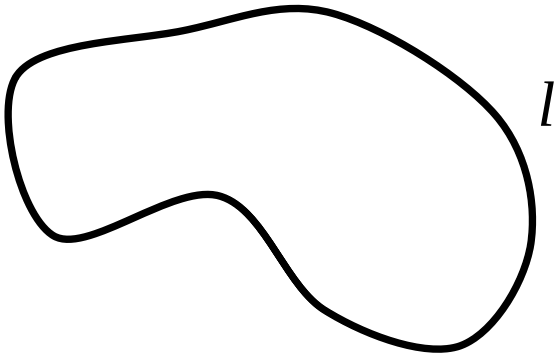
l is a closed curve



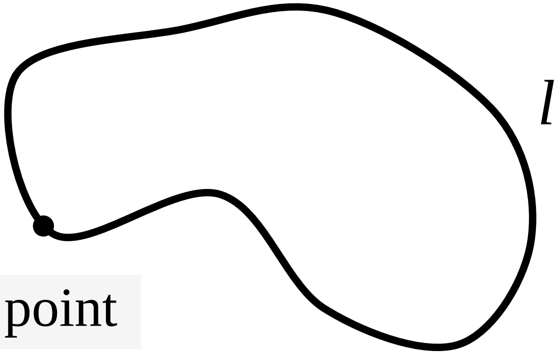
If a curve l is closed, we write the line integral as

$$\boxed{\oint_l \mathbf{F} \cdot d\mathbf{r}}$$

l is a closed curve

$$\boxed{\oint_l \mathbf{F}\cdot d\mathbf{r}}$$

$\mathbf{r}(b)$: terminal point



$\mathbf{r}(a)$: initial point

$\mathbf{F}$ is ***conservative*** if $\mathbf{F}=\nabla f$ for some f

(f is called a ***potential*** function for $\mathbf{F}$).

Fundamental Theorem for Line Integrals

$$\mathbf{F}=\nabla f$$

$$\begin{aligned}\oint_l \mathbf{F}\cdot d\mathbf{r} &= \int_C \nabla f\cdot d\mathbf{r}\\ &= f(\mathbf{r}(b)) - f(\mathbf{r}(a))\\ &= f(\mathbf{r}(a)) - f(\mathbf{r}(a))\\ &= 0\end{aligned}$$

l is a closed curve, so have the same initial point and terminal point.

Implications of Conservative Field

Fundamental Theorem for Line Integrals

$$\mathbf{F} = \nabla f$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r}$$

$$= f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

F is conservative

$\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path

$\oint_l \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed path l

Example

Let

$$\mathbf{F}(x, y) = (y^2 + 3x^2)\mathbf{i} + (2xy)\mathbf{j}.$$

Show that the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path and evaluate this integral over the curve C where C is

(i) given by $\mathbf{r}(t) = \cos t\ \mathbf{i} + e^t \sin t\ \mathbf{j},\ \ t \in [0, \boldsymbol{p}]$;

(ii) the unit circle.

To show that the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path.

$$\mathbf{F}(x, y) = (y^2 + 3x^2)\mathbf{i} + (2xy)\mathbf{j}$$

Here, $P = y^2 + 3x^2$ and $Q = 2xy$.

$$\frac{\partial Q}{\partial x} = 2y = \frac{\partial P}{\partial y}$$

$\Rightarrow \mathbf{F}$ is conservative.

By our earlier example, $\nabla f = \mathbf{F}$ where $f(x, y) = xy^2 + x^3$ is the potential function of $\mathbf{F}$. So $\mathbf{F}$ is conservative.

Hence, the line integral $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r}$ is independent of path.

$$\mathbf{F}(x,y)=(y^2+3x^2)\mathbf{i}+(2xy)\mathbf{j}$$

Fundamental Theorem for Line Integrals

$$\mathbf{F}=\nabla f$$

$$\int_C \mathbf{F}\cdot d\mathbf{r}=\int_C \nabla f\cdot d\mathbf{r}$$

$$= f(\mathbf{r}(b))-f(\mathbf{r}(a))$$

$$\nabla f=\mathbf{F} \text{ where } f(x,y)=xy^2+x^3$$

(i) $\mathbf{r}(t)=\cos t\ \mathbf{i}+e^t\sin t\ \mathbf{j},\quad 0\le t\le p$

$$\mathbf{r}(0)=(\cos 0)\mathbf{i}+(e^0\sin 0)\mathbf{j}=\mathbf{i}+0\mathbf{j}\quad\to\quad(1,0)$$

$$\mathbf{r}(p)=(\cos p)\mathbf{i}+(e^p\sin p)\mathbf{j}=-\mathbf{i}+0\mathbf{j}\quad\to\quad(-1,0)$$

$$\int_C \mathbf{F}\cdot d\mathbf{r}=\int_C \nabla f\cdot d\mathbf{r}$$

$$= f(\mathbf{r}(b))-f(\mathbf{r}(a))$$

$$= f(-1,0)-f(1,0)=-2$$

$\mathbf{F}$ is ***conservative*** if $\mathbf{F} = \nabla f$ for some f

(f is called a ***potential*** function for $\mathbf{F}$).

l is a closed curve, so have the same initial point and terminal point.

Fundamental Theorem for Line Integrals

$$\mathbf{F} = \nabla f$$

$$\begin{aligned}\oint_l \mathbf{F} \cdot d\mathbf{r} &= \int_C \nabla f \cdot d\mathbf{r} \\ &= f(\mathbf{r}(b)) - f(\mathbf{r}(a)) \\ &= f(\mathbf{r}(a)) - f(\mathbf{r}(a)) \\ &= 0\end{aligned}$$

(ii) Since the unit circle is a closed path and $\mathbf{F}$ is conservative, so we have

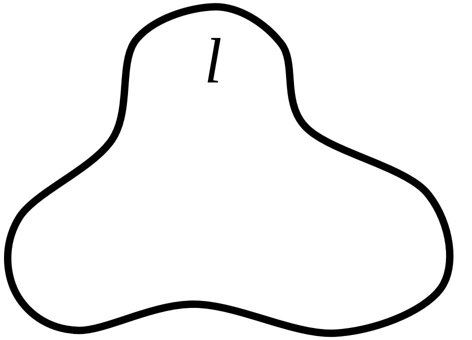
$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0$$

Green’s Theorem

F is conservative

$$\oint_l \mathbf{F} \cdot d\mathbf{r} = 0$$

for any closed path l



Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$.

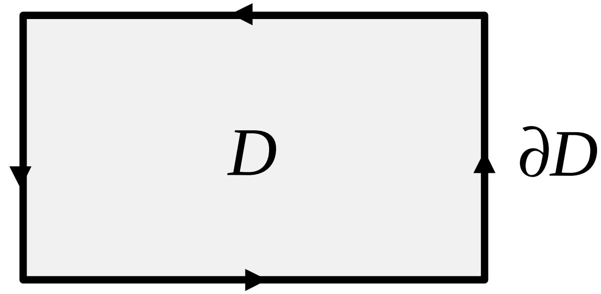
If $\mathbf{F}$ is ***conservative***, i.e., $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$, then

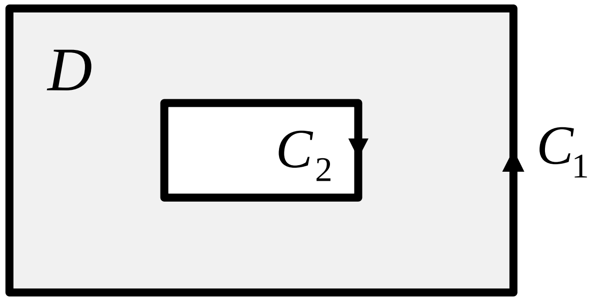
$$\oint_l P\ dx + Q\ dy = 0.$$

What can be said about $\oint_l P\ dx + Q\ dy$ if $\mathbf{F}$ is ***not*** conservative?

Positive Orientation

Let D : plane region with boundary ∂D





Positive orientation

positive orientation of ∂D:

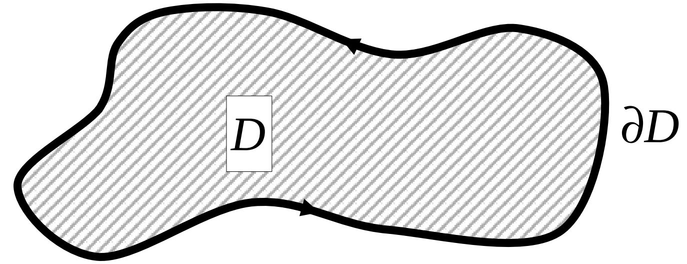
as one traverses along ∂D, the region D is on the LHS.

negative orientation of ∂D:

as one traverses along ∂D, the region D is on the RHS.

Green's Theorem

English mathematical physicist:
Sir George Green (1793-1841)



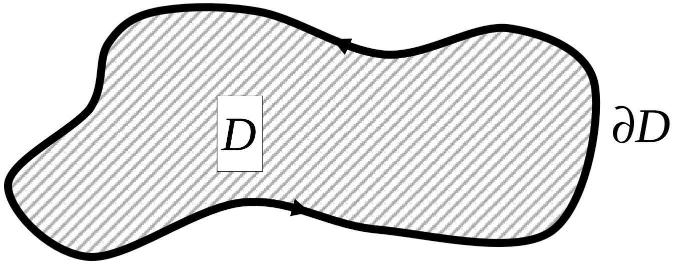
positive orientation of ∂D

Let D be a bounded region in the $xy-$plane and ∂D be the boundary of D. Suppose both $P(x, y)$ and $Q(x, y)$ have continuous partial derviatives on D. Then

$$\oint_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

∂D is oriented such that traversing ∂D in its positive direction keeps D to the left.

Green's Theorem



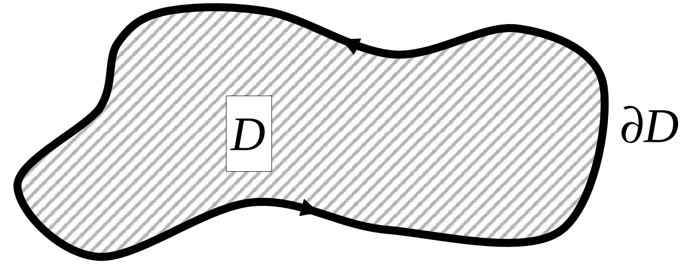
positive orientation of ∂D

$$\oint_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Line Integral = Double Integral

Green's Theorem

$$\oint_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$



positive orientation of ∂D

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$.

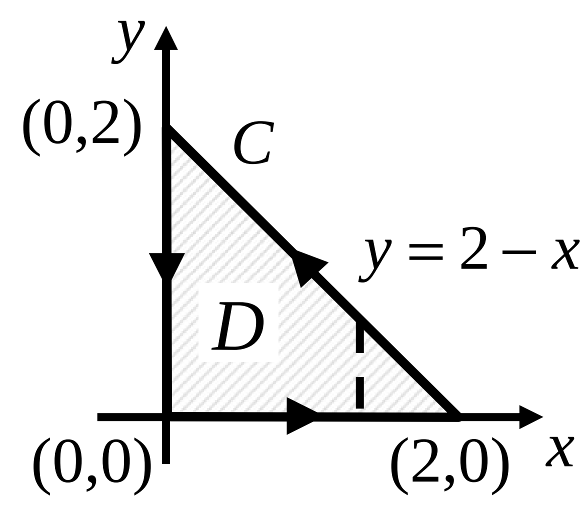
Note : If $\mathbf{F}$ is ***conservative***, i.e., $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$, then

$$\begin{aligned}\oint_l P\,dx + Q\,dy &= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \\ &= \iint_D 0\,dA \\ &= 0\end{aligned}$$

(A result which we have already observed earlier)

Green's Theorem - Example

Evaluate $\oint_C 2xy\,dx + xy^2 dy$, where C is the triangular curve consisting of the line segments from (0,0) to (2,0), from (2,0) to (0,2) and from (0,2) to (0,0).



The region D is given by: $0 \le y \le 2 - x,\ 0 \le x \le 2.$

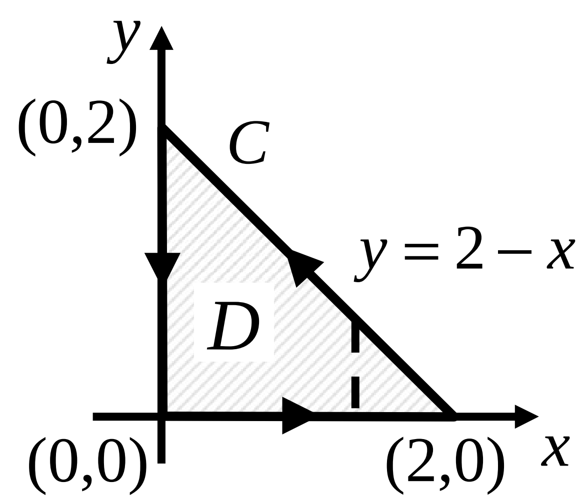
Green's Theorem - Example

Evaluate $\oint_C 2xy\,dx + xy^2 dy$, where C is the triangular curve consisting of the line segments from (0,0) to (2,0), from (2,0) to (0,2) and from (0,2) to (0,0).

Question

Without Green's Theorem

How many line integrals must you find ????



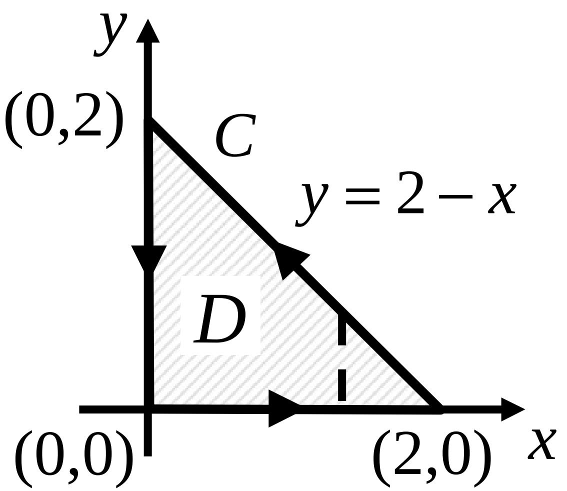
The region D is given by: $0 \le y \le 2 - x,\ 0 \le x \le 2.$

Evaluate $\oint_C 2xy\, dx + xy^2 dy$, where C is the triangular curve consisting of the line segments from (0,0) to (2,0), from (2,0) to (0,2) and from (0,2) to (0,0).

The functions

$$P(x, y) = 2xy \quad \text{and} \quad Q(x, y) = xy^2$$

have continuous partial derivatives on the xy-plane.

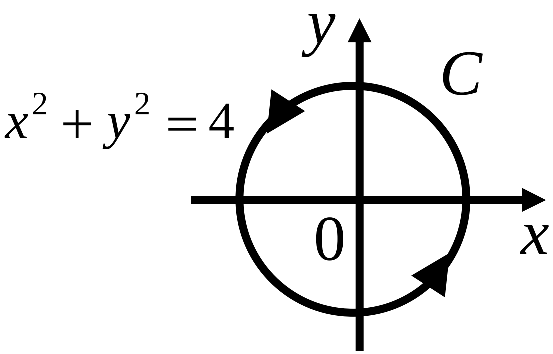


By Green's Theorem,

$$\oint_C 2xy\, dx + xy^2\, dy = \iint_D \left[\frac{\partial}{\partial x}(xy^2) - \frac{\partial}{\partial y}(2xy) \right] dA$$

$$= \int_0^2 \int_0^{2-x} (y^2 - 2x)\, dy\, dx = -\frac{4}{3}.$$

Green's Theorem - Example



Evaluate $\oint_C (4y - e^{x^2})\, dx + [9x + \sin(y^2 - 1)]\, dy$,

where C is the circle $x^2 + y^2 = 4$.

Note : C bounds the circular disk D of radius 2 and is given the positive orientation.

By Green's Theorem,

$$\oint_C (4y - e^{x^2})\, dx + [9x + \sin(y^2 - 1)]\, dy$$

$$= \iint_D \left\{ \frac{\partial}{\partial x}[9x + \sin(y^2 - 1)] - \frac{\partial}{\partial y}(4y - e^{x^2}) \right\} dA$$

$$= \iint_D 5\, dA$$

$$= 5\iint_D dA$$

$$= 5 \times (\text{Area of } D)$$

$$= 5(p\, 2^2) = 20p.$$

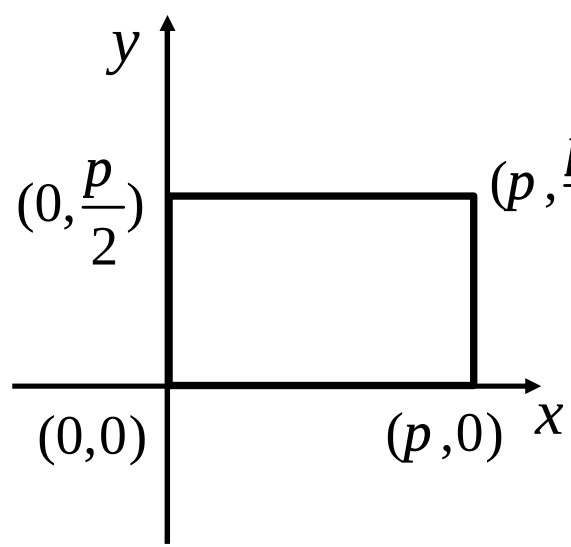
Green's Theorem - Exercise

Evaluate by Green's Theorem

$$\oint_C e^{-x} \sin y \, dx + e^{-x} \cos y \, dy$$

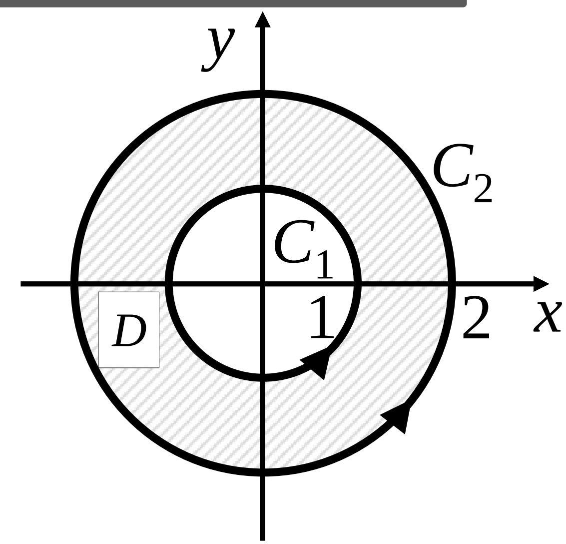
where C is the rectangle with vertices at $(0,0), (p,0), (p, \frac{p}{2})$ and $(0, \frac{p}{2})$.

$\text{Answer}: 2(e^{-p} - 1)$



Green's Theorem - Example

Let $\mathbf{F}(x, y) = y\mathbf{i} + y\mathbf{j}$ and D a region in xy-plane bounded by the two circles centered at the origin with radius 1 and 2. Verify Green's Theorem.



We shall verify Green's Theorem by :

(i) Computing $\int_{\partial D} \mathbf{F} \cdot d\mathbf{r}$ directly.

(ii) Computing $\int_{\partial D} \mathbf{F} \cdot d\mathbf{r}$ using Green's Theorem.

Show that the answers to (i) and (ii) are the same!!

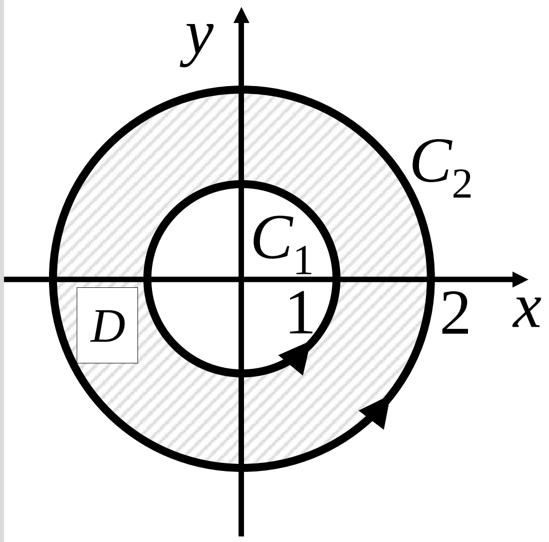
(i) Compute $\int_{\partial D} \mathbf{F} \cdot d\mathbf{r}$ directly:

The boundary of D is made up of two disjoint curves C_1 and C_2.

$$C_1 : \mathbf{r}_1 = \cos t \ \mathbf{i} + \sin t \ \mathbf{j} \qquad \text{and} \qquad C_2 : \mathbf{r}_2 = 2\cos t \ \mathbf{i} + 2\sin t \ \mathbf{j},$$

and we have $\partial D = C_2 - C_1$.

$$\begin{aligned}\int_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2p} (\sin t \ \mathbf{i} + \sin t \ \mathbf{j}) \cdot (-\sin t \ \mathbf{i} + \cos t \ \mathbf{j}) \, dt \\ &= \int_0^{2p} (-\sin^2 t + \sin t \cos t) \, dt \\ &= \int_0^{2p} \frac{1}{2}(\cos 2t - 1 + \sin 2t) \, dt \\ &= \frac{1}{2}\left[\frac{\sin 2t}{2} - t - \frac{\cos 2t}{2} \right]_0^{2p} = -p\end{aligned}$$



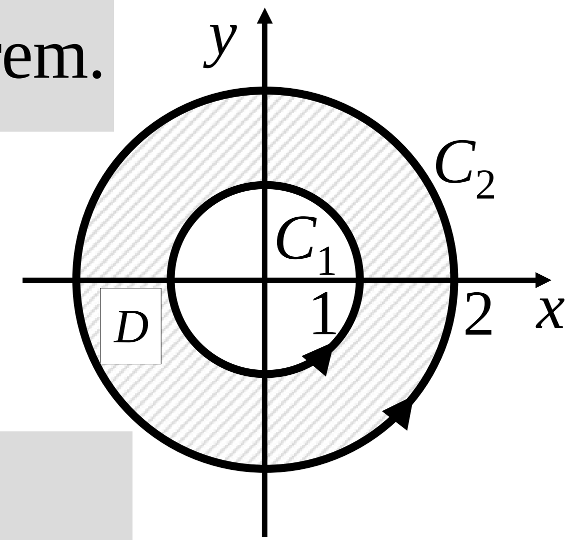
Similarly, $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = -4p$

(i) Compute $\int_{\partial D} \mathbf{F} \cdot d\mathbf{r}$ directly:

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = -p \qquad \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = -4p$$

$$\begin{aligned}
\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} &= \int_{C_2 - C_1} \mathbf{F} \cdot d\mathbf{r} \\
&= \int_{C_2} \mathbf{F} \cdot d\mathbf{r} - \int_{C_1} \mathbf{F} \cdot d\mathbf{r} \\
&= -4p - (-p) \\
&= -3p .
\end{aligned}$$

(ii) Computing $\int_{\partial D} \mathbf{F} \cdot d\mathbf{r}$ using Green's Theorem.



$$\mathbf{F}(x, y) = y\mathbf{i} + y\mathbf{j}$$

Here $P = Q = y$.

Using Green's Theorem, we have

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D \left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(y) \right] dA = \iint_D (-1)\ dA.$$

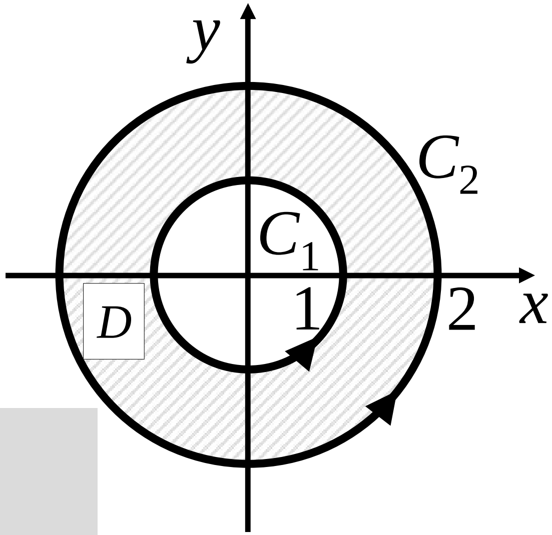
In polar coordinates, D is given by $1 \le r \le 2, \quad 0 \le q \le 2p$.

So we have

$$\iint_D (-1)\ dA = \int_0^{2p} \int_1^2 -r\ dr\ dq$$

$$= -3p$$

Note that the answers to (i) and (ii) are the same!!



Using Green's Theorem, we have

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D \left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(y) \right] dA = \iint_D (-1)\, dA.$$

Note that

$$\begin{aligned} \iint_D (-1)\, dA &= -\iint_D dA \\ &= -p\,(2^2 - 1^2) \\ &= -3p\,. \end{aligned}$$

Area of big circle − Area of small circle

End