# EE2011 Engineering Electromagnetics
## Semester II of Academic Year 2011/2012

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# Orthogonal Coordinates

three-dimensional space $\Rightarrow$ require 3 independent coordinates

(a) position vector to represent particular point P $(u_1, u_2, u_3)$

$$\overrightarrow{OP} = u_1 \hat{u}_1 + u_2 \hat{u}_2 + u_3 \hat{u}_3$$

(b) non-position vector to represent $\vec{F}$ (*e.g.* force)

$$\vec{F} = F_1 \hat{u}_1 + F_2 \hat{u}_2 + F_3 \hat{u}_3$$

advantageous to use orthogonal coordinate system

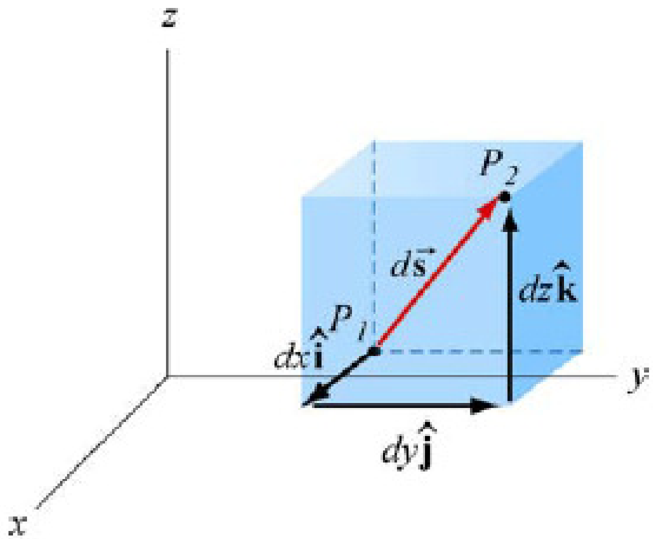
$$\hat{u}_1 \bullet \hat{u}_2 = \hat{u}_2 \bullet \hat{u}_3 = \hat{u}_3 \bullet \hat{u}_1 = 0$$

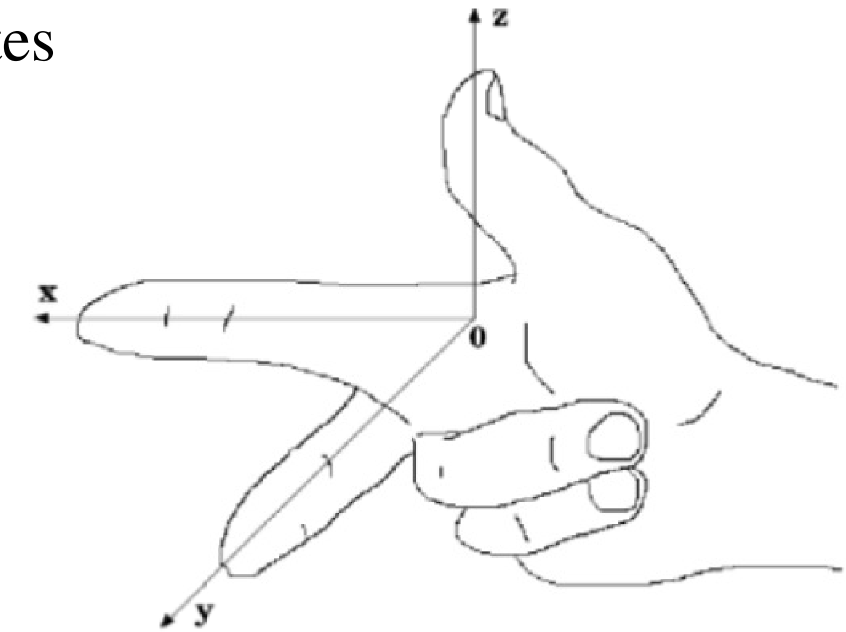
right-hand convention

$$\hat{u}_1 \times \hat{u}_2 = \hat{u}_3 \qquad \hat{u}_2 \times \hat{u}_3 = \hat{u}_1 \qquad \hat{u}_3 \times \hat{u}_1 = \hat{u}_2$$

# Orthogonal Coordinates

rectangular (Cartesian) coordinates



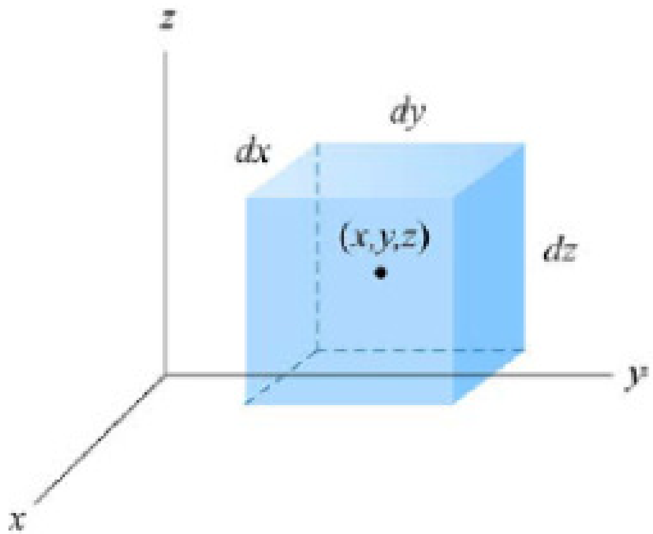

perturbation of P (from $P_1$ to $P_2$)

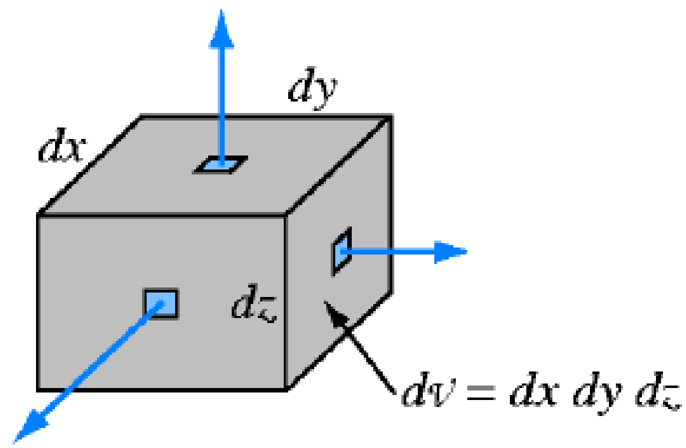
general need for three-dimensional displacement vector

$$d\vec{s} = dx\,\hat{\mathrm{i}} + dy\,\hat{\mathrm{j}} + dz\,\hat{\mathrm{k}}$$

# Orthogonal Coordinates

rectangular (Cartesian) coordinates




(scalar) elemental volume $dV = dx\,dy\,dz$

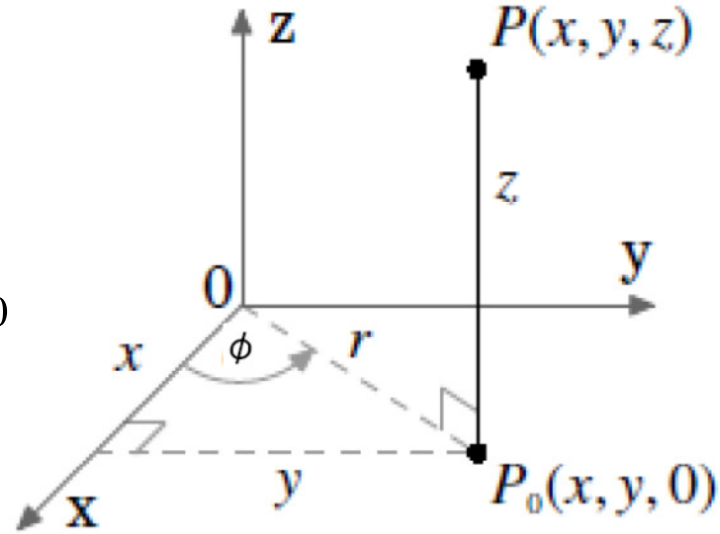
(vectorial) elemental areas $d\vec{A}_x = dy\,dz\,\hat{\mathrm{i}},\; d\vec{A}_y = dz\,dx\,\hat{\mathrm{j}},\; d\vec{A}_z = dx\,dy\,\hat{\mathrm{k}}$

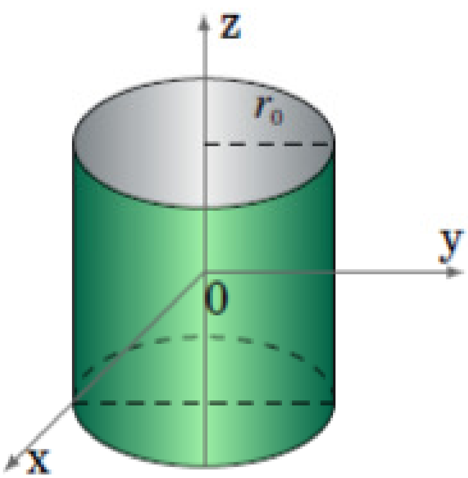
*N.B.*: need to follow sign convention for normal unit vectors

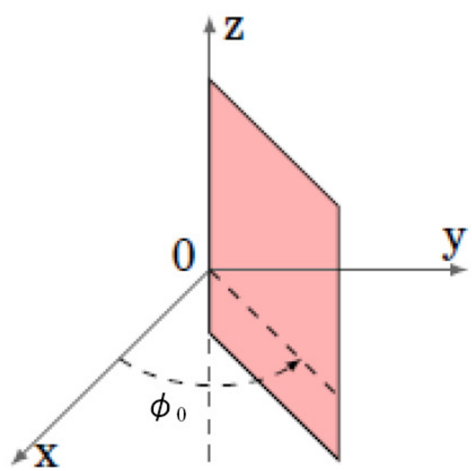
# Orthogonal Coordinates

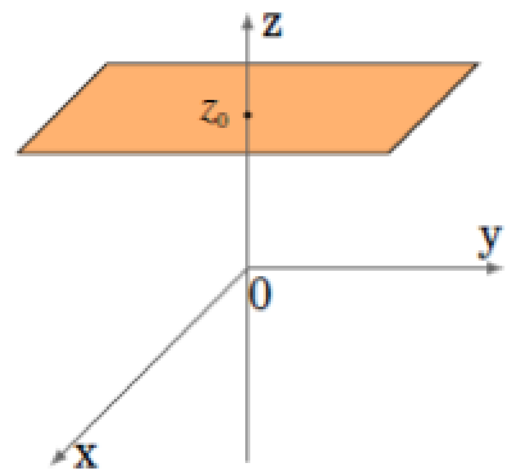
cylindrical coordinates

- same $z$ coordinate
- project $P$ onto $x$-$y$ plane to obtain $P_0$
- coordinate $r = |OP_0|$ instead of $|OP|$
- constant-coordinate surfaces



# Orthogonal Coordinates

cylindrical coordinates

- unit vectors

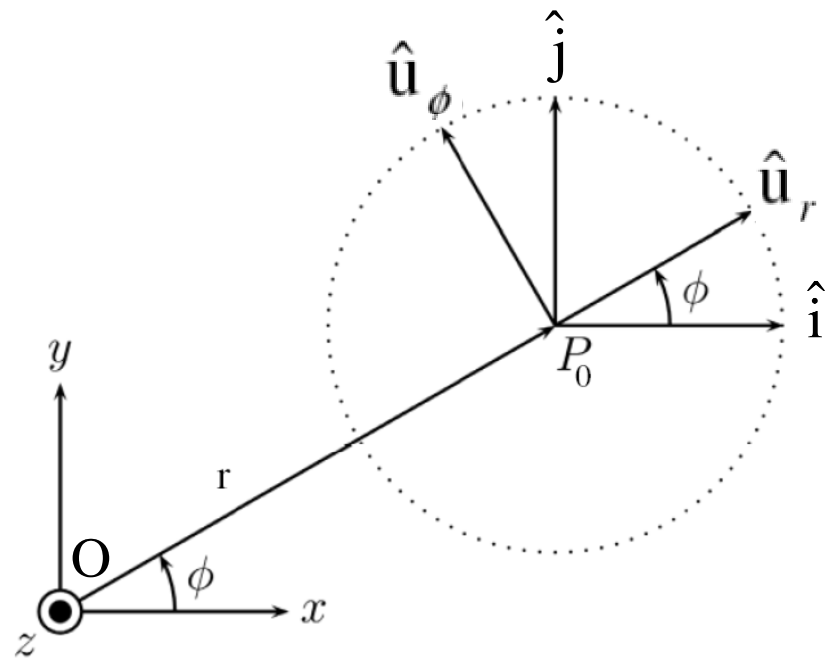
  $\hat{u}_z$ same as for Cartesian

  $\hat{u}_r$ in direction of $OP_0$

  $\hat{u}_\phi$ perpendicular to $OP_0$

- variable directions for $\hat{u}_r$ and $\hat{u}_\phi$

- caution for integration




*e.g.* $\int f(\phi)\, \hat{u}_r\, d\phi \neq \hat{u}_r \int f(\phi)\, d\phi$

$\int f(\phi)\, \hat{u}_z\, d\phi = \hat{u}_z \int f(\phi)\, d\phi$

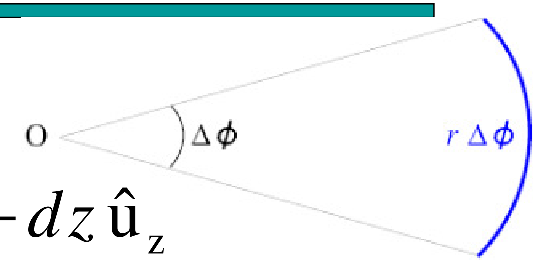
$$\hat{u}_r = \hat{i}\cos\phi + \hat{j}\sin\phi$$

$$\hat{u}_\phi = \hat{j}\cos\phi - \hat{i}\sin\phi$$

$$\hat{u}_z = \hat{k}$$

# Orthogonal Coordinates

cylindrical coordinates

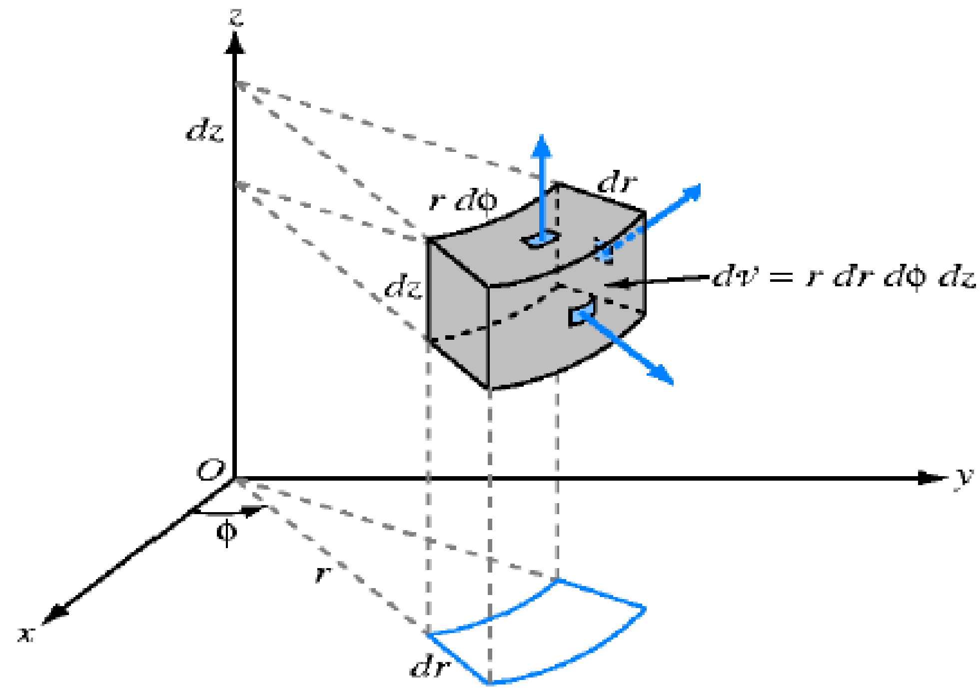



- (vectorial) line element $d\vec{s} = dr\,\hat{\mathrm{u}}_{\mathrm{r}} + rd\phi\,\hat{\mathrm{u}}_{\phi} + dz\,\hat{\mathrm{u}}_{\mathrm{z}}$
- (vectorial) elemental surfaces formed by slightly increasing any pair of coordinates

$$d\vec{A}_{\mathrm{r}} = rd\phi\,dz\;\hat{\mathrm{u}}_{\mathrm{r}}$$

$$d\vec{A}_{\phi} = dr\,dz\;\hat{\mathrm{u}}_{\phi}$$

$$d\vec{A}_{\mathrm{z}} = rdr\,d\phi\;\hat{\mathrm{u}}_{\mathrm{z}}$$




- (scalar) elemental volume

$$dV = rdr\,d\phi\,dz$$

# Orthogonal Coordinates

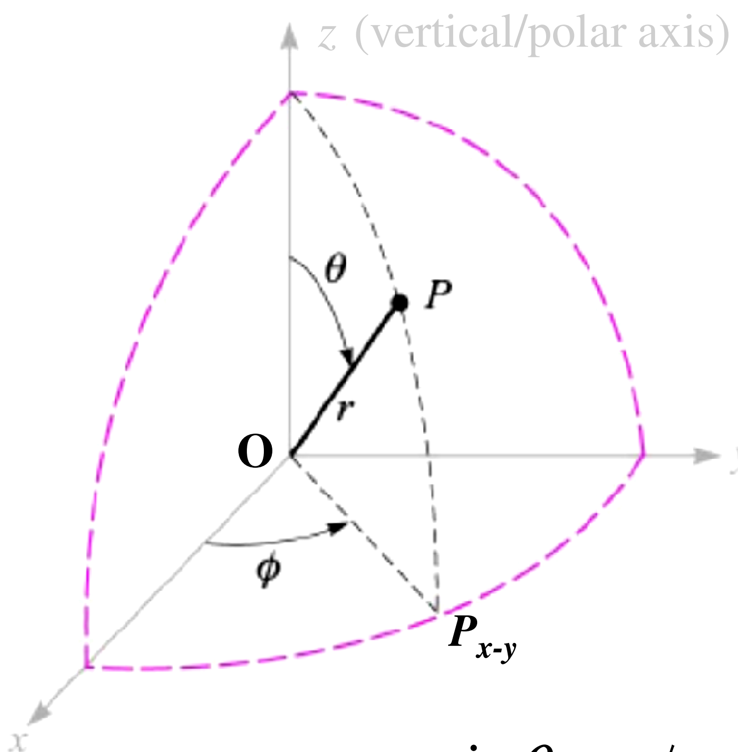
spherical coordinates

- original radial coordinate $r = |\mathrm{O}P|$
  (other notation such as $R$ and $\rho$)

- same azimuthal coordinate $\phi$

- new polar coordinate $\theta$
  (with reference to vertical $z$ axis)

- unit vectors with variable directions




$$\hat{u}_r = \hat{i}\sin\theta\cos\phi + \hat{j}\sin\theta\sin\phi + \hat{k}\cos\theta$$

$$\hat{u}_\theta = \hat{i}\cos\theta\cos\phi + \hat{j}\cos\theta\sin\phi - \hat{k}\sin\theta$$

$$\hat{u}_\phi = -\hat{i}\sin\phi + \hat{j}\cos\phi$$

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

# Orthogonal Coordinates

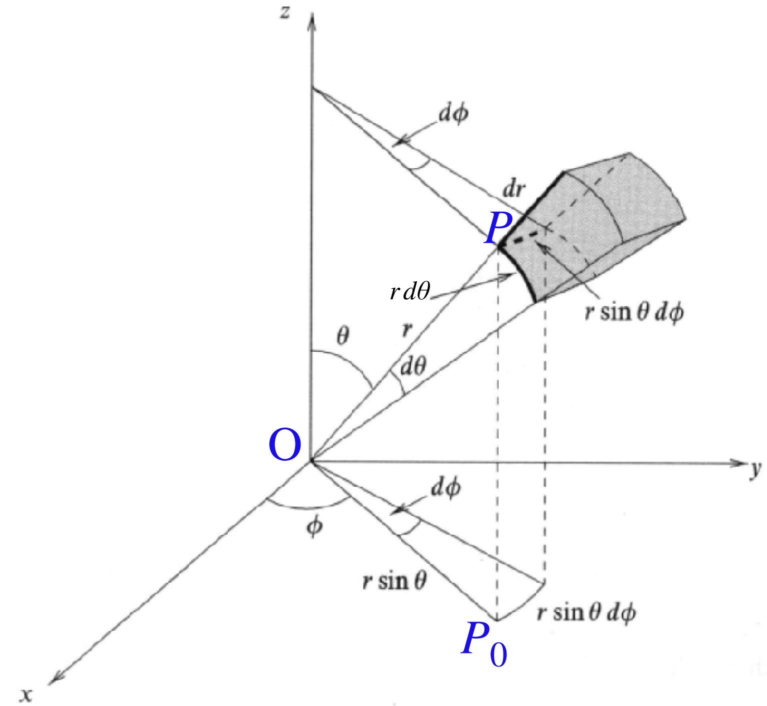
spherical coordinates

- (vectorial) line element $d\vec{s} = dr\,\hat{u}_r + rd\theta\,\hat{u}_\theta + r\sin\theta d\phi\,\hat{u}_\phi$
- (vectorial) elemental surfaces formed by slightly increasing any pair of coordinates

$$d\vec{A}_r = r^2 \sin\theta\, d\theta\, d\phi\ \hat{u}_r$$

$$d\vec{A}_\theta = r\sin\theta\, dr\, d\phi\ \hat{u}_\theta$$

$$d\vec{A}_\phi = rdr\, d\theta\ \hat{u}_\phi$$




- (scalar) elemental volume

$$dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$$

# Orthogonal Coordinates

transformation matrices for converting vector

(a) from cylindrical to Cartesian

$$\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_r \\ F_\phi \\ F_z \end{bmatrix}$$

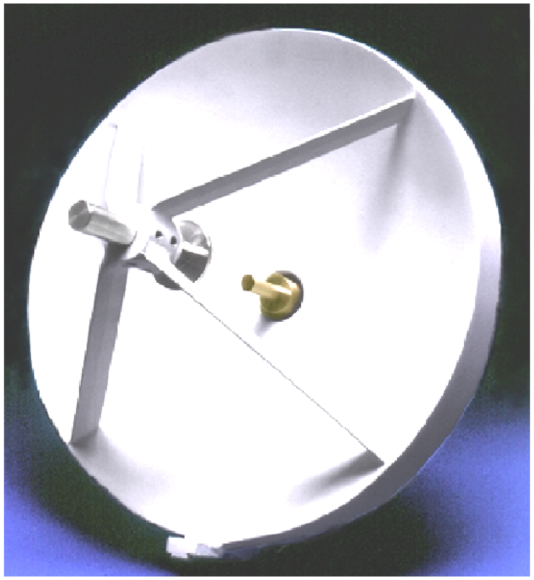
(b) from spherical to Cartesian

$$\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} F_r \\ F_\theta \\ F_\phi \end{bmatrix}$$

# Orthogonal Coordinates

need for additional coordinate systems
(8 other variations also in use)

elliptical waveguide

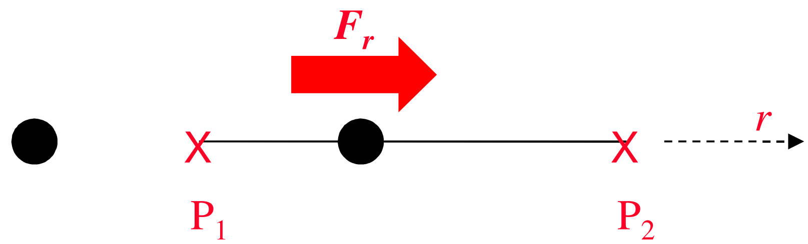


parabolic reflector
circular radiator

toroidal reflector

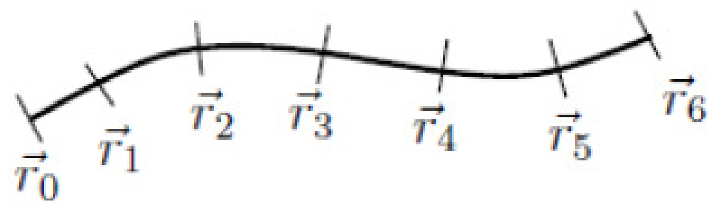
# Line Integrals

simple linear illustration: work done to move charge from $P_1$ to $P_2$




$$W = \int_{r_1}^{r_2} F_r \, dr$$

need to consider general case where $F_r \rightarrow \vec{F}$ and $dr \rightarrow d\vec{r}$




$$\lim_{|\Delta\vec{r}_k|\to 0} \sum_{k=1}^{N} \vec{F}(\vec{r}_k)\cdot\Delta\vec{r}_k = \int \vec{F}(\vec{r})\cdot d\vec{r}$$

summation of elemental contributions

not constrained by geometry or other special requirements

conducive for analytical or computational treatment

# Line Integrals

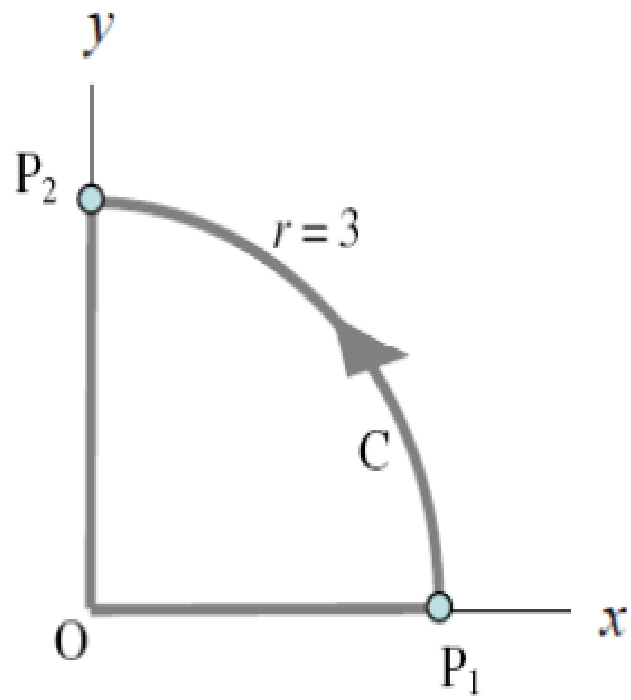
$e.g.\quad \int_{P_1}^{P_2} \vec{F} \bullet d\vec{s}$ along C where $\vec{F} = xy\,\hat{\imath} - 2x\,\hat{\jmath}$

cylindrical path where $d\vec{s} = r\,d\phi\,\hat{u}_\phi$ for elemental arc

$$F_\phi = -F_x \sin\phi + F_y \cos\phi$$

$$= -xy\sin\phi - 2x\cos\phi$$

$$= -9\sin^2\phi\cos\phi - 6\cos^2\phi$$

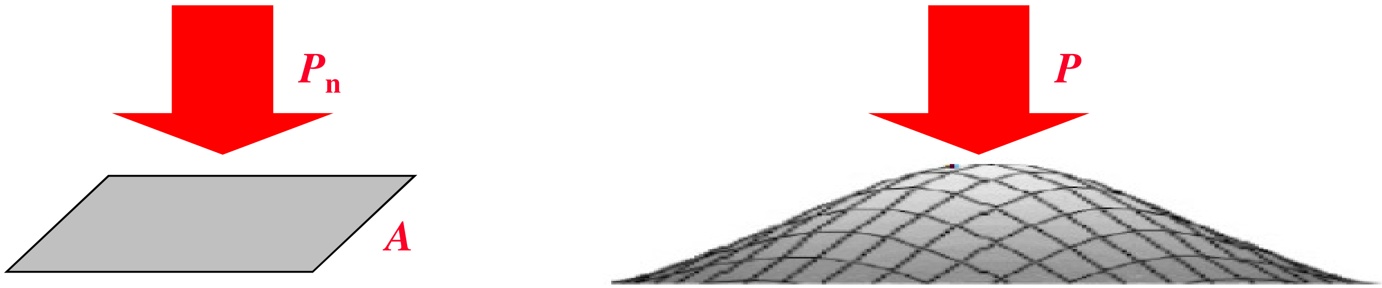



$$\int_{P_1}^{P_2} \vec{F} \bullet d\vec{s} = \int_{P_1}^{P_2} \begin{bmatrix} F_r \\ F_\phi \\ F_z \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 3d\phi \\ 0 \end{bmatrix} = 3\int_0^{\frac{\pi}{2}} F_\phi\, d\phi$$

$$= -23.14$$

# Surface Integrals

illustration: sunlight incident on (ideally flat) solar panel




need to consider general (warped) case where $P_n \to \vec{P}$ and $dA \to d\vec{A}$

$$W = \iint_A P_n \, dA = \iint_A \left( \vec{P} \bullet \hat{u}_n \right) dA = \iint_A \vec{P} \bullet \left( \hat{u}_n dA \right) = \iint_A \vec{P} \bullet d\vec{A}$$

not constrained by geometry or other special requirements

conducive for analytical or computational treatment

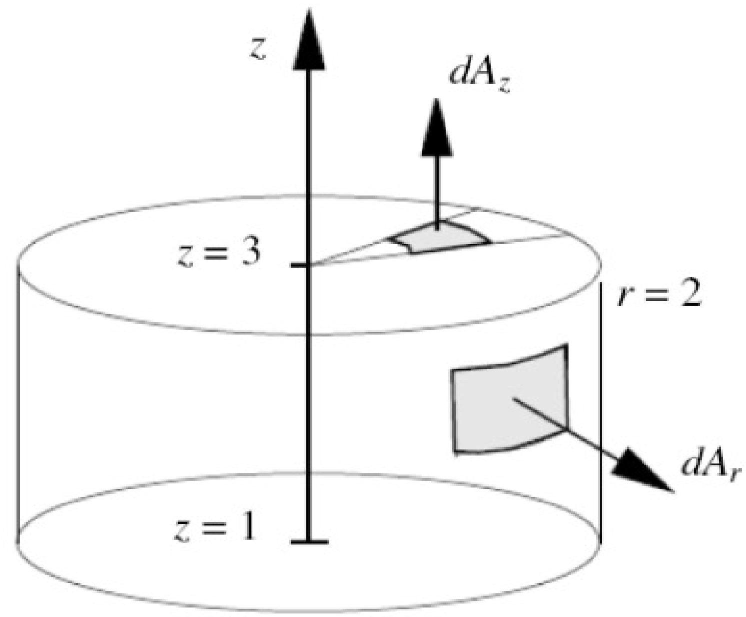
additional attention required for direction of $d\vec{A}$ (+ve convention)

# Surface Integrals

$e.g.\quad \oiint \vec{P} \bullet d\vec{A}$ for all surfaces of cylinder where $\vec{P} = \frac{3}{r}\,\hat{u}_r + 2z\,\hat{u}_z$

$$\oiint \vec{P} \bullet d\vec{A} = \oiint_{general} (\pm 1) \begin{bmatrix} P_r \\ 0 \\ P_z \end{bmatrix} \bullet \begin{bmatrix} r\,d\phi\,dz \\ dr\,dz \\ r\,dr\,d\phi \end{bmatrix}$$

$$= \iint_{top} (+1)(2\times 3)(r\,dr\,d\phi) + \iint_{bottom} (-1)(2\times 1)(r\,dr\,d\phi) + \iint_{cylinder} (+1)\tfrac{3}{2}(2\,d\phi\,dz)$$




$$= 6\int_0^2 r\,dr \int_0^{2\pi} d\phi - 2\int_0^2 r\,dr \int_0^{2\pi} d\phi + 3\int_1^3 dz \int_0^{2\pi} d\phi = 88.0$$

# Grad Operator

three-dimensional variation of, say, temperature $T(x, y, z)$ in room

change in $T$ with position $\quad \Delta T = \frac{\partial T}{\partial s_1}\Delta s_1 + \frac{\partial T}{\partial s_2}\Delta s_2 + \frac{\partial T}{\partial s_3}\Delta s_3$

coordinate-based distance
$$d\vec{s} = \begin{cases} 1dx\,\hat{u}_x + 1dy\,\hat{u}_y + 1dz\,\hat{u}_z \\ 1dr\,\hat{u}_r + rd\phi\,\hat{u}_\phi + 1dz\,\hat{u}_z \\ 1dr\,\hat{u}_r + rd\theta\,\hat{u}_\theta + r\sin\theta d\phi\,\hat{u}_\phi \end{cases}$$

define metric coefficients $\quad d\vec{s} = \lambda_1 du_1\hat{u}_1 + \lambda_2 du_2\hat{u}_2 + \lambda_3 du_3\hat{u}_3$

partial differentiation
$$\frac{\partial T}{\partial s_m} = \underbrace{\mathrm{Lim}\frac{\Delta T}{\Delta s_m}}_{\Delta s_m \to 0} = \frac{1}{\lambda_m}\underbrace{\mathrm{Lim}\frac{\Delta T}{\Delta u_m}}_{\Delta u_m \to 0} = \frac{1}{\lambda_m}\frac{\partial T}{\partial u_m}$$

# Grad Operator

three-dimensional variation of, say, temperature $T(x, y, z)$ in room

$$\Delta T = \frac{\partial T}{\partial s_1}\Delta s_1 + \frac{\partial T}{\partial s_2}\Delta s_2 + \frac{\partial T}{\partial s_3}\Delta s_3 = \begin{bmatrix} \frac{\partial T}{\partial s_1} \\ \frac{\partial T}{\partial s_2} \\ \frac{\partial T}{\partial s_3} \end{bmatrix} \bullet \begin{bmatrix} \Delta s_1 \\ \Delta s_2 \\ \Delta s_3 \end{bmatrix} = \nabla T \bullet \Delta \vec{s}$$

define grad operator

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial s_1} \\ \frac{\partial}{\partial s_2} \\ \frac{\partial}{\partial s_3} \end{bmatrix} = \begin{bmatrix} \frac{1}{\lambda_1}\frac{\partial}{\partial u_1} \\ \frac{1}{\lambda_2}\frac{\partial}{\partial u_2} \\ \frac{1}{\lambda_3}\frac{\partial}{\partial u_3} \end{bmatrix}$$

- grad (scalar) $\rightarrow$ vector
- $|\nabla T|$ = maximum rate of change of $T$ with position
- direction given by $\arg(\nabla T)$

# Grad Operator

examples:

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} (x^9 + xy^2z^3) = \begin{bmatrix} 9x^8 + y^2z^3 \\ 2xyz^3 \\ 3xy^2z^2 \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial z} \end{bmatrix} (r^9 + r\phi^2z^3) = \begin{bmatrix} 9r^8 + \phi^2z^3 \\ 2\phi z^3 \\ 3r\phi^2z^2 \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial \theta} \\ \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \end{bmatrix} (r^9 + r\phi^2\sin^3\theta) = \begin{bmatrix} 9r^8 + \phi^2\sin^3\theta \\ 3\phi^2\sin^2\theta\cos\theta \\ 2\phi\sin^2\theta \end{bmatrix}$$

# Div Operator

illustration: light emanating from source $\Omega_0$

$$W_1 = \oiint_{\Omega_1} \vec{P} \bullet d\vec{A} = \text{total radiated power}$$

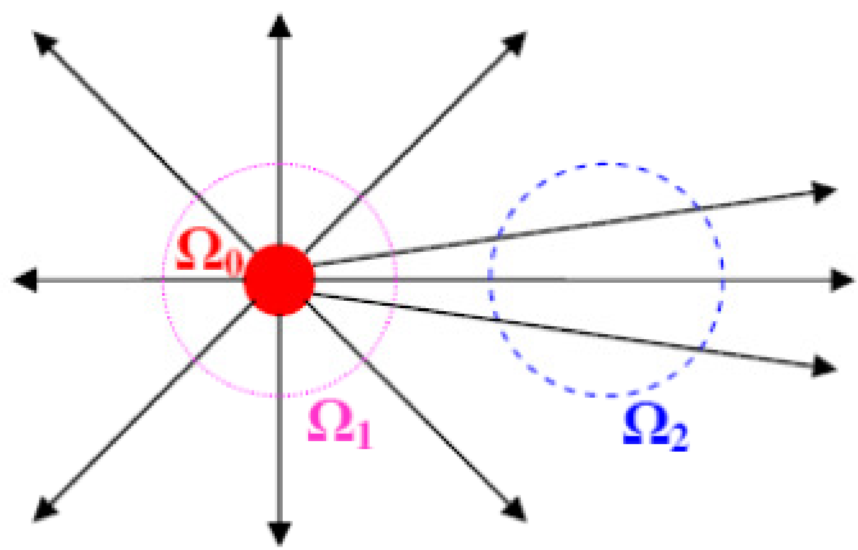
$$W_2 = \oiint_{\Omega_2} \vec{P} \bullet d\vec{A} = 0 \quad \text{since input power = output power}$$

macroscopic parameter $\oiint_{\Omega} \vec{P} \bullet d\vec{A}$ (*i.e.* need to specify surface $\Omega$)

not convenient for analysis

define differential equivalent
(*i.e.* applicable to any point)

convention: +ve for source
– ve for sink



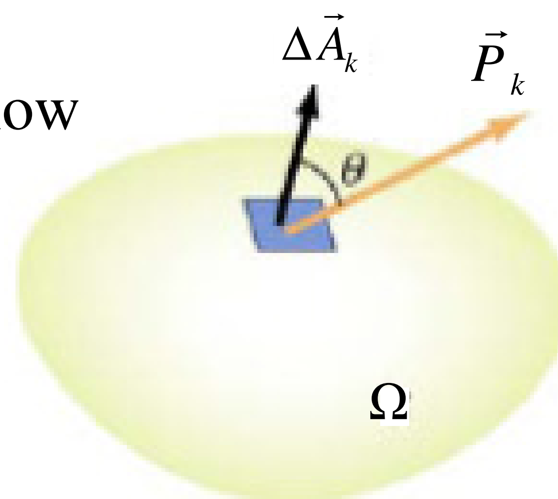
# Div Operator

general considerations:

(a) ought to normalize $\oiint_{\Omega} \vec{P} \bullet d\vec{A} \;\rightarrow\;$ divide by volume

(b) choose (elemental) volume in vicinity of point and then apply $\lim\limits_{V \to 0}$ operator

(c) apply ± convention for outward/inward flow




definition: $\operatorname{div} \vec{P} = \underbrace{\mathrm{Lim} \frac{\oiint_{\Delta V} \vec{P} \bullet d\vec{A}}{\Delta V}}_{\Delta V \to 0}$

notation: $\nabla \bullet \vec{P}$ (similar to dot-product format)

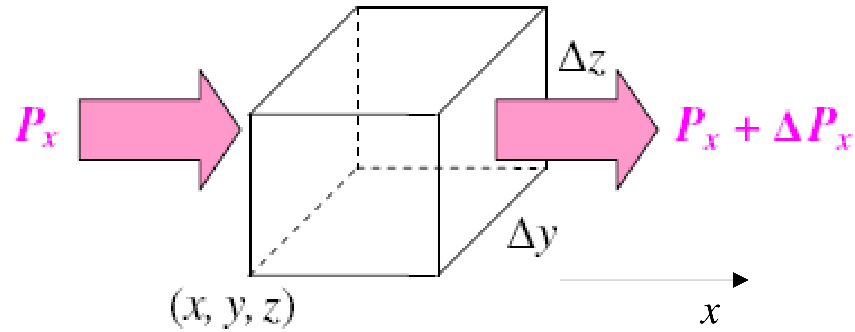
# Div Operator

div formulation in Cartesian coordinate system

small volume $\Delta V = \Delta x\ \Delta y\ \Delta z$

6 surfaces for $\oiint \vec{P} \bullet d\vec{A}$

first consider left and right $\Delta A_x$




$$\iint \vec{P} \bullet d\vec{A}_x = -\ P_x \Delta y\ \Delta z\ +\ (P_x + \Delta P_x)\ \Delta y\ \Delta z$$

$$= -\ P_x \Delta y\ \Delta z\ +\ P_x \Delta y\ \Delta z + \Delta P_x \Delta y\ \Delta z$$

$$= +\ \frac{\Delta P_x}{\Delta x} \Delta x\ \Delta y\ \Delta z$$

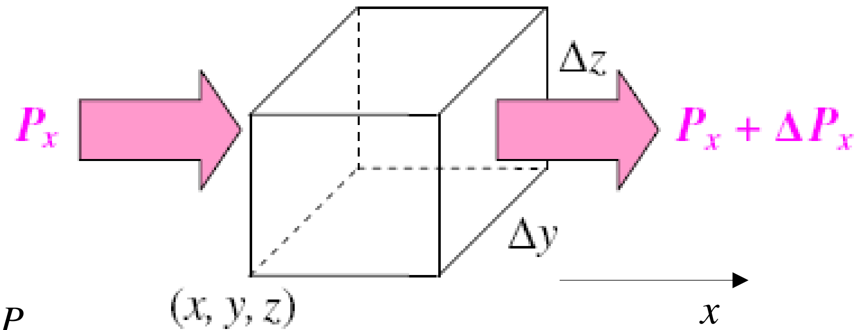
$$= +\ \frac{\partial P_x}{\partial x} \Delta V$$

similarly derive expressions for $\iint \vec{P} \bullet d\vec{A}_y$ and $\iint \vec{P} \bullet d\vec{A}_z$

# Div Operator

div formulation in Cartesian coordinate system (*continued*)

combine for all 6 surfaces




$$\oiint \vec{P} \bullet d\vec{A} = \left(\frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}\right)\Delta V$$

$$\text{div}\,\vec{P} = \underbrace{\text{Lim}\frac{\oiint \vec{P} \bullet d\vec{A}}{\Delta V}}_{\Delta V \to 0} = \frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}$$

$$\nabla \bullet \vec{P} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \bullet \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}$$

simple example:

$$\text{div}\left(x^2\hat{\text{i}} + xy\hat{\text{j}} + y^9\hat{\text{k}}\right)$$

$$= \tfrac{\partial}{\partial x}\left(x^2\right) + \tfrac{\partial}{\partial y}\left(xy\right) + \tfrac{\partial}{\partial z}\left(y^9\right)$$

$$= 3x$$

# Div Operator

div formulation in cylindrical coordinate system

small cylindrical volume $\Delta V = r\Delta r\, \Delta\phi\, \Delta z$

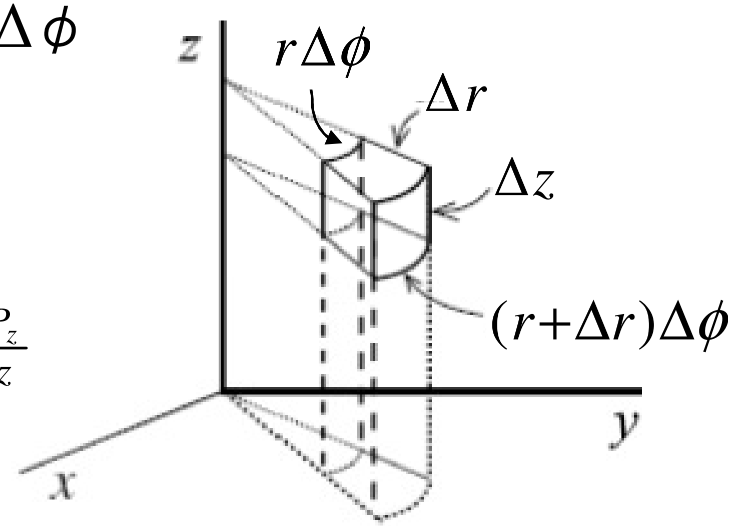
$$\oiint \vec{P} \bullet d\vec{A} = -P_r r\Delta\phi\,\Delta z + (P_r + \Delta P_r)(r + \Delta r)\Delta\phi\,\Delta z$$

$$-P_\phi \Delta r\Delta z + (P_\phi + \Delta P_\phi)\, \Delta r\Delta z$$

$$-P_z r\Delta r\Delta\phi + (P_z + \Delta P_z) r\Delta r\Delta\phi$$

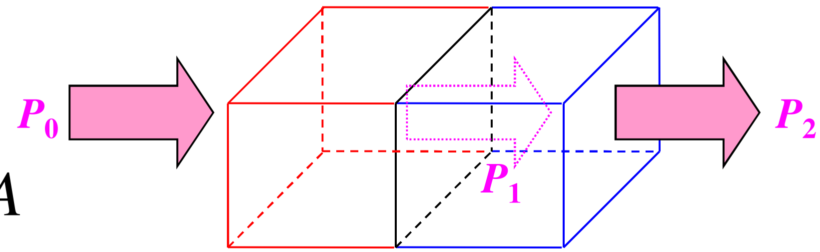
$$= \left(\frac{\Delta(rP_r)}{r\Delta r} + \frac{\Delta P_\phi}{r\Delta\phi} + \frac{\Delta P_z}{\Delta z}\right) r\Delta r\Delta\phi\Delta z$$

$$\Rightarrow \quad \text{div}\,\vec{P} = \underbrace{\text{Lim}\frac{\oiint \vec{P} \bullet d\vec{A}}{\Delta V}}_{\Delta V \to 0} = \frac{\partial(rP_r)}{r\partial r} + \frac{\partial P_\phi}{r\partial\phi} + \frac{\partial P_z}{\partial z}$$



# Divergence Theorem

combination of red and blue boxes



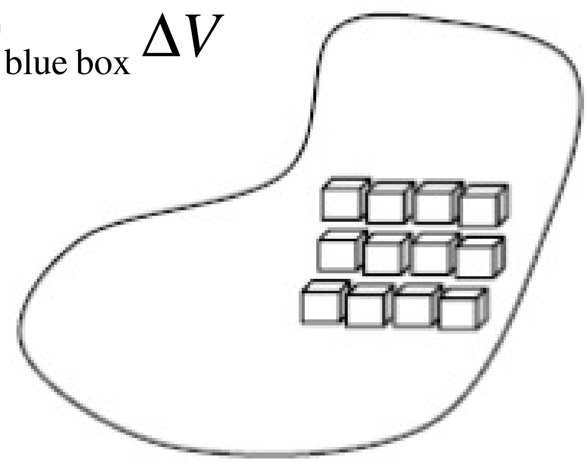

$$\oint\!\!\!\oint_{\text{both boxes}} \vec{P} \bullet d\vec{A} = -P_0 \Delta A + P_2 \Delta A$$

$$= (-P_0 \Delta A + P_1 \Delta A) + (-P_1 \Delta A + P_2 \Delta A)$$

$$= \oint\!\!\!\oint_{\text{red box}} \vec{P} \bullet d\vec{A} + \oint\!\!\!\oint_{\text{blue box}} \vec{P} \bullet d\vec{A}$$

$$= \left(\operatorname{div} \vec{P}\right)_{\text{red box}} \Delta V + \left(\operatorname{div} \vec{P}\right)_{\text{blue box}} \Delta V$$

extension to large number M of small boxes



$$\oint\!\!\!\oint_{\Omega} \vec{P} \bullet d\vec{A} = \sum_{m=1}^{M} \left(\operatorname{div} \vec{P}\right)_m \Delta V_m$$

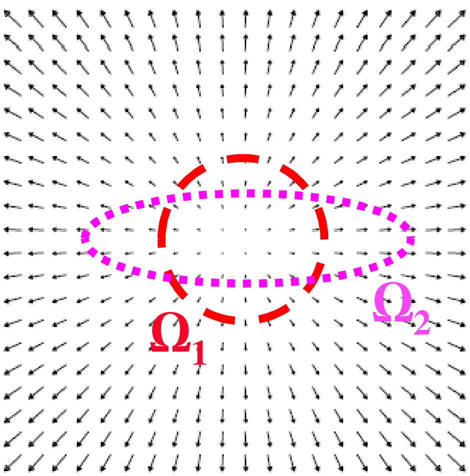
# Divergence Theorem

also known as Gauss's Divergence Theorem

$$\oiint_{\Omega} \vec{P} \bullet d\vec{A} = \iiint_{V} \nabla \bullet \vec{P}\, dV$$

LHS: view integral as flow out of enclosed surface $\Omega$

RHS: treat $\text{div}\,\vec{P}$ as source (or sink) at particular location
view volume integral as sum of sources/sinks in volume $V$

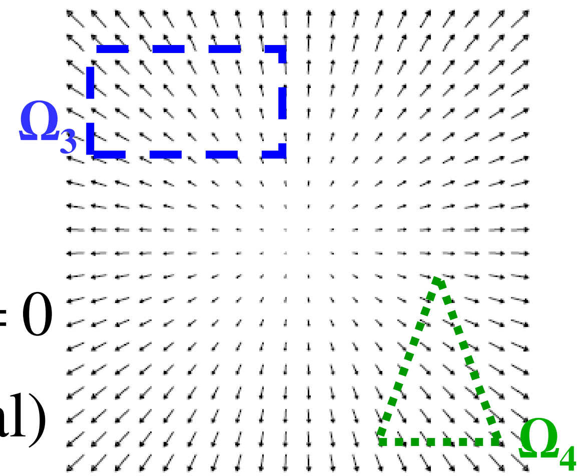



<u>special cases</u>:

$$\oiint_{\Omega_1} \vec{P} \bullet d\vec{A} = \oiint_{\Omega_2} \vec{P} \bullet d\vec{A}$$

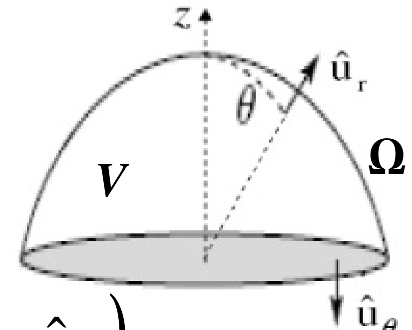
$$\oiint_{\Omega_3} \vec{P} \bullet d\vec{A} = \oiint_{\Omega_4} \vec{P} \bullet d\vec{A} = 0$$

(solenoidal)



# Divergence Theorem

verification example: hemisphere with radius = 2




$$\vec{D} = r^2\left(\hat{u}_r + \sin\theta\,\hat{u}_\theta + \sin\theta\sin\phi\,\hat{u}_\phi\right)$$

$$\oiint_\Omega \vec{D}\bullet d\vec{A} = \oiint_\Omega \left(D_r\hat{u}_r + D_\theta\hat{u}_\theta + D_\phi\hat{u}_\phi\right)\bullet\left(dA_r\hat{u}_r + dA_\theta\hat{u}_\theta\right)$$

$$= \iint_{\Omega_r}\left(r^2\right)r^2\sin\theta\,d\theta\,d\phi\Big|_{r=2} + \iint_{\Omega_\theta}\left(r^2\sin\theta\right)r\sin\theta\,dr\,d\phi\Big|_{\theta=\frac{1}{2}\pi}$$

$$= 16\int_0^{\frac{1}{2}\pi}\sin\theta\,\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi + \int_0^2 r^3\,dr\int_0^{2\pi}\mathrm{d}\phi \;=\; 64\pi$$

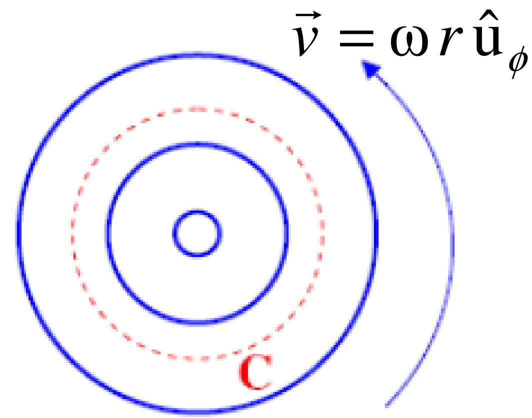
$$\iiint_V \nabla\bullet\vec{D}\,dV = \iiint_V\left\{\tfrac{1}{r^2}\tfrac{\partial}{\partial r}\left(r^2 D_r\right) + \tfrac{1}{r\sin\theta}\tfrac{\partial}{\partial\theta}\left(\sin\theta\,D_\theta\right) + \tfrac{1}{r\sin\theta}\tfrac{\partial}{\partial\phi}\left(D_\phi\right)\right\}dV$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\frac{1}{2}\pi}\int_{r=0}^{2}\left\{4r + 2r\cos\theta + r\cos\phi\right\}r^2\sin\theta\,dr\,d\theta\,d\phi = 64\pi$$

# Curl Operator

illustration: water flowing in circular paths

choose any circular path C




$\oint_C \vec{v} \bullet d\vec{s}$ macroscopic measure of circulation (*i.e.* need to specify contour C)

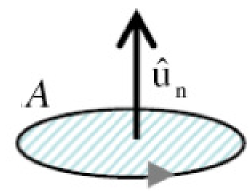
define differential equivalent (*i.e.* applicable to any point)

(a) normalize (divide by loop area)

(b) reduce (elemental) loop to point

(c) note direction convention

$$(\text{curl } \vec{v}) \bullet \hat{u}_n = \underbrace{\text{Lim} \frac{\oint_C \vec{v} \bullet d\vec{s}}{A}}_{A \to 0}$$




$$\text{water example: } (\text{curl } \vec{v})_z = \underbrace{\text{Lim} \frac{\oint_C (\omega\, r\, \hat{u}_\phi) \bullet (r\, \hat{u}_\phi)}{\pi\, r^2}}_{r \to 0} = \frac{\omega\, r^2 \int_0^{2\pi} d\phi}{\pi\, r^2} = 2\omega$$

# Curl Operator

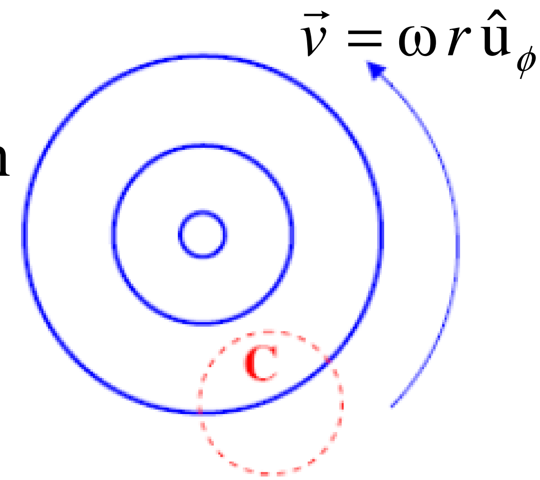
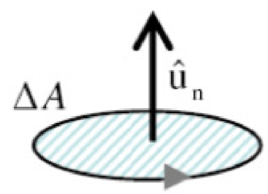



illustration: water flowing in circular paths

choose any circular path C not centred at origin

apply definition at point of interest

$$(\text{curl } \vec{v}) \bullet \hat{u}_n = \underbrace{\text{Lim} \frac{\oint_C \vec{v} \bullet d\vec{s}}{\Delta A}}_{\Delta A \to 0}$$

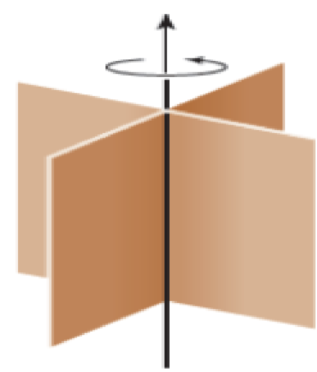



take all three directions into account for $\text{curl } \vec{v} \equiv \nabla \times \vec{v}$

simple (mental) test:

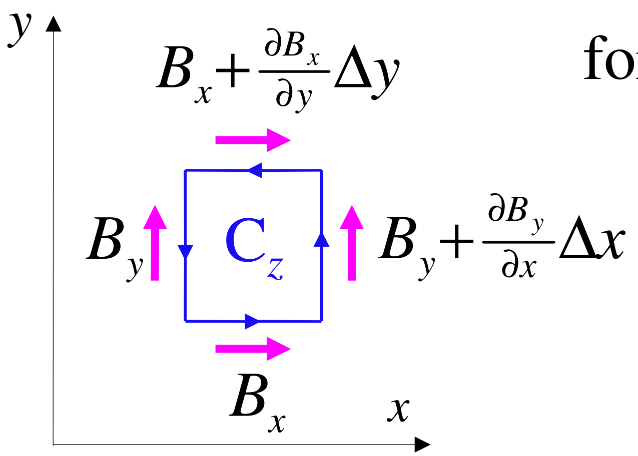
(a) place tiny paddle wheel at point of interest

(b) no spin $\Rightarrow$ curl $\vec{v} = \vec{0}$ (*i.e.* irrotational)



# Curl Operator

curl formulation in Cartesian coordinate system




for elemental loop $C_z$ with area $\Delta A_z = \Delta x \, \Delta y$

$$\oint_{C_z} \vec{B} \bullet d\vec{s} = +B_x \Delta x - \left(B_x + \frac{\partial B_x}{\partial y}\Delta y\right)\Delta x$$

$$- B_y \Delta y + \left(B_y + \frac{\partial B_y}{\partial x}\Delta x\right)\Delta y$$

$$= \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)\Delta x \, \Delta y$$

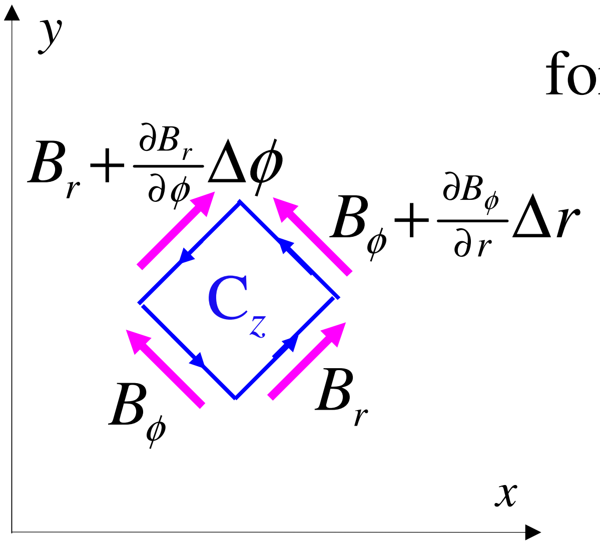
same process for $C_x$ and $C_y$

divide by respective areas

$$\text{curl } \vec{B} = \begin{bmatrix} \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \\ \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \\ \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \end{bmatrix} = \begin{vmatrix} \hat{\mathrm{i}} & \hat{\mathrm{j}} & \hat{\mathrm{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}$$

# Curl Operator

curl formulation in cylindrical coordinate system




for elemental loop $C_z$ with area $\Delta A_z = r\,\Delta r\,\Delta\phi$

$$\oint_{C_z} \vec{B} \bullet d\vec{s} = +B_r \Delta r - \left(B_r + \frac{\partial B_r}{\partial \phi}\Delta\phi\right)\Delta r - B_\phi r \Delta\phi + \left(B_\phi + \frac{\partial B_\phi}{\partial r}\Delta r\right)(r+\Delta r)\Delta\phi$$

$$= \left(\frac{1}{r}\frac{\partial (r B_\phi)}{\partial r} - \frac{1}{r}\frac{\partial B_r}{\partial \phi}\right) r\,\Delta r\,\Delta\phi$$

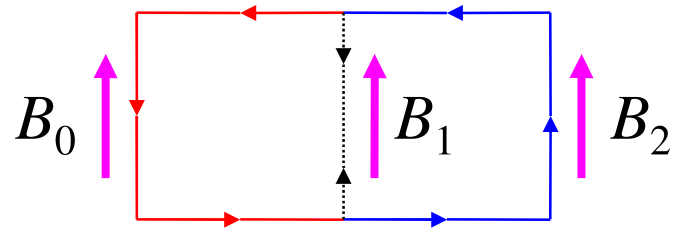
repeat for other directions

divide by respective areas

$$\text{curl } \vec{B} = \begin{bmatrix} \frac{1}{r}\frac{\partial B_z}{\partial \phi} - \frac{\partial B_\phi}{\partial z} \\ \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} \\ \frac{1}{r}\frac{\partial (r B_\phi)}{\partial r} - \frac{1}{r}\frac{\partial B_r}{\partial \phi} \end{bmatrix}$$

# Stoke's Theorem

combination of red and blue loops



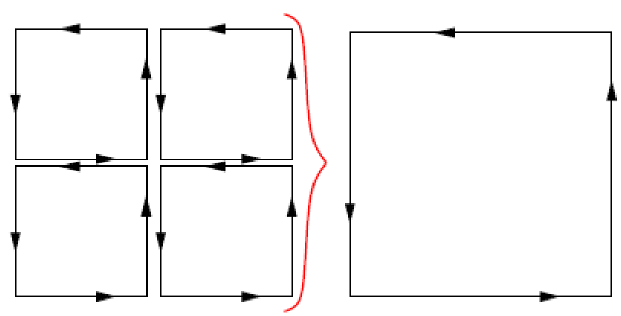

$$\oint_{\substack{\text{red and}\\\text{blue loops}}} \vec{B} \bullet d\vec{s} = -B_0 \Delta y + B_2 \Delta y$$

$$= (-B_0 \Delta y + B_1 \Delta y) + (-B_1 \Delta y + B_2 \Delta y)$$

$$= \oint_{\text{red loop}} \vec{B} \bullet d\vec{s} + \oint_{\text{blue loop}} \vec{B} \bullet d\vec{s}$$

$$= \left(\text{curl } \vec{B}\right)_{\text{red}} \Delta A + \left(\text{curl } \vec{B}\right)_{\text{blue}} \Delta A$$

extension to many small (planar) loops

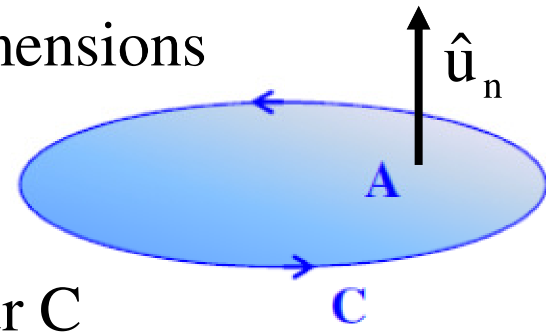


$$\oint_{\text{loop}} \vec{B} \bullet d\vec{s} = \sum_{m=1}^{M} \left(\text{curl}\,\vec{B}\right)_m \Delta A_m$$

# Stoke's Theorem

generalize in all three directions / dimensions

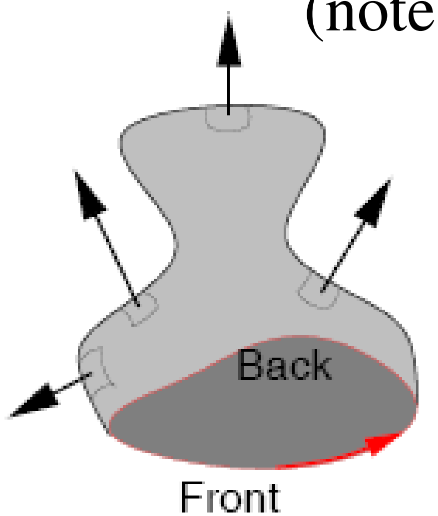
$$\oint_C \vec{B} \bullet d\vec{s} = \iint_A \nabla \times \vec{B} \bullet d\vec{A}$$

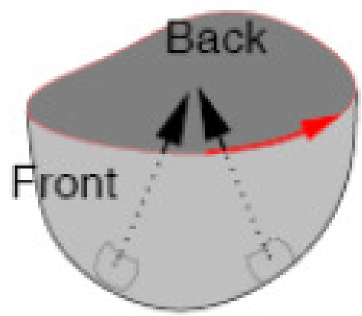


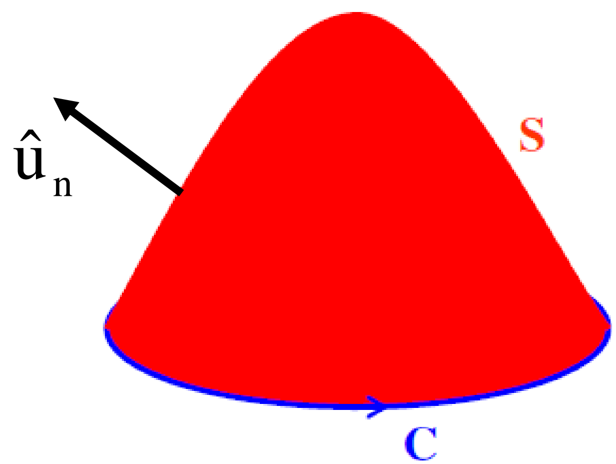

for planar area A enclosed by contour C

as well as for any surface S with C as boundary

(note: direction of S specified by sense of C)



# Stoke's Theorem

verification example: quarter-circle loop with $\vec{F} = xy\,\hat{\mathrm{i}} - 2x\,\hat{\mathrm{j}} + 0\,\hat{\mathrm{k}}$

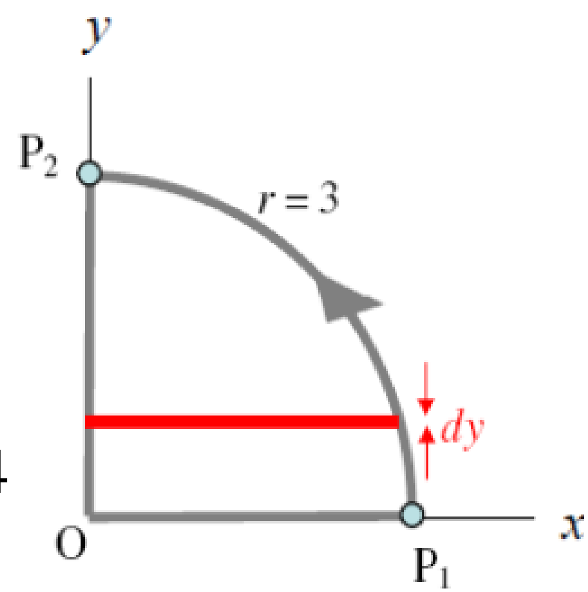
$$\oint \vec{F} \bullet d\vec{s} = \int_{P_1}^{P_2} \vec{F} \bullet d\vec{s} + \int_{P_2}^{O} \vec{F} \bullet d\vec{s} + \int_{O}^{P_1} \vec{F} \bullet d\vec{s}$$

$$= \int_0^{\frac{\pi}{2}} F_\phi (3\,d\phi) + \int_3^0 F_y dy \Big|_{x=0} + \int_0^3 F_x dx \Big|_{y=0}$$

$$= -23.14 + 0 + 0$$

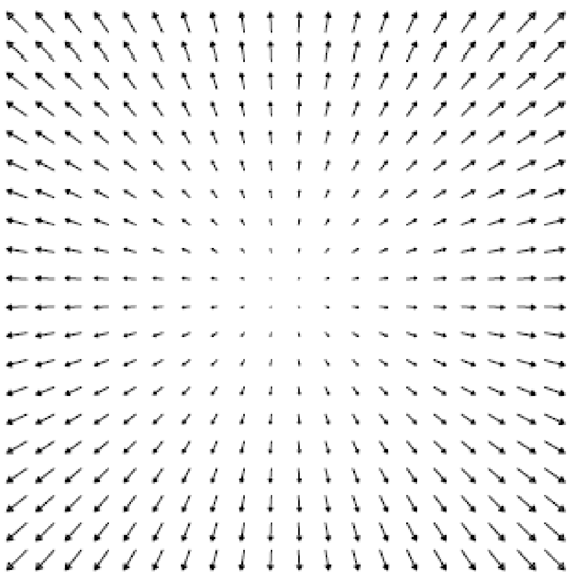
$$\iint \nabla \times F \bullet d\vec{A} = \iint \begin{vmatrix} \hat{\mathrm{i}} & \hat{\mathrm{j}} & \hat{\mathrm{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & -2x & 0 \end{vmatrix} \bullet d\vec{A}$$

$$= \int_0^3 \int_0^{\sqrt{9-y^2}} (-2-x)\,dx\,dy = -23.14$$



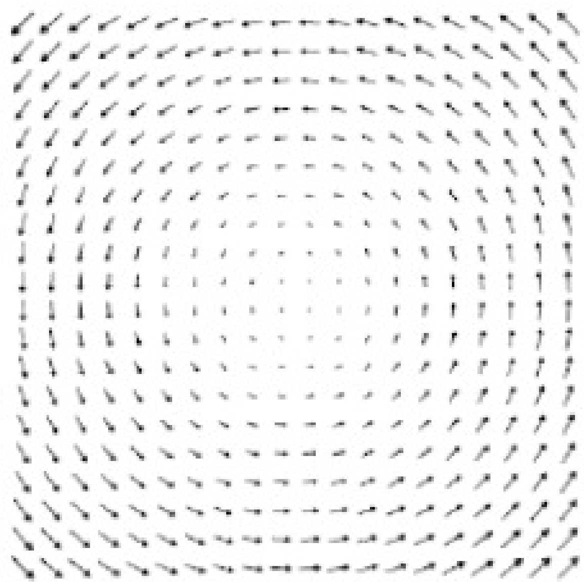
# Stoke's Theorem

examples of field patterns



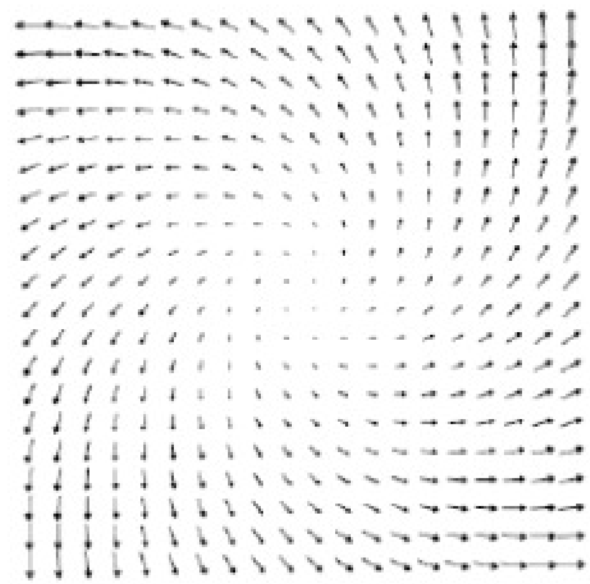
$$\nabla \bullet \vec{F} \neq 0$$

$$\nabla \times \vec{F} = \vec{0}$$



$$\nabla \bullet \vec{F} = 0$$

$$\nabla \times \vec{F} \neq \vec{0}$$



$$\nabla \bullet \vec{F} \neq 0$$

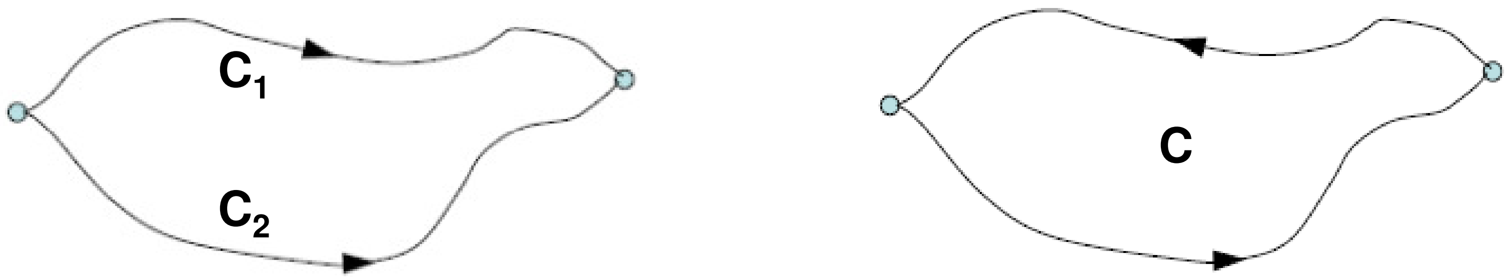
$$\nabla \times \vec{F} \neq \vec{0}$$

# Stoke's Theorem

Conservative Fields

*e.g.* gravitational or electrostatic force

same work to move mass or charge along any path from A to B




no work to move mass or charge along any closed path

$$\oint_C \vec{F} \bullet d\vec{s} = 0 \quad \Leftrightarrow \quad \iint_A \nabla \times \vec{F} \bullet d\vec{A} = 0$$

also need definition at any point $\qquad \nabla \times \vec{F} = \vec{0}$

# Important Null Identity #1

use Cartesian coordinate system for ease of understanding

$$\operatorname{curl}(\operatorname{grad} V) = \operatorname{curl}\begin{bmatrix} \frac{\partial}{\partial x}V \\ \frac{\partial}{\partial y}V \\ \frac{\partial}{\partial z}V \end{bmatrix} = \begin{vmatrix} \hat{\mathrm{i}} & \hat{\mathrm{j}} & \hat{\mathrm{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x}V & \frac{\partial}{\partial y}V & \frac{\partial}{\partial z}V \end{vmatrix} = \begin{bmatrix} \left(\frac{\partial^2}{\partial y \partial z} - \frac{\partial^2}{\partial z \partial y}\right)V \\ \left(\frac{\partial^2}{\partial z \partial x} - \frac{\partial^2}{\partial x \partial z}\right)V \\ \left(\frac{\partial^2}{\partial x \partial y} - \frac{\partial^2}{\partial y \partial x}\right)V \end{bmatrix} = \vec{0}$$

null result when using other coordinate systems as well

<u>implication</u>**:** used to define (scalar) potential for conservative field

$\vec{E} = \pm\nabla V$ subject to boundary conditions

otherwise not unique because of different possible $V$

$$\nabla \times (\nabla V_1 + \nabla V_2) = \nabla \times (\nabla V_1) + \nabla \times (\nabla V_2) = \vec{0}$$

# Important Null Identity #2

use Cartesian coordinate system for ease of understanding

$$\operatorname{div}(\operatorname{curl}\vec{A}) = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \bullet \begin{bmatrix} \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \\ \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \\ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \end{bmatrix} = \frac{\partial^2}{\partial x \partial y} A_z - \frac{\partial^2}{\partial x \partial z} A_y + \frac{\partial^2}{\partial y \partial z} A_x - \frac{\partial^2}{\partial y \partial x} A_z$$

$$+ \frac{\partial^2}{\partial z \partial x} A_y - \frac{\partial^2}{\partial z \partial y} A_x = \vec{0}$$

null result when using other coordinate systems as well

<u>implication</u>**:** used to define (vector) potential for solenoidal field

$\vec{B} = \nabla \times \vec{A}$ subject to boundary conditions

otherwise not unique because of different possible $V$

$$\nabla \bullet \left(\nabla \times \vec{A}_1 + \nabla \times \vec{A}_2\right) = \nabla \bullet \left(\nabla \times \vec{A}_1\right) + \nabla \bullet \left(\nabla \times \vec{A}_2\right) = 0$$