# Chapter 11. Partial Differential Equations

## 11.1 Differential Equations

Mathematical models of physical phenomena often involve differential equations with one or more independent variable. For instance, a model of heat conduction in a region (e.g. an ocean, the atmosphere, or a system of pipes) will normally involve three-space variables for coordinates of points in the region and one time variable, as well as other physical information.

### 11.1.1 Ordinary differential equation

An *ordinary differential equation* (o.d.e.) is an equation that involves an unknown function $y(x)$ of *exactly one* independent variable $x$ and derivatives of $y$. We also call $y$ the dependent variable.

#### 11.1.2 Example

(i) $y' - xy \;=\; 0$

where $y' = \frac{dy}{dx}$ is the derivative of $y$ w.r.t. $x$.

This is an o.d.e. that involves the function $y(x)$ with one independent variable $x$.

(ii) $y'' - 3y' + 2y \;=\; 0$

where $y' = \frac{dy}{dx}$ and $y'' = \frac{d^2y}{dx^2}$ .

Again, in this o.d.e., $x$ is the only independent variable of the function $y$.

### 11.1.3 Partial differential equation

A **partial differential equation (p.d.e.)** is an equation containing an unknown function $u(x, y, \ldots)$ of *two or more* independent variables $x, y, \ldots$ and its partial derivatives with respect to these variables. We also call $u$ the dependent variable.

### 11.1.4 Example

(i) $u_{xy} - 2x + y \; = \; 0$

This is a p.d.e. that involves the function $u(x, y)$ with two independent variables $x$ and $y$.

(ii) $w_{xy} + x(w_z)^2 \; = \; yz$

This is a p.d.e. that involves the function $w(x, y, z)$ with three independent variables $x$, $y$ and $z$.

### 11.1.5 Solutions of Differential Equations

A **solution** of a differential equation is any function which satisfies the equation identically.

There are usually one or more *family* of solutions for a differential equation. We call such a family of solutions a *general* solution of the differential equation.

A specific function from the general solution is called a *particular* solution of the differential equation.

### 11.1.6 Example

Let us substitute $y = e^{x^2/2}$ in the o.d.e of example 11.1.2 (i):

By differentiating $y = e^{x^2/2}$ using chain rule, we have

$$y' = e^{x^2/2}\frac{d}{dx}\left(\frac{x^2}{2}\right) = xe^{x^2/2}.$$

On the other hand, $xy = xe^{x^2/2}$.

So $y = e^{x^2/2}$ satisfies the o.d.e. and hence is a particular solution of the o.d.e.

In fact, we can easily check that $y = ke^{x^2/2}$ satisfies the o.d.e. for any constant $k$.

So a general solution of the o.d.e is $y = ke^{x^2/2}$.

### 11.1.7 Example

The function

$$u(x,y) = x^2y - \frac{1}{2}xy^2 + F(x) + G(y) \qquad (1)$$

is a general solution of the p.d.e. in example 11.1.4 (i). Here $F$ and $G$ can be any (arbitrary) single variable functions.

Indeed, by taking partial derivatives of (1):

$$\begin{aligned} u_x &= 2xy - \frac{1}{2}y^2 + F'(x) \text{ and} \\ u_{xy} &= 2x - y, \end{aligned}$$

we see that the function (1) satisfies the p.d.e.

If we set $F(x) = 3\sin x$ and $G(y) = 4y^5 - 6$, we get the particular solution

$$u(x,y) = x^2y - \frac{1}{2}xy^2 + 3\sin x + 4y^5 - 6.$$

Suppose we require the p.d.e. to also satisfy the conditions

$$u(x,0) = x^3 \text{ and } u(0,y) = \sin(3y).$$

Then using (1), we have

$$x^3 = u(x,0) = F(x) + G(0)$$

and

$$\sin(3y) = u(0,y) = F(0) + G(y).$$

In this case, we can simply take $F(x) = x^3$ and $G(y) = \sin(3y)$ and get the particular solution

$$u(x,y) = x^2y - \frac{1}{2}xy^2 + x^3 + \sin(3y)$$

which satisfy the additional conditions.

### 11.1.8 Exercise

Consider the o.d.e

$$ay'' + by' + cy = 0 - - - - (*)$$

(where $a, b, c$ are some real numbers).

Let $k_1$ and $k_2$ be the solutions of the quadratic equation $at^2 + bt + c = 0$.

Show that the o.d.e. $(*)$ has the following general solutions:

(i) If $k_1$ and $k_2$ are real and distinct, then

$$y = Ae^{k_1 x} + Be^{k_2 x}$$

is a general solution of $(*)$.

(ii) If $k_1$ and $k_2$ are real and equal, then

$$y = Ae^{k_1 x} + Bxe^{k_1 x}$$

is a general solution of $(*)$.

(iii) If $k_1 = \alpha + i\beta$ and $k_2 = \alpha - i\beta$ are complex, then

$$y = Ae^{\alpha x}\cos\beta x + Be^{\alpha x}\sin\beta x$$

is a general solution of $(*)$.

Here $A$ and $B$ are arbitrary constants.

### 11.1.9 Example

In general, the totality of solutions of a p.d.e. is very large.

The Laplace equation $u_{xx} + u_{yy} = 0$ has the following solutions

$$u(x, y) = x^2 - y^2, \quad u(x, y) = e^x \cos y,$$

$$u(x, y) = \ln(x^2 + y^2), \quad \text{etc}$$

which are entirely different from each other.

### 11.1.10 Order of Differential Equations

The **order** of the p.d.e. is the order of the highest derivative present.

Example 11.1.2 (i) is an o.d.e. of order 1 while (ii) is an o.d.e. of order 2.

Example 11.1.4 (i) is a p.d.e. of order 2 and (ii) is also a p.d.e. of order 2.

### 11.1.11 Linearity and Homogeneity

An order 1 *linear* o.d.e. has the form

$$Ay' + By = Z$$

and an order 2 *linear* o.d.e. has the form

$$Ay'' + By' + Cy = Z$$

where $A, B, C, Z$ are constants or functions of $x$ but not functions of $y$.

An order 1 *linear* p.d.e. has the form

$$Au_x + Bu_y + Cu = Z$$

and an order 2 *linear* p.d.e. has the form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = Z$$

where $A, B, C, D, E, F, Z$ are constants or functions of $x$ and $y$ but not functions of $u$.

An order 1 or 2 linear o.d.e. or p.d.e. is said to be *homogeneous* if the $Z$ term in the above form is 0.

Generalizations to differential equation of higher order and those with more than two independent variables can be easily made.

### 11.1.12 Example

| **p.d.e.** | **order** | **linear** | **homogeneous** |
|---|---|---|---|
| $4u_{xx} - u_t = 0$ | 2 | yes | yes |
| $x^2 R_{yyy} = y^3 R_{xx}$ | 3 | yes | yes |
| $tu_{tx} + 2u_x = x^2$ | 2 | yes | no |
| $4u_{xx} - uu_t = 0$ | 2 | no | n.a. |
| $(u_x)^2 + (u_y)^2 = 2$ | 1 | no | n.a. |

### 11.1.13 Superposition Principle

If $u_1$ and $u_2$ are any solutions of a linear homogeneous differential equation, then

$$u = c_1 u_1 + c_2 u_2,$$

where $c_1$ and $c_2$ are any constants, is also a solution of that equation.

### 11.1.14 Example

Referring to the particular solutions of Laplace equation $u_{xx} + u_{yy} = 0$ in Example 11.1.9, by superposition principle,

$$u(x,y) = 3(x^2 - y^2) - 7e^x \cos y + 10 \ln (x^2 + y^2)$$

is again a solution of the Laplace equation.

## 11.2 Solving Differential Equations

In this section, we will demonstrate some techniques to solve certain types of simple o.d.e. and p.d.e.

### 11.2.1 Separable O.D.E.

An order 1 o.d.e is separable if it has the form

$$A(y)y' + B(x) = 0.$$

We can *separate* the variables by rewriting the equation as

$$A(y)dy + B(x)dx = 0.$$

We can then solve the o.d.e. by integration:

$$\int A(y)dy + \int B(x)dx = c$$

where $c$ is an arbitrary constant.

### 11.2.2 Example

The o.d.e. $y' - xy = 0$ in example 11.1.2 (i) can be rewritten as

$$\frac{1}{y}y' - x = 0.$$

So it is a separable o.d.e. and can be solved as follows:

$$\int \frac{1}{y}dy - \int x dx = 0 \Rightarrow \ln|y| - \frac{x^2}{2} = c$$
$$\Rightarrow \ln|y| = c + \frac{x^2}{2} \Rightarrow |y| = e^c e^{x^2/2}$$
$$\Rightarrow y = Ke^{x^2/2}$$

where $K = \pm e^c$ is an arbitrary constant.

So we get the general solution as in example 11.1.6.

### 11.2.3 Reducing P.D.E. to O.D.E.

To solve a P.D.E., it is very common to first "reduce" it to an O.D.E. before solving it. Here we illustrate two situations where we can use this approach.

### 11.2.4 Example

(Absence of one partial derivative)

$$u_{xx} - u = 0 \tag{2}$$

Since there is no $y-$derivative in p.d.e. (2), we treat $y$ as constant and regard the p.d.e. as an o.d.e. in $x$:

$$u''(x) - u(x) \ = \ 0 \tag{3}$$

To solve this o.d.e., note that it is of the form

$$au'' + bu' + cu = 0$$

with $a = 1, b = 0, c = -1$.

The quadratic equation $t^2 - 1 = 0$ has solutions $t = \pm 1$. So by exercise 11.1.8, so a general solution of the o.d.e. (3) has the form

$$u(x) = Ae^{x} + Be^{-x},$$

where $A$ and $B$ are <u>constant w.r.t. $x$</u>. Thus the 'constants' $A$ and $B$ may in fact be functions of $y$. Hence, a general solution of the p.d.e. (2) is

$$u(x, y) = A(y)e^{x} + B(y)e^{-x}.$$

#### 11.2.5 **Example**

(Common "inner" derivative)

$$u_{xy} = -u_x \tag{4}$$

Set $u_x = p$ so that the p.d.e. (4) may be viewed as an o.d.e.

$$p_y = -p. \tag{5}$$

This is a separable o.d.e. So we can solve it as in 11.2.1:

$$\int \frac{1}{p} dp + \int y dy = 0 \Rightarrow \ln |p| + y = c$$

$$\Rightarrow \ln |p| = c - y \Rightarrow |p| = e^c e^{-y}$$

$$\Rightarrow p = K e^{-y} \tag{6}$$

which is a general solution of (5). Here $K$ is a constant w.r.t. $y$, so it may be a function of $x$.

Substituting $p$ by $u_x$ in (6) and integrate w.r.t. $x$, we obtain a general solution of the p.d.e. (4):

$$u(x, y) = \int K(x) e^{-y} dx = f(x) e^{-y} + g(y),$$

where $f(x) = \displaystyle\int K(x) dx$ and $g(y)$ is an arbitrary function in $y$.

### 11.2.6 Separation of Variables for P.D.E.

This method can be used to solve p.d.e. involving two independent variables, say $x$ and $y$, that can be 'separated' from each other in the p.d.e. There are similarities between this method and the technique of separating variables for o.d.e. in (11.2.1). We first make an observation:

Suppose $u(x,y) = X(x)Y(y)$.

Then

(i) $u_x(x,y) = X'(x)Y(y)$

(ii) $u_y(x,y) = X(x)Y'(y)$

(iii) $u_{xx}(x,y) = X''(x)Y(y)$

(iv) $u_{yy}(x,y) = X(x)Y''(y)$

(v) $u_{xy}(x,y) = X'(x)Y'(y)$

Notice that each derivative of $u$ remains 'separated' as a product of a function of $x$ and a function of $y$. We exploit this feature as follows:

### 11.2.7 Illustration of Separation of Variables

Consider a p.d.e. of the form

$$u_x \;=\; f(x)g(y)u_y.$$

If a solution of the form $u(x,y) = X(x)Y(y)$ exists, then we obtain

$$\begin{aligned} X'(x)Y(y) &= f(x)g(y)X(x)Y'(y) \\ \text{i.e.,} \qquad \frac{1}{f(x)} \cdot \frac{X'(x)}{X(x)} &= g(y)\frac{Y'(y)}{Y(y)}. \end{aligned}$$

LHS is a function of $x$ only while RHS is a function of $y$ only. We conclude that

$$\text{LHS} = \text{RHS} = \text{some constant } k.$$

Thus, we obtain two o.d.e.

$$\begin{aligned} \frac{1}{f(x)} \cdot \frac{X'(x)}{X(x)} &= k \Rightarrow X'(x) = kf(x)X(x) \qquad (7) \\ g(y)\frac{Y'(y)}{Y(y)} &= k \Rightarrow Y'(y) = \frac{k}{g(y)}Y(y) \qquad (8) \end{aligned}$$

Note that (7) is an o.d.e. with independent variable $x$ and dependent variable $X$ while (8) is an o.d.e. with independent variable $y$ and dependent variable $Y$.

By solving (7) and (8) respectively for $X(x)$ and $Y(y)$, we obtain the solution $u(x, y) = X(x)Y(y)$.

### 11.2.8 Example

Solve $u_x + xu_y = 0$.

**Solution:** If a solution $u(x,y) = X(x)Y(y)$ exists, then we obtain

$$X'(x)Y(y) + xX(x)Y'(y) = 0$$

$$\text{i.e.,} \qquad \frac{1}{x} \cdot \frac{X'(x)}{X(x)} = -\frac{Y'(y)}{Y(y)} \tag{9}$$

This gives two o.d.e.'s :

LHS of (9) $= k$ gives $X' = kxX$.

This o.d.e. has general solution

$$X(x) = Ae^{kx^2/2} \tag{a}$$

Similarly, RHS of (9) $= k$ gives $Y' = -kY$.

This o.d.e. has general solution

$$Y(y) = Be^{-ky} \qquad \text{(b)}$$

Multiplying (a) and (b), we obtain a general solution of the p.d.e.

$$u(x, y) = X(x)Y(y) = Ce^{k(x^2/2 - y)}.$$