Engineering Electromagnetics

EE2011

LECTURE 1

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Transmission Lines – Basic Theories

1. Introduction

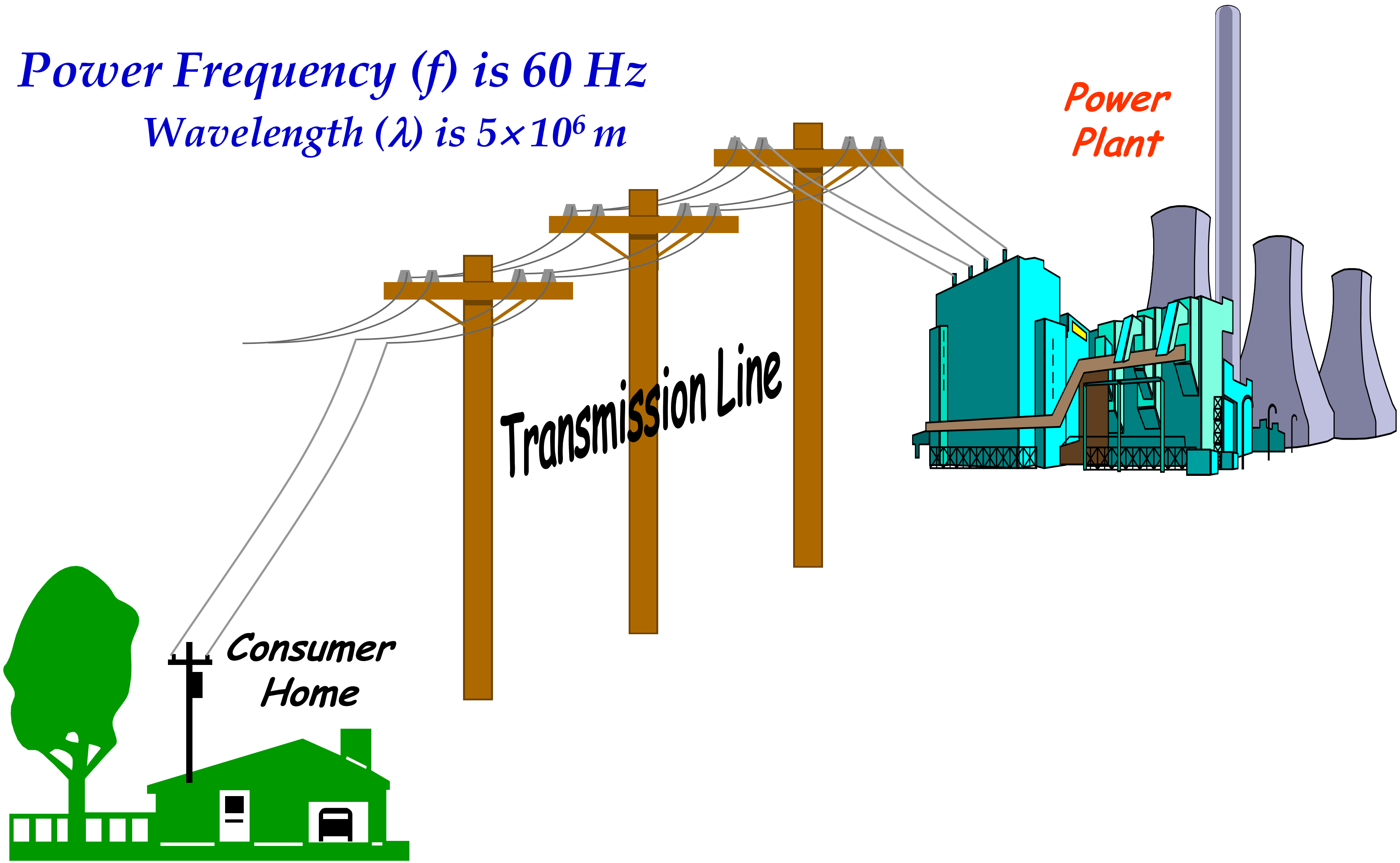
Transmission Line (T-Lines) may encompass all structures and media that serve to transfer electromagnetic energy (or signal) between two points.

Examples:

- Electric lines for delivering electricity from power plants
- Coaxial cables or telephone wires carrying video or audio signals
- Optical fibers carrying light waves

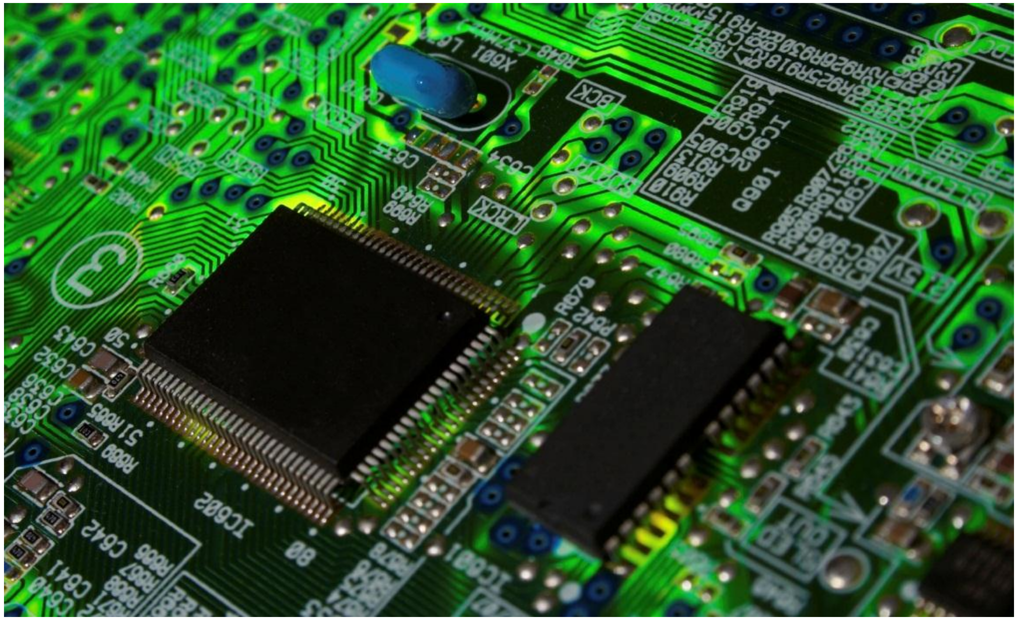
Power Frequency (f) is 60 Hz

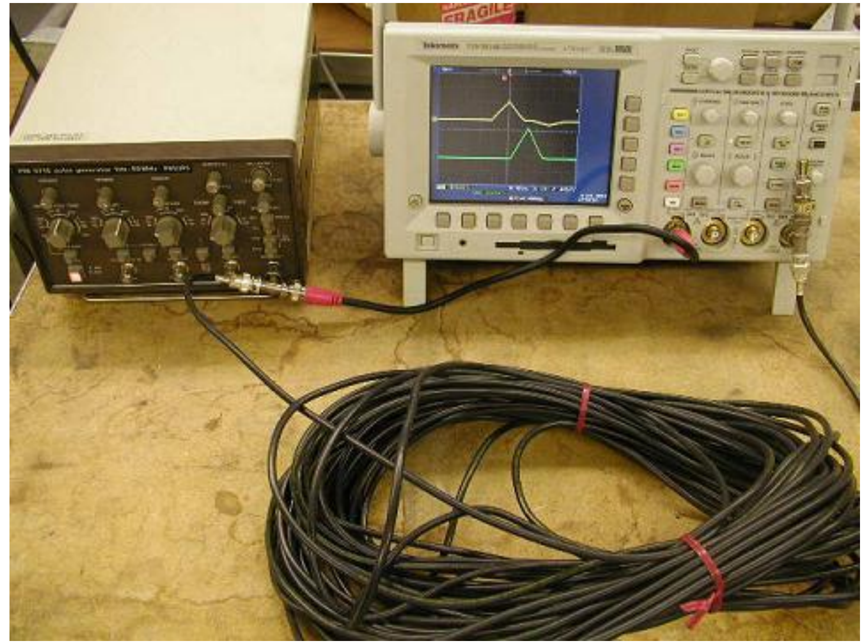
Wavelength (λ) is 5×10^6 m



Signal Frequency (f) is approaching 10 GHz

Wavelength (λ) is 3 cm

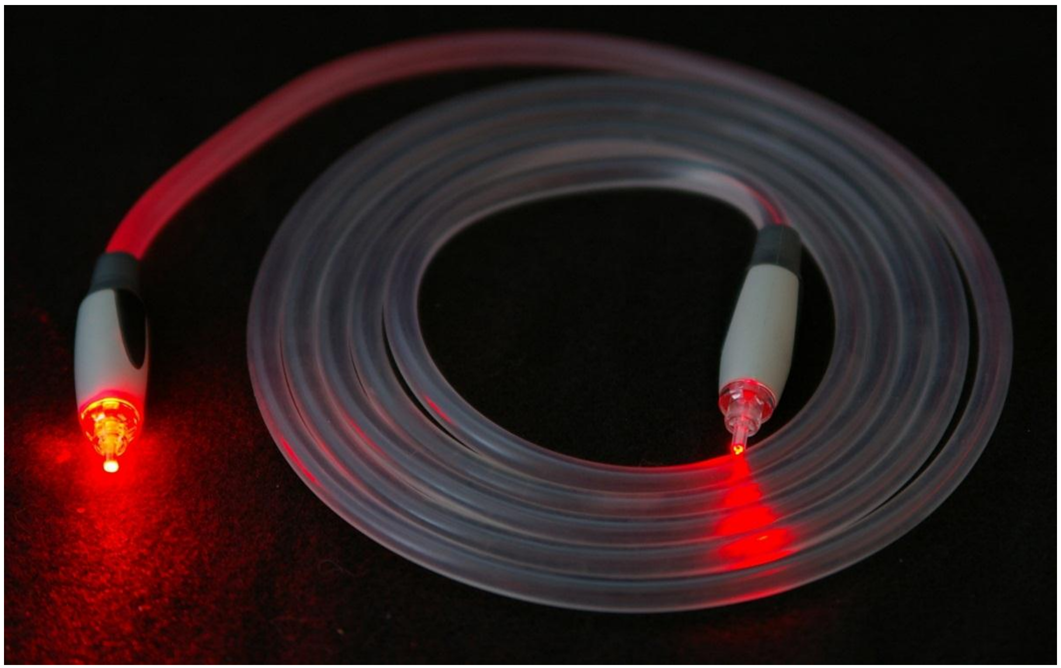


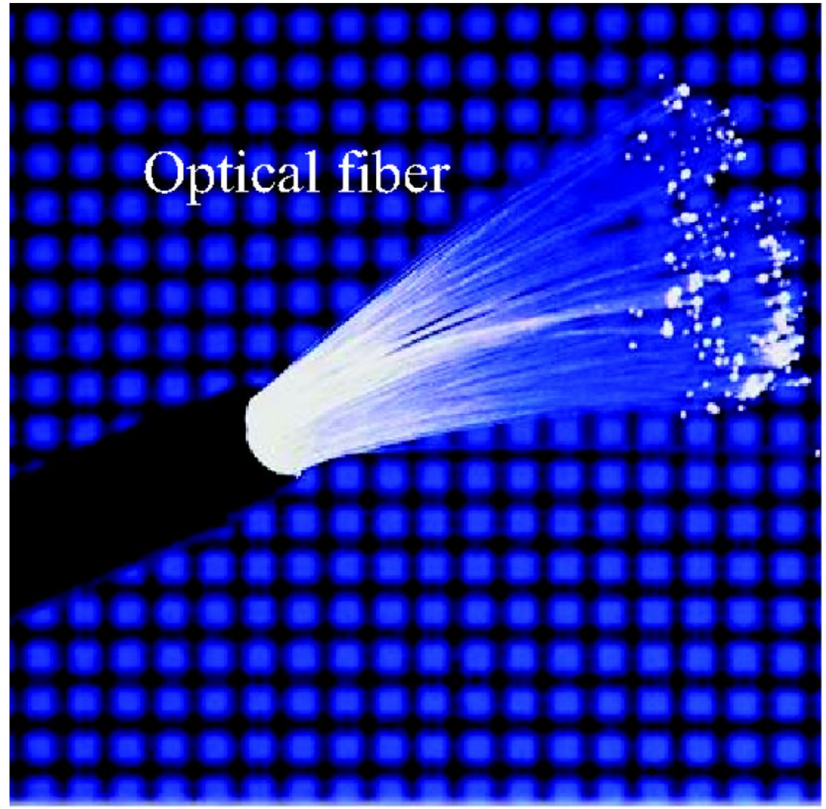


Microstrip Lines in Printed Circuit Board

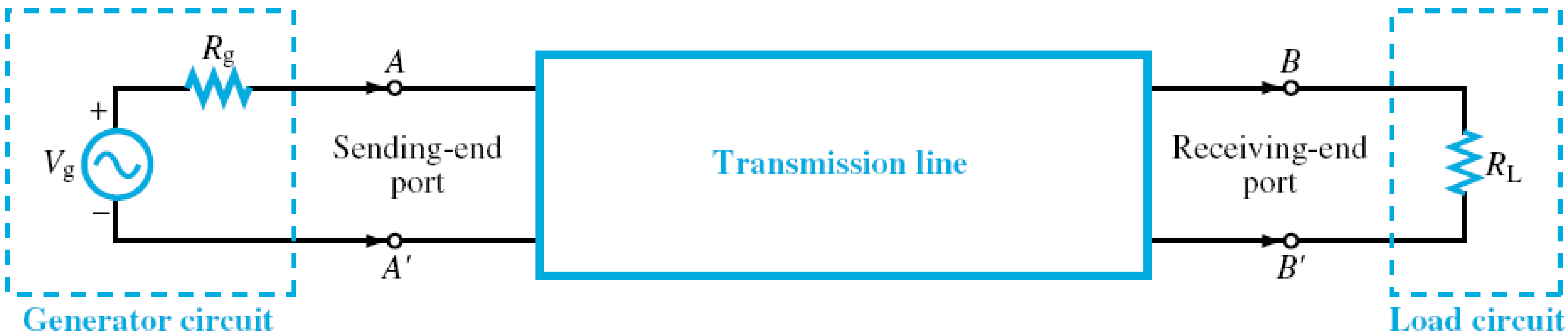
Coaxial Cables

Optical fiber



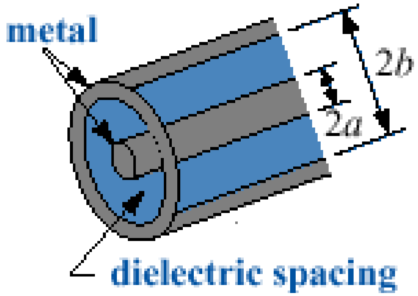


A transmission line is a two-port network connecting a generator circuit at the sending end to a load at the receiving end.

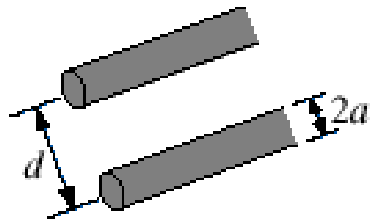


Note that the above symbol represents the functionality of the device, rather than its shape, size, material, or other attributes.

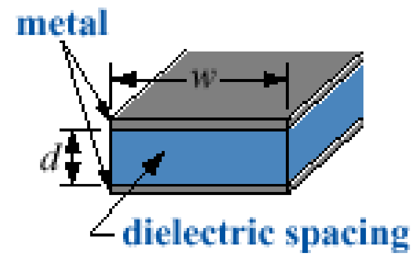
A few examples of Transmission Lines



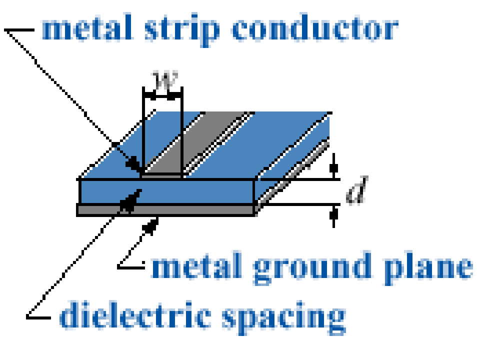
(a) Coaxial line



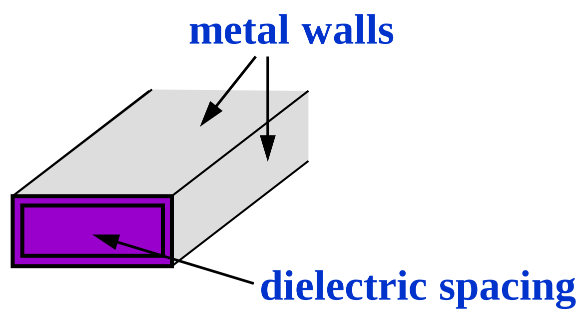
(b) Two-wire line



(c) Parallel-plate line

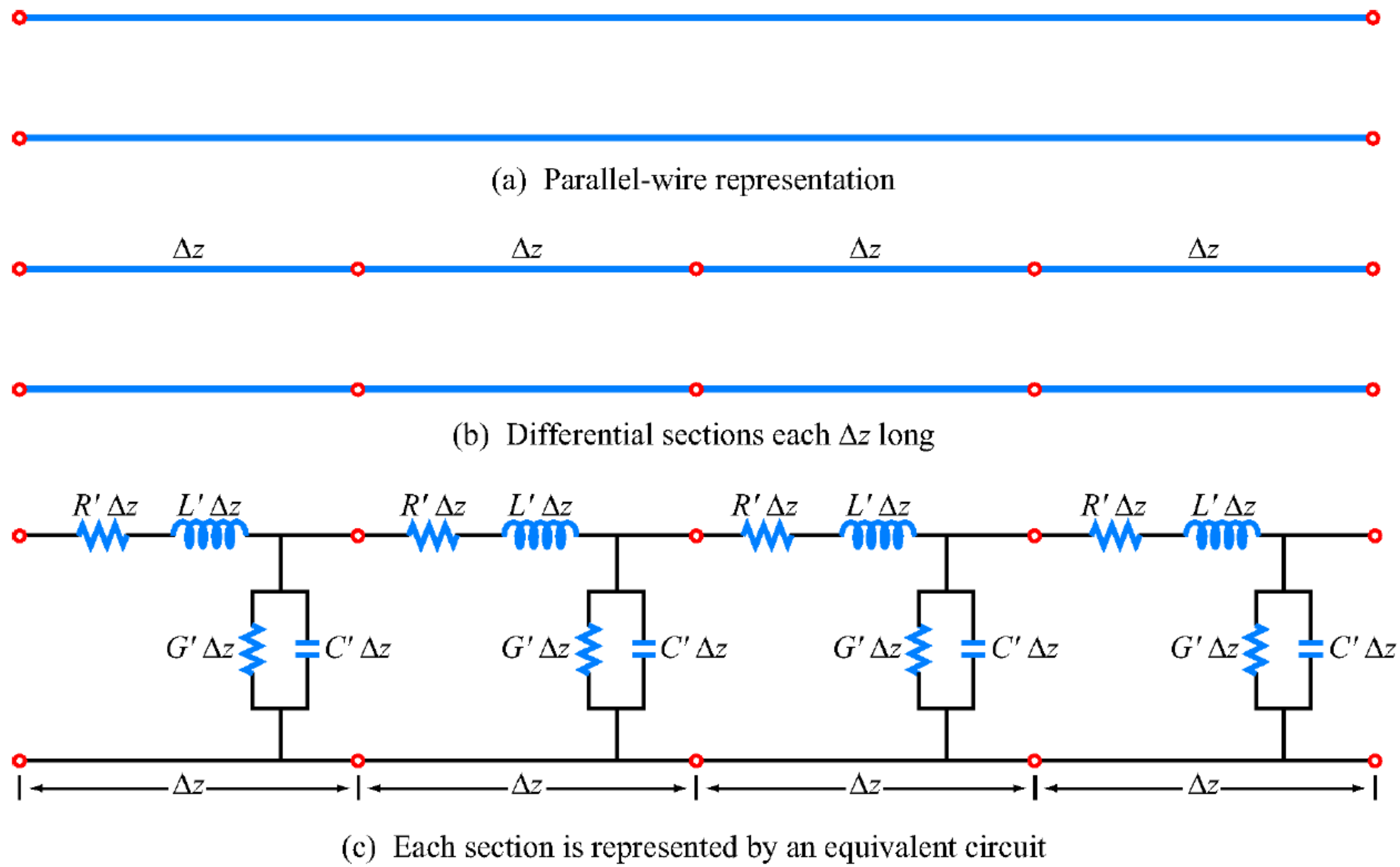


(d) Microstrip line



(e) Rectangular Waveguide

Circuit approach to T-Lines: Lumped-Element Model



(c) Each section is represented by an equivalent circuit

(a) A transmission line is represented by the parallel-wire configuration;
(b) The line is subdivided into **short** differential sections;
(c) Each of small differential sections is represented by an equivalent circuit.

Schematically, we use a parallel-wire symbol to denote T-Lines of arbitrary types.

T-Lines of different types differ from each other in the following parameters:

R' : Resistance per unit length, in Ω / m

L' : Inductance per unit length, in H / m

G' : Conductance per unit length, in S / m

C' : Capacitance per unit length, in F / m

The prime superscript is used as a reminder that the line parameters are differential quantities whose units are per unit length.

Note: Regarding the derivation of the above parameters, it will be taught in Part YSP, and it will not be covered or tested in Part CXD.

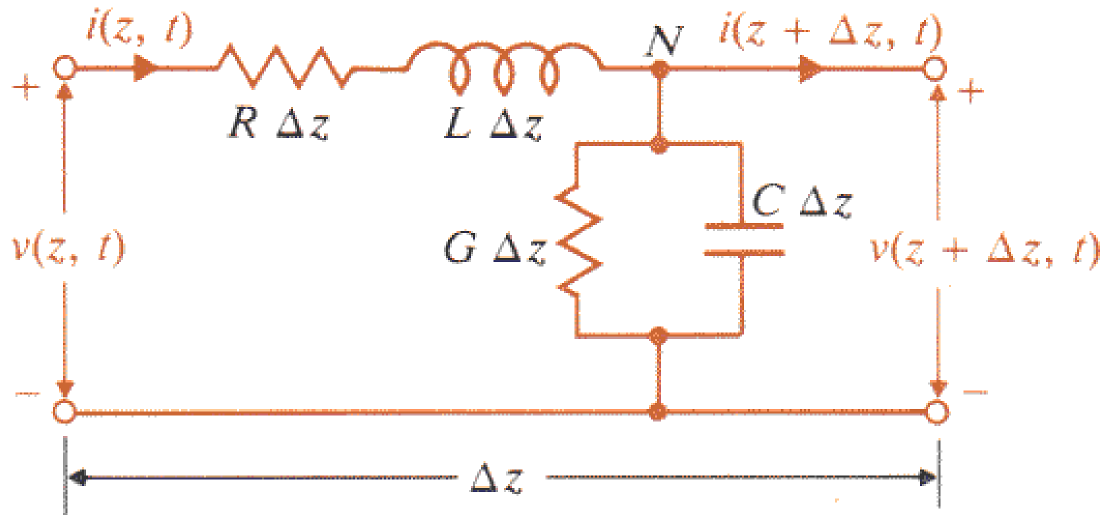
Table 2-1: Transmission-line parameters R', L', G', and C' for three types of lines.

Parameter	Coaxial	Two-Wire	Parallel-Plate	Unit
R'	$\frac{R_s}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right)$	$\frac{2R_s}{\pi d}$	$\frac{2R_s}{w}$	Ω/m
L'	$\frac{\mu}{2\pi}\ln(b/a)$	$\frac{\mu}{\pi}\ln\left[(D/d)+\sqrt{(D/d)^2-1}\right]$	$\frac{\mu h}{w}$	H/m
G'	$\frac{2\pi\sigma}{\ln(b/a)}$	$\frac{\pi\sigma}{\ln\left[(D/d)+\sqrt{(D/d)^2-1}\right]}$	$\frac{\sigma w}{h}$	S/m
C'	$\frac{2\pi\varepsilon}{\ln(b/a)}$	$\frac{\pi\varepsilon}{\ln\left[(D/d)+\sqrt{(D/d)^2-1}\right]}$	$\frac{\varepsilon w}{h}$	F/m

2. Equations and solutions

Consider a **short** section Δz of a transmission line (For convenience, dropping the primes on *R'*, *L'*, *G'*, *C'* hereafter) :

Generator (Source)



Load

Using **KVL** and **KCL** circuit theorems, we can derive the following differential equations for this section of T-Line.

KVL: $v(z,t) - R\Delta z\, i(z,t) - L\Delta z \dfrac{\partial i(z,t)}{\partial t} - v(z+\Delta z,t) = 0$

Upon dividing all terms by Δz and rearranging, we obtain

$$-\frac{v(z+\Delta z,t) - v(z,t)}{\Delta z} = R\, i(z,t) + L\frac{\partial i(z,t)}{\partial t}$$

By letting $\Delta z \to 0$, it becomes

$$\boxed{-\frac{\partial v(z,t)}{\partial z} = R i(z,t) + L\frac{\partial i(z,t)}{\partial t}}$$

KCL: $i(z,t) - G\Delta z\, v(z+\Delta z,t) - C\Delta z \dfrac{\partial v(z+\Delta z,t)}{\partial t} - i(z+\Delta z,t) = 0$

$$\boxed{-\frac{\partial i(z,t)}{\partial z} = G v(z,t) + C\frac{\partial v(z,t)}{\partial t}}$$

The two equations in red box are the Transmission Line Equations

In terms of phasors (as mentioned in Lecture 0, the ~ on the top is dropped off for convenience), the T-Line equations can be written as:

$$-\frac{dV(z)}{dz} = (R + j\omega L)I(z)$$

$$-\frac{dI(z)}{dz} = (G + j\omega C)V(z)$$

By eliminating $I(z)$ or $V(z)$, we obtain respectively

$$\frac{d^2V(z)}{dz^2} = \gamma^2 V(z)$$

$$\frac{d^2I(z)}{dz^2} = \gamma^2 I(z)$$

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \alpha + j\beta$$

$\boldsymbol{\gamma}$ is the complex propagation constant whose real part $\boldsymbol{\alpha}$ is the **attenuation constant** (Np/m) and whose imaginary part $\boldsymbol{\beta}$ is the **phase constant** (rad/m).

It is easily to verify that the solutions to transmission line equations are of the forms

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}$$

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z}$$

Forward travelling wave.

Backward travelling wave.

$V_0^+, V_0^-, I_0^+, I_0^-$: wave amplitudes in the forward and backward directions at $z = 0$. (They are complex numbers in general.)

The relationships between V_0^+ and I_0^+, V_0^- and I_0^-:

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z}$$

Since $-\dfrac{dI(z)}{dz} = (G + j\omega C)V(z)$

$$V(z) = \frac{1}{-(G + j\omega C)} \frac{\partial}{\partial z}\left(I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z}\right)$$

$$= \frac{1}{-(G + j\omega C)}\left(-\gamma I_0^+ e^{-\gamma z} + \gamma I_0^- e^{+\gamma z}\right)$$

$$= \frac{\gamma}{G + j\omega C}\left(I_0^+ e^{-\gamma z} - I_0^- e^{+\gamma z}\right)$$

It can be rewritten as

$$= V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}$$

$$\boxed{+\frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = \frac{\gamma}{G + j\omega C} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}}$$

3. Transmission Line Parameters

From the solutions to the transmission line equations, we have

$$+\frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = \frac{\gamma}{G+j\omega C} = \sqrt{\frac{R+j\omega L}{G+j\omega C}}$$

This ratio is defined as the characteristic impedance Z_0.

Z_0 and γ are the two most important parameters of a T-line. They depend on the distributed parameters ($RLGC$) of the line itself and ω.

$$Z_0 = \frac{R+j\omega L}{\gamma} = \frac{\gamma}{G+j\omega C} = \sqrt{\frac{R+j\omega L}{G+j\omega C}}$$

$$\gamma = \alpha + j\beta = \sqrt{(R+j\omega L)(G+j\omega C)}$$

Parameters for lossless transmission lines

In many practical situations, T-lines can be designed to exhibit very low losses, which can be well modelled as lossless T-Lines.

For lossless T-Lines, $R = G = 0$

$$\alpha = 0 \qquad \beta = \omega\sqrt{LC} \qquad \gamma = j\beta$$

$$Z_0 = \sqrt{L/C}$$

For lossless T-Lines, we often use k (wavenumber) instead of β. Thus,

$$\gamma = jk$$

Consequently, the wave solutions for $V(z)$ and $I(z)$ that are obtained earlier take the same format as the wave solutions introduced in Lecture 0.

$$V(z) = V_0^+ e^{-jkz} + V_0^- e^{+jkz}$$

$$I(z) = I_0^+ e^{-jkz} + I_0^- e^{+jkz}$$

From Lecture 0, we have

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{\omega\sqrt{LC}} = \frac{1}{f\sqrt{LC}} \quad \text{(wavelength along the T-Line)}$$

$$u_p = \frac{\omega}{k} = \frac{1}{\sqrt{LC}} = \frac{c}{\sqrt{\varepsilon_r}} \quad \text{(phase velocity)}$$

where c is the speed of light in free space, 3×10^8 m/s, and

ε_r is the relative permittivity of the dielectric material filling the T-Line

will be taught in Part YSP and later lectures of Part CXD

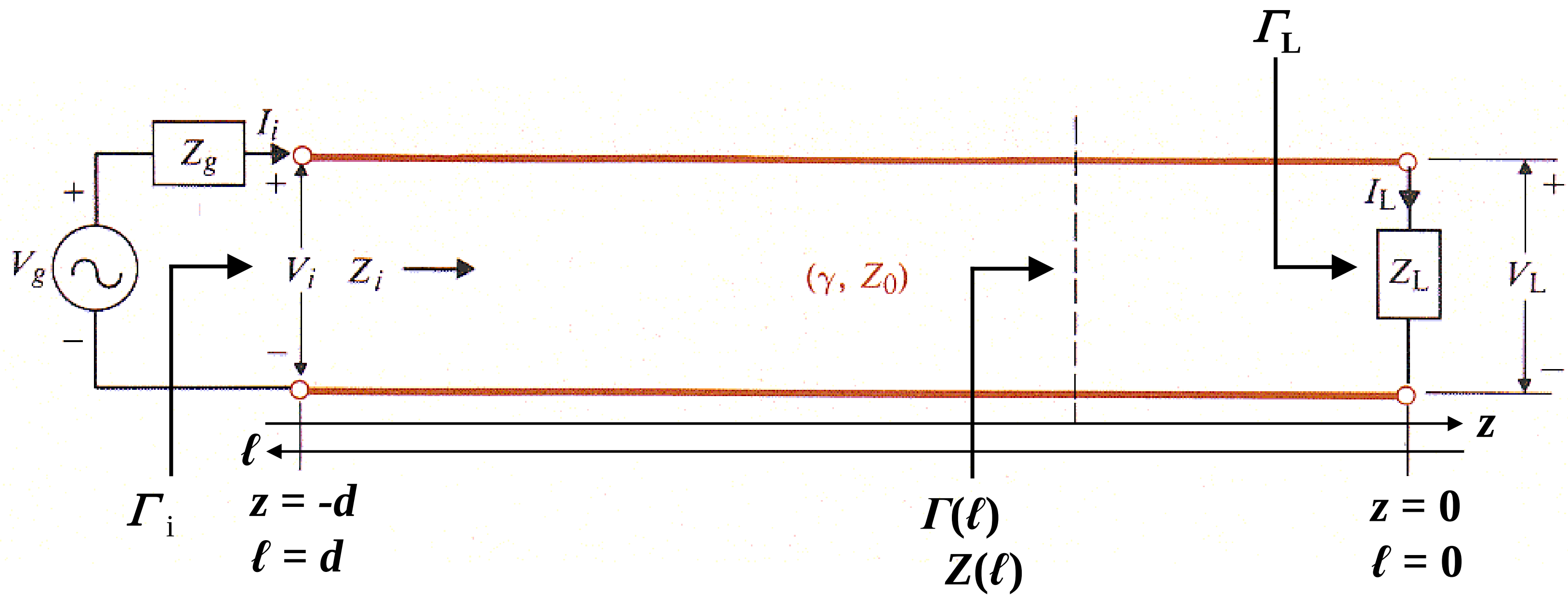
$\boxed{\text{From here onwards, we consider only lossless transmission lines}}$

Further discussion on the meaning of *k*

k in spatial domain
is the counterpart of
ω in temporal domain

Temporal domain	Spatial domain
$\omega = 2\pi / T$	$k = 2\pi / \lambda$
Unit: rad/s	Unit: rad/m
T is (temporal) period	λ is wavelength (or period in spatial domain)
ω means the number of revolutions in a period of 2π seconds, referred to as angular (temporal) frequency	k means the number of wavelengths in a distance of 2π meters, referred to as wavenumber (or spatial frequency)

4. Terminated Transmission Line



Note the two coordinate systems and their relation:

z = measuring from the left to the right

ℓ = measuring from the right to the left

$\boxed{\ell = -z}$

Note: Any position on T-Line has a nonnegative value of ℓ

Generally, Voltage and current along the line:

$$V(z) = V_0^+ e^{-jkz} + V_0^- e^{jkz}$$
$$I(z) = I_0^+ e^{-jkz} + I_0^- e^{jkz}$$

In the ℓ (ℓ = -z) coordinate system,

$$V_0^+ e^{jk\ell} + V_0^- e^{-jk\ell} = V(\ell)$$
$$I_0^+ e^{jk\ell} + I_0^- e^{-jk\ell} = I(\ell)$$

At the position of the load (ℓ = 0), we denote the voltage to be V_L and the current to be I_L. Then we have:

$$V_0^+ + V_0^- = V_L$$
$$\frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0} = I_L$$

$$\boxed{\frac{V_L}{I_L} = Z_L}$$

Two equations for two unknowns, the solution is given by:

$$V_0^+ = \frac{1}{2} I_L (Z_L + Z_0)$$

$$V_0^- = \frac{1}{2} I_L (Z_L - Z_0)$$

Putting the expressions for V_0^+ and V_0^- into the equations for the voltage and current, we have:

$$\boxed{\begin{aligned} V(\ell) &= \frac{1}{2} I_L \left[Z_L \left(e^{jk\ell} + e^{-jk\ell} \right) + Z_0 \left(e^{jk\ell} - e^{-jk\ell} \right) \right] \\ &= I_L \left[Z_L \cos(k\ell) + jZ_0 \sin(k\ell) \right] \end{aligned}}$$

$$\boxed{\begin{aligned} I(\ell) &= \frac{1}{2} \frac{I_L}{Z_0} \left[Z_L \left(e^{jk\ell} - e^{-jk\ell} \right) + Z_0 \left(e^{jk\ell} + e^{-jk\ell} \right) \right] \\ &= \frac{I_L}{Z_0} \left[Z_0 \cos(k\ell) + jZ_L \sin(k\ell) \right] \end{aligned}}$$

To summarize: Once knowing the current or voltage at the load, we then obtain current and voltage everywhere on T-Line.

Using $V(\ell)$ and $I(\ell)$, we can define the input impedance $Z(\ell)$ at an arbitrary point ℓ on the transmission line as:

$$\boxed{Z(\ell)=\frac{V(\ell)}{I(\ell)}=Z_0\frac{Z_L+jZ_0\tan(k\ell)}{Z_0+jZ_L\tan(k\ell)}}$$

Define a reflection coefficient at the load $l = 0$ as Γ_L:

$$\Gamma_L=\frac{\text{reflected voltage at } l=0}{\text{incident voltage at } l=0}$$

$$=\frac{V_0^- e^{-jk\times 0}}{V_0^+ e^{+jk\times 0}}=\frac{V_0^-}{V_0^+}=\frac{(1/2)I_L\left(Z_L-Z_0\right)}{(1/2)I_L\left(Z_L+Z_0\right)}$$

$$=\frac{Z_L-Z_0}{Z_L+Z_0}$$

In fact, we can further define a reflection coefficient $\Gamma(\ell)$ at any point ℓ on the transmission line by:

$$\Gamma(\ell) = \frac{\text{reflected voltage at point } \ell}{\text{incident voltage at point } \ell}$$

$$= \frac{V_0^- e^{-jk\ell}}{V_0^+ e^{jk\ell}} = \frac{V_0^-}{V_0^+} e^{-j2k\ell} = \Gamma_L e^{-j2k\ell}$$

Similar to the reflection coefficient at the load, we have

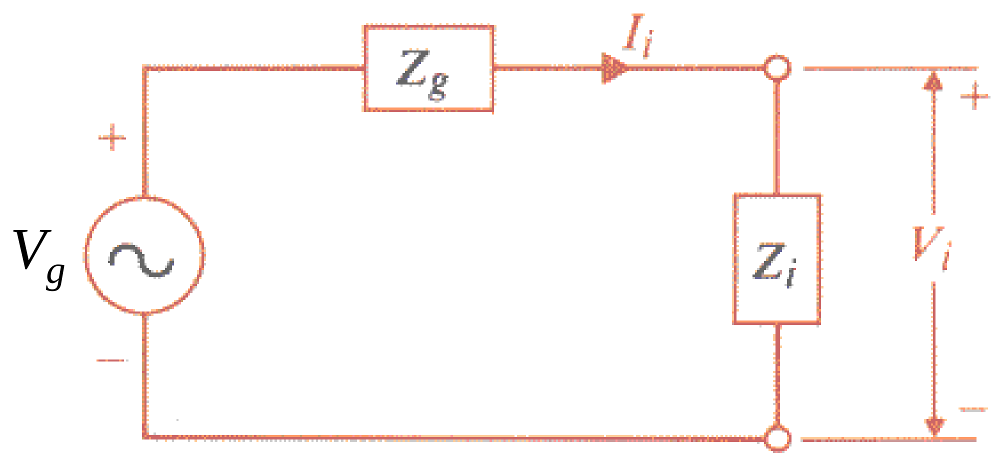
$$\boxed{\Gamma(\ell) = \frac{Z(\ell) - Z_0}{Z(\ell) + Z_0}}$$

At the position of the generator ($\ell = d$),

$$\boxed{Z_i = Z(\ell = d) = Z_0 \frac{Z_L + jZ_0 \tan(kd)}{Z_0 + jZ_L \tan(kd)}}$$

$$\boxed{\Gamma(\ell = d) = \Gamma_i = \frac{Z_i - Z_0}{Z_i + Z_0} = \Gamma_L e^{-j2kd}}$$

By using the input impedance Z_i, the T-Line problem becomes a traditional circuit problem: Voltage divider.



Applying KCL and KVL to circuit:

$$I(d) = I_i = \frac{V_g}{Z_g + Z_i}$$

$$V(d) = V_i = I_i Z_i = \frac{Z_i}{Z_g + Z_i} V_g$$

As obtained in Page 22, At an arbitrary position $z = l$,

$$\boxed{V(\ell) = I_L \left[Z_L \cos(k\ell) + jZ_0 \sin(k\ell) \right]}$$

$$\boxed{I(\ell) = \frac{I_L}{Z_0} \left[Z_0 \cos(k\ell) + jZ_L \sin(k\ell) \right]}$$

We have, letting $l=d$,

$$I_L = \frac{V_i}{Z_L \cos(kd) + jZ_0 \sin(kd)}$$

Thus, complete solutions for voltage and current on T-Line are obtained

To summarize:

The input impedance

$$\boxed{Z(\ell) = \frac{V(\ell)}{I(\ell)} = Z_0 \frac{Z_L + jZ_0 \tan(k\ell)}{Z_0 + jZ_L \tan(k\ell)}}$$

replaces the T-Line together with terminated load by an equivalent impedance.

Thus, the T-Line problem becomes a circuit problem, where KCL & KVL are applicable.

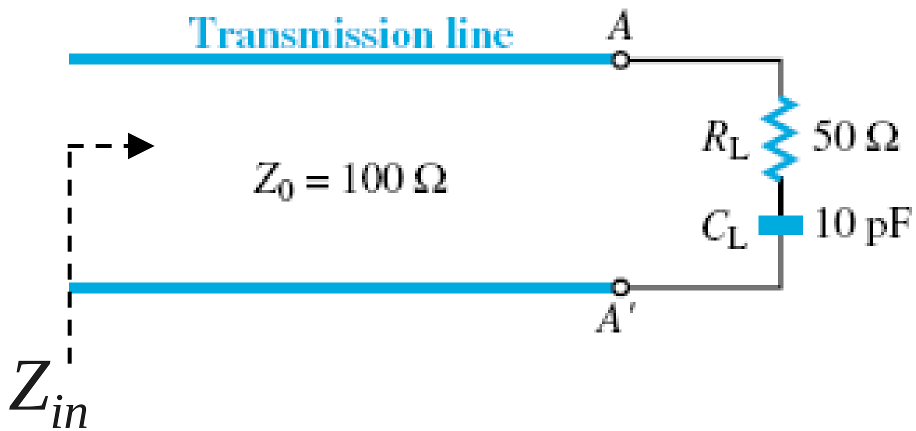
When there are multiple T-Lines, we apply input impedance multiple times, eventually yielding a circuit problem.

Example 1

A 100-Ω transmission line is connected to a load consisted of a 50-Ω resistor in series with a 10-pF capacitor.

(a) Find the reflection coefficient Γ_L at the load for a 100-MHz signal.

(b) Find the impedance Z_{in} at the input end of the transmission line if its length is 0.125λ.



Solutions

The following information is given

$$R_\mathrm{L} = 50\Omega,\quad C_\mathrm{L} = 10^{-11}\mathrm{F},\quad Z_0 = 100\Omega,\quad f = 100\mathrm{MHz} = 10^8\,\mathrm{Hz}$$

The load impedance is

$$Z_\mathrm{L} = R_\mathrm{L} - j/\omega C_\mathrm{L}$$

$$= 50 - j\frac{1}{2\pi\times 10^8\times 10^{-11}} = 50 - j159 \quad (\Omega)$$

(a) Reflection coefficient at the load is

$$\Gamma_L = \frac{Z_\mathrm{L}/Z_0 - 1}{Z_\mathrm{L}/Z_0 + 1} = \frac{0.5 - j1.59 - 1}{0.5 - j1.59 + 1} = 0.76\angle -60.70^\circ$$

(b) $d = 0.125\lambda$

Note: $k\lambda = 2\pi$

$$
\begin{aligned}
Z_{in} &= Z(\ell = 0.125\lambda) \\
&= Z_0 \frac{Z_L + jZ_0 \tan(\pi/4)}{Z_0 + jZ_L \tan(\pi/4)} \\
&= Z_0 \frac{Z_L + jZ_0}{Z_0 + jZ_L} \\
&= 14.3717 - j25.5544 \quad (\Omega) \\
&= 29.32\angle -60.65^\circ \quad (\Omega)
\end{aligned}
$$

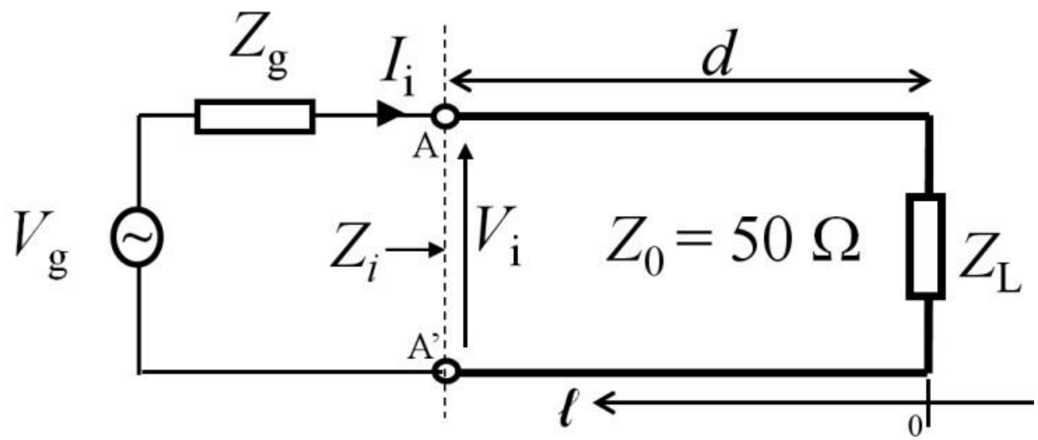
Example 2

A lossless transmission line with $Z_0 = 50\ \Omega$ and $d = 5.3\lambda$ connects a voltage V_g source to a terminal load of $Z_L = 100\ \Omega$. It is known that $V_g = 100$ V and $Z_g = 25\ \Omega$.

(a) Find the current I_i and voltage V_i at the left terminal of the transmission line.

(b) What are the current I_L and voltage V_L at the load?

(c) What are the current and voltage in the middle of transmission line?



Solutions

(a) The input impedance at the left terminal of the T-Line is

$$Z_i = Z_0 \frac{Z_L + jZ_0 \tan(kd)}{Z_0 + jZ_L \tan(kd)} = 50 \frac{100 + j50 \tan(5.3 \times 2\pi)}{50 + j100 \tan(5.3 \times 2\pi)} = 26.9 + j11.87\ \Omega$$

From the resultant voltage divider, we obtain

$$I_i = \frac{V_g}{Z_g + Z_i} = \frac{100}{25 + (26.9 + j11.87)} = 1.8301 - j0.4184\ \text{A}$$

$$V_i = I_i Z_i = 54.2480 + j10.4592\ \text{V}$$

(b) From Page 26, at $l=d$,

$$I_L = \frac{V_i}{Z_L \cos(kd) + jZ_0 \sin(kd)}$$

Plugging in numbers, we have

$$I_L = -0.3666 - j0.9026\ \text{A}$$

$$V_L = I_L Z_L = -36.6581 - j90.2576 \text{ V}$$

(c) Letting *l=d/2* and plugging numbers into

$$V(\ell) = I_L\left[Z_L\cos(k\ell) + jZ_0\sin(k\ell)\right]$$

$$I(\ell) = \frac{I_L}{Z_0}\left[Z_0\cos(k\ell) + jZ_L\sin(k\ell)\right]$$

we obtain

$$V(\ell = d/2) = -14.9629 + j67.8806 \text{ V}$$

$$I(\ell = d/2) = -1.2449 + j1.1237 \text{ A}$$

5. Voltage/current maxima and minima

$$
\begin{aligned}
V(\ell) &= V_0^+ e^{jk\ell} + V_0^- e^{-jk\ell} \\
&= V_0^+ e^{jk\ell}\left(1 + \frac{V_0^-}{V_0^+} e^{-j2k\ell}\right) \\
&= V_0^+ e^{jk\ell}\left(1 + \Gamma_L e^{-j2k\ell}\right)
\end{aligned}
$$

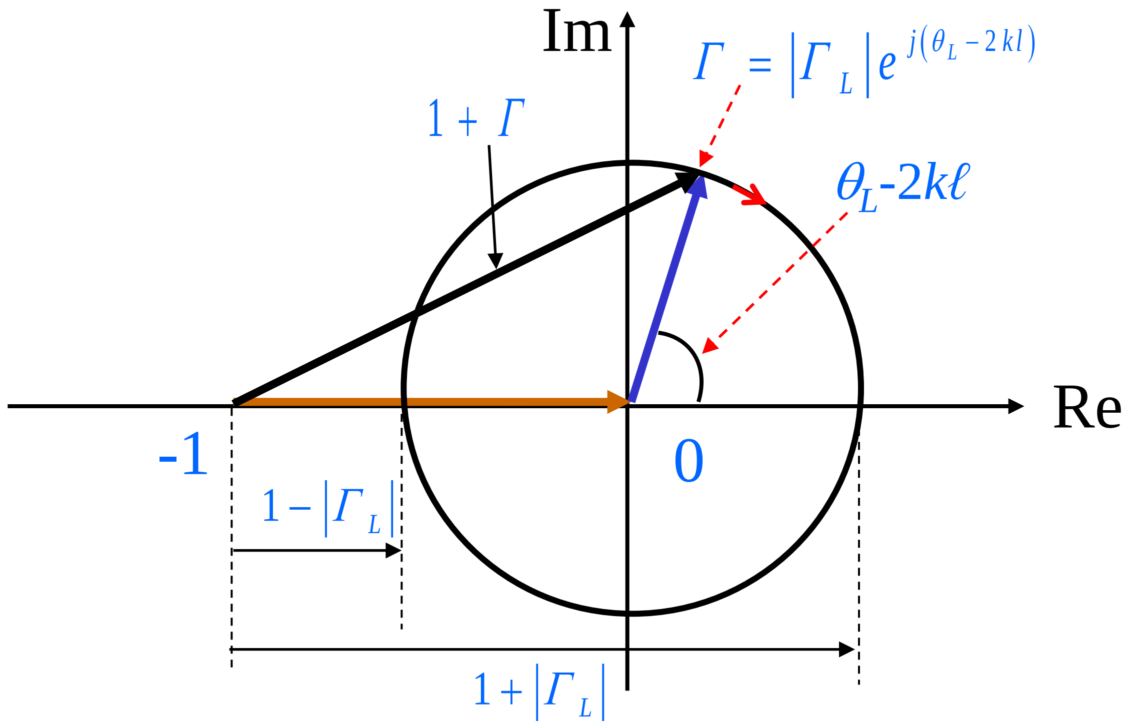
$$\boxed{\Gamma_L = |\Gamma_L| e^{j\theta_L}}$$

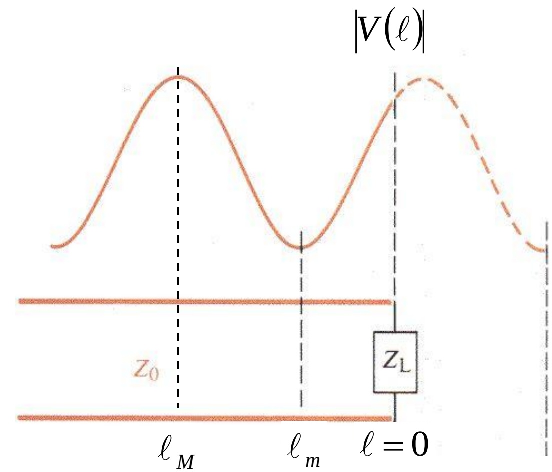
$$
\begin{aligned}
|V(\ell)| &= |V_0^+|\left|1 + \Gamma_L e^{-j2k\ell}\right| \\
&= |V_0^+|\left|1 + |\Gamma_L| e^{j(\theta_L - 2k\ell)}\right| \\
&= |V_0^+|\left|1 + \Gamma\right|
\end{aligned}
$$

$$\boxed{|\Gamma_L| \le 1}$$

$$\Gamma = |\Gamma_L| e^{j(\theta_L - 2k\ell)}$$

which is a complex number





$$\boxed{|1+\Gamma| = |\Gamma - (-1)| \text{ is the distance between } \Gamma \text{ and } -1 \text{ in complex plane}}$$

(1) When l increases, the angle $\theta_L - 2k\ell$ decreases, i.e., rotating clockwise.

(2) Since $k\lambda = 2\pi$, when l increases by λ, $|1+\Gamma|$ experiences 2 periods

since $2kl$ is in the exponent.

$|V(\ell)|$ is maximum when $|1+\Gamma| = (1+|\Gamma_L|)$

$$|V(\ell)|_{\max} \Rightarrow \quad \theta_L - 2k\ell = -2n\pi$$

$$\Rightarrow \quad \ell_M = \frac{\theta_L \lambda}{4\pi} + \frac{n\lambda}{2}, \quad \begin{cases} n = 0,1,2\ldots & \theta_L \geq 0 \\ n = 1,2\ldots & \theta_L < 0 \end{cases}$$

Note: θ_L has to be specified in the range $[-\pi, \pi)$.

$|V(\ell)|$ is minimum when $|1+\Gamma| = (1-|\Gamma_L|)$

$$|V(\ell)|_{\min} \Rightarrow \quad \theta_L - 2k\ell = -(2n+1)\pi$$

$$\Rightarrow \quad \ell_m = \frac{\theta_L \lambda}{4\pi} + \frac{(2n+1)\lambda}{4}, \quad n = 0,1,2,\cdots$$

Note: θ_L has to be specified in the range $[-\pi, \pi)$.

As current is
$$\left|I(\ell)\right| = \left|I_0^+\right|\left|1-\Gamma_L e^{-j2k\ell}\right|$$
$$= \left|\frac{V_0^+}{Z_0}\right|\left|1-\Gamma\right|$$

Current is maximum when voltage is minimum and minimum when voltage is maximum.

$$\boxed{\left|I(\ell)\right|_{\max} \text{ at } \ell_M = \frac{\theta_L \lambda}{4\pi} + \frac{(2n+1)\lambda}{4}, \quad n = 0, 1, 2, \cdots}$$

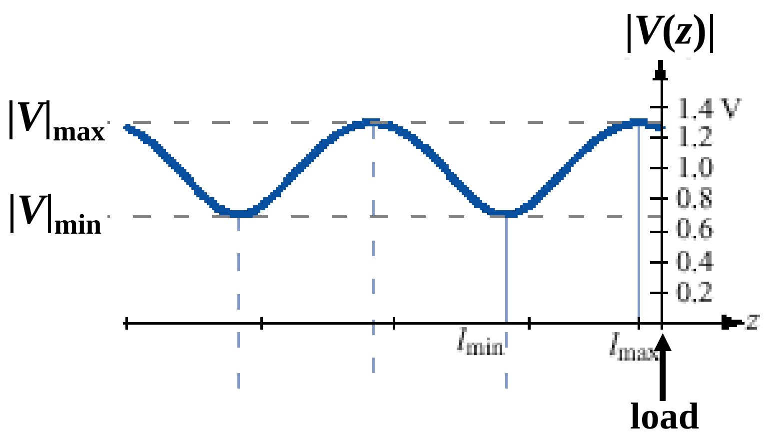
$$\boxed{\left|I(\ell)\right|_{\min} \text{ at } \ell_m = \frac{\theta_L \lambda}{4\pi} + \frac{n\lambda}{2}, \quad \begin{cases} n = 0, 1, 2 \ldots & \theta_L \geq 0 \\ n = 1, 2 \ldots & \theta_L < 0 \end{cases}}$$

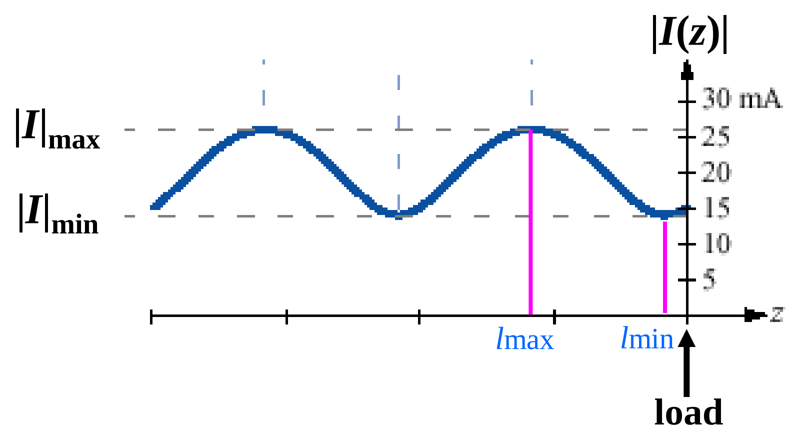
Define a voltage standing wave ratio (VSWR) as:

S = voltage standing wave ratio (VSWR)

$$= \frac{|V(\ell)|_{\max}}{|V(\ell)|_{\min}} = \frac{|V_0^+|(1+|\Gamma_L|)}{|V_0^+|(1-|\Gamma_L|)} = \frac{1+|\Gamma_L|}{1-|\Gamma_L|} \quad \text{(dimensionless)}$$

$$\boxed{|\Gamma_L| = \frac{S-1}{S+1}}$$





Special terminations

Γ_L	S	Z_L
0	1	$Z_L = Z_0$ (matched)
-1	∞	$Z_L = 0$ (short-circuited)
1	∞	$Z_L = \infty$ (open-circuited)

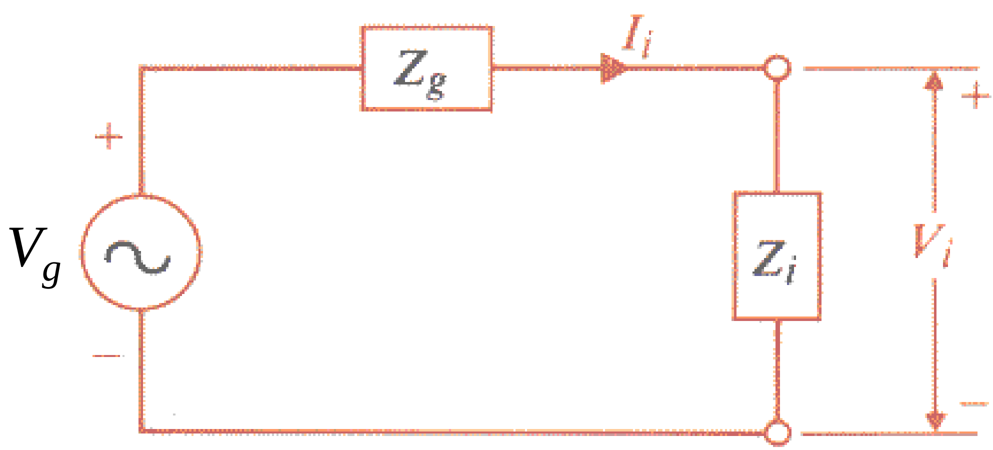
6. Power flow in a transmission line

Power flow at any point *z* on a transmission line is given by:

$$\boxed{P_{av}(z) = \frac{1}{2}\mathrm{Re}\{V(z)I^*(z)\}}$$

Since V and I correspond to E-field and H-field, the power flow corresponds to time-average Poynting power. It is not the power converted to heat by Joule's law.

At $z = -d$, the equivalent circuit is given by



$z = -d$

Power delivered by the source:

$$P_s = \frac{1}{2}\mathrm{Re}\left\{V_g I_i^*\right\}$$

Power dissipated in the source impedance Z_g:

$$P_{Z_g} = \frac{1}{2}\mathrm{Re}\left\{V_{Z_g} I_{Z_g}^*\right\} = \frac{1}{2}\mathrm{Re}\left\{Z_g I_i I_i^*\right\} = \frac{1}{2}\left|I_i\right|^2 \mathrm{Re}\left\{Z_g\right\}$$

Power input to the transmission line:

$$P_i = P_{av}\left(-d\right) = \frac{1}{2}\mathrm{Re}\left\{V\left(-d\right)I^*\left(-d\right)\right\}$$

$$= \frac{1}{2}\mathrm{Re}\left\{V_i I_i^*\right\} = \begin{cases} = \dfrac{1}{2}\mathrm{Re}\left\{Z_i I_i I_i^*\right\} = \dfrac{1}{2}\left|I_i\right|^2 \mathrm{Re}\left\{Z_i\right\} \\ = \dfrac{1}{2}\mathrm{Re}\left\{V_i \dfrac{V_i^*}{Z_i^*}\right\} = \dfrac{1}{2}\left|V_i\right|^2 \mathrm{Re}\left\{\dfrac{1}{Z_i}\right\} \end{cases}$$

Complex conjugate opertor: can be dropped off since

$$\mathrm{Re}\left\{\frac{1}{Z_i}\right\} = \frac{1}{\left|Z_i\right|}\mathrm{Re}\left\{\frac{1}{e^{j\theta_{Zi}}}\right\}$$

$$= \frac{1}{\left|Z_i\right|}\mathrm{Re}\left\{\cos\theta_{Z_i} - j\sin\theta_{Z_i}\right\}$$

$$= \frac{\cos\theta_{Z_i}}{\left|Z_i\right|}$$

and $\mathrm{Re}\left\{\dfrac{1}{Z_i^*}\right\}$

$$= \frac{1}{\left|Z_i\right|}\mathrm{Re}\left\{\cos\theta_{Z_i} + j\sin\theta_{Z_i}\right\}$$

$$= \frac{\cos\theta_{Z_i}}{\left|Z_i\right|}$$

Power dissipated in the terminal impedance:

$$P_L = P_{av}(0) = \frac{1}{2}\mathrm{Re}\left\{V(0)\,I^*(0)\right\}$$

$$= \frac{1}{2}\mathrm{Re}\left\{V_L I_L^*\right\} = \begin{cases} = \dfrac{1}{2}\left|I_L\right|^2 \mathrm{Re}\left\{Z_L\right\} \\ = \dfrac{1}{2}\left|V_L\right|^2 \mathrm{Re}\left\{\dfrac{1}{Z_L}\right\} \end{cases}$$

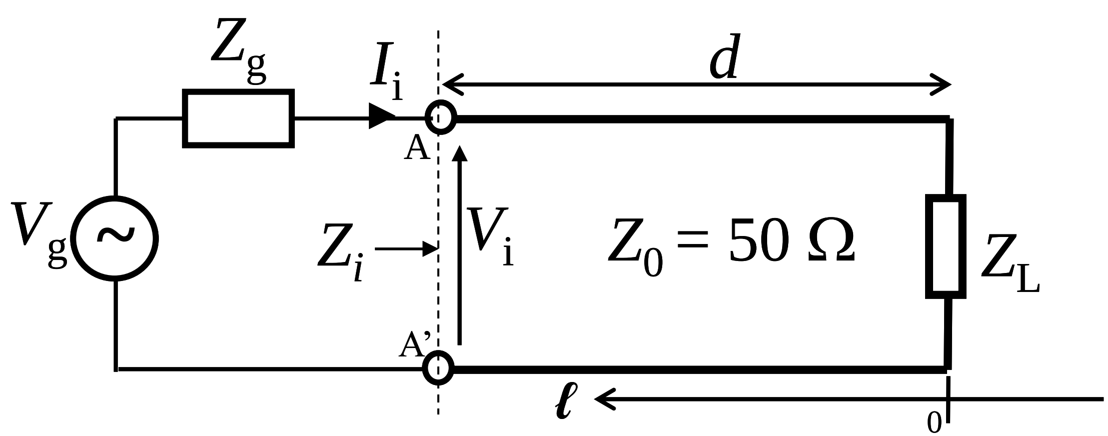
By the principle of conservation of power:

$$\boxed{\begin{aligned} P_s &= P_{Z_g} + P_i \\ P_i &= P_L \end{aligned}}$$

We consider only lossless T-lines

Example 3

A lossless transmission line with Z_0 = 50 Ω and d =1.5 m connects a voltage V_g source to a terminal load of Z_L = (50 + j50) Ω. If V_g = 60 V, operating frequency f = 100 MHz, and Z_g = 50 Ω, find the distance of the first voltage maximum ℓ_M from the load. What is the power delivered to the load P_L? Assume the speed of the wave along the transmission line equal to speed of light, c.



Solutions

The following information is given:

$$Z_0 = 50\Omega, \quad d = 1.5 \text{ m},$$

$$V_g = 60 \text{ V}, \;\; Z_g = 50\Omega, \;\; Z_L = 50 + j50\Omega,$$

$$f = 100\text{MHz} = 10^8\text{Hz}$$

$$u_p = c \quad \Rightarrow \quad \lambda = \frac{c}{10^8} = 3 \text{ m}$$

The reflection coefficient at the load is:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{50 + j50 - 50}{50 + j50 + 50} = 0.2 + j0.4 = 0.45e^{j1.11}$$

Therefore, $\left|\Gamma_L\right| = 0.45, \quad \theta_L = 1.11\,\text{rad}$

Then, $$\ell_M = \frac{\theta_L \lambda}{4\pi} + \frac{n\lambda}{2}, \quad \text{when } n = 0$$

$$= \frac{1.11\lambda}{4\pi} = 0.09\lambda = 0.27\,\text{m} \quad \text{(from the load)}$$

The input impedance Z_i looking at the input to the transmission line is:

$$Z_i = Z_0 \frac{Z_L + jZ_0 \tan(kd)}{Z_0 + jZ_L \tan(kd)}$$

$$Z_i = 50\frac{50 + j50 + j50\tan\left(\frac{2\pi}{3}\times 1.5\right)}{50 + j(50 + j50)\tan\left(\frac{2\pi}{3}\times 1.5\right)} = 50 + j50\Omega$$

The current at the input to the transmission line is :

$$I_i = \frac{V_g}{Z_g + Z_i} = \frac{60}{50 + 50 + j50} = 0.48 - j0.24\text{ A}$$

As the transmission line is lossless, power delivered to the load P_L is equal to the power input to the transmission line P_i. Hence,

$$P_L = P_i = \frac{1}{2}\left|I_i\right|^2 \mathrm{Re}\{Z_i\} = \frac{1}{2}\times 0.288\times 50 = 7.2\text{ W}$$

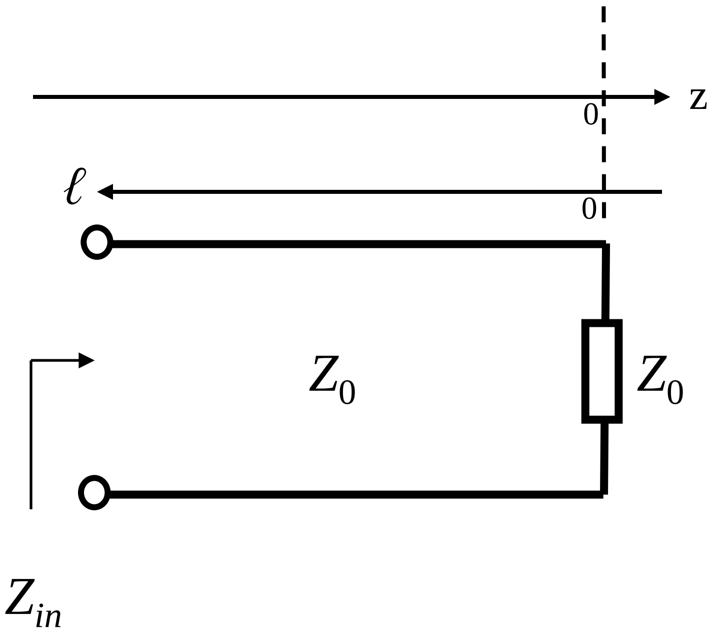
7. Special Cases of Terminations in a Transmission Line

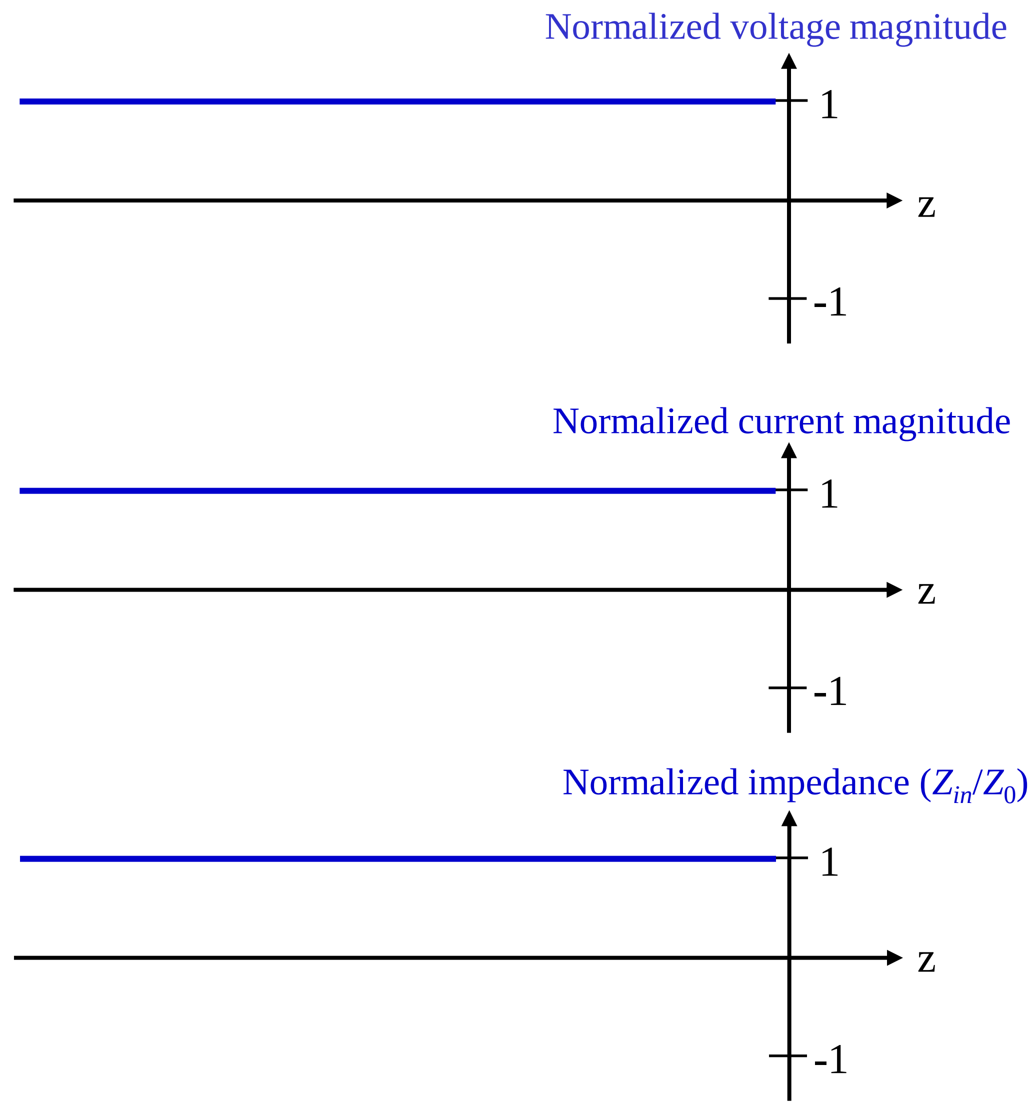
7.1 Matched line

For a matched line, $Z_L = Z_0$. Then,

$$\left.\begin{aligned} Z(\ell) &= Z_0 \frac{Z_0 + jZ_0 \tan(k\ell)}{Z_0 + jZ_0 \tan(k\ell)} = Z_0 \\ \Gamma(\ell) &= \frac{Z(\ell) - Z_0}{Z(\ell) + Z_0} = 0 \end{aligned}\right\} \text{for any length } \ell \text{ of the line}$$

Thus, there is no reflection on a matched line. There is only an incident voltage.





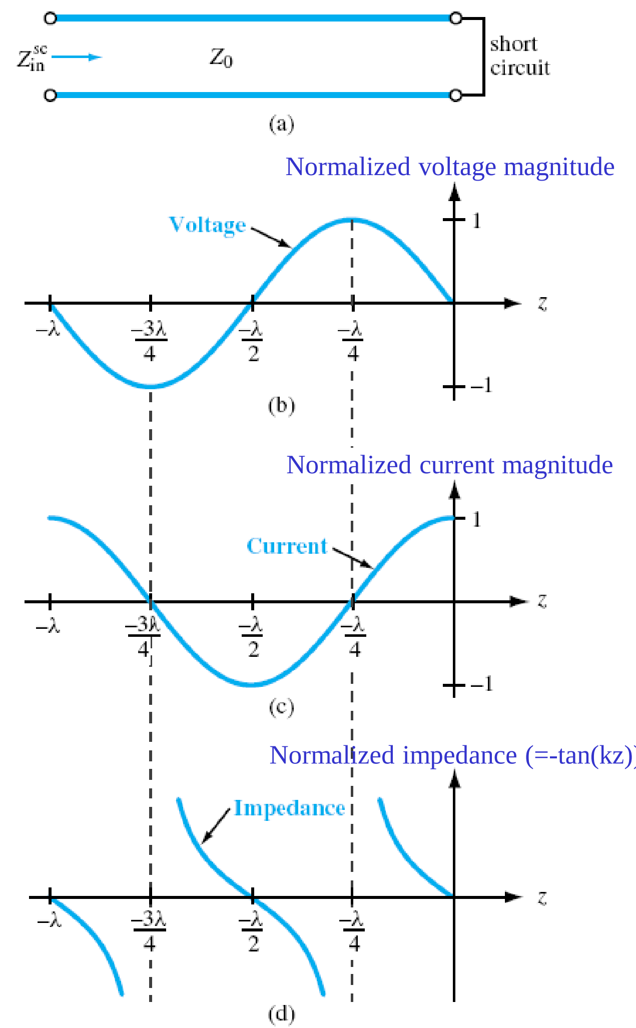
Note:
Normalized voltage = voltage/max. |voltage|
Normalized current = current/max. |current|

7.2 Short-circuited line

For a short circuit, $Z_L = 0$. Then

$$Z_{\text{in}}^{\text{sc}} = jZ_0 \tan(k\ell) = -jZ_0 \tan(kz)$$

Note ℓ =-z

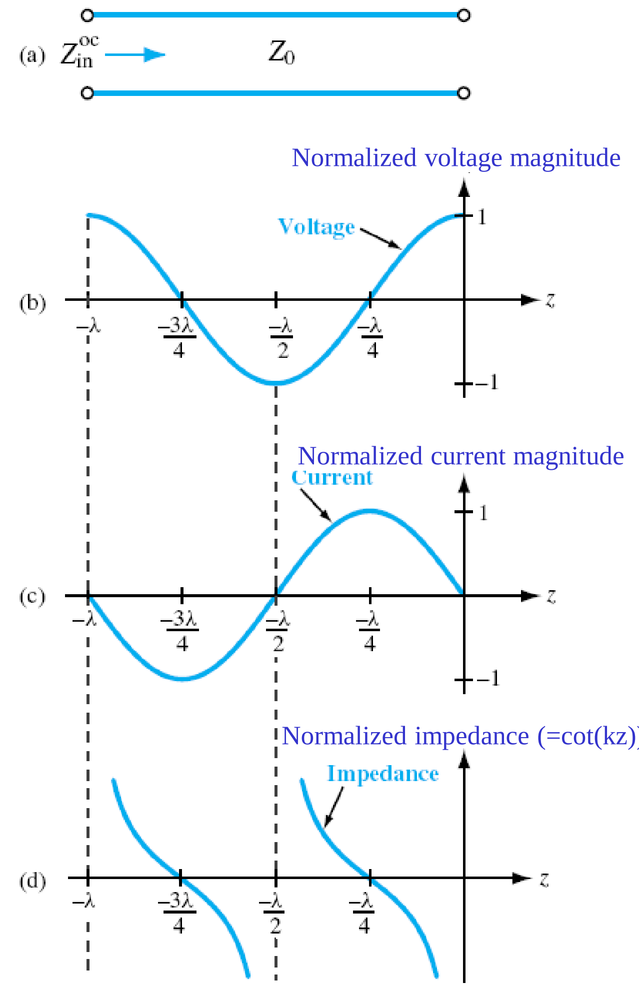


7.3 Open-circuited line

For an open circuit, $Z_L = \infty$. Then

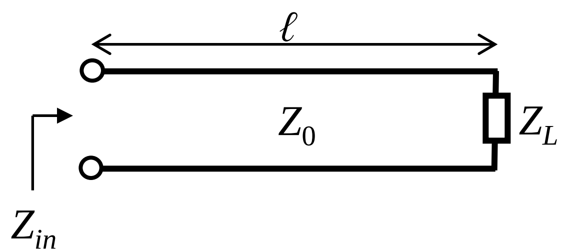
$$Z_{\text{in}}^{\text{oc}} = -jZ_0 \cot(k\ell) = jZ_0 \cot(kz)$$

$\boxed{\text{Note } \ell = -z}$



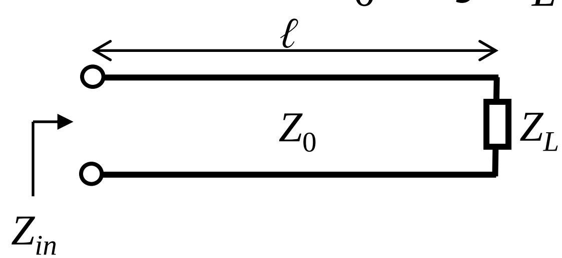
7.4 λ/4 transmission line terminated in Z_L

$$Z_{in} = Z(\ell = \lambda/4) = Z_0 \frac{Z_L + jZ_0 \tan(\pi/2)}{Z_0 + jZ_L \tan(\pi/2)} = \frac{Z_0^2}{Z_L}$$



7.5 λ/2 transmission line terminated in Z_L

$$Z_{in} = Z(\ell = \lambda/2) = Z_0 \frac{Z_L + jZ_0 \tan(\pi)}{Z_0 + jZ_L \tan(\pi)} = Z_L$$



Example 4

The open-circuit and short-circuit impedances measured at the input terminals of a lossless transmission line of length 1.5 m (which is less than a quarter wavelength) are $-j54.6\ \Omega$ and $j103\ \Omega$, respectively.

(a) Find Z_0 and k of the line.

(b) Without changing the operating frequency, find the input impedance of a short-circuited line that is twice the given length.

(c) How long should the short-circuited line be in order for it to appear as an open circuit at the input terminals?

Solution

The given quantities are

$$Z_{\text{in}}^{\text{oc}} = -j54.6\,\Omega \qquad Z_{\text{in}}^{\text{sc}} = j103\,\Omega \qquad \ell = 1.5\text{m}$$

(a) $Z_{\text{in}}^{\text{sc}} = jZ_0 \tan(k\ell)$

$$Z_{\text{in}}^{\text{oc}} = -jZ_0 \cot(k\ell)$$

$$Z_0 = \sqrt{Z_{\text{in}}^{\text{oc}} Z_{\text{in}}^{\text{sc}}} = 75\,\Omega$$

$$k = \frac{1}{\ell}\tan^{-1}\sqrt{-Z_{\text{in}}^{\text{sc}}/Z_{\text{in}}^{\text{oc}}} = 0.628 \ \ \text{rad/m}$$

$$\lambda = \frac{2\pi}{k} = 10\text{m}$$

(b) For a line twice as long, ℓ = 3 m and $k\,\ell$ =1.884 rad,

$$Z_{\text{in}}^{\text{sc}} = jZ_0 \tan k\ell = -j232\,\Omega$$

(c) Short circuit input impedance

$$Z_{\text{in}}^{\text{sc}} = jZ_0 \tan\left(k\ell\right)$$

For open circuit $Z_{\text{in}}^{\text{sc}} = \infty, \quad \Rightarrow \quad k\ell = \pi/2 + n\pi, \quad n = 0, 1, 2, \cdots$

$$\ell = \frac{\pi/2 + n\pi}{k} = \frac{2n+1}{4}\lambda$$

❑ Textbooks:

– *Fundamentals of Applied Electromagnetics*

F. T. Ulaby, E. Michielssen, U. Ravaioli,

Pearson Education, 2010, 6th edition

Suggested reading [textbook]:

– Page 62
– Section 2-2: Lumped-Element Model
– Section 2-3: Transmission Line Equations
– Section 2-4: Wave Propagation on a Transmission Line
– Section 2-6: The Lossless Transmission Line: General Considerations
– Section 2-7: Wave Impedance of the Lossless Line
– Section 2-8: Special Cases of the Lossless Line

Optional reading [textbook]:

– Section 2-1.1: The Role of Wavelength
– Section 2-1.2: Propagation Modes