3 – IMAGE TRANSFORMS (B)

THE DISCRETE FOURIER TRANSFORM

Suppose that a continuous function $f(x)$ is discretized into a sequence

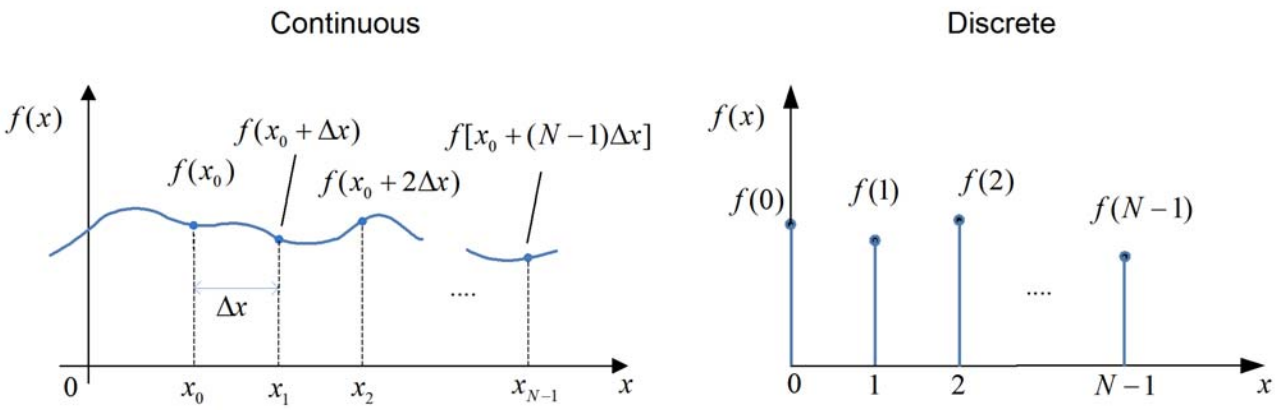
$$\{f(x_0), f(x_0+\Delta x), f(x_0+2\Delta x), \ldots, f(x_0+[N-1]\Delta x)\}$$

by taking N samples Δx units apart.
Note that x may be used as either a discrete or continuous variable, depending on context. In the discrete case, we define

$$f(x) = f(x_0 + x\Delta x) \qquad x = 0, 1, 2, \ldots, N-1$$

i.e.,

$$\begin{aligned} f(0) &= f(x_0) \\ f(1) &= f(x_0+\Delta x) \\ f(2) &= f(x_0+2\Delta x) \\ f(3) &= f(x_0+3\Delta x) \\ \cdots & \\ f(N-1) &= f[x_0+(N-1)\Delta x] \end{aligned}$$



The discrete Fourier transform (DFT) pair is given by

$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\exp[-j2\pi ux/N]; \quad u = 0,1,2,\ldots,N-1 \tag{1}$$

$$f(x) = \sum_{u=0}^{N-1} F(u)\exp[j2\pi ux/N]; \quad x = 0,1,2,\ldots,N-1 \tag{2}$$

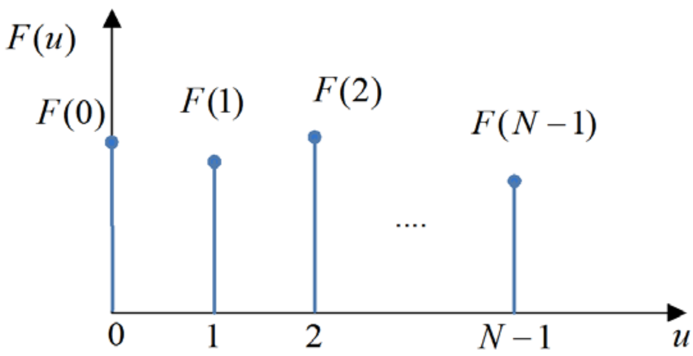
The terms Δu and Δx are related by

$$\Delta u = \frac{1}{N\Delta x} \tag{3}$$

$$F(u) = F(u\Delta u) \qquad u = 0,1,2,\ldots,N-1$$

i.e.,

$$\begin{aligned} F(0) &= F(0) \\ F(1) &= F(\Delta u) \\ F(2) &= F(2\Delta u) \\ F(3) &= F(3\Delta u) \\ &\cdots \\ f(N-1) &= f((N-1)\Delta u) \end{aligned}$$



In the 2D case, the DFT pair is

$$F(u,v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) \exp[-j2\pi(ux/M + vy/N)]; \tag{4}$$

$$u = 0,1,2,\ldots,M-1,\ v = 0,1,2,\ldots,N-1,$$

and

$$f(x,y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) \exp[j2\pi(ux/M + vy/N)]; \tag{5}$$

$$\text{for } x = 0,1,2,\ldots,M-1,\ y = 0,1,2,\ldots,N-1.$$

The sampling increments in the spatial and frequency domains are related by

$$\Delta u = \frac{1}{M\Delta x}, \qquad \Delta v = \frac{1}{N\Delta y} \tag{6}$$

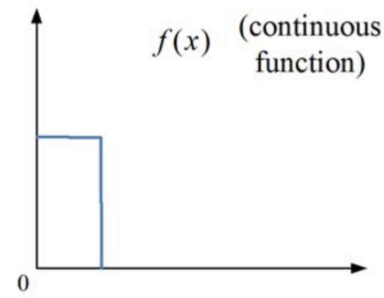
The Fourier spectrum, phase, and power spectrum of 1D and 2D discrete functions are computed as for the continuous case.

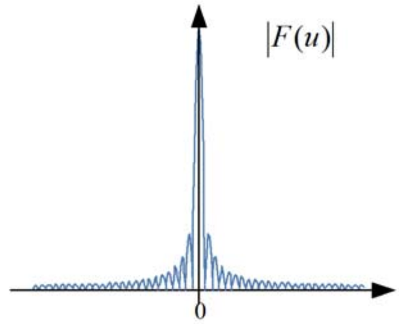
$$\begin{array}{lll} \text{Fourier spectrum:} & |F(u,v)| & = [R^2(u,v) + I^2(u,v)]^{1/2} \\ \text{Phase spectrum:} & \phi(u,v) & = \tan^{-1}[I(u,v)/R(u,v)] \\ \text{Power spectrum:} & P(u,v) & = |F(u,v)|^2 = R^2(u,v) + I^2(u,v) \end{array}$$

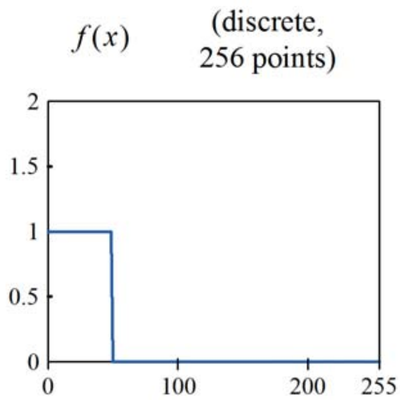
The direct computation of an N-point DFT requires of the order of N^2 operations. For an $M \times N$ array, M^2N^2 operations are required. This can be considerably reduced by the fast Fourier transform (FFT) algorithm to $MN \log_2 M \log_2 N$ operations. Suppose $M = N = 2^9$. Then

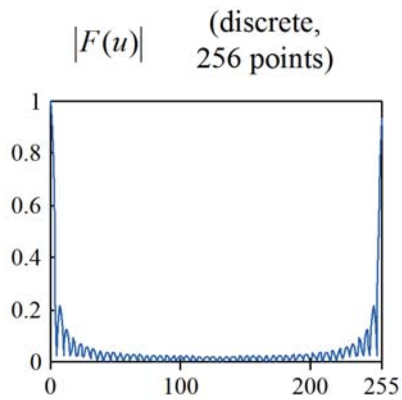
$$\begin{array}{lrl} \text{Direct DFT:} & M^2N^2 & = 69 \times 10^9 \\ \text{FFT:} & MN \log_2 M\ \log_2 N & = 21 \times 10^6 \end{array}$$

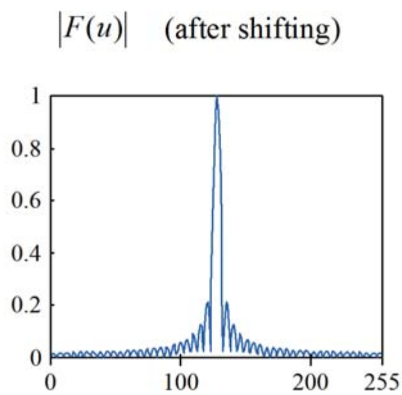
Example (1D DFT)



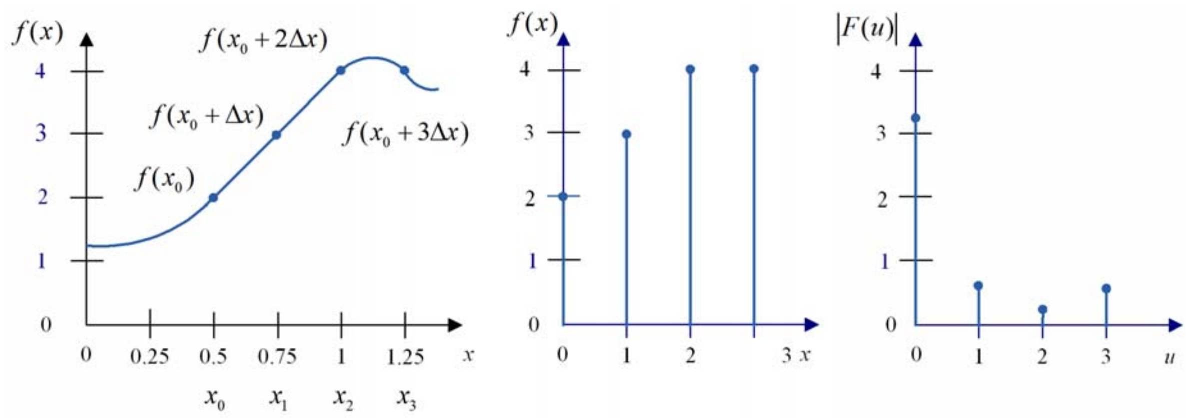








Example (1D DFT)



Sampling takes place at $x_0 = 0.5$, $x_1 = 0.75$, $x_2 = 1.0$, $x_2 = 1.25$, i.e., $f(0) = 2$, $f(1) = 3$, $f(2) = 4$, $f(3) = 4$.

$$
\begin{aligned}
F(u) &= \frac{1}{N}\sum_{x=0}^{N-1} f(x)\exp[-j2\pi ux/N] = \frac{1}{4}\sum_{x=0}^{3} f(x)\exp[-j2\pi ux/4] \\
F(0) &= \frac{1}{4}\sum_{x=0}^{3} f(x)\exp(0) \\
&= \frac{1}{4}[f(0)+f(1)+f(2)+f(3)] = \frac{1}{4}[2+3+4+4] \\
&= 3.25 \quad \text{or} \quad 3.25\angle 0^\circ \\
F(1) &= \frac{1}{4}\sum_{x=0}^{3} f(x)\exp[-j2\pi x/4] \\
&= \frac{1}{4}[f(0)\exp(0)+f(1)\exp(-j\pi/2)+f(2)\exp(-j\pi)+f(3)\exp(-j3\pi/2)] \\
&= \frac{1}{4}(-2+j) \quad \text{or} \quad 0.56\angle 153^\circ
\end{aligned}
$$

Similarly,

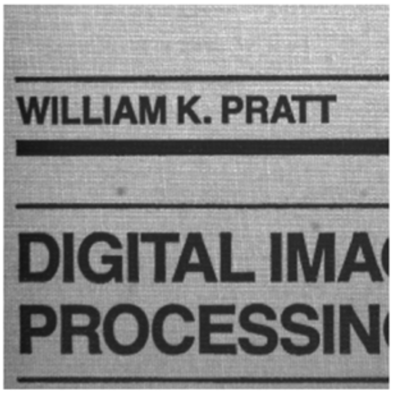
$$F(2) = -\frac{1}{4}(1+j\times 0) \quad \text{or} \quad 0.25\angle 180^\circ, \qquad F(3) = -\frac{1}{4}(2+j) \quad \text{or} \quad 0.56\angle{-153^\circ}$$

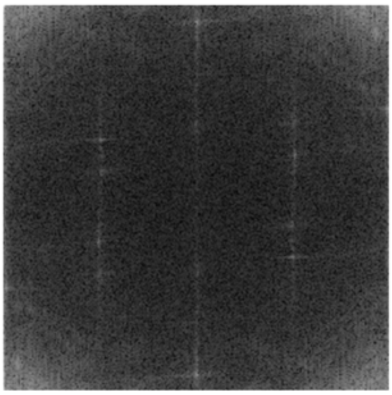
Fourier spectrum:

$$|F(0)| = 3.25, \quad |F(1)| = 0.56 \quad |F(2)| = 0.25 \quad |F(3)| = 0.56$$

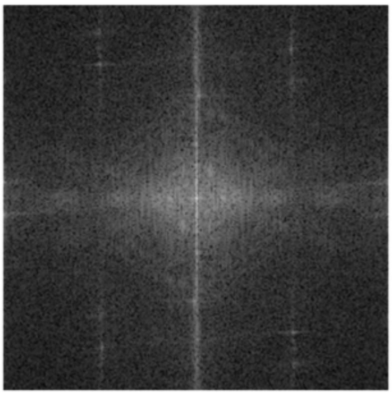
In this example,
$N = 4, \qquad \Delta x = 0.25, \qquad \Delta u = 1/(N\Delta x) = 1$

Examples of 2D DFT

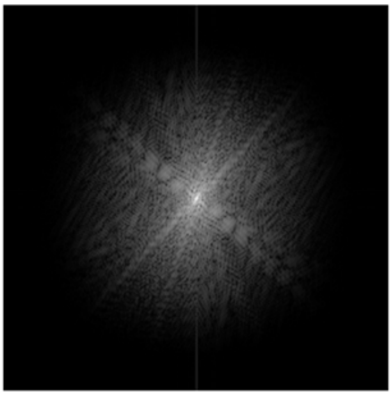


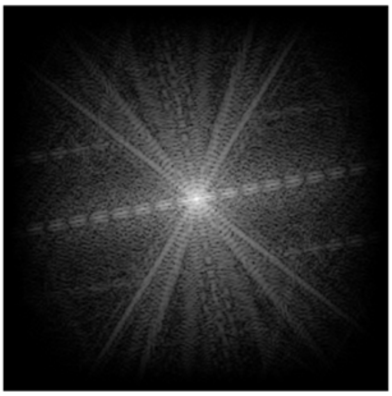


Without shifting



With shifting





Some Properties of the 2D DFT

Separability

We can write Eq. (4) in the separable form

$$\begin{aligned} F(u,v) &= \frac{1}{M}\sum_{x=0}^{M-1}\exp[-j2\pi ux/M]\times \\ &\qquad \left\{\frac{1}{N}\sum_{y=0}^{N-1} f(x,y)\exp[-j2\pi vy/N]\right\} \quad (7)\\ &= \frac{1}{M}\sum_{x=0}^{M-1}\exp[-j2\pi ux/M]\times\{F_r(x,v)\} \quad (8)\\ &= \frac{1}{M}\sum_{x=0}^{M-1} F_r(x,v)\exp[-j2\pi ux/M] \quad (9)\end{aligned}$$

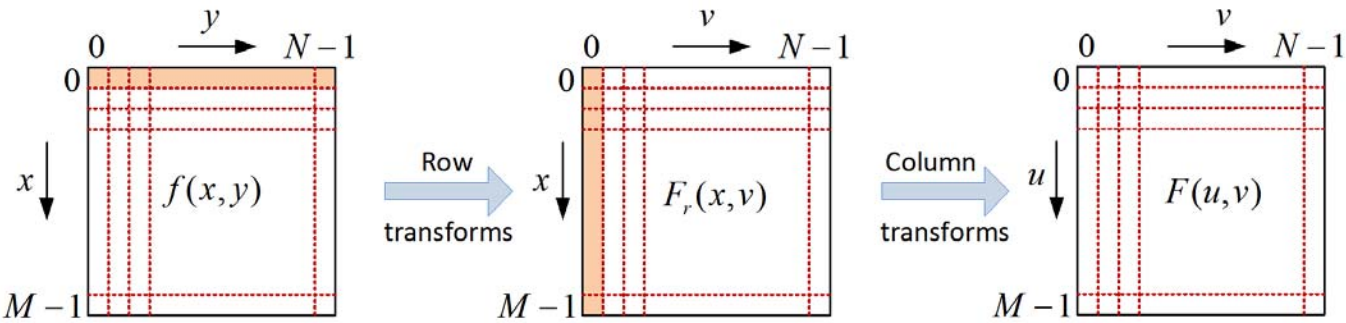
where

$$F_r(x,v) = \frac{1}{N}\sum_{y=0}^{N-1} f(x,y)\exp[-j2\pi vy/N] \quad (10)$$

Because of the separability property, $F(u,v)$ (or $f(x,y)$) can be obtained in two steps by successive applications of the 1D Fourier transform (or its inverse).

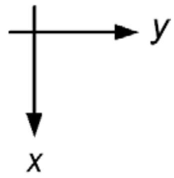
For each value of x, the expression inside the brackets is a 1D transform, with frequency values $v = 0, 1, \ldots, N-1$. Therefore the 2D function $F(x,v)$ is obtained by taking a transform along each row of $f(x,y)$. The desired result $F(u,v)$ is then obtained by taking a transform along each column of $F(x,v)$.

The same results may be obtained by first taking transforms along the columns of $f(x,y)$ and then along the rows of that result.



Example

In general, for DFT computation, we will use this convention: x axis points down, y axis to the right, origin at the top left corner.



$$f(x,y) = \begin{bmatrix} 4 & 2 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Taking the row transforms:

$$F_r(x,v) \quad = \quad \frac{1}{4}\begin{bmatrix} 9 & 2-j & 3 & 2+j \\ 4 & 1-j & 2 & 1+j \\ 2 & 1-j & 0 & 1+j \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Taking the column transforms:

$$F(u,v) \quad = \quad \frac{1}{16}\begin{bmatrix} 16 & 5-j3 & 6 & 5+j3 \\ 7-j3 & 0 & 3-j & 2 \\ 6 & 1-j1 & 0 & 1+j \\ 7+j3 & 2 & 3+j & 0 \end{bmatrix}$$

Average Value

The average value of the digital image $f(x, y)$ is

$$\bar{f}(x, y) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \tag{11}$$

It is easily shown that

$$\bar{f}(x, y) = F(0, 0) \tag{12}$$

Translation

The translation property is

$$f(x - a, y - b) \leftrightarrow F(u, v) \exp[-j2\pi(ua/M + vb/N)] \tag{13}$$

For example, for $M = 100, N = 100$, and $a = 20, b = 40$,

$$\begin{aligned}
f_1(x, y) &= f(x - 20, y - 40) &(14)\\
F_1(u, v) &= \mathcal{F}\{f(x - 20, y - 40)\} &(15)\\
&= F(u, v)\ \exp[-j2\pi(u/5 + 2v/5)] &(16)\\
&= F(u, v)\ \exp(-j2\pi u/5)\ \exp(-j4\pi v/5) &(17)
\end{aligned}$$

Note that a shift in $f(x, y)$ does not affect the magnitude of its Fourier transform since

$$|F(u, v) \exp[-j2\pi(ua/M + vb/N)]| = |F(u, v)| \tag{18}$$

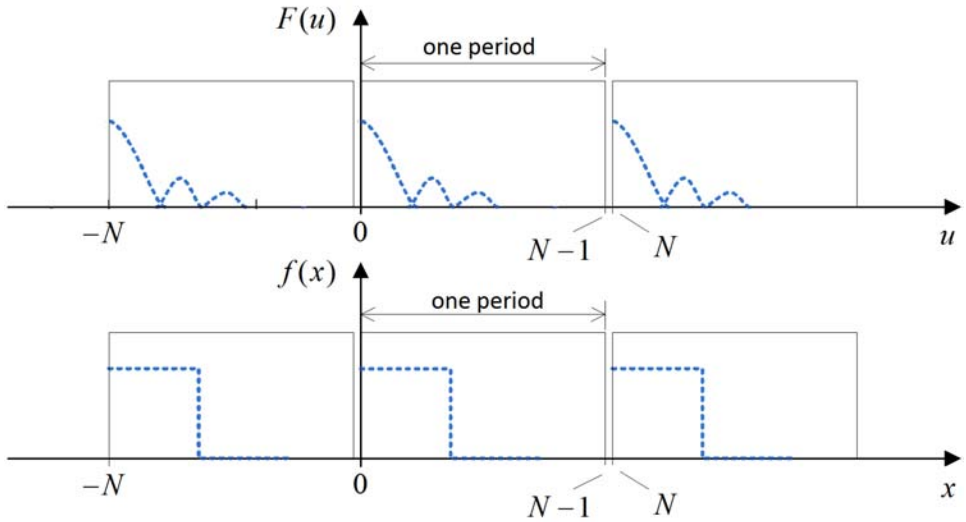
Periodicity and Conjugate Symmetry

From the definition of the DFT, we can show that $F(u)$ is periodic with period N:

$$F(u) = F(u + kN) \quad k = 0, \pm 1, \pm 2, \ldots \tag{19}$$

This periodicity property also applies to the inverse of $F(u)$, i.e., $f(x)$ computed from Eq. 2 is periodic:

$$f(x) = f(x + kN) \quad k = 0, \pm 1, \pm 2, \ldots \tag{20}$$

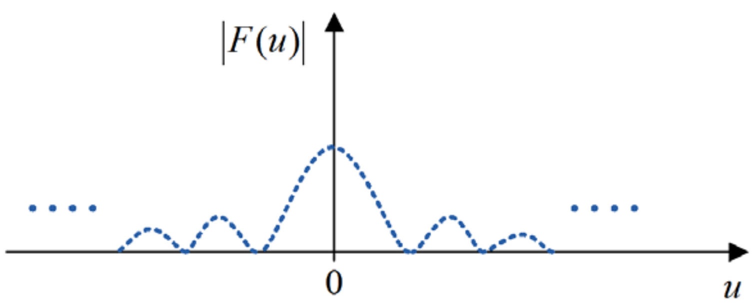


Furthermore, if $f(x)$ is real, the DFT also exhibits conjugate symmetry:

$$F(u) = F^*(-u)$$

The DFT magnitude is then symmetrical about $u = 0$ since

$$|F(u)| = |F(-u)|$$



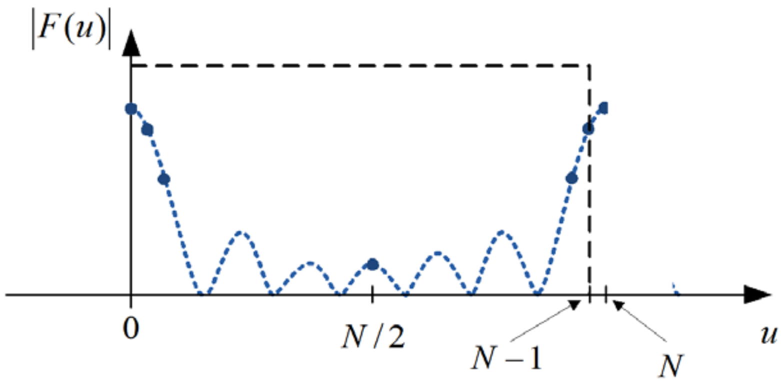
Hence, for the DFT magnitude function $|F(u)|$, we have

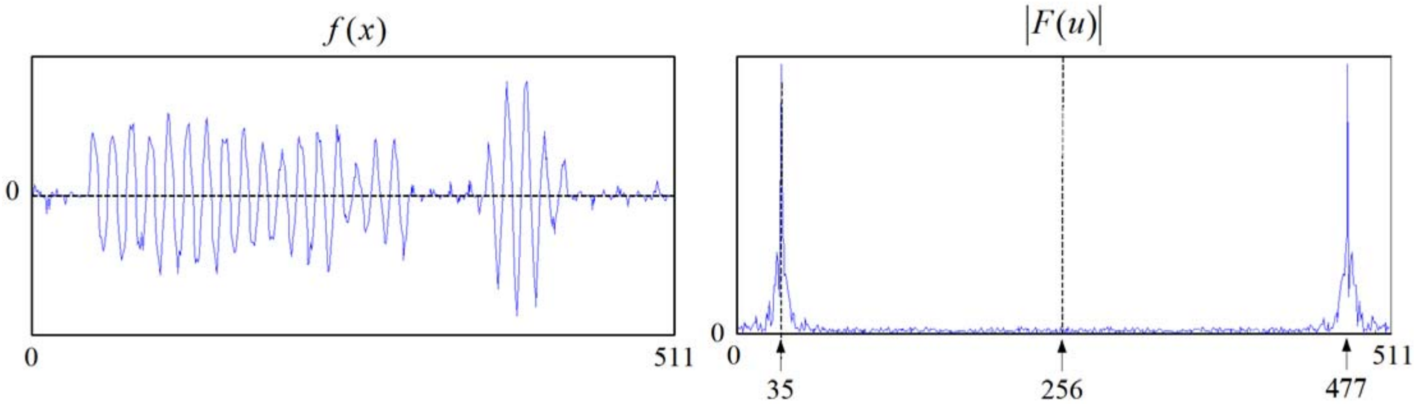
$$
\begin{aligned}
|F(u)| &= |F(-u)| \quad \text{(symmetry)} && (21)\\
&= |F(-u+N)| \quad \text{(periodicity)} && (22)
\end{aligned}
$$

In the window $u = 0, 1, \ldots, N-1$,

$$
\begin{aligned}
|F(0)| &= |F(N)| \\
|F(1)| &= |F(N-1)| \\
|F(2)| &= |F(N-2)| \\
&\cdots \\
|F(N/2)| &= |F(N/2)
\end{aligned}
$$

i.e., $|F(u)|$ is symmetrical about $u = N/2$.

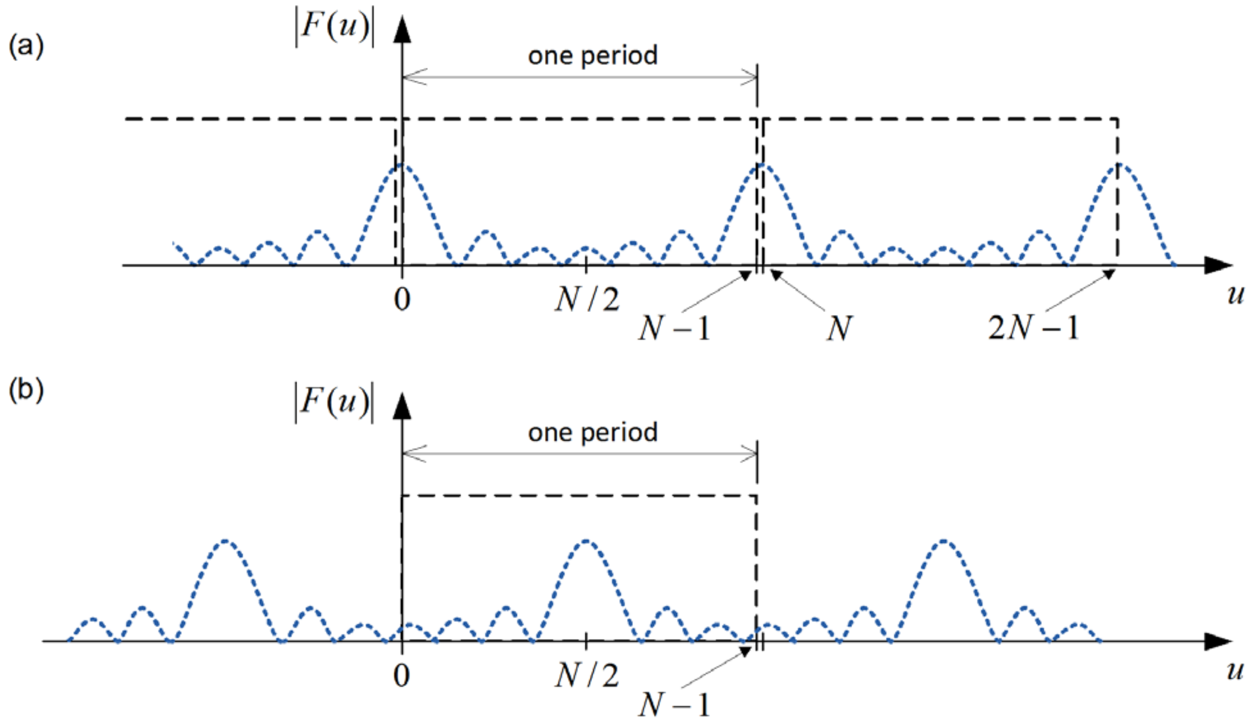




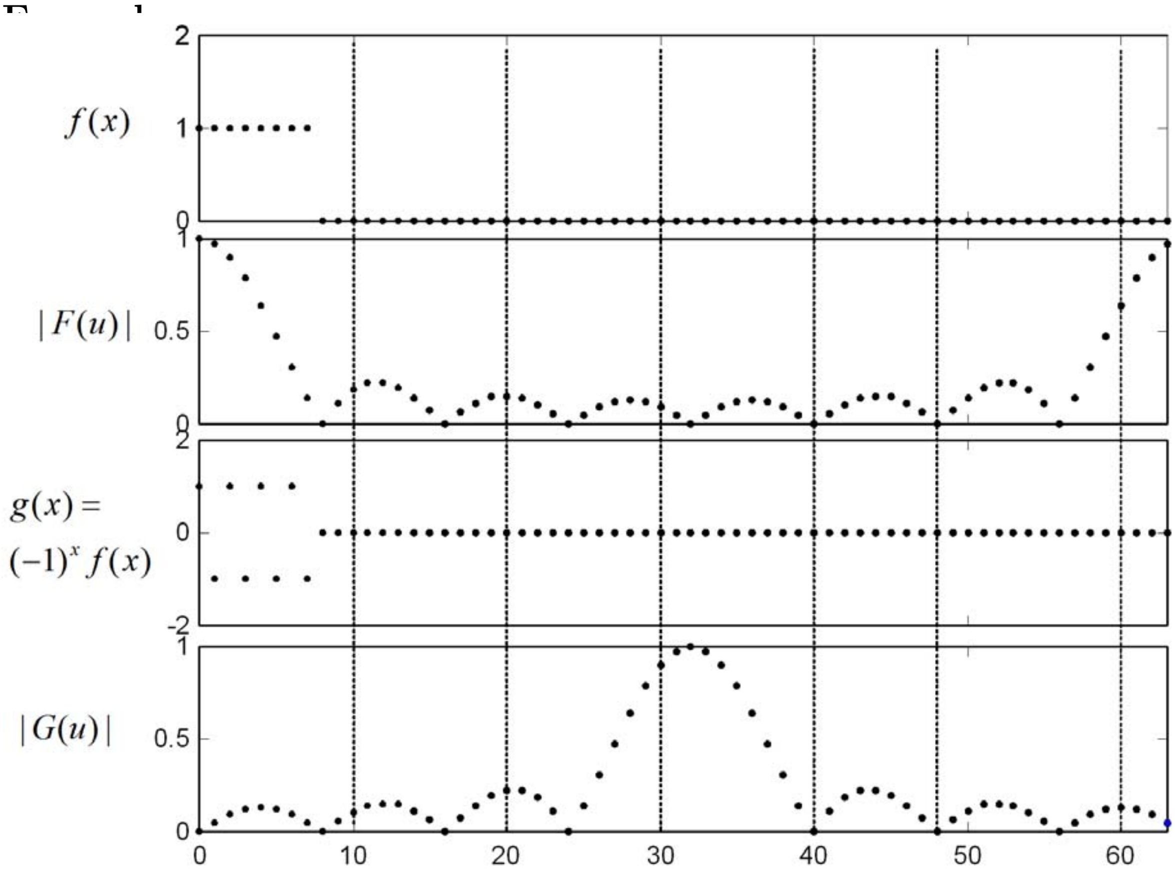
Cecause of the periodicity property, the values of $|F(u)|$ in this window are repeated along the u axis.

For viewing purposes, it would be better to centre $|F(u)|$, i.e., move the origin of the transform to the point $u = N/2$. This is done by multiplying $f(x)$ by $(-1)^x$ prior to taking the transform, i.e.,

$$f'(x) = f(x)\,(-1)^x$$



(a) Fourier spectrum showing back-to-back half periods in the window [0, N - 1];
(b) Shifted spectrum showing a full period in the same window.



Proof of Periodicity and Conjugate Symmetry

$$
\begin{aligned}
F(u) &= (1/N)\sum_{x=0}^{N-1} f(x)\exp[-j2\pi ux/N] \\
F(u+N) &= (1/N)\sum_{x} f(x)\exp[-j2\pi(u+N)x/N] \\
&= (1/N)\sum_{x} f(x)\exp[-j2\pi ux/N - j2\pi Nx/N] \\
&= (1/N)\sum_{x} f(x)\exp[-j2\pi ux/N]\exp[-j2\pi x] \\
&= (1/N)\sum_{x} f(x)\exp[-j2\pi ux/N] \quad \text{since } \exp[-j2\pi x] \equiv 1 \\
&= F(u)
\end{aligned}
$$

$$
\begin{aligned}
F(-u) &= (1/N)\sum_{x} f(x)\exp[j2\pi ux/N] \\
F^*(-u) &= (1/N)\sum_{x} f^*(x)\exp[-j2\pi ux/N] \\
&= (1/N)\sum_{x} f(x)\exp[-j2\pi ux/N] \quad \text{for real } f \\
&= F(u)
\end{aligned}
$$

For the 2D case involving an $N \times N$ image,

$$F(u, v) = F(u + kN, v + lN) \quad k, l = 0, \pm 1, \pm 2, \ldots \tag{23}$$

If $f(x, y)$ is real, the DFT also exhibits conjugate symmetry:

$$F(u, v) = F^*(-u, -v) \tag{24}$$

Hence, the DFT magnitude function $|F(u, v)|$ is symmetrical about the origin:

$$|F(u, v)| = |F(-u, -v)| \tag{25}$$

In the window $u, v = 0, 1, \ldots, N - 1$, $|F(u, v)|$ is symmetrical about $(u, v) = (N/2, N/2)$, i.e.,

$$|F(u, v)| = |F(N - u, N - v)| \tag{26}$$

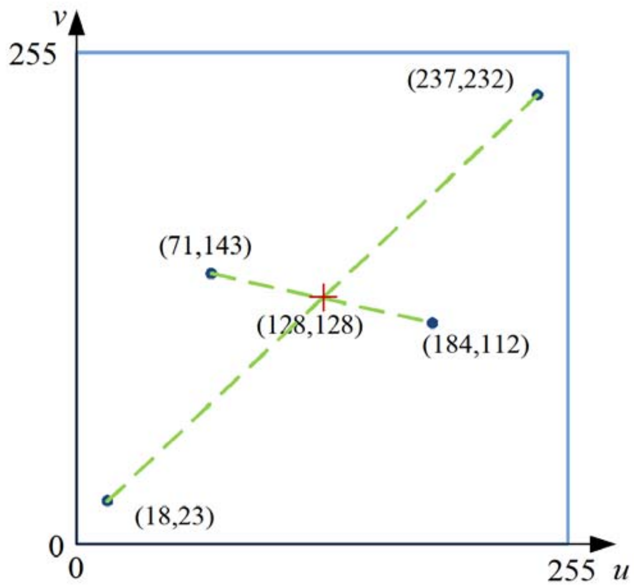
To move the origin of the transform to the point $(u, v) = (N/2, N/2)$, we multiply $f(x, y)$ by $(-1)^{x+y}$ prior to taking the transform, i.e.,

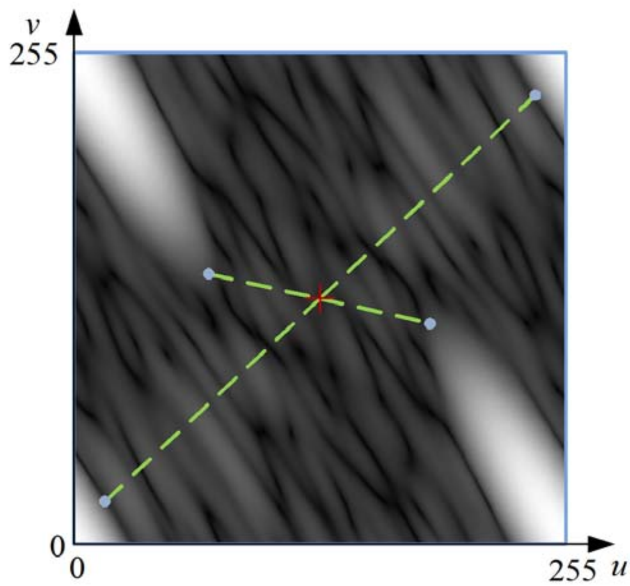
$$f'(x, y) = f(x, y)\,(-1)^{x+y}$$

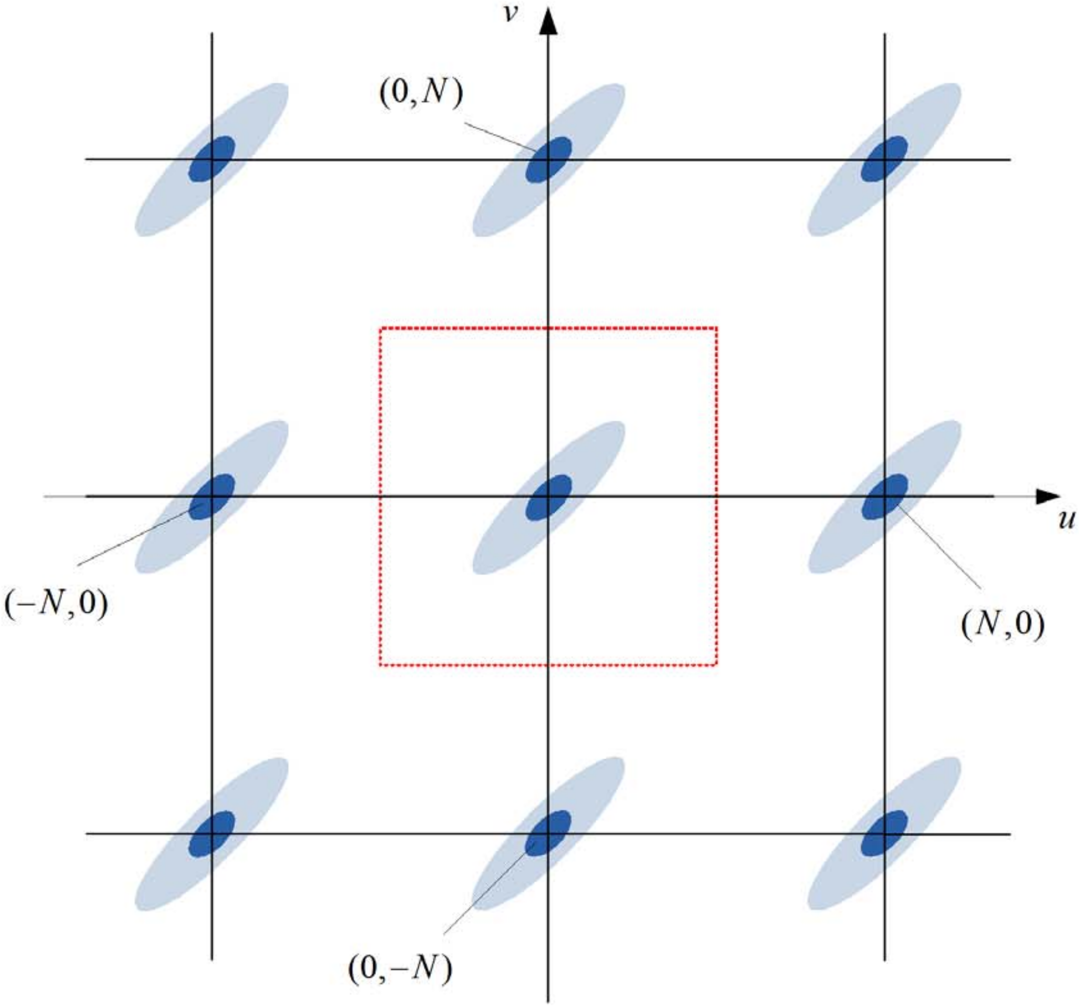
Example

Consider a square image of size $N = 256$. $|F(u, v)|$ is symmetrical about $(u, v) = (128, 128)$.

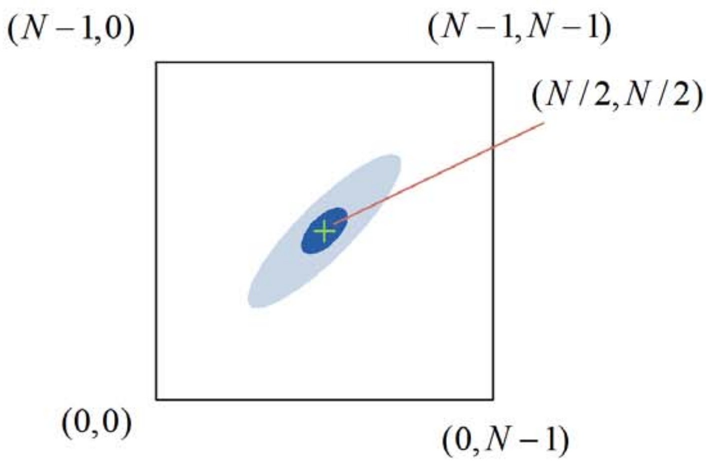
$$\begin{aligned} |F(u, v)| &= |F(255 - u, 255 - v)| \\ |F(18, 23)| &= |F(237, 232)| \\ |F(71, 143)| &= |F(184, 112)| \end{aligned}$$



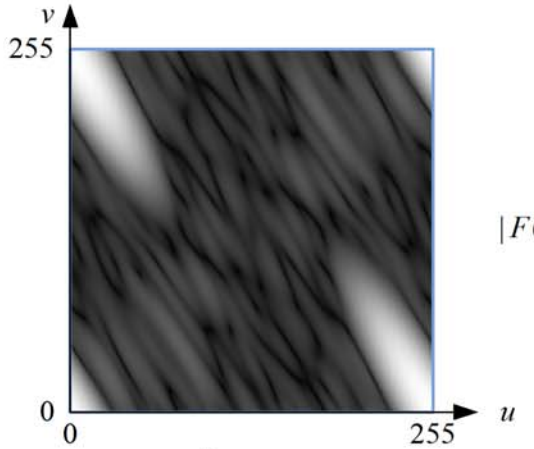




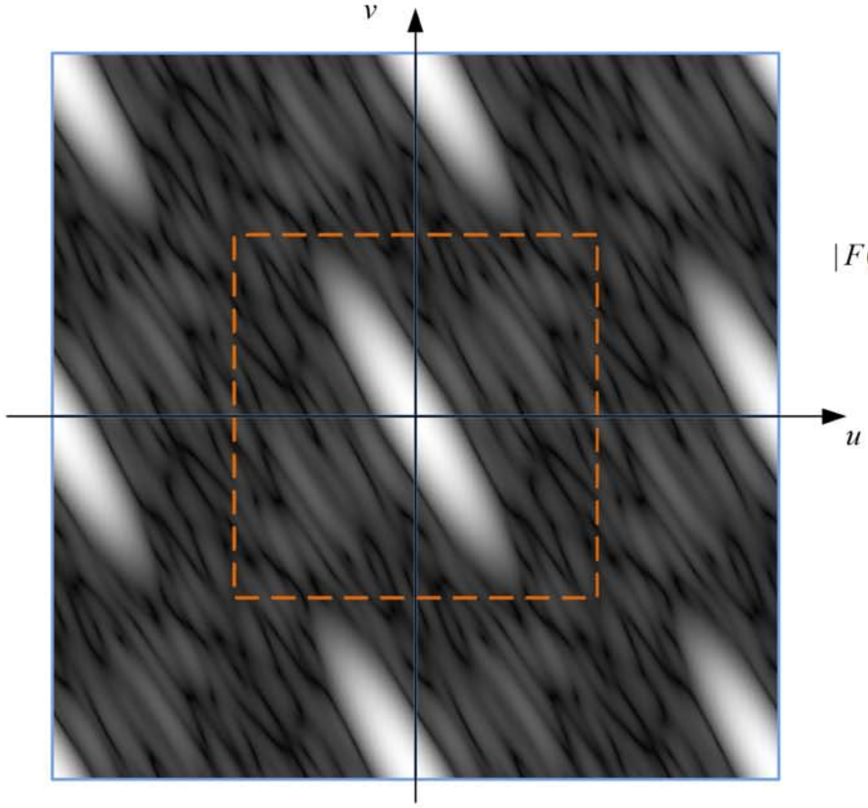
Schematic depiction of $|F(u,v)|$ for real $f(x,y)$



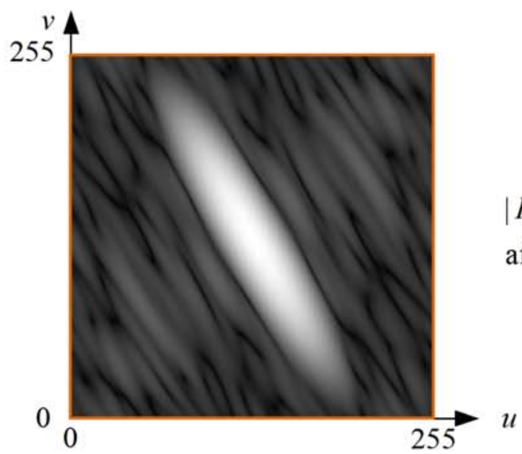
$|F(u,v)|$ centred at $(N/2,N/2)$



$|F(u,v)|$ in the window $u, v \in [0,255]$



$|F(u,v)|$ in the uv plane



$|F(u,v)|$ in the window $u, v \in [0,255]$
after shifting