2. Spectrum of Continuous-time Signals

2.1 What is a Spectrum in the Context of Signals?

- The spectrum of a time-domain signal is a representation of that signal in the frequency domain.
- Any time-domain energy or power signal has a corresponding spectrum. This includes familiar signals such as visible light (color), musical notes and radio/TV transmissions.
- When time-domain signals are represented in the form of a spectrum certain physical descriptions of their characteristics become much simpler. For example:

  Why are infrared sensors blind to ultraviolet light?
  Ans: ____________________

  How do we describe Julie Andrews' coloratura soprano voice before it was damaged by a throat operation in 1997?
  Ans: ____________________

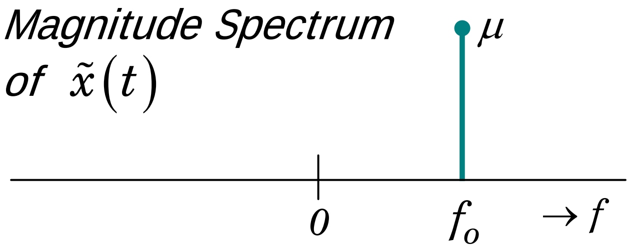
  How do we explain why Internet and Cable TV signals can be concurrently brought to a subscriber home using the same cable without interfering with each other.
  Ans: ____________________

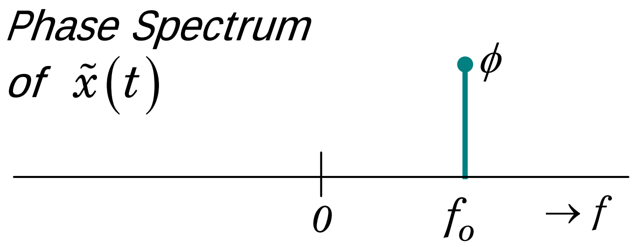
- The mathematical model of a spectrum is, in general, a *complex* function of frequency.
- The graphical representation of a spectrum always consists of two plots: the *magnitude spectrum* and the *phase spectrum*. In some cases, these may be combined into a single spectral plot.

2.1.1 Spectrum of a Sinusoid

- ***Spectrum of a complex exponential signal***

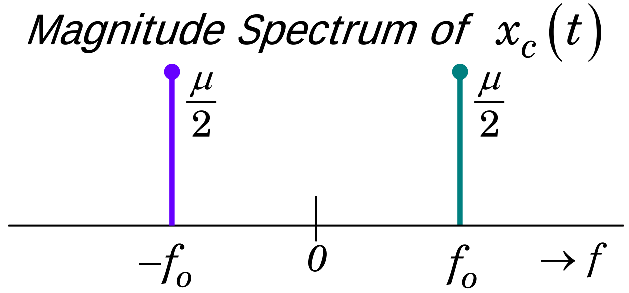
$$\tilde{x}(t) = \mu \exp\left[j\left(2\pi f_o t + \phi\right)\right] = \underbrace{\mu \exp\left(j\phi\right)}_{X_o} \exp\left(j2\pi f_o t\right) \quad \begin{cases} \text{Magnitude: } & \left|X_o\right| = \mu \\ \text{Phase: } & \angle X_o = \phi \\ \text{Frequency: } & f_o \end{cases}$$

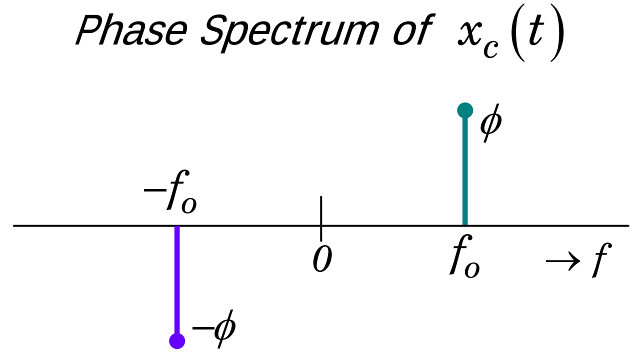




- ***Spectrum of a cosine signal***

$$x_c(t) = \mu \cos\left(2\pi f_o t + \phi\right) = \frac{1}{2}\underbrace{\mu \exp\left[j\left(2\pi f_o t + \phi\right)\right]}_{\tilde{x}(t)} + \frac{1}{2}\underbrace{\mu \exp\left[-j\left(2\pi f_o t + \phi\right)\right]}_{\tilde{x}^*(t)} = \frac{1}{2}\tilde{x}(t) + \frac{1}{2}\tilde{x}^*(t)$$

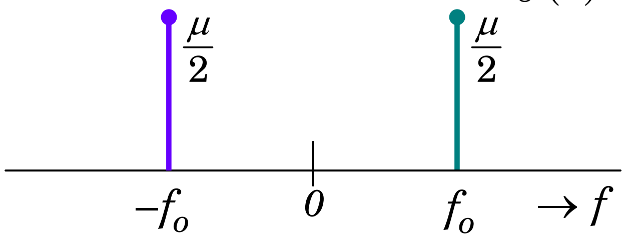




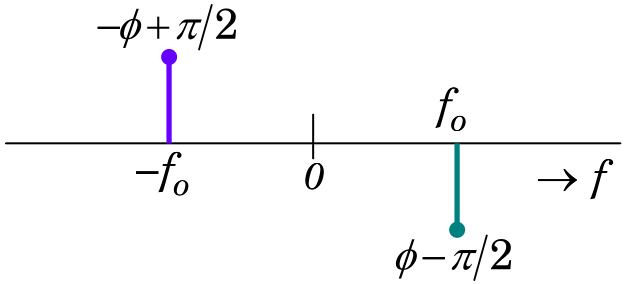
- ***Spectrum of a sine signal***

$$x_s(t) = \mu \sin(2\pi f_o t + \phi) = -\frac{j}{2}\underbrace{\mu \exp\left[j(2\pi f_o t + \phi)\right]}_{\tilde{x}(t)} + \frac{j}{2}\underbrace{\mu \exp\left[-j(2\pi f_o t + \phi)\right]}_{\tilde{x}^*(t)} = \underbrace{-\frac{j}{2}\tilde{x}(t) + \frac{j}{2}\tilde{x}^*(t)}_{Note: j = \exp(j\pi/2)}$$

Magnitude Spectrum of $x_s(t)$



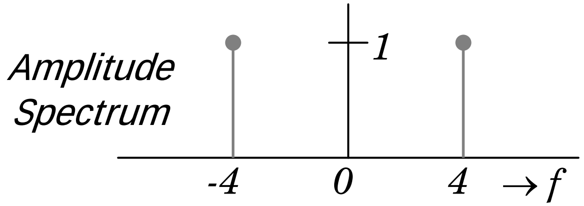
Phase Spectrum of $x_s(t)$

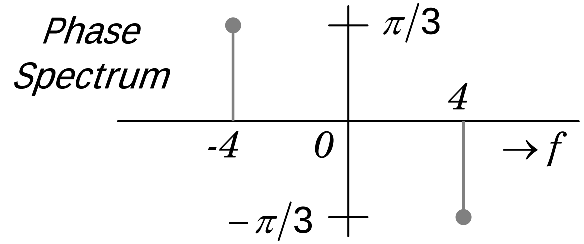


Example 2-1:

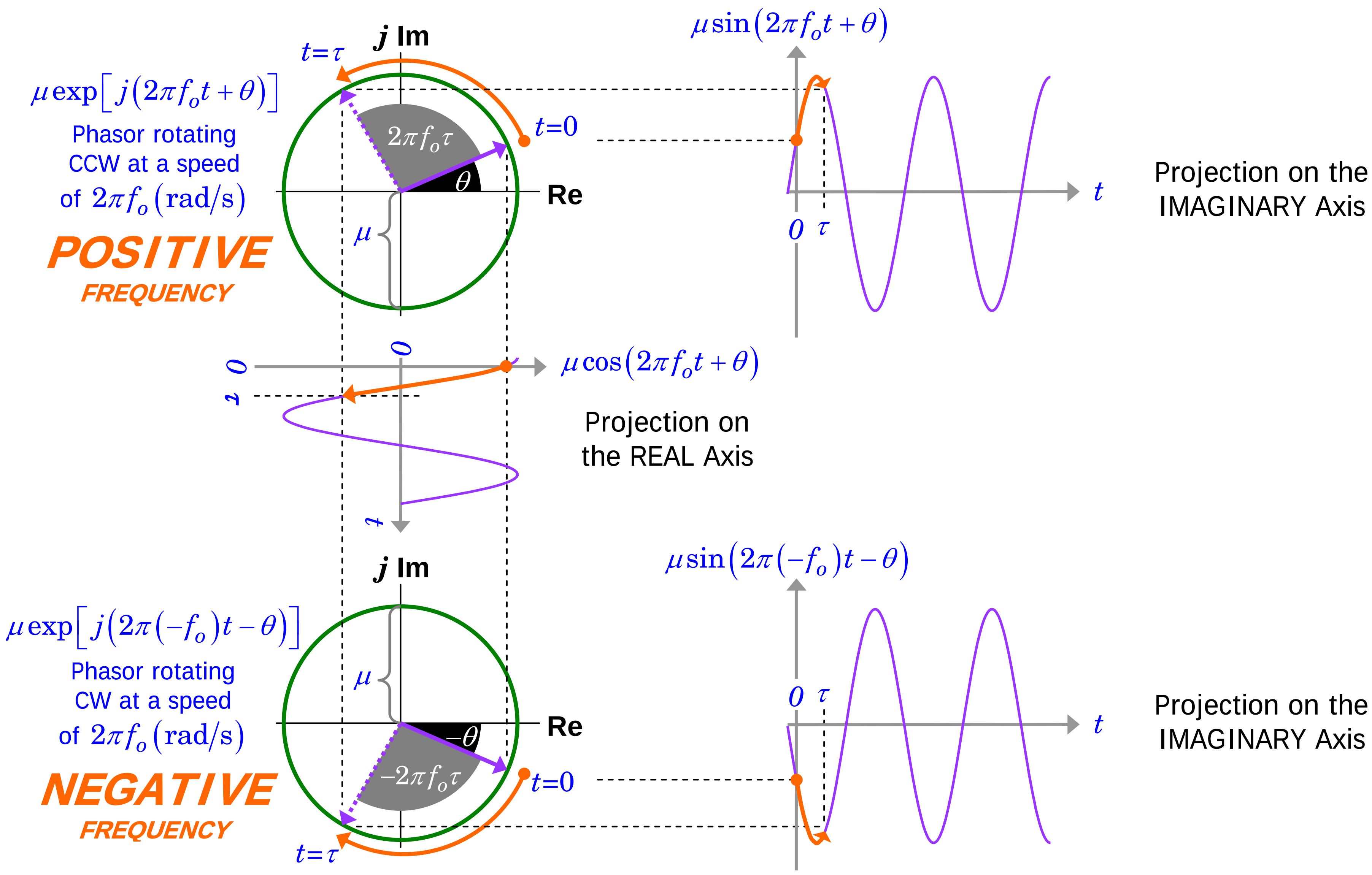
Amplitude and phase spectra of $\boldsymbol{x(t) = 2\sin(8\pi t + \pi/6)}$

$$\begin{aligned} x(t) &= 2\sin(8\pi t + \pi/6) \\ &= 2\cdot\frac{1}{j2}\left\{\exp\left[j(2\pi(4)t + \pi/6)\right] - \exp\left[-j(2\pi(4)t + \pi/6)\right]\right\} \\ &= \begin{cases} \exp(-j\pi/2)\exp(j\pi/6)\exp(j2\pi(4)t) + \\ \exp(j\pi/2)\exp(-j\pi/6)\exp(j2\pi(-4)t) \end{cases} \\ &= \exp(-j\pi/3)\exp(j2\pi(4)t) + \exp(j\pi/3)\exp(j2\pi(-4)t) \end{aligned}$$





2.1.2 Complex Exponentials and Phasors (The concept of negative frequency)



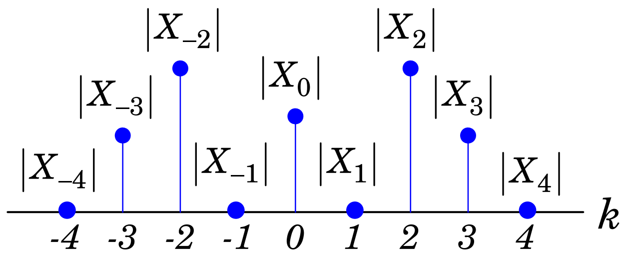
2.1.3 Spectrum of Non-Sinusoidal Signals

Unlike sinusoidal signals, the spectrum of a non-sinusoidal signal may not be determined simply by inspection in most cases. Transformation techniques are thus required to transform a time-domain signal into its frequency-domain (or spectral) representation.

The two *time-frequency* transformation techniques for continuous-time signals are:

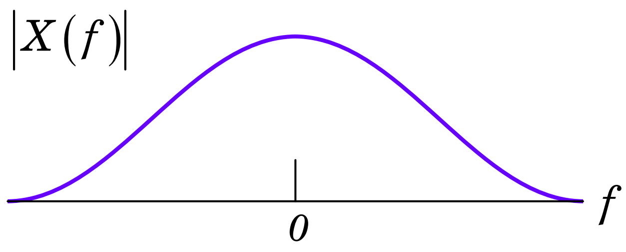
- ***Fourier Series:***

$$\begin{pmatrix} \boldsymbol{x(t)} \\ \textbf{CONTINUOUS-TIME} \\ \textbf{PERIODIC} \end{pmatrix} \rightleftarrows \begin{pmatrix} \boldsymbol{X_k} \\ \textbf{DISCRETE-FREQUENCY} \\ \textbf{NON-PERIODIC} \end{pmatrix}$$



- ***Fourier Transform:***

$$\begin{pmatrix} \boldsymbol{x(t)} \\ \textbf{CONTINUOUS-TIME} \end{pmatrix} \rightleftarrows \begin{pmatrix} \boldsymbol{X(f)} \\ \textbf{CONTINUOUS-FREQUENCY} \\ \textbf{NON-PERIODIC} \end{pmatrix}$$



Note: The symbol $\rightleftarrows$ is used to denote a transform pair.

2.2 Fourier Series

(Spectrum of **continuous-time, periodic** signals) ***Signal period*** $= T_p$

- The Fourier series theorem states that any *periodic signal*, $x_p(t)$, can be represented by a sum of *harmonically related* sinusoids:

$$\boldsymbol{x_p(t)} = \underbrace{\sum_{k=-\infty}^{\infty} X_k \exp\left(j2\pi \frac{k}{T_p} t \right)}_{\textbf{Fourier series expansion}} \tag{2.1}$$

where $1/T_p$ is the fundamental frequency and k/T_p is the k^{th} harmonic frequency of $x_p(t)$.

- X_k are called *Fourier series coefficients* of $x_p(t)$. They constitute the *discrete-frequency spectrum* of $x_p(t)$.

- $x_p(t)$ is a power signal:

$$\begin{cases} \text{Average power}: & P = \dfrac{1}{T_p} \displaystyle\int_{t_o}^{t_o+T_p} \left|x_p(t)\right|^2 dt = \sum_{k=-\infty}^{\infty} \left|X_k\right|^2 \\ \text{Total energy}: & E = \infty \end{cases}$$

- Given $x_p(t)$, how do we compute the k^{th} Fourier series coefficient, X_k?

To compute X_k, we multiply $x_p(t)$ by $\exp\left(-j2\pi kt/T_p\right)$ and integrate the product over any one period:

$$\int_{t_o}^{t_o+T_p} x_p(t)\exp\left(-\frac{j2\pi kt}{T_p}\right)dt = \int_{t_o}^{t_o+T_p} \exp\left(-\frac{j2\pi kt}{T_p}\right)\sum_{m=-\infty}^{\infty} X_m \exp\left(j2\pi\frac{m}{T_p}t\right)dt$$

$$= \sum_{m=-\infty}^{\infty} X_m \underbrace{\int_{t_o}^{t_o+T_p} \exp\left(-\frac{j2\pi kt}{T_p}\right)\exp\left(\frac{j2\pi mt}{T_p}\right)dt}_{=\begin{cases}T_p, & if\ m=k\\ 0, & if\ m\neq k\end{cases}}$$

$$= X_k T_p$$

This yields:

$$\therefore X_k = \frac{1}{T_p}\int_{t_o}^{t_o+T_p} x_p(t)\exp\left(-\frac{j2\pi kt}{T_p}\right)dt, \qquad k = 0, \pm 1, \pm 2, \cdots \tag{2.2}$$

- The Fourier series expansion of $x_p(t)$ can also be expressed in terms of cosine and sine functions:

$$\begin{aligned}
x_p(t) &= \sum_{k=-\infty}^{\infty} X_k \exp\left(\frac{j2\pi kt}{T_p}\right) \\
&= \sum_{k=-\infty}^{-1} X_k \exp\left(\frac{j2\pi kt}{T_p}\right) + X_0 + \sum_{k=1}^{\infty} X_k \exp\left(\frac{j2\pi kt}{T_p}\right) \\
&= X_0 + \sum_{k=1}^{\infty}\left[X_{-k}\exp\left(-\frac{j2\pi kt}{T_p}\right) + X_k \exp\left(\frac{j2\pi kt}{T_p}\right)\right] \\
&= X_0 + \sum_{k=1}^{\infty}\left[\left(X_k + X_{-k}\right)\cos\left(\frac{2\pi kt}{T_p}\right) + j\left(X_k - X_{-k}\right)\sin\left(\frac{2\pi kt}{T_p}\right)\right] \\
&= a_0 + 2\sum_{k=1}^{\infty}\left[a_k\cos\left(\frac{2\pi kt}{T_p}\right) + b_k \sin\left(\frac{2\pi kt}{T_p}\right)\right]
\end{aligned}$$

where

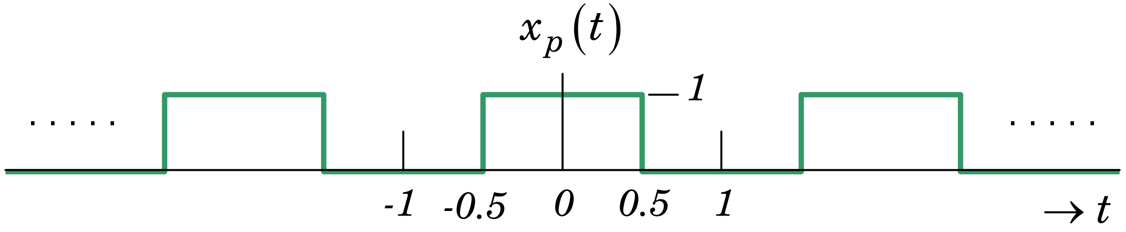
$$\left(\begin{aligned}
a_k &= \frac{X_{-k} + X_k}{2} = \frac{1}{T_p}\int_{t_o}^{t_o+T_p} x_p(t)\cos\left(\frac{2\pi kt}{T_p}\right)dt; \quad k \geq 0 \\
b_k &= \frac{X_{-k} - X_k}{j2} = \frac{1}{T_p}\int_{t_o}^{t_o+T_p} x_p(t)\sin\left(\frac{2\pi kt}{T_p}\right)dt; \quad k > 0
\end{aligned}\right)$$

Table 2-1

SUMMARY	
Fourier Series (*complex* **exp** kernel) a.k.a. **Complex Exponential Fourier Series**	Fourier Analysis (forward transform) $$X_k = \frac{1}{T_p}\int_{t_o}^{t_o+T_p} x_p(t)\exp\left(-j2\pi\frac{k}{T_p}t\right)dt, \qquad k = 0, \pm 1, \pm 2, \cdots$$ Fourier Synthesis (inverse transform) $$x_p(t) = \sum_{k=-\infty}^{\infty} X_k \exp\left(j2\pi\frac{k}{T_p}t\right)$$
Fourier Series (**cos & sin** kernels) a.k.a. **Trigonometric Fourier Series**	Fourier Analysis (forward transform) $$a_k = \frac{1}{T_p}\int_{t_o}^{t_o+T_p} x_p(t)\cos\left(2\pi\frac{k}{T_p}t\right)dt; \quad k \geq 0$$ $$b_k = \frac{1}{T_p}\int_{t_o}^{t_o+T_p} x_p(t)\sin\left(2\pi\frac{k}{T_p}t\right)dt; \quad k > 0$$ Fourier Synthesis (inverse transform) $$x_p(t) = a_0 + 2\sum_{k=1}^{\infty}\left[a_k\cos\left(2\pi\frac{k}{T_p}t\right) + b_k\sin\left(2\pi\frac{k}{T_p}t\right)\right]$$

Example 2-2:

Spectrum of a square wave, $\boldsymbol{x_p(t)}$.



Period: $T_p = 2$. *Fundamental frequency,* $f_p = 1/T_p = 0.5$.

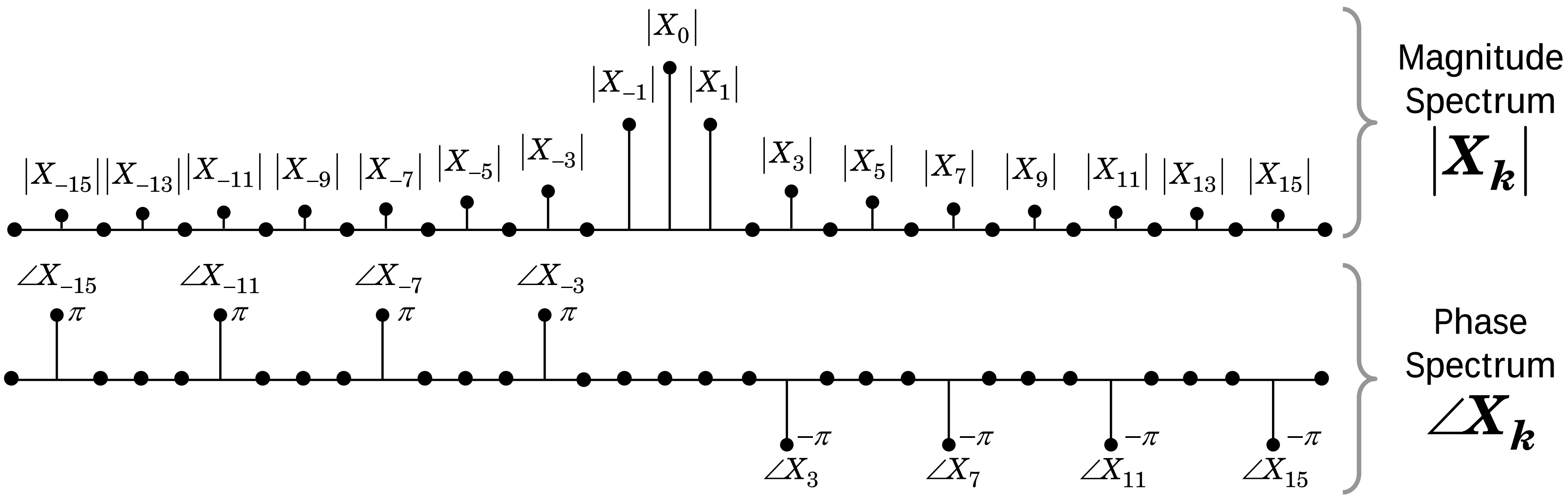
The complex exponential Fourier series expansion of $x_p(t)$ *is given by*

$$x_p(t) = \sum_{k=-\infty}^{\infty} X_k \exp(j\pi k t)$$

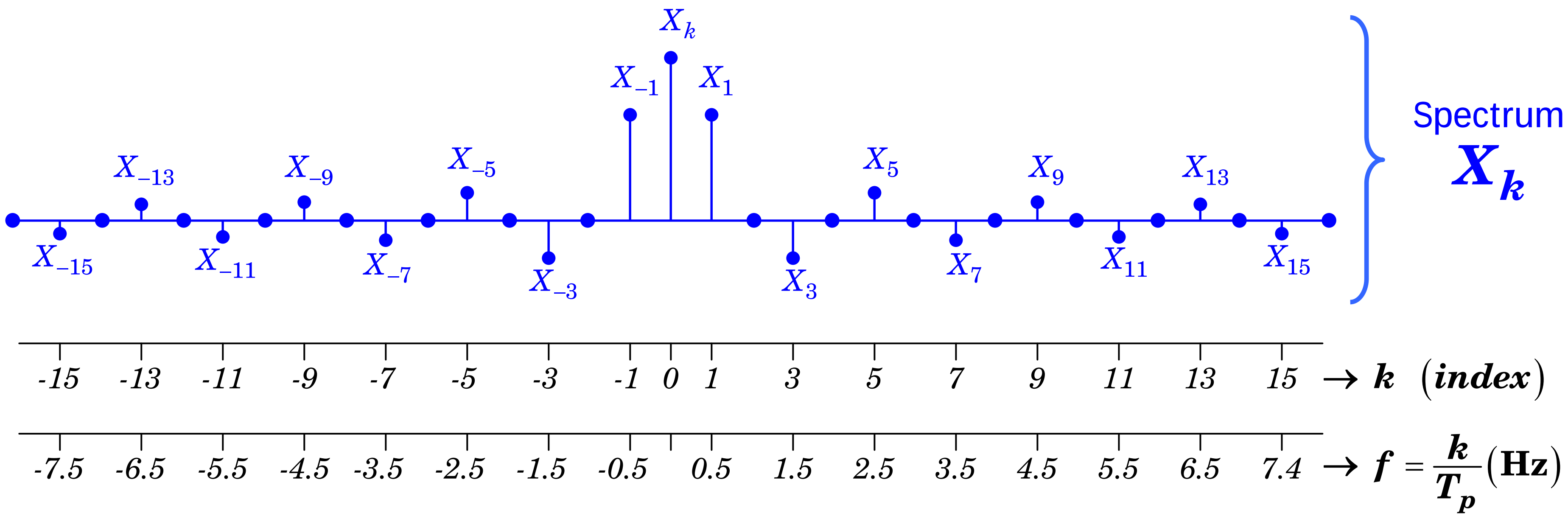
where

$$X_k = \frac{1}{2}\int_{-1}^{1} x_p(t)\exp(-j\pi k t)\,dt = \frac{1}{2}\int_{-0.5}^{0.5} \exp(-j\pi k t)\,dt = \frac{1}{2}\left[\frac{\exp(-j\pi k t)}{-j\pi k}\right]_{-0.5}^{0.5}$$

$$= \frac{1}{j2\pi k}\left[\exp\left(j\frac{\pi k}{2}\right) - \exp\left(-j\frac{\pi k}{2}\right)\right] = \begin{pmatrix} \frac{1}{2}\left(\frac{\sin(0.5\pi k)}{0.5\pi k}\right); & k \neq 0 \\ \frac{1}{2}; & k = 0 \end{pmatrix} = \frac{1}{2}\operatorname{sinc}(0.5k)$$



Since X_k is real, the magnitude and phase spectra can be combined into a ***single*** *spectral plot:*



Since X_k *is* ***real*** $\left(X_k = X_k^*\right)$ *and* ***symmetric*** $\left(X_k = X_{-k}\right)$, *the Fourier series expansion of* $x_p(t)$ *reduces to*

$$x_p(t) = \underbrace{X_0}_{\upsilon_0} + \sum_{k=1}^{\infty} \underbrace{2X_k \cos(\pi k t)}_{\upsilon_k(t)} = \upsilon_0 + \sum_{k=1}^{\infty} \upsilon_k(t)$$

where

$$\left\{\begin{array}{lll} \upsilon_0 = X_0 & : & \textit{DC component} \\ \upsilon_1(t) = 2X_1 \cos\left(2\pi \frac{1}{2} t\right) & : & \textit{Fundamental frequency component} \\ \upsilon_k(t) = 2X_k \cos\left(2\pi \frac{k}{2} t\right) & : & k^{th} \textit{ - harmonic;} \quad k > 1 \end{array}\right\}$$

in which $X_k = \frac{1}{2}\mathrm{sinc}(0.5k)$.

Note that $\left[X_0 = 0.5\right]$ *and* $\left[X_k = 0;\ k = \pm 2, \pm 4, \pm 6, \cdots\right]$. *The latter implies that* $x_p(t)$ *has no even harmonics.*

goto GOLDWAVE Demo-1

2.3 Fourier Transform

(Spectrum of **continuous-time, non-periodic** signals)

- Let $x(t)$ be a continuous-time, non-periodic signal, which has a *continuous-frequency spectrum* denoted by $X(f)$.

- The relationship between $x(t)$ and $X(f)$ is called ***Fourier transform***.

$$\left[\begin{array}{ll} \left.\begin{array}{c}\textit{forward}\\ \text{FOURIER TRANSFORM}\end{array}\right\} & X(f)=\int_{-\infty}^{\infty} x(t)\exp(-j2\pi ft)\,dt \\ \left.\begin{array}{c}\textit{inverse}\\ \text{FOURIER TRANSFORM}\end{array}\right\} & x(t)=\int_{-\infty}^{\infty} X(f)\exp(j2\pi ft)\,df \end{array}\right] \qquad (2.3)$$

- ***Dirichlet's Conditions***

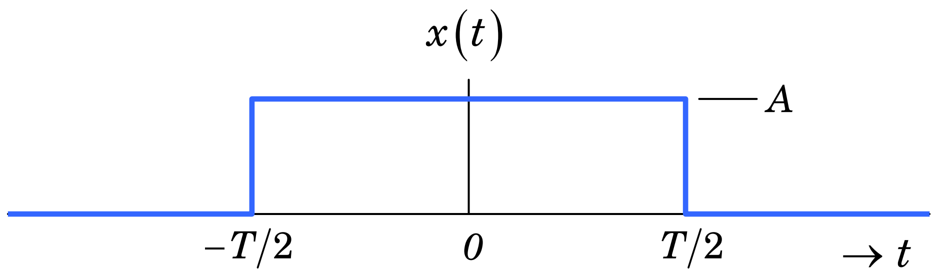
  For the Fourier transform of $x(t)$ to exist, the following conditions must be satisfied:

  1. $x(t)$ *is single-valued within any finite time interval.*
  2. $x(t)$ *have at most a finite number of maxima and minima in any finite time interval.*
  3. $x(t)$ *have at most a finite number of discontinuities in any finite time interval.*
  4. $x(t)$ *is absolutely integrable, i.e.* $\int_{-\infty}^{\infty} |x(t)|\,dt < \infty$, *i.e. an energy signal*

Example 2-3(a):

Spectrum of a rectangular pulse, $x(t)$.

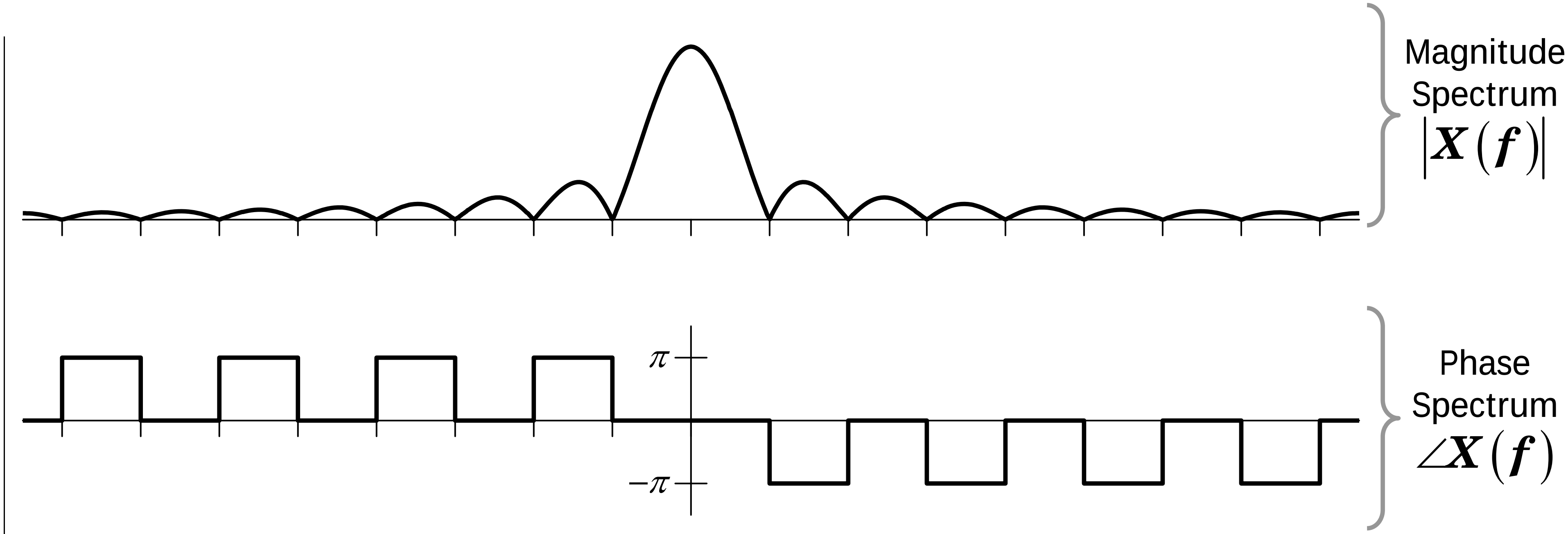
$$x(t) = A \cdot \mathrm{rect}\left(\frac{t}{T}\right) = \begin{cases} A; & |t| \le T/2 \\ 0; & |t| > T/2 \end{cases}$$



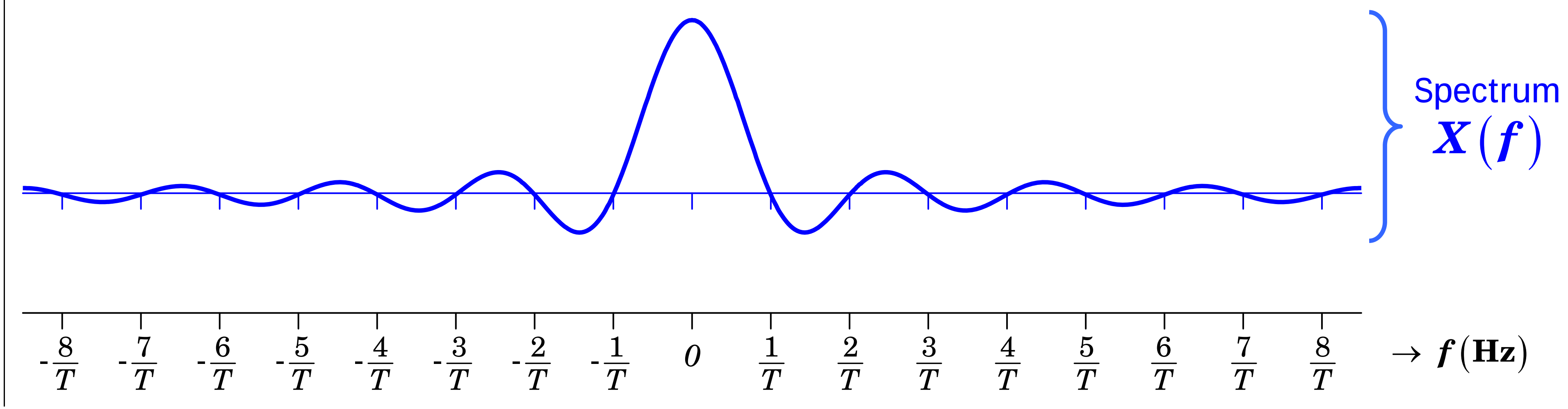
The Fourier transform of $x(t)$ is given by

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t)\exp(-j2\pi ft)\,dt \\ &= \int_{-T/2}^{T/2} A\exp(-j2\pi ft)\,dt \\ &= \begin{pmatrix} AT\dfrac{\sin(\pi fT)}{\pi fT}; & f \ne 0 \\ AT; & f = 0 \end{pmatrix} \\ &= AT\,\mathrm{sinc}(fT) \end{aligned}$$

$$\left[\boldsymbol{A \cdot \mathrm{rect}\left(\frac{t}{T}\right) \rightleftarrows AT\,\mathrm{sinc}(fT)} \right]$$



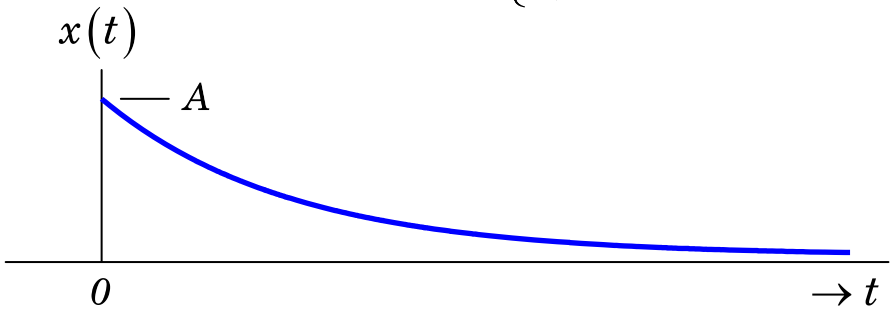
Since $X(f)$ is real, the magnitude and phase spectra can be combined into a ***single*** *spectral plot:*



Example 2-3(b):

Spectrum of an exponentially decaying pulse, $x(t)$.

$$x(t) = A\exp(-\alpha t)u(t) = \begin{cases} A\exp(-\alpha t); & t \geq 0 \\ 0; & t < 0 \end{cases}$$



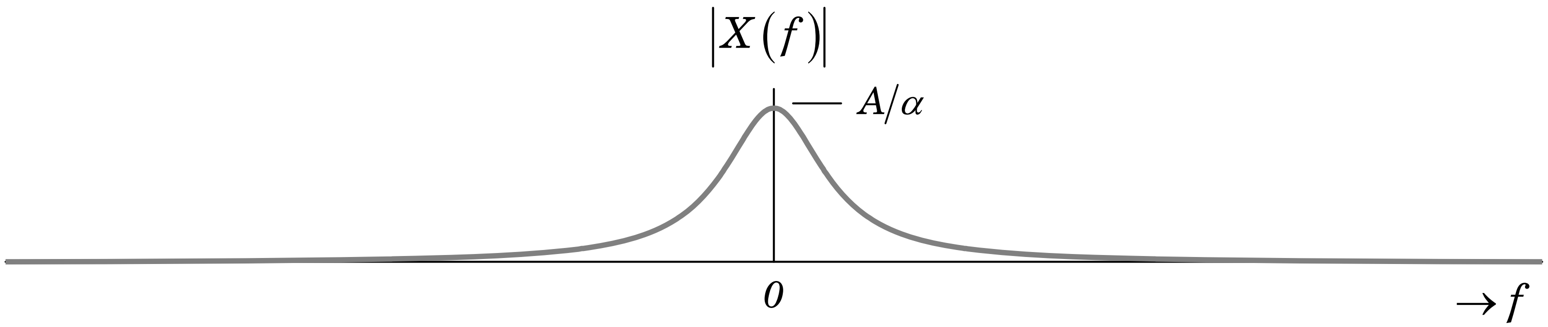
The Fourier transform of $x(t)$ is given by

$$X(f) = \int_{-\infty}^{\infty} x(t)\exp(-j2\pi ft)\,dt = \int_{0}^{\infty} A\exp(-\alpha t)\exp(-j2\pi ft)\,dt$$

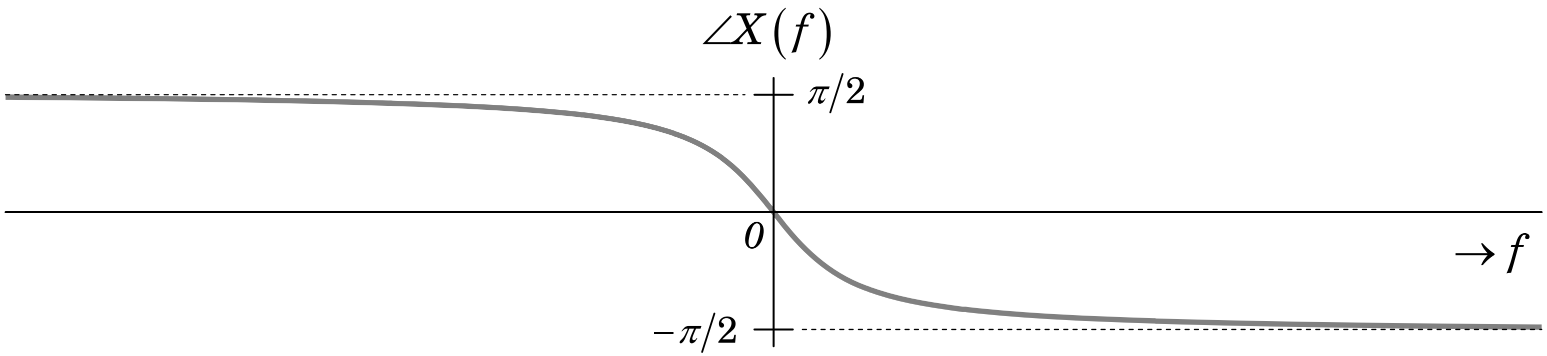
$$= \int_{0}^{\infty} A\exp\big(-(\alpha + j2\pi f)t\big)\,dt = A\left[\frac{\exp\big(-(\alpha + j2\pi f)t\big)}{-(\alpha + j2\pi f)}\right]_0^{\infty}$$

$$= \frac{A}{\alpha + j2\pi f}$$

Magnitude Spectrum: $|X(f)| = \left[X(f)X^*(f)\right]^{1/2} = \left(\dfrac{A^2}{\alpha^2 + 4\pi^2 f^2}\right)^{1/2}$



Phase Spectrum: $\angle X(f) = \tan^{-1}\left(\dfrac{\mathrm{Im}\left[X(f)\right]}{\mathrm{Re}\left[X(f)\right]}\right) = -\tan^{-1}\left(\dfrac{2\pi f}{\alpha}\right)$



Since $X(f)$ is complex, the magnitude and phase spectra **cannot** be combined into a single spectral plot.

2.3.1 Properties of Fourier Transform

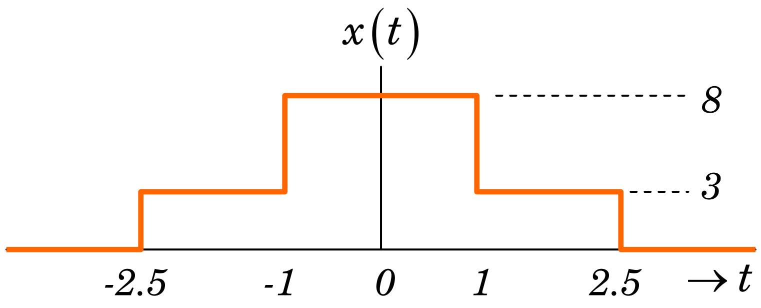
Let $\begin{cases} X(f) = \Im\{x(t)\} & \textit{denote the Fourier transform of } x(t) \\ x(t) \rightleftarrows X(f) & \textit{denote a Fourier transform pair} \end{cases}$

A. *Linearity*

$$\alpha x_1(t) + \beta x_2(t) \rightleftarrows \alpha X_1(f) + \beta X_2(f) \tag{2.4}$$

Example 2-4(A)

Consider $x(t) = \begin{cases} 0; & |t| \geq 2.5 \\ 3; & 1 \leq |t| < 2.5 \\ 8; & |t| < 1 \end{cases}$ ➔



$x(t)$ can be modeled as

$$x(t) = 3 \cdot \mathrm{rect}\left(\frac{t}{5}\right) + 5 \cdot \mathrm{rect}\left(\frac{t}{2}\right).$$

Applying the LINEARITY property:

$$X(f) = \Im\{x(t)\} = \Im\left\{3 \cdot \mathrm{rect}\left(\frac{t}{5}\right)\right\} + \Im\left\{5 \cdot \mathrm{rect}\left(\frac{t}{2}\right)\right\} = 15 \cdot \mathrm{sinc}(5f) + 10 \cdot \mathrm{sinc}(2f).$$

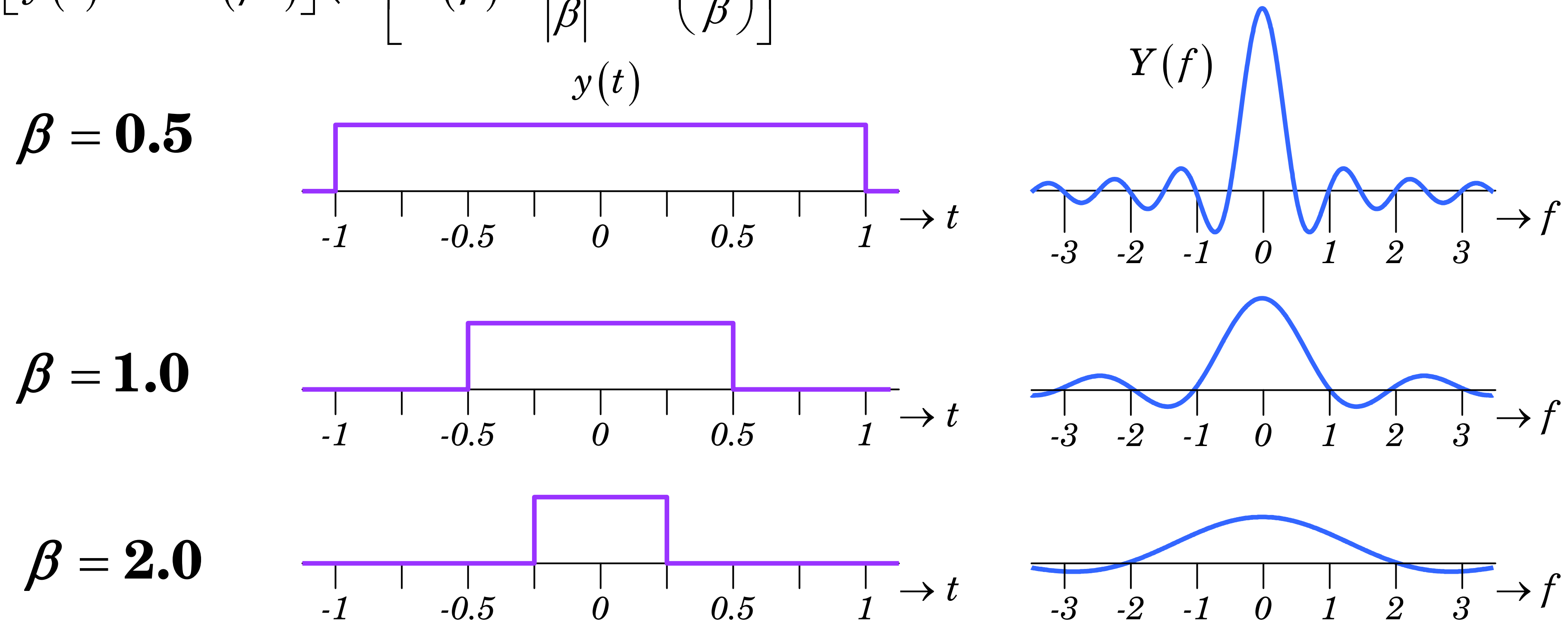
B. *Time Scaling*

$$x(\beta t) \rightleftarrows \frac{1}{|\beta|} X\left(\frac{f}{\beta}\right) \tag{2.5}$$

Example 2-4(B):

Consider the Fourier transform pair $\left[x(t) = \mathrm{rect}(t)\right] \rightleftarrows \left[X(f) = \mathrm{sinc}(f)\right]$. *Then,*

$$\left[y(t) = \mathrm{rect}(\beta t)\right] \rightleftarrows \left[Y(f) = \frac{1}{|\beta|}\mathrm{sinc}\left(\frac{f}{\beta}\right)\right]$$



Wide pulses** are called **narrowband pulses.
Narrow pulses** are called **wideband pulses.

C. *Duality*

$$\boldsymbol{X(t) \rightleftarrows x(-f)} \tag{2.6}$$

Example 2-4(C):

Consider the sinc function: $x(t) = \alpha \ \mathrm{sinc}(2Bt)$ *where* α *and* B *are positive constant. Find* $\Im\{x(t)\}$.

Start with the Fourier transform pair

$$\left[\tilde{x}(t) = A \ \mathrm{rect}\left(\frac{t}{T}\right)\right] \rightleftarrows \left[\tilde{X}(f) = AT \ \mathrm{sinc}(Tf)\right]$$

and substitute $T = 2B$ *and* $A = \dfrac{\alpha}{2B}$ *to get*

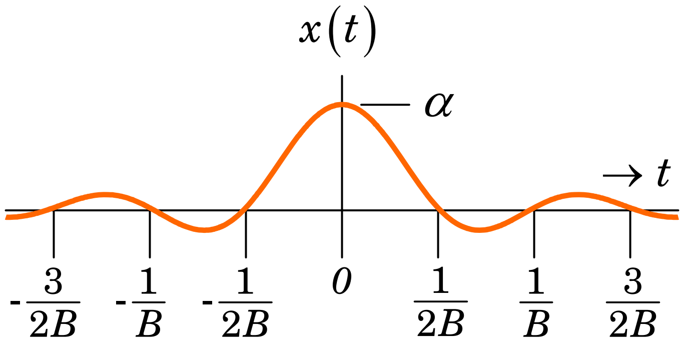
$$\left[\tilde{x}(t) = \frac{\alpha}{2B} \ \mathrm{rect}\left(\frac{t}{2B}\right)\right] \rightleftarrows \left[\tilde{X}(f) = \alpha \ \mathrm{sinc}(2Bf)\right].$$

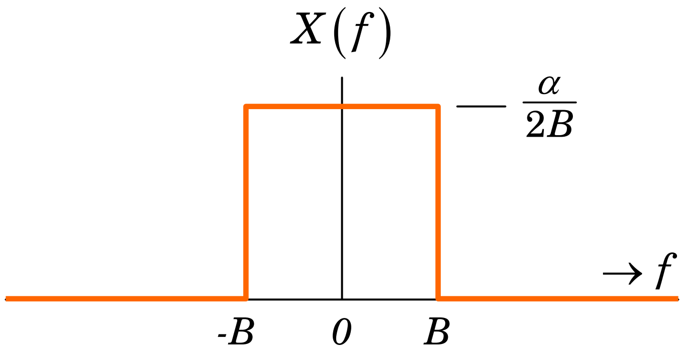
Applying the DUALITY property:

$$\left[\tilde{x}(-f) = \frac{\alpha}{2B} \ \mathrm{rect}\left(-\frac{f}{2B}\right)\right] \rightleftarrows \left[\tilde{X}(t) = \alpha \ \mathrm{sinc}(2Bt)\right].$$

Hence,

$$\Im\{\alpha \ \mathrm{sinc}(2Bt)\} = \frac{\alpha}{2B} \ \mathrm{rect}\left(-\frac{f}{2B}\right) = \frac{\alpha}{2B} \ \mathrm{rect}\left(\frac{f}{2B}\right).$$





D. *Time Shifting*

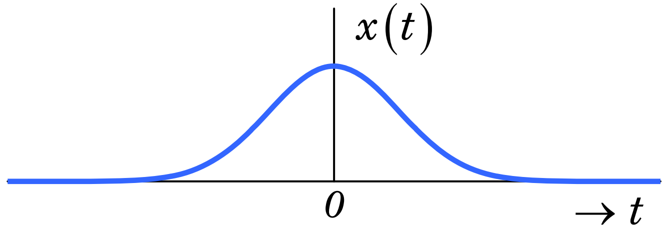
$$\boldsymbol{x(t-t_0) \rightleftarrows X(f)\exp(-j2\pi f t_0)} \qquad (2.7)$$

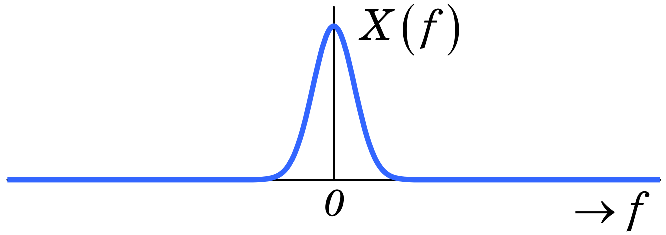
Example 2-4(D):

Let $x(t)$ *be a normalized Gaussian pulse:*

$$\left[x(t)=(2\pi)^{-0.5}\exp\left(-\frac{t^2}{2}\right)\right] \rightleftarrows \left[X(f)=\exp\left(-2\pi^2 f^2\right)\right].$$

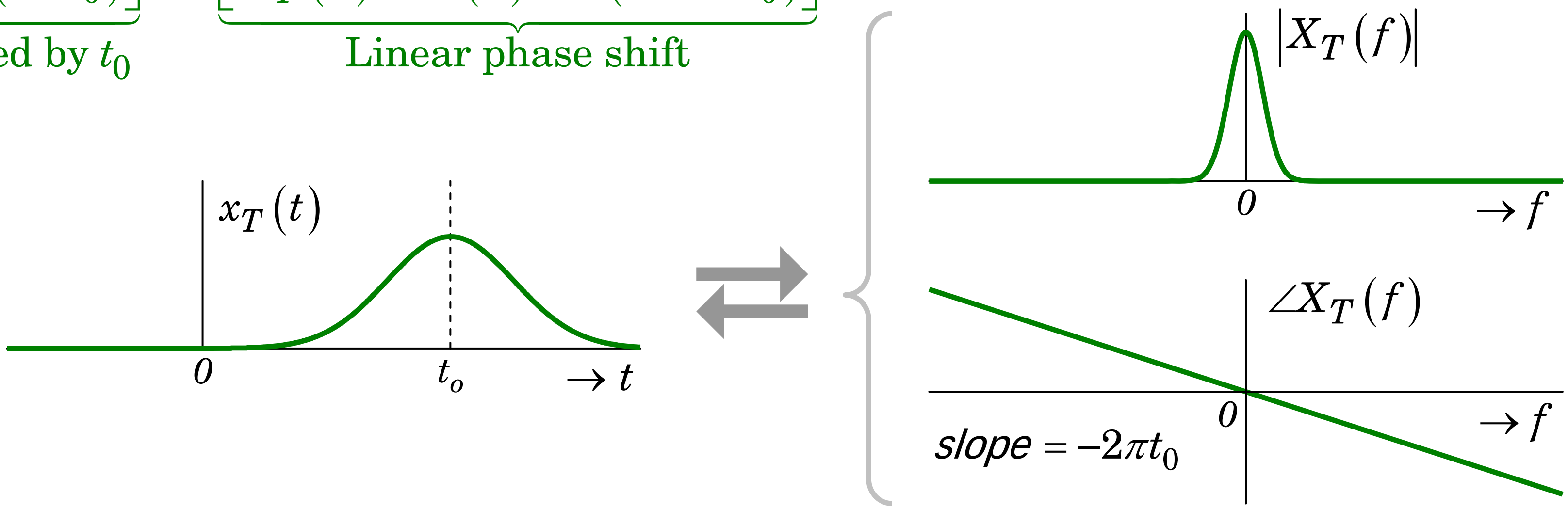
Find $\Im\{x(t-t_0)\}$.





Applying the TIME-SHIFTING property:

$$\underbrace{\left[x_T(t)=x(t-t_0)\right]}_{\text{Time-shifted by } t_0} \rightleftarrows \underbrace{\left[X_T(f)=X(f)\exp(-j2\pi f t_0)\right]}_{\text{Linear phase shift}}$$



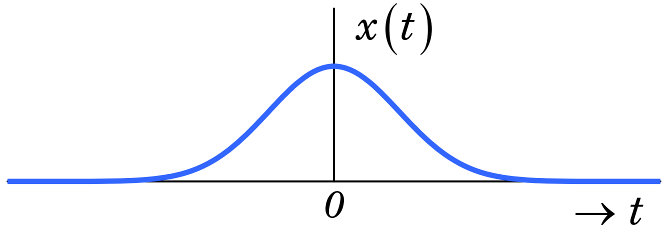
E. *Frequency Shifting (Modulation)*

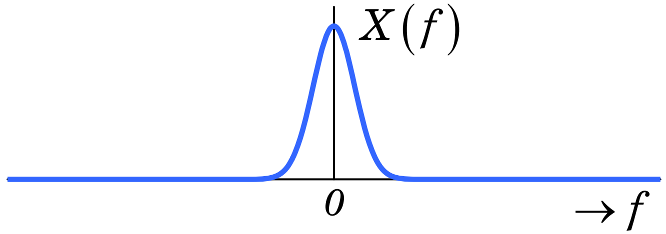
$$\boldsymbol{x(t)\exp(j2\pi f_0 t) \rightleftarrows X(f-f_0)} \tag{2.8}$$

Example 2-4(E):

Let $x(t)$ *be a normalized Gaussian pulse:*

$$\left[x(t)=(2\pi)^{-0.5}\exp\left(-\frac{t^2}{2}\right)\right] \rightleftarrows \left[X(f)=\exp\left(-2\pi^2 f^2\right)\right].$$

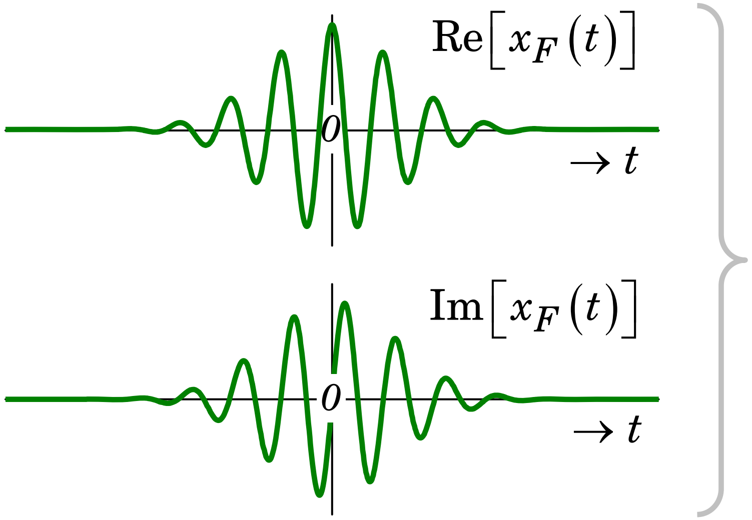


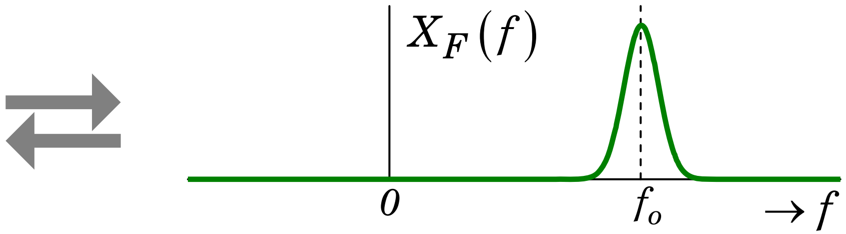


Find $\Im\{x(t)\exp(j2\pi f_o t)\}$.

Applying the FREQUENCY-SHIFTING property:

$$\underbrace{\left[x_F(t)=x(t)\exp(j2\pi f_0 t)\right]}_{\text{Modulation}} \rightleftarrows \underbrace{\left[X_F(f)=X(f-f_0)\right]}_{\text{Frequency-shifted by } f_0}.$$





Notice that $x_F(t)$ *is complex and* $X_F(f)$ *is not symmetric*

F. *Differentiation in the Time Domain*

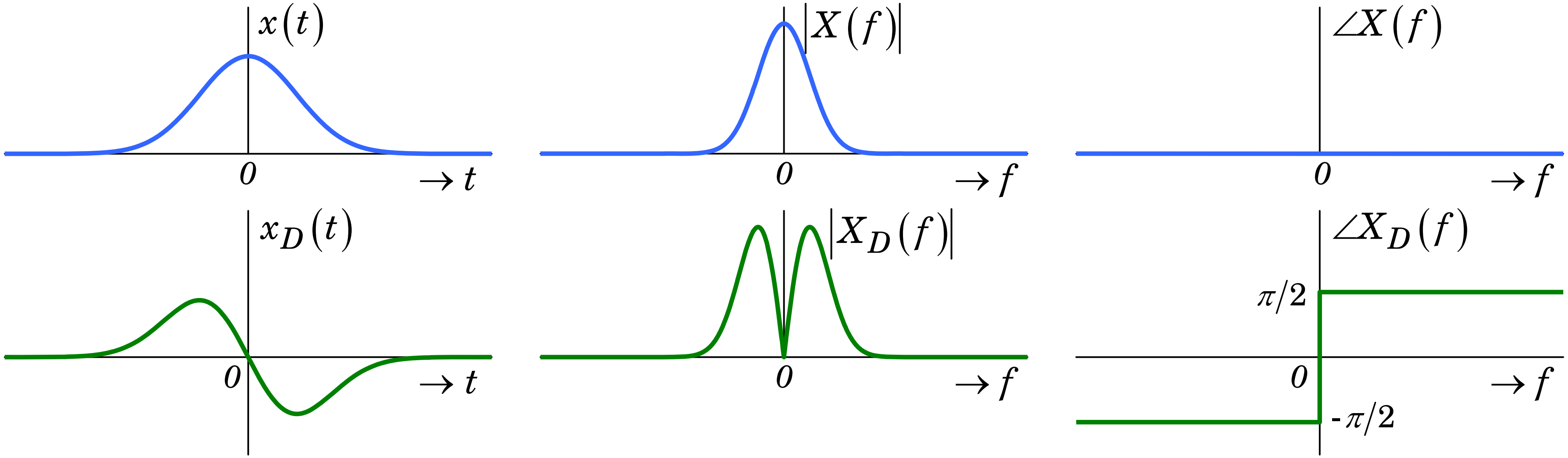
$$\frac{d}{dt}x(t) \rightleftarrows j2\pi f X(f) \tag{2.9}$$

Example 2-4(F):

Let $x(t)$ be a normalized Gaussian pulse:

Notice that $x_D(t)$ is odd and $X_F(f)$ is pure imaginary

$$\left(\begin{array}{c}\left[x(t)=(2\pi)^{-0.5}\exp\left(-t^2/2\right)\right] \\ \downarrow\uparrow \\ \left[X(f)=\exp\left(-2\pi^2 f^2\right)\right]\end{array}\right) \xrightarrow[\text{in time-domain}]{\text{Differentiation}} \left(\begin{array}{c}\left[x_D(t)=-(2\pi)^{-0.5}\,t\exp\left(-t^2/2\right)\right] \\ \downarrow\uparrow \\ \left[X_D(f)=j2\pi f\exp\left(-2\pi^2 f^2\right)\right]\end{array}\right)$$



SUMMARY: $\left\{|X_D(f)| = 2\pi|X(f)||f| \qquad \angle X_D(f) = \angle X(f) + \frac{\pi}{2}\operatorname{sgn}(f)\right\}$

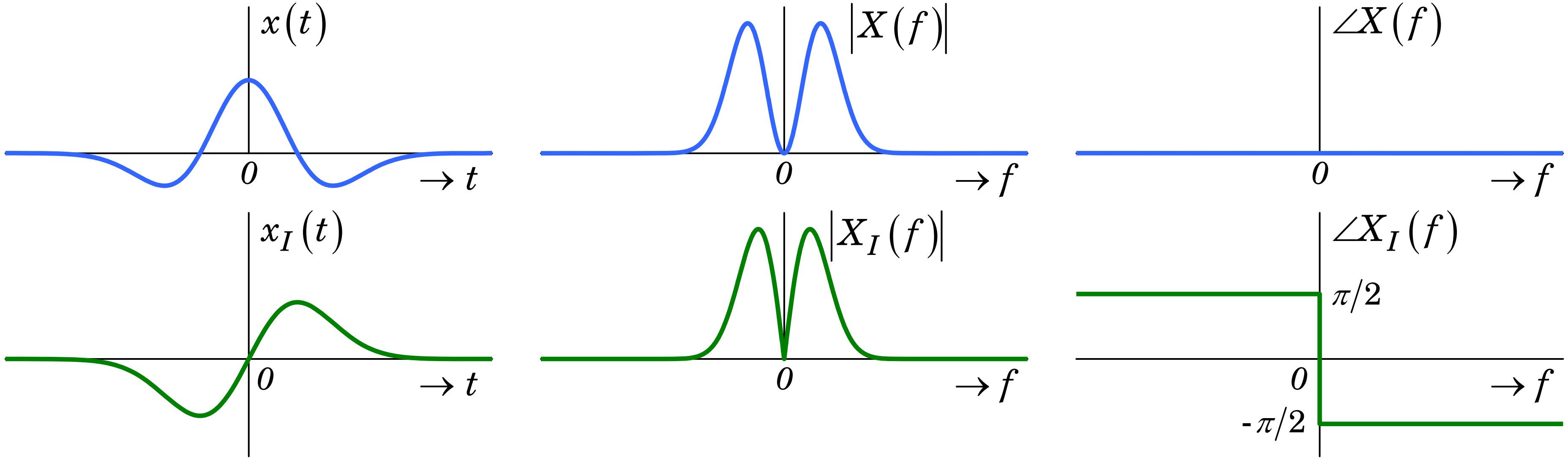
G. *Integration in the Time Domain*

$$\int_{-\infty}^{t} x(\tau)d\tau \rightleftarrows \frac{1}{j2\pi f}X(f) \cdots \text{ iff } \left(X(0) = \int_{-\infty}^{\infty} x(t)dt = 0 \right) \qquad (2.10)$$

Example 2-4(G):

Let $x(t)$ be the second derivative of an inverted normalized Gaussian pulse:

$$\left(\left[x(t) = (2\pi)^{-0.5}\left(1-t^2\right)\exp\left(-\frac{t^2}{2}\right) \right] \downarrow\uparrow \left[X(f) = (2\pi f)^2 \exp\left(-2\pi^2 f^2\right) \right] \right) \xrightarrow{\text{Integration in time-domain}} \left(\left[x_I(t) = (2\pi)^{-0.5}\, t \exp\left(-\frac{t^2}{2}\right) \right] \downarrow\uparrow \left[X_I(f) = -j2\pi f \exp\left(-2\pi^2 f^2\right) \right] \right)$$



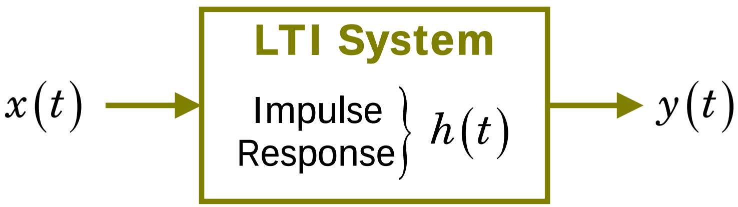
SUMMARY: $\left\{ \left|X_I(f)\right| = \left|X(f)\right| / \left|f\right| \qquad \angle X_I(f) = \angle X(f) - \frac{\pi}{2}\operatorname{sgn}(f) \right\}$

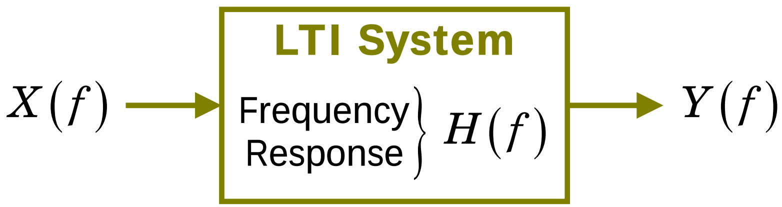
H. *Convolution in the Time Domain* (*or Multiplication in the Frequency-Domain*)

$$\int_{-\infty}^{\infty} x_1(\zeta)x_2(t-\zeta)d\zeta \rightleftarrows X_1(f)X_2(f) \tag{2.11}$$

Example 2-4(H):

Application in ***Linear Time-invariant (LTI)*** *systems:*





TIME-DOMAIN

$$y(t) = \underset{\substack{\uparrow \\ \textit{Convolution}}}{h(t)*x(t)} = \underbrace{\int_{-\infty}^{\infty} h(\zeta)x(t-\zeta)d\zeta}_{\text{Convolution Integral}}$$

FREQUENCY-DOMAIN

$$Y(f) = H(f)X(f)$$

$$\left\{\begin{matrix} x(t) \rightleftarrows X(f) \\ h(t) \rightleftarrows H(f) \\ y(t) \rightleftarrows Y(f) \end{matrix}\right\} \textit{ are Fourier transform pairs.}$$

Proof* (2.11)*:

$$X_1(f)X_2(f)=\int_{-\infty}^{\infty}x_1(\zeta)\exp(-j2\pi f\zeta)d\zeta\int_{-\infty}^{\infty}x_2(v)\exp(-j2\pi fv)dv$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_1(\zeta)x_2(v)\exp\big(-j2\pi f(\zeta+v)\big)d\zeta dv$$

$\cdots\cdots$ *letting* $t=\zeta+v$ *and* $\therefore dt=dv$

$$=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}x_1(\zeta)x_2(t-\zeta)d\zeta\right]\exp(-j2\pi ft)dt$$

$$=\Im\left\{\int_{-\infty}^{\infty}x_1(\zeta)x_2(t-\zeta)d\zeta\right\}=\Im\{x_1(t)*x_2(t)\}$$

Remarks:

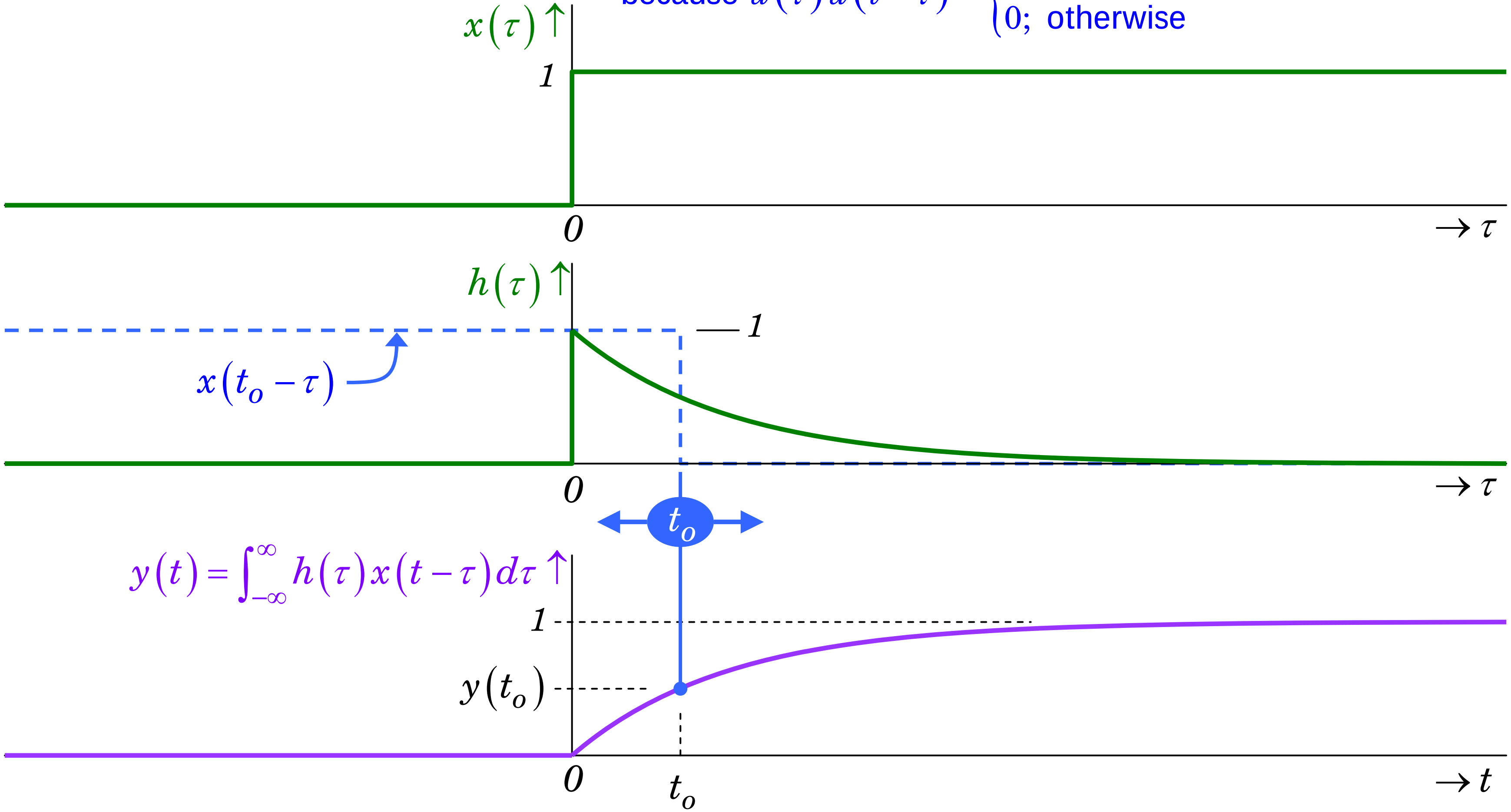
- $x_1(t)*x_2(t)=\int_{-\infty}^{\infty}x_1(\tau)x_2(t-\tau)d\tau$ is defined as the *convolution* of $x_1(t)$ and $x_2(t)$.
- Convolution is **Commutative** : $x_1(t)*x_2(t)=x_2(t)*x_1(t)$.
- Convolution is **Associative** : $[x_1(t)*x_2(t)]*x_3(t)=x_1(t)*[x_2(t)*x_3(t)]$
- Convolution is **Distributive** : $x_1(t)*[x_2(t)+x_3(t)]=x_1(t)*x_2(t)+x_1(t)*x_3(t)$
- This Fourier transform property plays a central role in the analysis and design of continuous-time LTI systems.

Illustration (Convolution): www.jhu.edu/~signals/index.html

Suppose $x(t)=u(t)$ *and* $h(t)=\exp(-t)u(t)$. *How do we evaluate* $y(t)=h(t)*x(t)$*?*

$$y(t)=\int_{-\infty}^{\infty}h(\tau)x(t-\tau)d\tau=\underbrace{\int_{-\infty}^{\infty}\exp(-\tau)u(\tau)u(t-\tau)d\tau=\int_{0}^{t}\exp(-\tau)d\tau}=1-\exp(-\tau)$$

because $u(\tau)u(t-\tau)=\begin{cases}1; & 0\le\tau\le t\\ 0; & \text{otherwise}\end{cases}$

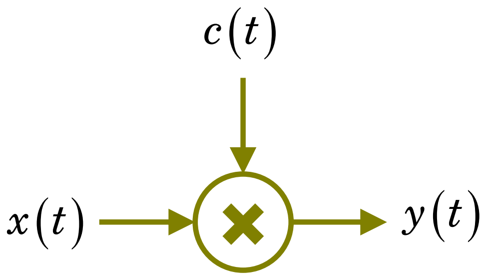


I. *Multiplication in the Time Domain* (*or Convolution in the Frequency-Domain*)

$$x_1(t)x_2(t) \rightleftarrows \int_{-\infty}^{\infty} X_1(\zeta)X_2(f-\zeta)d\zeta \tag{2.12}$$

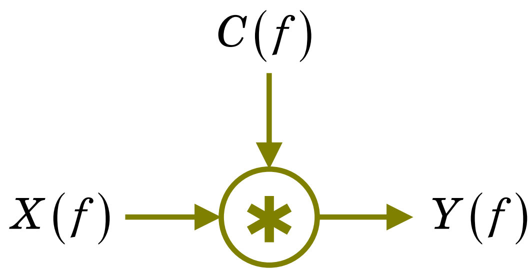
Example 2-4(I):

Application in ***Multiplier (or Mixer)*** *circuits:*



TIME-DOMAIN

$$y(t) = x(t)c(t)$$



FREQUENCY-DOMAIN

$$Y(f) = X(f) \underset{\substack{\uparrow \\ \textit{Convolution}}}{\ast} C(f) = \underbrace{\int_{-\infty}^{\infty} X(\zeta)C(f-\zeta)d\zeta}_{\text{Convolution Integral}}$$

$$\left\{\begin{matrix} x(t) & \rightleftarrows & X(f) \\ c(t) & \rightleftarrows & C(f) \\ y(t) & \rightleftarrows & Y(f) \end{matrix}\right\}$$ *are Fourier transform pairs.*

***Proof* (2.12):**

Similar to Proof (2.11).

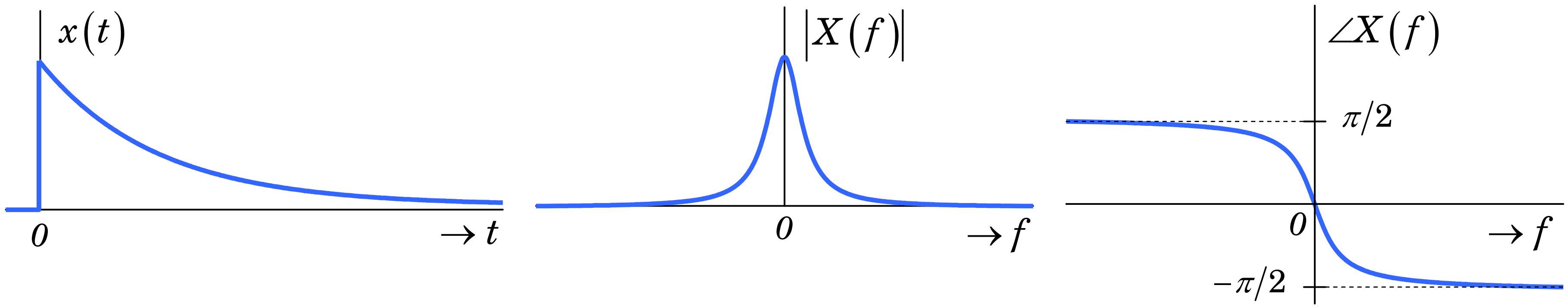
2.4 Spectral Properties of a REAL Signal

- $\boldsymbol{x(t)}$ ***is REAL:*** $\left[\boldsymbol{x^*(t) = x(t)}\right]$

$$\left.\begin{array}{l} X(f) = \int_{-\infty}^{\infty} x(t)\exp(-j2\pi ft)\,dt \\ \left(\begin{array}{l} X(-f) = \int_{-\infty}^{\infty} x(t)\exp(j2\pi ft)\,dt \\ X^*(f) = \int_{-\infty}^{\infty} \underbrace{x^*(t)}_{x(t)}\exp(j2\pi ft)\,dt \end{array}\right. \end{array}\right\} \Rightarrow \left\{\begin{array}{c} \underbrace{X^*(f) = X(-f)}_{} \\ \boldsymbol{X(f)} \textbf{\textit{ is Conjugate Symmetric}} \\ \Downarrow \\ \underbrace{|X(f)| = |X(-f)|}_{\textbf{\textit{EVEN Symmetry}}} \quad \textit{and} \quad \underbrace{\angle X(f) = -\angle X(-f)}_{\textbf{\textit{ODD Symmetry}}} \end{array}\right. \tag{2.13}$$

Example 2-8:

$$\left[x(t) = \exp(-4t)u(t)\right] \rightleftarrows \left[X(f) = (4 + j2\pi f)^{-1}\right]$$



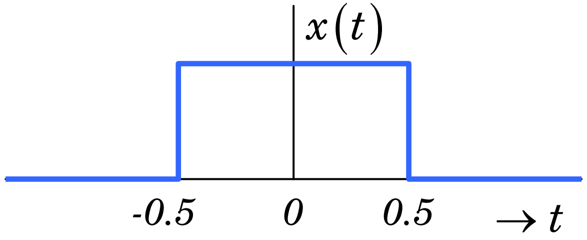
- ***$x(t)$ is REAL and EVEN:*** $\left[\boldsymbol{x}^*(\boldsymbol{t}) = \boldsymbol{x}(\boldsymbol{t}) \;\; \textbf{\textit{and}} \;\; \boldsymbol{x}(\boldsymbol{t}) = \boldsymbol{x}(-\boldsymbol{t})\right]$

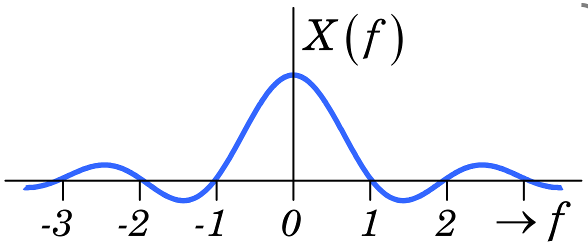
$$\left.\begin{array}{l} \left(\underbrace{x^*(t) = x(t) \;\; \Rightarrow \;\; X^*(f) = X(-f)}_{x(t)\text{ is REAL, see (2.13)}}\right) \\ \left(\begin{array}{l} x(-t) \rightleftarrows X(-f) \;\; \cdots \;\; \text{Scaling Property} \\ \underbrace{x(t) = x(-t)}_{x(t)\text{ is EVEN}} \;\; \Rightarrow \;\; X(f) = X(-f) \end{array}\right) \end{array}\right\} \Rightarrow \underbrace{\overbrace{X^*(f) = X(f)}^{\text{Real}} \;\; \text{and} \;\; \overbrace{X(f) = X(-f)}^{\text{Even}}}_{\boldsymbol{X(f)}\textbf{\textit{ is REAL and EVEN}}} \qquad (2.14)$$

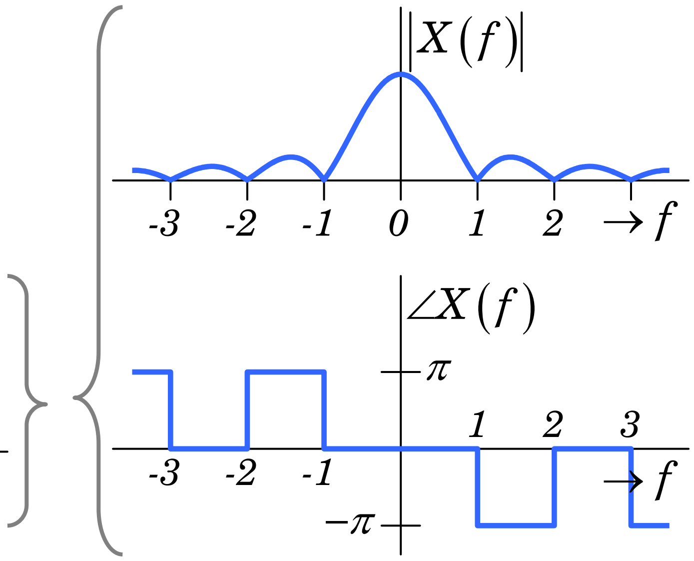
$$\angle X(f) = \begin{cases} 0; & X(f) \ge 0 \\ \pm\pi; & X(f) < 0 \end{cases}$$

Example 2-9:

$$\left[x(t) = \text{rect}(t)\right] \rightleftarrows \left[X(f) = \text{sinc}(f)\right]$$







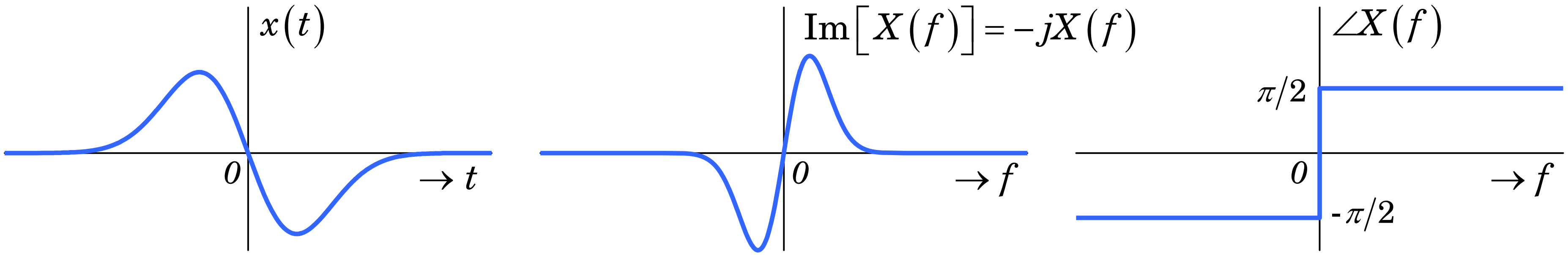
- ***$x(t)$ is REAL and ODD:*** $\left[x^*(t)=x(t) \text{ and } x(-t)=-x(t)\right]$

$$\left.\begin{array}{l}\left(\underbrace{x^*(t)=x(t) \;\rightarrow\; X^*(f)=X(-f)}_{x(t)\text{ is REAL, see (2.13)}}\right. \\ \left(\begin{array}{l} x(-t)\rightleftarrows X(-f) \;\cdots\; \text{Scaling Property} \\ \underbrace{x(t)=-x(-t)}_{x(t)\text{ is ODD}} \;\rightarrow\; X(f)=-X(-f)\end{array}\right.\end{array}\right\} \rightarrow \underbrace{\overbrace{X^*(f)=-X(f)}^{\text{Imaginary}} \text{ and } \overbrace{X(f)=-X(-f)}^{\text{Odd}}}_{\boldsymbol{X(f)}\textbf{ is IMAGINARY and ODD}} \quad (2.15)$$

$$\angle X(f)=\begin{cases}\pi/2; & -jX(f)\geq 0\\ -\pi/2; & -jX(f)<0\end{cases} = \frac{\pi}{2}\operatorname{sgn}\left(\operatorname{Im}\left[X(f)\right]\right) = \frac{\pi}{2}\operatorname{sgn}\left(-jX(f)\right)$$

Example 2-10:

$\left[x(t)=-(2\pi)^{-0.5}t\exp\left(-t^2/2\right)\right] \rightleftarrows \left[X(f)=j2\pi f\exp\left(-2\pi^2f^2\right)\right]$ *---- see Example 2-4(F)*



The above results are also applicable to the Fourier series coefficients of periodic signals, as summarized below:

- $\boldsymbol{x(t)}$ ***is REAL:*** $$\left(\underbrace{X_k^* = X_{-k}}_{\boldsymbol{X_k} \textbf{ is Conjugate Symmetric}} \qquad \underbrace{|X_k| = |X_{-k}|}_{\textbf{EVEN Symmetry}} \qquad \underbrace{\angle X_k = -\angle X_{-k}}_{\textbf{ODD Symmetry}} \right) \tag{2.16}$$

- $\boldsymbol{x(t)}$ ***is REAL and EVEN:*** $$\left(\underbrace{\overbrace{X_k^* = X_k}^{\text{Real}} \quad \text{and} \quad \overbrace{X_k = X_{-k}}^{\text{Even}}}_{\boldsymbol{X_k} \textbf{ is REAL and EVEN}} \qquad \angle X_k = \begin{cases} 0; & X_k \ge 0 \\ \pm\pi; & X_k < 0 \end{cases} \right) \tag{2.17}$$

- $\boldsymbol{x(t)}$ ***is REAL and ODD:*** $$\left(\underbrace{\overbrace{X_k^* = -X_k}^{\text{Imaginary}} \quad \text{and} \quad \overbrace{X_k = -X_{-k}}^{\text{Odd}}}_{\boldsymbol{X_k} \textbf{ is IMAGINARY and ODD}} \qquad \angle X_k = \frac{\pi}{2}\operatorname{sgn}(-jX_k) \right) \tag{2.18}$$

REMARKS: Either the *positive frequency* portion or the *negative frequency* portion of a spectrum would suffice to specify a real signal completely in the frequency domain because one can be derived from the other through the conjugate symmetry property of the spectrum. **Spectrum analyzers** usually display only the positive frequency portion of a spectrum.

2.5 The Dirac-δ and Spectrum of Periodic Signals

2.5.1 The Continuous-time Unit Impulse (Dirac-δ function)

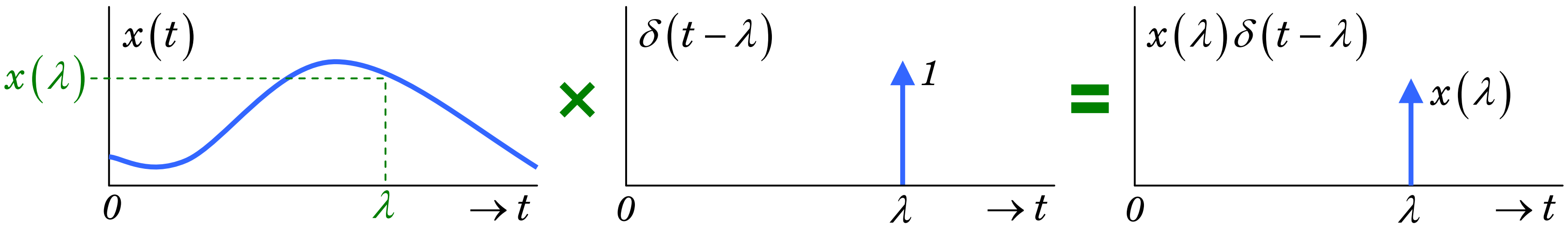
The ***continuous-time unit impulse***, also known as the ***Dirac-δ function***, is defined as (see Chapter 1, Section 1.2)

$$\delta(t) = \begin{cases} \infty; & t = 0 \\ 0; & t \neq 0 \end{cases} \quad \text{and} \quad \int_{0^-}^{0^+} \delta(t)\, dt = 1 \tag{2.19}$$

- **Properties of $\delta(t)$**

1. ***Symmetry:*** $\delta(t) = \delta(-t)$ (2.20)

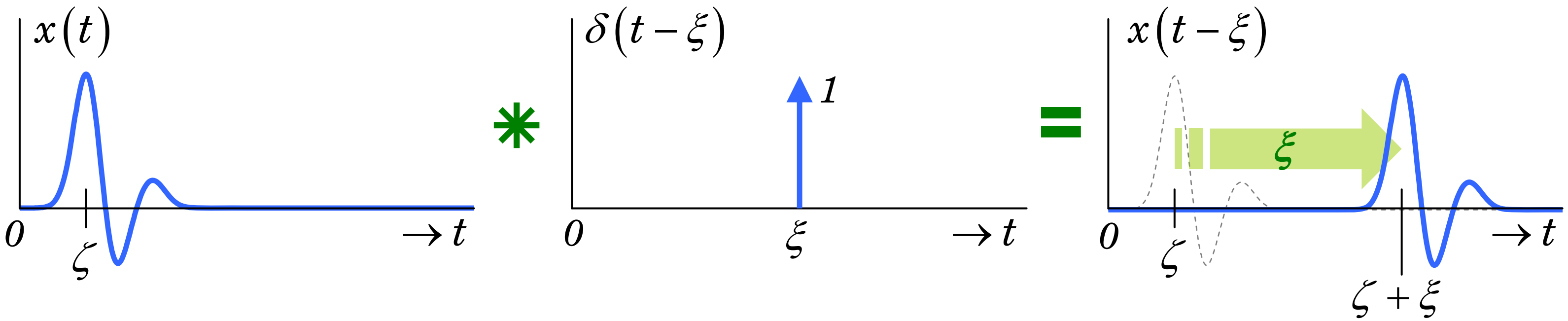
2. ***Sampling:*** $x(t)\delta(t-\lambda) = x(\lambda)\delta(t-\lambda)$ (2.21)



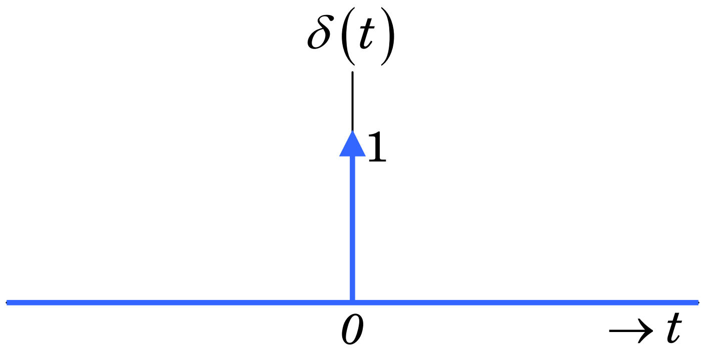
3. ***Sifting:*** $$\underbrace{\int_{-\infty}^{\infty} x(t)\delta(t-\lambda)dt = x(\lambda)\int_{-\infty}^{\infty}\delta(t-\lambda)dt}_{\textit{from the sampling property of } \delta(\bullet)} = x(\lambda) \quad (2.22)$$

4. ***Replication:***

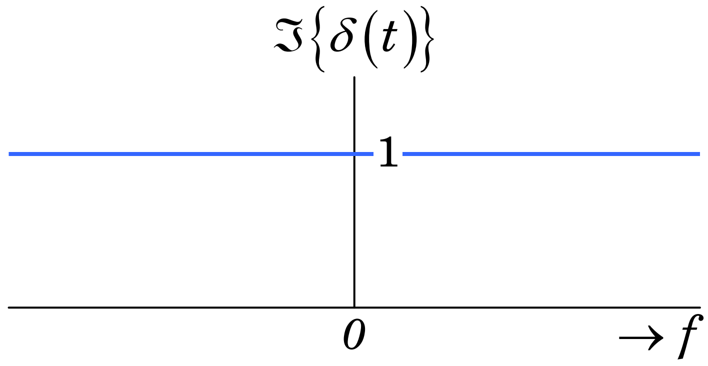
$$\begin{cases} x(t)*\delta(t-\xi) = \overbrace{\int_{-\infty}^{\infty} x(\zeta)\delta(t-\xi-\zeta)d\zeta = \int_{-\infty}^{\infty} x(\zeta)\delta(\zeta-(t-\xi))d\zeta}^{\textit{apply } \textbf{symmetry} \textit{ property}} = x(t-\xi) \\ \text{(the last integral and result: } \underbrace{\int_{-\infty}^{\infty} x(\zeta)\delta(\zeta-(t-\xi))d\zeta = x(t-\xi)}_{\textit{apply } \textbf{sifting} \textit{ property}}) \\ \textbf{Note: } x(t)*\delta(t) = x(t) \end{cases} \quad (2.23)$$



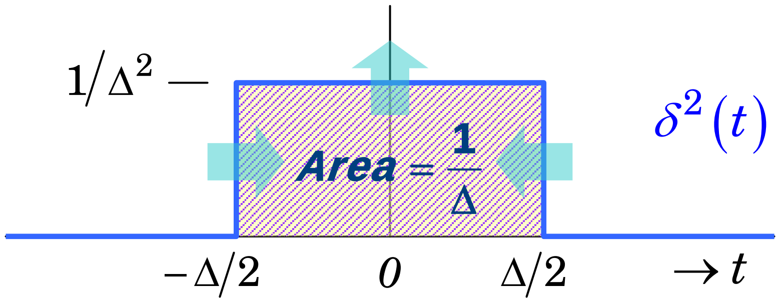
5. ***White Spectrum:*** $\left\{ \Im\{\delta(t)\} = \underbrace{\int_{-\infty}^{\infty} \delta(t)\exp(-j2\pi ft)\,dt = 1}_{\textit{apply } \textbf{\textit{sifting}} \textit{ property}} \right.$ (2.24)



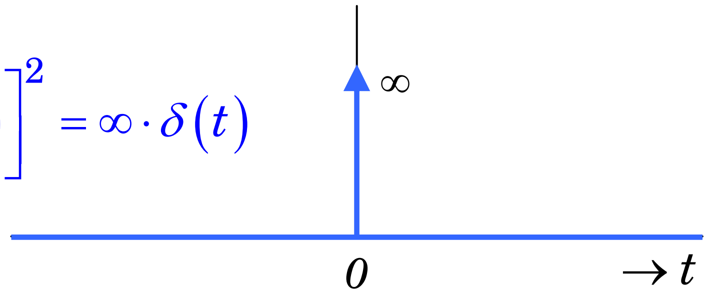
$\rightleftarrows$



6. δ ***-Square:*** $\delta^2(t) = \infty \cdot \delta(t)$ (2.25)



$$\delta^2(t) = \lim_{\Delta\to 0}\left[\frac{1}{\Delta}\operatorname{rect}\left(\frac{t}{\Delta}\right)\right]^2 = \infty \cdot \delta(t)$$



Example 2-11 (Spectra of Signum and Unit Step):

From the definitions of the unit step function, $u(t)$, *and unit impulse function,* $\delta(t)$, *we establish the relationships:*

$$\left[\int_{-\infty}^{t}\delta(\tau)d\tau = u(t)\right] \text{ or } \left[\frac{d}{dt}u(t)=\delta(t)\right]$$

The signum function $\operatorname{sgn}(t)=\begin{cases}1, & t\ge 0\\ -1, & t<0\end{cases}$ *can be expressed in terms of* $u(t)$ *as*

$$\operatorname{sgn}(t)=2u(t)-1. \quad \cdots\cdots \quad (\clubsuit)$$

Differentiating $(\clubsuit)$ *w.r.t. t and then taking Fourier transform, we get*

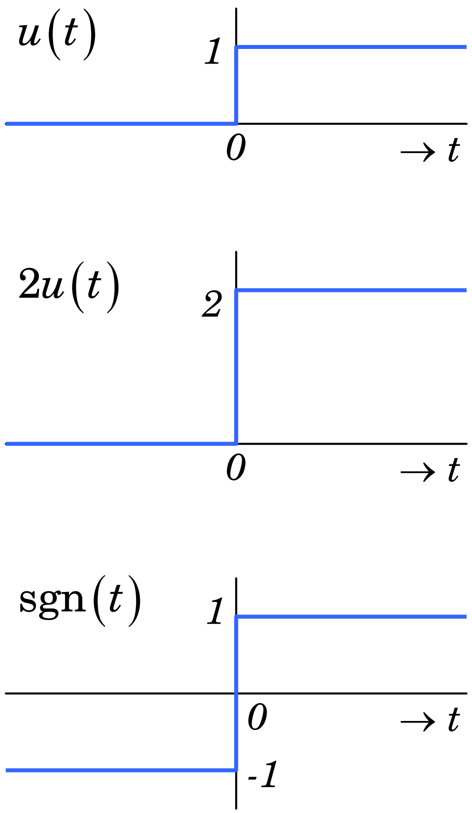
$$\left[\frac{d}{dt}\operatorname{sgn}(t)=2\delta(t)\right] \xrightarrow[\text{Transform}]{\text{Fourier}} \left[j2\pi f\,\Im\{\operatorname{sgn}(t)\}=2\right].$$

which yields

$$\Im\{\operatorname{sgn}(t)\}=\frac{1}{j\pi f}.$$

From $(\clubsuit)$, *we have* $u(t)=\frac{1}{2}(\operatorname{sgn}(t)+1)$. *Therefore,*

$$\Im\{u(t)\}=\frac{1}{j2\pi f}+\frac{1}{2}\delta(f)$$



2.5.2 Spectrum of Periodic Signals

Recap:

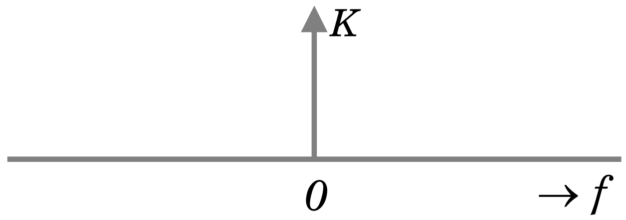
- Fourier series analysis of a periodic signal leads to a discrete-frequency spectrum.
- Dirichlet's 4th condition: A signal must be absolutely integrable for its Fourier transform to exist. *(This will exclude the application of Fourier transform to periodic signals, which have infinite total energy.)*

With the help of the unit impulse function, Fourier transform can be applied to a periodic signal (violating Dirichlet's 4th condition) ***to obtain its continuous-frequency spectrum.***

- ***DC***: $\left\{x(t) = K\right\}$

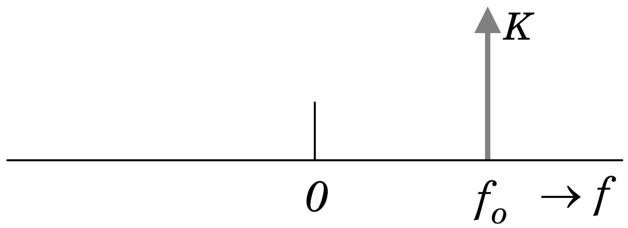
  (A dc signal may be viewed as a periodic signal of arbitrary period)

$$\left(\begin{array}{c}\Im\left\{K\delta(t)\right\} = K \xrightarrow{\text{Duality Property}} \Im\left\{K\right\} = K\delta(f) \\ \left[x(t) = K\right] \rightleftarrows \left[X(f) = K\delta(f)\right]\end{array}\right)$$



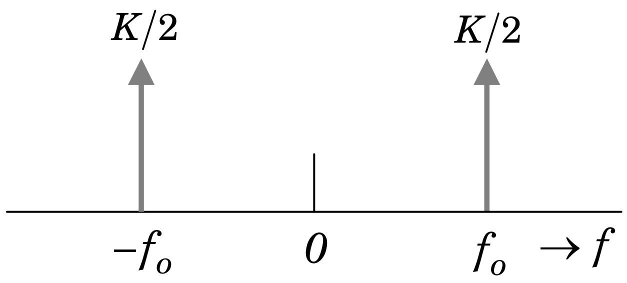
- ***Complex Exponential:*** $\left\{x(t) = K\exp\left(j2\pi f_o t\right)\right\}$

$$\left(\underbrace{\left[x(t) = K\exp\left(j2\pi f_o t\right)\right] \rightleftarrows \left[X(f) = K\delta\left(f - f_o\right)\right]}_{\text{Frequency-shifting (or Modulation) Property}}\right)$$



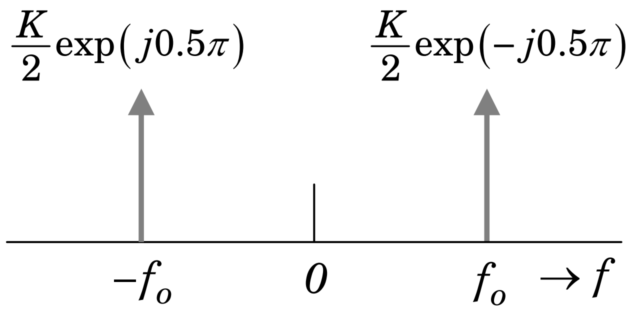
- ***Cosine:*** $\left\{x(t) = K\cos(2\pi f_o t)\right\}$

$$\left(\begin{array}{c} x(t) = \dfrac{K}{2}\exp(j2\pi f_o t) + \dfrac{K}{2}\exp(-j2\pi f_o t) \\ \left[x(t) = K\cos(2\pi f_o t)\right] \rightleftarrows \left[X(f) = \dfrac{K}{2}\delta(f-f_o) + \dfrac{K}{2}\delta(f+f_o)\right] \end{array}\right)$$



- ***Sine:*** $\left\{x(t) = K\sin(2\pi f_o t)\right\}$

$$\left(\begin{array}{c} x(t) = \dfrac{K}{j2}\exp(j2\pi f_o t) - \dfrac{K}{j2}\exp(-j2\pi f_o t) \\ \left[x(t) = \sin(2\pi f_o t)\right] \rightleftarrows \left[\begin{array}{l} X(f) = \dfrac{K}{j2}\delta(f-f_o) - \dfrac{K}{j2}\delta(f+f_o) \\ = \left(\begin{array}{l} \dfrac{K}{2}\exp\left(-j\dfrac{\pi}{2}\right)\delta(f-f_o) \\ +\dfrac{K}{2}\exp\left(j\dfrac{\pi}{2}\right)\delta(f+f_o) \end{array}\right) \end{array}\right] \end{array}\right)$$



- ***Arbitrary periodic signals:*** $\left\{x_p(t):\ Period = T_p\right\}$

Fourier series: $$\left[\underbrace{X_k = \frac{1}{T_p}\int_{t_0}^{t_0+T_p} x_p(t)\exp\left(-j2\pi\frac{k}{T_p}t\right)dt}_{\text{ANALYSIS}} \qquad \underbrace{x_p(t) = \sum_{k=-\infty}^{\infty} X_k \exp\left(j2\pi\frac{k}{T_p}t\right)}_{\text{SYNTHESIS}}\right]$$

Applying Fourier transform to $x_p(t)$:

$$X_p(f) = \Im\left\{x_p(t)\right\} = \underbrace{\Im\left\{\sum_{k=-\infty}^{\infty} X_k \exp\left(j2\pi\frac{k}{T_p}t\right)\right\} = \sum_{k=-\infty}^{\infty} X_k \Im\left\{\exp\left(j2\pi\frac{k}{T_p}t\right)\right\}}_{\text{Linearity property of Fourier transform}} \tag{2.26}$$

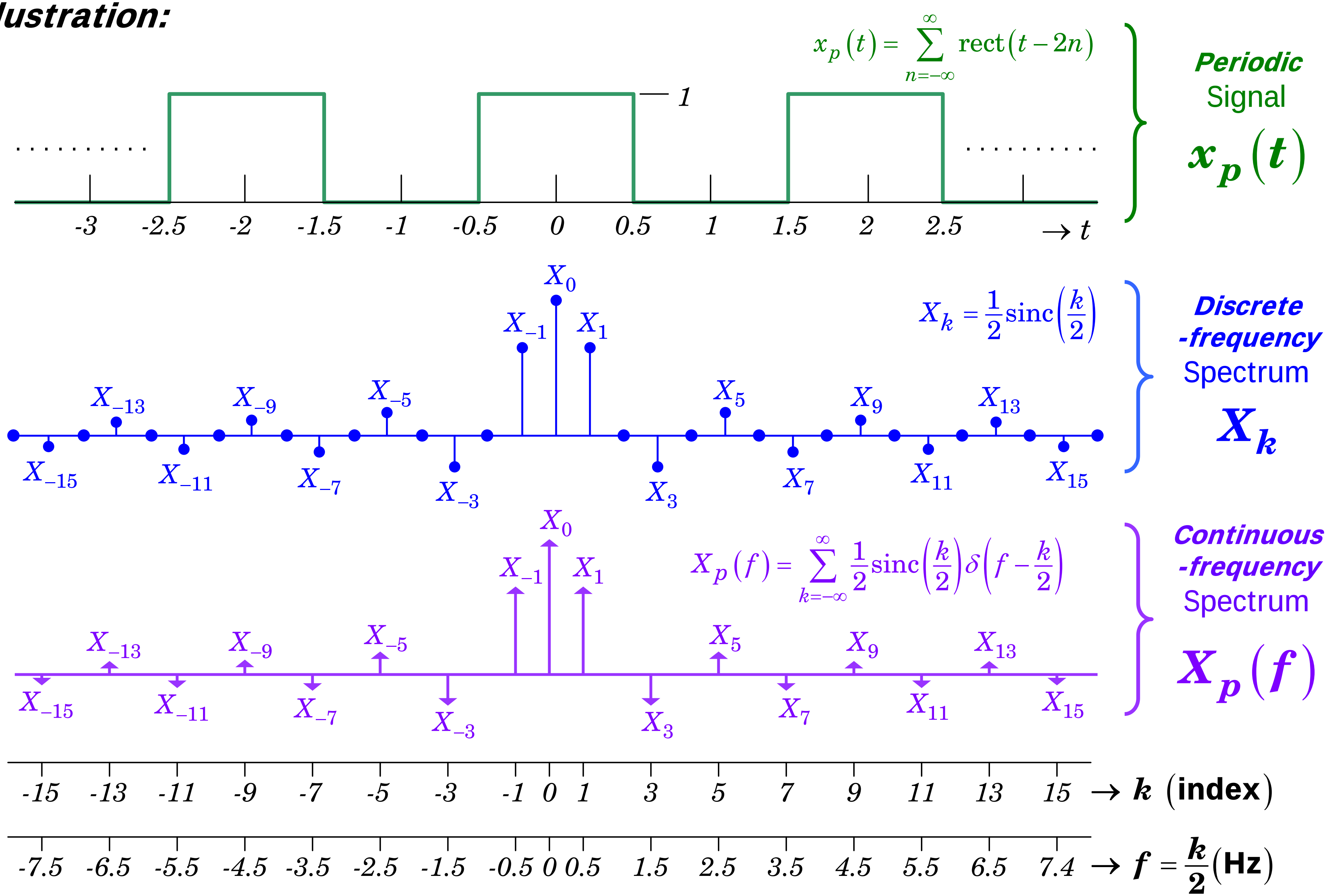
Substituting $\Im\left\{\exp\left(j2\pi\left(\frac{k}{T_p}\right)t\right)\right\} = \delta\left(f-\frac{k}{T_p}\right)$ into (2.26) yields:

$$X_p(f) = \sum_{k=-\infty}^{\infty} X_k \delta\left(f-\frac{k}{T_p}\right) \tag{2.27}$$

Inference:

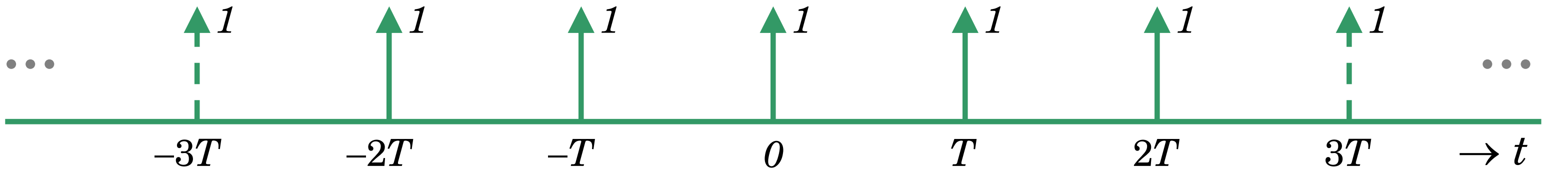
The Fourier transform, $X_p(f)$, of a periodic signal, $x_p(t)$, can be obtained by first computing the Fourier series coefficients, X_k, of $x_p(t)$ and then substituting them into (2.27).

Illustration:



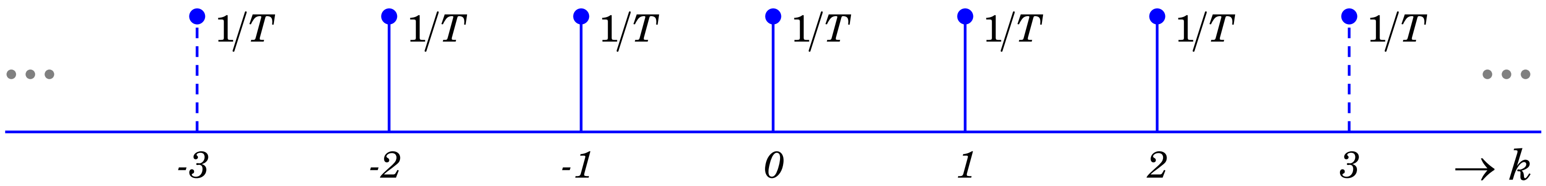
Example 2-12:

Spectrum of a COMB FUNCTION, $\xi_T(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT)$



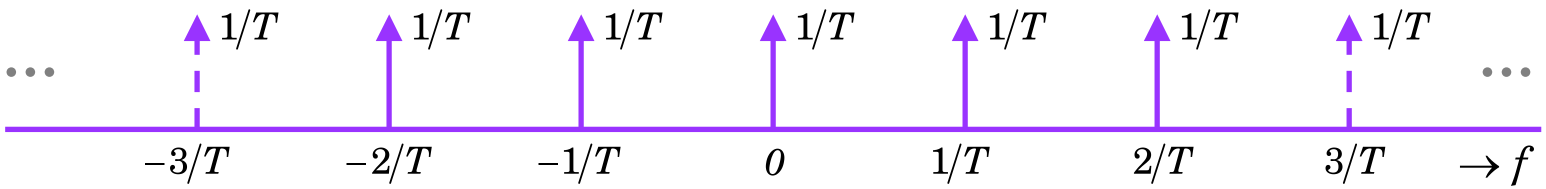
Discrete-frequency spectrum, $\Xi_{T,k}$ $\left[\textbf{Fourier series coefficients of } \xi_T(t)\right]$*:*

$$\Xi_{T,k}=\frac{1}{T}\int_{-T/2}^{T/2}\xi_T(t)\exp\left(-j2\pi\left(\frac{k}{T}\right)t\right)dt=\frac{1}{T}\int_{-\infty}^{\infty}\delta(t)\exp\left(-j2\pi\left(\frac{k}{T}\right)t\right)dt=\frac{1}{T}$$

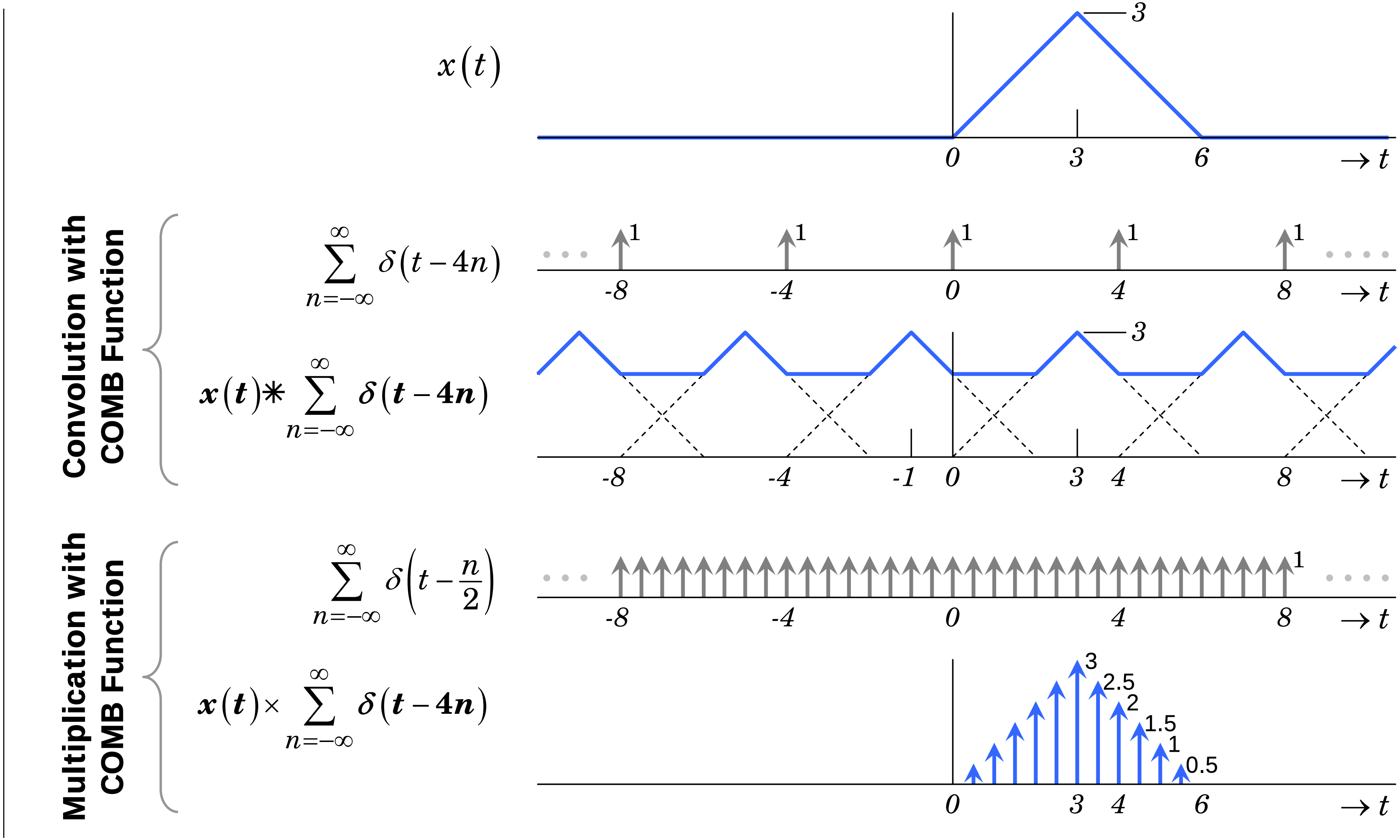


Continuous-frequency spectrum, $\Xi_T(f)$ $\left[\textbf{Fourier transform of } \xi_T(t)\right]$*:*

$$\Xi_T(f)=\sum_{k=-\infty}^{\infty}\Xi_{T,k}\delta\left(f-\frac{k}{T}\right)=\frac{1}{T}\sum_{k=-\infty}^{\infty}\delta\left(f-\frac{k}{T}\right)$$



Example 2-13:



$x(t)$
3
0
3
6
$\rightarrow t$
Convolution with COMB Function
$\sum_{n=-\infty}^{\infty} \delta(t-4n)$
1
-8
-4
0
4
8
$x(t) * \sum_{n=-\infty}^{\infty} \delta(t-4n)$
-1
Multiplication with COMB Function
$\sum_{n=-\infty}^{\infty} \delta\left(t-\frac{n}{2}\right)$
$x(t) \times \sum_{n=-\infty}^{\infty} \delta(t-4n)$
2.5
2
1.5
0.5