin{aligned} V_F &= \tfrac{1}{3}\pi h\left(\frac{r_1}{r_1 - r_2}(r_1^2 - r_2^2) + r_2^2\right) \\ &= \tfrac{1}{3}\pi h\left(\frac{r_1(r_1 - r_2)(r_1 + r_2)}{r_1 - r_2} + r_2^2\right) \\ V_F &= \tfrac{1}{3}\pi h(r_1^2 + r_2^2 + r_1 r_2) \end{aligned}$$

$$\boxed{\text{Volume of a right conical frustum } = \tfrac{1}{3}\pi h(r_1^2 + r_2^2 + r_1 r_2)}$$

3.6.10 Proof: Surface area of a sphere



Strategy: Use the integral formula for computing the surface area of revolution.

The equation of the 2D curve is $y = \sqrt{r^2 - x^2}$

The general equation for the surface area of revolution is

$$S = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\,dx$$

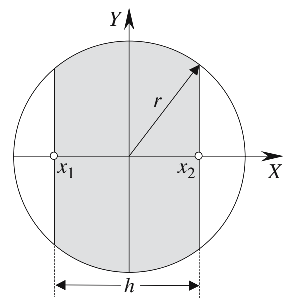
therefore, the surface area of a sphere is

$$
\begin{aligned}
S &= 2\pi\int_{-r}^{r}\sqrt{r^2-x^2}\sqrt{1+\frac{d}{dx}\left[\sqrt{r^2-x^2}\right]^2}\,dx \\
&= 2\pi\int_{-r}^{r}\sqrt{r^2-x^2}\sqrt{1+\left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2}\,dx \\
&= 2\pi\int_{-r}^{r}\sqrt{r^2-x^2}\,\frac{r}{\sqrt{r^2-x^2}}\,dx \\
&= 2\pi r\int_{-r}^{r}dx \\
&= 2\pi r\Big[x\Big]_{-r}^{r} = 4\pi r^2
\end{aligned}
\qquad (1)
$$

$$\boxed{\text{Surface area of a sphere} = 4\pi r^2}$$

Surface area of a spherical segment

We can compute the surface area of a spherical segment by integrating equation (1) above over different limits. The limit range is determined by the segment thickness h and the new limits become x_1 to x_2:

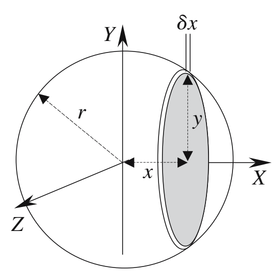


$$
\begin{aligned}
\text{Surface area of segment} &= 2\pi r[x]_{x_1}^{x_2} \\
&= 2\pi rh
\end{aligned}
$$

$$\boxed{\text{Surface area of a spherical segment} = 2\pi rh}$$

3.6.11 Proof: Volume of a sphere

Strategy: Use integral calculus to find the volume of a sphere by summing vertical cross-sections.



$$\text{Area of disk} = \pi y^2$$

$$\text{Volume of disk} = \pi y^2\delta x$$

but $$y^2 = r^2 - x^2$$

$$\text{Volume of disk} = \pi(r^2 - x^2)\delta x$$

$$\text{Volume of sphere } V = \int_{-r}^{r} \pi(r^2 - x^2)dx$$

$$V = \pi\left[r^2 x - \frac{x^3}{3}\right]_{-r}^{r}$$

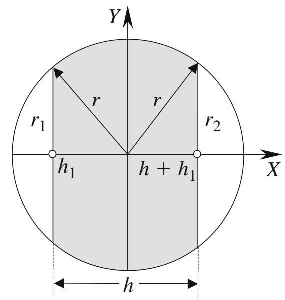
$$= \pi\left(r^3 - \frac{r^3}{3} + r^3 - \frac{r^3}{3}\right) \quad (1)$$

$$= \pi\left(2r^3 - \frac{2r^3}{3}\right)$$

$$\boxed{\text{Volume of a sphere } = \tfrac{4}{3}\pi r^3}$$

Volume of a spherical segment

The volume of a spherical segment is computed by integrating equation (1) above over different limits. The limit range is determined by the segment thickness h and the radii of the circular ends r_1 and r_2. The limits become h_1 to $h + h_1$:



$$V = \pi\left[r^2 x - \frac{x^3}{3}\right]_{h_1}^{h+h_1}$$

$$= \pi(r^2(h+h_1) - \tfrac{1}{3}(h+h_1)^3 - r^2 h_1 + \tfrac{1}{3}h_1^3)$$

$$= \tfrac{1}{3}\pi(3r^2 h - (h+h_1)^3 + h_1^3) \quad (2)$$

$$= \tfrac{1}{3}\pi(3r^2 h - h^3 - 3h^2 h_1 - 3hh_1^2)$$

$$= \tfrac{1}{3}\pi h(3r^2 - h^2 - 3hh_1 - 3h_1^2)$$

but $\quad r^2 = r_2^2 + (h + h_1)^2 \quad (3)$

and $\quad r^2 = r_1^2 + h_1^2 \quad (4)$

Subtract (4) from (3) $\quad hh_1 = \tfrac{1}{2}(r_1^2 - r_2^2 - h^2) \quad (5)$

Substitute (5) in (2) $\quad V = \tfrac{1}{3}\pi h(3r^2 - h^2 - \tfrac{3}{2}(r_1^2 - r_2^2 - h^2) - 3h_1^2)$

$$= \tfrac{1}{6}\pi h(6r^2 + h^2 + 3r_2^2 - 3r_1^2 - 6h_1^2)$$

but $\quad h_1^2 = r^2 - r_1^2$

$$V = \tfrac{1}{6}\pi h(6r^2 + h^2 + 3r_2^2 - 3r_1^2 - 6r^2 + 6r_1^2)$$

$$V = \tfrac{1}{6}\pi h(3r_1^2 + 3r_2^2 + h^2)$$

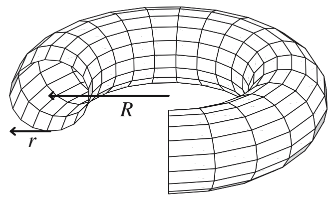
$$\boxed{\text{Volume of a spherical segment } = \tfrac{1}{6}\pi h(3r_1^2 + 3r_2^2 + h^2)}$$

If one of the radii is zero the volume becomes

$$\boxed{\text{Volume of a spherical segment } = \tfrac{1}{6}\pi h(3r_1^2 + h^2)}$$

3.6.12 Proof: Area and volume of a torus

Strategy: Guldin's first rule states that the area of a surface of revolution is the product of the arc length of the generating curve and the distance traveled by its centroid. Guldin's second rule states that the volume of a surface of revolution is the product of the cross-sectional area and the distance traveled by the area's centroid.



Surface area

$$\text{Length of the cross-section} = 2\pi r$$
$$\text{Path of the centroid} = 2\pi R$$

$$\boxed{\text{Surface area} = 4\pi^2 rR}$$

Volume

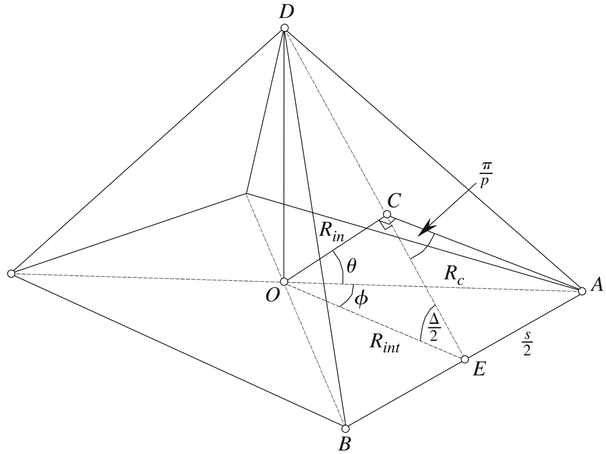
$$\text{Area of the cross-section} = \pi r^2$$
$$\text{Path of the centroid} = 2\pi R$$

$$\boxed{\text{Volume of torus} = 2\pi^2 r^2 R}$$

3.6.13 Proof: Radii of the spheres associated with the Platonic solids

Strategy: Each Platonic solid is constructed from a common regular polygon. The resulting symmetry ensures that every vertex lies on a circumsphere. Similarly, a mid-sphere exists which touches the mid-point of each edge. Thirdly, an in-sphere exists which lies on the mid-point of every face. The radii of these spheres can be calculated by considering the geometry associated with a portion of a single Platonic object: an octahedron.

Let $q =$ number of edges associated with a vertex
$p =$ number of edges associated with a face



R_c = radius of the circumsphere touching every vertex
R_{int} = radius of the mid-sphere touching the mid-point of each edge
R_{in} = radius of the in-sphere touching the mid-point of each face

Let s = length of an edge
O = center of the octahedron
E = mid-point of the edge AB
C = mid-point of the face ABD
R_c = radius of the circumsphere
R_{int} = radius of the mid-sphere
R_{in} = radius of the inner sphere
$\frac{\Delta}{2}$ = half the dihedral angle

$\triangle EOA$, $\triangle EDA$, $\triangle COE$, $\triangle COA$ are right-angled triangles.
$\triangle DAB$ is an equilateral triangle.

Therefore $\quad \angle DAB = \dfrac{\pi}{p} \qquad \angle CAB = \dfrac{\pi}{2p} \qquad \angle ECA = \dfrac{\pi}{p}$

Let $\quad \angle AOE = \phi \qquad \angle COA = \theta$

but $\quad \angle AOB = \dfrac{2\pi}{q}$

therefore $\quad \phi = \dfrac{\pi}{q}$

The objective of the proof is to express R_c, R_{in}, R_{int} in terms of p, q and s.

Let us introduce two intermediate equations

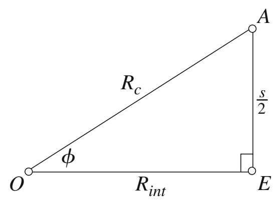
$$\sin^2\left(\frac{\pi}{p}\right) + \cos^2\left(\frac{\pi}{p}\right) = 1$$

$$\sin^2\left(\frac{\pi}{q}\right) + \cos^2\left(\frac{\pi}{q}\right) = 1$$

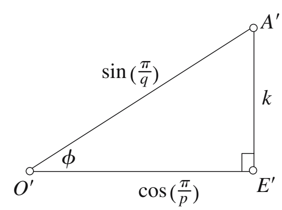
therefore
$$\sin^2\left(\frac{\pi}{p}\right) - \cos^2\left(\frac{\pi}{q}\right) = \sin^2\left(\frac{\pi}{q}\right) - \cos^2\left(\frac{\pi}{p}\right) = k^2$$

where k is some constant.

We already have a triangle $\triangle EOA$ as follows



but a similar triangle $\triangle E'O'A'$ can be created if we make $\sin(\phi) = \dfrac{k}{\sin\left(\frac{\pi}{q}\right)}$



Comparing the two similar triangles we discover that

$$\frac{k}{\sin\left(\frac{\pi}{q}\right)} = \frac{s/2}{R_c}$$

therefore
$$\frac{R_c}{s} = \frac{1}{2k}\sin\left(\frac{\pi}{q}\right)$$

and
$$\frac{k}{\cos\left(\frac{\pi}{p}\right)} = \frac{s/2}{R_{int}}$$

Therefore
$$\frac{R_{int}}{s} = \frac{1}{2k}\cos\left(\frac{\pi}{p}\right)$$

but
$$k = \sqrt{\sin^2\left(\frac{\pi}{q}\right) - \cos^2\left(\frac{\pi}{p}\right)}$$

therefore
$$\frac{R_c}{s} = \frac{\sin\left(\frac{\pi}{q}\right)}{2\sqrt{\sin^2\left(\frac{\pi}{q}\right) - \cos^2\left(\frac{\pi}{p}\right)}}$$

and
$$\boxed{\frac{R_{int}}{s} = \frac{\cos\left(\frac{\pi}{p}\right)}{2\sqrt{\sin^2\left(\frac{\pi}{q}\right) - \cos^2\left(\frac{\pi}{p}\right)}}}$$

From $\triangle COE$
$$R_{in}^2 = R_{int}^2 - \left(\frac{s}{2}\right)^2 \cot^2\left(\frac{\pi}{p}\right)$$

therefore
$$\boxed{\frac{R_{in}}{s} = \frac{\cot\left(\frac{\pi}{p}\right)\cos\left(\frac{\pi}{q}\right)}{2\sqrt{\sin^2\left(\frac{\pi}{q}\right) - \cos^2\left(\frac{\pi}{p}\right)}}}$$

We can also express R_c in terms of R_{in} as follows:
$$\frac{R_c}{s} = \frac{\sin\left(\frac{\pi}{q}\right)}{2k}$$

and
$$\frac{R_{in}}{s} = \frac{\cot\left(\frac{\pi}{p}\right)\cos\left(\frac{\pi}{q}\right)}{2k}$$

therefore
$$\boxed{R_c = R_{in}\tan\left(\frac{\pi}{p}\right)\tan\left(\frac{\pi}{q}\right)}$$

Compute R_{in}, R_{int} and R_c for the five Platonic objects.

Tetrahedron $\quad p = 3 \qquad q = 3 \qquad s = 1$

$$R_{in} = \frac{\sqrt{6}}{12} = 0.204124$$
$$R_{int} = \frac{\sqrt{2}}{4} = 0.353554$$

$$\boxed{R_c = \frac{\sqrt{6}}{4} = 0.612372}$$

Cube $p = 4 \quad q = 3 \quad s = 1$

$$R_{in} = \tfrac{1}{2} = 0.5$$

$$R_{int} = \frac{\sqrt{2}}{2} = 0.707107$$

$$\boxed{R_c = \frac{\sqrt{3}}{2} = 0.866025}$$

Octahedron $p = 3 \quad q = 4 \quad s = 1$

$$R_{in} = \frac{\sqrt{6}}{6} = 0.408248$$

$$R_{int} = \tfrac{1}{2} = 0.5$$

$$\boxed{R_c = \frac{\sqrt{2}}{2} = 0.707107}$$

Dodecahedron $p = 5 \quad q = 3 \quad s = 1$

$$R_{in} = \tfrac{1}{20}\sqrt{250 + 110\sqrt{5}} = 1.113516$$

$$R_{int} = \tfrac{1}{4}\sqrt{14 + 6\sqrt{5}} = 1.309017$$

$$\boxed{R_c = \tfrac{1}{4}\sqrt{18 + 6\sqrt{5}} = 1.401259}$$

Icosahedron $p = 5 \quad q = 5 \quad s = 1$

$$R_{in} = \tfrac{1}{12}\sqrt{42 + 18\sqrt{5}} = 0.755761$$

$$R_{int} = \tfrac{1}{4}\sqrt{6 + 2\sqrt{5}} = 0.809017$$

$$\boxed{R_c = \tfrac{1}{4}\sqrt{10 + 2\sqrt{5}} = 0.951057}$$

Calculating the dihedral angles

From $\triangle COE$ we see that $\dfrac{R_{in}}{R_{int}} = \sin\left(\frac{\Delta}{2}\right)$ where Δ is the dihedral angle.

$$\frac{R_{in}}{s} = \frac{\cot\left(\frac{\pi}{p}\right)\cos\left(\frac{\pi}{q}\right)}{2\sqrt{\sin^2\left(\frac{\pi}{q}\right) - \cos^2\left(\frac{\pi}{p}\right)}}$$

$$\frac{R_{int}}{s} = \frac{\cos\left(\frac{\pi}{p}\right)}{2\sqrt{\sin^2\left(\frac{\pi}{q}\right) - \cos^2\left(\frac{\pi}{p}\right)}}$$

therefore

$$\boxed{\frac{R_{in}}{R_{int}} = \frac{\cos\left(\frac{\pi}{q}\right)}{\sin\left(\frac{\pi}{p}\right)} = \sin\left(\frac{\Delta}{2}\right)}$$

Tetrahedron $$\Delta = 2\sin^{-1}\left(\frac{\cos 60^\circ}{\sin 60^\circ}\right) = 70.528878^\circ$$

Cube $$\Delta = 2\sin^{-1}\left(\frac{\cos 60^\circ}{\sin 45^\circ}\right) = 90^\circ$$

Octahedron $$\Delta = 2\sin^{-1}\left(\frac{\cos 45^\circ}{\sin 60^\circ}\right) = 109.471221^\circ$$

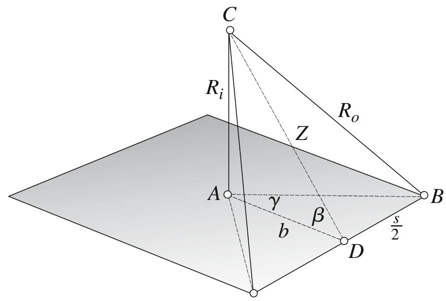
Dodecahedron $$\Delta = 2\sin^{-1}\left(\frac{\cos 60^\circ}{\sin 36^\circ}\right) = 116.565051^\circ$$

Icosahedron $$\Delta = 2\sin^{-1}\left(\frac{\cos 36^\circ}{\sin 60^\circ}\right) = 138.189685^\circ$$

3.6.14 Proof: Inner and outer radii for the Platonic solids

Strategy: Each Platonic solid is constructed from a common regular polygon. The vertices of each solid lie on a sphere whose radius R_o is calculated as shown below.

Using the geometry of a cube as an illustration, a parametric formula is derived which can be applied to each solid in turn. The outer radius is expressed as a ratio to the edge length s.



Let C = center of the cube

s = edge length

β = half the dihedral angle

R_i = radius of the inner sphere

R_o = radius of the outer sphere

$\triangle ACD$ $\quad R_i^2 + b^2 = Z^2$

$\triangle DCB$ $\quad Z^2 + (s/2)^2 = R_o^2$

therefore $\quad R_i^2 + b^2 + (s/2)^2 = R_o^2$

$$\tan\beta = \frac{R_i}{b}$$

and $\quad R_i = b\tan\beta$

therefore
$$b^2\tan^2\beta + b^2 + (s/2)^2 = R_o^2$$
$$b^2(\tan^2\beta + 1) + (s/2)^2 = R_o^2$$
$$\tan\gamma = \frac{s/2}{b}$$

therefore
$$b = \frac{s/2}{\tan\gamma}$$
$$\frac{(s/2)^2}{\tan^2\gamma}(\tan^2\beta + 1) + \left(\frac{s}{2}\right)^2 = R_o^2$$
$$\frac{\tan^2\beta + 1}{\tan^2\gamma} + 1 = \frac{R_o^2}{(s/2)^2}$$

therefore
$$\frac{R_o}{s/2} = \sqrt{1 + \frac{1 + \tan^2\beta}{\tan^2\gamma}}$$

and
$$\boxed{\frac{R_o}{s} = \tfrac{1}{2}\sqrt{1 + \frac{1 + \tan^2\beta}{\tan^2\gamma}}}$$

Tetrahedron $\quad \gamma = 60° \qquad \beta = 70.528779°/2$

$$\frac{R_o}{s} = \frac{1}{2}\sqrt{1 + \frac{1 + 0.5}{3}} = \frac{\sqrt{1.5}}{2}$$

$$\boxed{\frac{R_o}{s} = 0.612372}$$

Cube $\gamma = 45°$ $\beta = 90°/2$

$$\frac{R_o}{s} = \tfrac{1}{2}\sqrt{1 + \frac{1+1}{3}} = \frac{\sqrt{3}}{2}$$

$$\boxed{\frac{R_o}{s} = 0.866025}$$

Octahedron $\gamma = 60°$ $\beta = 109.47122°/2$

$$\frac{R_o}{s} = \tfrac{1}{2}\sqrt{1 + \frac{1+2}{3}} = \frac{\sqrt{2}}{2}$$

$$\boxed{\frac{R_o}{s} = 0.707107}$$

Dodecahedron $\gamma = 36°$ $\beta = 116.56505°/2$

$$\frac{R_o}{s} = \tfrac{1}{2}\sqrt{1 + \frac{1+2.618}{0.527864}}$$

$$\boxed{\frac{R_o}{s} = 1.4012585}$$

Icosahedron $\gamma = 60°$ $\beta = 138.189685°/2$

$$\frac{R_o}{s} = \tfrac{1}{2}\sqrt{1 + \frac{1+6.854102}{3}}$$

$$\boxed{\frac{R_o}{s} = 0.9510565}$$

The outer sphere of radius R_o intersects all the vertices, whereas the inner sphere of radius R_i touches the center of each face.

Using the original diagram

$\triangle ACD$ $R_i^2 + b^2 = Z^2$

but $\dfrac{b}{Z} = \cos\beta$

therefore $$Z = \frac{b}{\cos\beta}$$

therefore $$R_i^2 + b^2 = \frac{b^2}{\cos^2\beta}$$

$$R_i^2 = \frac{b^2}{\cos^2\beta} - b^2 = b^2\left(\frac{1}{\cos^2\beta} - 1\right)$$

But $$\frac{s/2}{b} = \tan\gamma$$

therefore $$b = \frac{s/2}{\tan\gamma}$$

therefore $$R_i^2 = \frac{(s/2)^2}{\tan^2\gamma}\left(\frac{1}{\cos^2\beta} - 1\right)$$

$$R_i^2 = \frac{s^2}{4\tan^2\gamma}\tan^2\beta$$

$$\frac{R_i^2}{s^2} = \frac{\tan^2(\beta)}{4\tan^2(\gamma)}$$

$$\boxed{\frac{R_i}{s} = \frac{\tan\beta}{2\tan\gamma}}$$

Tetrahedron $\gamma = 60°$ $\beta = 70.528779°/2$

$$\boxed{\frac{R_i}{s} = \frac{\tan 35.264389°}{2\tan 60°} = 0.204124}$$

Cube $\gamma = 45°$ $\beta = 90°/2$

$$\boxed{\frac{R_i}{e} = \frac{\tan 45°}{2\tan 45°} = 0.5}$$

Octahedron $\gamma = 60°$ $\beta = 109.47122°/2$

$$\boxed{\frac{R_i}{e} = \frac{\tan 54.73561°}{2\tan 60°} = 0.408248}$$

Dodecahedron $\gamma = 36°$ $\beta = 116.56505°/2$

$$\boxed{\frac{R_i}{e} = \frac{\tan 58.28253°}{2 \tan 36°} = 1.113516}$$

Icosahedron $\gamma = 60°$ $\beta = 138.189685°/2$

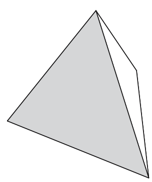
$$\boxed{\frac{R_i}{e} = \frac{\tan 69.094843°}{2 \tan 60°} = 0.755761}$$

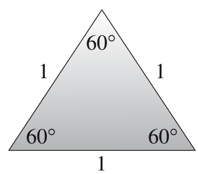
3.6.15 Proof: Dihedral angles for the Platonic solids

Strategy: Each Platonic solid is constructed from a collection of identical regular polygons. The tetrahedron, octagon and icosahedron are constructed from equilateral triangles; the cube from squares; and the dodecahedron from pentagons. The angle between two faces sharing a common edge is called the dihedral angle. This angle is different for each Platonic solid.

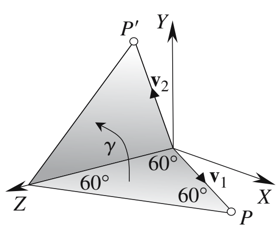
To compute the dihedral angle, imagine one face lying on the ground plane with one common edge aligned with the negative z-axis. A vector $\mathbf{v}_1$ forms a neighboring edge. The face containing $\mathbf{v}_1$ is rotated such that $\mathbf{v}_1$ becomes $\mathbf{v}_2$. The angle between $\mathbf{v}_1$ and $\mathbf{v}_2$ becomes the dihedral angle.

Tetrahedron





An equilateral triangle: one side of a tetrahedron



$$P(x, y, z) = P(\cos 30°, 0, \sin 30°)$$

but $\|\mathbf{v}_1\| = \|\mathbf{v}_2\| = 1$

and
$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 30° \\ 0 \\ \sin 30° \end{bmatrix}$$

therefore
$$x' = \cos\gamma \cos 30°$$
$$y' = \sin\gamma \cos 30°$$
$$z' = \sin 30°$$

Also
$$\mathbf{v}_1 \bullet \mathbf{v}_2 = \|\mathbf{v}_1\| \cdot \|\mathbf{v}_2\| \cos\theta = xx' + yy' + zz'$$

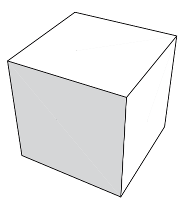
therefore
$$\cos\theta = \cos\gamma \cos^2 30° + \sin^2 30°$$

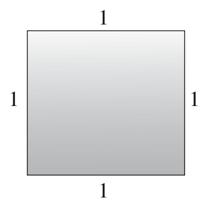
θ equals 60° (internal angle of an equilateral triangle)

therefore
$$\cos\gamma = \frac{\cos 60° - \sin^2 30°}{\cos^2 30°} = \tfrac{1}{3}$$

$$\boxed{\text{Dihedral angle } \gamma = 70.52878°}$$

Cube



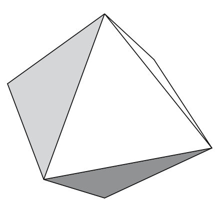


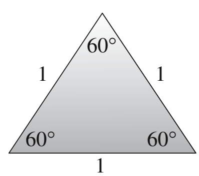
A square: one side of a cube

By inspection

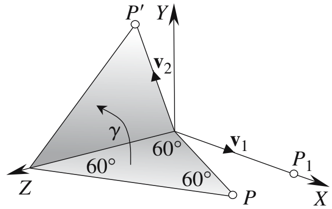
$$\boxed{\text{Dihedral angle } \gamma = 90°}$$

Octahedron





An equilateral triangle: one side of an octahedron



$P(x, y, z) = P(\cos 30°, 0, \sin 30°)$

but $\|\mathbf{v}_1\| = \|\mathbf{v}_2\| = 1$

and $P_1(1, 0, 0)$

$\mathbf{v}_1$ is aligned with one side of the square cross-section

and
$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 30° \\ 0 \\ \sin 30° \end{bmatrix}$$

therefore
$$x' = \cos\gamma \cos 30°$$
$$y' = \sin\gamma \cos 30°$$
$$z' = \sin 30°$$

But $\mathbf{v}_1 = \mathbf{i}$

and $\mathbf{v}_1 \bullet \mathbf{v}_2 = \|\mathbf{v}_1\| \cdot \|\mathbf{v}_2\| \cos\theta = xx' + yy' + zz'$

therefore $\cos\theta = \cos\gamma \cos 30°$

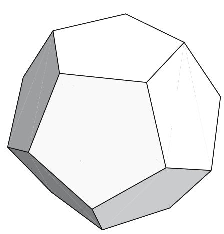
θ equals 60° (internal angle of an equilateral triangle)

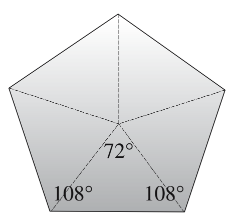
therefore
$$\cos\gamma = \frac{\cos 60°}{\cos 30°} = \frac{\sqrt{3}}{3}$$

$\gamma = 54.73561°$ [γ is half the dihedral angle]

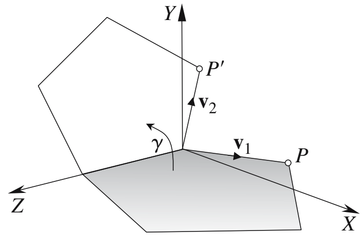
$$\boxed{\text{Dihedral angle} = 2\gamma = 109.47122°}$$

Dodecahedron





A pentagon: one side of a dodecahedron



$$P(x, y, z) = P(\sin 72°, 0, -\cos 72°)$$

but $\|\mathbf{v}_1\| = \|\mathbf{v}_2\| = 1$

and
$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin 72° \\ 0 \\ -\cos 72° \end{bmatrix}$$

therefore
$$x' = \cos\gamma \sin 72°$$
$$y' = \sin\gamma \sin 72°$$
$$z' = -\cos 72°$$

and
$$\mathbf{v}_1 \bullet \mathbf{v}_2 = \|\mathbf{v}_1\| \cdot \|\mathbf{v}_2\| \cos\theta = xx' + yy' + zz'$$

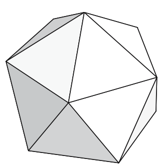
therefore
$$\cos\theta = \cos\gamma \sin^2 72° + \cos^2 72°$$

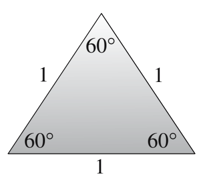
θ equals 108° (internal angle of a regular pentagon)

therefore
$$\cos\gamma = \frac{\cos 108° - \cos^2 72°}{\sin^2 72°} = \frac{\cos 72°}{\cos 72° - 1}$$

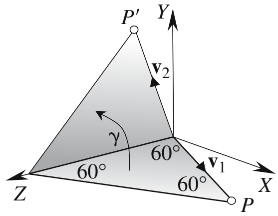
$$\boxed{\text{Dihedral angle } \gamma = 116.56505°}$$

Icosahedron





An equilateral triangle: one side of an icosahedron



$$P(x, y, z) = P(\cos 30°, 0, \sin 30°)$$

but $$\|\mathbf{v}_1\| = \|\mathbf{v}_2\| = 1$$

and $$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 30° \\ 0 \\ \sin 30° \end{bmatrix}$$

therefore $$x' = \cos\gamma \cos 30°$$
$$y' = \sin\gamma \cos 30°$$
$$z' = \sin 30°$$

$$\mathbf{v}_1 \bullet \mathbf{v}_2 = \|\mathbf{v}_1\| \cdot \|\mathbf{v}_2\| \cos\theta = xx' + yy' + zz'$$

therefore $$\cos\theta = \cos\gamma \cos^2 30° + \sin^2 30°$$

θ equals $2 \times 54° = 108°$ (internal angle of a regular pentagon)

therefore $$\cos\gamma = \frac{\cos 108° - \sin^2 30°}{\cos^2 30°}$$

$$\boxed{\text{Dihedral angle } \gamma = 138.189685°}$$

3.6.16 Proof: Surface area and volume of the Platonic solids

Surface area

Strategy: Each Platonic solid is constructed from a common regular polygon. The tetrahedron, octagon and icosahedron are built from equilateral triangles; the cube from squares; and the dodecahedron from pentagons.

The area of a regular polygon with p edges of length s is given by

$$\text{Area} = \tfrac{1}{4} ps^2 \cot\left(\tfrac{\pi}{p}\right)$$

The total surface area for f sides is

$$A = \tfrac{1}{4} f ps^2 \cot\left(\tfrac{\pi}{p}\right)$$

or we can express the surface area A as a ratio to s^2

$$\boxed{\frac{A}{s^2} = \tfrac{1}{4} f p \cot\left(\tfrac{\pi}{p}\right)}$$

Tetrahedron	$\frac{A}{s^2} = \frac{1}{4} \times 4 \times 3 \cot 60°$	1.732051
Cube	$\frac{A}{s^2} = \frac{1}{4} \times 6 \times 4 \cot 45°$	6
Octahedron	$\frac{A}{s^2} = \frac{1}{4} \times 8 \times 3 \cot 60°$	3.464102
Dodecahedron	$\frac{A}{s^2} = \frac{1}{4} \times 12 \times 5 \cot 36°$	20.645728
Icosahedron	$\frac{A}{s^2} = \frac{1}{4} \times 20 \times 3 \cot 60°$	8.660254

Volume

Strategy: A Platonic solid can be visualized as a collection of pyramids with a base at each face and a height R_{in} (radius of the inner sphere).

Volume of a pyramid $\qquad V_p = \frac{1}{3} Area_{base} R_{in}$

Volume of a Platonic solid $\qquad V = f V_p$

$$V = \tfrac{1}{3} f \times Area_{base} R_{in}$$

but $\qquad Area_{base} = \frac{1}{4} p s^2 \cot\left(\frac{\pi}{p}\right)$

$$\boxed{\frac{V}{s^3} = \tfrac{1}{12} fp \cot\left(\tfrac{\pi}{p}\right) \frac{R_{in}}{s}}$$

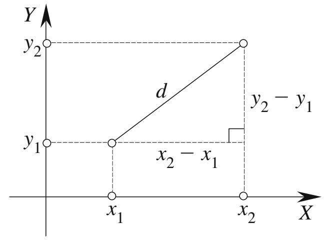
Tetrahedron	$\frac{V}{s^3} = \frac{4 \times 3}{12} \cot 60° \frac{\sqrt{6}}{12}$	0.117851
Cube	$\frac{V}{s^3} = \frac{6 \times 4}{12} \cot 45° \frac{1}{2}$	1

Octahedron	$\frac{V}{s^3} = \frac{8 \times 3}{12} \cot 60^\circ$	0.471405
Dodecahedron	$\frac{V}{s^3} = \frac{12 \times 5}{12} \cot 36^\circ \frac{1}{20} \sqrt{250 + 110\sqrt{5}}$	7.663119
Icosahedron	$\frac{V}{s^3} = \frac{20 \times 3}{12} \cot 60^\circ \frac{1}{12} \sqrt{42 + 18\sqrt{5}}$	2.181695

3.7 Coordinate systems

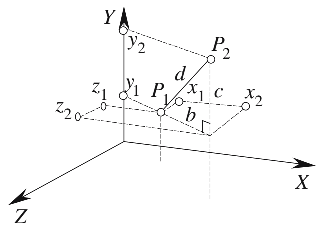
3.7.1 Cartesian coordinates

Distance in $\mathbb{R}^2$
From the diagram and using the Pythagorean theorem



$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

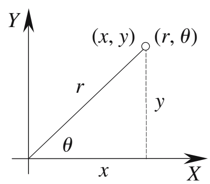
Distance in $\mathbb{R}^3$
From the diagram and using the Pythagorean theorem



$$b^2 = (x_2 - x_1)^2 + (z_2 - z_1)^2$$
$$d^2 = b^2 + c^2$$
$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

3.7.2 Polar coordinates

Given a point with Cartesian coordinates (x, y), then from the diagram and using the Pythagorean theorem



$$r^2 = x^2 + y^2$$
$$r = \sqrt{x^2 + y^2}$$

and $$\tan\theta = \frac{y}{x}$$

$$\theta = \tan^{-1}\frac{y}{x} \quad \text{(1st and 4th quadrants only)}$$

The polar coordinates are (r, θ)
Given a point with polar coordinates (r, θ)

then
$$x = r\cos\theta$$
$$y = r\sin\theta$$

Distance in $\mathbb{R}^2$
Given two points (r_1, θ_1) and (r_2, θ_2) then using their equivalent Cartesian coordinates

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

then
$$d = \sqrt{(r_2 \cos\theta_2 - r_1 \cos\theta_1)^2 + (r_2 \sin\theta_2 - r_1 \sin\theta_1)^2}$$

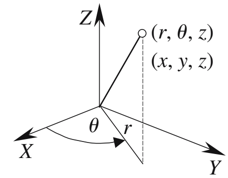
$$d = \sqrt{\begin{aligned}&(r_2^2 \cos^2\theta_2 + r_1^2 \cos^2\theta_1 - 2r_1r_2 \cos\theta_1 \cos\theta_2\\ &+ r_2^2 \sin^2\theta_2 + r_1^2 \sin^2\theta_1 - 2r_1r_2 \sin\theta_1 \sin\theta_2)\end{aligned}}$$

$$d = \sqrt{r_2^2 + r_1^2 - 2r_1r_2(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2)}$$

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_2 - \theta_1)}$$

3.7.3 Cylindrical coordinates

Given a point with Cartesian coordinates (x, y, z), then from the diagram and using the Pythagorean theorem



$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\frac{y}{x} \quad \text{(1st and 4th quadrants only)}$$

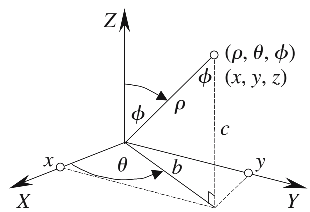
$$z = z$$

Given a point with cylindrical coordinates (r, θ, z)

then
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

3.7.4 Spherical coordinates

Given a point with Cartesian coordinates (x, y, z), then from the diagram and using the Pythagorean theorem



$$\rho = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1}\frac{y}{x} \quad \text{(1st and 4th quadrants only)}$$

$$\phi = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

N.B. The z-axis is normally taken as the vertical axis.

Given a point with spherical coordinates (ρ, θ, ϕ), then from the diagram

$$\sin\phi = \frac{b}{\rho}$$

$$b = \rho \sin \phi \tag{1}$$

$$\cos \phi = \frac{z}{\rho}$$

$$z = \rho \cos \phi$$

but $$\frac{x}{b} = \cos \theta$$

Substituting (1) $$x = \rho \sin \phi \cos \theta$$

Similarly $$\frac{y}{b} = \sin \theta$$

$$y = \rho \sin \phi \sin \theta$$

The Cartesian coordinates are

$$x = \rho \sin \phi \cos \theta$$
$$y = \rho \sin \phi \sin \theta$$
$$z = \rho \cos \phi$$

3.8 Vectors

3.8.1 Proof: Magnitude of a vector

A vector represents a directed line segment whose magnitude is defined by its length. The length of a line segment is given by

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

therefore, given $\mathbf{a} = x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k}$

then $||\mathbf{a}|| = \sqrt{x_a^2 + y_a^2 + z_a^2}$

3.8.2 Proof: Normalizing a vector to a unit length

A vector is normalized to a unit length by dividing each component by its magnitude.

If $\mathbf{a} = x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k}$

then $||\mathbf{a}|| = \sqrt{x_a^2 + y_a^2 + z_a^2}$

therefore $\hat{\mathbf{a}} = \frac{x_a}{||\mathbf{a}||}\mathbf{i} + \frac{y_a}{||\mathbf{a}||}\mathbf{j} + \frac{z_a}{||\mathbf{a}||}\mathbf{k}$

Check the magnitude of $\hat{\mathbf{a}}$ to prove that its length is 1.

$$||\hat{\mathbf{a}}|| = \sqrt{\frac{x_a^2}{||\mathbf{a}||^2} + \frac{y_a^2}{||\mathbf{a}||^2} + \frac{z_a^2}{||\mathbf{a}||^2}}$$

$$||\hat{\mathbf{a}}|| = \frac{1}{||\mathbf{a}||}\sqrt{x_a^2 + y_a^2 + z_a^2}$$

$$||\hat{\mathbf{a}}|| = \frac{||\mathbf{a}||}{||\mathbf{a}||} = 1$$

3.8.3 Proof: Scalar (dot) product

The scalar product is defined as $\mathbf{a} \bullet \mathbf{b} = ||\mathbf{a}|| \cdot ||\mathbf{b}|| \cos\alpha$
where α is the angle between vectors **a** and **b**.

Let $\mathbf{a} = x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k}$

and $\mathbf{b} = x_b\mathbf{i} + y_b\mathbf{j} + z_b\mathbf{k}$

therefore $\mathbf{a} \bullet \mathbf{b} = (x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k}) \bullet (x_b\mathbf{i} + y_b\mathbf{j} + z_b\mathbf{k})$

$$\mathbf{a} \bullet \mathbf{b} = x_a x_b \mathbf{i} \bullet \mathbf{i} + x_a y_b \mathbf{i} \bullet \mathbf{j} + x_a z_b \mathbf{i} \bullet \mathbf{k} + y_a x_b \mathbf{j} \bullet \mathbf{i} + y_a y_b \mathbf{j} \bullet \mathbf{j} + y_a z_b \mathbf{j} \bullet \mathbf{k} + z_a x_b \mathbf{k} \bullet \mathbf{i} + z_a y_b \mathbf{k} \bullet \mathbf{j} + z_a z_b \mathbf{k} \bullet \mathbf{k}$$

but $\mathbf{i} \bullet \mathbf{i} = \mathbf{j} \bullet \mathbf{j} = \mathbf{k} \bullet \mathbf{k} = 1$

and $\mathbf{i} \bullet \mathbf{j} = \mathbf{i} \bullet \mathbf{k} = \mathbf{j} \bullet \mathbf{i} = \mathbf{j} \bullet \mathbf{k} = \mathbf{k} \bullet \mathbf{i} = \mathbf{k} \bullet \mathbf{j} = 0$

therefore $\mathbf{a} \bullet \mathbf{b} = x_a x_b + y_a y_b + z_a z_b = \|\mathbf{a}\| \cdot \|\mathbf{b}\| \cos\alpha$

3.8.4 Proof: Commutative law of the scalar product

$$\mathbf{b} \bullet \mathbf{a} = (x_b \mathbf{i} + y_b \mathbf{j} + z_b \mathbf{k}) \bullet (x_a \mathbf{i} + y_a \mathbf{j} + z_a \mathbf{k})$$

$$\mathbf{b} \bullet \mathbf{a} = x_b x_a \mathbf{i} \bullet \mathbf{i} + x_b y_a \mathbf{i} \bullet \mathbf{j} + x_b z_a \mathbf{i} \bullet \mathbf{k} + y_b x_a \mathbf{j} \bullet \mathbf{i} + y_b y_a \mathbf{j} \bullet \mathbf{j} + y_b z_a \mathbf{j} \bullet \mathbf{k} + z_b x_a \mathbf{k} \bullet \mathbf{i} + z_b y_a \mathbf{k} \bullet \mathbf{j} + z_b z_a \mathbf{k} \bullet \mathbf{k}$$

then $\mathbf{b} \bullet \mathbf{a} = x_b x_a + y_b y_a + z_b z_a$

therefore $\mathbf{b} \bullet \mathbf{a} = \mathbf{a} \bullet \mathbf{b}$

3.8.5 Proof: Associative law of the scalar product

$$\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = (x_a \mathbf{i} + y_a \mathbf{j} + z_a \mathbf{k}) \bullet ((x_b \mathbf{i} + y_b \mathbf{j} + z_b \mathbf{k}) + (x_c \mathbf{i} + y_c \mathbf{j} + z_c \mathbf{k}))$$

$$\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = (x_a \mathbf{i} + y_a \mathbf{j} + z_a \mathbf{k}) \bullet ((x_b + x_c)\mathbf{i} + (y_b + y_c)\mathbf{j} + (z_b + z_c)\mathbf{k})$$

$$\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = x_a(x_b + x_c) + y_a(y_b + y_c) + z_a(z_b + z_c)$$

$$\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = x_a x_b + x_a x_c + y_a y_b + y_a y_c + z_a z_b + z_a z_c$$

$$\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = x_a x_b + y_a y_b + z_a z_b + x_a x_c + y_a y_c + z_a z_c$$

therefore $\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}$

Prove $\mathbf{a} \bullet \mathbf{a} = \|\mathbf{a}\|^2$

Given $\mathbf{a} \bullet \mathbf{a} = \|\mathbf{a}\| \cdot \|\mathbf{a}\| \cos\alpha$

but $\alpha = 0^\circ \qquad \cos\alpha = 1$

therefore $\mathbf{a} \bullet \mathbf{a} = \|\mathbf{a}\|^2$

Prove $\mathbf{a} \bullet \mathbf{b} = 0 \quad \Leftrightarrow \quad \mathbf{a} \perp \mathbf{b}$

If $\mathbf{a} \perp \mathbf{b} \quad \Leftrightarrow \quad \alpha = 90^\circ$

then $\mathbf{a} \bullet \mathbf{b} = \|\mathbf{a}\| \cdot \|\mathbf{b}\| \cos 90^\circ$

therefore $\mathbf{a} \bullet \mathbf{b} = 0$

3.8.6 Proof: Angle between two vectors

Let $\mathbf{a} = x_a \mathbf{i} + y_a \mathbf{j} + z_a \mathbf{k}$

and $\mathbf{b} = x_b \mathbf{i} + y_b \mathbf{j} + z_b \mathbf{k}$

then $\mathbf{a} \bullet \mathbf{b} = \|\mathbf{a}\| \cdot \|\mathbf{b}\| \cos\alpha$

therefore $$\cos\alpha = \frac{x_a x_b + y_a y_b + z_a z_c}{\|\mathbf{a}\| \cdot \|\mathbf{b}\|}$$

and $$\alpha = \cos^{-1}\left(\frac{x_a x_b + y_a y_b + z_a z_c}{\|\mathbf{a}\| \cdot \|\mathbf{b}\|}\right)$$

3.8.7 Proof: Vector (cross) product

The vector product is defined as follows:

$$\mathbf{a} \times \mathbf{b} = \mathbf{c} \quad \text{where} \quad \|\mathbf{c}\| = \|\mathbf{a}\| \cdot \|\mathbf{b}\| \sin\alpha \text{ and } \mathbf{c} \text{ is orthogonal to } \mathbf{a} \text{ and } \mathbf{b}.$$

Let $\mathbf{a} = x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k}$

and $\mathbf{b} = x_b\mathbf{i} + y_b\mathbf{j} + z_b\mathbf{k}$

then $\mathbf{a} \times \mathbf{b} = (x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k}) \times (x_b\mathbf{i} + y_b\mathbf{j} + z_b\mathbf{k})$

$$\begin{aligned}\mathbf{a} \times \mathbf{b} = {} & x_a x_b\mathbf{i} \times \mathbf{i} + x_a y_b\mathbf{i} \times \mathbf{j} + x_a z_b\mathbf{i} \times \mathbf{k} + y_a x_b\mathbf{j} \times \mathbf{i} + y_a y_b\mathbf{j} \times \mathbf{j} \\ & + y_a z_b\mathbf{j} \times \mathbf{k} + z_a x_b\mathbf{k} \times \mathbf{i} + z_a y_b\mathbf{k} \times \mathbf{j} + z_a z_b\mathbf{k} \times \mathbf{k}\end{aligned}$$

but $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$

and $\mathbf{i} \times \mathbf{j} = \mathbf{k} \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j} \quad \mathbf{j} \times \mathbf{i} = -\mathbf{k} \quad \mathbf{j} \times \mathbf{k} = \mathbf{i} \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}$
$\mathbf{k} \times \mathbf{j} = -\mathbf{i}$

Then $\mathbf{a} \times \mathbf{b} = (y_a z_b - z_a y_b)\mathbf{i} + (z_a x_b - x_a z_b)\mathbf{j} + (x_a y_b - y_a x_b)\mathbf{k}$

therefore $$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} y_a & z_a \\ y_b & z_b \end{vmatrix}\mathbf{i} + \begin{vmatrix} z_a & x_a \\ z_b & x_b \end{vmatrix}\mathbf{j} + \begin{vmatrix} x_a & y_a \\ x_b & y_b \end{vmatrix}\mathbf{k}$$

or $$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_a & y_a & z_a \\ x_b & y_b & z_b \end{vmatrix}$$

3.8.8 Proof: The non-commutative law of the vector product

Let $\mathbf{a} = x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k}$

and $\mathbf{b} = x_b\mathbf{i} + y_b\mathbf{j} + z_b\mathbf{k}$

then $\mathbf{b} \times \mathbf{a} = (x_b\mathbf{i} + y_b\mathbf{j} + z_b\mathbf{k}) \times (x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k})$

$$\begin{aligned}\mathbf{b} \times \mathbf{a} = {} & x_b x_a\mathbf{i} \times \mathbf{i} + x_b y_a\mathbf{i} \times \mathbf{j} + x_b z_a\mathbf{i} \times \mathbf{k} + y_b x_a\mathbf{j} \times \mathbf{i} + y_b y_a\mathbf{j} \times \mathbf{j} \\ & + y_b z_a\mathbf{j} \times \mathbf{k} + z_b x_a\mathbf{k} \times \mathbf{i} + z_b y_a\mathbf{k} \times \mathbf{j} + z_b z_a\mathbf{k} \times \mathbf{k}\end{aligned}$$

$$\mathbf{b} \times \mathbf{a} = (y_b z_a - z_b y_a)\mathbf{i} + (z_b x_a - x_b z_a)\mathbf{j} + (x_b y_a - y_b x_a)\mathbf{k}$$

$$\mathbf{b} \times \mathbf{a} = -(z_b y_a - y_b z_a)\mathbf{i} - (x_b z_a - z_b x_a)\mathbf{j} - (y_b x_a - x_b y_a)\mathbf{k}$$

therefore $$\mathbf{b} \times \mathbf{a} = -\begin{vmatrix} y_a & z_a \\ y_b & z_b \end{vmatrix}\mathbf{i} - \begin{vmatrix} z_a & x_a \\ z_b & x_b \end{vmatrix}\mathbf{j} - \begin{vmatrix} x_a & y_a \\ x_b & y_b \end{vmatrix}\mathbf{k} = -\mathbf{a} \times \mathbf{b}$$

3.8.9 Proof: The associative law of the vector product

Let $\mathbf{a} = x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k} \qquad \mathbf{b} = x_b\mathbf{i} + y_b\mathbf{j} + z_b\mathbf{k} \qquad \mathbf{c} = x_c\mathbf{i} + y_c\mathbf{j} + z_c\mathbf{k}$

then $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k}) \times ((x_b\mathbf{i} + y_b\mathbf{j} + z_b\mathbf{k}) + (x_c\mathbf{i} + y_c\mathbf{j} + z_c\mathbf{k}))$

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k}) \times ((x_b + x_c)\mathbf{i} + (y_b + y_c)\mathbf{j} + (z_b + z_c)\mathbf{k})$$

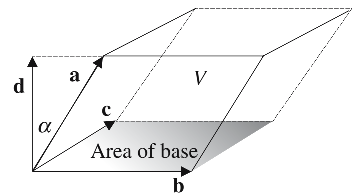
$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \begin{vmatrix} y_a & z_a \\ (y_b + y_c) & (z_b + z_c) \end{vmatrix}\mathbf{i} + \begin{vmatrix} z_a & x_a \\ (z_b + z_c) & (x_b + x_c) \end{vmatrix}\mathbf{j} + \begin{vmatrix} x_a & y_a \\ (x_b + x_c) & (y_b + y_c) \end{vmatrix}\mathbf{k}$$

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \begin{vmatrix} y_a & z_a \\ y_b & z_b \end{vmatrix}\mathbf{i} + \begin{vmatrix} y_a & z_a \\ y_c & z_c \end{vmatrix}\mathbf{i} + \begin{vmatrix} z_a & x_a \\ z_b & x_b \end{vmatrix}\mathbf{j} + \begin{vmatrix} z_a & x_a \\ z_c & x_c \end{vmatrix}\mathbf{j} + \begin{vmatrix} x_a & y_a \\ x_b & y_b \end{vmatrix}\mathbf{k} + \begin{vmatrix} x_a & y_a \\ x_c & y_c \end{vmatrix}\mathbf{k}$$

therefore $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$

3.8.10 Proof: Scalar triple product

$$[\mathbf{a}, \quad \mathbf{b}, \quad \mathbf{c}] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} x_a & y_a & z_a \\ x_b & y_b & z_b \\ x_c & y_c & z_c \end{vmatrix}$$



Let $\mathbf{d} = \mathbf{b} \times \mathbf{c}$ where $\mathbf{d}$ is orthogonal to $\mathbf{b}$ and $\mathbf{c}$.

Volume of parallelpiped V = Area of base $\times$ orthogonal height

= Area of base $\times$ $|\mathbf{a}| \cos\alpha$

therefore $V = \|\mathbf{d}\| \cdot \|\mathbf{a}\| \cos\alpha = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$

3.9 Quaternions

3.9.1 Definition of a quaternion

This is an explanation rather than a proof of the background to quaternions. Quaternions are a natural extension of complex numbers where a real number is paired with an imaginary component to make $(a + ib)$. A quaternion has three imaginary components: $(s + ia + jb + kc)$. In fact, any number of imaginary components can be considered, however, the problem is interpreting the result.

William Rowan Hamilton discovered quaternions on 16 October 1843, and his friend, John Graves, discovered octonions in 1845. Arthur Cayley had also been investigating octonions, which is why they are also known as Cayley numbers. An octonion has the form $(s + ai + bj + ck + dl + em + fn + go)$ [Fenn, 2001].

Let us investigate the multiplication of two quaternions and see how they give rise to vectors, the scalar and vector products.

Given $\quad \mathbf{q}_1 = (s_1, x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k})$

and $\quad \mathbf{q}_2 = (s_2, x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k})$

then $\quad \mathbf{q}_1\mathbf{q}_2 = (s_1, x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k})(s_2, x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k})$

$$\begin{aligned}\mathbf{q}_1\mathbf{q}_2 = (&s_1s_2, s_1x_2\mathbf{i} + s_1y_2\mathbf{j} + s_1z_2\mathbf{k} + s_2x_1\mathbf{i} + x_1x_2\mathbf{i}^2 + x_1y_2\mathbf{ij}\\ &+ x_1z_2\mathbf{ik} + s_2y_1\mathbf{j} + y_1x_2\mathbf{ji} + y_1y_2\mathbf{j}^2 + y_1z_2\mathbf{jk} + s_2z_1\mathbf{k}\\ &+ x_2z_1\mathbf{ki} + z_1y_2\mathbf{kj} + z_1z_2\mathbf{k}^2)\end{aligned}$$

$$\begin{aligned}\mathbf{q}_1\mathbf{q}_2 = (&s_1s_2,(s_1x_2 + s_2x_1)\mathbf{i} + (s_1y_2 + s_2y_1)\mathbf{j} + (s_1z_2 + s_2z_1)\mathbf{k}\\ &+ x_1x_2\mathbf{i}^2 + y_1y_2\mathbf{j}^2 + z_1z_2\mathbf{k}^2 + x_1y_2\mathbf{ij} + y_1z_2\mathbf{jk} + x_2z_1\mathbf{ki}\\ &+ y_1x_2\mathbf{ji} + z_1y_2\mathbf{kj} + x_1z_2\mathbf{ik})\end{aligned}$$

Interpreting this result was the stumbling block for Hamilton as it was necessary to interpret the meaning of $\mathbf{i}^2, \mathbf{j}^2, \mathbf{k}^2, \mathbf{ij}, \mathbf{jk}, \mathbf{ki}, \mathbf{ji}, \mathbf{kj}$ and $\mathbf{ik}$. In a stroke of genius he thought of the following rules:

$$\begin{aligned}&\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1\\ &\mathbf{ij} = \mathbf{k} \quad \mathbf{jk} = \mathbf{i} \quad \mathbf{ki} = \mathbf{j}\\ &\mathbf{ji} = -\mathbf{k} \quad \mathbf{kj} = -\mathbf{i} \quad \mathbf{ik} = -\mathbf{j}\end{aligned}$$

or summarized as

$$\begin{array}{c c} & \begin{array}{ccc}\mathbf{i} & \mathbf{j} & \mathbf{k}\end{array}\\ \begin{array}{c}\mathbf{i}\\ \mathbf{j}\\ \mathbf{k}\end{array} & \begin{pmatrix}-1 & \mathbf{k} & -\mathbf{j}\\ -\mathbf{k} & -1 & \mathbf{i}\\ \mathbf{j} & -\mathbf{i} & -1\end{pmatrix}\end{array}$$

If we apply these rules to the last equation we get

$$\begin{aligned}\mathbf{q}_1\mathbf{q}_2 = (&s_1s_2 - x_1x_2 - y_1y_2 - z_1z_2, (s_1x_2 + s_2x_1)\mathbf{i} + (s_1y_2 + s_2y_1)\mathbf{j}\\ &+ (s_1z_2 + s_2z_1)\mathbf{k} + x_1y_2\mathbf{k} + y_1z_2\mathbf{i} + x_2z_1\mathbf{j} - y_1x_2\mathbf{k}\\ &- z_1y_2i - x_1z_2\mathbf{j})\end{aligned}$$

simplifying to
$$\mathbf{q}_1\mathbf{q}_2 = (s_1s_2 - (x_1x_2 + y_1y_2 + z_1z_2), s_1(x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}) + s_2(x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}) + (y_1z_2 - z_1y_2)\mathbf{i} + (x_2z_1 - x_1z_2)\mathbf{j} + (x_1y_2 - y_1x_2)\mathbf{k})$$

This equation now only contains real and imaginary components derived from the original quaternions.

We can see that $s_1s_2 - (x_1x_2 + y_1y_2 + z_1z_2)$ is a real quantity

$s_1(x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k})$ is the product of s_1 and the imaginary part of $\mathbf{q}_2$

and $s_2(x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k})$ is the product of s_2 and the imaginary part of $\mathbf{q}_1$

The last part $(y_1z_2 - z_1y_2)\mathbf{i} + (x_2z_1 - x_1z_2)\mathbf{j} + (x_1y_2 - y_1x_2)\mathbf{k}$ can be rewritten as

$$\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}\mathbf{i} + \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}\mathbf{k}$$

which we recognize as the vector product of $(x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}) \times (x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k})$

Similarly $x_1x_2 + y_1y_2 + z_1z_2$ is the scalar product of $(x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}) \bullet (x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k})$

So if we describe the original quaternions as a scalar and vector:

$$\mathbf{q}_1 = (s_1, \mathbf{v}_1) \quad \text{and} \quad \mathbf{q}_2 = (s_2, \mathbf{v}_2)$$

we obtain
$$\mathbf{q}_1\mathbf{q}_2 = (s_1s_2 - \mathbf{v}_1 \bullet \mathbf{v}_2, s_1\mathbf{v}_2 + s_2\mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2)$$

One very important difference between quaternions and complex numbers is that the multiplication of quaternions is non-commutative: $\mathbf{q}_1\mathbf{q}_2 \neq \mathbf{q}_2\mathbf{q}_1$

Rooney [1977] explores the development of quaternions as a tool for performing rotations and considers the product of a quaternion with a vector:

given
$$\mathbf{q} = (q_s, q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k}) = (q_s, \mathbf{q}_v)$$

and the vector $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ which can be represented as a quaternion using

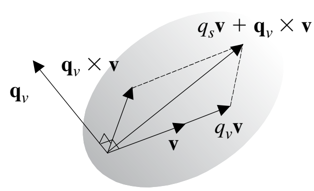
$$\mathbf{r} = (0, \mathbf{v})$$

then
$$\mathbf{qr} = (q_s, \mathbf{q}_v)(0, \mathbf{v})$$

equals
$$\mathbf{qr} = (-\mathbf{q}_v \bullet \mathbf{v}, q_s\mathbf{v} + \mathbf{q}_v \times \mathbf{v}) \tag{1}$$

We can see from (1) that the vector component of $\mathbf{qr}$, i.e. $q_s\mathbf{v} + \mathbf{q}_v \times \mathbf{v}$ is the sum of the scaled vector $q_s\mathbf{v}$ and $\mathbf{q}_v \times \mathbf{v}$.

If $\mathbf{q}_v$ and $\mathbf{v}$ are orthogonal then we obtain the situation shown in the diagram:



and
$$\mathbf{qr} = q_s\mathbf{v} + \mathbf{q}_v \times \mathbf{v} \quad \text{i.e. a vector}$$

Vector $\mathbf{v}$ has been rotated in the plane orthogonal to $\mathbf{q}_v$ but it has been stretched. This is how quaternions can be used to rotate a vector, but somehow we need to avoid the stretching.

If we make $\quad \mathbf{q} = (\cos\theta, \sin\theta(l\mathbf{i} + m\mathbf{j} + n\mathbf{k}))$

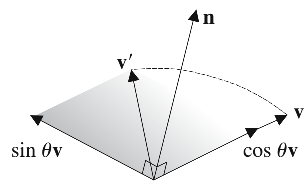
where $\quad l^2 + m^2 + n^2 = 1$

then $\quad \mathbf{q} = (\cos\theta, \mathbf{n}\sin\theta)$

where $\quad \mathbf{n} = (l\mathbf{i} + m\mathbf{j} + n\mathbf{k}) \quad \text{and} \quad \|\mathbf{n}\| = 1$

then $\quad \mathbf{qr} = (\cos\theta, \mathbf{n}\sin\theta)(0, \mathbf{v})$

and $\quad \mathbf{qr} = \sin\theta(\mathbf{n}\times\mathbf{v}) + \cos\theta\mathbf{v} \qquad (2)$



The result of $\sin\theta(\mathbf{n}\times\mathbf{v})$ is a vector with magnitude $\sin\theta\|\mathbf{v}\|$ in a plane containing $\mathbf{v}$ and orthogonal to $\mathbf{n}$. When this is added to $\cos\theta\mathbf{v}$ we obtain the rotated vector $\mathbf{v}'$:

then
$$\|\mathbf{v}'\|^2 = \sin^2\theta\|\mathbf{v}\|^2 + \cos^2\theta\|\mathbf{v}\|^2$$
$$\|\mathbf{v}'\|^2 = \|\mathbf{v}\|^2(\sin^2\theta + \cos^2\theta)$$
$$\|\mathbf{v}'\| = \|\mathbf{v}\|$$

Thus $\mathbf{v}$ is rotated to $\mathbf{v}'$. But the problem with this strategy is that in order to rotate a vector we must arrange that the quaternion is orthogonal to the vector, which is not convenient. Brand [Brand, 1947] proposed an alternative approach using half-angles, where

$$\mathbf{q} = (\cos\frac{\theta}{2}, \mathbf{n}\sin\frac{\theta}{2})$$

and $\quad \mathbf{n} = (l\mathbf{i} + m\mathbf{j} + n\mathbf{k})$ and is a unit vector

and $\quad \mathbf{v}' = \mathbf{qvq}^{-1}$

where $\mathbf{q}^{-1}$ is the inverse of $\mathbf{q}$ given by $\mathbf{q}^{-1} = q_s - \mathbf{q}_v$ (for a unit quaternion).
If we now rotate $\mathbf{v}$ using this technique we obtain:

$$\mathrm{v}' = (\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\mathbf{n})(0, \mathbf{v})(\cos\frac{\theta}{2}, -\sin\frac{\theta}{2}\mathbf{n})$$

Let $\quad c = \cos\frac{\theta}{2} \quad \text{and} \quad s = \sin\frac{\theta}{2}$

then $\quad \mathbf{v}' = (\mathrm{c}, s\mathbf{n})(0, \mathbf{v})(\mathrm{c}, -s\mathbf{n})$

Multiplying the first two quaternions

$$\mathbf{v}' = (-s(\mathbf{n}\cdot\mathbf{v}), c\mathbf{v} + s(\mathbf{n}\times\mathbf{v}))(c, -s\mathbf{n})$$

Multiply these quaternions

$$\begin{aligned}\mathbf{v}' = &-cs(\mathbf{n}\cdot\mathbf{v}) + (c\mathbf{v} + s(\mathbf{n}\times\mathbf{v}))\cdot(s\mathbf{n}) + s^2(\mathbf{n}\cdot\mathbf{v})\mathbf{n}\\ &+ c^2\mathbf{v} + cs(\mathbf{n}\times\mathbf{v}) + (c\mathbf{v} + s(\mathbf{n}\times\mathbf{v}))\times(-s\mathbf{n})\end{aligned}$$

$$\mathbf{v}' = -cs(\mathbf{n}\cdot\mathbf{v}) + cs(\mathbf{n}\cdot\mathbf{v}) + s^2(\mathbf{n}\times\mathbf{v})\cdot\mathbf{n} + s^2(\mathbf{n}\cdot\mathbf{v})\mathbf{n} + c^2\mathbf{v} + cs(\mathbf{n}\times\mathbf{v}) - cs(\mathbf{v}\times\mathbf{n}) - s^2(\mathbf{n}\times\mathbf{v})\times\mathbf{n}$$

but $(\mathbf{n}\times\mathbf{v})\cdot\mathbf{v} = 0$ $\quad \mathbf{v}' = s^2(\mathbf{n}\cdot\mathbf{v})\mathbf{n} + c^2\mathbf{v} + 2cs(\mathbf{n}\times\mathbf{v}) - s^2(\mathbf{n}\times\mathbf{v})\times\mathbf{n}$

but $(\mathbf{n}\times\mathbf{v})\times\mathbf{n} = \mathbf{v}(\mathbf{n}\cdot\mathbf{n}) - \mathbf{n}(\mathbf{v}\cdot\mathbf{n}) = \mathbf{v} - \mathbf{n}(\mathbf{v}\cdot\mathbf{n})$

therefore $\quad \mathbf{v}' = s^2(\mathbf{n}\cdot\mathbf{v})\mathbf{n} + c^2\mathbf{v} + 2cs(\mathbf{n}\times\mathbf{v}) - s^2\mathbf{v} + s^2(\mathbf{v}\cdot\mathbf{n})\mathbf{n}$

but $2cs = 2\cos\frac{\theta}{2}\sin\frac{\theta}{2} = \sin\theta$

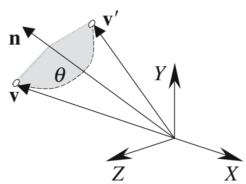
$$\mathbf{v}' = 2s^2(\mathbf{n}\cdot\mathbf{v})\mathbf{n} + \mathbf{v}(c^2 - s^2) + \sin\theta(\mathbf{n}\times\mathbf{v})$$

but $c^2 - s^2 = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \cos\theta$

$$\mathbf{v}' = 2s^2(\mathbf{n}\cdot\mathbf{v})\mathbf{n} + \cos\theta\,\mathbf{v} + \sin\theta(\mathbf{n}\times\mathbf{v})$$

therefore $\quad \mathbf{v}' = \sin\theta(\mathbf{n}\times\mathbf{v}) + \cos\theta\,\mathbf{v} + 2\sin^2\frac{\theta}{2}(\mathbf{n}\cdot\mathbf{v})\mathbf{n} \quad (3)$

This is very similar to (2) and confirms that the vector is still being rotated. The diagram clarifies what is happening.



Let us test (3) by rotating the point (0, 1, 1) 90° about the y-axis.

Therefore $\quad \mathbf{q} = (\cos\frac{90°}{2}, \sin\frac{90°}{2}\mathbf{j})$ and $\mathbf{r} = (0, \mathbf{j} + \mathbf{k})$

then $\quad \mathbf{v}' = \sin 90°(\mathbf{j}\times(\mathbf{j}+\mathbf{k})) + \cos 90°(\mathbf{j}+\mathbf{k}) + 2\sin^2 45°\mathbf{j}\cdot(\mathbf{j}+\mathbf{k})\mathbf{j}$

$$\mathbf{v}' = (\mathbf{j}\times(\mathbf{j}+\mathbf{k})) + \mathbf{j}\cdot(\mathbf{j}+\mathbf{k})\mathbf{j}$$

$$\mathbf{v}' = \mathbf{i} + \mathbf{j}$$

which points to (1, 1, 0), which is correct.

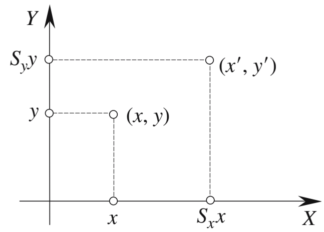
Naturally, we would obtain the same result if we had evaluated this using pure quaternions.

3.10 Transformations

3.10.1 Proof: Scaling in $\mathbb{R}^2$

Scaling relative to the origin

A point (x, y) is scaled relative to the origin by factors S_x and S_y to a new position (x', y') by

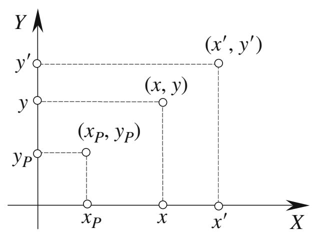


$$x' = S_x x$$

$$y' = S_y y$$

or as a homogeneous matrix $$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scaling relative to a point



A point (x, y) is scaled relative to a point $P(x_P, y_P)$ by factors S_x and S_y to a new position (x', y') in the following steps:

1. Translate (x, y) by $(-x_P, -y_P)$.
2. Scale the translated point by S_x and S_y.
3. Translate the scaled point (x_P, y_P).

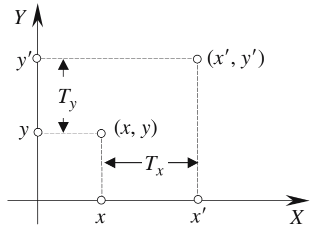
Therefore $$x' = S_x(x - x_P) + x_P = S_x x + x_P(1 - S_x)$$

$$y' = S_y(y - y_P) + y_P = S_y y + y_P(1 - S_y)$$

or as a homogeneous matrix $$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & x_P(1 - S_x) \\ 0 & S_y & y_P(1 - S_x) \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

3.10.2 Proof: Translation in $\mathbb{R}^2$

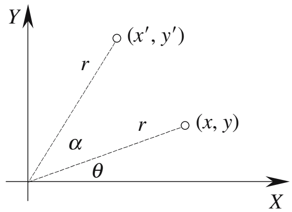
A point (x, y) is translated by distances T_x and T_y to a new position (x', y') by



$$x' = x + T_x$$
$$y' = y + T_y$$

or as a homogeneous matrix $\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & T_x \\ 0 & 1 & T_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

3.10.3 Proof: Rotation in $\mathbb{R}^2$

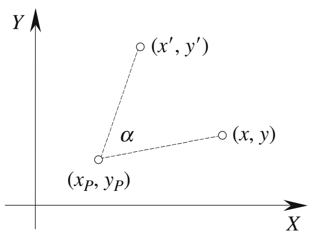


A point (x, y) is rotated about the origin by angle α to a new position (x', y') by

$$x' = r\cos(\theta + \alpha) = r(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$$
$$y' = r\sin(\theta + \alpha) = r(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$$
$$x' = r\left(\frac{x}{r}\cos\alpha - \frac{y}{r}\sin\alpha\right) = x\cos\alpha - y\sin\alpha$$
$$y' = r\left(\frac{y}{r}\cos\alpha + \frac{x}{r}\sin\alpha\right) = y\cos\alpha + x\sin\alpha$$

or as a homogeneous matrix $\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

Rotation about a point



A point (x, y) is rotated about a point (x_P, y_P) by angle α to a new position (x', y') in the following steps:

1. Translate (x, y) by $(-x_P, -y_P)$.
2. Rotate the translated point about the origin by angle α.
3. Translate the rotated point by (x_P, y_P).

Therefore

$$\begin{aligned} x_1 &= x - x_P \\ y_1 &= y - y_P \\ x_2 &= x_1 \cos\alpha - y_1 \sin\alpha \\ y_2 &= x_1 \sin\alpha + y_1 \cos\alpha \\ x' &= (x - x_P)\cos\alpha - (y - y_P)\sin\alpha + x_P \\ y' &= (x - x_P)\sin\alpha + (y - y_P)\cos\alpha + y_P \\ x' &= x\cos\alpha - y\sin\alpha + x_P(1 - \cos\alpha) + y_P\sin\alpha \\ y' &= x\sin\alpha + y\cos\alpha + y_P(1 - \cos\alpha) - x_P\sin\alpha \end{aligned}$$

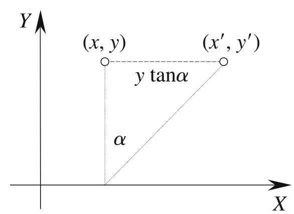
or as a homogeneous matrix

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha & x_P(1-\cos\alpha) + y_P\sin\alpha \\ \sin\alpha & \cos\alpha & y_P(1-\cos\alpha) - x_P\sin\alpha \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

3.10.4 Proof: Shearing in $\mathbb{R}^2$

Shear along the *x*-axis

A point (x, y) is sheared by angle α along the x-axis to a new position (x', y') by



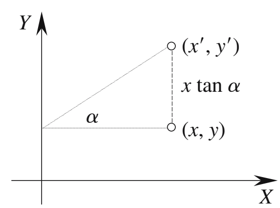
$$\begin{aligned} x' - x &= y\tan\alpha \\ x' &= x + y\tan\alpha \\ y' &= y \end{aligned}$$

or as a homogeneous matrix

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \tan\alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Shear along the *y*-axis

A point (x, y) is sheared by angle α along the y-axis to a new position (x', y') by



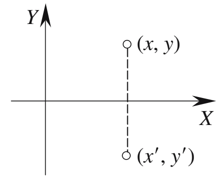
$$\begin{aligned} y' - y &= x\tan\alpha \\ y' &= y + x\tan\alpha \\ x' &= x \end{aligned}$$

or as a homogeneous matrix $\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \tan\alpha & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

3.10.5 Proof: Reflection in $\mathbb{R}^2$

Reflection about the *x*-axis

A point (x, y) is reflected about the x-axis to (x', y') by

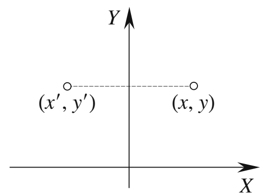


$$x' = x$$
$$y' = -y$$

or as a homogeneous matrix $\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

Reflection about the *y*-axis

A point (x, y) is reflected about the y-axis to (x', y') by

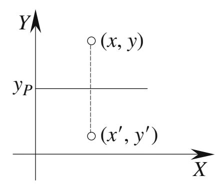


$$x' = -x$$
$$y' = y$$

or as a homogeneous matrix $\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

Reflection about a line parallel with the *x*-axis

A point is reflected about a line in the following steps:



1. Translate the point $(0, -y_P)$.
2. Perform the reflection.
3. Translate the reflected point $(0, y_P)$.

Therefore

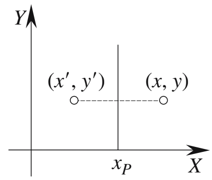
$$x' = x$$
$$y' = -(y - y_P) + y_P = 2y_P - y$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 2y_P \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Reflection about a line parallel with the *y*-axis

A point is reflected about a line in the following steps:

1. Translate the point $(-x_P, 0)$.
2. Perform the reflection.
3. Translate the reflected point $(x_P, 0)$.



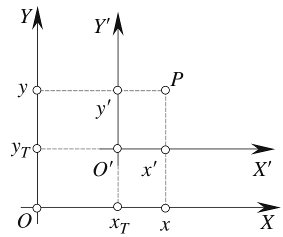
Therefore

$$x' = -(x - x_P) + x_P = 2x_P - x$$
$$y' = y$$
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2x_P \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

3.10.6 Proof: Change of axes in $\mathbb{R}^2$

Translated axes

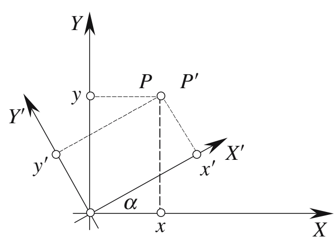
Translating the axes by (x_T, y_T) is equivalent to translating the point by $(-x_T, -y_T)$:



$$x' = x - x_T$$
$$y' = y - y_T$$

or as a homogeneous matrix $$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -x_T \\ 0 & 1 & -y_T \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotated axes by angle α about the origin



Rotating the axes by α is equivalent to rotating the point by $-\alpha$.

Therefore

$$x' = x \cos\alpha + y \sin\alpha$$
$$y' = y \cos\alpha - x \sin\alpha$$

or as a homogeneous matrix $$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

3.10.7 Proof: Identity matrix in $\mathbb{R}^2$

The identity matrix does not alter the coordinates being transformed.

Therefore

$$x' = x$$
$$y' = y$$

or as a homogeneous matrix

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

3.10.8 Proof: Scaling in $\mathbb{R}^3$

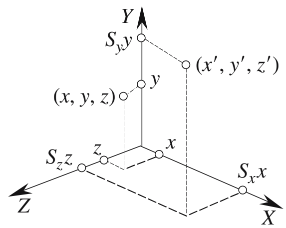
Scaling relative to the origin

A point (x, y, z) is scaled relative to the origin by factors S_x, S_y and S_z to a new position (x', y', z') by

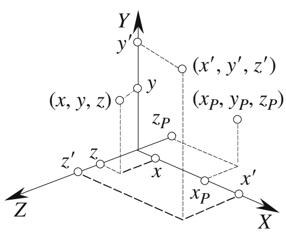
$$x' = S_x x$$
$$y' = S_y y$$
$$z' = S_z z$$

or as a homogeneous matrix

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Scaling relative to a point



A point (x, y, z) is scaled relative to another point (x_P, y_P, z_P) by factors S_x, S_y and S_z to a new position (x', y', z') in the following steps:

1. Translate (x, y, z) by $(-x_P, -y_P, -z_P)$.
2. Scale the translated point by S_x, S_y and S_z.
3. Translate the scaled point (x_P, y_P, z_P).

Therefore

$$x' = S_x(x - x_P) + x_P = S_x x + x_P(1 - S_x)$$
$$y' = S_y(y - y_P) + y_P = S_y y + y_P(1 - S_y)$$
$$z' = S_z(z - z_P) + z_P = S_z z + z_P(1 - S_z)$$

or as a homogeneous matrix $\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 & x_P(1-S_x) \\ 0 & S_y & 0 & y_P(1-S_y) \\ 0 & 0 & S_z & z_P(1-S_z) \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$

3.10.9 Proof: Translation in $\mathbb{R}^3$

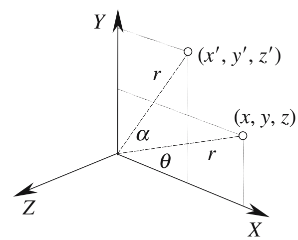
A point (x, y, z) is translated by distances T_x, T_y and T_z to a new position (x', y', z') by

$$x' = x + T_x$$
$$y' = y + T_y$$
$$z' = z + T_z$$

or as a homogeneous matrix $\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$

3.10.10 Proof: Rotation in $\mathbb{R}^3$

Rotation about the *z*-axis



A point (x, y, z) is rotated about the z-axis by the *roll* angle α to a new position (x', y', z') by

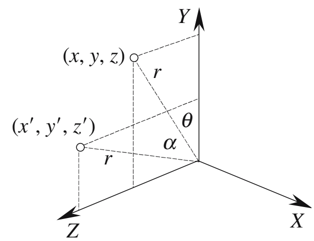
$$x' = r\cos(\theta + \alpha) = r(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$$
$$y' = r\sin(\theta + \alpha) = r(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$$
$$z' = z$$

$$x' = r\left(\frac{x}{r}\cos\alpha - \frac{y}{r}\sin\alpha\right) = x\cos\alpha - y\sin\alpha$$

$$y' = r\left(\frac{y}{r}\cos\alpha + \frac{x}{r}\sin\alpha\right) = y\cos\alpha + x\sin\alpha$$

or as a homogeneous matrix $$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Rotation about the *x*-axis

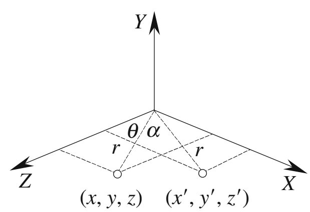


A point (x, y, z) is rotated about the x-axis by the *pitch* angle α to a new position (x', y', z') by

$$x' = x$$
$$y' = r\cos(\theta + \alpha) = r(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$$
$$z' = r\sin(\theta + \alpha) = r(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$$
$$y' = r\left(\frac{y}{r}\cos\alpha - \frac{z}{r}\sin\alpha\right) = y\cos\alpha - z\sin\alpha$$
$$z' = r\left(\frac{z}{r}\cos\alpha + \frac{y}{r}\sin\alpha\right) = z\cos\alpha + y\sin\alpha$$

or as a homogeneous matrix $$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Rotation about the *y*-axis



A point (x, y, z) is rotated about the y-axis by the *yaw* angle α to a new position (x', y', z') by
$x' = r\sin(\theta + \alpha) = r(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$

$$y' = y$$

$$z' = r\cos(\theta + \alpha) = r(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$$

$$x' = r\left(\frac{x}{r}\cos\alpha + \frac{z}{r}\sin\alpha\right) = x\cos\alpha + z\sin\alpha$$

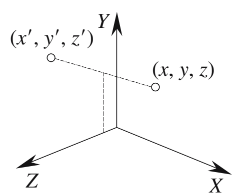
$$z' = r\left(\frac{z}{r}\cos\alpha - \frac{x}{r}\sin\alpha\right) = z\cos\alpha - x\sin\alpha$$

or as a homogeneous matrix $\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$

3.10.11 Proof: Reflection in $\mathbb{R}^3$

Reflection about the *yz*-plane

A point (x, y, z) is reflected about the yz-plane to (x', y', z') by



$$x' = -x$$

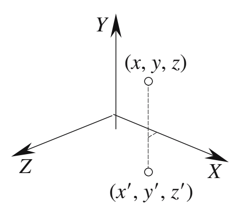
$$y' = y$$

$$z' = z$$

or as a homogeneous matrix $\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$

Reflection about the *zx*-plane

A point (x, y, z) is reflected about the zx-plane to (x', y', z') by



$$x' = x$$

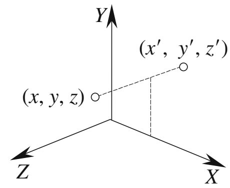
$$y' = -y$$

$$z' = z$$

or as a homogeneous matrix $\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$

Reflection about the *xy*-plane

A point (x, y, z) is reflected about the xy-plane to (x', y', z') by

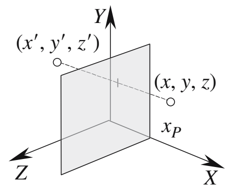


$$x' = x$$
$$y' = y$$
$$z' = -z$$

or as a homogeneous matrix $$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Reflection about a plane parallel with the *yz*-plane

A point is reflected about a plane in the following steps:



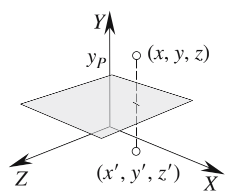
1. Translate the point $(-x_P, 0, 0)$.
2. Perform the reflection.
3. Translate the reflected point $(x_P, 0, 0)$.

Therefore $$x' = -(x - x_P) + x_P = 2x_P - x$$
$$y' = y$$
$$z' = z$$

or as a homogeneous matrix $$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 2x_P \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Reflection about a plane parallel with the *zx*-plane

A point is reflected about a plane in the following steps:



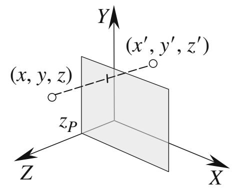
1. Translate the point $(0, -y_P, 0)$.
2. Perform the reflection.
3. Translate the reflected point $(0, y_P, 0)$.

Therefore $$x' = x$$
$$y' = -(y - y_P) + y_P = 2y_P - y$$
$$z' = z$$

or as a homogeneous matrix $$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2y_P \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Reflection about a plane parallel with the *xy*-plane

A point is reflected about a plane in the following steps:



1. Translate the point $(0, 0, -z_P)$.
2. Perform the reflection.
3. Translate the reflected point $(0, 0, z_P)$.

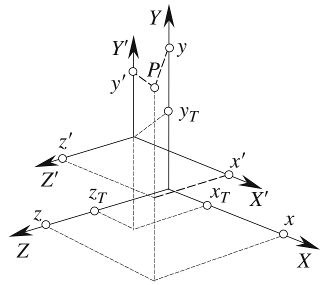
Therefore
$$x' = x$$
$$y' = y$$
$$z' = -(z - z_P) + z_P = 2z_P - z$$

or as a homogeneous matrix
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2z_P \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

3.10.12 Proof: Change of axes in $\mathbb{R}^3$

Translated axes

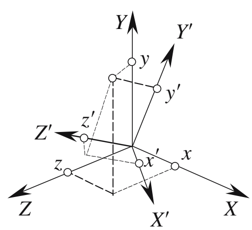
Translating the axes by (x_T, y_T, z_T) is equivalent to translating the point by $(-x_T, -y_T, -z_T)$.



Therefore
$$x' = x - x_T$$
$$y' = y - y_T$$
$$z' = z - z_T$$

or as a homogeneous matrix
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -x_T \\ 0 & 1 & 0 & -y_T \\ 0 & 0 & 1 & -z_T \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Rotated axes about the origin



Direction cosines are used for calculating coordinates in rotated frames of reference:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

where r_{11}, r_{12} and r_{13} are the direction cosines of the secondary x-axis
r_{21}, r_{22} and r_{23} are the direction cosines of the secondary y-axis
r_{31}, r_{32} and r_{33} are the direction cosines of the secondary z-axis.

3.10.13 Proof: Identity matrix in $\mathbb{R}^3$

The identity matrix does not alter the coordinates being transformed.

Therefore

$$x' = x$$
$$y' = y$$
$$z' = z$$

or as a homogeneous matrix

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

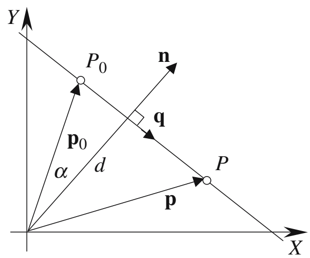
3.11 Two-dimensional straight lines

Equation to a line

Various line characteristics can be used to develop the equation of a straight line, such as specific Cartesian coordinates, the line's slope, its intercepts with the Cartesian axes, the perpendicular distance to the origin, polar coordinates, or even vectors. We will develop equations for six forms: the normal, general, determinant, parametric, Cartesian and Hessian normal form.

3.11.1 Proof: Cartesian form of the line equation

Strategy: Let $\mathbf{n}$ be a nonzero vector normal to a line, and $P(x, y)$ be a point on the line, which also contains a point $P_0(x_0, y_0)$. Use vector analysis to derive the general form of the line equation.



Let the vector normal to the line be $\mathbf{n} = a\mathbf{i} + b\mathbf{j}$
Let $\mathbf{p}$ and $\mathbf{p}_0$ be the position vectors for P and P_0 respectively

where $\qquad \mathbf{p}_0 = x_0\mathbf{i} + y_0\mathbf{j}$
and $\qquad \mathbf{p} = x\mathbf{i} + y\mathbf{j}$

Therefore the line's direction vector is

$$\mathbf{q} = \mathbf{p} - \mathbf{p}_0$$

As $\mathbf{n}$ is orthogonal to $\mathbf{q}$ $\qquad \mathbf{n} \cdot \mathbf{q} = 0$
therefore $\qquad \mathbf{n} \cdot (\mathbf{p} - \mathbf{p}_0) = 0$
and $\qquad \mathbf{n} \cdot \mathbf{p} = \mathbf{n} \cdot \mathbf{p}_0 \qquad (1)$
therefore $\qquad ax + by = ax_0 + by_0$
The line equation is $\qquad ax + by = c$
where $\qquad c = ax_0 + by_0$

However, the value of c also has this interpretation:

from the diagram $\qquad d = \|\mathbf{p}_0\| \cos\alpha$
but $\qquad \mathbf{n} \cdot \mathbf{p}_0 = \|\mathbf{n}\| \cdot \|\mathbf{p}_0\| \cos\alpha = d\,\|\mathbf{n}\|$

Therefore the line equation is $\qquad ax + by = c \qquad (2)$
where $\qquad c = d\|\mathbf{n}\| \quad \text{or} \quad ax_0 + by_0$

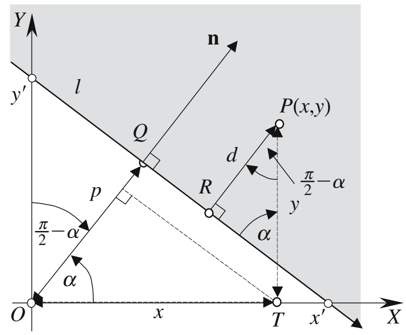
Dividing (2) by $\|\mathbf{n}\|$ we obtain the normalized Cartesian line equation.
The normalized Cartesian line equation is

$$\frac{a}{\|\mathbf{n}\|}x + \frac{b}{\|\mathbf{n}\|}y = d$$

Note that this equation depends upon the line being oriented with its normal vector pointing to the left of its direction.

3.11.2 Proof: Hessian normal form (after Otto Hesse (1811–1874))

The Hessian normal form of the equation of a line develops the Cartesian form and is used to partition the xy-plane in two. The division is determined by an oriented line l, such that when looking along the line's direction, points to the left are classified as positive, points to the right negative, and points on the line zero.



Strategy: Develop a general equation for the perpendicular distance of an arbitrary point $P(x, y)$ from a line l, taking into account the signs of angles associated with the geometry.

Q is a point on line l such that $\overrightarrow{OQ} = p$ and is perpendicular to l.

α is the angle between the x-axis and $\overrightarrow{OQ}$.

R is a point on line l such that $\overrightarrow{RP} = d$ and is perpendicular to l.

T is a point on the x-axis such that $\overrightarrow{TP}$ is perpendicular to the x-axis.

The diagram shows the resulting angles.

The vector path from the origin O to P has two routes:

$$\overrightarrow{OQ} + \overrightarrow{QR} + \overrightarrow{RP} = \overrightarrow{OT} + \overrightarrow{TP}$$

But rather than compute these individual vectors, compute their projections on the normal **n**:

therefore $$p + 0 + d = x\cos\alpha + y\sin\alpha$$

and $$d = x\cos\alpha + y\sin\alpha - p$$

$$\begin{array}{ll} d > 0 & \text{to the left of } l \\ d = 0 & \text{on the line } l \\ d < 0 & \text{to the right of } l \end{array}$$

where the sign of d provides space partitioning.

The Hessian normal form is expressed as

$$x\cos\alpha + y\sin\alpha = p$$

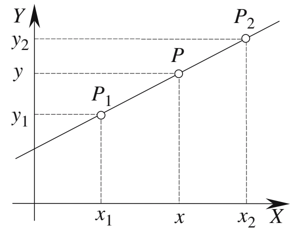
The axis intercepts are

$$x' = \frac{p}{\cos\alpha} \qquad y' = \frac{p}{\sin\alpha}$$

3.11.3 Proof: Equation of a line from two points

Strategy: Given two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ create an extra point $P(x, y)$ and equate the slopes between pairs of points.

Normal form of the line equation



From the diagram
$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$

therefore
$$y-y_1=\left(\frac{y_2-y_1}{x_2-x_1}\right)(x-x_1)$$

and
$$y=\left(\frac{y_2-y_1}{x_2-x_1}\right)x+y_1-x_1\left(\frac{y_2-y_1}{x_2-x_1}\right)$$

The normal form is
$$y=mx+c$$

where
$$m=\left(\frac{y_2-y_1}{x_2-x_1}\right)\qquad c=y_1-x_1\left(\frac{y_2-y_1}{x_2-x_1}\right)$$

General form of the line equation

From the diagram
$$\frac{y-y_1}{x-x_1}=\frac{y-y_1}{x-x_1}$$
$$(x_2-x_1)\,(y-y_1)=(y_2-y_1)\,(x-x_1)$$
$$(y_2-y_1)x-(y_2-y_1)x_1=(x_2-x_1)y-(x_2-x_1)y_1$$
$$(y_2-y_1)x+(x_1-x_2)y=x_1y_2-x_2y_1 \tag{1}$$

The general form is $Ax+By+C=0$

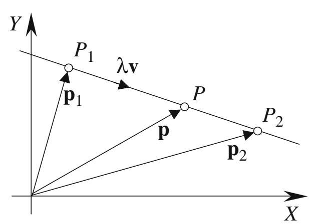
where $A=y_2-y_1 \qquad B=x_1-x_2 \qquad C=-(x_1y_2-x_2y_1)$

Determinant form of the line equation

Determinants can be used to describe (1)

$$\begin{vmatrix}1 & y_1\\ 1 & y_2\end{vmatrix}x+\begin{vmatrix}x_1 & 1\\ x_2 & 1\end{vmatrix}y=\begin{vmatrix}x_1 & y_1\\ x_2 & y_2\end{vmatrix}$$

Parametric form of the line equation



P_1 and P_2 are the two points and $\mathbf{p}_1$ and $\mathbf{p}_2$ their respective position vectors.

Let $\mathbf{v}=\mathbf{p}_2-\mathbf{p}_1$

therefore $\mathbf{p}=\mathbf{p}_1+\lambda\mathbf{v}$ where λ is a scalar.

P is between P_1 and P_2 for $\lambda\in[0,1]$.

If $||\mathbf{v}||=1$, λ corresponds to the linear distance along $\mathbf{v}$.

3.11.4 Proof: Point of intersection of two straight lines

General form of the line equation

Strategy: Solve the pair of simultaneous linear equations describing the straight lines.

Let the two lines be
$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

Let $P(x_P, y_P)$ be the point of intersection of the two lines.

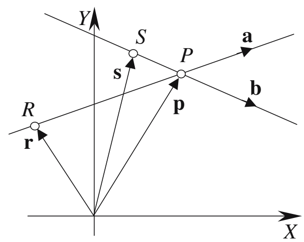
Therefore
$$a_1x_P + b_1y_P = -c_1$$
$$a_2x_P + b_2y_P = -c_2$$

therefore
$$\frac{x_P}{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}} = \frac{y_P}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}} = \frac{-1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Coordinates of P
$$x_P = \frac{c_2b_1 - c_1b_2}{a_1b_2 - a_2b_1} \qquad y_P = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}$$

The lines are parallel if the denominator $a_1b_2 - a_2b_1 = 0$

Parametric form of the line equation



Strategy: Equate the two parametric line equations and determine the values of λ and ε.

Let the line equations be $\mathbf{p} = \mathbf{r} + \lambda\mathbf{a}$
and $\mathbf{p} = \mathbf{s} + \varepsilon\mathbf{b}$

Let $P(x_P, y_P)$ be the point of intersection for the two lines and $\mathbf{p}$ its position vector.

Therefore
$$\mathbf{r} + \lambda\mathbf{a} = \mathbf{s} + \varepsilon\mathbf{b}$$

and
$$x_R + \lambda x_a = x_S + \varepsilon x_b \quad (1)$$
$$y_R + \lambda y_a = y_S + \varepsilon y_b \quad (2)$$

From (1)
$$\lambda = \frac{x_S - x_R + \varepsilon x_b}{x_a}$$

Substitute λ in (2)
$$y_R + y_a\left(\frac{x_S - x_R + \varepsilon x_b}{x_a}\right) = y_S + \varepsilon y_b$$

Expanding
$$x_a y_r + x_S y_a - x_R y_a + \varepsilon x_b y_a = x_a y_S + \varepsilon x_a y_b$$

Rearranging
$$\varepsilon(x_b y_a - x_a y_b) = x_a y_S - x_a y_R - x_S y_a + x_R y_a$$

We obtain
$$\varepsilon = \frac{x_a(y_S - y_R) - y_a(x_S - x_R)}{x_b y_a - x_a y_b}$$

Similarly
$$\lambda = \frac{x_b(y_S - y_R) - y_b(x_S - x_R)}{x_b y_a - x_a y_b}$$

or in determinant form

$$\varepsilon = \frac{\begin{vmatrix} x_a & (x_S - x_R) \\ y_a & (y_S - y_R) \end{vmatrix}}{\begin{vmatrix} x_b & y_b \\ x_a & y_a \end{vmatrix}} \qquad \lambda = \frac{\begin{vmatrix} x_b & (x_S - x_R) \\ y_b & (y_S - y_R) \end{vmatrix}}{\begin{vmatrix} x_b & y_b \\ x_a & y_a \end{vmatrix}}$$

Coordinates of P $\quad x_P = x_R + \lambda x_a \qquad y_P = y_R + \lambda y_a$

The lines are parallel if $\mathbf{a} \cdot \mathbf{b} = 0$

3.11.5 Proof: Angle between two straight lines

General form of the line equation

Strategy: Derive the normal vectors to the lines and compute the scalar product to reveal the cosine of the enclosed angle. Derive the sine and tangent of the angle from the cosine function.

Let the two lines be $\quad a_1x + b_1y + c_1 = 0$

and $\quad a_2x + b_2y + c_2 = 0$

The normal vectors are $\quad \mathbf{n} = a_1\mathbf{i} + b_1\mathbf{j}$ and $\mathbf{m} = a_2\mathbf{i} + b_2\mathbf{j}$

therefore $\quad \mathbf{n} \cdot \mathbf{m} = \|\mathbf{n}\| \cdot \|\mathbf{m}\| \cos\alpha$

Angle between the lines $\quad \alpha = \cos^{-1}\left(\dfrac{\mathbf{n} \cdot \mathbf{m}}{\|\mathbf{n}\| \cdot \|\mathbf{m}\|}\right)$

If $\|\mathbf{n}\| = \|\mathbf{m}\| = 1 \quad \alpha = \cos^{-1}(\mathbf{n} \cdot \mathbf{m})$

Normal form of the line equation

Strategy: Use the $\tan(A - B)$ function to reveal the enclosed angle α.

Let the two lines be $\quad y = m_1x + c_1$

$$y = m_2x + c_2$$

where $\quad m_1 = \tan\alpha_1$ and $m_2 = \tan\alpha_2$

$$\tan\alpha = \tan(\alpha_1 - \alpha_2) = \frac{\tan\alpha_1 - \tan\alpha_2}{1 + \tan\alpha_1 \tan\alpha_2}$$

$$\tan\alpha = \frac{m_1 - m_2}{1 + m_1m_2}$$

Angle between the lines $\quad \alpha = \tan^{-1}\left(\dfrac{m_1 - m_2}{1 + m_1m_2}\right)$

Note that if the lines are interchanged $\tan\alpha' = \tan(\alpha_2 - \alpha_1) = -\alpha$

If $m_1 m_2 = -1$ the lines are perpendicular.
To compute $\cos\alpha$

$$\tan\alpha_1 = m_1$$

but $$1 + \tan^2\alpha = \sec^2\alpha$$

therefore $$\cos\alpha_1 = \frac{1}{\sqrt{1+m_1^2}} \quad \text{and} \quad \sin\alpha_1 = \frac{m_1}{\sqrt{1+m_1^2}}$$

Similarly $$\cos\alpha_2 = \frac{1}{\sqrt{1+m_2^2}} \quad \text{and} \quad \sin\alpha_2 = \frac{m_2}{\sqrt{1+m_2^2}}$$

therefore $$\cos\alpha = \cos(\alpha_2 - \alpha_1) = \cos\alpha_2\cos\alpha_1 + \sin\alpha_2\sin\alpha_1$$

$$\cos\alpha = \frac{1}{\sqrt{1+m_2^2}}\frac{1}{\sqrt{1+m_1^2}} + \frac{m_2}{\sqrt{1+m_2^2}}\frac{m_1}{\sqrt{1+m_1^2}}$$

$$\cos\alpha = \frac{1+m_1 m_2}{\sqrt{1+m_1^2}\sqrt{1+m_2^2}}$$

Angle between the lines $$\alpha = \cos^{-1}\left(\frac{1+m_1 m_2}{\sqrt{1+m_1^2}\sqrt{1+m_2^2}}\right)$$

Note that this solution is not sensitive to the order of the lines.

Parametric form of the line equation

Strategy: Use the scalar product of the two line vectors to reveal the enclosed angle.

Let the two lines be $$\mathbf{p} = \mathbf{r} + \lambda\mathbf{a}$$
$$\mathbf{q} = \mathbf{s} + \varepsilon\mathbf{b}$$

The angle between the two lines is the angle between the vectors **a** and **b**, which is given by

$$\mathrm{a} \bullet \mathrm{b} = \|\mathbf{a}\| \cdot \|\mathbf{b}\| \cos\alpha$$

Angle between the lines $$\alpha = \cos^{-1}\left(\frac{\mathbf{a} \bullet \mathbf{b}}{\|\mathbf{a}\| \cdot \|\mathbf{b}\|}\right)$$

If $\|\mathbf{a}\| = \|\mathbf{b}\| = 1$ $$\alpha = \cos^{-1}(\mathbf{a} \bullet \mathbf{b})$$

3.11.6 Proof: Three points lie on a straight line

Strategy: If two vectors are created from the three points the vectors must be linearly related for the points to lie on a straight line.
Given three points P_1, P_2, P_3

Let $$\mathbf{r} = \overrightarrow{P_1P_2} \quad \text{and} \quad \mathbf{s} = \overrightarrow{P_1P_3}$$

therefore $\mathbf{s} = \lambda\mathbf{r}$ for the points to lie on a straight line.

3.11.7 Proof: Parallel and perpendicular straight lines

General form of the line equation

Let the lines be $a_1x + b_1y + c_1 = 0$
and $a_2x + b_2y + c_2 = 0$

Parallel lines

The normal vectors are $\mathbf{n} = a_1\mathbf{i} + b_1\mathbf{j}$ and $\mathbf{m} = a_1\mathbf{i} + b_1\mathbf{j}$ respectively.
n and **m** are parallel if $\mathbf{n} = \lambda\mathbf{m}$ where λ is a scalar.

Perpendicular lines

The lines are mutually perpendicular when $\mathbf{n} \cdot \mathbf{m} = 0$

Normal form of the line equation

Let the lines be $y = m_1x + c_1$
and $y = m_2x + c_2$

Parallel lines

m_1 and m_2 are the respective slopes of the two lines therefore the two lines are parallel when $m_1 = m_2$

Perpendicular lines

$$m_1 = \tan\alpha$$
$$m_2 = \tan(90° + \alpha)$$
but $m_2 = \tan(90° + \alpha) = -\cot\alpha$
therefore $m_1m_2 = \tan\alpha(-\cot\alpha) = -1$
$$m_1m_2 = -1$$

Parametric form of the line equation

Let the lines be $\mathbf{p} = \mathbf{r} + \lambda\mathbf{a}$
and $\mathbf{q} = \mathbf{s} + \varepsilon\mathbf{b}$

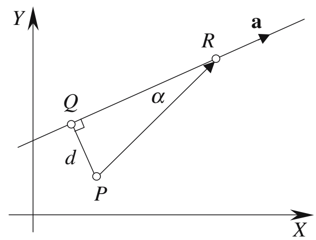
Parallel lines

p and **q** are parallel if $\mathbf{a} = k\mathbf{b}$ where k is a scalar.

Perpendicular lines

$$\mathbf{a} \cdot \mathbf{b} = 0$$

3.11.8 Proof: Shortest distance to a line



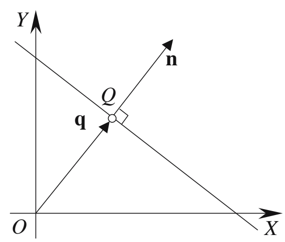
Strategy: Postulate that the shortest distance is a normal to a line and prove that other lines are longer.

Let P be an arbitrary point not on line **a**.
Let Q be a point on **a** such that PQ is orthogonal to **a**.
For any other point R on $PR = \dfrac{PQ}{\sin\alpha}$ therefore $PR > PQ$ for $\alpha \neq 90°$.
$PR = PQ$ when $\alpha = 90°$, therefore, PQ is the shortest distance from P to the line **a**.

Obviously, the same reasoning applies for a 3D line and a plane.

3.11.9 Proof: Position and distance of a point on a line perpendicular to the origin

General form of the line equation



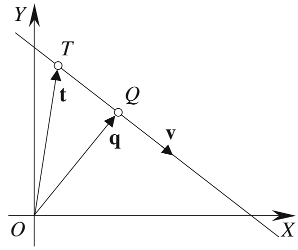
Strategy: Express the general form of the line equation as the scalar product of two vectors and use vector analysis to identify the point Q on the perpendicular to the origin.

Let the equation of the line be $ax + by + c = 0$
Q is the nearest point on the line to O and **q** is its position vector.

Let	$\mathbf{n} = a\mathbf{i} + b\mathbf{j}$
and	$\mathbf{q} = x\mathbf{i} + y\mathbf{j}$
therefore	$\mathbf{n} \cdot \mathbf{q} = -c$
Let	$\mathbf{q} = \lambda\mathbf{n}$
therefore	$\mathbf{n} \cdot \mathbf{q} = \lambda\mathbf{n} \cdot \mathbf{n} = -c$
and	$\lambda = \dfrac{-c}{\mathbf{n} \cdot \mathbf{n}}$
If $\lVert\mathbf{n}\rVert = 1$	$\lambda = -c$
position vector	$\mathbf{q} = \lambda\mathbf{n}$
distance	$OQ = \lVert\mathbf{q}\rVert$

Parametric form of the line equation

Strategy: Express the parametric form of line equation as the scalar product of two vectors and use vector analysis to identify the point on the perpendicular to the origin.



Let $\mathbf{q} = \mathbf{t} + \lambda\mathbf{v}$ (1)

Q is nearest to O when $\mathbf{q}$ is perpendicular to $\mathbf{v}$

therefore $\mathbf{v} \bullet \mathbf{q} = 0$

Take the scalar product of (1) with $\mathbf{v}$

$$\mathbf{v} \bullet \mathbf{q} = \mathbf{v} \bullet \mathbf{t} + \lambda \mathbf{v} \bullet \mathbf{v}$$

therefore $\lambda = \dfrac{-\mathbf{v} \bullet \mathbf{t}}{\mathbf{v} \bullet \mathbf{v}}$

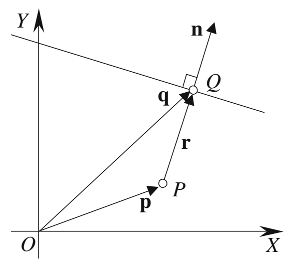
If $\|\mathbf{v}\| = 1$ $\lambda = -\mathbf{v} \bullet \mathbf{t}$

position vector $\mathbf{q} = \mathbf{t} + \lambda\mathbf{v}$

distance $OQ = \|\mathbf{q}\|$

3.11.10 Proof: Position and distance of the nearest point on a line to a point

General form of the line equation



Strategy: Express the general form of the line equation as the scalar product of two vectors and use vector analysis to identify the point Q on the perpendicular from P to the line.

Let the equation of the line be $ax + by + c = 0$ and $Q(x, y)$ be the nearest point on the line to P.

Let $\mathbf{n} = a\mathbf{i} + b\mathbf{j}$

and $\mathbf{q} = x\mathbf{i} + y\mathbf{j}$

therefore $\mathbf{n} \bullet \mathbf{q} = -c$ (1)

$\mathbf{r}$ is parallel to $\mathbf{n}$, therefore $\mathbf{r} = \lambda\mathbf{n}$ (2)

and $\mathbf{n} \bullet \mathbf{r} = \lambda \mathbf{n} \bullet \mathbf{n}$ (3)

but $\mathbf{r} = \mathbf{q} - \mathbf{p}$

therefore $\mathbf{n} \bullet \mathbf{r} = \mathbf{n} \bullet \mathbf{q} - \mathbf{n} \bullet \mathbf{p}$ (4)

Substitute (1) and (3) in (4) $\lambda \mathbf{n} \bullet \mathbf{n} = -c - \mathbf{n} \bullet \mathbf{p}$ (5)

therefore $\lambda = \dfrac{-(\mathbf{n} \bullet \mathbf{p} + c)}{\mathbf{n} \bullet \mathbf{n}}$

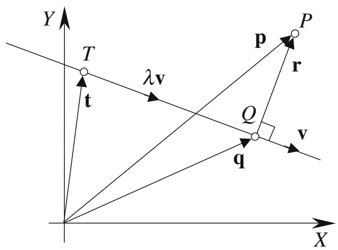
If $\|\mathbf{n}\| = 1$ $\lambda = -(\mathbf{n} \bullet \mathbf{p} + c)$

but $\mathbf{q} = \mathbf{p} + \mathbf{r}$ (6)

Substitute (2) in (6) $\mathbf{q} = \mathbf{p} + \lambda\mathbf{n}$

distance $PQ = \|\mathbf{r}\| = \|\lambda\mathbf{n}\|$

Parametric form of the line equation



Strategy: Express the parametric form of the line equation as the scalar product of two vectors and use vector analysis to identify the point Q on the perpendicular from P to the line.

Let the equation of the line be $\mathbf{q} = \mathbf{t} + \lambda\mathbf{v}$ (7)

Let Q be the nearest point on the line to P

but $\mathbf{p} = \mathbf{q} + \mathbf{r}$

therefore $\mathbf{v} \bullet \mathbf{p} = \mathbf{v} \bullet \mathbf{q} + \mathbf{v} \bullet \mathbf{r}$

$\mathbf{r}$ is orthogonal to $\mathbf{v}$, therefore $\mathbf{v} \bullet \mathbf{r} = 0$

and $\mathbf{v} \bullet \mathbf{p} = \mathbf{v} \bullet \mathbf{q}$

From (7) $\mathbf{v} \bullet \mathbf{q} = \mathbf{v} \bullet \mathbf{t} + \lambda\mathbf{v} \bullet \mathbf{v}$

therefore $\lambda = \dfrac{\mathbf{v} \bullet (\mathbf{p} - \mathbf{t})}{\mathbf{v} \bullet \mathbf{v}}$

If $\|\mathbf{v}\| = 1$ $\lambda = \mathbf{v} \bullet (\mathbf{p} - \mathbf{t})$

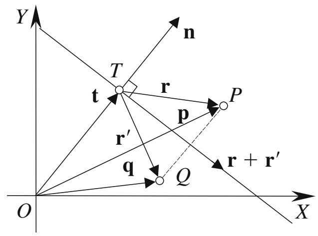
position vector $\mathbf{q} = \mathbf{t} + \lambda\mathbf{v}$

distance $PQ = \|\mathbf{r}\| = \|\mathbf{p} - \mathbf{q}\| = \|\mathbf{p} - (\mathbf{t} + \lambda\mathbf{v})\|$

3.11.11 Proof: Position of a point reflected in a line

General form of the line equation

Strategy: Exploit the fact that a line connecting a point and its reflection is parallel to the line's normal.



Let the equation of the line be $ax + by + c = 0$

$T(x, y)$ is the nearest point on the line to O and $\mathbf{t} = x\mathbf{i} + y\mathbf{j}$ is its position vector.

let $\quad \mathbf{n} = a\mathbf{i} + b\mathbf{j}$

therefore $\quad \mathbf{n} \cdot \mathbf{t} = -c \qquad (1)$

P is an arbitrary point and Q is its reflection; $\mathbf{p}$ and $\mathbf{q}$ are their respective position vectors.

$\mathbf{r} + \mathbf{r}'$ is orthogonal to $\mathbf{n}$

therefore $\quad \mathbf{n} \cdot (\mathbf{r} + \mathbf{r}') = 0$

$$\mathbf{n} \cdot \mathbf{r} + \mathbf{n} \cdot \mathbf{r}' = 0 \qquad (2)$$

$\mathbf{p} - \mathbf{q}$ is parallel with $\mathbf{n}$

therefore $\quad \mathbf{p} - \mathbf{q} = \mathbf{r} - \mathbf{r}' = \lambda \mathbf{n}$

and $\quad \lambda = \dfrac{\mathbf{r} - \mathbf{r}'}{\mathbf{n}} \qquad (3)$

but $\quad \mathbf{r} = \mathbf{p} - \mathbf{t} \qquad (4)$

Substitute (1) in (4) $\quad \mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{p} - \mathbf{n} \cdot \mathbf{t} = \mathbf{n} \cdot \mathbf{p} + c \qquad (5)$

Substitute (2) and (5) in (3) $\quad \lambda = \dfrac{\mathbf{n} \cdot \mathbf{r} - \mathbf{n} \cdot \mathbf{r}'}{\mathbf{n} \cdot \mathbf{n}} = \dfrac{2\mathbf{n} \cdot \mathbf{r}}{\mathbf{n} \cdot \mathbf{n}}$

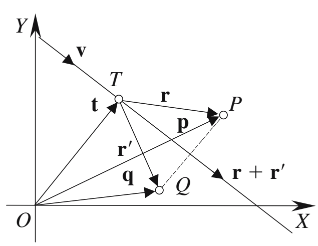
$$\lambda = \frac{2(\mathbf{n} \cdot \mathbf{p} + c)}{\mathbf{n} \cdot \mathbf{n}}$$

If $\|\mathbf{n}\| = 1 \quad \lambda = 2(\mathbf{n} \cdot \mathbf{p} + c)$

position vector $\quad \mathbf{q} = \mathbf{p} - \lambda \mathbf{n}$

Parametric form of the line equation

Strategy: Exploit the fact that the line's direction vector is orthogonal to the line connecting a point and its reflection.



P is an arbitrary point and Q is its reflection; $\mathbf{p}$ and $\mathbf{q}$ are their respective position vectors.

Let the equation of the line be $\quad \mathbf{s} = \mathbf{t} + \lambda \mathbf{v}$

therefore $\quad \mathbf{p} = \mathbf{t} + \mathbf{r}$

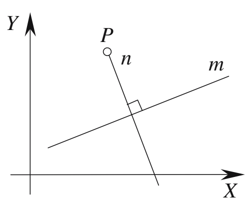
and $\quad \mathbf{q} = \mathbf{t} + \mathbf{r}'$

therefore	$\mathbf{p}+\mathbf{q}=2\mathbf{t}+\mathbf{r}+\mathbf{r}'$	(6)
$\mathbf{r}-\mathbf{r}'$ is orthogonal to $\mathbf{v}$		
therefore	$\mathbf{v}\bullet(\mathbf{r}-\mathbf{r}')=0$	
	$\mathbf{v}\bullet\mathbf{r}=\mathbf{v}\bullet\mathbf{r}'$	(7)
$\mathbf{r}+\mathbf{r}'$ is parallel to $\mathbf{v}$		
therefore	$\mathbf{r}+\mathbf{r}'=\varepsilon\mathbf{v}$	(8)
and	$\mathbf{v}\bullet(\mathbf{r}+\mathbf{r}')=\varepsilon\mathbf{v}\bullet\mathbf{v}$	
where	$\varepsilon=\dfrac{\mathbf{v}\bullet\mathbf{r}+\mathbf{v}\bullet\mathbf{r}'}{\mathbf{v}\bullet\mathbf{v}}$	(9)
Substitute (7) in (9)	$\varepsilon=\dfrac{2\mathbf{v}\bullet\mathbf{r}}{\mathbf{v}\bullet\mathbf{v}}$	
but	$\mathbf{r}=\mathbf{p}-\mathbf{t}$	
therefore	$\varepsilon=\dfrac{2\mathbf{v}\bullet(\mathbf{p}-\mathbf{t})}{\mathbf{v}\bullet\mathbf{v}}$	
If $\lVert\mathbf{v}\rVert=1$	$\varepsilon=2\mathbf{v}\bullet(\mathbf{p}-\mathbf{t})$	
Substitute (8) in (6)	$\mathbf{p}+\mathbf{q}=2\mathbf{t}+\varepsilon\mathbf{v}$	
position vector	$\mathbf{q}=2\mathbf{t}+\varepsilon\mathbf{v}-\mathbf{p}$	

3.11.12 Proof: Normal to a line through a point

Strategy: Given a line m and a point P the object is to identify a line n that passes through P and is normal to m. This is achieved by finding the perpendicular form of the line equation.

General form of the line equation

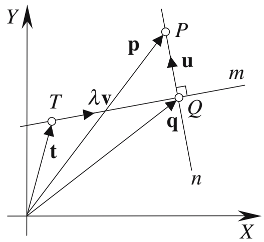


Given the line m	$ax+by+c=0$

Let the line n be perpendicular to m passing through the point $P(x_P, y_P)$.

Let the line be	$a_n x+b_n y+c_n=0$
n is perpendicular to m when	$a_n=-b \qquad b_n=a \qquad c_n=-(ay_P-bx_P)$
The line equation for n is	$-bx+ay+bx_P-ay_P=0$

Parametric form of the line equation



Given the line m	$\mathbf{q} = \mathbf{t} + \lambda\mathbf{v}$	(1)

there exists a point Q such that $\mathbf{v}$ is normal to $\mathbf{u}$.

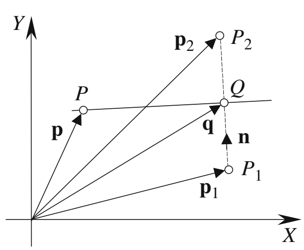
Also	$\mathbf{q} = \mathbf{p} - \mathbf{u}$	(2)
From (1) and (2)	$\mathbf{t} + \lambda\mathbf{v} = \mathbf{p} - \mathbf{u}$	(3)
therefore	$\mathbf{v}\cdot\mathbf{t} + \lambda\mathbf{v}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{p} - \mathbf{v}\cdot\mathbf{u}$	
but	$\mathbf{v}\cdot\mathbf{u} = 0$	
therefore	$\lambda = \dfrac{\mathbf{v}\cdot(\mathbf{p}-\mathbf{t})}{\mathbf{v}\cdot\mathbf{v}}$	
If $\|\|\mathbf{v}\|\| = 1$	$\lambda = \mathbf{v}\cdot(\mathbf{p}-\mathbf{t})$	
From (3)	$\mathbf{u} = \mathbf{p} - (\mathbf{t} + \lambda\mathbf{v})$	
therefore, line n is	$\mathbf{n} = \mathbf{p} + \varepsilon\mathbf{u}$ where ε is a scalar.	

3.11.13 Proof: Line equidistant from two points

Given two distinct points we require to identify a line passing between them such that any point on the line is equidistant to the points.

Strategy: The key to this solution is that the normal of the line is parallel to the line joining the two points.

General form of the line equation



Let the equation of the line equidistant to $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be $ax + by + c = 0$
$P(x, y)$ is a point on this line which contains Q equidistant to P_1 and P_2.

Let $\mathbf{n} = a\mathbf{i} + b\mathbf{j} = \mathbf{p}_2 - \mathbf{p}_1$ (1)

and $\mathbf{q} = \mathbf{p}_1 + \frac{1}{2}\mathbf{n} = \frac{1}{2}(\mathbf{p}_2 - \mathbf{p}_1)$ (2)

then $\mathbf{n} \bullet (\mathbf{p} - \mathbf{q}) = 0$

therefore $\mathbf{n} \bullet \mathbf{p} = \mathbf{n} \bullet \mathbf{q}$

But the line equation is $\mathbf{n} \bullet \mathbf{p} + \mathbf{c} = 0$

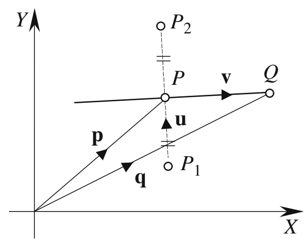
therefore $c = -\mathbf{n} \bullet \mathbf{p} = -\mathbf{n} \bullet \mathbf{q}$ (3)

Substituting (1) and (2) in (3) $c = -(\mathbf{p}_2 - \mathbf{p}_1) \bullet (\mathbf{p}_1 + \frac{1}{2}(\mathbf{p}_2 - \mathbf{p}_1)) = -\frac{1}{2}(\mathbf{p}_2 - \mathbf{p}_1) \bullet (\mathbf{p}_2 + \mathbf{p}_1)$

The line equation is $(\mathbf{p}_2 - \mathbf{p}_1) \bullet (\mathbf{p} - \frac{1}{2}(\mathbf{p}_2 + \mathbf{p}_1)) = 0$

or $(x_2 - x_1)x + (y_2 - y_1)y - \frac{1}{2}(x_2^2 - x_1^2 + y_2^2 - y_1^2) = 0$

Parametric form of the line equation



Let $P(x_P, y_P)$ be a point equidistant between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

Let $\mathbf{u}$ be the vector $\overrightarrow{P_1P_2}$

P is also on the line $\mathbf{q} = \mathbf{p} + \lambda\mathbf{v}$, which is perpendicular to $\mathbf{u}$.

$\mathbf{p}$ and $\mathbf{q}$ are the position vectors for P and Q respectively.

Therefore $\mathbf{p} = \frac{1}{2}(x_1 + x_2)\mathbf{i} + \frac{1}{2}(y_1 + y_2)\mathbf{j}$

also $\mathbf{u} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j}$

As $\mathbf{v}$ is perpendicular to $\mathbf{u}$ $\mathbf{v} = -(y_2 - y_1)\mathbf{i} + (x_2 - x_1)\mathbf{j}$

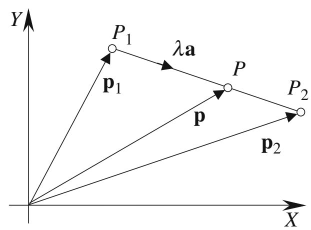
therefore $\mathbf{q} = \frac{1}{2}(x_1 + x_2)\mathbf{i} + \frac{1}{2}(y_1 + y_2)\mathbf{j} + \lambda(-(y_2 - y_1)\mathbf{i} + (x_2 - x_1)\mathbf{j})$

$$\mathbf{q} = (\tfrac{1}{2}(x_1 + x_2) - \lambda(y_2 - y_1))\mathbf{i} + (\tfrac{1}{2}(y_1 + y_2) + \lambda(x_2 - x_1))\mathbf{j}$$

where λ is a scalar.

3.11.14 Proof: Equation of a two-dimensional line segment

Parametric form of the line equation



Strategy: The parametric form of the straight-line equation is the most practical basis for manipulating straight-line segments. The value of the parameter can then be used to determine the position of a point along the segment.

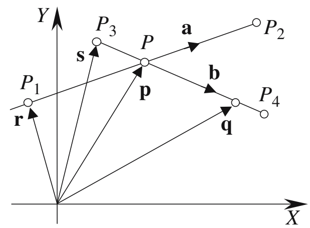
$P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ define the line segment and $\mathbf{p}_1$ and $\mathbf{p}_2$ are their respective position vectors and $P(x_P, y_P)$ is a point on the line segment.

Let	$\mathbf{a} = \mathbf{p}_2 - \mathbf{p}_1$
Position vector of P	$\mathbf{p} = \mathbf{p}_1 + \lambda\mathbf{a}$
Coordinates of P	$x_P = x_1 + \lambda(x_2 - x_1) \qquad y_P = y_1 + \lambda(y_2 - y_1)$

P is between P_1 and P_2 for $\lambda \in [0, 1]$.

3.11.15 Proof: Point of intersection of two two-dimensional line segments

Strategy: The parametric proof for calculating the intersection of two straight lines can be used to determine the spatial relationship between two line segments. The values of the parameters controlling the direction vectors determine whether the line segments touch or intersect.



Let the two line segments be defined by $P_1(x_1, y_1) \rightarrow P_2(x_2, y_2)$ and $P_3(x_3, y_3) \rightarrow P_4(x_4, y_4)$ where $P(x_P, y_P)$ is the point of intersection.

Let	$\mathbf{a} = x_a\mathbf{i} + y_a\mathbf{j}$	
where	$x_a = x_2 - x_1$ and $y_a = y_2 - y_1$	
and	$\mathbf{b} = x_b\mathbf{i} + y_b\mathbf{j}$	
where	$x_b = x_4 - x_3$ and $y_b = y_4 - y_3$	
The line equations are	$\mathbf{p} = \mathbf{r} + \lambda\mathbf{a}$	
and	$\mathbf{q} = \mathbf{s} + \varepsilon\mathbf{b}$	
For intersection	$\mathbf{r} + \lambda\mathbf{a} = \mathbf{s} + \varepsilon\mathbf{b}$	
where	$x_1 + \lambda x_a = x_3 + \varepsilon x_b$	(1)
and	$y_1 + \lambda y_a = y_3 + \varepsilon y_b$	(2)
From (1)	$\lambda = \dfrac{x_3 - x_1 + \varepsilon x_b}{x_a}$	
Substitute λ in (2)	$y_1 + y_a\left(\dfrac{x_3 - x_1 + \varepsilon x_b}{x_a}\right) = y_3 + \varepsilon y_b$	
therefore	$x_a y_1 + x_3 y_a - x_1 y_a + \varepsilon x_b y_a = x_a y_3 + \varepsilon x_a y_b$	
and	$\varepsilon(x_b y_a - x_a y_b) = x_a y_3 - x_a y_1 - x_3 y_a + x_1 y_a$	
	$\varepsilon = \dfrac{x_a(y_3 - y_1) - y_a(x_3 - x_1)}{x_b y_a - x_a y_b}$	
Similarly	$\lambda = \dfrac{x_b(y_3 - y_1) - y_b(x_3 - x_1)}{x_b y_a - x_a y_b}$	

In determinant form
$$\varepsilon = \frac{\begin{vmatrix} x_a & (x_3 - x_1) \\ y_a & (y_3 - y_1) \end{vmatrix}}{\begin{vmatrix} x_b & y_b \\ x_a & y_a \end{vmatrix}}$$

and
$$\lambda = \frac{\begin{vmatrix} x_b & (x_3 - x_1) \\ y_b & (y_3 - y_1) \end{vmatrix}}{\begin{vmatrix} x_b & y_b \\ x_a & y_a \end{vmatrix}}$$

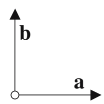
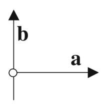
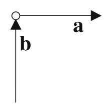
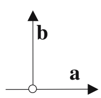
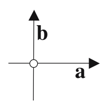
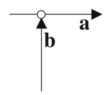
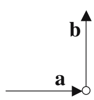
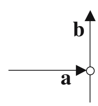
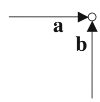
If $0 \leqslant \lambda \leqslant 1$ and $0 \leqslant \varepsilon \leqslant 1$ the lines intersect or touch one another.

Coordinates of P $\quad x_P = x_1 + \lambda x_a \quad y_P = y_1 + \lambda y_a$

or $\quad x_P = x_3 + \varepsilon x_b \quad y_P = y_3 + \varepsilon\lambda y_b$

The line segments are parallel if the denominator is zero $x_b y_a - x_a y_b = 0$

The table below illustrates the relative positions of the line segments for different values of λ and ε.

λ	ε		ε		ε	
0	0		$0 < \varepsilon < 1$		1	
$0 < \lambda < 1$	0		$0 < \varepsilon < 1$		1	
1	0		$0 < \varepsilon < 1$		1	

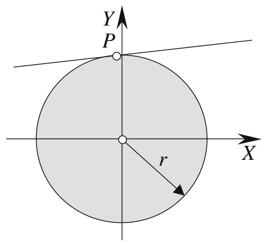
3.12 Lines and circles

3.12.1 Proof: Line and a circle

There are three scenarios: the line intersects, touches or misses the circle.

Strategy: The cosine rule proves very useful in setting up a geometric condition that identifies the above scenarios, which are readily solved using vector analysis. We also explore different approaches governed by the type of equation used.

General form of the line equation



A circle with radius r is centered at the origin

therefore its equation is $\qquad x^2 + y^2 = r^2 \qquad (1)$

The normalized line equation is $ax + by + c = 0$

where $\qquad a^2 + b^2 = 1$

therefore $\qquad x = \dfrac{-c - by}{a} \qquad (2)$

Substituting (2) in (1) $\qquad \left(\dfrac{-c - by}{a}\right)^2 + y^2 = r^2$

we have $\qquad c^2 + 2bcy + b^2y^2 + a^2y^2 = a^2r^2$

therefore $\qquad (a^2 + b^2)y^2 + 2bcy + c^2 - a^2r^2 = 0$

But $a^2 + b^2 = 1$, therefore $\qquad y^2 + 2bcy + c^2 - a^2r^2 = 0 \qquad (3)$

(3) is a quadratic in y where $\qquad y = -bc \pm \sqrt{c^2(b^2 - 1) + a^2r^2} \qquad (4)$

Similarly $\qquad x = -ac \pm \sqrt{c^2(a^2 - 1) + b^2r^2} \qquad (5)$

The discriminant of (4) or (5) determines whether the line intersects, touches or misses the circle:

Miss condition	$c^2(b^2 - 1) + a^2r^2 < 0$	(complex roots)
Touch condition	$c^2(b^2 - 1) + a^2r^2 = 0$	(equal roots)
Intersect condition	$c^2(b^2 - 1) + a^2r^2 > 0$	(real roots)

When either x or y is evaluated, the other variable is found by substituting the known variable in (2).

The above proof is for a circle centered at the origin, which is probably rare, and if the circle is positioned at (x_C, y_C) the associated formulas become rather fussy. To avoid this problem it is useful to leave the circle centered at the origin and translate the line by $(-x_C, -y_C)$ and add (x_C, y_C) to the final solution.

The circle is located at the origin:

therefore $\qquad x^2 + y^2 = r^2 \qquad (6)$

but the line equation is translated $(-x_C, -y_C)$

therefore $\qquad a(x - (-x_C)) + b(y - (-y_C)) + c = 0$

and $\qquad ax + by + (ax_C + by_C + c) = 0$

which becomes $\qquad ax + by + c_T = 0 \qquad (7)$

where $\qquad c_T = ax_C + by_C + c$

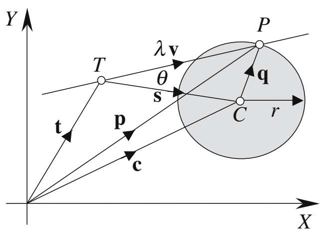
Substituting (7) in (6) we obtain similar equations to those derived above:

$$x = -ac_T \pm \sqrt{c_T^2(a^2 - 1) + b^2r^2}$$
$$y = -bc_T \pm \sqrt{c_T^2(b^2 - 1) + a^2r^2}$$

but these have to be extended to accommodate the original translation to the line:

Coordinates of P	$x = x_C - ac_T \pm \sqrt{c_T^2(a^2 - 1) + b^2r^2}$
	$y = y_C - bc_T \pm \sqrt{c_T^2(b^2 - 1) + a^2r^2}$
where	$c_T = ax_C + by_C + c$
Miss condition	$c_T^2(b^2 - 1) + a^2r^2 < 0$
Touch condition	$c_T^2(b^2 - 1) + a^2r^2 = 0$
Intersect condition	$c_T^2(b^2 - 1) + a^2r^2 > 0$

Parametric form of the line equation



A circle with radius r is located at $C(x_C, y_C)$ with position vector $\mathbf{c} = x_C\mathbf{i} + y_C\mathbf{j}$

The equation of the line is $\qquad \mathbf{p} = \mathbf{t} + \lambda\mathbf{v}$

where $\qquad \|\mathbf{v}\| = 1 \qquad (8)$

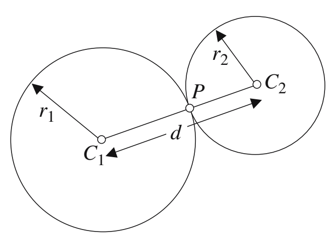
for an intersection at P	$\lVert\mathbf{q}\rVert = r \quad \text{or} \quad \lVert\mathbf{q}\rVert^2 = r^2 \quad \text{or} \quad \lVert\mathbf{q}\rVert^2 - r^2 = 0$	
Using the cosine rule	$\lVert\mathbf{q}\rVert^2 = \lVert\lambda\mathbf{v}\rVert^2 + \lVert\mathbf{s}\rVert^2 - 2\,\lVert\lambda\mathbf{v}\rVert \cdot \lVert\mathbf{s}\rVert\cos\theta$	
	$\lVert\mathbf{q}\rVert^2 = \lambda^2\,\lVert\mathbf{v}\rVert^2 + \lVert\mathbf{s}\rVert^2 - 2\,\lVert\mathbf{v}\rVert \cdot \lVert\mathbf{s}\rVert\lambda\cos\theta$	(9)
Substituting (8) in (9)	$\lVert\mathbf{q}\rVert^2 = \lambda^2 + \lVert\mathbf{s}\rVert^2 - 2\,\lVert\mathbf{s}\rVert\lambda\cos\theta$	(10)
Identify $\cos\theta$	$\mathbf{s}\bullet\mathbf{v} = \lVert\mathbf{s}\rVert \cdot \lVert\mathbf{v}\rVert\cos\theta$	
therefore	$\cos\theta = \dfrac{\mathbf{s}\bullet\mathbf{v}}{\lVert\mathbf{s}\rVert}$	(11)
Substitute (11) in (10)	$\lVert\mathbf{q}\rVert^2 = \lambda^2 - 2\mathbf{s}\bullet\mathbf{v}\lambda + \lVert\mathbf{s}\rVert^2$	
therefore	$\lVert\mathbf{q}\rVert^2 - r^2 = \lambda^2 - 2\mathbf{s}\bullet\mathbf{v}\lambda + \lVert\mathbf{s}\rVert^2 - r^2 = 0$	(12)
(12) is a quadratic where	$\lambda = \mathbf{s}\bullet\mathbf{v} \pm \sqrt{(\mathbf{s}\bullet\mathbf{v})^2 - \lVert\mathbf{s}\rVert^2 + r^2}$	(13)
and	$\mathbf{s} = \mathbf{c} - \mathbf{t}$	

The discriminant of (13) determines whether the line intersects, touches or misses the circle.

Coordinates of P	$x_P = x_T + \lambda x_v$
	$y_P = y_T + \lambda y_v$
where	$\lambda = \mathbf{s}\bullet\mathbf{v} \pm \sqrt{(\mathbf{s}\bullet\mathbf{v})^2 - \lVert\mathbf{s}\rVert^2 + r^2}$
	$\mathbf{s} = \mathbf{c} - \mathbf{t}$
Miss condition	$(\mathbf{s}\bullet\mathbf{v})^2 - \lVert\mathbf{s}\rVert^2 + r^2 < 0$
Touch condition	$(\mathbf{s}\bullet\mathbf{v})^2 - \lVert\mathbf{s}\rVert^2 + r^2 = 0$
Intersect condition	$(\mathbf{s}\bullet\mathbf{v})^2 - \lVert\mathbf{s}\rVert^2 + r^2 > 0$

3.12.2 Proof: Touching and intersecting circles

There are basically five scenarios associated with a pair of circles: first, they are totally separate; second, they touch as solid objects; third, their boundaries intersect; fourth, they touch when one circle is inside the other; and fifth, one circle is inside the other or possibly coincident. This proof examines two strategies: one to detect when two circles intersect, touch as solid objects or are separate, the other to provide the points of intersection.



Strategy 1: Use basic coordinate geometry to identify the touch condition.

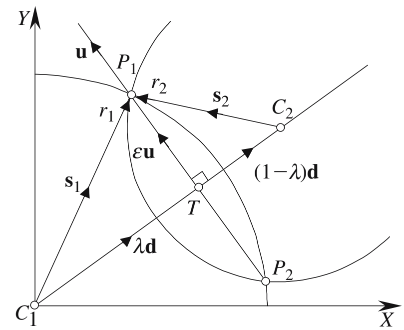
The diagram shows two circles with radii r_1 and r_2 centered at $C_1(x_{C1}, y_{C1})$ and $C_2(x_{C2}, y_{C2})$ respectively, touching at $P(x_P, y_P)$.

For a touch condition the distance d between C_1 and C_2 must equal $r_1 + r_2$:

$$d = \sqrt{(x_{C2} - x_{C1})^2 + (y_{C2} - y_{C1})^2}$$

Touch condition	$d = r_1 + r_2$
Intersect condition	$r_1 + r_2 > d > \lvert r_1 - r_2 \rvert$
Separate condition	$d > r_1 + r_2$
Touch point	$x_P = x_{C1} + \frac{r_1}{d}(x_{C2} - x_{C1})$ and $y_P = y_{C1} + \frac{r_1}{d}(y_{C2} - y_{C1})$

Strategy 2: Use vector analysis to identify the points of intersection.



This strategy assumes that the circles intersect.

The diagram shows a circle with radius r_1 centered at the origin and a second circle with radius r_2 centered at C_2 (x_{C2}, y_{C2}).

$\mathbf{s}_1$ is the position vector of the intersection point $P_1(x_{P1}, y_{P1})$ and will be used to identify the coordinates of P_1.

$\mathbf{d}$ is the position vector of C_2 and $d = \|\mathbf{d}\|$ is the distance between the circles' centers.

T is a point on $\mathbf{d}$ determined by the common chord passing through the two intersection points.

$\mathbf{u}$ is the vector $\overrightarrow{TP_1}$.

Euclidean geometry confirms that a line connecting the centers of two circles is perpendicular to a common chord, hence $\mathbf{u}$ is perpendicular to $\mathbf{d}$.

Let	$\mathbf{d} = x_d\mathbf{i} + y_d\mathbf{j}$ represent the vector $\overrightarrow{C_1C_2}$	
then	$\mathbf{u} = -y_d\mathbf{i} + x_d\mathbf{j}$	
and	$d = \|\mathbf{d}\| = \|\mathbf{u}\|$	(1)
	$\|\mathbf{s}_1\| = r_1$ and $\|\mathbf{s}_2\| = r_2$	(2)
Let	$\lambda\mathbf{d}$ represent the vector $\overrightarrow{C_1T}$	
and	$(1 - \lambda)\mathbf{d}$ represent the vector $\overrightarrow{TC_2}$	
Therefore	$\|\mathbf{s}_1\|^2 = \lambda^2 \|\mathbf{d}\|^2 + \varepsilon^2 \|\mathbf{u}\|^2$	(3)
and	$\|\mathbf{s}_2\|^2 = (1 - \lambda)^2 \|\mathbf{d}\|^2 + \varepsilon^2 \|\mathbf{u}\|^2$	(4)
Subtracting (4) from (3)	$\|\mathbf{s}_1\|^2 - \|\mathbf{s}_2\|^2 = 2\lambda \|\mathbf{d}\|^2 - \|\mathbf{d}\|^2$	(5)
Substituting (1) and (2) in (5)	$\lambda = \dfrac{r_1^2 - r_2^2 + d^2}{2d^2}$	

From (3)
$$\varepsilon^2 = \frac{r_1^2 - \lambda^2 d^2}{\|\mathbf{u}\|^2} = \frac{r_1^2}{d^2} - \lambda^2$$

$$\varepsilon = \pm\sqrt{\frac{r_1^2}{d^2} - \lambda^2} \tag{6}$$

and
$$\mathbf{s}_1 = \lambda\mathbf{d} + \varepsilon\mathbf{u}$$

However, the coordinates of P_1 must be translated by (x_{C1}, y_{C1}) as one circle was centered at the origin.

	$d = \sqrt{(x_{C2} - x_{C1})^2 + (y_{C2} - y_{C1})^2}$
Touch condition	$d = r_1 + r_2$
Touch point	$x_P = x_{C1} + \frac{r_1}{d}(x_{C2} - x_{C1})$ and $y_P = y_{C1} + \frac{r_1}{d}(y_{C2} - y_{C1})$
Miss condition	$d > r_1 + r_2$
Intersect condition	$d < r_1 + r_2$
Point(s) of intersection	$x_{P1} = x_{C1} + \lambda x_d - \varepsilon y_d$ $\quad y_{P1} = y_{C1} + \lambda y_d + \varepsilon x_d$
	$x_{P2} = x_{C1} + \lambda x_d + \varepsilon y_d$ $\quad y_{P2} = y_{C1} + \lambda y_d - \varepsilon x_d$
where	$\lambda = \dfrac{r_1^2 - r_2^2 + d^2}{2d^2}$
and	$\varepsilon = \left\lvert\sqrt{\dfrac{r_1^2}{d^2} - \lambda^2}\right\rvert$

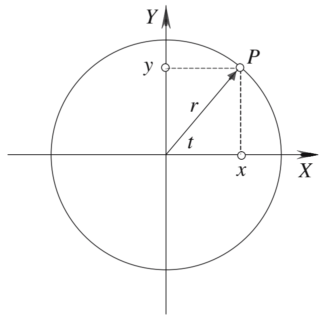
3.13 Second degree curves

3.13.1 Circle

General equation

The general equation of a circle is based upon the Pythagorean theorem, where a point $P(x, y)$ on the circle is related to the radius:

$$x^2 + y^2 = r^2$$



If the circle's center is offset from the origin, the x and y-coordinates are offset to accommodate the translation:

center (x_c, y_c) $\qquad (x - x_c)^2 + (y - y_c)^2 = r^2$

Parametric equation

By making the angle of rotation a parameter, the x and y-coordinates can be written as:

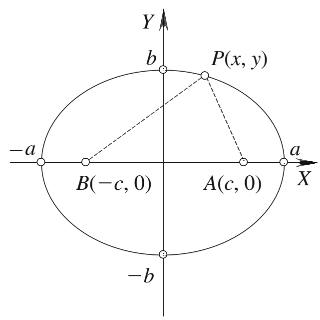
Center origin $\qquad \left.\begin{array}{l} x = r\cos t \\ y = r\sin t \end{array}\right\} \quad 0 \le t \le 2\pi$

or with an offset center (x_c, y_c) $\qquad \left.\begin{array}{l} x = x_c + r\cos t \\ y = y_c + r\sin t \end{array}\right\} \quad 0 \le t \le 2\pi$

3.13.2 Ellipse

General equation

Let the two foci be $(c, 0)$ and $(-c, 0)$, and $P(x, y)$ be a point on the ellipse.



Distance $\qquad |AP| = \sqrt{(x - c)^2 + (y - 0)^2}$

and $\qquad |BP| = \sqrt{(x + c)^2 + (y - 0)^2}$

However, an ellipse is defined such that $2a = |AP| + |BP|$

therefore $\qquad \sqrt{(x - c)^2 + y^2} = \sqrt{(x + c)^2 + y^2} = 2a$

and $$\sqrt{x^2 - 2cx + c^2 + y^2} = 2a - \sqrt{x^2 + 2cx + c^2 + y^2}$$

then $$x^2 - 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2cx + c^2 + y^2$$

$$-4cx = 4a^2 - 4a\sqrt{(x+c)^2 + y^2}$$

$$a^2 + cx = a\sqrt{(x+c)^2 + y^2}$$

Squaring both sides $$a^4 + 2a^2cx + c^2x^2 = a^2((x+c)^2 + y^2)$$

$$a^4 + 2a^2cx + c^2x^2 = a^2x^2 + 2a^2cx + a^2c^2 + a^2y^2$$

$$a^2(a^2 - c^2) = x^2(a^2 - c^2) + a^2y^2$$

but $$a^2 = b^2 + c^2 \quad \text{or} \quad b^2 = a^2 - c^2$$

therefore $$a^2b^2 = x^2b^2 + a^2y^2$$

and $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

If the center is offset by (x_c, y_c) the equation becomes

$$\frac{(x - x_c)^2}{a^2} + \frac{(y - y_c)^2}{b^2} = 1$$

Parametric equation

By making the angle of rotation a parameter the x and y-coordinates can be written as

Center origin $$\left.\begin{aligned} x &= a\cos t \\ y &= b\sin t \end{aligned}\right\} \quad 0 \le t \le 2\pi$$

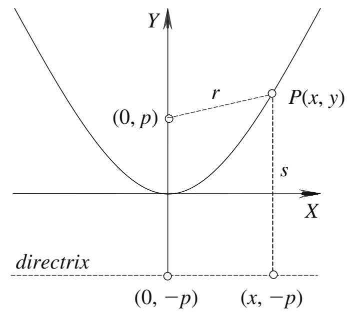
or with an offset center (x_c, y_c)

$$\left.\begin{aligned} x &= x_c + a\cos t \\ y &= y_c + b\sin t \end{aligned}\right\} \quad 0 \le t \le 2\pi$$

3.13.3 Parabola

General equation

By definition, the parabola maintains $r = s$ where $(0, p)$ is the focus.



Then $\quad r = \sqrt{x^2 + (y-p)^2} \quad$ and $\quad s = y + p$

and
$$x^2 + (y-p)^2 = (y+p)^2$$
$$x^2 + y^2 - 2yp + p^2 = y^2 + 2yp + p^2$$
$$x^2 = 4py$$
or if the axes are reversed $y^2 = 4px$

If the center is offset by (x_c, y_c) the equation becomes

$$(x - x_c)^2 = 4p(y - y_c)$$

or $\quad (y - y_c)^2 = 4p(x - x_c)$

Parametric equation

If we make $\quad y = t^2$

and $\quad x = 2\sqrt{pt}$

then $\quad t = \sqrt{y} \quad$ and $\quad t = \dfrac{x}{2\sqrt{p}}$

and $\quad \sqrt{y} = \dfrac{x}{2\sqrt{p}}$

therefore $\quad y = \dfrac{x^2}{4p}$

$$x^2 = 4py$$

Therefore, the parametric equations are

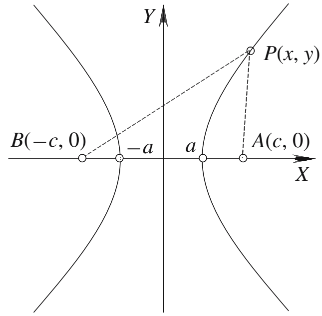
$$x = 2\sqrt{pt} \qquad y = t^2$$

If the axes area reversed

$$x = t^2 \qquad y = 2\sqrt{pt}$$

To offset the parametric equations, add (x_c, y_c).

3.13.4 Hyperbola



General equation

By definition, the hyperbola maintains $|BP| - |PA| = 2a$

where $|BP| = \sqrt{(x+c)^2 + y^2}$

$$|PA| = \sqrt{(x-c)^2 + y^2}$$

therefore $\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = 2a$

$$\sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2}$$

Squaring both sides $x^2 + 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + x^2 - 2cx + c^2 + y^2$

$$cx - a^2 = -a\sqrt{(x-c)^2 + y^2}$$

Squaring both sides $c^2x^2 - 2a^2cx + a^4 = a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2$

$$(c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2)$$

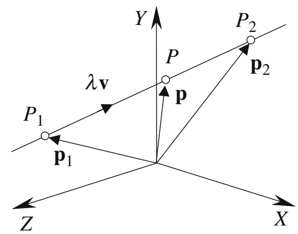
Let $b = \sqrt{c^2 - a^2}$

then $b^2x^2 - a^2y^2 = a^2b^2$

therefore $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

3.14 Three-dimensional straight lines

3.14.1 Proof: Straight-line equation from two points



Strategy: Create a vector from two points and use a parameter to identify any point on the vector.

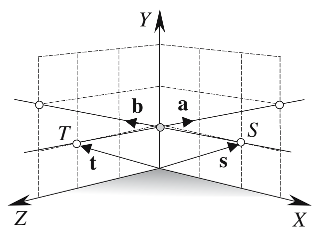
P_1 and P_2 are the two points and $\mathbf{p}_1$ and $\mathbf{p}_2$ their respective position vectors.

Let $\quad \mathbf{v} = \mathbf{p}_2 - \mathbf{p}_1$

therefore $\quad \mathbf{p} = \mathbf{p}_1 + \lambda\mathbf{v} \quad$ where λ is a scalar.

P is between P_1 and P_2 for $\lambda \in [0, 1]$.
If $\|\mathbf{v}\| = 1$, λ corresponds to the linear distance along $\mathbf{v}$.

3.14.2 Proof: Intersection of two straight lines



Strategy: Step 1: Ensure that the two lines are not parallel.
Step 2: Ensure that the two lines touch.
Step 3: Compute the intersection point.

Given two lines $\mathbf{p} = \mathbf{t} + \lambda\mathbf{a}$ and $\mathbf{q} = \mathbf{s} + \varepsilon\mathbf{b}$

where $\mathbf{t} = x_t\mathbf{i} + y_t\mathbf{j} + z_t\mathbf{k}$ and $\mathbf{s} = x_s\mathbf{i} + y_s\mathbf{j} + z_s\mathbf{k}$

$\mathbf{a} = x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k}$ and $\mathbf{b} = x_b\mathbf{i} + y_b\mathbf{j} + z_b\mathbf{k}$

Step 1: If $\mathbf{a} \times \mathbf{b} = 0$ the lines are parallel and do not intersect.

Step 2: The distance between two skew lines is given by $d = \dfrac{\|(\mathbf{t} - \mathbf{s}) \cdot (\mathbf{a} \times \mathbf{b})\|}{\|\mathbf{a} \times \mathbf{b}\|}$

If $(\mathbf{t} - \mathbf{s}) \cdot (\mathbf{a} \times \mathbf{b}) \neq 0$ the lines do not intersect.

Step 3: Equate the two line equations:

$$(x_t\mathbf{i} + y_t\mathbf{j} + z_t\mathbf{k}) + \lambda(x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k}) = (x_s\mathbf{i} + y_s\mathbf{j} + z_s\mathbf{k}) + \varepsilon(x_b\mathbf{i} + y_b\mathbf{j} + z_b\mathbf{k})$$

Collect up the components

$$(x_t - x_s + \lambda x_a - \varepsilon x_b)\mathbf{i} + (y_t - y_s + \lambda y_a - \varepsilon y_b)\mathbf{j} + (z_t - z_s + \lambda z_a - \varepsilon z_b)\mathbf{k} = 0$$

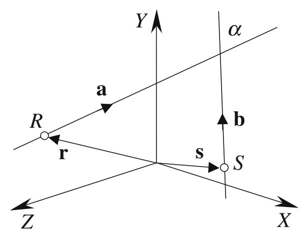
For this vector to be null, its components must vanish. Therefore, we have

$$\begin{aligned}
\lambda x_a - \varepsilon x_b &= x_s - x_t \\
\lambda y_a - \varepsilon y_b &= y_s - y_t \\
\lambda z_a - \varepsilon z_b &= z_s - z_t
\end{aligned}$$

which provide values for λ and ε which, when substituted in the original line equations reveal the intersection point.

3.14.3 Proof: Angle between two straight lines

Strategy: Use the scalar product of the two line vectors to reveal the enclosed angle.



Let the line equations be $\mathbf{p} = \mathbf{r} + \lambda\mathbf{a}$
and $\mathbf{q} = \mathbf{s} + \varepsilon\mathbf{b}$

The angle between the two lines is the angle between the vectors **a** and **b** and is given by

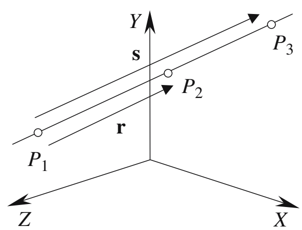
$$\mathbf{a} \bullet \mathbf{b} = \|\mathbf{a}\| \cdot \|\mathbf{b}\| \cos\alpha$$

$$\alpha = \cos^{-1}\left(\frac{\mathbf{a} \bullet \mathbf{b}}{\|\mathbf{a}\| \cdot \|\mathbf{b}\|}\right)$$

If $\|\mathbf{a}\| = \|\mathbf{b}\| = 1 \qquad \alpha = \cos^{-1}(\mathbf{a} \bullet \mathbf{b})$

3.14.4 Proof: Three points lie on a straight line

Strategy: If two vectors are created from the three points, the vectors must be linearly related for the points to lie on a straight line.



Given three points P_1, P_2, P_3

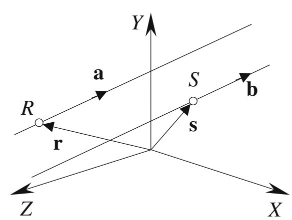
let $\mathbf{r} = \overrightarrow{P_1P_2}$ and $\mathbf{s} = \overrightarrow{P_1P_3}$

therefore $\mathbf{s} = \lambda\mathbf{r}$

for the points to lie on a straight line, where λ is a scalar.

3.14.5 Proof: Parallel and perpendicular straight lines

Let the line equations be $\mathbf{p} = \mathbf{r} + \mu\mathbf{a}$
and $\mathbf{q} = \mathbf{s} + \varepsilon\mathbf{b}$

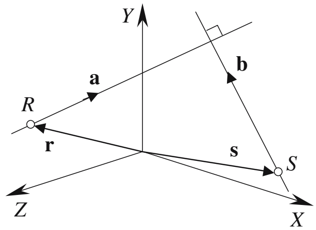


Parallel lines

p and **q** are parallel if $\mathbf{a} = \lambda\mathbf{b}$ where λ is a scalar.

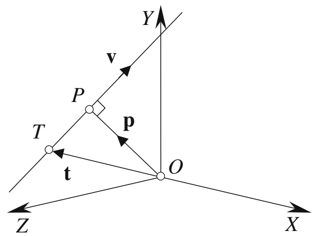
Perpendicular lines

$\mathbf{p}$ and $\mathbf{q}$ are perpendicular if $\mathbf{a} \cdot \mathbf{b} = 0$



3.14.6 Proof: Position and distance of a point on a line perpendicular to the origin

Strategy: The nearest point to the origin forms a perpendicular to the origin.



Let the line equation be $\quad \mathbf{p} = \mathbf{t} + \lambda\mathbf{v} \quad (1)$

Let P be such that $\mathbf{p}$ is perpendicular to $\mathbf{v}$

therefore $\quad \mathbf{v} \cdot \mathbf{p} = 0 \quad (2)$

Derive $\mathbf{v} \cdot \mathbf{p}$ using (1) $\quad \mathbf{v} \cdot \mathbf{p} = \mathbf{v} \cdot (\mathbf{t} + \lambda\mathbf{v}) = \mathbf{v} \cdot \mathbf{t} + \mathbf{v} \cdot \mathbf{v}\lambda = 0 \quad (3)$

Substitute (2) in (3) $\quad \mathbf{v} \cdot \mathbf{v}\lambda = -\mathbf{v} \cdot \mathbf{t}$

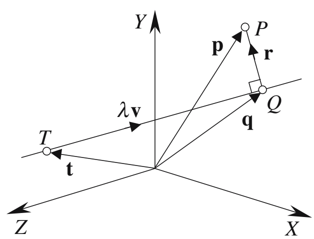
therefore $\quad \lambda = \dfrac{-\mathbf{v} \cdot \mathbf{t}}{\mathbf{v} \cdot \mathbf{v}}$

If $\|\mathbf{v}\| = 1 \quad \lambda = -\mathbf{v} \cdot \mathbf{t}$

Position vector $\quad \mathbf{p} = \mathbf{t} + \lambda\mathbf{v}$

Distance $\quad OP = \|\mathbf{p}\|$

3.14.7 Proof: Position and distance of the nearest point on a line to a point



Strategy: The shortest distance from a point to a straight line is a perpendicular to the line. Use vector analysis to determine the distance.

Let the line equation be $\mathbf{q} = \mathbf{t} + \lambda\mathbf{v}$ (1)

and Q be the nearest point on the line to P

therefore $\mathbf{p} = \mathbf{q} + \mathbf{r}$

and $\mathbf{v} \bullet \mathbf{p} = \mathbf{v} \bullet \mathbf{q} + \mathbf{v} \bullet \mathbf{r}$

r is orthogonal to **v**, therefore $\mathbf{v} \bullet \mathbf{r} = 0$

and $\mathbf{v} \bullet \mathbf{p} = \mathbf{v} \bullet \mathbf{q}$

From (1) $\mathbf{v} \bullet \mathbf{q} = \mathbf{v} \bullet \mathbf{t} + \lambda\mathbf{v} \bullet \mathbf{v}$

therefore $$\lambda = \frac{\mathbf{v} \bullet (\mathbf{p} - \mathbf{t})}{\mathbf{v} \bullet \mathbf{v}}$$

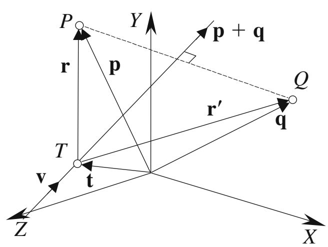
If $\|\mathbf{v}\| = 1$ $\lambda = \mathbf{v} \bullet (\mathbf{p} - \mathbf{t})$

Position vector $\mathbf{q} = \mathbf{t} + \lambda\mathbf{v}$

Distance $PQ = \|\mathbf{r}\| = \|\mathbf{p} - \mathbf{q}\| = \|\mathbf{p} - (\mathbf{t} + \lambda\mathbf{v})\|$

3.14.8 Proof: Position of a point reflected in a line

Strategy: Exploit the fact that the line's direction vector is orthogonal to the line connecting a point and its reflection. Note that this strategy is identical to the 2D case.



P is an arbitrary point and Q is its reflection with **p** and **q** their respective position vectors.

Let the line equation be $\mathbf{s} = \mathbf{t} + \lambda\mathbf{v}$

therefore $\mathbf{p} = \mathbf{t} + \mathbf{r}$

and $\mathbf{q} = \mathbf{t} + \mathbf{r}'$

therefore $\mathbf{p} + \mathbf{q} = 2\mathbf{t} + \mathbf{r} + \mathbf{r}'$ (1)

$\mathbf{r} - \mathbf{r}'$ is orthogonal to **v**, therefore $\mathbf{v} \bullet (\mathbf{r} - \mathbf{r}') = 0$

$\mathbf{v} \bullet \mathbf{r} = \mathbf{v} \bullet \mathbf{r}'$ (2)

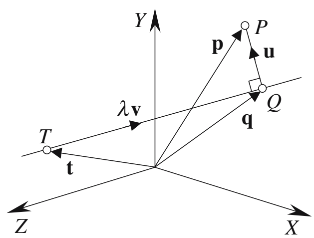
$\mathbf{r} + \mathbf{r}'$ is parallel to **v**, therefore $\mathbf{r} + \mathbf{r}' = \varepsilon\mathbf{v}$ (3)

Substitute (3) in (1) $\mathbf{p} + \mathbf{q} = 2\mathbf{t} + \varepsilon\mathbf{v}$

therefore $\mathbf{q} = 2\mathbf{t} + \varepsilon\mathbf{v} - \mathbf{p}$

From (3)	$\mathbf{v} \cdot (\mathbf{r} + \mathbf{r}') = \varepsilon \mathbf{v} \cdot \mathbf{v}$	
and	$\varepsilon = \dfrac{\mathbf{v} \cdot \mathbf{r} + \mathbf{v} \cdot \mathbf{r}'}{\mathbf{v} \cdot \mathbf{v}}$	(4)
Substitute (2) in (4)	$\varepsilon = \dfrac{2\mathbf{v} \cdot \mathbf{r}}{\mathbf{v} \cdot \mathbf{v}}$	
but	$\mathbf{r} = \mathbf{p} - \mathbf{t}$	
therefore	$\varepsilon = \dfrac{2\mathbf{v} \cdot (\mathbf{p} - \mathbf{t})}{\mathbf{v} \cdot \mathbf{v}}$	(5)
If $\lVert\mathbf{v}\rVert = 1$	$\varepsilon = 2\mathbf{v} \cdot (\mathbf{p} - \mathbf{t})$	
Position vector	$\mathbf{q} = 2\mathbf{t} + \varepsilon \mathbf{v} - \mathbf{p}$	

3.14.9 Proof: Normal to a line through a point

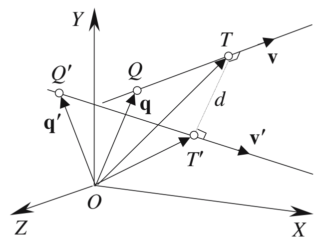


Let the line equation be	$\mathbf{q} = \mathbf{t} + \lambda \mathbf{v}$	(1)

Given a point P, there exists a point Q such that vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

Therefore	$\mathbf{q} = \mathbf{p} - \mathbf{u}$	(2)
From (1) and (2)	$\mathbf{t} + \lambda \mathbf{v} = \mathbf{p} - \mathbf{u}$	
therefore	$\mathbf{v} \cdot \mathbf{t} + \lambda \mathbf{v} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{p} + \mathbf{v} \cdot \mathbf{u}$	
v and **u** are orthogonal	$\mathbf{v} \cdot \mathbf{u} = 0$	
therefore	$\lambda = \dfrac{\mathbf{v} \cdot (\mathbf{p} - \mathbf{t})}{\mathbf{v} \cdot \mathbf{v}}$	
If $\lVert\mathbf{v}\rVert = 1$	$\lambda = \mathbf{v} \cdot (\mathbf{p} - \mathbf{t})$	
From (1) and (2)	$\mathbf{u} = \mathbf{p} - (\mathbf{t} + \lambda \mathbf{v})$	
The line equation for the normal is	$\mathbf{p} + \varepsilon \mathbf{u}$	

3.14.10 Proof: Shortest distance between two skew lines



Strategy: The nearest point to a line will lie on a perpendicular to the line. Therefore, given two skew lines (lines that do not intersect and are not parallel) the shortest distance between the lines will be on a mutually perpendicular to both lines. This means that the cross-product of the two lines will be a vector parallel to the perpendicular and can be exploited by vector analysis. A parametric approach provides an elegant solution to the problem.

Let the line equations be $\mathbf{p} = \mathbf{q} + \lambda\mathbf{v}$

and $\mathbf{p}' = \mathbf{q}' + \tau\mathbf{v}'$

The shortest distance d between the lines is the magnitude of the vector $\overrightarrow{TT'}$ which is perpendicular to both lines.

Therefore $$\overrightarrow{OT} = \mathbf{q} + \lambda_1\mathbf{v} \tag{1}$$

and $$\overrightarrow{OT'} = \mathbf{q}' + \tau_1\mathbf{v}' \tag{2}$$

But $\overrightarrow{TT'}$ is perpendicular to $\mathbf{v}$ and $\mathbf{v}'$ and parallel to $\mathbf{v} \times \mathbf{v}'$

therefore $$\overrightarrow{TT'} = \frac{d \cdot (\mathbf{v} \times \mathbf{v}')}{\|\mathbf{v} \times \mathbf{v}'\|}$$

but $$\overrightarrow{OT'} = \overrightarrow{OT} + \overrightarrow{TT'}$$

therefore $$\overrightarrow{OT'} = \overrightarrow{OT} + \frac{d \cdot (\mathbf{v} \times \mathbf{v}')}{\|\mathbf{v} \times \mathbf{v}'\|} \tag{3}$$

Take the scalar product of (3) with $\mathbf{v} \times \mathbf{v}'$

$$(\mathbf{v} \times \mathbf{v}') \bullet \overrightarrow{OT'} = (\mathbf{v} \times \mathbf{v}') \bullet \overrightarrow{OT} + (\mathbf{v} \times \mathbf{v}') \bullet \frac{d \cdot (\mathbf{v} \times \mathbf{v}')}{\|\mathbf{v} \times \mathbf{v}'\|}$$

$$(\mathbf{v} \times \mathbf{v}') \bullet \overrightarrow{OT'} = (\mathbf{v} \times \mathbf{v}') \bullet \overrightarrow{OT} + d \cdot \|\mathbf{v} \times \mathbf{v}'\| \tag{4}$$

Substitute (1) and (2) in (4)

$$(\mathbf{v} \times \mathbf{v}') \bullet (\mathbf{q}' + \tau_1\mathbf{v}') = (\mathbf{v} \times \mathbf{v}') \bullet (\mathbf{q} + \lambda\mathbf{v}) + d \cdot \|\mathbf{v} \times \mathbf{v}'\|$$
$$\mathbf{q}' \bullet (\mathbf{v} \times \mathbf{v}') + \tau_1\mathbf{v}' \bullet (\mathbf{v} \times \mathbf{v}') = \mathbf{q} \bullet (\mathbf{v} \times \mathbf{v}') + \lambda_1\mathbf{v} \bullet (\mathbf{v} \times \mathbf{v}') + d \cdot \|\mathbf{v} \times \mathbf{v}'\|$$

But $\tau_1\mathbf{v}' \bullet (\mathbf{v} \times \mathbf{v}') = 0$ and $\lambda_1\mathbf{v} \bullet (\mathbf{v} \times \mathbf{v}')$ as $\mathbf{v}, \mathbf{v}'$ and $\mathbf{v} \times \mathbf{v}'$ are mutually perpendicular.

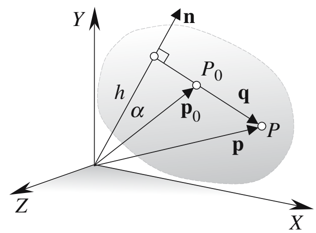
Therefore $$(\mathbf{q}' - \mathbf{q}) \bullet (\mathbf{v} \times \mathbf{v}') = d \cdot \|\mathbf{v} \times \mathbf{v}'\|$$

therefore the shortest distance is $$d = \frac{\left|(\mathbf{q} - \mathbf{q}') \bullet (\mathbf{v} \times \mathbf{v}')\right|}{\|\mathbf{v} \times \mathbf{v}'\|}$$

3.15 Planes

3.15.1 Proof: Equation to a plane

Cartesian form of the plane equation



Strategy: Let $\mathbf{n}$ be a nonzero vector normal to the plane and $P(x, y, z)$ be a point on the plane, which also contains a point $P_0(x_0, y_0, z_0)$. Use vector analysis to derive the plane equation. Note that the strategy is similar to that used for the equation of a line.

Let	$\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$
and	$\mathbf{p}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$
and	$\mathbf{p} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
therefore	$\mathbf{q} = \mathbf{p} - \mathbf{p}_0$
As $\mathbf{n}$ is orthogonal to $\mathbf{q}$	$\mathbf{n} \bullet \mathbf{q} = 0$
therefore	$\mathbf{n} \bullet (\mathbf{p} - \mathbf{p}_0) = 0$
and	$\mathbf{n} \bullet \mathbf{p} = \mathbf{n} \bullet \mathbf{p}_0 \qquad (1)$
therefore	$ax + by + cz = ax_0 + by_0 + cz_0$

But $ax_0 + by_0 + cz_0$ is a scalar quantity associated with the plane and can be replaced by d

$$ax + by + cz = d$$

where $\quad d = ax_0 + by_0 + cz_0$

The value of d also has the interpretation:

from the diagram $\quad h = \|\mathbf{p}_0\| \cos\alpha$

therefore $\quad \mathbf{n} \bullet \mathbf{p}_0 = \|\mathbf{n}\| \cdot \|\mathbf{p}_0\| \cos\alpha = h\|\mathbf{n}\|$

Therefore the plane equation can be expressed as

$$ax + by + cz = h\|\mathbf{n}\| \tag{2}$$

Dividing (2) by $\|\mathbf{n}\|$ we have $\quad \frac{a}{\|\mathbf{n}\|}x + \frac{b}{\|\mathbf{n}\|}y + \frac{c}{\|\mathbf{n}\|}z = h$

where h is the perpendicular from the origin to the plane, and $\|\mathbf{n}\| = \sqrt{a^2 + b^2 + c^2}$

General form of the plane equation

The general form of the equation is expressed as

$$Ax + By + Cz + D = 0$$

which means that the Cartesian form is translated into the general form by making

$$A = a, \qquad B = b, \qquad C = c, \qquad D = -d$$

The individual values of A, B, C, D have no absolute geometric meaning as it is possible to multiply the equation by any scalar quantity to produce another equation describing the same plane. However, as there is a direct relationship between the Cartesian form and the general form, the values of A, B, C can be associated with a vector normal to the plane, but the direction of the vector can be in one of two directions: directed from one side of the plane or the other side. The orientation of this normal vector is resolved by the Hessian normal form.

Hessian normal form of the plane equation

The Hessian normal form of the plane equation scales the general form plane equation by a factor to make the magnitude of the plane's normal vector equal to 1, i.e. a unit vector.

For the plane equation $\quad Ax + By + Cz + D = 0$

the scale factor is $\dfrac{1}{\sqrt{A^2 + B^2 + C^2}}$

therefore

$$\frac{Ax}{\sqrt{A^2 + B^2 + C^2}} + \frac{By}{\sqrt{A^2 + B^2 + C^2}} + \frac{Cz}{\sqrt{A^2 + B^2 + C^2}} + \frac{D}{\sqrt{A^2 + B^2 + C^2}} = 0$$

Let

$$n_1 = \frac{A}{\sqrt{A^2 + B^2 + C^2}} \qquad n_2 = \frac{B}{\sqrt{A^2 + B^2 + C^2}}$$

$$n_3 = \frac{C}{\sqrt{A^2 + B^2 + C^2}} \qquad p = \frac{D}{\sqrt{A^2 + B^2 + C^2}}$$

which allows us to write the Hessian normal form of the plane equation as

$$n_1x + n_2y + n_3z + p = 0$$

This can also be expressed using vectors:

if $\quad \mathbf{p} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \quad$ (a point on the plane)

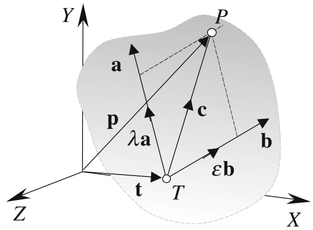
and $\quad \mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k} \quad$ (the unit normal vector of the plane)

then $\quad \mathbf{n} \cdot \mathbf{p} = -p$

The positive and negative values of $\sqrt{A^2 + B^2 + C^2}$ provide the two potential directions of the unit normal vector. However, by convention, only the positive value of $\sqrt{A^2 + B^2 + C^2}$ is considered. Furthermore, the side of the plane that lies in the direction of $\mathbf{n}$ is declared the positive side whilst the other side of the plane is declared the negative side. This partitioning of space creates two half-spaces.

We have seen above that $\mathbf{n} \cdot \mathbf{p} = -p$, where p is the perpendicular distance from the plane to the origin. Therefore, if $p > 0$ the origin lies in the positive half-space, and if $p < 0$ it lies in the negative half-space. If $p = 0$ the origin lies on the plane.

Parametric form of the plane equation



Let vectors **a** and **b** be parallel to the plane and the point $T(x_T, y_T, z_T)$ be on the plane.

Therefore $\quad \mathbf{c} = \lambda\mathbf{a} + \varepsilon\mathbf{b}$

and $\quad \mathbf{p} = \mathbf{t} + \mathbf{c}$

therefore
$$x_P = x_T + \lambda x_a + \varepsilon x_b$$
$$y_P = y_T + \lambda y_a + \varepsilon y_b$$
$$z_P = z_T + \lambda z_a + \varepsilon z_b$$

If **a** and **b** are unit vectors and are mutually perpendicular, i.e. $\mathbf{a} \cdot \mathbf{b} = 0$, λ and ε become linear measurements along the **a** and **b** axes relative to T.

Converting from the parametric form to the general form

Strategy: First compute the values of λ and ε that identify a point P perpendicular to the origin, then determine the individual components of the plane equation.

$$\mathbf{c} = \lambda\mathbf{a} + \varepsilon\mathbf{b}$$
$$\mathbf{p} = \mathbf{t} + \mathbf{c}$$

therefore
$$\mathbf{p} = \mathbf{t} + \lambda\mathbf{a} + \varepsilon\mathbf{b} \tag{3}$$

But **a** and **b** are perpendicular to **p**

therefore $\quad \mathbf{a} \cdot \mathbf{p} = 0 \quad \text{and} \quad \mathbf{b} \cdot \mathbf{p} = 0$

Compute $\mathbf{a} \cdot \mathbf{p}$ using (3)
$$\mathbf{a} \cdot \mathbf{p} = \mathbf{a} \cdot \mathbf{t} + \lambda\mathbf{a} \cdot \mathbf{a} + \varepsilon\mathbf{a} \cdot \mathbf{b} = 0 \tag{4}$$

Compute $\mathbf{b} \cdot \mathbf{p}$ using (3)
$$\mathbf{b} \cdot \mathbf{p} = \mathbf{b} \cdot \mathbf{t} + \lambda\mathbf{a} \cdot \mathbf{b} + \varepsilon\mathbf{b} \cdot \mathbf{b} = 0 \tag{5}$$

From (4)
$$\mathbf{a} \cdot \mathbf{t} + \lambda||\mathbf{a}||^2 + \varepsilon\mathbf{a} \cdot \mathbf{b} = 0 \tag{6}$$

From (5)
$$\mathbf{b} \cdot \mathbf{t} + \lambda\mathbf{a} \cdot \mathbf{b} + \varepsilon||\mathbf{b}||^2 = 0 \tag{7}$$

To eliminate ε multiply (6) by $||\mathbf{b}||^2$ and (7) by $\mathbf{a} \cdot \mathbf{b}$ and subtract

$$(\mathbf{a} \cdot \mathbf{t})||\mathbf{b}||^2 + \lambda||\mathbf{a}||^2||\mathbf{b}||^2 + \varepsilon(\mathbf{a} \cdot \mathbf{b})||\mathbf{b}||^2 = 0$$
$$(\mathbf{a} \cdot \mathbf{b})(\mathbf{b} \cdot \mathbf{t}) + \lambda(\mathbf{a} \cdot \mathbf{b})^2 + \varepsilon(\mathbf{a} \cdot \mathbf{b})||\mathbf{b}||^2 = 0$$
$$(\mathbf{a} \cdot \mathbf{t})||\mathbf{b}||^2 + \lambda||\mathbf{a}||^2||\mathbf{b}||^2 - (\mathbf{a} \cdot \mathbf{b})(\mathbf{b} \cdot \mathbf{t}) - \lambda(\mathbf{a} \cdot \mathbf{b})^2 = 0$$

$$\lambda = \frac{(\mathbf{a} \cdot \mathbf{b})(\mathbf{b} \cdot \mathbf{t}) - (\mathbf{a} \cdot \mathbf{t})||\mathbf{b}||^2}{||\mathbf{a}||^2||\mathbf{b}||^2 - (\mathbf{a} \cdot \mathbf{b})^2}$$

To eliminate λ multiply (6) by $\mathbf{a} \cdot \mathbf{b}$ and (7) by $||\mathbf{a}||^2$ and subtract

$$(\mathbf{a} \cdot \mathbf{b})\ (\mathbf{a} \cdot \mathbf{t}) + \lambda(\mathbf{a} \cdot \mathbf{b})||\mathbf{a}||^2 + \varepsilon(\mathbf{a} \cdot \mathbf{b})^2 = 0$$
$$(\mathbf{b} \cdot \mathbf{t})\ ||\mathbf{a}||^2 + \lambda(\mathbf{a} \cdot \mathbf{b})||\mathbf{a}||^2 + \varepsilon||\mathbf{a}||^2||\mathbf{b}||^2 = 0$$
$$(\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{t}) + \varepsilon(\mathbf{a} \cdot \mathbf{b})^2 - (\mathbf{b} \cdot \mathbf{t})||\mathbf{a}||^2 - \varepsilon||\mathbf{a}||^2||\mathbf{b}||^2 = 0$$

$$\varepsilon = \frac{(\mathbf{a} \bullet \mathbf{b})(\mathbf{a} \bullet \mathbf{t}) - (\mathbf{b} \bullet \mathbf{t})\|\mathbf{a}\|^2}{\|\mathbf{a}\|^2 \|\mathbf{b}\|^2 - (\mathbf{a} \bullet \mathbf{b})^2}$$

Substitute λ and ε in (3) to identify the point $P(x_P, y_P, z_P)$ perpendicular to the origin. If vectors **a** and **b** had been unit vectors, λ and ε would have been greatly simplified:

$$\lambda = \frac{(\mathbf{a} \bullet \mathbf{b})(\mathbf{b} \bullet \mathbf{t}) - \mathbf{a} \bullet \mathbf{t}}{1 - (\mathbf{a} \bullet \mathbf{b})^2}$$

$$\varepsilon = \frac{(\mathbf{a} \bullet \mathbf{b})(\mathbf{a} \bullet \mathbf{t}) - \mathbf{b} \bullet \mathbf{t}}{1 - (\mathbf{a} \bullet \mathbf{b})^2}$$

P's position vector **p** is also the plane's normal vector.

Then

$$x_P = x_T + \lambda x_a + \varepsilon x_b$$
$$y_P = y_T + \lambda y_a + \varepsilon y_b$$
$$z_P = z_T + \lambda z_a + \varepsilon z_b$$

The normal vector is $\mathbf{p} = x_P\mathbf{i} + y_P\mathbf{j} + z_P\mathbf{k}$

and because $\|\mathbf{p}\|$ is the perpendicular distance from the plane to the origin we can state

$$\frac{x_P}{\|\mathbf{p}\|}x + \frac{y_P}{\|\mathbf{p}\|}y + \frac{z_P}{\|\mathbf{p}\|}z = \|\mathbf{p}\|$$

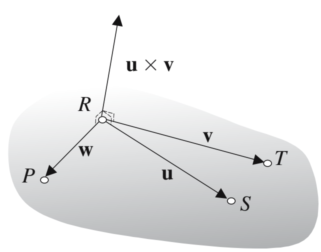
or in the general form of the plane equation

$$Ax + By + Cz + D = 0$$

where $A = \dfrac{x_P}{\|\mathbf{p}\|} \qquad B = \dfrac{y_P}{\|\mathbf{p}\|} \qquad C = \dfrac{z_P}{\|\mathbf{p}\|} \qquad D = -\|\mathbf{p}\|$

3.15.2 Proof: Plane equation from three points

Strategy: Given three points R, S and T create two vectors $\mathbf{u} = \overrightarrow{RS}$ and $\mathbf{v} = \overrightarrow{RT}$. The vector product $\mathbf{u} \times \mathbf{v}$ provides a vector normal to the plane containing the points. Take another point $P(x, y, z)$ and form a vector $\mathbf{w} = \overrightarrow{RP}$. The scalar product $\mathbf{w} \bullet (\mathbf{u} \times \mathbf{v}) = 0$ if P is in the plane containing the original points. This condition can be expressed as a determinant and converted into the general equation of a



plane. The three points are assumed to be in a counter-clockwise sequence viewed from the direction of the surface normal.

Let the three points R, S, T and a fourth point $P(x, y, z)$ lie on the same plane.

Let $\quad \mathbf{u} = \overrightarrow{RS} \quad \text{and} \quad \mathbf{v} = \overrightarrow{RT}$

then
$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix}$$

Let $\quad \mathbf{w} = \overrightarrow{RP}$

As $\mathbf{w}$ is perpendicular to $\mathbf{u} \times \mathbf{v}$

$$\mathbf{w} \bullet (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} x_w & y_w & z_w \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} = 0$$

Expanding the determinant we obtain

$$x_w \begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix} + y_w \begin{vmatrix} z_u & x_u \\ z_v & x_v \end{vmatrix} + z_w \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} = 0$$

which becomes

$$(x - x_R)\begin{vmatrix} y_S - y_R & z_S - z_R \\ y_T - y_R & z_T - z_R \end{vmatrix} + (y - y_R)\begin{vmatrix} z_S - z_R & x_S - x_R \\ z_T - z_R & x_T - x_R \end{vmatrix}$$
$$+ (z - z_R)\begin{vmatrix} x_S - x_R & y_S - y_R \\ x_T - x_R & y_T - y_R \end{vmatrix} = 0$$

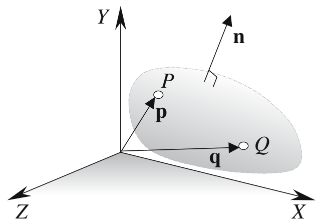
This can be arranged in the form $ax + by + cz + d = 0$

where
$$a = \begin{vmatrix} y_S - y_R & z_S - z_R \\ y_T - y_R & z_T - z_R \end{vmatrix} \qquad b = \begin{vmatrix} z_S - z_R & x_S - x_R \\ z_T - z_R & x_T - x_R \end{vmatrix}$$
$$c = \begin{vmatrix} x_S - x_R & y_S - y_R \\ x_T - x_R & y_T - y_R \end{vmatrix} \qquad d = -(ax_R + by_R + cz_R)$$

or

$$a = \begin{vmatrix} 1 & y_R & z_R \\ 1 & y_S & z_S \\ 1 & y_T & z_T \end{vmatrix} \qquad b = \begin{vmatrix} x_R & 1 & z_R \\ x_S & 1 & z_S \\ x_T & 1 & z_T \end{vmatrix}$$
$$c = \begin{vmatrix} x_R & y_R & 1 \\ x_S & y_S & 1 \\ x_T & y_T & 1 \end{vmatrix} \qquad d = -(ax_R + by_R + cz_R)$$

3.15.3 Proof: Plane through a point and normal to a line



Strategy: Use the general equation of a plane as this incorporates a surface normal and recognizes points on the plane.

Let the plane equation be $ax + by + cz + d = 0$

where $P(x, y, z)$ is any point on the plane

and $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

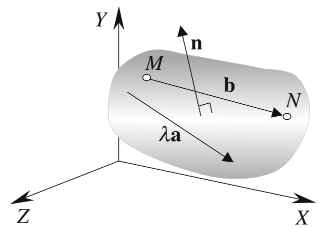
therefore $\mathbf{n} \cdot \mathbf{p} + d = 0$

Given $Q(x_Q, y_Q, z_Q)$ $\mathbf{n} \cdot \mathbf{q} + d = 0$

therefore $\mathbf{n} \cdot \mathbf{p} - \mathbf{n} \cdot \mathbf{q} = 0$

and $ax + by + cz - (ax_Q + by_Q + cz_Q) = 0$

3.15.4 Proof: Plane through two points and parallel to a line



Strategy: Create one vector from the two points and another from the line. The vector product of these vectors will be normal to the associated plane.

Let the line be $\mathbf{p} = \mathbf{r} + \lambda\mathbf{a}$

where $\mathbf{a} = x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k}$

and the two points are $M(x_M, y_M, z_M)$ and $N(x_N, y_N, z_N)$

therefore $\mathbf{b} = (x_N - x_M)\mathbf{i} + (y_N - y_M)\mathbf{j} + (z_N - z_M)\mathbf{k}$

but $\mathbf{a} \times \mathbf{b} = \mathbf{n}$

where $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

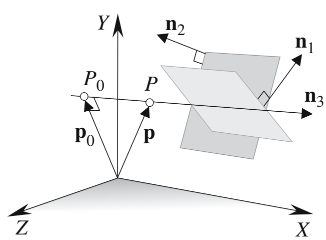
and $a = \begin{vmatrix} y_a & z_a \\ y_b & z_b \end{vmatrix} \qquad b = \begin{vmatrix} z_a & x_a \\ z_b & x_b \end{vmatrix} \qquad c = \begin{vmatrix} x_a & y_a \\ x_b & y_b \end{vmatrix}$

Let the plane equation be $ax + by + cz + d = 0$
As the point M is on the plane

$$ax_M + by_M + cz_M + d = 0$$

The plane equation is $ax + by + cz - (ax_M + by_M + cz_M) = 0$

3.15.5 Proof: Intersection of two planes



Strategy: Two non-parallel planes will intersect and form a straight line, which is parallel to both planes. The vector product of the planes' surface normals reveals the direction vector of the intersection line, but a point on the line is required to secure a unique line equation. A convenient point is perpendicular to the origin. Three simultaneous equations are now available to reveal the line equation.

Let the plane equations be $\mathbf{n}_1 \cdot \mathbf{p} + d_1 = 0 \qquad \mathbf{n}_2 \cdot \mathbf{p} + d_2 = 0$

where $\mathbf{n}_1 = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k} \qquad \mathbf{n}_2 = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$

and $\mathbf{p} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

Let the line of intersection be $\mathbf{p} = \mathbf{p}_0 + \lambda\mathbf{n}_3$

where $\mathbf{p}$ is the position vector for any point P on the line
$\mathbf{p}_0$ is the position vector for a known point P_0 on the line
$\mathbf{n}_3$ is the direction vector for the line of intersection
λ is a scalar.

The direction vector is $\mathbf{n}_3 = a_3\mathbf{i} + b_3\mathbf{j} + c_3\mathbf{k} = \mathbf{n}_1 \times \mathbf{n}_2$

P_0 must satisfy both plane equations, therefore

$$\mathbf{n}_1 \cdot \mathbf{p}_0 = -d_1 \tag{1}$$

and

$$\mathbf{n}_2 \cdot \mathbf{p}_0 = -d_2 \tag{2}$$

P_0 is such that $\mathbf{p}_0$ is orthogonal to $\mathbf{n}_3$

therefore

$$\mathbf{n}_3 \cdot \mathbf{p}_0 = 0 \tag{3}$$

Equations (1), (2) and (3) form three simultaneous equations, which reveal the point P_0.

$$\begin{bmatrix} -d_1 \\ -d_2 \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

or

$$\begin{bmatrix} d_1 \\ d_2 \\ 0 \end{bmatrix} = -\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

Therefore

$$\frac{x_0}{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{vmatrix}} = \frac{y_0}{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & 0 & c_3 \end{vmatrix}} = \frac{z_0}{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & 0 \end{vmatrix}} = \frac{-1}{DET}$$

$$x_0 = \frac{d_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} - d_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}{DET}$$

$$y_0 = \frac{d_2\begin{vmatrix} a_3 & c_3 \\ a_1 & c_1 \end{vmatrix} - d_1\begin{vmatrix} a_3 & c_3 \\ a_2 & c_2 \end{vmatrix}}{DET}$$

$$z_0 = \frac{d_2\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} - d_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}{DET}$$

where $$DET = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

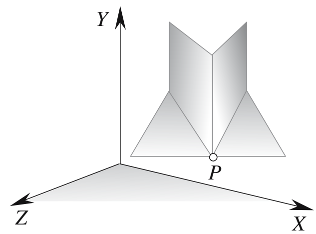
The line of intersection is $\mathbf{p} = \mathbf{p}_0 + \lambda \mathbf{n}_3$

If $DET = 0$ the line and plane are parallel.

3.15.6 Proof: Intersection of three planes

Strategy: Solve the three simultaneous plane equations using determinants.

The diagram shows three planes intersecting at the point $P(x, y, z)$.



Given three planes
$$a_1x + b_1y + c_1z + d_1 = 0$$
$$a_2x + b_2y + c_2z + d_2 = 0$$
$$a_3x + b_3y + c_3z + d_3 = 0$$

they can be rewritten as
$$\begin{bmatrix} -d_1 \\ -d_2 \\ -d_3 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

or
$$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = -\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

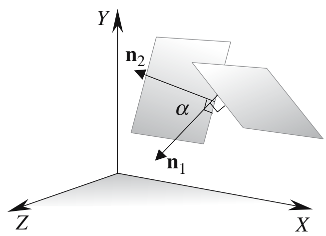
$$\frac{x}{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}} = \frac{y}{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}} = \frac{-1}{DET}$$

where $$DET = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

therefore
$$x = -\frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{DET} \quad y = -\frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{DET} \quad z = -\frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{DET}$$

If $DET = 0$, two of the planes, at least, are parallel.

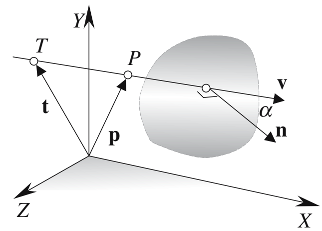
3.15.7 Proof: Angle between two planes



Strategy: Use the dot product to find the angle between the planes' normals.

Given the plane equations	$ax_1 + by_1 + cz_1 + d_1 = 0$
and	$ax_2 + by_2 + cz_2 + d_2 = 0$
where	$\mathbf{n}_1 = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$
and	$\mathbf{n}_2 = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$
then	$\mathbf{n}_1 \bullet \mathbf{n}_2 = \lVert\mathbf{n}_1\rVert \cdot \lVert\mathbf{n}_2\rVert \cos\alpha$
and	$\alpha = \cos^{-1}\left(\dfrac{\mathbf{n}_1 \bullet \mathbf{n}_2}{\lVert\mathbf{n}_1\rVert \cdot \lVert\mathbf{n}_2\rVert}\right)$
If $\lVert\mathbf{n}_1\rVert = \lVert\mathbf{n}_2\rVert = 1$	$\alpha = \cos^{-1}(\mathbf{n}_1 \bullet \mathbf{n}_2)$

3.15.8 Proof: Angle between a line and a plane

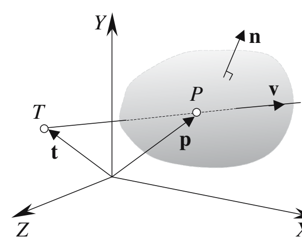


Strategy: Use the dot product to find the angle between the plane's normal and the line's direction vector.

Given the plane equation	$ax + by + cz + d = 0$
where	$\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$
and the line equation	$\mathbf{p} = \mathbf{t} + \lambda\mathbf{v}$
therefore	$\mathbf{n} \bullet \mathbf{v} = \lVert\mathbf{n}\rVert \cdot \lVert\mathbf{v}\rVert \cos\alpha$
and	$\alpha = \cos^{-1}\left(\dfrac{\mathbf{n} \bullet \mathbf{v}}{\lVert\mathbf{n}\rVert \cdot \lVert\mathbf{v}\rVert}\right)$
If $\lVert\mathbf{n}\rVert = \lVert\mathbf{v}\rVert = 1$	$\alpha = \cos^{-1}(\mathbf{n} \bullet \mathbf{v})$

When the line is parallel with the plane $\mathbf{n} \bullet \mathbf{v} = 0$

3.15.9 Proof: Intersection of a line and a plane



Strategy: Solve a parametric line equation with the general equation for a plane.

Let the plane equation be	$ax + by + cz + d = 0$
where	$\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

P is a point on the plane with position vector

$$\mathbf{p} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

therefore	$\mathbf{n} \bullet \mathbf{p} + d = 0$

Let the line equation be $\mathbf{p} = \mathbf{t} + \lambda\mathbf{v}$

where $\mathbf{t} = x_T\mathbf{i} + y_T\mathbf{j} + z_T\mathbf{k}$ and $\mathbf{v} = x_v\mathbf{i} + y_v\mathbf{j} + z_v\mathbf{k}$

They intersect for some λ $\mathbf{n} \cdot (\mathbf{t} + \lambda\mathbf{v}) + d = \mathbf{n} \cdot \mathbf{t} + \lambda\mathbf{n} \cdot \mathbf{v} + d = 0$

therefore $\lambda = \dfrac{-(\mathbf{n} \cdot \mathbf{t} + d)}{\mathbf{n} \cdot \mathbf{v}}$ for the intersection point.

If $\|\mathbf{n}\| = \|\mathbf{v}\| = 1$ $\lambda = -(\mathbf{n} \cdot \mathbf{t} + d)$

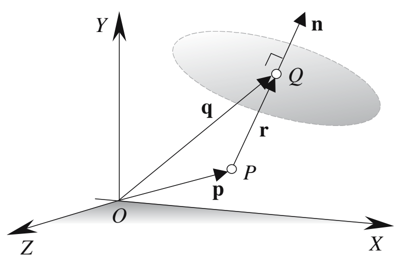
The position vector for P is $\mathbf{p} = \mathbf{t} + \lambda\mathbf{v}$

If $\mathbf{n} \cdot \mathbf{v} = 0$ the line and plane are parallel.

3.15.10 Proof: Position and distance of the nearest point on a plane to a point

General form of the plane equation

Strategy: Express the plane equation as the scalar product of two vectors and use vector analysis to identify a point Q on the perpendicular from a point P to the plane.



Let Q be the nearest point on the plane to P.

Let the plane equation be $ax + by + cz + d = 0$

where $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

and $\mathbf{q} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

therefore $\mathbf{n} \cdot \mathbf{q} = -d$ (1)

r is parallel to **n**, therefore $\mathbf{r} = \lambda\mathbf{n}$

and $\mathbf{n} \cdot \mathbf{r} = \lambda\mathbf{n} \cdot \mathbf{n}$ (2)

but $\mathbf{r} = \mathbf{q} - \mathbf{p}$

therefore $\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{q} - \mathbf{n} \cdot \mathbf{p}$ (3)

Substitute (1) and (2) in (3) $\lambda\mathbf{n} \cdot \mathbf{n} = -(\mathbf{n} \cdot \mathbf{p} + d)$

therefore $\lambda = \dfrac{-(\mathbf{n} \cdot \mathbf{p} + d)}{\mathbf{n} \cdot \mathbf{n}}$

If $\|\mathbf{n}\| = 1$ $\lambda = -(\mathbf{n} \cdot \mathbf{p} + d)$

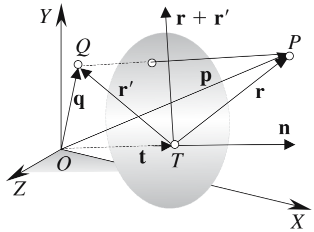
but $\mathbf{q} = \mathbf{p} + \mathbf{r}$

Position vector of Q $\mathbf{q} = \mathbf{p} + \lambda\mathbf{n}$

Distance of Q $PQ = \|\mathbf{r}\| = \|\lambda\mathbf{n}\|$

If $\|\mathbf{n}\| = 1$ $PQ = |\lambda|$

3.15.11 Proof: Reflection of a point in a plane



Strategy: Exploit the fact that a line connecting a point and its reflection is parallel to the plane's normal.

Let the equation of the plane be $ax + by + cz + d = 0$
T is the nearest point on the plane to O and $\mathbf{t}$ is its position vector.

If $\quad \mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

then $\quad \mathbf{n} \cdot \mathbf{t} = -d \qquad (1)$

P is an arbitrary point and Q is its reflection, with their respective position vectors $\mathbf{p}$ and $\mathbf{q}$.
$\mathbf{r} + \mathbf{r}'$ is orthogonal to $\mathbf{n}$

therefore $\quad \mathbf{n} \cdot (\mathbf{r} + \mathbf{r}') = 0$

and $\quad \mathbf{n} \cdot \mathbf{r} + \mathbf{n} \cdot \mathbf{r}' = 0 \qquad (2)$

$\mathbf{p} - \mathbf{q}$ is parallel with $\mathbf{n}$

therefore $\quad \mathbf{p} - \mathbf{q} = \mathbf{r} - \mathbf{r}' = \lambda\mathbf{n} \qquad (3)$

where $\quad \lambda = \dfrac{\mathbf{r} - \mathbf{r}'}{\mathbf{n}} \qquad (4)$

but $\quad \mathbf{r} = \mathbf{p} - \mathbf{t} \qquad (5)$

Substitute (1) in (5) $\quad \mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{p} - \mathbf{n} \cdot \mathbf{t} = \mathbf{n} \cdot \mathbf{p} + d \qquad (6)$

Substitute (2) and (6) in (4) $\quad \lambda = \dfrac{\mathbf{n} \cdot \mathbf{r} - \mathbf{n} \cdot \mathbf{r}'}{\mathbf{n} \cdot \mathbf{n}} = \dfrac{2\mathbf{n} \cdot \mathbf{r}}{\mathbf{n} \cdot \mathbf{n}}$

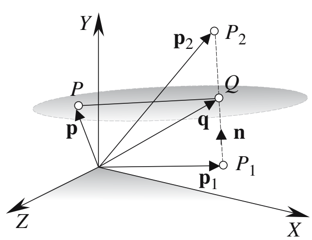
$$\lambda = \frac{2(\mathbf{n} \cdot \mathbf{p} + d)}{\mathbf{n} \cdot \mathbf{n}}$$

If $\|\mathbf{n}\| = 1 \quad \lambda = 2(\mathbf{n} \cdot \mathbf{p} + d)$

Substitute λ in (3) $\quad \mathbf{p} - \mathbf{q} = \lambda\mathbf{n}$

Position vector of Q is $\quad \mathbf{q} = \mathbf{p} - \lambda\mathbf{n}$

3.15.12 Proof: Plane equidistant from two points



Given two distinct points we require to identify a plane such that any point on the plane is equidistant to the points.

Strategy: The key to this solution is that the normal of the plane is parallel to the line joining the two points.

Let the plane equation equidistant to $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be $ax + by + cz + d = 0$

$P(x, y, z)$ is any point on this plane which contains Q equidistant to P_1 and P_2.

Let $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = \mathbf{p}_2 - \mathbf{p}_1$ (1)

and $\mathbf{q} = \mathbf{p}_1 + \frac{1}{2}\mathbf{n} = \frac{1}{2}(\mathbf{p}_2 - \mathbf{p}_1)$ (2)

then $\mathbf{n} \cdot (\mathbf{p} - \mathbf{q}) = 0$

therefore $\mathbf{n} \cdot \mathbf{p} = \mathbf{n} \cdot \mathbf{q}$

But the plane equation is $\mathbf{n} \cdot \mathbf{p} + d = 0$

therefore $d = -\mathbf{n} \cdot \mathbf{p} = -\mathbf{n} \cdot \mathbf{q}$ (3)

Substituting (1) and (2) in (3)

$$d = -(\mathbf{p}_2 - \mathbf{p}_1)\cdot(\mathbf{p}_1 + \tfrac{1}{2}(\mathbf{p}_2 - \mathbf{p}_1)) = -\tfrac{1}{2}(\mathbf{p}_2 - \mathbf{p}_1)\cdot(\mathbf{p}_2 + \mathbf{p}_1)$$

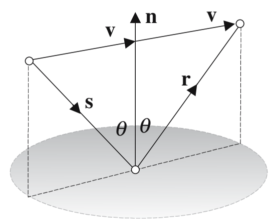
The plane equation is

$$(\mathbf{p}_2 - \mathbf{p}_1)\cdot(\mathbf{p} - \tfrac{1}{2}(\mathbf{p}_2 + \mathbf{p}_1)) = 0$$

or $$(x_2 - x_1)x + (y_2 - y_1)y + (z_2 - z_1)z - \tfrac{1}{2}(x_2^2 - x_1^2 + y_2^2 - y_1^2 + z_2^2 - z_1^2) = 0$$

3.15.13 Proof: Reflected ray on a surface

Strategy: Invoke the law of reflection using vectors: The law of reflection states that the angle of incidence equals the angle of reflection. The incident ray, reflected ray and the surface normal all lie in a common plane.



Let $\mathbf{n}$ be the surface normal vector
$\mathbf{s}$ be the incident ray
$\mathbf{r}$ be the reflected ray
θ be the angle of incidence and reflection

then $\mathbf{v} = \mathbf{s} + \lambda\mathbf{n}$

and $\mathbf{r} = \mathbf{v} + \lambda\mathbf{n}$

therefore $\mathbf{r} - \lambda\mathbf{n} = \mathbf{s} + \lambda\mathbf{n}$

and $\mathbf{r} = \mathbf{s} + 2\lambda\mathbf{n}$ (1)

Take the dot product of (1) $\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{s} + 2\lambda\mathbf{n} \cdot \mathbf{n}$ (2)

but by symmetry

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot (-\mathbf{s}) = -\mathbf{n} \cdot \mathbf{s} \quad (3)$$

Substitute (3) in (2)

$$-\mathbf{n} \cdot \mathbf{s} = \mathbf{n} \cdot \mathbf{s} + 2\lambda\mathbf{n} \cdot \mathbf{n}$$

then $$\lambda = \frac{-\mathbf{n}\cdot\mathbf{s}}{\mathbf{n}\cdot\mathbf{n}}$$

If $\|\mathbf{n}\| = 1$ $\lambda = -2\mathbf{n} \cdot \mathbf{s}$

If $\theta = 90°$ $\mathbf{r} = \mathbf{s}$

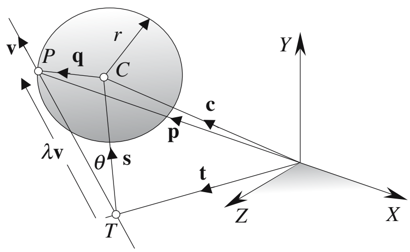
3.16 Lines, planes and spheres

3.16.1 Proof: Line intersecting a sphere

There are three scenarios: the line intersects, touches or misses the sphere.

Strategy: The cosine rule proves very useful in setting up a geometric condition that identifies the above scenarios, which are readily solved using vector analysis.

Parametric equation of a line



A sphere with radius r is located at C with position vector $\mathbf{c} = x_C\mathbf{i} + y_C\mathbf{j} + z_C\mathbf{k}$

The equation of the line is $\quad \mathbf{p} = \mathbf{t} + \lambda\mathbf{v}$

where $\quad ||\mathbf{v}|| = 1 \qquad (1)$

For an intersection at P $\quad ||\mathbf{q}|| = r \quad \text{or} \quad ||\mathbf{q}||^2 = r^2 \quad \text{or} \quad ||\mathbf{q}||^2 - r^2 = 0$

Using the cosine rule $\quad ||\mathbf{q}||^2 = ||\lambda\mathbf{v}||^2 + ||\mathbf{s}||^2 - 2||\lambda\mathbf{v}|| \cdot ||\mathbf{s}||\cos\theta$

$$||\mathbf{q}||^2 = \lambda^2||\mathbf{v}||^2 + ||\mathbf{s}||^2 - 2||\mathbf{v}|| \cdot ||\mathbf{s}||\lambda\cos\theta \qquad (2)$$

Substituting (1) in (2) $\quad ||\mathbf{q}||^2 = \lambda^2 + ||\mathbf{s}||^2 - 2\,||\mathbf{s}||\,\lambda\cos\theta \qquad (3)$

Identify $\cos\theta$ $\quad \mathbf{s}\bullet\mathbf{v} = ||\mathbf{s}|| \cdot ||\mathbf{v}||\cos\theta$

Therefore $\quad \cos\theta = \dfrac{\mathbf{s}\bullet\mathbf{v}}{||\mathbf{s}||} \qquad (4)$

Substitute (4) in (3) $\quad ||\mathbf{q}||^2 = \lambda^2 - 2\mathbf{s}\bullet\mathbf{v}\lambda + ||\mathbf{s}||^2$

Therefore $\quad ||\mathbf{q}||^2 - r^2 = \lambda^2 - 2\mathbf{s}\bullet\mathbf{v}\lambda + ||\mathbf{s}||^2 - r^2 = 0 \qquad (5)$

(5) is a quadratic where $\quad \lambda = \mathbf{s}\bullet\mathbf{v} \pm \sqrt{(\mathbf{s}\bullet\mathbf{v})^2 - ||\mathbf{s}||^2 + r^2} \qquad (6)$

and $\quad \mathbf{s} = \mathbf{c} - \mathbf{t}$

The discriminant of (6) determines whether the line intersects, touches or misses the sphere.

Position vector for P $\quad \mathbf{p} = \mathbf{t} + \lambda\mathbf{v}$

where $\quad \lambda = \mathbf{s} \bullet \mathbf{v} \pm \sqrt{(\mathbf{s} \bullet \mathbf{v})^2 - \|\mathbf{s}\|^2 + r^2}$

$\mathbf{s} = \mathbf{c} - \mathbf{t}$

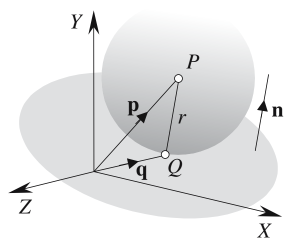
Miss condition $\quad (\mathbf{s} \bullet \mathbf{v})^2 - \|\mathbf{s}\|^2 + r^2 < 0$

Touch condition $\quad (\mathbf{s} \bullet \mathbf{v})^2 - \|\mathbf{s}\|^2 + r^2 = 0$

Intersect condition $\quad (\mathbf{s} \bullet \mathbf{v})^2 - \|\mathbf{s}\|^2 + r^2 > 0$

3.16.2 Proof: Sphere touching a plane

Strategy: A sphere will touch a plane if the perpendicular distance from its center to the plane equals its radius. The geometry describing this condition is identical to finding the position and distance of the nearest point on a plane to a point.



Given the plane $\quad ax + by + cz + d = 0$

where $\quad \mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

The nearest point Q on the plane to a point P is given by

$$\mathbf{q} = \mathbf{p} + \lambda\mathbf{n} \tag{1}$$

where $\quad \lambda = -\dfrac{\mathbf{n} \bullet \mathbf{p} + d}{\mathbf{n} \bullet \mathbf{n}}$

The distance $\quad PQ = \|\lambda\mathbf{n}\|$

If P is the center of the sphere with radius r, and position vector $\mathbf{p}$ the touch point is also given by (1)

when $\quad PQ = \|\lambda\mathbf{n}\| = r$

If $\|\mathbf{n}\| = 1$ $\quad \lambda = -(\mathbf{n} \bullet \mathbf{p} + d)$

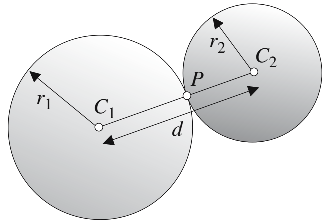
3.16.3 Proof: Touching spheres

Strategy: Use basic coordinate geometry to identify the touch condition.

The diagram shows two spheres with radii r_1 and r_2 centered at $C_1(x_{C1}, y_{C1}, z_{C1})$ and $C_2(x_{C2}, y_{C2}, z_{C2})$ respectively, touching at $P(x_P, y_P, z_P)$.

For a touch condition the distance d between C_1 and C_2 must equal $r_1 + r_2$:

$$d = \sqrt{(x_{C2} - x_{C1})^2 + (y_{C2} - y_{C1})^2 + (z_{C2} - z_{C1})^2}$$



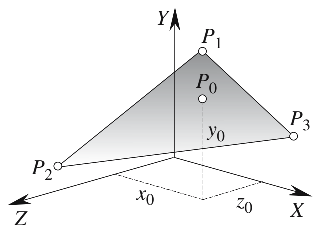
Touch condition	$d = r_1 + r_2$
Intersect condition	$r_1 + r_2 > d > \lvert r_1 - r_2 \rvert$
Separate condition	$d > r_1 + r_2$
Touch point	$x_P = x_{C1} + \frac{r_1}{d}(x_{C2} - x_{C1})$
	$y_P = y_{C1} + \frac{r_1}{d}(y_{C2} - y_{C1})$
	$z_P = z_{C1} + \frac{r_1}{d}(z_{C2} - z_{C1})$

3.17 Three-dimensional triangles

3.17.1 Proof: Point inside a triangle

Strategy: A point $P_0(x_0, y_0, z_0)$ within the boundary of the triangle can be located using barycentric coordinates.

Let $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$ and $P_3(x_3, y_3, z_3)$ be the vertices of a triangle.



Using barycentric coordinates we can write

$$x_0 = \varepsilon x_1 + \lambda x_2 + \beta x_3$$
$$y_0 = \varepsilon y_1 + \lambda y_2 + \beta y_3$$
$$z_0 = \varepsilon z_1 + \lambda z_2 + \beta z_3$$

where $\varepsilon + \lambda + \beta = 1$

P_0 is within the boundary of the triangle if $\varepsilon + \lambda + \beta = 1$ and $(\varepsilon, \lambda, \beta) \in [0, 1]$.

3.17.2 Proof: Unknown coordinate value inside a triangle

Strategy: Given a triangle with vertices P_1, P_2, P_3 and a point $P_0(x_0, y_0, z_0)$, where only two of the coordinates are known, the third coordinate can be determined within the boundary of the triangle using barycentric coordinates.

For example, if x_0 and z_0 are known we can find y_0 using barycentric coordinates:

$$x_0 = \varepsilon x_1 + \lambda x_2 + \beta x_3$$

where $\varepsilon + \lambda + \beta = 1$

Therefore $x_0 - x_3 = \varepsilon(x_1 - x_3) + \lambda(x_2 - x_3) \quad (1)$

Similarly $z_0 - z_3 = \varepsilon(z_1 - z_3) + \lambda(z_2 - z_3) \quad (2)$

Using (1) and (2) we can write

$$\frac{\varepsilon}{\begin{vmatrix} x_0 - x_3 & x_2 - x_3 \\ z_0 - z_3 & z_2 - z_3 \end{vmatrix}} = \frac{\lambda}{\begin{vmatrix} x_1 - x_3 & x_0 - x_3 \\ z_1 - z_3 & z_0 - z_3 \end{vmatrix}} = \frac{1}{\begin{vmatrix} x_1 - x_3 & x_2 - x_3 \\ z_1 - z_3 & z_2 - z_3 \end{vmatrix}}$$

and

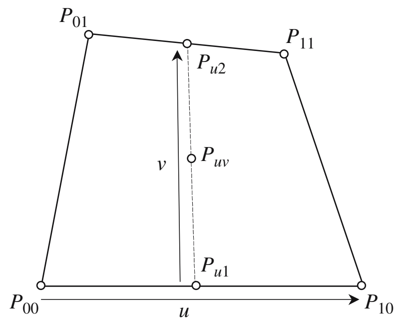
$$\frac{\varepsilon}{\begin{vmatrix} x_0 & z_0 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix}} = \frac{\lambda}{\begin{vmatrix} x_0 & z_0 & 1 \\ x_3 & z_3 & 1 \\ x_1 & z_1 & 1 \end{vmatrix}} = \frac{1}{\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix}}$$

Therefore $y_0 = \varepsilon y_1 + \lambda y_2 + \beta y_3$

P_0 is within the boundary of the triangle if $\varepsilon + \lambda + \beta = 1$ and $(\varepsilon, \lambda, \beta) \in [0, 1]$. Similar formulas can be derived for other combinations of coordinates.

3.18 Parametric curves and patches

3.18.1 Proof: Planar surface patch



Strategy: Locate the position of a point on a patch by linearly interpolating across the patch.

Given four points $P_{00}, P_{10}, P_{11}, P_{01}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ that form a patch

$$P_{u1} = (1-u)P_{00} + uP_{10}$$

where $u \in [0,1]$

$$P_{u2} = (1-u)P_{01} + uP_{11}$$

$$P_{uv} = (1-v)[(1-u)P_{00} + uP_{10}] + v[(1-u)P_{01} + uP_{11}] \quad \text{where } v \in [0,1]$$

Or in matrix form

$$P_{uv} = [u \quad 1]\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} v \\ 1 \end{bmatrix}$$

3.18.2 Proof: Bézier curves in $\mathbb{R}^2$ and $\mathbb{R}^3$

Linear interpolation

Two scalars V_1 and V_2 can be linearly interpolated using

$$V = (1-t)V_1 + tV_2, \quad t \in [0,1]$$

where the sum of the interpolating terms $((1-t)+t) = 1$ (1)

Quadratic interpolation using Bernstein polynomials

From (1) $\quad ((1-t)+t)^n = 1$ (2)

and when $n = 2 \quad ((1-t)+t)^2 = (1-t)^2 + 2t(1-t) + t^2 = 1$

which produces the quadratic interpolant:

$$V = (1-t)^2V_1 + 2t(1-t) + t^2V_2$$

The individual terms are called quadratic Bernstein polynomials and are generated by

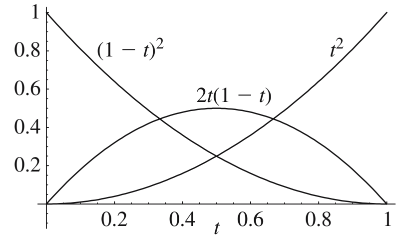
$$B_{k,2}(t) = \frac{2!}{k!(2-k)!}t^k(1-t)^{2-k}, \quad t \in [0,1]$$

giving

$$B_{0,2}(t) = (1-t)^2 = 1 - 2t + t^2$$
$$B_{1,2}(t) = 2t(1-t) = 2t - 2t^2$$
$$B_{2,2}(t) = t^2$$

The graphs of the three polynomials are shown in the diagram.

The central term $2t(1 - t) = 0$ when $t = 0$ and $t = 1$, and therefore does not influence the start and end values of the interpolated value. Furthermore, the central term can be used to influence the nature of the interpolant for $0 < t < 1$.



The complete quadratic interpolant becomes

$V(t) = (1 - t)^2 V_1 + 2t(1 - t)V_C + t^2 V_2$

where V_C is some arbitrary control value.

Quadratic Bézier curve in and $\mathbb{R}^2$ and $\mathbb{R}^3$

A quadratic Bézier curve employs the above quadratic Bernstein polynomials to interpolate the coordinates of two points using a control point $\mathbf{p}_C$

$$\mathbf{p}(t) = (1 - t)^2\mathbf{p}_1 + 2t(1 - t)\mathbf{p}_C + t^2\mathbf{p}_2$$

or in matrix form

$$\mathbf{p}(t) = [t^2 \quad t \quad 1]\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_C \\ \mathbf{p}_2 \end{bmatrix}$$

Cubic Bézier curve in $\mathbb{R}^2$ and $\mathbb{R}^3$

When $n = 3$ in (2) $\quad ((1 - t) + t)^3 = (1 - t)^3 + 3t(1 - t)^2 + 3t^2(1 - t) + t^3 = 1$

The individual terms are called cubic Bernstein polynomials and are generated by

$$B_{k,3}(t) = \frac{3!}{k!(3-k)!}t^k(1-t)^{3-k}, \quad t \in [0,1]$$

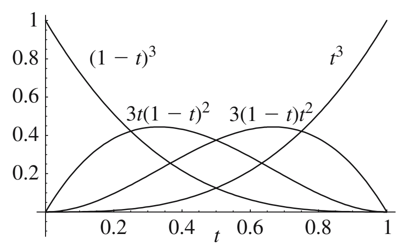
giving

$$\begin{aligned} B_{0,3}(t) &= (1 - t)^3 = 1 - 3t + 3t^2 - t^3 \\ B_{1,3}(t) &= 3t(1 - t)^2 = 3(t - 2t^2 + t^3) \\ B_{2,3}(t) &= 3t^2(1 - t) = 3(t^2 - t^3) \\ B_{3,3}(t) &= t^3 \end{aligned}$$

The graphs are shown in the following diagram.

The central terms $3t(1 - t)^2 = 0$ and $3t^2(1 - t) = 0$ when $t = 0$ and $t = 1$, and therefore do not influence the start and end values of the interpolated value. Furthermore, these terms can be used to influence the nature of the interpolant for $0 < t < 1$.

The complete cubic interpolant becomes



$$V(t) = (1-t)^3V_1 + 3t(1-t)^2V_{C1} + 3t^2(1-t)V_{C2} + t^3V_2$$

Therefore, a cubic Bézier curve has the following form:

$$\mathbf{p}(t) = (1-t)^3\mathbf{p}_1 + 3t(1-t)^2\mathbf{p}_{C1} + 3t^2(1-t)\mathbf{p}_{C2} + t^3\mathbf{p}_2$$

or in matrix form

$$\mathbf{p}(t) = [t^3 \quad t^2 \quad t \quad 1]\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_{c1} \\ \mathbf{p}_{c2} \\ \mathbf{p}_2 \end{bmatrix}$$

In general, a Bézier curve has the form:

$$\mathbf{p}(t) = \binom{n}{i} t^i (1-t)^{n-i}\,\mathbf{p}_i \quad \text{for } 0 \le i \le n$$

or

$$\mathbf{p}(t) = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i}\,\mathbf{p}_i$$

or

$$\mathbf{p}(t) = \sum_{i=0}^{n} \binom{n}{i} B_{i,n}(t)\,\mathbf{p}_i \quad \text{where } B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

3.18.3 Proof: Bézier surface patch in $\mathbb{R}^3$

A Bézier surface patch is defined as

$$\mathbf{p}(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n} B_{i,m}(u)\,B_{j,n}(v)\,\mathbf{p}_{i,j}$$

where

$$B_{i,m}(t) = \binom{m}{i} t^i (1-t)^{m-i} \quad \text{and} \quad B_{j,n}(t) = \binom{n}{j} t^j (1-t)^{n-j}$$

Quadratic Bézier surface patch in $\mathbb{R}^3$

A quadratic Bézier surface patch is defined as

$$\mathbf{p}(u,v) = \sum_{i=0}^{2}\sum_{j=0}^{2} B_{i,2}(u)B_{j,2}(v)\,\mathbf{p}_{i,j}$$

where

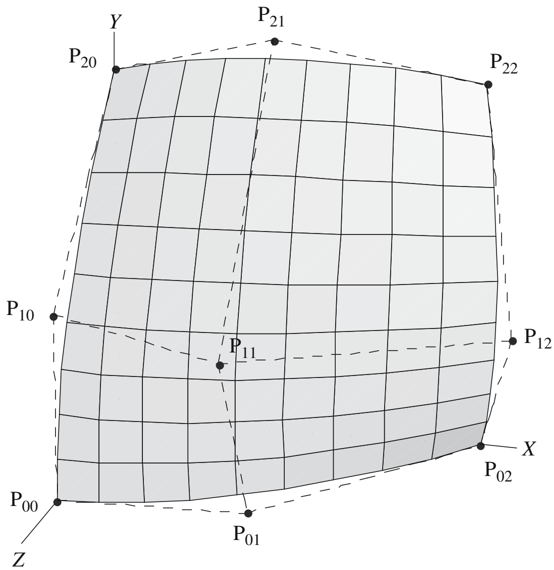
$$B_{i,2}(u) = \binom{2}{i} u^i (1-u)^{2-i} \quad \text{and} \quad B_{j,2}(v) = \binom{2}{j} v^j (1-v)^{2-j}$$

which means that $\mathbf{p}_{i,j}$ is a 3×3 matrix of 3D control points.

Or in matrix form $\mathbf{p}(u,v) = [(1-u)^2 \;\; 2u(1-u) \;\; u^2]\begin{bmatrix} \mathbf{p}_{00} & \mathbf{p}_{01} & \mathbf{p}_{02} \\ \mathbf{p}_{10} & \mathbf{p}_{11} & \mathbf{p}_{12} \\ \mathbf{p}_{20} & \mathbf{p}_{21} & \mathbf{p}_{22} \end{bmatrix}\begin{bmatrix} (1-v)^2 \\ 2v(1-v) \\ v^2 \end{bmatrix}$

or $\mathbf{p}(u,v) = [u^2 \quad u \quad 1]\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{p}_{00} & \mathbf{p}_{01} & \mathbf{p}_{02} \\ \mathbf{p}_{10} & \mathbf{p}_{11} & \mathbf{p}_{12} \\ \mathbf{p}_{20} & \mathbf{p}_{21} & \mathbf{p}_{22} \end{bmatrix}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} v^2 \\ v \\ 1 \end{bmatrix}$

The diagram shows an example.



Cubic Bézier surface patch in $\mathbb{R}^3$

A cubic Bézier surface patch is defined as

$$\mathbf{p}(u,v) = \sum_{i=0}^{3}\sum_{j=0}^{3} B_{i,3}(u)B_{j,3}(v)\mathbf{p}_{i,j}$$

where $B_{i,3}(u) = \binom{3}{i} u^i(1-u)^{3-i}$ and $B_{j,2}(v) = \binom{2}{j} v^j(1-v)^{2-j}$

which means that $\mathbf{p}_{i,j}$ is a 4×4 matrix of 3D control points.

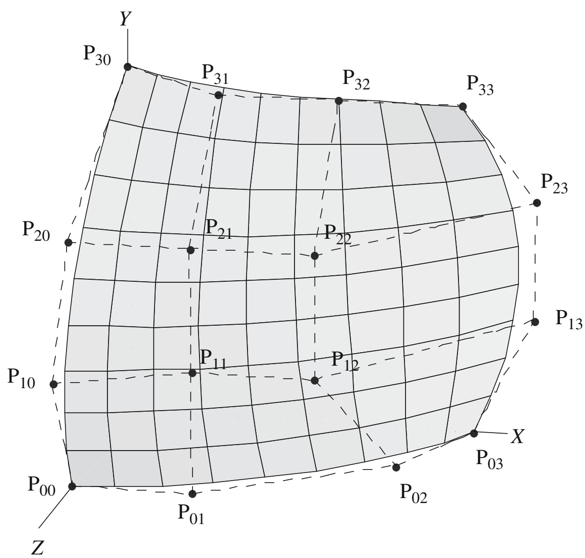
Or in matrix form

$$\mathbf{p}(u,v)=[(1-u)^3 \quad 3u(1-u)^2 \quad 3u^2(1-u) \quad u^3]\begin{bmatrix}\mathbf{p}_{00} & \mathbf{p}_{01} & \mathbf{p}_{02} & \mathbf{p}_{03}\\ \mathbf{p}_{10} & \mathbf{p}_{11} & \mathbf{p}_{12} & \mathbf{p}_{13}\\ \mathbf{p}_{20} & \mathbf{p}_{21} & \mathbf{p}_{22} & \mathbf{p}_{23}\\ \mathbf{p}_{30} & \mathbf{p}_{31} & \mathbf{p}_{32} & \mathbf{p}_{33}\end{bmatrix}\begin{bmatrix}(1-v)^3\\ 3v(1-v)^2\\ 3v^2(1-v)\\ v^3\end{bmatrix}$$

or

$$\mathbf{p}(u,v)=[u^3 \; u^2 \; u \; 1]\begin{bmatrix}-1 & 3 & -3 & 1\\ 3 & -6 & 3 & 0\\ -3 & 3 & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{p}_{00} & \mathbf{p}_{01} & \mathbf{p}_{02} & \mathbf{p}_{03}\\ \mathbf{p}_{10} & \mathbf{p}_{11} & \mathbf{p}_{12} & \mathbf{p}_{13}\\ \mathbf{p}_{20} & \mathbf{p}_{21} & \mathbf{p}_{22} & \mathbf{p}_{23}\\ \mathbf{p}_{30} & \mathbf{p}_{31} & \mathbf{p}_{32} & \mathbf{p}_{33}\end{bmatrix}\begin{bmatrix}-1 & 3 & -3 & 1\\ 3 & -6 & 3 & 0\\ -3 & 3 & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}v^3\\ v^2\\ v\\ 1\end{bmatrix}$$

The diagram shows an example.



4 Glossary

abscissa The x-coordinate of the ordered pair (x, y).

acute angle An angle between 0° and 90°.

acute triangle A triangle that has all interior angles <90°.

adjacent (angle, point, side, plane) Lying next to another angle, point, side, plane.

affine transformation A function with domain and codomain $\mathbb{R}^2$, with a rule of the form $\mathbf{x} \mapsto \mathbf{A}\mathbf{x} + \mathbf{a}$, where $\mathbf{a}$ is a vector with two components and $\mathbf{A}$ is a 2×2 matrix.

altitude (of a geometric figure) The perpendicular from a vertex to the opposite side, or the extended opposite side.

angle (between two lines) The smallest of the two angles formed between two intersecting lines.

angle (between two planes) The dihedral angle formed by two planes, which is also the angle between the planes' normals.

angle (of depression) The angle between a reference horizontal line from the observer's eye and the line of sight to an object below the observer.

angle (of elevation) The angle between a reference horizontal line from the observer's eye and the line of sight to an object above the observer.

angle (of inclination) The positive angle between 0° and 180° that a line makes with the x-axis.

annulus The region bounded by two concentric co-planar concentric circles.

apex The point that is the greatest distance from an edge or plane.

apothem (of a regular polygon) The perpendicular from the center of a polygon to a side.

arc The part of a circle between two points on the circle.

arclength The length of an arc of a circle.

arccosine The inverse function of the trigonometric cosine function with domain $[0, \pi]$.

arcsine The inverse function of the trigonometric sine function with domain $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$.

arctangent The inverse function of the trigonometric tangent function with domain $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$.

area (of a geometric solid) The total area of all the solid's faces.

Argand diagram Represents complex numbers as points on a plane such that $z = x + yi$ represents the point (x, y).

astroid A hypercycloid of four cusps.

asymptote A straight line to which a curve approximates but never touches.

auxiliary line A line introduced to a geometric figure to clarify a proof.

axiom An unproven mathematical statement, e.g. Two straight lines may intersect at one point only.

axis A line of reference for measuring distances (x-axis) or a straight line that divides a plane or solid figure.

axis of symmetry A straight line reference used to describe the symmetric properties of a shape or figure.

Barycentric coordinates A set of numbers locating a point in space relative to a set of fixed points.

base angles The two angles formed by a base line and two sides, as found in an isosceles triangle.

binomial expansion The expansion of a binomial expression of the form $(a + b)^n$.

bisect To divide into two equal parts.

bisector A point, line or plane that divides a figure into two equal parts.

bisector (of an angle) The line that divides an angle into two equal angles.

cardioid The locus of a point on a circle in $\mathbb{R}^2$ that rolls on an equal, fixed circle. The equation is given by $x^2 + y^2 + ax = a\sqrt{x^2 + y^2}$.

Cartesian coordinate system A system where a pair of coordinates (x, y) define a point in $\mathbb{R}^2$ or three coordinates (x, y, z) define a point in $\mathbb{R}^3$.

Cartesian unit vector A unit vector aligned with the x-, y- or z-axis.

catenary The curve of a heavy cable hanging in a gravitational field.

catenoid The surface of revolution formed by rotating a catenary about a vertical axis.

central angle (of a regular polygon) The angle formed at a polygon's center by two radii to an angle.

center (of an ellipse or hyperbola) The point of intersection of the axes of symmetry of the conic.

centroid A point in a shape representing the arithmetic mean of the coordinates.

chord A line segment joining two points on a curve.

circle The set of points in a plane that are a fixed distance (radius) from a specified point (center) in the plane.

circle of curvature The circle whose radius equals the radius of curvature of a curve.

circular functions The trigonometric functions: sine, cosine, tangent, cosecant, secant and cotangent.

circumcenter The common point of intersection of the perpendicular bisectors of the sides of a triangle.

circumcircle See *circumscribed circle*.

circumference The length of a circle's boundary.

circumscribed circle The circle which intersects all the vertices of a polygon.

co-linear points Two or more points intersected by a common line.

complementary angles Two angles whose sum equals 90°.

complex number A number of the form $a + bi$ where $i = \sqrt{-1}$ and a and b are real numbers.

component (of a vector) See *vector*.

component form (of a vector) Representing a vector **a** in terms of its Cartesian unit vectors **i** and **j**: $\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

concave polygon A polygon which contains one or more angles greater than 180°.

concentric Means that two circles or spheres share a common center.

concurrent lines Three or more lines passing through a common center.

cone A solid figure formed by a closed curve base and a separate vertex through which lines intersect with points on the closed curve.

congruent Identical.

congruent triangles Identical triangles.

conic sections The curves obtained as cross-sections when a double cone is sliced by a plane. See also *ellipse*, *hyperbola*, and *parabola*.

contour plot A set of contours for a given function.

convex polygon A polygon whose angles are all less than 180°.

coordinate A scalar used within a coordinate system to locate a point. See *Cartesian coordinate system*, *cylindrical coordinate system*, and *spherical coordinate system*.

corresponding angles Two angles in the same relative position when two lines are intersected by a third line. When the two lines are parallel, the corresponding angles are equal.

cosecant (of an angle α) A trigonometric function representing $1/\sin\alpha$, provided that $\sin\alpha \neq 0$.

cosine (of an angle α) A trigonometric function representing the ratio of the adjacent side to the hypotenuse in a right-angled triangle.

cosine rule A rule relating the three sides and one angle of a triangle.

cotangent (of an angle α) A trigonometric function representing $1/\tan\alpha$, provided that $\tan\alpha \neq 0$.

cross product See *vector product*.

cube A platonic object having six square faces (hexahedron).

cubic A mathematical expression of the form $ax^3 + bx^2 + cx + d$ where $a \neq 0$.

cubic expression A polynomial of the form $ax^3 + bx^2 + cx + d$ where $a \neq 0$.

cusp A double point on a curve at which two tangents are coincident.

cylinder A solid formed by a closed cylindrical surface bounded by two planes.

cylindrical coordinate system A system of coordinates where a point is located in space with reference to its height above a ground plane and its polar coordinates on this plane.

derivative of a function For a function $f(x)$ its derivative $f'(x)$ is the gradient of the graph at point x.

determinant of a matrix A scalar quantity derived from the terms of a matrix. If

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \det \mathbf{A} = ad - bc.$$

diagonal A line joining two nonadjacent vertices.

diameter A chord through the center of a circle or sphere.

dihedral group The group of order $2n$ formed by the symmetries of a regular n-gon.

direction cosines The angles formed between a line and the x-, y- and z-axes.

directrix A line associated with a conic. See also *eccentricity*.

discriminant (of a quadratic equation) The term $b^2 - 4ac$.

dodecahedron A Platonic object that has 12 faces, each of which is a regular pentagon.

domain of a function The set of allowable input values for a function. See also *function*.

dot product See *scalar product*.

eccentricity The ratio of the distances from a point on a conic to the focus of the conic and from that point to the directrix of the conic.

edge A line joining two vertices.

ellipse A conic having eccentricity between 0 and 1.

equidistant Having equal distance from a reference point.

equilateral Having sides of equal length.

equilateral triangle A triangle that has sides of equal length.

Euclidean space Represented by the symbol $\mathbb{R}^n$ where n is the spatial dimension.

exterior angle (of a polygon) The external angle of a polygon.

face A planar region bounding a polyhedron.

focus A point associated with a conic. See also *eccentricity*.

frustum Part of a solid figure cut off by two parallel planes.

function A rule which assigns to each element of one set one element of another set. For example, $f(x) = x + 1$.

geometric form (of a vector) Representing a vector $\mathbf{a}$ in terms of its magnitude $||\mathbf{a}||$ and direction θ.

golden ratio The constant $\phi = \frac{1}{2}(1 + \sqrt{5}) = 1.618\ldots$

golden rectangle A rectangle with sides m (long side) and n (short side) such that m/n equals the golden ratio.

gradient (of a graph at a point) The gradient of the tangent to the graph at that point.

gradient (of a line) See *slope (of a line)*.

hexagon A six-sided polygon.

hexahedron A polyhedron that has six faces (a cube).

hyperbola A conic having eccentricity greater than 1.

hypotenuse The side opposite the right-angle in a right-angled triangle.

i-component (of a vector) The scalar x in the component form of the vector $\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

icosahedron A polyhedron that has twenty faces.

identity matrix A matrix, whose function performs a null operation.

imaginary part (of a complex number) The scalar term associated with the i term in a complex number. See also *complex number*.

inclined plane A plane that is not horizontal.

intercept The point where a line or surface meets the x-, y- or z-axis.

interior angle The angle between two sides of a polygon.

inverse trigonometric functions The functions $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$, $\csc^{-1}$, $\sec^{-1}$, and $\cot^{-1}$.

isogonal Having equal angles.

isometric Having equal lengths.

isoperimetric Having equal perimeters.

isosceles triangle A triangle with two equal sides only.

j-component (of a vector) The scalar y in the component form of the vector $\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

k-component (of a vector) The scalar z in the component form of the vector $\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

linear A first degree equation, expression, etc., such as $x + 2y + 3z = 4$.

linear transformation A function having the same domain and codomain such that $\mathbf{x} \mapsto \mathbf{A}\mathbf{x}$, where the linear transformation is determined by matrix $\mathbf{A}$.

locus A curve defined by a particular property.

magnitude (of a vector a) The length of the line segment representing the vector, and written as $||\mathbf{a}||$.

major axis (of an ellipse) The line segment from $(-a, 0)$ to $(a, 0)$ for the ellipse $x^2/a^2 + y^2/b^2 = 1$, where $a \geqslant b > 0$.

matrix A rectangular array of numbers.

minor axis (of an ellipse) The line segment from $(0, -b)$ to $(0, b)$ for the ellipse $x^2/a^2 + y^2/b^2 = 1$, where $a \geqslant b > 0$.

***n*-gon** A regular polygon with n sides.

oblique angle An angle that is not a multiple of 90°.

oblique pyramid A pyramid whose vertex is not perpendicular to the center of its base.

obtuse angle An angle between 90° and 180°.

octagon An eight-sided polygon.

octahedron A polyhedron with eight faces.

ordered pair A set with a first and second element, e.g. (x, y).

ordinate The y-coordinate of a point as used in Cartesian coordinates.

origin A point of reference from which distances are measured.

orthogonal At right angles.

parabola A conic having eccentricity 1.

parallelepiped A prism whose faces are parallelograms.

parallelogram A quadrilateral constructed from two pairs of parallel sides.

parameter A variable used when defining a function or curve.

parametric equation Equations that generate the coordinates of a point on a curve using a common variable (parameter), e.g. $x = \cos(t), y = \sin(t)$.

Pascal's triangle The triangle of numbers used to generate binomial coefficients.

pentagon A five-sided polygon.

pentahedron A polyhedron with five faces.

perimeter The length of a closed curve.

perpendicular A line/plane that is at right angles to another line/plane.

perpendicular bisector (of a line segment) The line that cuts the line segment halfway along its length and is at right angles to the line.

plane A surface where a line joining any two points on the surface is also on the surface.

point A point in space that has position but no spatial extension.

polar coordinates (of a point *P*) The numbers r and θ for the point P with Cartesian coordinates $(r\cos\theta, r\sin\theta)$.

polygon A figure constructed from three or more straight sides.

polyhedral angle The solid angle between three or more faces of a polyhedron.

polyhedron A figure constructed from plane polygonal faces.

position vector A vector representing the line segment from the origin to a point.

prism A solid figure constructed from two congruent polygons where corresponding vertices are connected with straight edges.

pyramid A solid figure constructed from a polygonal base and lateral triangular faces.

Pythagoras' theorem For a right-angled triangle with sides a, b and c then $a^2 = b^2 + c^2$ where a is the hypotenuse.

quadrant One of the four regions defined by the Cartesian coordinate system.

quadratic curve A curve represented by an equation of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + f = 0$, where A, B, C are not all zero.

quadrilateral A plane figure constructed from four edges.

quaternion A four-tuple of the form $(s, \mathbf{v})$ where s is a scalar and $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

radian A unit of angular measure such that $2\pi[rad] = 360°$.

radius (of a circle) The distance from the center of the circle to any point on the circle's circumference.

rectangle A quadrilateral with all interior angles right angles.

rectangular hyperbola A hyperbola for which the asymptotes are at right angles.

reflex angle An angle between 180° and 360°.

regular polygon A polygon with equal interior angles and equal sides.

regular polyhedron A polyhedron with congruent polyhedral angles and regular congruent faces.

regular prism A right prism that has regular polygons as bases.

right angle An angle equal to 90°.

right-angled triangle A triangle with one interior angle equal to a right angle.

right circular cone A cone for which the cross-sections obtained by slicing the cone with planes at right angles to the axis are circles.

scalar A single number, as opposed to a vector.

scalar product A vector operation also known as the dot product, where given two vectors $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a} \bullet \mathbf{b} = ||\mathbf{a}|| \cdot ||\mathbf{b}|| \cos\alpha$, where α is the angle between the vectors.

scalene triangle A triangle constructed from three unequal sides.

secant (of an angle α) The secant of α is $1/\cos\alpha$, provided that $\cos\alpha \neq 0$.

sector (of a circle) The region between two radii of a circle.

segment (of a circle) The region between a chord of a circle and the arc determined by the chord's ends.

semicircle Half a circle.

similar Two shapes are similar if one is an enlargement of the other.

sine (of an angle α) A trigonometric function representing the ratio of the side opposite α to the hypotenuse in a right-angled triangle.

sine rule A rule that relates pairs of sides and the corresponding opposite angles of a triangle.

slope (of a line) The gradient of a line expressed as a ratio of the y rise divided by the x run between two points.

spherical coordinate system A polar coordinate system where a point P is defined as $P = (r, \theta, \phi)$, where r is a radius, θ and ϕ are angles.

square A quadrilateral with four equal sides and interior angles are right angles.

supplementary angles Two angles whose sum equals 180°.

surface of revolution A surface created by rotating a contour about an axis.

tangent A line whose slope equals that of a curve where it touches the curve.

tangent (of an angle α) A trigonometric function representing the ratio of the side opposite α to the adjacent side in a right-angled triangle.

tetrahedron A solid figure constructed from four triangular faces.

transformation Another name for a function.

trapezium A quadrilateral that has one pair of opposite sides parallel.

triangle A closed, three-sided figure.

triple product The product of three vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$: the triple scalar product is $\mathbf{A} \bullet (\mathbf{B} \times \mathbf{C})$ and the triple vector product is $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$.

unit circle The circle with radius 1 and center at the origin.

unit square The square in $\mathbb{R}^2$ with vertices $(0, 0), (0, 1), (1, 1)$ and $(0, 1)$.

vector A single column matrix.

vector product A vector operation also known as the cross product, where given two vectors $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a} \times \mathbf{b} = \mathbf{c}$, where $||\mathbf{c}|| = ||\mathbf{a}|| \cdot ||\mathbf{b}|| \sin \alpha$ and α is the angle between the vectors.

zero vector A vector in which every component is equal to zero.

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