Solutions to Tutorial 7

5.72 (a) The independent random variables X_1 and X_2 have the same probability distribution $b(x; 2, .4)$. Hence the joint probability distribution of X_1 and X_2 is

$$\begin{aligned} f(x_1, x_2) &= b(x_1; 2, .4) \cdot b(x_2; 2, .4) = \binom{2}{x_1} .4^{x_1} .6^{2-x_1} \cdot \binom{2}{x_2} .4^{x_2} .6^{2-x_2} \\ &= \binom{2}{x_1}\binom{2}{x_2} .4^{x_1+x_2} .6^{4-x_1-x_2} \end{aligned}$$

where $x_1 = 0, 1, 2$, and $x_2 = 0, 1, 2$.

(b)

$$\begin{aligned} P(X_1 < X_2) &= f(0,1) + f(0,2) + f(1,2) \\ &= \binom{2}{0}\binom{2}{1} .4^1 .6^3 + \binom{2}{0}\binom{2}{2} .4^2 .6^2 \\ &\quad + \binom{2}{1}\binom{2}{2} .4^3 .6^1 \\ &= .1728 + .0576 + .0768 = .3072 \end{aligned}$$

5.111 (a) P(error will be between -0.03 and 0.04)

$$= \int_{-.03}^{.04} 25dx = \int_{-.02}^{.02} 25dx = 1$$

(b) P(error will be between -0.005 and 0.005) $= 25(.01) = .25$

6.3 (a) A typical member of the population does not vacation on a luxury cruise. The sample taken would be biased.

(b) This sample will very likely be biased. Those will high incomes will tend to respond while those with low incomes will tend to not respond.

(c) Everyone feels that unfair things should be stopped. The way the question is phrased biases the responses.

6.5 (a) The number of samples (given that order does not matter) is

$$\binom{7}{2} = \frac{7 \cdot 6}{2 \cdot 1} = 21.$$

(b) The number of samples (given that order does not matter) is

$$\binom{24}{2} = \frac{24 \cdot 23}{2 \cdot 1} = 276.$$

Solutions to Tutorial 7

6.7 (a) The probability of each of the numbers is given in the table:

Number	-4	-3	-2	-1	0	1	2	3	4
Probability	$1/6$	$2/15$	$1/10$	$1/15$	$1/15$	$1/15$	$1/10$	$2/15$	$1/6$

The mean of the distribution is

$$\frac{1}{6}(-4)+\frac{2}{15}(-3)+\cdots+\frac{2}{15}(3)+\frac{1}{6}(4)$$
$$=\frac{1}{6}(4-4)+\frac{2}{15}(3-3)+\frac{1}{10}(2-2)+\frac{1}{15}(1-1)+\frac{1}{15}(0)=0.$$

The variance is

$$\frac{2}{6}(16)+\frac{4}{15}(9)+\frac{2}{10}(4)+\frac{2}{15}(1)+\frac{1}{15}(0)=8.667.$$

(b) The 50 samples are shown in Table 6.1 along with their means.

Table 6.1. 50 Samples of Size 10 Taken Without Replacement

obs.	obs.	obs.	obs.	obs.	obs.	obs.	obs.	obs.	obs.	mean
1	3	-1	4	4	4	-4	-4	-4	-2	0.1
-2	3	3	-2	2	-4	4	-4	-3	-4	-0.7
4	4	-2	-3	-3	2	1	3	4	0	1.0
3	3	-3	4	2	1	-2	0	4	0	1.2
4	0	-1	-2	3	1	2	4	-1	1	1.1
1	-2	-3	-4	2	4	-2	3	4	-4	-0.1
-1	2	1	-1	-4	-3	0	-4	3	1	-0.6
-4	-2	2	1	-4	-3	3	2	-2	1	-0.6
1	-3	2	-4	2	4	3	-4	-3	-2	-0.4
1	4	4	4	-4	3	-1	-2	2	-4	0.7
4	0	-1	1	-3	1	4	-3	4	2	0.9
-4	4	2	-4	2	-2	-2	-1	4	3	0.2
2	-1	-2	4	-4	0	-4	1	3	-2	-0.3
-4	-1	-3	4	0	-4	-3	1	2	4	-0.4
-2	4	4	-1	-4	1	4	1	-4	-3	0.0
-3	0	-3	-2	0	-4	4	-1	1	-2	-1.0
4	-4	3	4	-4	-1	3	2	0	1	0.8
3	3	-3	4	-3	1	3	-3	-4	-1	0.0
3	-4	-4	4	4	3	4	-3	2	-2	0.7
4	0	2	-3	-3	-3	-2	3	-4	-4	-1.0
2	-1	4	-3	2	3	4	4	-4	4	1.5
3	2	3	4	4	4	3	1	2	-2	2.4
4	-3	0	-1	-3	-3	0	-4	4	-4	-1.0
-3	3	-4	3	0	1	-4	-4	-3	-4	-1.5
-2	-4	-2	-3	4	4	3	-3	1	-3	-0.5
3	-1	-2	-3	1	-4	-2	3	4	2	0.1
4	2	-4	3	-2	2	-2	1	4	3	1.1
0	-2	4	-2	0	-1	2	3	-3	-4	-0.3
1	-4	4	4	-2	-3	-4	-4	0	3	-0.5
2	4	3	-2	3	1	0	0	-4	-3	0.4
4	0	2	-1	1	-4	-4	-4	-3	4	-0.5
4	-3	-1	-3	-3	0	4	4	3	1	0.6
-4	-1	0	-3	4	-3	2	1	3	4	0.3
0	4	-4	3	-2	2	0	4	2	-4	0.5
3	2	-2	4	-4	4	3	4	0	-3	1.1
-4	2	0	-2	-3	2	-1	-2	-4	-2	-1.4
-3	-1	-2	4	2	4	0	1	2	2	0.9
2	-4	-2	-3	-1	3	1	-2	-4	-3	-1.3
-4	3	1	0	-1	0	-3	-3	2	4	-0.1
2	2	4	-4	-4	4	4	-3	-3	-2	0.0
-4	1	4	4	3	2	-2	-3	4	-3	0.6
0	-4	-4	0	-4	1	4	-3	1	4	-0.5

-4	4	-3	1	4	-1	-4	-3	-4	-2	-1.2
4	-3	-3	4	4	-3	1	2	3	-1	0.8
-2	3	2	-4	-1	3	-4	-1	-3	1	-0.6
4	2	4	-3	-4	-4	-1	-3	-4	-2	-1.1
-3	-1	1	2	-3	0	-4	-2	4	2	-0.4
0	-1	4	-4	2	-2	3	-3	-2	3	0.0
-4	2	1	-2	-2	-1	-3	-4	3	-3	-1.3
-4	4	4	4	-3	2	3	3	3	4	2.0

(c) The mean of the 50 sample means is .034. The sample variance of the 50 sample means is .8096.

(d) According to Theorem 6.1, the distribution of the means has mean 0 and variance

$$\frac{\sigma^2}{n} \cdot \frac{N-n}{N-1} = \frac{8.667}{10} \cdot \frac{20}{29} = .5977.$$

The sample values in part (c) compare well with these theoretical values.

6.10 The 25 means are

3.80, 4.25, 4.25, 5.05, 4.85, 4.20, 4.05, 4.15, 4.85, 4.20, 3.00, 5.20, 4.30,
4.45, 3.75, 5.40, 5.60, 5.70, 4.00, 5.10, 3.15, 4.50, 3.40, 5.00, 4.50

The sample mean of these 25 means is 4.428.

The sample standard deviation of these 25 means is .714.

The population mean is:

$$0\frac{1}{10} + 1\frac{1}{10} + \cdots + 9\frac{1}{10} = 4.5.$$

The population variance is:

$$0^2\frac{1}{10} + 1^2\frac{1}{10} + \cdots + 9^2\frac{1}{10} - (4.5)^2 = 8.25,$$

so the population standard deviation is 2.87228. Theorem 6.1 says that the mean of the distribution of $\bar{X}$ is 4.5 and the variance is $8.25/20 = .4125$ Thus the theoretical standard deviation is .642. These compare well to the sample results.

6.11 The variance of the sample mean $\bar{X}$, based on a sample of size n, is σ^2/n. Thus the standard deviation, or standard error of the mean is $\sigma/\sqrt{n}$.

(a) The standard deviation for a sample of size 50 is $\sigma/\sqrt{50}$. The standard deviation for a sample is size 200 is $\sigma/\sqrt{200}$. That is, the ratio of standard errors is

$$\frac{\sigma/\sqrt{200}}{\sigma/\sqrt{50}} = \frac{\sqrt{50}}{\sqrt{200}} = \frac{1}{2},$$

so the standard error is halved.

(b) The ratio of standard errors is

$$\frac{\sigma/\sqrt{900}}{\sigma/\sqrt{400}} = \frac{\sqrt{400}}{\sqrt{900}} = \frac{2}{3},$$

so the standard error for a sample size 900 is 2/3rd's that for sample size 400.

(c) The standard error for a sample size 25 is

$$\frac{\sqrt{225}}{\sqrt{25}} = 3$$

times as large as that for sample size 225.

(d) The standard error for a sample size 40 is

$$\frac{\sqrt{640}}{\sqrt{40}} = 4$$

times as large as that for sample size 640.

6.13 We need to find $P(|\bar{X} - \mu| < .6745 \cdot \sigma/\sqrt{n})$. Since the standard deviation of the mean is $\sigma/\sqrt{n}$, the standardized variable $(\bar{X} - \mu)/(\sigma/\sqrt{n})$ is approximately a normal random variable for large n (central limit theorem). Thus , we need to find:

$$P(|\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}| < .6745).$$

Now, interpolating in Table 3 gives

$$P(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < .6745) = .75.$$

Thus,

$$P(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq -.6745) = P(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \geq .6745) = .25$$

so

$$P(|\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}| < .6745) = .75 - .25 = .50.$$

The probability that the mean of a random sample of size n , from a population with standard deviation σ, will differ from μ by less than $(.6745)(\sigma/\sqrt{n})$ is approximately .5 for sufficiently large n.

6.17 We need to find

$$\begin{aligned} P(\sum_{i=1}^{36} X_i > 6,000) &= P(\bar{X} > 166.67) = P(\bar{X} - 163 > 3.67) \\ &= P(\frac{\bar{X} - 163}{18/6} > 1.222). \end{aligned}$$

Since $n = 36$ is relatively large, we use the central limit theorem to approximate this probability by

$$1 - F(1.222) = .111.$$