Tutorial 4

Q1

(a) Recall : An equilibrium point (solution) of a given ODE is a solution which is constant.

If x_E is an equilibrium pt of $\ddot{x} = f(x)$,

then $\underset{\substack{\| \\ 0}}{\ddot{x}_E} = f(x_E)$

$\therefore \quad f(x_E) = 0$

So finding equilibrium point is finding roots of $f(x) = 0$

(b) Stability of equilibrium pt

To discuss stability of equilibrium pt, we just need to look at those x near the equilibrium pt

Hence we want to find the approximate value of $f(x)$, where x near the equilibrium pt

We can use the tangent line at equilibrium pt to approximate $f(x)$

$$f'(x_E) = \lim_{x \to x_E} \frac{f(x) - f(x_E)}{x - x_E}$$

$$\approx \frac{f(x) - f(x_E)}{x - x_E}$$

$$f(x) \approx \underbrace{f(x_E)}_{0} + f'(x_E)(x - x_E)$$

$$= f'(x_E)\,x - f'(x_E)\,x_E$$

So near the equilibrium pt, the given ODE can be approximated by

$$\ddot{x} = f(x) \approx f'(x_E)\,x - f'(x_E)\,x_E$$

$\therefore$ We just need to discuss the stability of

$$\ddot{x} = f'(x_E)\,x - f'(x_E)\,x_E$$

$$\ddot{x} - f'(x_E)\,x = -f'(x_E)\,x_E$$

2nd order nonhomogeneous ODE

Case 1 $\quad f'(x_E) < 0$. Let $w^2 = -f'(x_E)$

$$x = x_h + x_p$$

$$= \underbrace{A\cos wt + B\sin wt}_{\text{SHM}} + \underset{\text{constant}}{x_E}$$

If the initial value is close to the equilibrium pt x_E, then x oscillates near x_E.

Hence the equilibrium pt x_E is stable.

Case 2 $\quad f'(x_E) > 0$. Let $w^2 = f'(x_E)$

$$x = x_h + x_p$$

$$= Ae^{wt} + Be^{-wt} + x_E$$

x will move away from equilibrium pt. x_E. Hence x_E is not stable

<u>Case 3</u> $f'(x_E) = 0$

$\therefore \quad \ddot{x} = f'(x_E)\,x + x_E = x_E$

$\therefore \quad \dot{x} = x_E t + A$

$x = \frac{1}{2} x_E t^2 + At + B$

$\therefore$ x will move away from x_E

$\therefore$ x_E not stable

(i) $\ddot{x} = \cosh x = \dfrac{e^x + e^{-x}}{2}$

$f(x) = \dfrac{e^x + e^{-x}}{2} \neq 0$ for all x

$\therefore$ $\ddot{x} = \cosh x$ has No equilibrium pt

(ii) $\ddot{x} = \cos x$

$f(x) = \cos x$

$f(x_E) = 0 \Leftrightarrow x_E = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

$f'(x) = -\sin x$

$f'(x_E) = -\sin x_E = \begin{cases} + & \text{if } x_E = \frac{3\pi}{2} \\ - & \text{if } x_E = \frac{\pi}{2} \end{cases}$

$\therefore$ At $x_E = \frac{\pi}{2}$ stable

$x_E = \frac{3\pi}{2}$ unstable

when $x_E = \frac{\pi}{2}$

$$\omega^2 = -f'(x_E) = -(-\sin x_E)$$

$$= \sin x_E$$

$$= \sin \frac{\pi}{2} = 1$$

$$\omega = 1$$

(iii) $\ddot{x} = \tan(\sin x)$

$\therefore$ $f(x) = \tan(\sin x)$

$f(x_E) = 0 \iff x_E = 0, \pi, \cdots$

$f'(x) = [\sec^2(\sin x)] \cos x$

$f'(0) = 1$ unstable

$f'(\pi) = -1$ stable

$$\omega^2 = -f'(\pi) = 1$$

$$\omega = 1$$

Q3

Q3
Amplitude $A(\alpha)$ is given by

$$A(\alpha) = \frac{F_0/m}{\sqrt{(w^2-\alpha^2)^2 + \frac{b^2}{m^2}\alpha^2}}$$

Let $f(\alpha) = (w^2-\alpha^2)^2 + \frac{b^2}{m^2}\alpha^2$

Hence $A(\alpha)$ local maxi (local mini)

$\Leftrightarrow$ $f(\alpha)$ local mini (local maxi)

Find $f'(\alpha)$

$$f'(\alpha) = 2(w^2-\alpha^2)(-2\alpha) + \frac{b^2}{m^2}2\alpha$$
$$= 4\alpha\left[\alpha^2 - \left(w^2 - \frac{b^2}{2m^2}\right)\right]$$
$$f''(\alpha) = 12\alpha^2 - \left(w^2 - \frac{b^2}{2m^2}\right)4$$

Two cases:

<u>Case 1</u> $\omega^2 - \frac{b^2}{2m^2} \geqslant 0$

$$\left(\omega^2 \geqslant \frac{b^2}{2m^2} \right)$$

$f'(\alpha) = 0 \Leftrightarrow \alpha = 0$ or $\alpha^2 = \omega^2 - \frac{b^2}{2m^2}$

<u>Case 2</u> $\omega^2 - \frac{b^2}{2m^2} < 0$

$$\left(\omega^2 < \frac{b^2}{2m^2} \right)$$

$$f'(\alpha) = 0 \Leftrightarrow \alpha = 0$$

(Note that when $\omega^2 - \frac{b^2}{2m^2} < 0$

then $\alpha^2 - \left(\omega^2 - \frac{b^2}{2m^2}\right) \neq 0$)

Case 1 $\quad \omega^2 \geq \dfrac{b^2}{2m^2}$

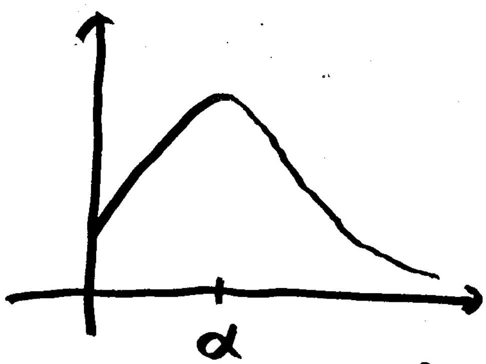
By 2nd derivative test,

At $\alpha = 0$, $\quad f(\alpha)$ is maxi (local)

$\alpha^2 = \omega^2 - \dfrac{b^2}{2m^2}$, $\quad f(\alpha)$ is mini (local)

$\therefore$ At $\alpha = 0$, $\quad A(\alpha)$ is mini (local)

$\alpha^2 = \omega^2 - \dfrac{b^2}{2m^2}$, $\quad A(\alpha)$ is maxi (local)



where $\alpha^2 = \omega^2 - \dfrac{b^2}{2m^2}$

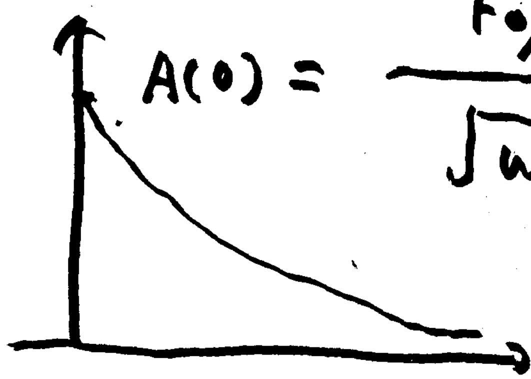
Case 2 $\qquad \omega^2 < \frac{b^2}{2m^2}$

By 2nd derivative test,

At $\alpha = 0$, $\quad f(\alpha)$ is mini (local)

$\therefore$ At $\alpha = 0$, $\quad A(\alpha)$ is maxi (local)

$$A(0) = \frac{F_0/m}{\sqrt{\omega^4}} = \frac{F_0}{\omega^2 m}$$



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when b is small

so $\quad \omega^2 \geq \frac{b^2}{2m^2} \quad$ case 1

$$A(\alpha) = \frac{F_0/m}{\sqrt{(\omega^2-\alpha^2)^2 + \frac{b^2}{m^2}\alpha^2}} \qquad \alpha^2 = \omega^2 - \frac{b^2}{2m^2}$$

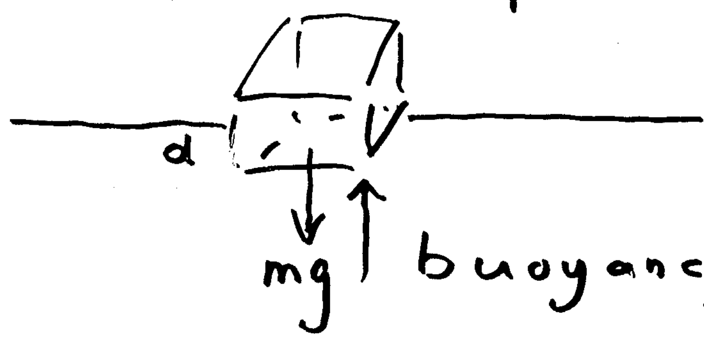
$$= \frac{F_0/m}{\sqrt{\left(\frac{b^2}{2m}\right)^2 + \frac{b^2}{m^2}\left(\omega^2 - \frac{b^2}{2m^2}\right)}}$$

$$A(\alpha) = \frac{F_0/m}{\sqrt{\frac{b^2\omega^2}{m^2} - \frac{1}{4}\frac{b^4}{m^4}}}$$

$$= \frac{F_0/m}{\frac{b\omega}{m}\sqrt{1-\frac{1}{4}\frac{b^2}{m^2\omega^2}}}$$

$$\approx \frac{F_0/m}{\frac{b\omega}{m}} = \frac{F_0}{b\omega}$$

Q4



mg = buoyancy force

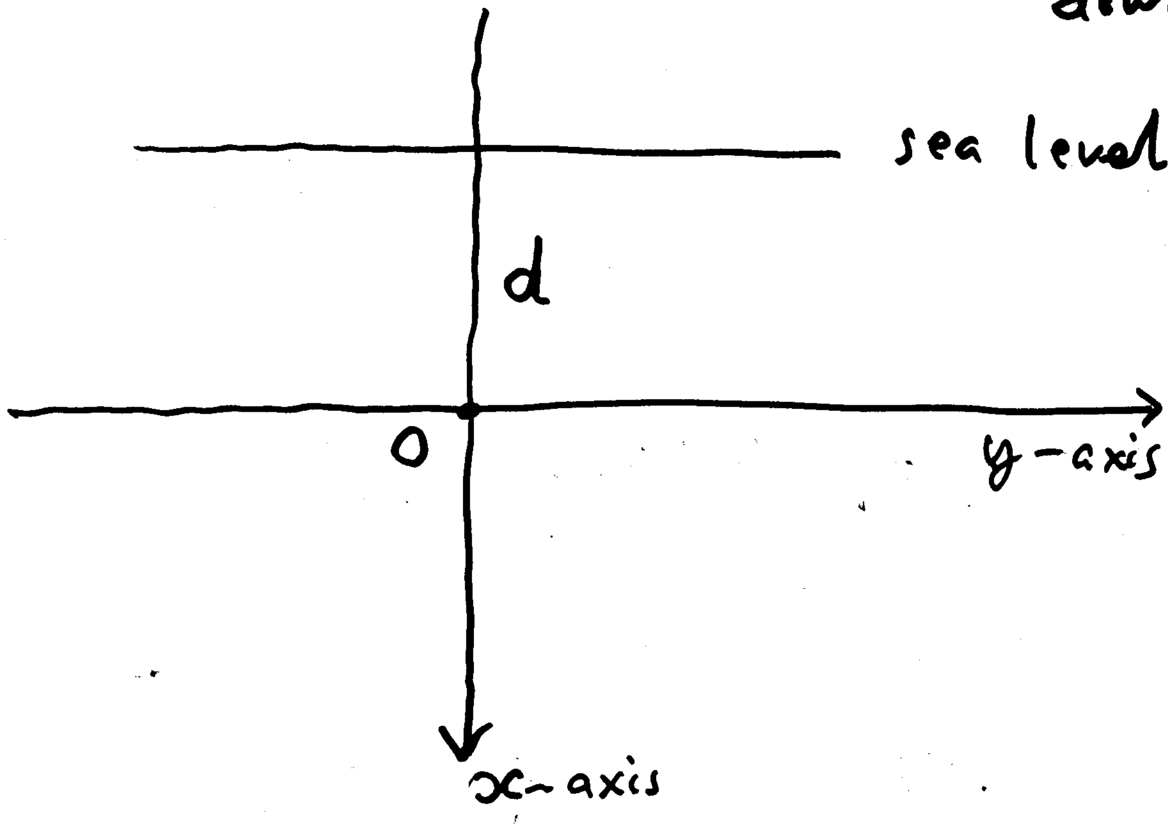
= weight of displaced water

$mg = \rho$ (volume of displaced water) g

$= \rho A d g = (\rho A g)\, d$

spring (restoring) constant

ship moving up and down



SHM

$$m\ddot{x} = -\rho A x g = -(\rho A g)x$$

spring (restorin
constant

why?

weight of displaced water

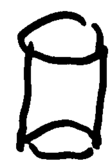
$$m\ddot{x} = mg - \rho A(x+d)g$$

$$= mg - \rho A d g - \rho A x g$$

$$= -\rho A x g$$

$\therefore \quad m\ddot{x} = -kx \quad$ where $k = \rho A g$

This formula holds for any right prism with cross-section area A. e.g cylinder



$$\ddot{x} = -\frac{k}{m} x$$

where

$$\frac{k}{m} = \frac{\rho A g}{m}$$

$$\text{or} = \frac{g}{d} \qquad \text{since } mg = (\rho A g) d$$

$$\omega^2 = \frac{\rho A g}{m} \quad \text{or} \quad \frac{g}{d} \quad \left(\omega^2 = \frac{k}{m}\right)$$

$$\omega = \sqrt{\frac{\rho A g}{m}} \quad \text{or} \quad \sqrt{\frac{g}{d}}$$

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos(\alpha t)$$

friction

external force due to wave

$$k = \rho A g$$

Ref: 2.5 Forced Damped Oscillators

general soln $x(t)$

= particular soln + general soln of $m\ddot{x} + b\dot{x} + kx = 0$

tends to 0 rapidly (transient soln)

$\simeq$ particular soln (called steady-state soln)

$$= \frac{\frac{1}{m} F_0 \cos(\alpha t - \gamma)}{\sqrt{(\omega^2 - \alpha^2)^2 + \frac{b^2}{m^2}\alpha^2}}$$

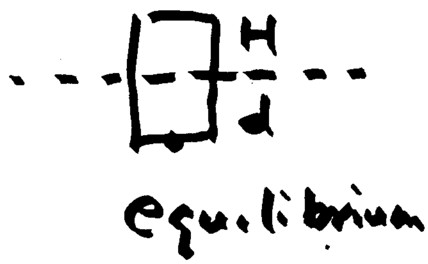
$$\omega = \sqrt{\frac{k}{m}}$$

Amplitude

$$A(\alpha) = \frac{F_0/m}{\sqrt{(\omega^2-\alpha^2)^2 + \frac{b^2}{m^2}\alpha^2}}$$

we want

$$A(\alpha) < H + d$$



For convenience, we want

$$A(\alpha) < H \quad \text{for all } \alpha$$

$$A(\alpha) \text{ maxi} \iff \alpha^2 = \omega^2 - \frac{b^2}{2m^2}$$

most dangerous α

$$A_{maxi} = \frac{F_0/m}{\sqrt{\frac{b^2\omega^2}{m^2} - \frac{1}{4}\frac{b^4}{m^4}}} \qquad \omega = \sqrt{\frac{\rho A g}{m}}$$

$$A_{maxi} = \frac{2mF_0}{b\sqrt{4\rho mAg - b^2}} < H$$

Choose m and A such that above holds