Why we use hyperbolic functions

To make our life easier

Examples:

(1) In Chapter 2, (Gp B slide 55),

we have

$$\theta = -\frac{\varepsilon}{2}\left[e^{(\sqrt{g/L})t} + e^{-(\sqrt{g/L})t} \right] + \pi$$

Now if $\theta = \pi - 2\varepsilon$ find t.

Subst above into the equation, get

$$2 = \frac{1}{2}\left[e^{(\sqrt{g/L})t} + e^{-(\sqrt{g/L})t} \right]$$

Then we need to solve the above equation to find t.

Suppose that we use hyperbolic function, we get

$$2 = \cosh(t\sqrt{g / L})$$

$$t = (\sqrt{L / g})\cosh^{-1}(2) = (\sqrt{L / g})1.3170$$

So it is easier.

(2) In Chapter 2,(GP B slides 36, 37)

$$x = \frac{1}{2}\alpha(e^{\omega t} + e^{-\omega t}) \quad y = \frac{1}{2}\omega\alpha(e^{\omega t} - e^{-\omega t})$$

We can verify that

$$\left(\frac{x}{\alpha}\right)^2 - \left(\frac{y}{\alpha\omega}\right)^2 = 1$$

However If we use hyperbolic function, then it is easier.

$$\frac{x}{\alpha} = \sinh(\omega t) \qquad \frac{y}{\alpha\omega} = \cosh(\omega t)$$

By formula $(\cosh(\omega t))^2 - (\sinh(\omega t))^2 = 1$

we have
$$\left(\frac{x}{\alpha}\right)^2 - \left(\frac{y}{\alpha\omega}\right)^2 = 1$$

(3) $$\int \frac{1}{\sqrt{u^2+a^2}}du = \sinh^{-1}\left(\frac{u}{a}\right)$$

$$\int \frac{1}{\sqrt{u^2+a^2}}du = \ln(x+\sqrt{x^2+1})$$ 2nd version

$$\int \frac{1}{\sqrt{u^2-a^2}}du = \cosh^{-1}\left(\frac{u}{a}\right)$$

$$\int \frac{1}{\sqrt{u^2-a^2}}du = \ln(x+\sqrt{x^2-1})$$ 2nd version

To evaluate the above integrals,

use the version of hyperbolic functions is easier.

$$\int_2^3 \frac{1}{\sqrt{u^2-1}}du = \cosh^{-1}(3) - \cosh^{-1}(2)$$

$$= 1.7627 - 1.3170 = 0.4457$$

(4) $\ddot{x} - \omega^2 x = 0$

has two linearly indep solutions, namely,

$$e^{\omega t} \qquad e^{-\omega t}$$

Hence $\dfrac{1}{2}e^{\omega t} \qquad \dfrac{1}{2}e^{-\omega t}$

are again solutions.

Thus, by superposition principle,

$$\sinh \omega t = \frac{1}{2}(e^{\omega t} - e^{-\omega t}) \qquad \cosh \omega t = \frac{1}{2}(e^{\omega t} + e^{-\omega t})$$

are two solutions.

In fact they are linearly indep.,

i.e., the curves of these two solutions are not parallel.

Hence every solution can be represented by

$$x = A \sinh \omega t + B \cosh \omega t$$

This is another version of the general solution of

$$\ddot{x} - \omega^2 x = 0$$

Suppose $x(0) = \alpha, \dot{x}(0) = 0$

Then $\alpha = A\sinh 0 + B\cosh 0 = B$

$$\dot{x} = A\omega\cosh\omega t + B\omega\sinh\omega t$$

$$0 = A\omega\cosh 0 + B\omega\sinh 0 = A\omega$$

So $A = 0$

Hence $x = \alpha\cosh\omega t \qquad \dot{x} = \alpha\omega\sinh\omega t$

Then $\left(\dfrac{x}{\alpha}\right)^2 - \left(\dfrac{\dot{x}}{\alpha\omega}\right)^2 = (\cosh\omega t)^2 - (\sinh\omega t)^2 = 1$