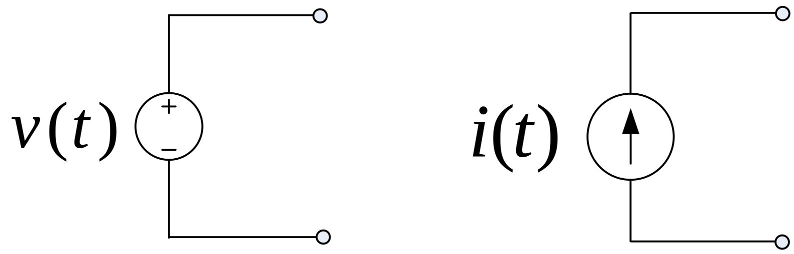
Lecture 6

Steady-state sinusoidal analysis

Time-dependent signal sources

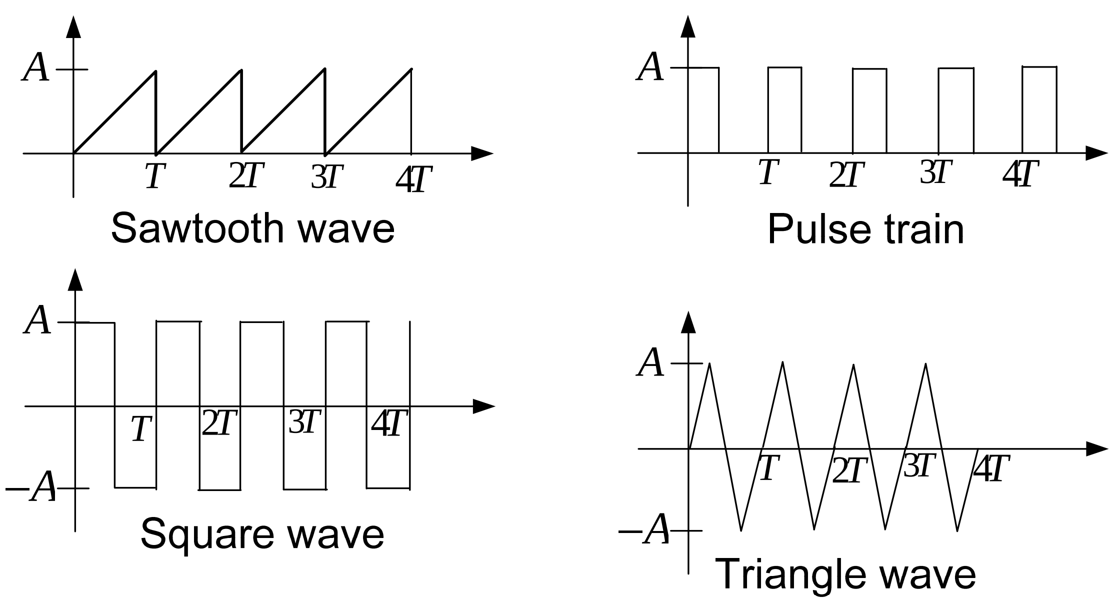
Sometimes, the ideal voltage and current sources can be time-dependent signals. This means, their voltage and current outputs respectively are a function of time. They are not affected by the load connected to them.



Of the time-dependent signals, the **periodic** signals are the most important, given as:
$x(t) = x(t + nT), \quad n = 1,2,3,...$

Here T is the period of the time signal. The signal is a repetition of a part of the signal over time. The part that repeats itself is also called one cycle of the signal.

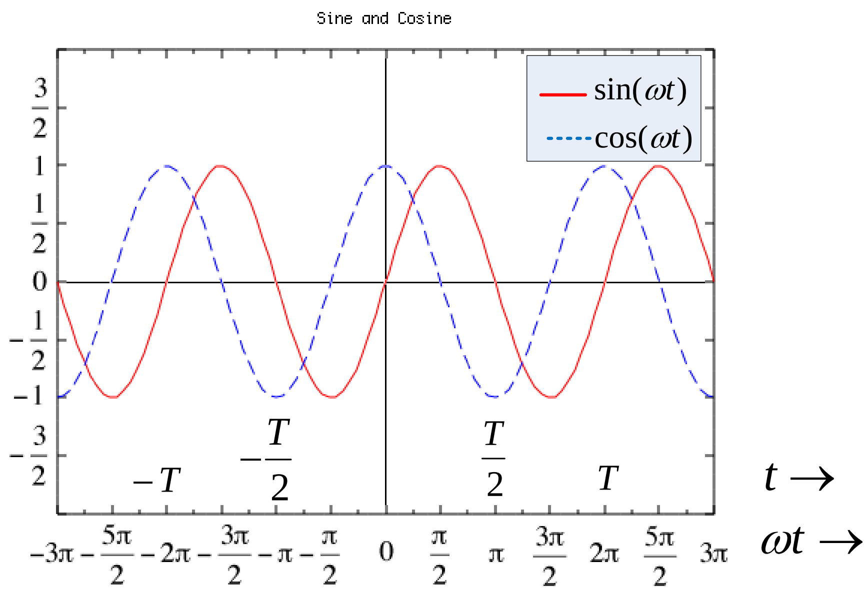
The common period signals are given below. These can be outputted from a lab equipment called signal generator.



The most commonly encountered periodic signal in electrical engineering is the **sinusoidal signal.**

Why sinusoids?

Sinusoidal sources have many important applications. Electric power supply is sinusoidal in form. Sinusoidal signals have many uses in Radio communications. From Fourier analysis it is known that all periodic signals can be represented as sum of sinusoidal signals.



The **time period** of the signal is defined as the time taken to complete one cycle (part of the signal that repeats itself over time.).

The **frequency** of the periodic signal is the number of cycles completed in one second.

$$f = \frac{1}{T}$$

The units of frequency are hertz (Hz).

Angular frequency (radians per second) $\omega = \frac{2\pi}{T} = 2\pi f$, as one period in time corresponds to 2π radian.

Sinusoidal Currents and Voltages

A general sinusoidal voltage is given by
$v(t) = V_m \cos(\omega t + \theta)$, where V_m is the peak value of the voltage, and θ is the phase angle.

Sinusoidal functions can be represented by both the cosine and the sine forms, which are inter changeable as $\sin(z) = \cos(z - 90^0)$ and $\cos(z) = \sin(z + 90^0)$. Hence, even though a bunch of sinusoidal signals may be represented by both sine and cosine forms, they are all converted to either cosine or sine for sake of uniformity. This also helps in reading their relative phase angles if all of them are same frequency.

Root-mean-square values

If a periodic voltage $v(t)$ is applied across a resistor, then the power dissipated in it will be

$$p(t) = \frac{v^2(t)}{R}$$

Thus power also is a function of time.

Energy dissipated in the resistor per one period is given by

$$E_T = \int_0^T p(t)dt = \int_0^T \frac{v^2(t)}{R}dt$$

The average power dissipated in the resistor will be

$$P_{avg} = \frac{E_T}{T} = \frac{1}{T}\int_0^T p(t)dt = \frac{1}{T}\int_0^T \frac{v^2(t)}{R}dt = \frac{\frac{1}{T}\int_0^T v^2(t)dt}{R}$$

$$\frac{1}{T}\int_0^T v^2(t)dt = V_{rms}^2$$

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T v^2(t)dt}$$

$$P_{avg} = \frac{V_{rms}^2}{R}$$

The average power in a resistor when a dc voltage V_{dc} is applied across a resistor would be $P_{avg} = \frac{V_{dc}^2}{R}$

The root-mean-square for a periodic signal helps in calculation of average power in a resistor.

RMS of a sinusoid

Consider the sinusoidal voltage $v(t) = V_m \cos(\omega t + \theta)$.

The RMS value of this voltage will be

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T v^2(t)dt} = \sqrt{\frac{1}{T}\int_0^T V_m^2 \cos^2(\omega t + \theta)dt}$$
$$= \sqrt{\frac{1}{T}\frac{V_m^2}{2}\int_0^T (1 + \cos 2(\omega t + \theta))dt} = \sqrt{\frac{V_m^2}{2}} = \frac{V_m}{\sqrt{2}}$$

The cosine function integrated over one time period will contribute zero to the integration.

Usually in discussing sinusoids, the RMS value is given rather than the peak value.

For example when we say the PUB supply in Singapore is 230V, we mean that the RMS value of the PUB supply is 230V and its peak value would be $V_m = \sqrt{2}V_{rms} = 230 \times \sqrt{2} = 230 \times 1.414 = 325V$

Phasors

Sinusoidal voltages and currents can be represented as vectors in a complex plane. These are called Phasors and are very useful in steady-state analysis of sinusoidal voltages and currents.

Handling of phasors requires familiarity with complex-number arithmetic.

As seen in earlier chapters on KVL and KCL, we need to add voltages and currents to solve a given DC circuit. However, when the voltage and current are sinusoidal, arithmetic becomes quite involved requiring repeated application of trigonometric substitutions. However, with phasors, the process is quite simplified.

Phasor Definition : To represent sinusoidal signals as complex numbers.

For a sinusoidal voltage of $v_1(t) = V_1 \cos(\omega t + \theta)$, we define the phasor as $\mathrm{V}_1 = V_1 \angle \theta_1$.
Thus, phasor of a sinusoid is a complex number having a magnitude equal to the peak value and having the same phase angle as the sinusoid.

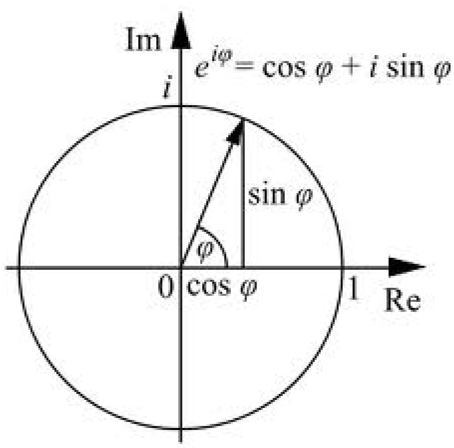
It is worth mentioning that in some conventions, the magnitude of phasor is taken to be the RMS value of the sinusoid instead of the peak value. It is important that the same convention is followed consistently.

Phasor is just a definition. This gives rise to mathematical convenience. It has no physical significance.

If the sinusoid is in given in sine form as $v_2(t) = V_2 \sin(\omega t + \theta)$, then the we first convert it to cosine function first as $v_2(t) = V_2 \sin(\omega t + \theta) = V_2 \cos(\omega t + \theta - 90^0)$. Its phasor would then be $\mathrm{V}_2 = V_2 \angle \theta - 90^0$.

The same applies to sinusoidal currents as well.

Euler's formula



Euler's identity corresponds to the polar form of complex numbers to its rectangular form.

$$e^{j\varphi} = \cos\varphi + j\sin\varphi$$

$$\cos\varphi = \mathrm{Re}(e^{j\varphi})$$

Using this relationship, any sinusoid can be represented as:

$$V_m \cos(\omega t + \theta) = \mathrm{Re}(V_m e^{j(\omega t + \theta)})$$

Phasor as a rotating vector

$V_m e^{j(\omega t + \theta)}$ represents a vector in the complex plane whose magnitude is V_m and whose angle with the real axis is $\omega t + \theta$. The angle changes with time or in other words the vector keeps rotating at the angular speed of ω rad/sec. and its angle at t=0 was θ. The projection of the vector along the real axis will be the original sinusoidal function $V_m \cos(\omega t + \theta)$.

The phasor for $\cos(\omega t+\theta)$ is $V_m e^{j\theta}$ which can be thought of as a snap shot of the rotating vector at t=0.

Phase relationships

Consider two voltages as

$v_1(t)=3\cos(\omega t+40^0)$

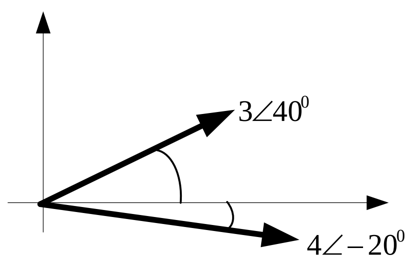
$v_2(t)=4\cos(\omega t-20^0)$

The two phasors would be:

$3\angle 40^0$

$4\angle -20^0$

We can draw the phasor diagram for these two.



Leading lagging

The signal that reaches the peak (the phase angle to become 0 degrees as cosine is maximum at phase angle 0) earlier is said to be leading.

Another way find the phase difference is to find the difference $\theta_1-\theta_2$. If it is positive and less than 180 degree, then v1 is leading v2. Else, v1 is lagging v2.

Impedances (also known as complex resistance, frequency-dependent resistance)

With understanding of impedances for the circuit elements like resistance, inductance and capacitance, the sinusoidal steady-state analysis will be same as analysis of purely resistive circuits under DC supply.

Inductance

Considering an inductance in which a sinusoidal current is flowing as

$i_L(t) = I_m \sin(\omega t + \theta) = I_m \cos(\omega t + \theta - 90^0)$

We can express the voltage across the inductance as

$v_L(t) = L\dfrac{di_L(t)}{dt} = L\omega I_m \cos(\omega t + \theta)$

Putting both current and voltage in phasor current will be

$\mathrm{I_L} = I_m \angle \theta - 90^0$

Voltage phasor will be

$\mathrm{V_L} = \omega L I_m \angle \theta = V_m \angle \theta$

We can rewrite the voltage equation as

$\mathrm{V_L} = \omega L I_m \angle \theta = \omega L \angle 90^0 \times I_m \angle \theta - 90^0 = \mathrm{Z_L} \times \mathrm{I_L}$.

$Z_L = \omega L \angle 90^0 = j\omega L$ is the impedance of the inductance.

And $V_L = Z_L I_L$ is the Ohm's law in phasor form.

Impedance in general is a complex number where as resistance is a real number. Impedances which are purely imaginary are called reactances.

Capacitance

Similar to inductances, for capacitances, if current and voltages are sinusoidal, the phasors are related by

$V_c = Z_c I_c$

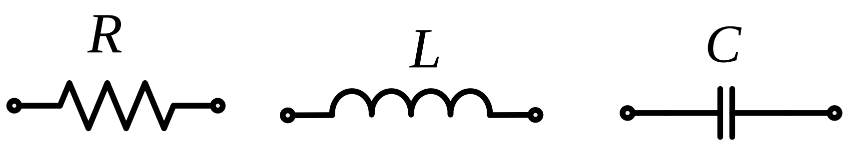
It can be shown that $Z_c = -j\dfrac{1}{\omega C} = \dfrac{1}{j\omega C} = \dfrac{1}{\omega C}\angle -90^0$.

The impedance of capacitance is also purely imaginary.

Resistance

For a resistance, the phasors are related by $V_R = RI_R$.

This implies, the sinusoidal voltage and current for a resistor will be in phase.



$$Z_R = R \qquad Z_L = j\omega L \qquad Z_C = \frac{1}{j\omega C}$$

Circuit Analysis with phasors and Complex Impedances

Kirchoff's laws in Phasor form

For a circuit with sinusoidal voltages, the sum of voltages will be equal to zero.

$v_1(t) + v_2(t) - v_3(t) = 0$

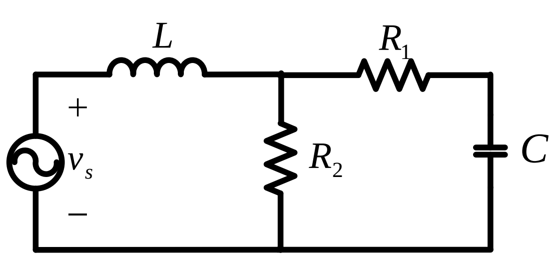
This can be rewritten in terms of their phasors as:

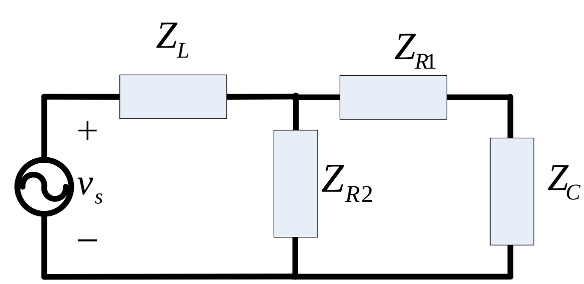
$V_1 + V_2 - V_3 = 0$.

The resulting equations will have complex numbers.

Step-by-step procedures for steady-state analysis of circuits with sinusoidal sources:

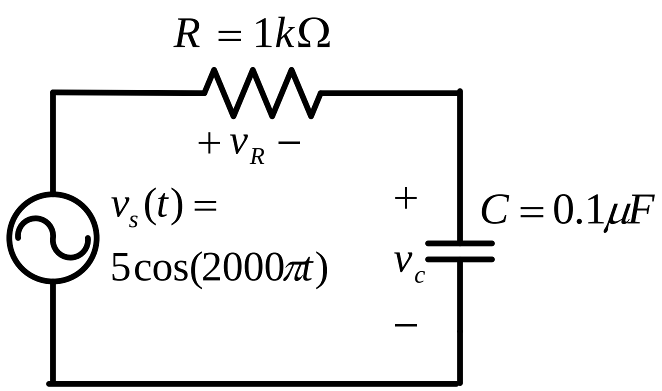
1. Replace the time descriptions of the voltage and current sources with the corresponding phasors. (All sources must have the same frequency).
2. Replace the inductance with its impedance $Z_L = j\omega L$ and capacitance with its impedance of $Z_c = \frac{j}{\omega C}$. Resistances have the same impedance as their resistance.
3. Analyze the circuits as before with DC sources and resistances only.
4. Convert the final results in phasor to time-domain form





Example:

Find the voltages across the resistor and the capacitor using the complex impedances and the phasor.



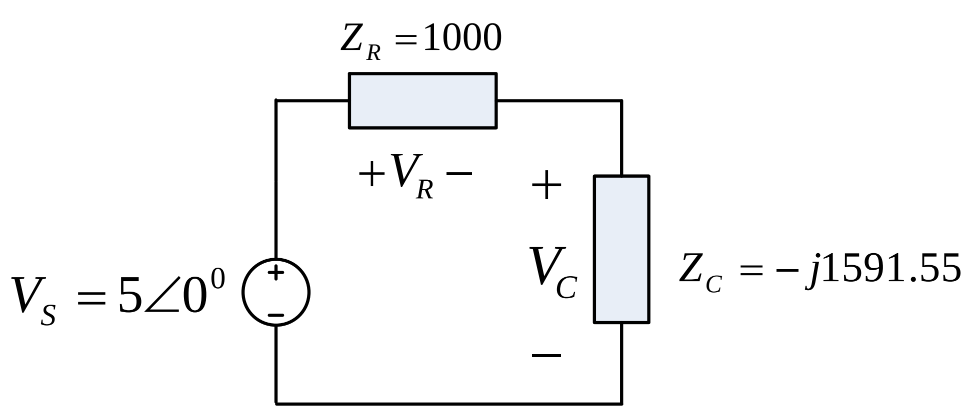
Solution:

We first need to convert the source voltage into its phasor and the elements into their complex impedances.

The voltage source $v_s(t) = 5\cos(2000\pi t)$ has maximum value of 5, angular frequency of 2000π and phase angle of 0^0. Hence its phasor will be $V_s = 5\angle 0^0$.

The impedance of the resistor will be $Z_R = R = 1000$.

The impedance of the capacitor will be $Z_C = \dfrac{1}{j\omega C} = \dfrac{1}{j2000\pi \times 0.1 \times 10^{-6}} = -j1591.55$



Then, we shall treat the equivalent circuit as a DC circuit and apply the DC circuit principles to solve the circuit.

We can now write the KVL in terms of the phasors of the voltage drops across the elements:

$$V_S - V_R - V_C = 0$$
$$V_S = V_R + V_C$$

We can apply the voltage divider rule and find the voltage across each element:

$$V_C = \frac{Z_C}{Z_R + Z_C} V_S$$

Of course, we have to deal with complex algebra here. We have alternate between rectangular and polar form for the complex numbers to do the addition and dvision.

$$\frac{Z_C}{Z_R + Z_C} = \frac{-j1591.55}{1000 - j1591.55} = \frac{1591.55\angle -90^0}{1879.64\angle -57.86^0} = 0.847\angle -32.14^0$$

Finally the phasor of the capacitor voltage will be $V_C = 0.847\angle -32.40^0 \times 5\angle 0^0 = 4.235\angle -27.40^0$.

Comparing source voltage and the capacitor voltage phasors:

$$V_S = 5\angle 0^0$$
$$V_C = 4.235\angle -27.40^0$$

The equivalent time domain signals would be:

$$V_S = 5\angle 0^0 \Rightarrow v_s(t) = 5\cos(2000\pi t)$$
$$V_C = 4.235\angle -27.40^0 = 4.235\cos(2000\pi t - 27.40^0)$$

Thus, the voltage across the capacitor will be smaller in magnitude than the source voltage and will be lagging the source voltage.

Similarly, we can find the voltage across the resistor as:

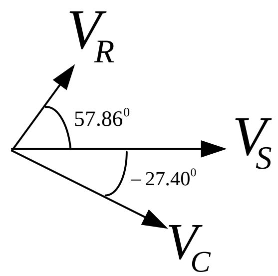
$$V_R = \frac{Z_R}{Z_R + Z_C} V_S$$

$$\frac{Z_C}{Z_R + Z_C} = \frac{1000}{1000 - j1591.55} = \frac{1000\angle 0^0}{1879.64\angle -57.86^0} = 0.532\angle 57.86^0$$

Finally the phasor of the resistor voltage will be $V_R = 0.532\angle 57.86^0 \times 5\angle 0^0 = 2.66\angle 57.86^0$.
The equivalent time domain signal would be:
$v_R(t) = 2.66\cos(2000\pi t + 57.86^0)$.

The three voltage phasors are:
$V_S = 5\angle 0^0$
$V_C = 4.235\angle -27.40^0$
$V_R = 2.66\angle 57.86^0$

Putting them in a phasor diagram:



When we measure the RMS values of the voltages across the source, resistor and capacitor, we get the magnitudes only. The KVL should not then hold with the magnitudes alone as it does not contain the phase information. AC signals are time-dependent, and KVL (or KCL) hold at each time instant. This is also equivalent to saying that KVL (or KCL) also holds with the phasors.