Bode Plots (ESSENTIALS)

Bode Straight-Line Plots

(A) Bode plots of LTI Systems with non-zero Real Poles and Zeros, assuming no Integrator and Differentiator in the system.

In constructing or reading Bode plots, it is often more convenient to express the system transfer function as

$$G(s) = K_{dc} \frac{\left(\frac{s}{z_1}+1\right)\left(\frac{s}{z_2}+1\right)\cdots\left(\frac{s}{z_M}+1\right)}{\left(\frac{s}{p_1}+1\right)\left(\frac{s}{p_2}+1\right)\cdots\left(\frac{s}{p_N}+1\right)}$$

where

$$\left(\frac{s}{z_m}+1\right) \text{ is a zero factor of } G(s)$$

$$\left(\frac{s}{p_n}+1\right) \text{ is a pole factor of } G(s)$$

$$K_{dc} \text{ is the DC (or static) gain of } G(s)$$

i. Bode plots of a zero factor: $G_z(s) = \left(\frac{s}{z_m} + 1\right)$

Zero Location : $s = -z_m$

Frequency Response : $G_z(j\omega) = \left(\frac{j\omega}{z_m} + 1\right)$

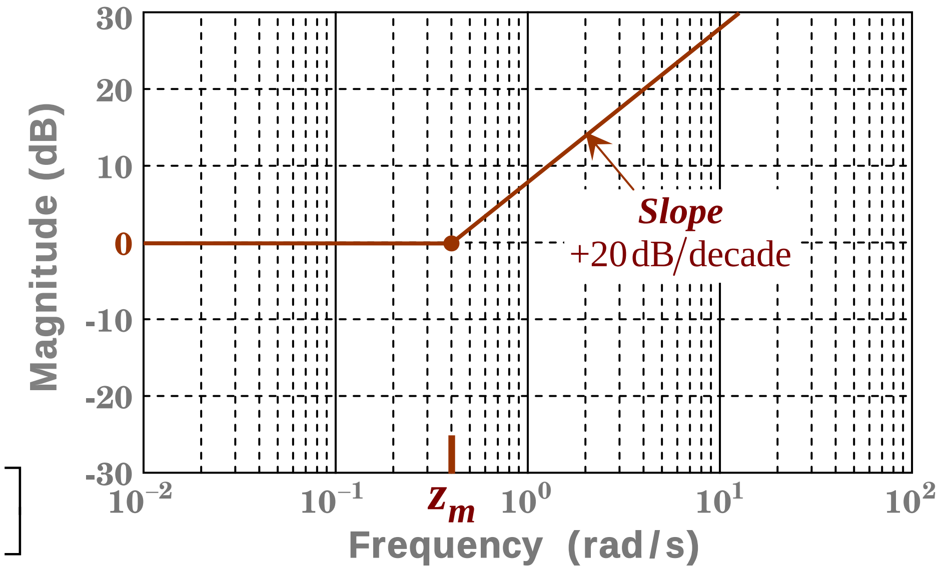
Magnitude Response :
$$\begin{cases} |G_z(j\omega)| = \left(\omega^2 / z_m^2 + 1\right)^{0.5} \\ |G_z(j\omega)|_{dB} = 20\log_{10}\left[\left(\omega^2 / z_m^2 + 1\right)^{0.5}\right] \end{cases}$$

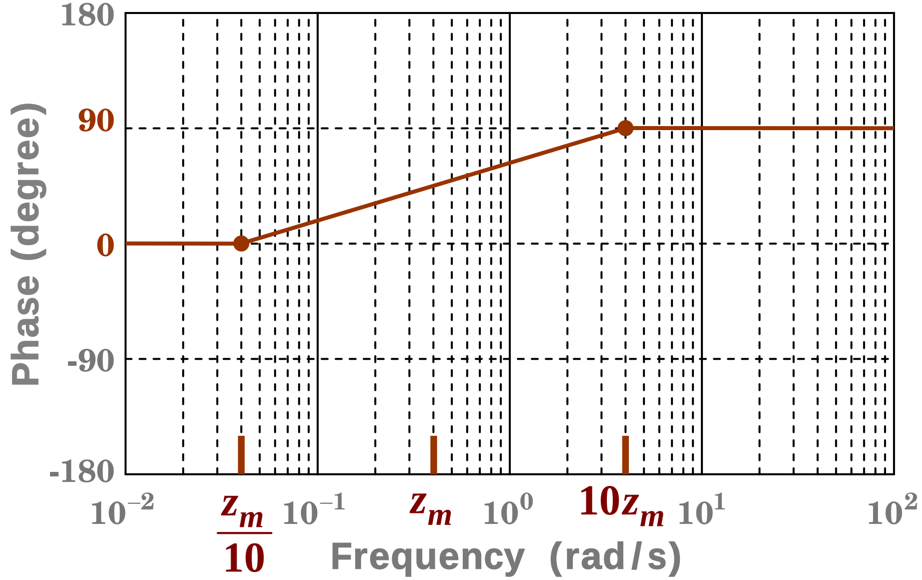
Corner Frequency : $\omega = z_m$ rad/s

DC Gain :
$$\begin{cases} |G_z(j0)| = \left|\frac{j0}{z_m} + 1\right| = 1 \\ |G_z(j0)|_{dB} = 0 \end{cases}$$

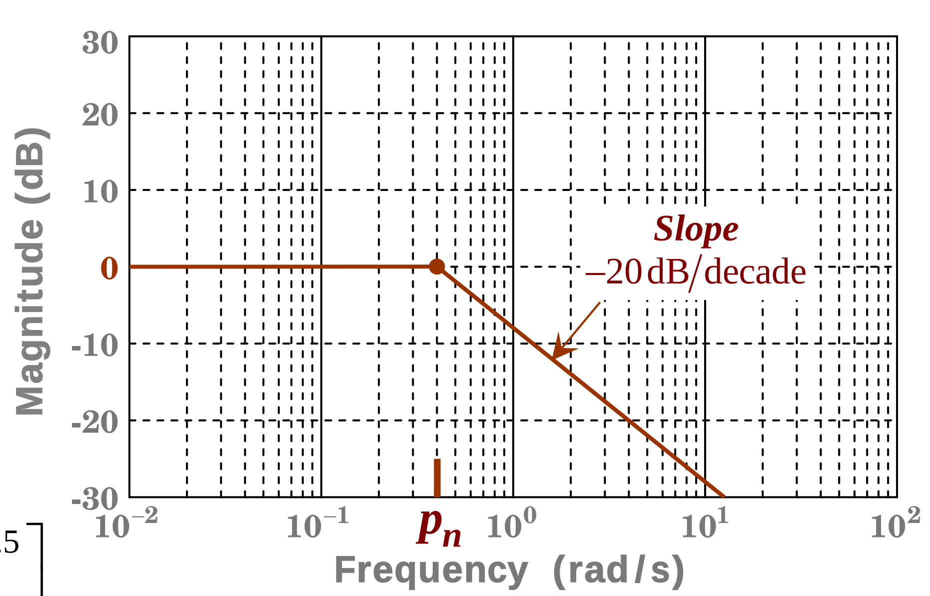
LO-Frequency Phase : $\lim_{\omega \to 0} \tan^{-1}\left(\frac{\omega}{z_m}\right) = 0°$

HI-Frequency Phase : $\lim_{\omega \to \infty} \tan^{-1}\left(\frac{\omega}{z_m}\right) = 90°$





ii. Bode plots of a pole factor: $G_p(s) = \left(\dfrac{s}{p_n} + 1\right)^{-1}$

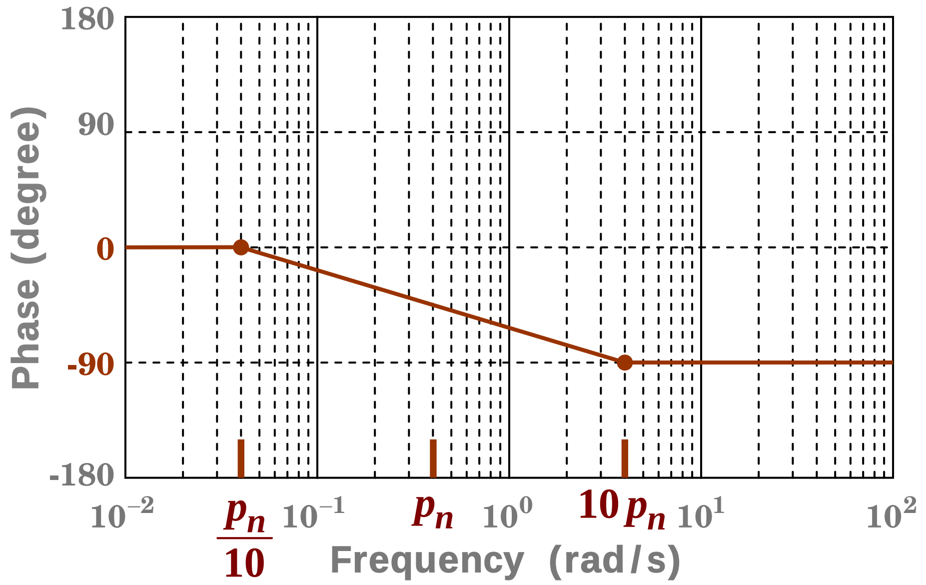


Pole Location : $s = -p_n$

Frequency Response : $G_p(j\omega) = \left(\dfrac{j\omega}{p_n} + 1\right)^{-1}$

Magnitude Response : $\begin{cases} \left|G_p(j\omega)\right| = \left(\omega^2/p_n^2 + 1\right)^{-0.5} \\ \left|G_p(j\omega)\right|_{dB} = -20\log_{10}\left[\left(\omega^2/p_n^2 + 1\right)^{0.5}\right] \end{cases}$

Corner Frequency : $\omega = p_n$ rad/s

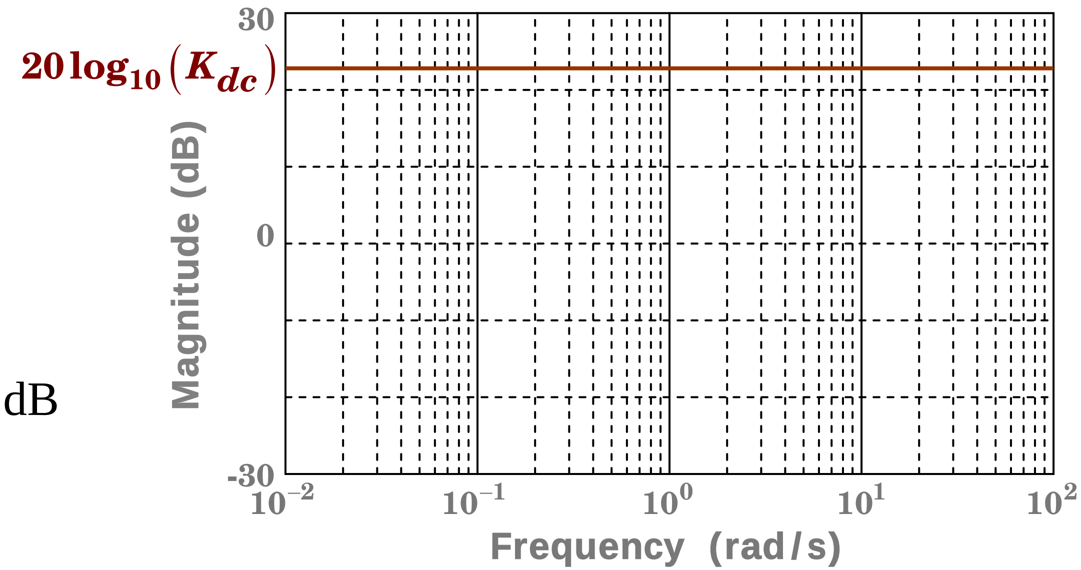


DC Gain : $\begin{cases} \left|G_p(j0)\right| = \left|\dfrac{j0}{p_n} + 1\right|^{-1} = 1 \\ \left|G_p(j0)\right|_{dB} = 0 \text{ dB} \end{cases}$

LO-Frequency Phase : $\lim\limits_{\omega \to 0} -\tan^{-1}\left(\dfrac{\omega}{p_n}\right) = 0°$

HI-Frequency Phase : $\lim\limits_{\omega \to \infty} -\tan^{-1}\left(\dfrac{\omega}{p_n}\right) = -90°$

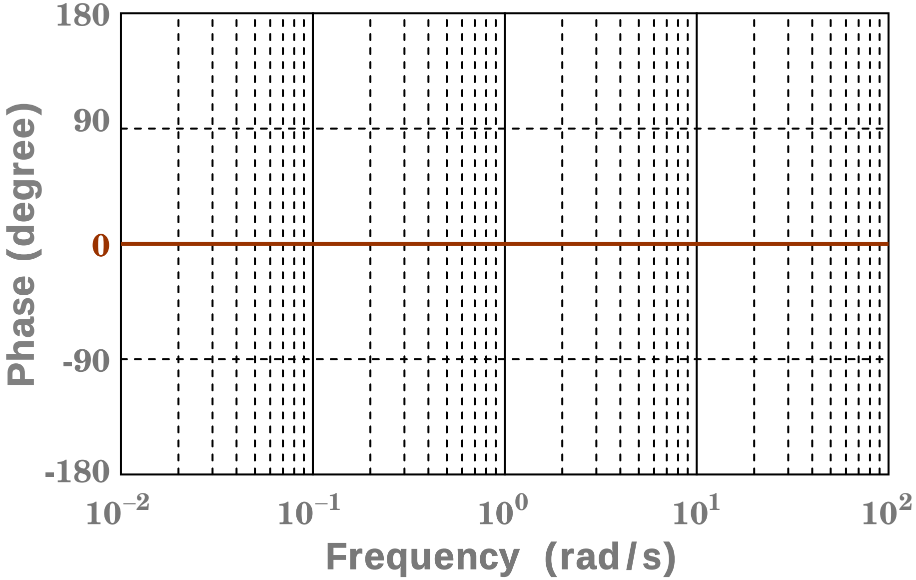
iii. Bode plots of DC gain: $G_{dc}(s) = K_{dc}$



Frequency Response : $G_{dc}(j\omega) = K_{dc}$

Magnitude Response : $\begin{cases} \left|G_{dc}(j\omega)\right| = K_{dc} \\ \left|G_{dc}(j\omega)\right|_{dB} = 20\log_{10}(K_{dc}) \text{ dB} \end{cases}$

Phase Response : $\angle G_{dc}(j\omega) = 0°$

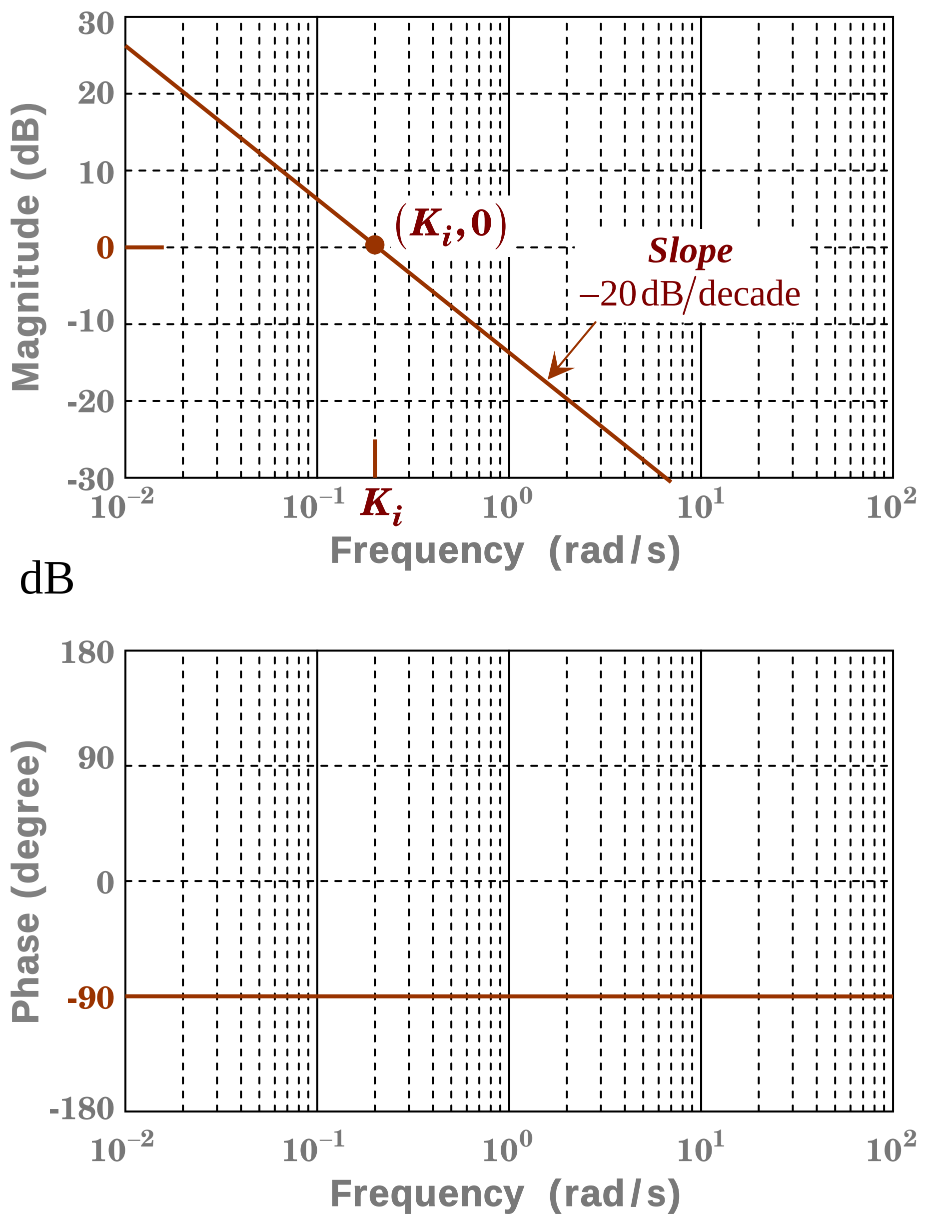


The magnitude and phase responses are both straight lines with zero gradient.

(B) Bode plots of Integrator: $G_i(s) = \dfrac{K_i}{s}$

Frequency Response : $G_i(j\omega) = \dfrac{K_i}{j\omega}$

Magnitude Response :
$$\begin{cases} \left|G_i(j\omega)\right| = \dfrac{K_i}{\omega} \\ \left|G_i(s)\right|_{dB} = 20\log_{10}(K_i) - 20\log_{10}(\omega) \text{ dB} \end{cases}$$

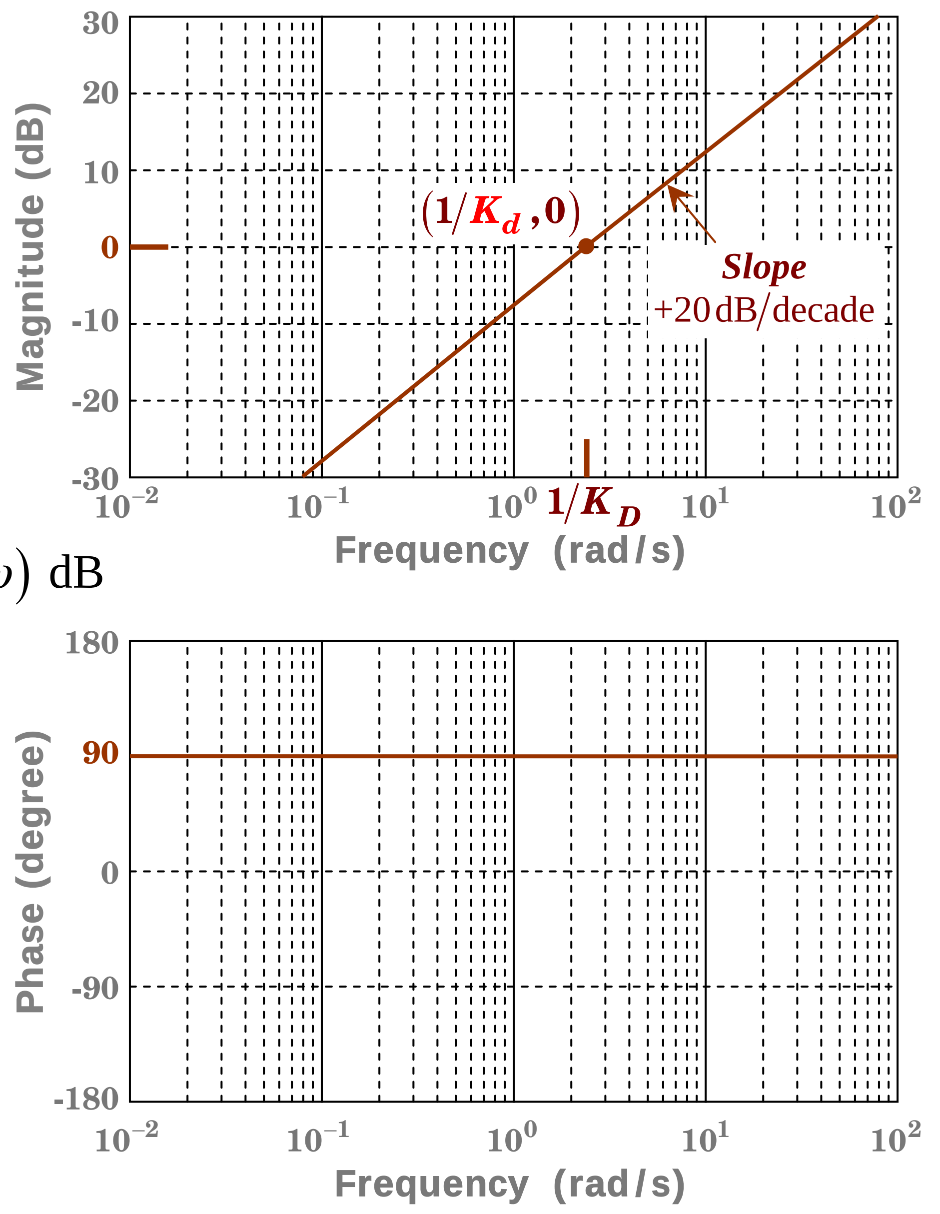


The magnitude response is a straight line with slope -20 dB/decade. At $\omega = K_I$ rad/s, its value is 0 dB.

Phase Response :
$$\begin{cases} \angle G_i(j\omega) = -\tan^{-1}\left(\dfrac{\omega}{0}\right) \\ \qquad = -90^\circ \end{cases}$$

The phase response is a straight line with zero gradient.

(C) Bode plots of Differentiator: $G_d(s) = K_d s$



Frequency Response : $G_d(j\omega) = jK_d\omega$

Magnitude Response : $\begin{cases} |G_d(j\omega)| = K_d\omega \\ |G_d(s)|_{dB} = 20\log_{10}(K_d) + 20\log_{10}(\omega) \text{ dB} \end{cases}$

The magnitude response is a straight line with slope 20 dB/decade. At $\omega = \dfrac{1}{K_d}$ rad/s, its value is 0 dB.

Phase Response : $\begin{cases} \angle G_d(j\omega) = \tan^{-1}\left(\dfrac{\omega}{0}\right) \\ \qquad\qquad = 90^\circ \end{cases}$

The phase response is a straight line with zero gradient.

(D) Bode magnitude plot for second-order systems with <u>complex poles</u> and <u>unity DC gain</u>:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}; \quad 0 \le \zeta < 1$$

To construct the straight-line approximation, set $\zeta = 1$ so that

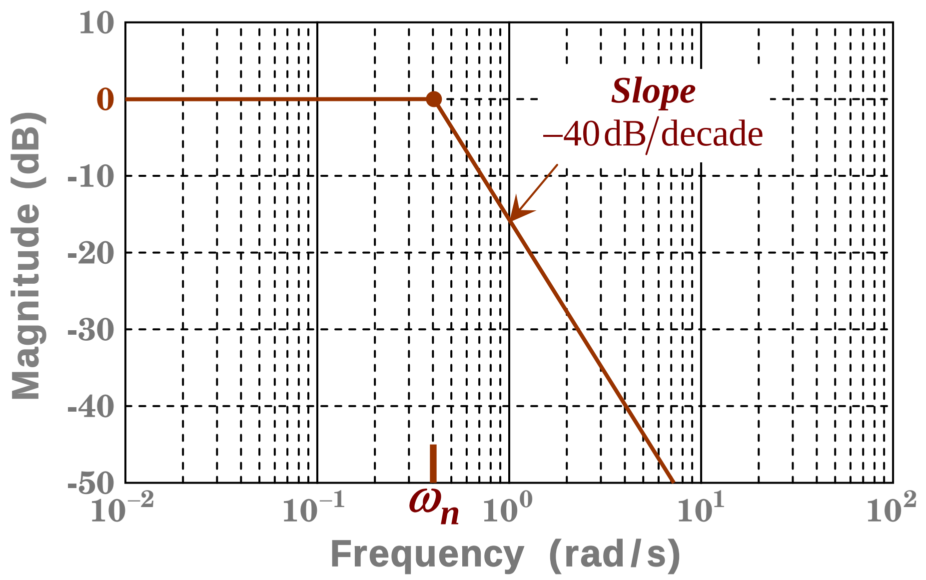
$$G(s)\Big|_{\zeta=1} = \frac{\omega_n^2}{(s+\omega_n)^2} = \left(\frac{s}{\omega_n}+1\right)^{-1}\left(\frac{s}{\omega_n}+1\right)^{-1}$$

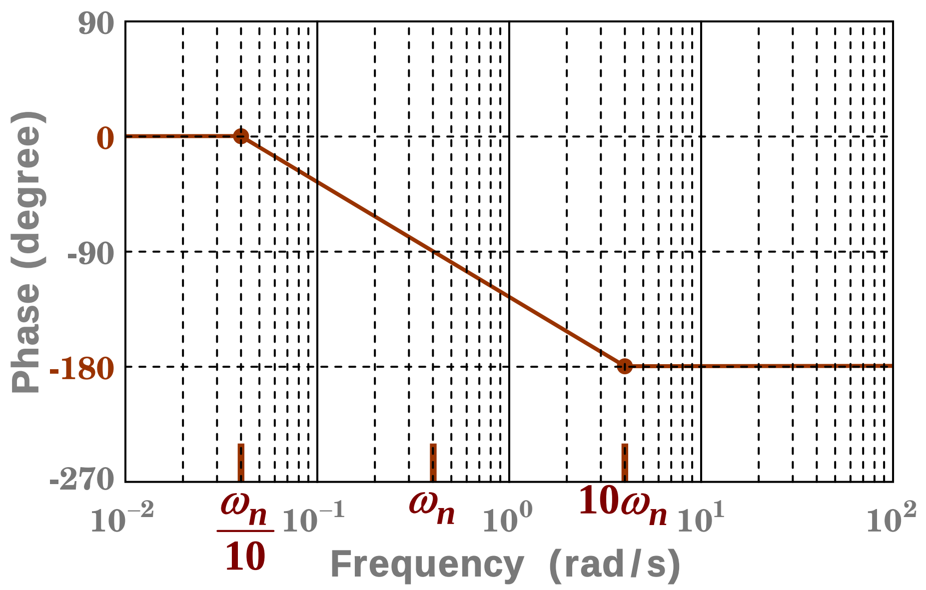
is a product of two pole factors, with each having the features given in **(A)(ii)**.

LO-Frequency Phase : $\lim_{\omega\to 0} -2\tan^{-1}\left(\frac{\omega}{p_n}\right) = 0°$

HI-Frequency Phase : $\lim_{\omega\to \infty} -2\tan^{-1}\left(\frac{\omega}{\omega_n}\right) = -180°$

Note that the value of ζ <u>cannot</u> be determined from the Bode straight-line plot. Additional information is needed to evaluate ζ.

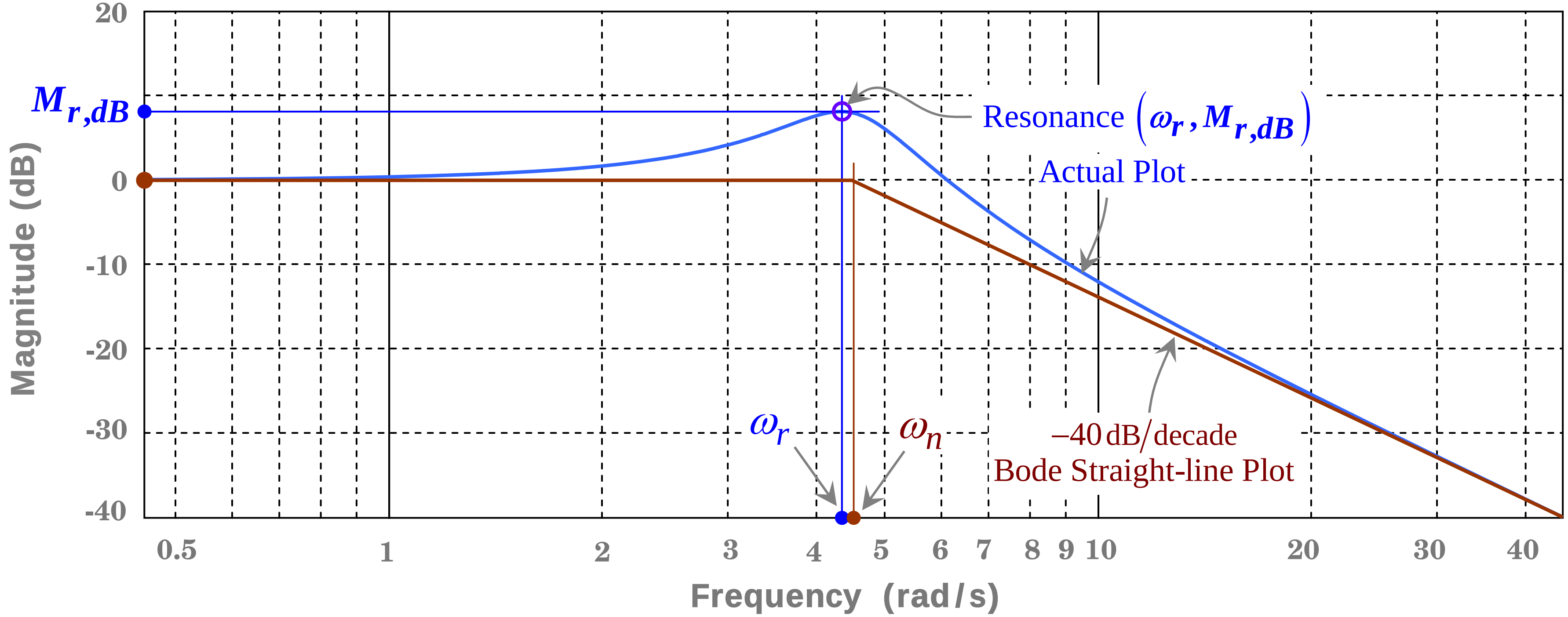




$$\left.\begin{cases}\text{Resonance frequency: } & \omega_r = \omega_n\sqrt{1-2\zeta^2} \\ \text{Resonance peak: } & M_r = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}}\end{cases}\right\}\cdots \text{ iff } \zeta < \frac{1}{\sqrt{2}} = 0.7071$$

REMARKS: (1) If $G(s) = \dfrac{\boldsymbol{K}\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$, then $M_r = \dfrac{\boldsymbol{K}}{2\zeta\sqrt{1-\zeta^2}}$

(2) Both ω_r and M_r do not exist if $\zeta \geq \dfrac{1}{\sqrt{2}} = 0.7071$



(E) Rewriting $G(s)$ to facilitate plotting and interpretation of Bode Straight-Line diagram

i. System without Integrator or Differentiator: $\boldsymbol{G(s) = K\frac{(s+z_1)(s+z_2)\cdots(s+z_M)}{(s+p_1)(s+p_2)\cdots(s+p_N)}}$

$$G(s) = K_{dc}\frac{\left(\frac{s}{z_1}+1\right)\left(\frac{s}{z_2}+1\right)\cdots\left(\frac{s}{z_M}+1\right)}{\left(\frac{s}{p_1}+1\right)\left(\frac{s}{p_2}+1\right)\cdots\left(\frac{s}{p_N}+1\right)}; \qquad \underbrace{K_{dc} = K\cdot\frac{z_1 z_2\cdots z_M}{p_1 p_2\cdots p_N}}_{\text{DC Gain}}$$

The Bode straight-line plot of $G(s)$ always begins with a horizontal line at low frequency with $\left|G(j\omega)\right|_{dB} = 20\log_{10}(K_{dc})$.

ii. System with N INTEGRATORS: $\boldsymbol{G(s) = \frac{K_i}{s^N}\cdot K\frac{(s+z_1)(s+z_2)\cdots(s+z_M)}{(s+p_1)(s+p_2)\cdots(s+p_N)}}$

[K_i is the combined gain of the N cascaded integrators]

$$G(s) = \frac{K_I}{s^N}\cdot\frac{\left(\frac{s}{z_1}+1\right)\left(\frac{s}{z_2}+1\right)\cdots\left(\frac{s}{z_M}+1\right)}{\left(\frac{s}{p_1}+1\right)\left(\frac{s}{p_2}+1\right)\cdots\left(\frac{s}{p_N}+1\right)}; \qquad \underbrace{K_I = K_i K\cdot\frac{z_1 z_2\cdots z_M}{p_1 p_2\cdots p_N}}_{\text{Modified Integrator Gain}}$$

The Bode straight-line plot of $G(s)$ always begins with a line of slope $-20N$ dB/decade at low frequency, and $K_I = \tilde{\omega}^N \left|G(j\tilde{\omega})\right| = \tilde{\omega}^N \cdot 10^{\left|G(j\tilde{\omega})\right|_{dB}/20}$ where $\left(\tilde{\omega},\ \left|G(j\tilde{\omega})\right|_{dB}\right)$ is a point on this line.

Since $G(0) = \infty$, the term DC gain does not make particular sense.

iii. System with N DIFFERENTIATORS: $\boldsymbol{G(s) = K_d s^N \cdot K \dfrac{(s+z_1)(s+z_2)\cdots(s+z_M)}{(s+p_1)(s+p_2)\cdots(s+p_N)}}$

$[K_d$ is the combined gain of the N cascaded differentiators$]$

$$G(s) = K_D s^N \cdot \frac{\left(\frac{s}{z_1}+1\right)\left(\frac{s}{z_2}+1\right)\cdots\left(\frac{s}{z_M}+1\right)}{\left(\frac{s}{p_1}+1\right)\left(\frac{s}{p_2}+1\right)\cdots\left(\frac{s}{p_N}+1\right)}; \qquad \underbrace{K_D = K_d K \cdot \frac{z_1 z_2 \cdots z_M}{p_1 p_2 \cdots p_N}}_{\text{Modified Differentiator Gain}}$$

The Bode straight-line plot of $G(s)$ always begins with a line of slope $+20N$ dB/decade at low frequency, and $K_D = \dfrac{\left|G(j\tilde{\omega})\right|}{\tilde{\omega}^N} = \dfrac{10^{\left|G(j\tilde{\omega})\right|_{dB}/20}}{\tilde{\omega}^N}$ where $\left(\tilde{\omega},\ \left|G(j\tilde{\omega})\right|_{dB}\right)$ is a point on this line.

Since $G(0) = 0$, the term DC gain does not make particular sense.

(F) Asymptotic Phase $\left[\angle G(j\omega)\right]$

$$\lim_{\omega\to\infty}\angle G(j\omega)=\underbrace{\left[\text{Number of POLES - Number of ZEROS}\right]}_{\textit{Inclusive of Integrators and Differentiators}}\times\left(-90^\circ\right)$$

$$\lim_{\omega\to 0}\angle G(j\omega)=\left[\text{Number of Integrators - Number of Differentiators}\right]\times\left(-90^\circ\right)$$

(G) Asymptotic <u>SLOPE</u> of dB-Gain $\left[\left|G(j\omega)\right|_{dB}\right]$

$$\lim_{\omega\to\infty}\left[\textit{slope of }\left|G(j\omega)\right|_{dB}\right]=\underbrace{\left[\text{Number of POLES - Number of ZEROS}\right]}_{\textit{Inclusive of Integrators and Differentiators}}\times\left(-20\ \text{dB/decade}\right)$$

$$\lim_{\omega\to 0}\left[\textit{slope of }\left|G(j\omega)\right|_{dB}\right]=\left[\text{Number of Integrators - Number of Differentiators}\right]\times\left(-20\ \text{dB/decade}\right)$$