Question:

(1) How to show $\delta(\beta x)=\frac{1}{\beta}\delta(x)$? (2) How do we convert between $\delta(f)$ and $\delta(\omega)$?

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(1) Answer:

From Tutorial 1 Q.6

Let $\boldsymbol{\beta} > \mathbf{0}$.

We note that $\delta(\beta x)$ is essentially the unit impulse $\delta(x)$ with its time-width compressed by a factor of β. Since the area under $\delta(x)$ is 1, the area under $\delta(\beta x)$ must then be $1/\beta$. This leads directly to the relationship

$$\boldsymbol{\delta(\beta x)=\frac{1}{\beta}\delta(x)}.$$

Mathematically:

$\delta(\beta x)=\begin{cases}\infty; & x=0\\ 0; & x\neq 0\end{cases}$ implies that $\delta(\beta x)$ is an impuse function. We may thus write $\delta(\beta x)=A\delta(x)$ where $\delta(x)$ is the unit impulse function, and A is the area under $\delta(\beta x)$ given by

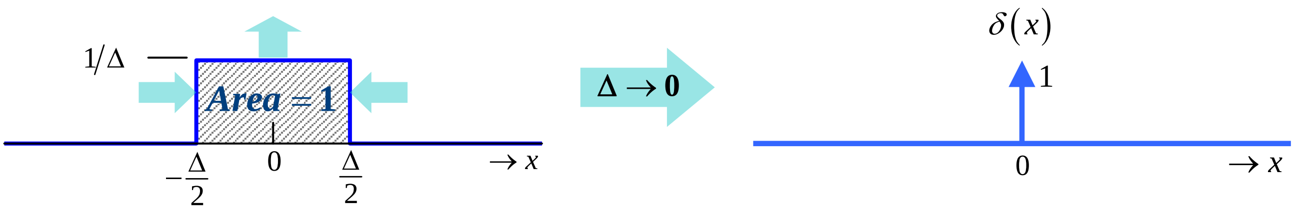
$$A=\underbrace{\int_{-\infty}^{\infty}\delta(\beta x)dx=\int_{-\infty}^{\infty}\frac{1}{\beta}\delta(\zeta)d\zeta}_{\text{apply change of variable: }\zeta=\beta x}=\frac{1}{\beta}$$

This shows that $\boldsymbol{\delta(\beta x)=\frac{1}{\beta}\delta(x)}$.

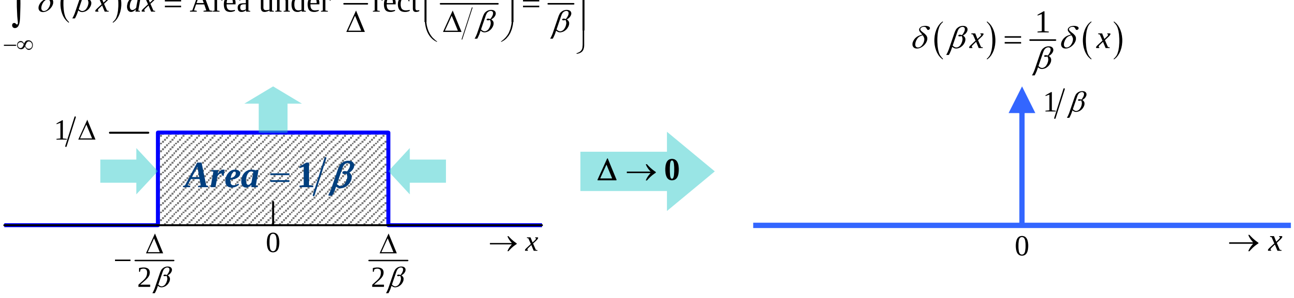
Alternate Explanation

Let $\boldsymbol{\beta} > \mathbf{0}$ and $\Delta > \mathbf{0}$.

$\delta(x)=\lim_{\Delta\to 0}\left(\frac{1}{\Delta}\text{rect}\left(\frac{x}{\Delta}\right)\right)=\begin{cases}\infty; & x=0\\ 0; & x\neq 0\end{cases}$ and $\int_{-\infty}^{\infty}\delta(x)dx=\text{Area under }\frac{1}{\Delta}\text{rect}\left(\frac{x}{\Delta}\right)=1$.



$$\left.\begin{aligned}&\delta(\beta x)=\lim_{\Delta\to 0}\left(\frac{1}{\Delta}\text{rect}\left(\frac{x}{\Delta/\beta}\right)\right)=\begin{cases}\infty; & x=0\\ 0; & x\neq 0\end{cases}\\ &\int_{-\infty}^{\infty}\delta(\beta x)dx=\text{Area under }\frac{1}{\Delta}\text{rect}\left(\frac{x}{\Delta/\beta}\right)=\frac{1}{\beta}\end{aligned}\right\}\text{ implies that }\delta(\beta x)=\frac{1}{\beta}\delta(x)$$



(2) Answer:

Domain-scaling of $\delta(\cdot)$ is often encountered in transforming a spectrum containing $\delta(\cdot)$ between cyclic-frequency (f) and radian frequency (ω) domain:

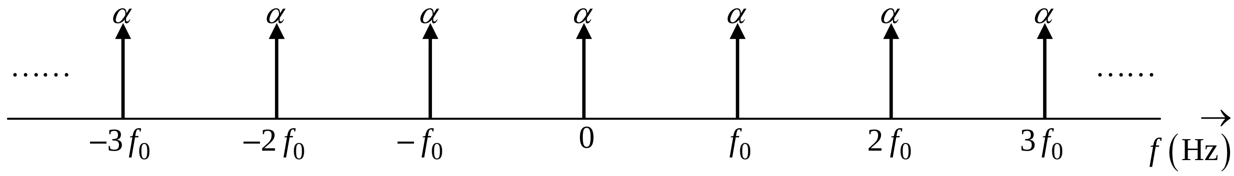
Using the result $\delta(\beta x) = \frac{1}{\beta}\delta(x)$, we get

$$\left[\delta(\omega) = \delta(2\pi f) = \frac{1}{2\pi}\delta(f)\right] \text{ or } \left[\delta(f) = \delta\left(\frac{\omega}{2\pi}\right) = 2\pi\delta(\omega)\right].$$

♦ ***Example:***

Converting the spectrum $\alpha\sum_k \delta(f - kf_0)$ from cyclic frequency (f) to angular frequency (ω).

$$\alpha\sum_k \delta(f - kf_0)$$



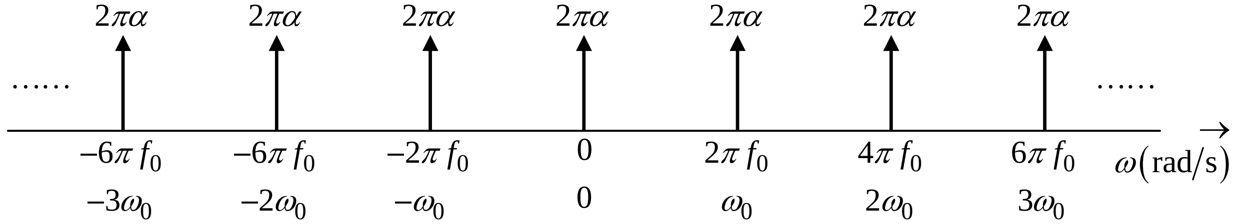
$$\alpha\sum_k \delta(f - kf_0) = \alpha\sum_k \delta\left(\frac{\omega}{2\pi} - kf_0\right) \quad \leftarrow \cdots\cdots \textit{ letting } f = \frac{\omega}{2\pi}$$

$$= \alpha\sum_k \delta\left(\frac{\omega - k2\pi f_0}{2\pi}\right)$$

$$\cdots\cdots \textit{ using } \quad \delta(\beta x) = \frac{1}{\beta}\delta(x)$$

$$= 2\pi\alpha\sum_k \delta(\omega - k2\pi f_0) = 2\pi\alpha\sum_k \delta(\omega - k\omega_0)$$

$$2\pi\alpha\sum_k \delta(\omega - k\omega_0)$$



♦ **Interpretation of $\boldsymbol{\delta(f) = 2\pi\delta(\omega)}$ and $\boldsymbol{\delta(\omega) = \frac{1}{2\pi}\delta(f)}$**

<u>CORRECT</u> INTERPRETATION

$\boldsymbol{\delta(f) = 2\pi\delta(\omega)}$**:** A unit impulse on the f-*axis* is equal to an impulse of strength 2π on the ω-*axis*.

$\boldsymbol{\delta(\omega) = \frac{1}{2\pi}\delta(f)}$**:** A unit impulse on the ω-*axis* is equal to an impulse of strength $\frac{1}{2\pi}$ on the f-*axis*.

<u>WRONG</u> INTERPRETATION

$\boldsymbol{\delta(f) = 2\pi\delta(\omega)}$: If $\delta(\omega)$ is a unit impulse on the ω-*axis*, then $2\pi\delta(\omega)$ is an impulse of strength 2π on the f-*axis*. This interpretation obviously contradicts the equation because the left-hand-side of the equation is $\delta(f)$, which is a unit impulse on the f-*axis*.

$\boldsymbol{\delta(\omega) = \frac{1}{2\pi}\delta(f)}$: If $\delta(f)$ is a unit impulse on the f-*axis*, then $\frac{1}{2\pi}\delta(f)$ is an impulse of strength $\frac{1}{2\pi}$ on the ω-*axis*. This interpretation obviously contradicts the equation because the left-hand-side of the equation is $\delta(\omega)$, which is a unit impulse on the ω-*axis*.
