Chapter 8. Multiple Integrals

8.1 Double Integrals

8.1.1 Definition

The definition of definite integral in chapter 3 can be extended to functions of two variables.

Let R be a plane region in the xy-plane.

Subdivide R into subrectangles $R_i \quad (i = 1, ..., n)$.

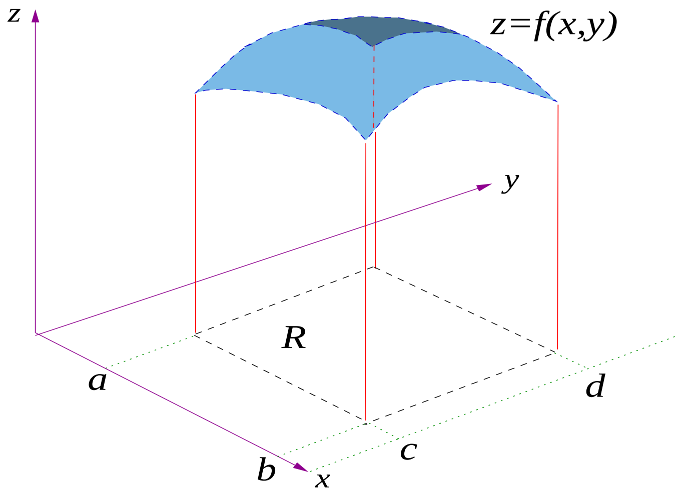
Let ΔA_i be the area of R_i and (x_i, y_i) be a point in R_i.

Let $f(x, y)$ be a function of two variables. Then the **double integral** of f over R is

$$\iint_R f(x, y)\, dA = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i)\Delta A_i.$$

8.1.2 Geometrical meaning

Geometrically, if $f(x, y) \geq 0$ for all $(x, y) \in R$, the definite integral $\iint_R f(x, y)\, dA$ is equal to the *volume* under the surface $z = f(x, y)$ and above the xy-plane over the region R as shown in the following diagram.



Summing over all the rectangles,

$$\sum_{i=1}^{n} f(x_i, y_i) A_i$$

gives the *approximate* volume of the solid under the surface and above R.

By letting n go to ∞, (i.e. making the subdivision more refined), the above sum will approach the *exact* volume of the solid.

8.1.3 Properties of Double Integrals

(1) $$\iint_R (f(x,y)+g(x,y))\ dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA.$$

(2) $\displaystyle\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA$, where c is a constant.

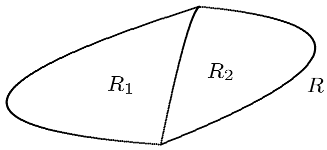
(3) If $f(x,y) \geq g(x,y)$ for all $(x,y) \in R$, then $\displaystyle\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$.

(4) $\displaystyle\iint_R dA \left(= \iint_R 1\,dA\right) = A(R)$, the area of R.

(5) $$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA,$$

where $R = R_1 \cup R_2$

and R_1, R_2 do not overlap except perhaps on their boundary.



(6) If $m \le f(x,y) \le M$ for all $(x,y) \in R$, then

$$mA(R) \le \iint_R f(x,y)\,dA \le MA(R).$$

8.2 Evaluation of double integrals

We shall discuss how to derive an efficient way to evaluate double integrals over certain plane regions. The key is to describe the given region in terms of the coordinates.

8.2.1 Rectangular regions

A rectangular region R in the xy-plane can be described in terms of inequalities:

$$a \leq x \leq b, \qquad c \leq y \leq d.$$

Then

$$\iint_R f(x,y)\,dA = \int_c^d \left[\int_a^b f(x,y)\ dx\right]\ dy.$$

The RHS is called an **iterated integral**. i.e. repeating the integration for each variable, one at a time.

We can also change the order of the variables of integration (without changing the value):

$$\iint_R f(x,y)\,dA = \int_a^b \left[\int_c^d f(x,y)\ dy\right]\ dx.$$

Note that we can do away with the square brackets in the integrated integrals.

8.2.2 **Example**

Evaluate the iterated integrals:

(a) $\int_0^3 \int_1^2 (x+2y)\, dydx$, (b) $\int_1^2 \int_0^3 (x+2y)\, dxdy$.

Solution:

$$
\begin{aligned}
(a) \quad \int_0^3 \int_1^2 (x+2y)\, dydx &= \int_0^3 \left[xy+y^2\right]_{y=1}^{y=2} dx \\
= \quad \int_0^3 (x+3)\, dx &= \left[\frac{x^2}{2}+3x\right]_{x=0}^{x=3} = 27/2.
\end{aligned}
$$

$$
\begin{aligned}
(b) \quad \int_1^2 \int_0^3 (x+2y)\, dxdy &= \int_1^2 \left[\frac{x^2}{2}+2xy\right]_{x=0}^{x=3} dy \\
= \quad \int_1^2 \left[\frac{9}{2}+6y\right] dy &= \left[\frac{9y}{2}+3y^2\right]_{y=1}^{y=2} = 27/2.
\end{aligned}
$$

8.2.3 Example

Let R be the rectangular region

$$0 \leq x \leq 4, \quad 1 \leq y \leq 2.$$

Evaluate $\displaystyle\iint_R x^2 y \, dA$.

Solution:

$$\begin{aligned}\iint_R x^2 y \, dA &= \int_0^4 \int_1^2 x^2 y \, dy dx \\ &= \left(\int_0^4 x^2 \, dx \right) \left(\int_1^2 y \, dy \right) \\ &= \frac{64}{3} \times \frac{3}{2} = 32.\end{aligned}$$

8.2.4 Remark

In general, if $f(x, y) = g(x)h(y)$, then

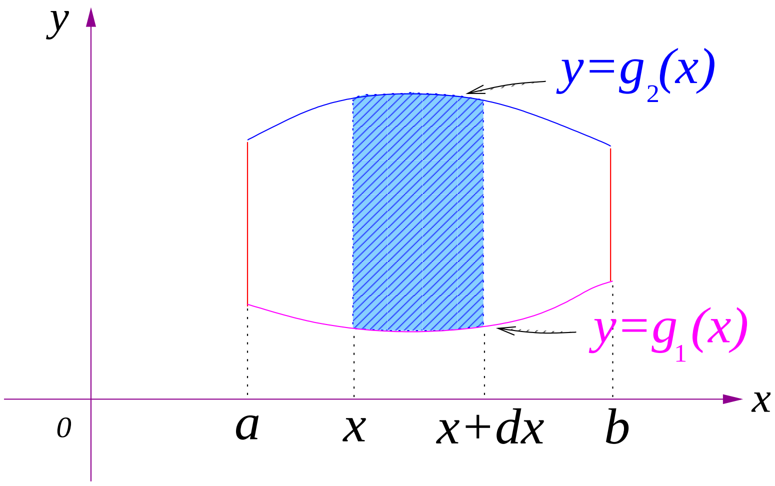
$$\iint_R g(x)h(y) \, dA = \left(\int_a^b g(x) \, dx \right) \left(\int_c^d h(y) \, dy \right)$$

where R is the rectangular region

$$a \leq x \leq b, \quad c \leq y \leq d.$$

8.2.5 General regions - Type A

Bottom and top boundaries are curves given by $y = g_1(x)$ and $y = g_2(x)$ respectively. i.e. they need not be straight lines. Left and right boundaries are straight lines given by $x = a$ and $x = b$ respectively.



If the top and bottom curves happen to intersect at $x = a$ or b, then the left or right side may reduce to just a point.

The region R is given by

$$R: \quad g_1(x) \leq y \leq g_2(x), \quad a \leq x \leq b.$$

The cross-sectional area of the slice between x and $x + dx$ is given by

$$c(x) = \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy$$

The total volume is the sum of the volumes of all the slices:

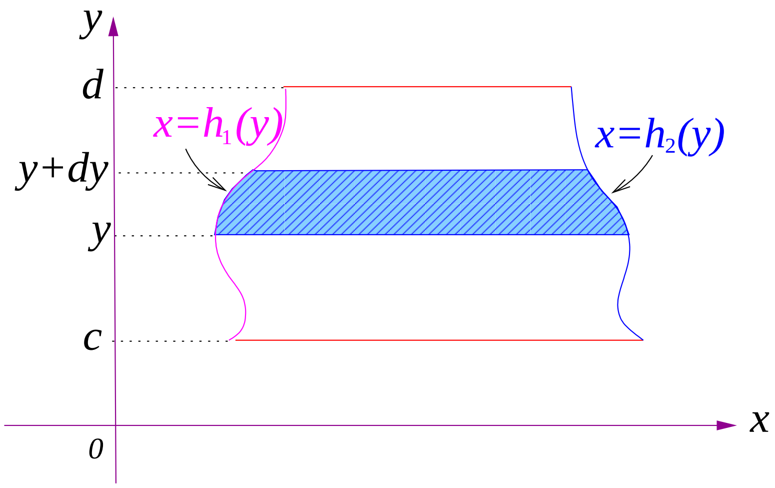
$$\int_a^b c(x)\,dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right] dx.$$

In other words, the double integral of f over Type A region R can be computed by an iterated integral first w.r.t. dy followed by dx:

$$\iint_R f(x,y)\,dA = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right] dx$$

8.2.6 General regions - Type B

Left and right boundaries are curves given by $x = h_1(y)$ and $x = h_2(y)$ respectively, i.e. they need not be straight lines. Bottom and top boundaries are straight lines (given by $y = c$ and $y = d$ resp.



If the left and right curves happen to intersect at $y = c$ or d, then the bottom or top side may reduce to just a point.

The region R is given by

$$R: \quad h_1(y) \le x \le h_2(y), \quad c \le y \le d.$$

The double integral of f over Type B region R can be computed by an iterated integral first w.r.t. dx followed by dy:

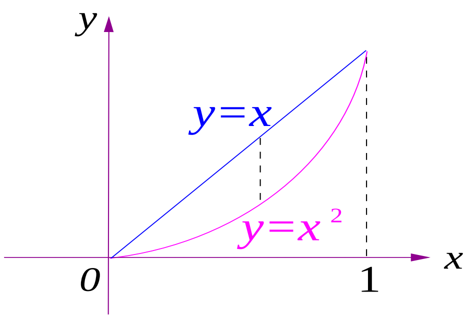
$$\iint_R f(x,y)\, dA = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\, dx \right] dy$$

8.2.7 Example

If R is bounded by $y = x$ and $y = x^2$, find $\iint_R 30xy\, dA$.

Solution: Treat R as a Type A region:

$$R: \quad x^2 \le y \le x, \quad 0 \le x \le 1.$$

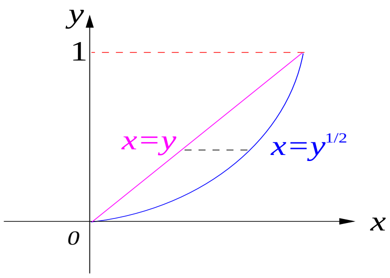


Then

$$\iint_R 30xy\, dA = \int_0^1 \left[\int_{x^2}^{x} 30xy\, dy\right] dx$$
$$= \int_0^1 \left[15xy^2\right]_{y=x^2}^{y=x} dx = \int_0^1 15x(x^2 - x^4)\, dx$$
$$= \left[\frac{15x^4}{4} - \frac{15x^6}{6}\right]_{x=0}^{x=1} = \frac{5}{4}.$$

If we treat R as a type B region:

$$R: \quad y \le x \le \sqrt{y}, \quad 0 \le y \le 1.$$



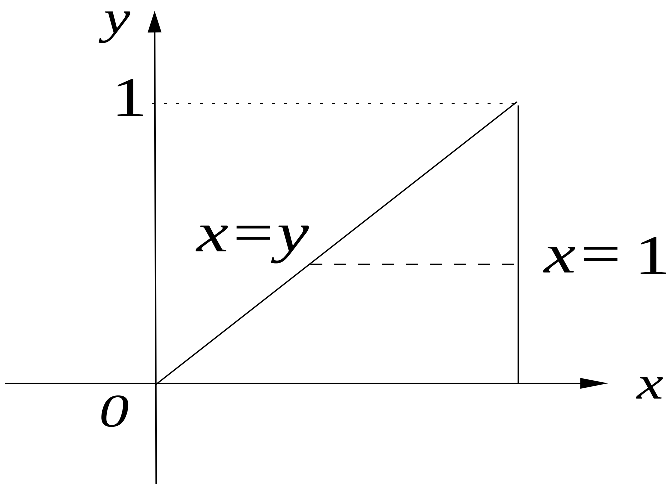
$$\iint_R 30xy\, dA = \int_0^1 \left[\int_{y}^{\sqrt{y}} 30xy\, dx\right] dy$$
$$= \int_0^1 \left[15x^2y\right]_{x=y}^{x=\sqrt{y}} dy = \int_0^1 \left(15y^2 - 15y^3\right) dy$$
$$= \left[\frac{15y^3}{3} - \frac{15y^4}{4}\right]_{y=0}^{y=1} = \frac{5}{4}.$$

8.2.8 **Example**

Calculate $\displaystyle\iint_R \frac{\sin\ x}{x}\ dA$, where R is the triangle in the xy-plane bounded by the x-axis, the line $y = x$ and the line $x = 1$.

Solution: If we treat R as type B:

$$R: \quad y \le x \le 1, \quad 0 \le y \le 1.$$

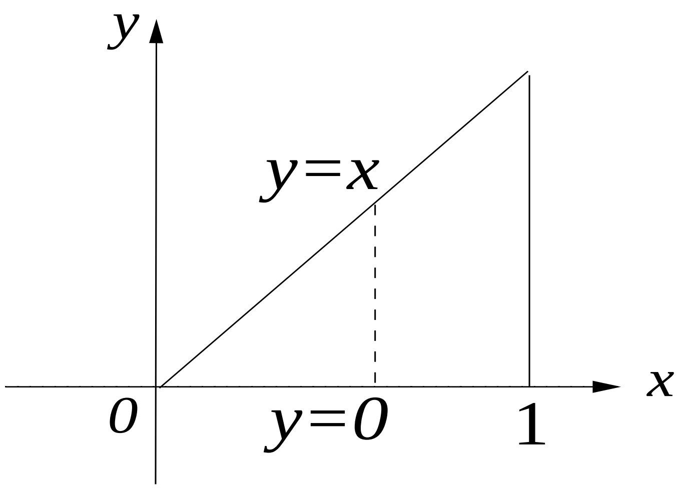


$$\iint_R \frac{\sin\ x}{x}\ dA = \int_0^1 \left[\int_y^1 \frac{\sin\ x}{x}\ dx \right]\ dy,$$

which cannot be evaluated by elementary means.

If we treat R as type A:

$$R: \quad 0 \le y \le x, \quad 0 \le x \le 1.$$



$$\iint_R \frac{\sin\ x}{x}\ dA = \int_0^1 \left[\int_0^x \frac{\sin\ x}{x}\ dy\right]\ dx$$

$$= \int_0^1 \left[y\frac{\sin\ x}{x}\right]_{y=0}^{y=x}\ dx = \int_0^1 (\sin\ x - 0)\ dx$$

$$= [-\cos\ x]_{x=0}^{x=1} = 1 - \cos\ 1.$$

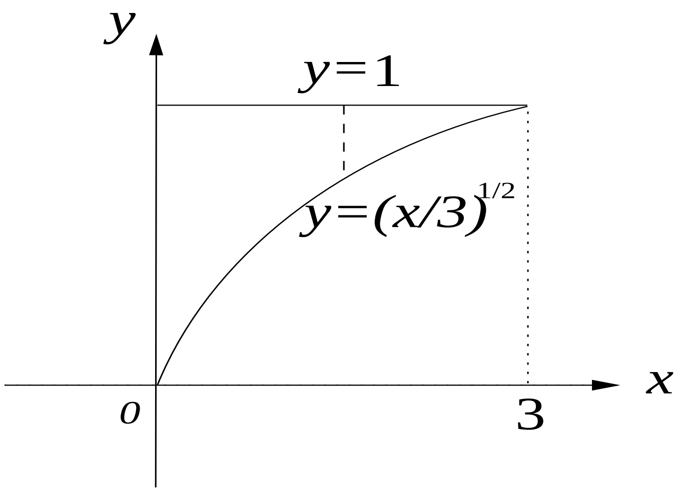
8.2.9 Example

Evaluate $\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3} \, dy \;\; dx.$

Solution: R may be described as :

$$R : \quad \sqrt{\frac{x}{3}} \le y \le 1, \quad 0 \le x \le 3.$$

Hence the Type A region looks like



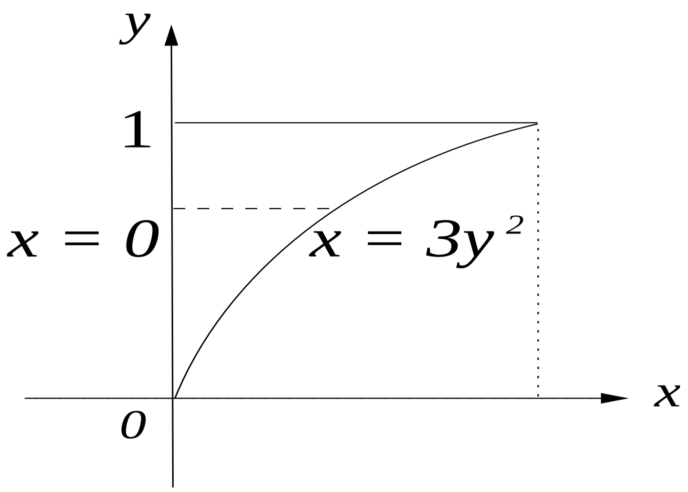
However, it is difficult to integrate e^{y^3} directly.

Now we treat R as type B:

Note that $y = \sqrt{\frac{x}{3}} \Longrightarrow x = 3y^2.$

So the region R is given by

$$R: \quad 0 \le x \le 3y^2, \quad 0 \le y \le 1.$$



$$\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3}\ dy\ \ dx = \int_0^1 \left[\int_0^{3y^2} e^{y^3}\ dx\right]\ dy$$

$$= \int_0^1 \left[xe^{y^3}\right]_{x=0}^{x=3y^2}\ dy = \int_0^1 3y^2 e^{y^3}\ dy$$

$$= \int_0^1 e^u du \qquad = [e^u]_{u=0}^{u=1} = e - 1.$$

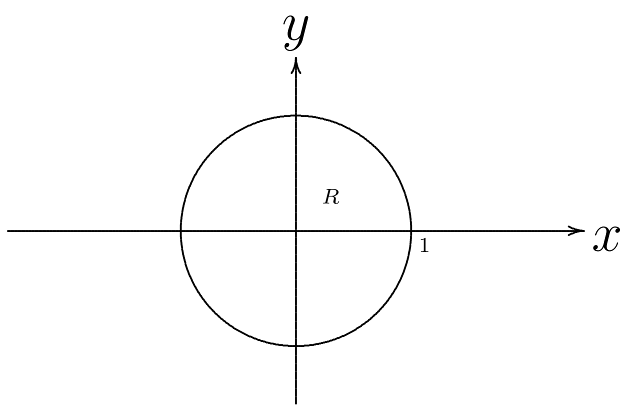
[Here we have used a substitution $u = y^3$.]

8.3 Double integral in polar coordinates

Certain regions (see examples below) can be described more simply using polar coordinates r and θ. Hence, it is more straightforward to evaluate double integrals over such regions using polar coordinates.

Instead of giving the ranges for x and y, we give the ranges for r, distance from origin to a point in the region, and θ, angle of elevation of a point from the x-axis.

8.3.1 Circle



In Cartesian coordinates, the circle can be regarded as Type A region with the upper and lower semicircles as the upper and lower boundaries. i.e.

$$R: \quad -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}, \quad -1 \le x \le 1.$$

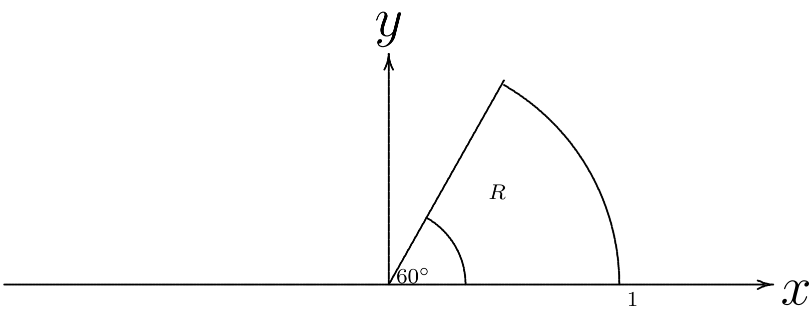
Alternatively, we can regard it as Type B region with the left and right semicircles as the left and right boundaries. i.e.

$$R: \quad -\sqrt{1-y^2} \le x \le \sqrt{1-y^2}, \quad -1 \le y \le 1.$$

In polar coordinates, we describe the circle in terms of the ranges of r and θ:

$$R: \quad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi$$

8.3.2 Sector of a circle



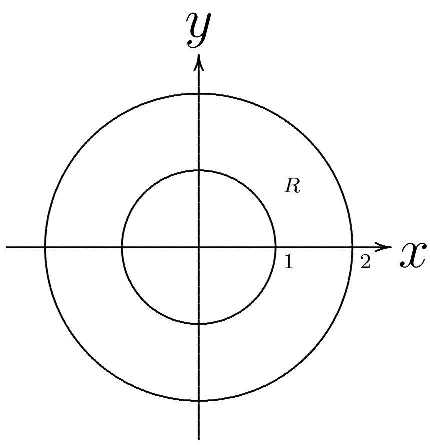
In Cartesian coordinates, the sector can be regarded as Type B region with the line segment as the left boundary and the arc of the circle as the right boundary. i.e.

$$R: \quad \frac{1}{\sqrt{3}}y \leq x \leq \sqrt{1-y^2}, \quad 0 \leq y \leq \frac{\sqrt{3}}{2}.$$

In polar coordinates, the sector is given by

$$R: \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq \pi/3$$

8.3.3 Ring



This region is neither Type A nor Type B. To use Cartesian coordinates, you need to partition the ring into smaller regions which are either Type A or Type B.

In polar coordinates, the ring is given by

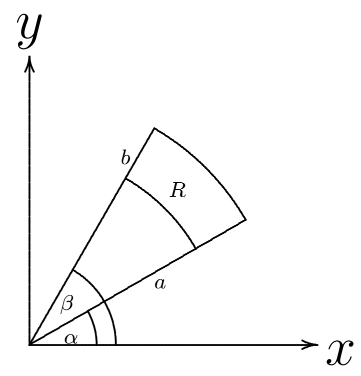
$$1 \le r \le 2, \quad 0 \le \theta \le 2\pi$$

8.3.4 Polar rectangle

In general, a region R described in polar coordinates:

$$R: \quad a \le r \le b, \quad \alpha \le \theta \le \beta$$

is shown in the following diagram. We call such a region a polar rectangle.



8.3.5 Change of variables

When we transform from the Cartesian coordinates to polar coordinates, we are performing change of variables from (x, y) to (r, θ):

$$x = r\cos\theta, \quad y = r\sin\theta.$$

In this case, dA will be changed from $dxdy$ (or $dydx$) to $r\ dr\ d\theta$.

Suppose a region R (in xy-plane) is given in polar coordinates by

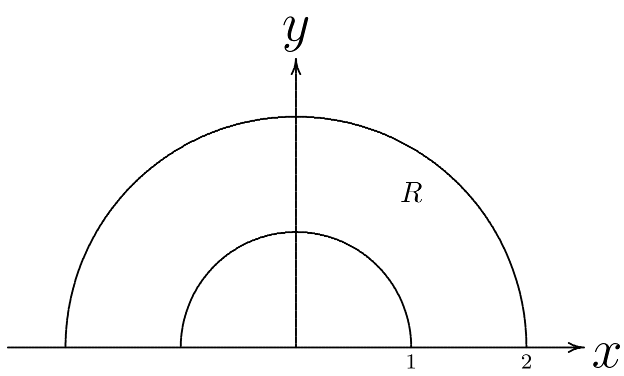
$$R: \quad a \le r \le b, \quad \alpha \le \theta \le \beta,$$

then we have

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r dr d\theta.$$

8.3.6 Example

Evaluate $\displaystyle\iint_R (3x+4y^2)\,dA$, where R is the semicircular ring in the upper half-plane between the semicircles $x^2+y^2=1$ and $x^2+y^2=4$.



Solution: The region R is given by

$$R: \quad 1 \le r \le 2, \quad 0 \le \theta \le \pi.$$

So

$$\begin{aligned}
\iint_R (3x + 4y^2)\, dA &= \int_0^\pi \int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\ r dr d\theta \\
&= \int_0^\pi \left[r^3\cos\theta + r^4\sin^2\theta)\right]_{r=1}^{r=2} d\theta \\
&= \int_0^\pi \left(7\cos\theta + 15\sin^2\theta\right) d\theta \\
&= \int_0^\pi \left(7\cos\theta + \frac{15}{2}(1 - \cos 2\theta)\right) d\theta \\
&= \left[7\sin\theta + \frac{15}{2}(\theta - \frac{\sin 2\theta}{2})\right]_{\theta=0}^{\theta=\pi} \\
&= \frac{15\pi}{2}
\end{aligned}$$

8.4 Applications of Double Integrals

8.4.1 Volume

Suppose D is a solid region under the surface of a function $f(x, y)$ over a plane region R. Then, as we have seen in 8.1.2, the volume of D is given by

$$\iint_R f(x, y) dA.$$

8.4.2 Example

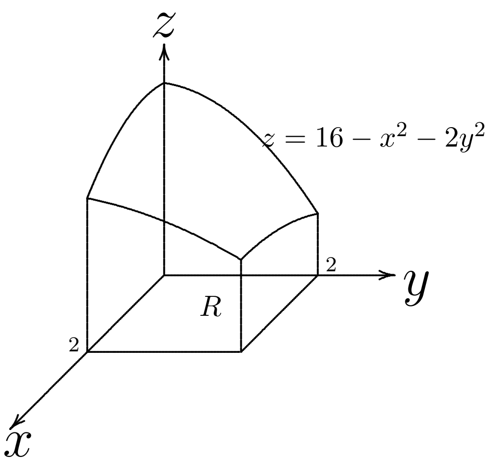
Find the volume of the solid D that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$, $y = 2$, and the 3-coordinate planes.

Solution: The solid region D is under the surface represented by the function $f(x, y) = 16 - x^2 - 2y^2$

and is above the rectangular region

$$R: \quad 0 \leq x \leq 2, \quad 0 \leq y \leq 2.$$

So volume of D is



$$\iint_R (16 - x^2 - 2y^2)dA$$
$$= \int_0^2 \int_0^2 (16 - x^2 - 2y^2)dxdy$$
$$= 48.$$

8.4.3 Example

Find the volume of the solid enclosed laterally by the circular cylinder about z-axis of radius 3 and bounded on top by the plane $x + z = 20$ and below by the paraboloid $z = x^2 + y^2$.

Solution: The volume can be computed as

$$V = \iint_R f_1(x,y)dA - \iint_R f_2(x,y)dA$$

where f_1 and f_2 are the functions of the top and bottom surfaces respectively and R is the projection of the solid on the xy-plane.

From the equations of the plane and paraboloid, we have

$$f_1(x,y) = 20 - x \quad \text{and} \quad f_2(x,y) = x^2 + y^2.$$

It is also clear that R is the circle of radius 3 centred at origin which has polar coordinates

$$0 \le r \le 3, \quad 0 \le \theta \le 2\pi.$$

So the volume of the solid is

$$V = \iint_R (20 - x)dA - \iint_R (x^2 + y^2)dA$$

and can be computed as

$$\begin{aligned}
V &= \int_0^{2\pi} \int_0^3 (20 - r\cos\theta) r \; dr \; d\theta - \int_0^{2\pi} \int_0^3 (r^2) r \; dr \; d\theta \\
&= \int_0^{2\pi} \int_0^3 20r - r^2 \cos\theta - r^3 \; dr \; d\theta \\
&= \int_0^{2\pi} \left[10r^2 - \frac{r^3}{3} \cos\theta - \frac{r^4}{4} \right]_0^3 \; d\theta \\
&= \int_0^{2\pi} 90 - 9\cos\theta - \frac{81}{4} \; d\theta \\
&= \left[\frac{279}{4}\theta - 9\sin\theta \right]_0^{2\pi} = \frac{279}{2}\pi
\end{aligned}$$

8.4.4 Surface area

If f has continuous first partial derivatives on a closed region R of the xy-plane, then the area S of that portion of the surface $z = f(x, y)$ that projects onto

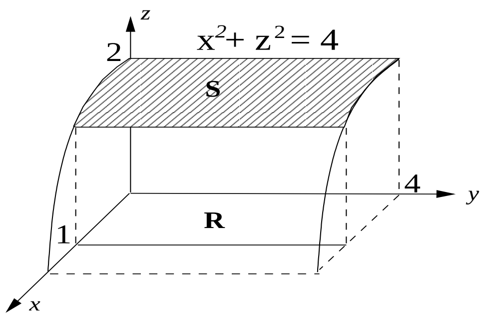
R is

$$S = \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA.$$

8.4.5 Example

Find the surface area of the portion of the cylinder $x^2 + z^2 = 4$ above the rectangle

$$R: \quad 0 \le x \le 1, \quad 0 \le y \le 4.$$



Solution: The portion of the cylinder $x^2 + z^2 = 4$ that lies above the xy-plane has the equation $z = \sqrt{4 - x^2}$.

So the surface is given by the function $f(x, y) =$

$\sqrt{4-x^2}$.

$$\begin{aligned} S &= \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\; dA \\ &= \iint_R \sqrt{\left(-\frac{x}{\sqrt{4-x^2}}\right)^2 + 0^2 + 1}\; dA \\ &= \int_0^4 \left[\int_0^1 \frac{2}{\sqrt{4-x^2}}\, dx\right]\, dy \\ &= 2\int_0^4 \left[\sin^{-1}(x/2)\right]_{x=0}^{x=1}\, dy \\ &= 2\int_0^4 \frac{\pi}{6}\, dy = \frac{4\pi}{3}. \end{aligned}$$

Note that $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\, dx = \sin^{-1}(x/a) + C.$

8.4.6 Mass and center of gravity

If a lamina with a continuous **density function** $\delta(x,y)$ occupies a region R in the xy-plane, its **total**

mass M is given by the double integral

$$M = \iint_R \delta(x, y)\ dA,$$

and its **center of gravity** $(\bar{x}, \bar{y})$ is

$$\bar{x} = \frac{\iint_R x\delta(x,y)\ dA}{\iint_R \delta(x,y)\ dA} = \frac{\iint_R x\delta(x,y)\ dA}{M},$$
$$\bar{y} = \frac{\iint_R y\delta(x,y)\ dA}{\iint_R \delta(x,y)\ dA} = \frac{\iint_R y\delta(x,y)\ dA}{M}.$$

Note that if $\delta(x, y)$ is a constant, then the center of gravity of the lamina is:

$$\bar{x} = \frac{\iint_R x\ dA}{\iint_R 1\ dA} = \frac{\iint_R x\ dA}{\text{Area of } R}, \quad \bar{y} = \frac{\iint_R y\ dA}{\iint_R 1\ dA} = \frac{\iint_R y\ dA}{\text{Area of } R}.$$

8.4.7 Example

Find the center of gravity of the triangular lamina with vertices $(0, 0)$, $(0, 1)$ and $(1, 0)$, and density function $\delta(x, y) = xy$.

Solution: The triangular lamina has boundaries

$x = 0, \quad y = 0, \quad y = -x + 1.$

It is described by $R: \ \ 0 \le y \le -x+1, \quad 0 \le x \le 1.$

The mass of the lamina is

$$\iint_R \delta(x, y)\ dA = \iint_R xy\ dA$$
$$= \int_0^1 \int_0^{-x+1} xy\ dy\ dx = \int_0^1 \left[\frac{1}{2}xy^2\right]_{y=0}^{y=-x+1} dx$$
$$= \int_0^1 \left[\frac{1}{2}x^3 - x^2 + \frac{1}{2}x\right] dx = \frac{1}{24}.$$

For the center of gravity,

$$\iint_R x\delta(x, y)\ dA = \iint_R x^2 y\ dA$$
$$= \int_0^1 \int_0^{-x+1} x^2 y\ dy\ dx = \int_0^1 \left[\frac{1}{2}x^2 y^2\right]_{y=0}^{y=-x+1} dx$$
$$= \int_0^1 \left[\frac{1}{2}x^4 - x^3 + \frac{1}{2}x^2\right] dx = \frac{1}{60},$$

$$\iint_R y\delta(x,y)\ dA = \iint_R xy^2\ dA$$
$$= \int_0^1 \int_0^{-x+1} xy^2\ dy\ dx = \int_0^1 \left[\frac{1}{3}xy^3\right]_{y=0}^{y=-x+1}\ dx$$
$$= \int_0^1 \left[-\frac{1}{3}x^4 + x^3 - x^2 + \frac{1}{3}x\right]\ dx = \frac{1}{60}.$$

So

$$\bar{x} = \frac{\iint_R x\delta(x,y)\ dA}{\iint_R \delta(x,y)\ dA} = \frac{1/60}{1/24} = \frac{2}{5}$$

and

$$\bar{y} = \frac{\iint_R y\delta(x,y)\ dA}{\iint_R \delta(x,y)\ dA} = \frac{1/60}{1/24} = \frac{2}{5}$$

8.5 Triple integral

We can also define integration on functions of three variables over solid region in xyz-space.

Let D be a solid region in the xyz space. Subdivide D into smaller cubic region $D_i \quad (i = 1, ..., n)$.

Let ΔV_i be the volume of V_i and (x_i, y_i, z_i) be a point in D_i. Let $f(x, y, z)$ be a function of three variables. Then the **triple integral** of f over D is

$$\iiint_D f(x,y,z)\,dV = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta V_i.$$

8.5.1 Physical meaning

There is no direct geometrical meaning for triple integral on a general function. However, when the function f represents certain physical quantity, then integrating f over a solid region may have some physical meaning.

For example, when f is the constant function 1, then

$$\iiint_D 1\,dV = \text{ volume of } D.$$

8.5.2 Example

Given a solid object D with volume V and uniform density δ, the mass M of D is given by

$$M = \delta \times V.$$

However, suppose the density actually varies (continuously) at different points of D, i.e. the density is a function $\delta(x, y, z)$ where (x, y, z) are the coordinates of a point in D.

Divide D into subregions D_i as before and let M_i be the mass of the subregion D_i.

Then

$$M_i \approx \delta(x_i, y_i, z_i) \times \Delta V_i.$$

Hence

$$M = \lim_{n\to\infty} \sum_{i=1}^{n} \delta(x_i, y_i, z_i)\Delta V_i = \iiint_D \delta(x, y, z)\, dV.$$

8.5.3 Evaluation of triple integral

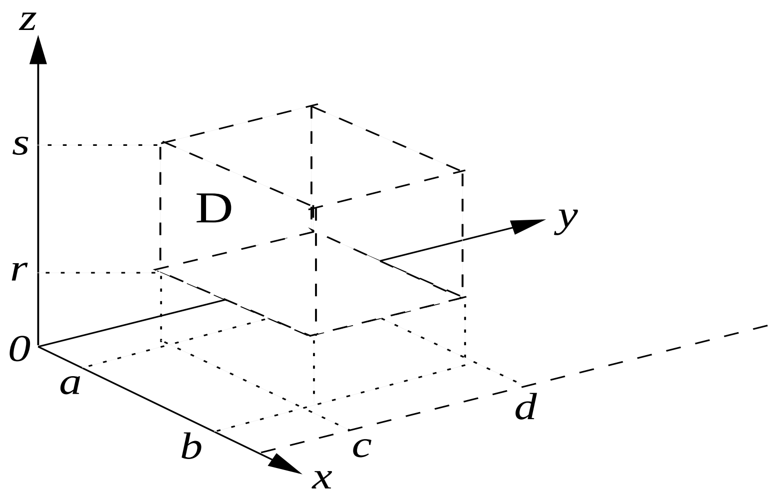
Similar to double integral, we need to describe the solid region D in terms of coordinates and evaluate the triple integral as a "triple" iterated integral.

We will only consider the rectangular region in this chapter.

8.5.4 Rectangular region

Suppose D is the rectangular box consisting of points (x, y, z) such that

$$D: \quad a \le x \le b, \quad c \le y \le d, \quad r \le z \le s.$$



Then we have the iterated integral

$$\iiint_D f(x,y,z)\,dV = \int_a^b \int_c^d \int_r^s f(x,y,z)\,dz\,dy\,dx.$$

As in the case of double integrals, the order of integration with respect to the three variables does not affect the answer of the triple integrals.