Chapter 9
Line Integrals

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- <u>Vector Fields</u>
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  - Gradient Fields
  - Conservative Fields
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Overview

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  - Line Integrals of Scalar Functions
  - Evaluation of Line Integral
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  - Line Integrals of Vector Fields
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  - Line Integrals in Component Form
  - Fundamental Theorem for Line Integrals
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Overview

- <u>Green's Theorem</u>

Line Integrals

$$\int_a^b f(x)\,dx$$

Integration of single variable scalar function $f(x)$ over the straight line segment $[a, b]$.

Generalisation

Integration of scalar function $f(x, y)$ or $f(x, y, z)$ over a curve C

Line integrals of scalar functions.

Integration of vector function $F(x, y)$ or $F(x, y, z)$ over a curve C

Line integrals of vector fields

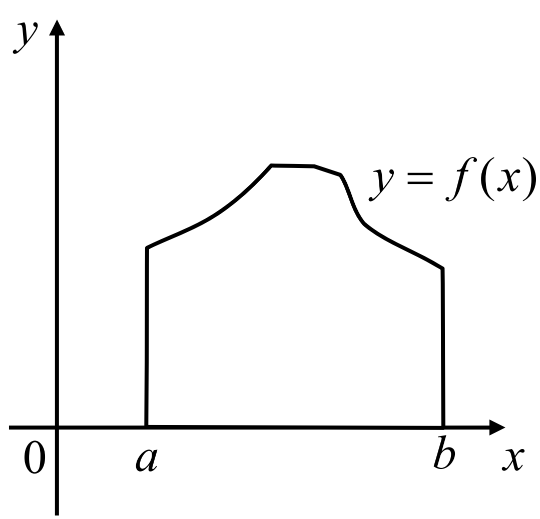
Line Integrals of Scalar Functions

$$\int_a^b f(x)\,dx$$

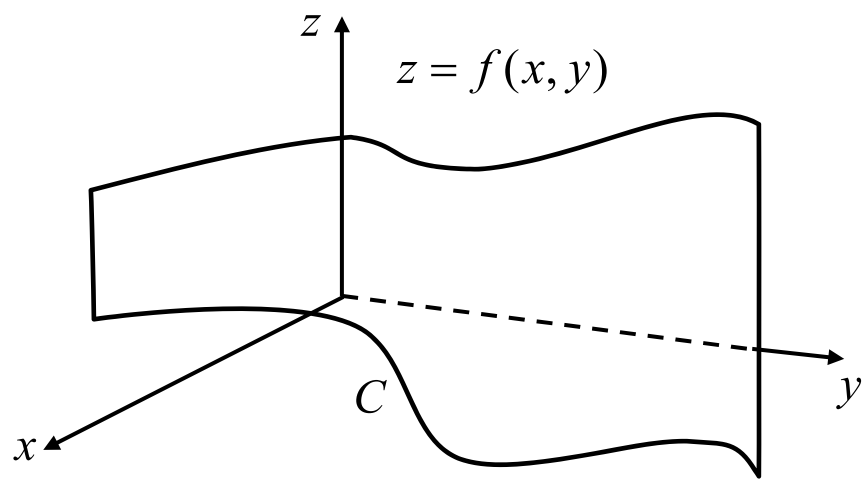
$$\int_C f(x,y)\,ds$$

curve in *xy*-plane

area under the graph of $f(x)$ above the line segment $[a, b]$



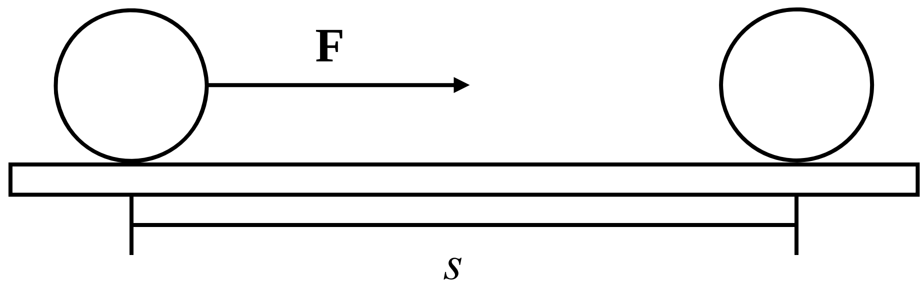
area of lateral vertical surface under the graph of $f(x, y)$ above the curve C



a fence with a curve base and variable height

Work Done

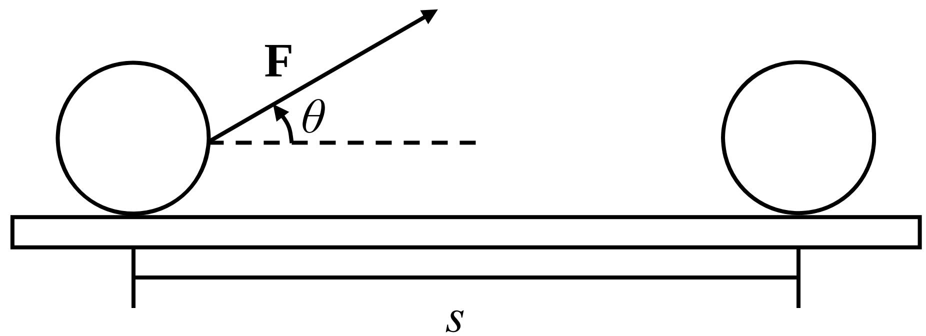
Work Done I Scenario 1



(i) Let **F** be a constant force acting on a particle in the displacement direction as shown. Suppose the distance moved by the particle is s. The work done is given by

$$W = \|\mathbf{F}\| \times s.$$

Work Done I Scenario 2

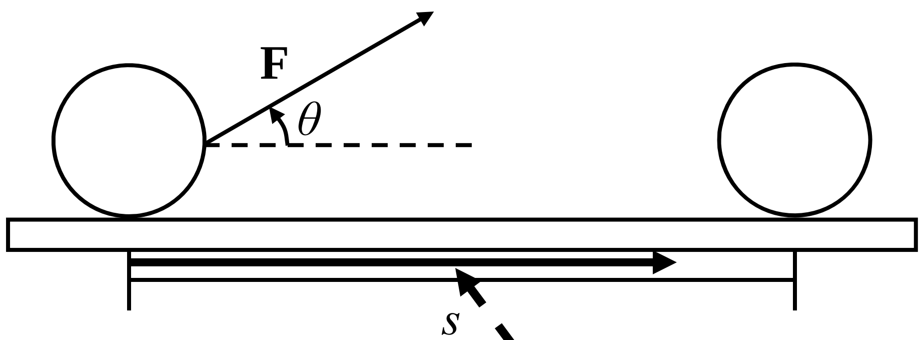


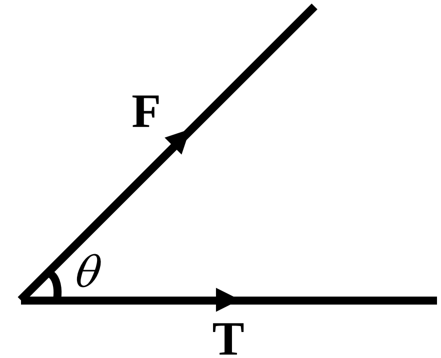
(ii) Let $\mathbf{F}$ be a constant force acting on a particle in the direction which form an angle θ against the displacement direction as shown. Suppose the distance moved by the particle is s.

The work done is given by

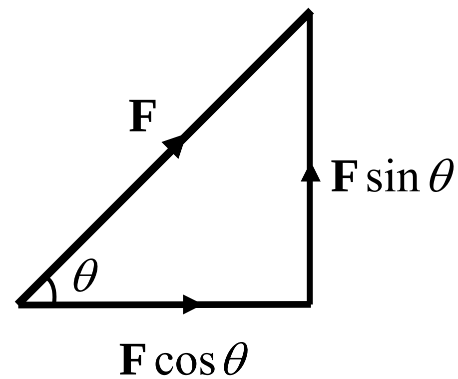
$$W = \|\mathbf{F}\| \cos\theta \times s = (\mathbf{F}\cdot\mathbf{T}) \times s = \mathbf{F}\cdot s\mathbf{T}$$

where $\mathbf{T}$ is the unit vector in the displacement direction.





Let $\mathbf{T}$ be the unit vector in the displacement direction.



$$\boxed{\begin{aligned}\mathbf{F}\cdot\mathbf{T} &= \|\mathbf{F}\|\,\|\mathbf{T}\|\cos\theta \\ &= \|\mathbf{F}\|\cos\theta\end{aligned}}$$

$$\boxed{\begin{aligned}W &= \|\mathbf{F}\|\cos\theta\times s \\ &= (\mathbf{F}\cdot\mathbf{T})\times s \\ &= \mathbf{F}\cdot s\mathbf{T}\end{aligned}}$$

displacement vector

$$\boxed{W = \mathbf{F}\cdot\text{displacement vector}}$$

Note : $s\mathbf{T}$ = displacement vector and

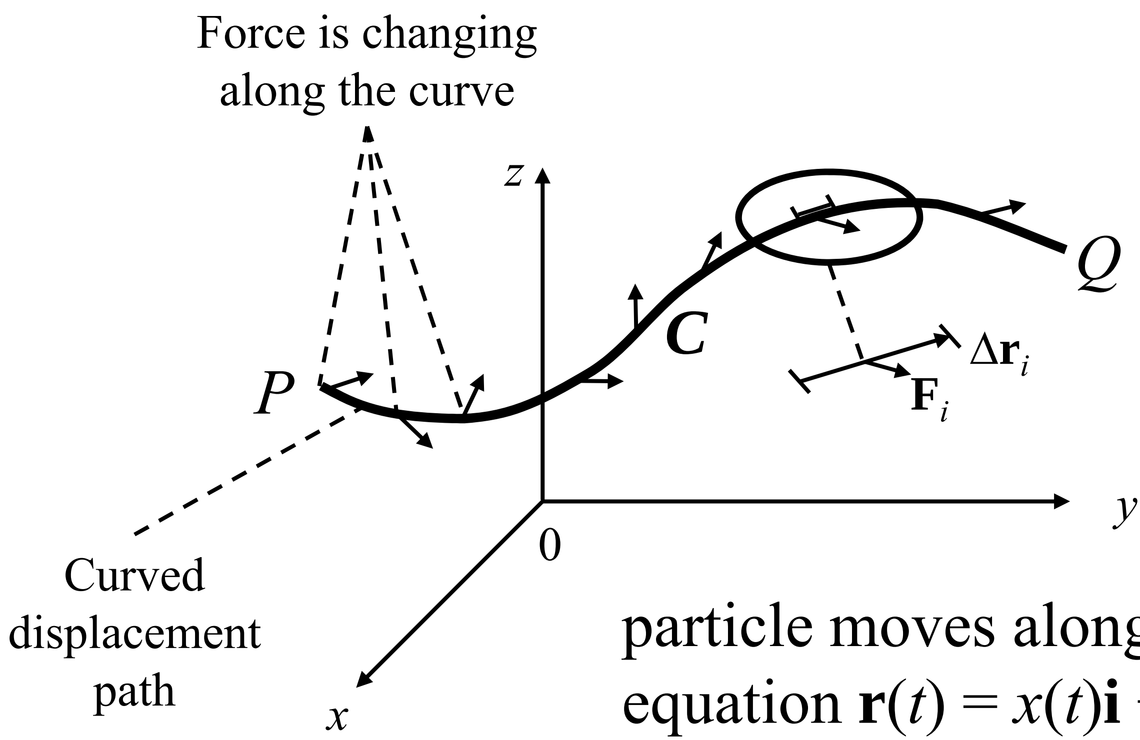
s = distance moved.

Work Done II Scenario 3 (General Case)

Let $\mathbf{F}(x, y, z)$ be a variable force acting on a particle which moves along the curve C with vector equation

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}.$$

Suppose the particle moves from point P to point Q. What is the work done?

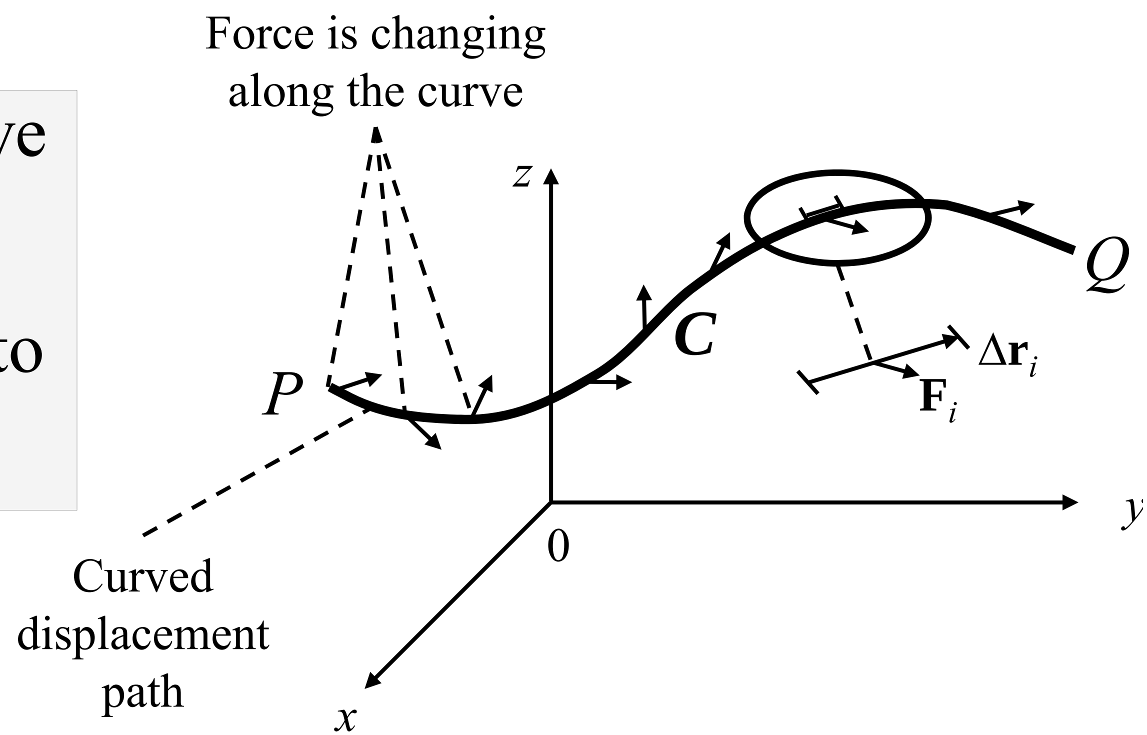


$\mathbf{F}$ a variable force

particle moves along curve C with vector equation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$

Work Done II

To find workdone to move the particle from P to Q, we divide the curve C into n segments.



If a segment i is small enough, we may assume :

(i) a straight line segment and

(ii) constant force

Work Done II

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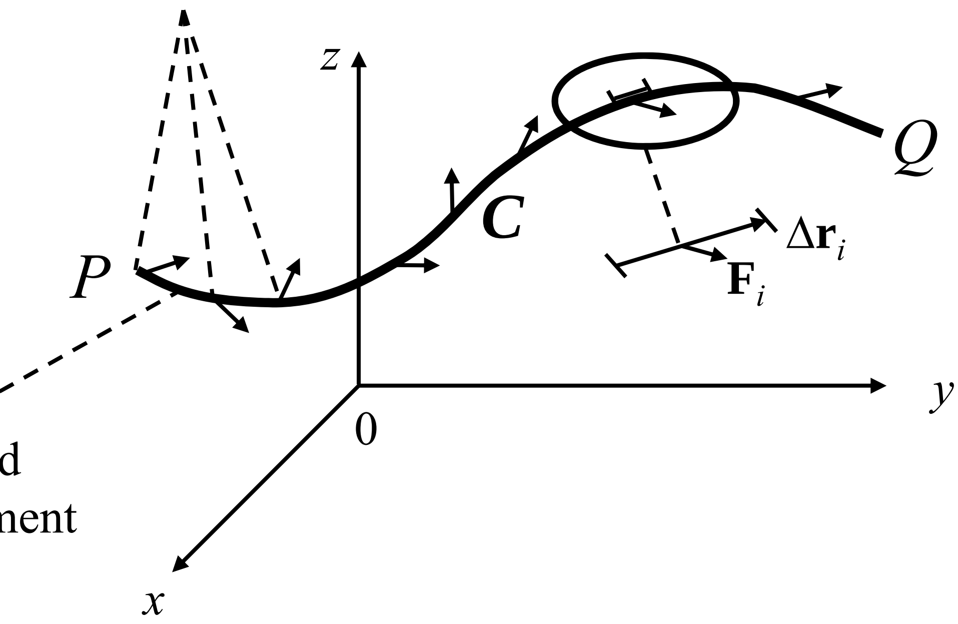


$$\boxed{W = \mathbf{F} \cdot \text{displacement vector}}$$

$$W_i \approx \mathbf{F}_i \cdot \Delta\mathbf{r}_i$$

constant force along this segment = $\mathbf{F}_i$

displacement vector along this segment = $\Delta\mathbf{r}_i$

Total workdone : $W_{\text{total}} \approx \sum_{i=1}^{n} \mathbf{F}_i \cdot \Delta\mathbf{r}_i$

$n \to \infty \qquad W_{\text{total}} \approx \boxed{\int_C \mathbf{F} \cdot d\mathbf{r}}$

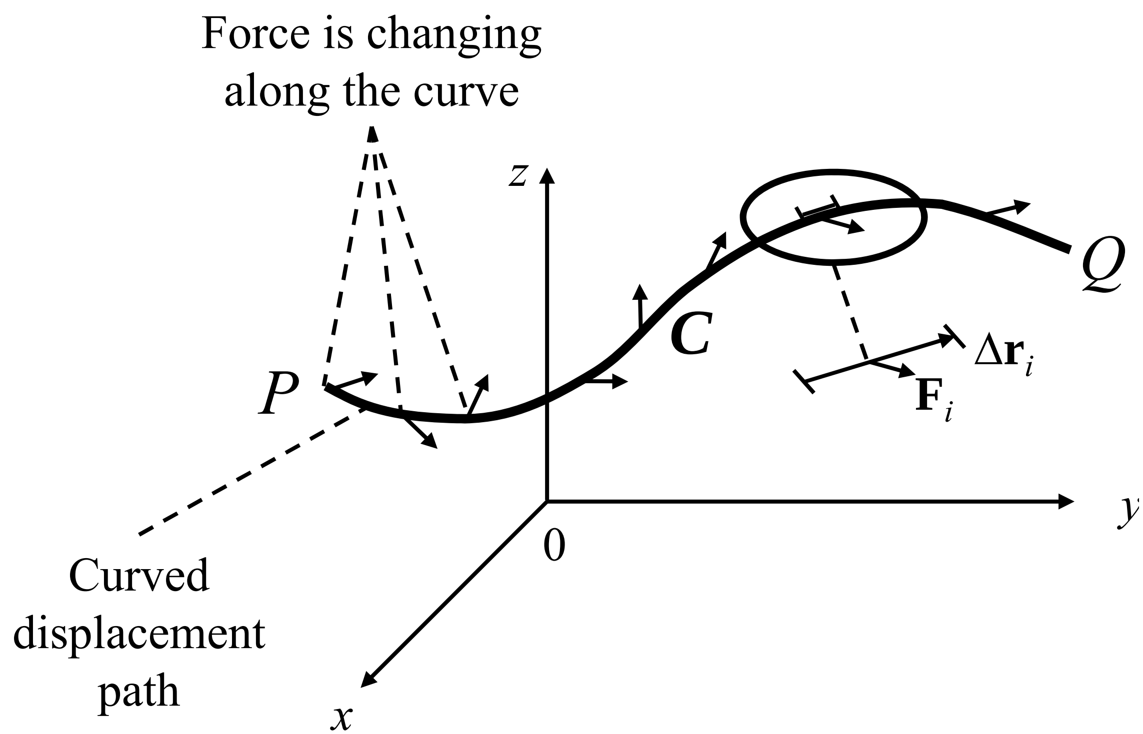
vector field

line integral of vector field

Work Done II

The work done for a segment is approximately

$$W_i \approx \mathbf{F}_i \cdot \Delta \mathbf{r}_i$$



The total work done is approximately

$$W_{\text{total}} \approx \sum_{i=1}^{n} \mathbf{F}_i \cdot \Delta \mathbf{r}_i.$$

As $n \to \infty$, we write this as

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

which gives the actual total work done.

The vector function $\mathbf{F}$ is called in general a ***vector field*** and the integral is called the ***line integral of*** $\mathbf{F}$ ***along the curve*** C.

Vector Fields

Scalar Function

A point (x, y) in xy-plane $\longrightarrow$ $f(x, y)$ $\longrightarrow$ a real number

A point (x, y, z) in xyz-space $\longrightarrow$ $f(x, y, z)$ $\longrightarrow$ a real number

---

Vector Function / Vector Field

A point (x, y) in xy-plane $\longrightarrow$ $\mathbf{F}(x, y)$ $\longrightarrow$ a *vector* in xy-plane

A point (x, y, z) in xyz-space $\longrightarrow$ $\mathbf{F}(x, y, z)$ $\longrightarrow$ a *vector* in xyz-space

Two Variables (Vector fields)

Let R be a region in xy-plane.

A ***vector field*** on R is a vector function $\mathbf{F}$ that assigns (maps) each point (x, y) in R to a two-dimensional vector $\mathbf{F}(x, y)$.

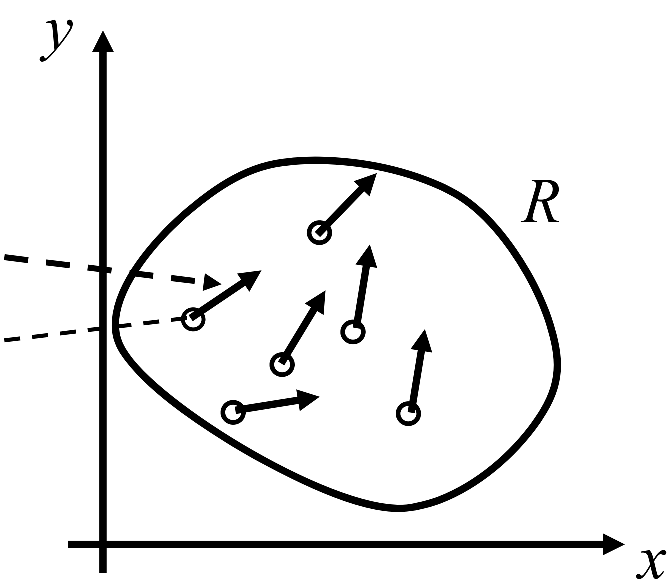
Notation

$$\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$$

$$\Uparrow$$

$$(x, y)$$

Graph



A representative set of outcome vectors

Two Variables (Vector fields)

We may write $\mathbf{F}(x, y)$ in terms of its component functions. That is

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$$

or simply

$$\mathbf{F} = P\mathbf{i} + Q\mathbf{j}.$$

Three Variables (Vector fields)

Let D be a solid region in xyz-space. A ***vector field*** on D is a vector function $\mathbf{F}$ that assigns to each point (x, y, z) in D a three-dimensional vector $\mathbf{F}(x, y, z)$. That is,

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}.$$

Let $\mathbf{F}(x, y) = (-y)\mathbf{i} + x\mathbf{j}$ be a vector field in xy-plane.

$$\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$$

$$\mathbf{F}(1,0) = 0\mathbf{i} + 1\mathbf{j}$$

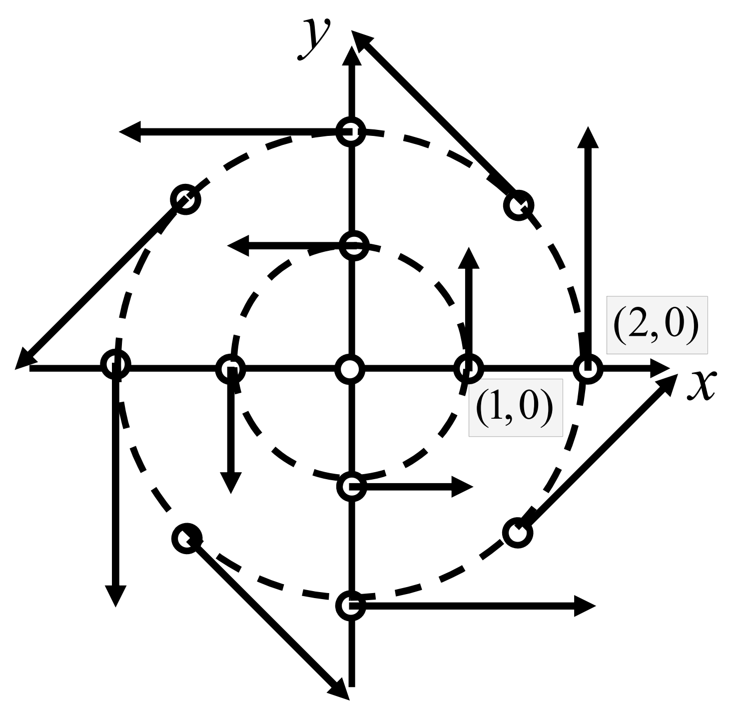
$$\mathbf{F}(2,0) = 0\mathbf{i} + 2\mathbf{j}$$

$$\mathbf{F}(0,1) = -1\mathbf{i} + 0\mathbf{j}$$

$$\mathbf{F}(0,2) = -2\mathbf{i} + 0\mathbf{j}$$

$$\mathbf{F}(\sqrt{2}, \sqrt{2}) = -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}$$

$$\mathbf{F}(0,0) = 0\mathbf{i} + 0\mathbf{j}$$



The diagram shows the vector field $\mathbf{F}$.

A vector field in xy-plane is defined by

$$\mathbf{F}(x, y) = (-y)\mathbf{i} + x\mathbf{j}.$$

Show that $\mathbf{F}(x, y)$ is always perpendicular to the position vector of the point (x, y).

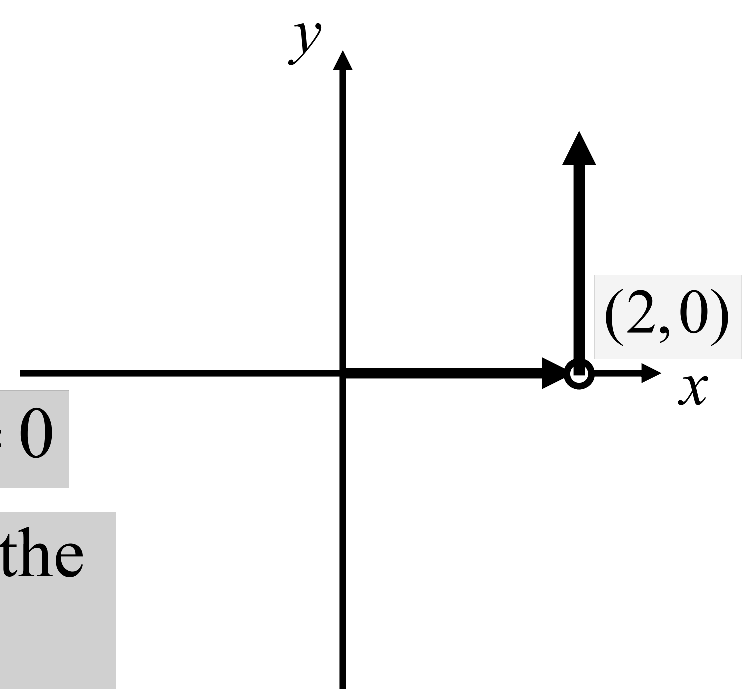
At a point (x, y):

position vector $= x\mathbf{i} + y\mathbf{j}$

$\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$

$(x\mathbf{i} + y\mathbf{j}) \cdot \mathbf{F}(x, y) = (x\mathbf{i} + y\mathbf{j}) \cdot (-y\mathbf{i} + x\mathbf{j}) = 0$

Thus, $\mathbf{F}(x, y)$ is always perpendicular to the position vector of the point (x, y)



At the point $(2, 0)$:

position vector $= 2\mathbf{i} + 0\mathbf{j}$

$\mathbf{F}(2, 0) = 0\mathbf{i} + 2\mathbf{j}$

A vector field in xy-plane is defined by

$$\mathbf{F}(x, y) = (-y)\mathbf{i} + x\mathbf{j}.$$

Show that $\mathbf{F}(x, y)$ is always perpendicular to the position vector of the point (x, y).

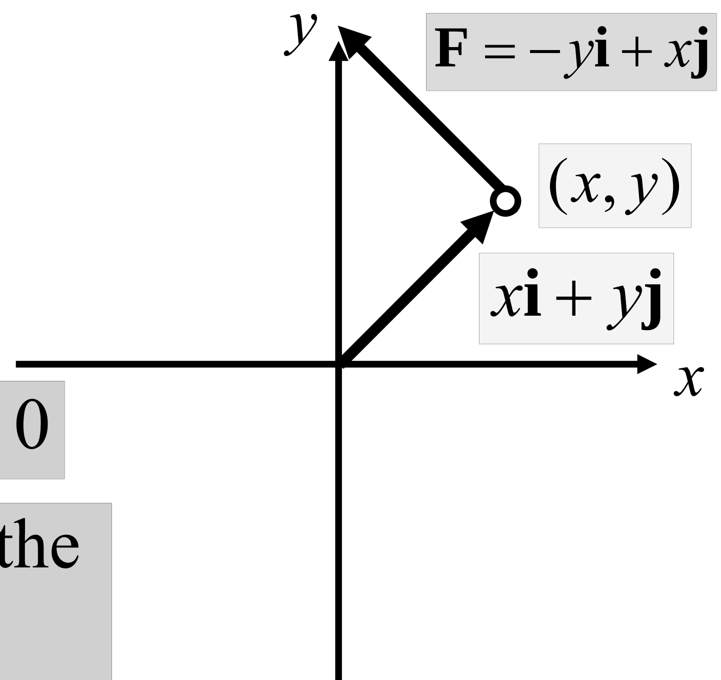
At a point (x, y):

position vector $= x\mathbf{i} + y\mathbf{j}$

$$\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$$

$$(x\mathbf{i} + y\mathbf{j}) \cdot \mathbf{F}(x, y) = (x\mathbf{i} + y\mathbf{j}) \cdot (-y\mathbf{i} + x\mathbf{j}) = 0$$

Thus, $\mathbf{F}(x, y)$ is always perpendicular to the position vector of the point (x, y)



At the point $(2,0)$:

position vector $= 2\mathbf{i} + 0\mathbf{j}$

$$\mathbf{F}(2,0) = 0\mathbf{i} + 2\mathbf{j}$$

Gradient Fields

Gradient Fields

If $f(x, y)$ is a function of two variables, then

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$$

is a vector field in the xy-plane and it is called the ***gradient*** (***field***) of f.

Similarly, if $f(x, y, z)$ is a function of three variables, then

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}$$

is a vector field in the xyz-space and it is called the ***gradient*** (***field***) of f.

Gradient Fields - Example

The gradient field of $f(x, y) = xy^2 + x^3$ is

$$\begin{aligned}\nabla f(x, y) &= f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} \\ &= (y^2 + 3x^2)\mathbf{i} + (2xy)\mathbf{j}\end{aligned}$$

Note (Directional derivatives) (Chapter 7)

Another expression for directional derivatives:

$$\boxed{\begin{aligned} D_{\mathbf{u}} f(a,b) &= f_x(a,b)u_1 + f_y(a,b)u_2 \\ &= \nabla f(a,b) \cdot (u_1\mathbf{i} + u_2\mathbf{j}) \\ &= \nabla f(a,b) \cdot \mathbf{u} \end{aligned}}$$

Conservative Fields

A vector field $\mathbf{F}$ is called a ***conservative*** vector field if it is the gradient of some (scalar) function. In other words, there is a function f such that $\mathbf{F} = \nabla f$. In this case, f is called a ***potential*** function for $\mathbf{F}$.

$$\mathbf{F} = \nabla f$$

conservative

potential
function for **F**

Conservative Fields - Example

The vector field $\mathbf{F}(x, y) = (y^2 + 3x^2)\mathbf{i} + (2xy)\mathbf{j}$ is conservative since it has a potential function

$$f(x, y) = xy^2 + x^3.$$

Note that $\nabla f = \mathbf{F}.$

Questions on : $\nabla f = \mathbf{F}$

For a given vector field $\mathbf{F}$:

1. How to check which $\mathbf{F}$ we can find a scalar function f such that $\nabla f = \mathbf{F}$???

2. How to find f such that $\nabla f = \mathbf{F}$???

Let $\mathbf{F}(x,y)=(3+2xy)\mathbf{i}+(x^2-3y^2)\mathbf{j}$. Find a potential function f for $\mathbf{F}$.

Want to find f such that $\nabla f=\mathbf{F}$

$\nabla f(x,y)=f_x(x,y)\mathbf{i}+f_y(x,y)\mathbf{j}$

Thus, $f_x(x,y)\mathbf{i}+f_y(x,y)\mathbf{j} = (3+2xy)\mathbf{i}+(x^2-3y^2)\mathbf{j}$

Comparing $\mathbf{i}$ component : $f_x(x,y)=3+2xy$

Integrate with respect to x, treat y as constant

$f(x,y)=3x+x^2y+g(y)$

partial differentiate with respect to y, treat x as constant

$f_y(x,y)=x^2+g'(y)$

Hence, $x^2+g'(y)=x^2-3y^2$

Thus, $g'(y)=-3y^2$

$g(y)=-y^3+K$, K is a constant

Final answer : $f(x,y)=3x+x^2y+g(y)=3x+x^2y-y^3+K$

Conservative Fields - Example

The gravitational field given by

$$\mathbf{G} = \frac{-m_1 m_2 K x}{(x^2+y^2+z^2)^{3/2}}\mathbf{i} + \frac{-m_1 m_2 K y}{(x^2+y^2+z^2)^{3/2}}\mathbf{j} + \frac{-m_1 m_2 K z}{(x^2+y^2+z^2)^{3/2}}\mathbf{k}$$

is conservative because it is the gradient of the gravitational potential function

$$g(x,y,z) = \frac{m_1 m_2 K}{\sqrt{(x^2+y^2+z^2)}},$$

where K is the gravitational constant, m_1 and m_2 are the masses of two objects.

Check that $\nabla g = g_x(x,y,z)\mathbf{i} + g_y(x,y,z)\mathbf{j} + g_z(x,y,z)\mathbf{k} = \mathbf{G}$

Question :

How to check for conservative fields ?????

1. How to check which $\mathbf{F}$ we can find a scalar function f such that $\nabla f = \mathbf{F}$???

Criteria of Conservative Fields

(a) Let $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ be a vector field on the xy-plane. If

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},$$

then $\mathbf{F}$ is conservative.

(b) Let $\mathbf{F}(x, y) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x,y,z)\mathbf{k}$ be a vector field on the xyz-plane. If

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y},$$

then $\mathbf{F}$ is conservative.

The converses of (a) and (b) also hold.

(a) Let $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ be a vector field on the xy-plane. If

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},$$

then $\mathbf{F}$ is conservative.

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$$

$$\nabla f = f_x \, \mathbf{i} + f_y \, \mathbf{j}$$

Suppose $\nabla f = \mathbf{F}$.

Then $f_x = P$ and $f_y = Q$.

Thus, $f_{xy} = \dfrac{\partial P}{\partial y}$ and $f_{yx} = \dfrac{\partial Q}{\partial x}$

Recall that $f_{xy} = f_{yx}$ (Result from Chapter 7)

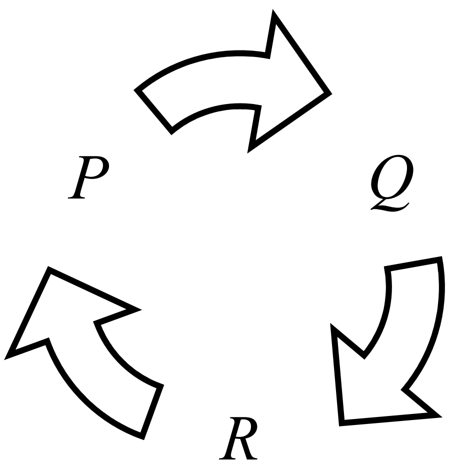
Hence, $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

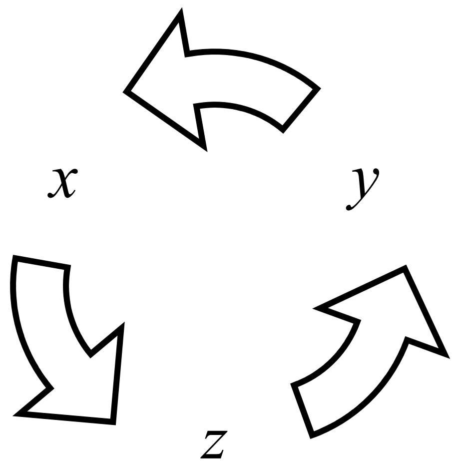
$\mathbf{F}$ is conservative

(b) Let $\mathbf{F}(x, y) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x,y,z)\mathbf{k}$ be a vector field on the xyz-plane. If

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y},$$

then $\mathbf{F}$ is conservative.





(b) Let $\mathbf{F}(x, y) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x,y,z)\mathbf{k}$ be a vector field on the xyz-plane. If

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y},$$

then $\mathbf{F}$ is conservative.

To check $\mathbf{F}$ is ***conservative***, we must check

(1) $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$, (2) $\dfrac{\partial P}{\partial z} = \dfrac{\partial R}{\partial x}$, (3) $\dfrac{\partial Q}{\partial z} = \dfrac{\partial R}{\partial y}$ ***ALL*** hold !!

To check $\mathbf{F}$ is NOT conservative, we just need to show,

either (1) $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$, (2) $\dfrac{\partial P}{\partial z} = \dfrac{\partial R}{\partial x}$ or (3) $\dfrac{\partial Q}{\partial z} = \dfrac{\partial R}{\partial y}$

is not true (does not hold).

Example

Consider the vector field

$$\mathbf{F}(x, y) = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}.$$

$$P = (3 + 2xy) \text{ and } Q = x^2 - 3y^2$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x^2 - 3y^2) = 2x$$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(3 + 2xy) = 2x$$

Thus, $\mathbf{F}$ is conservative.

Example

Show that $\mathbf{F}(x, y, z) = xz\mathbf{i} + xyz\mathbf{j} - y^2\mathbf{k}$ is not conservative.

$$P(x, y, z) = xz, \quad Q(x, y, z) = xyz \quad \text{and} \quad R(x, y, z) = -y^2.$$

$$\frac{\partial Q}{\partial x} = \frac{\partial (xyz)}{\partial x} = yz$$

$$\frac{\partial P}{\partial y} = \frac{\partial (xz)}{\partial y} = 0$$

Thus, $\dfrac{\partial P}{\partial y} \neq \dfrac{\partial Q}{\partial x}$ and hence $\mathbf{F}$ is not conservative.

To check $\mathbf{F}$ is NOT conservative, we just need to show,

either (1) $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$, (2) $\dfrac{\partial P}{\partial z} = \dfrac{\partial R}{\partial x}$ or (3) $\dfrac{\partial Q}{\partial z} = \dfrac{\partial R}{\partial y}$

is not true (does not hold).

Line Integrals

Line Integrals

$\int_a^b f(x)\,dx$

Integration of single variable scalar function $f(x)$ over the straight line segment $[a,b]$.

Generalisation

Integration of scalar function $f(x,y)$ or $f(x,y,z)$ over a curve C

Line integrals of scalar functions.

Integration of vector function $F(x,y)$ or $F(x,y,z)$ over a curve C

Line integrals of vector fields

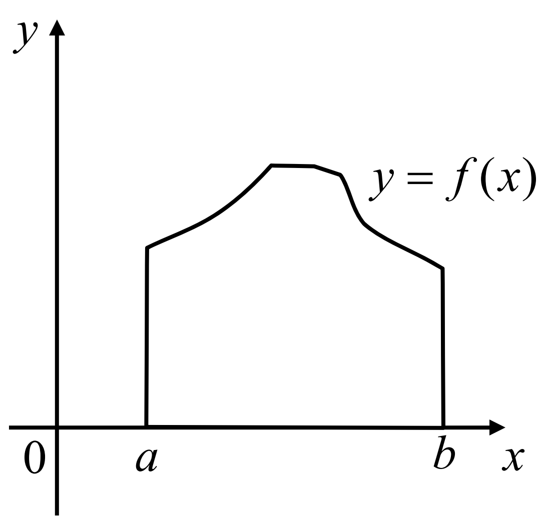
Line Integrals of Scalar Functions

$$\int_a^b f(x)\,dx$$

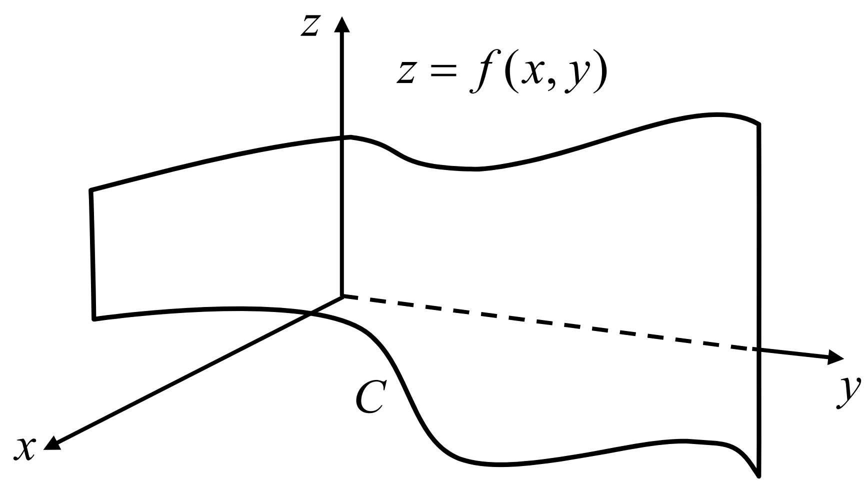
$$\int_C f(x, y)\,ds$$

curve in *xy*-plane

area under the graph of $f(x)$ above the line segment $[a, b]$

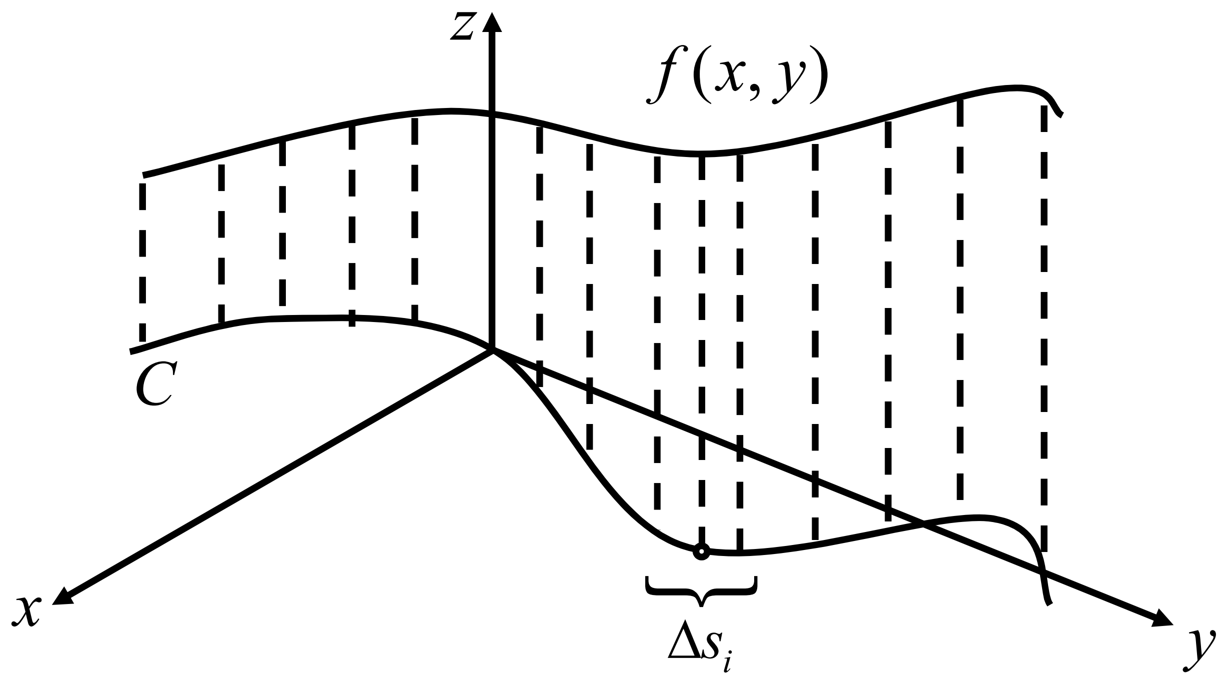


area of lateral vertical surface under the graph of $f(x, y)$ above the curve C



a fence with a curve base and variable height

Line Integrals of Scalar Functions



$$\text{The surface area} = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

Line Integrals of Scalar Functions

The line integral of the (scalar) function f along C is given by

$$\int_C f(x,y)\,ds = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

How do we compute $\int_C f(x,y)\,ds$?

Arc Length of a Space Curve (Recall from Chapter 6)

$$S = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\ dt$$
$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

A more compact formula of both arc length formulas is

$$S = \int_a^b \|\mathbf{r}'(t)\|\ dt$$

$$\text{Recall}:\ \mathbf{v} = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$$

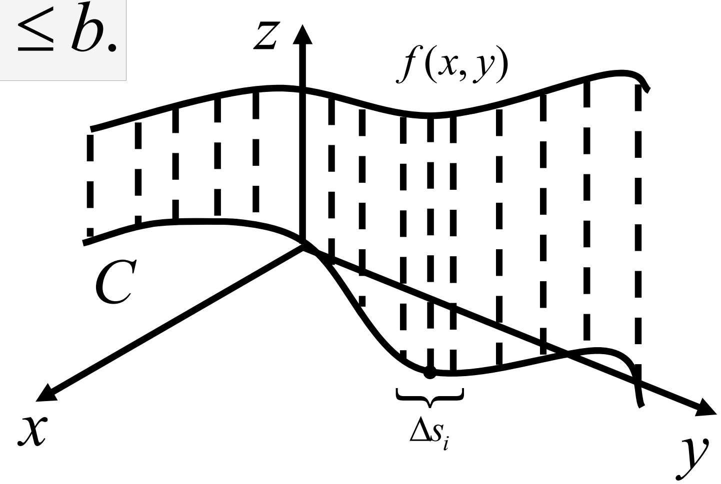
$$\|\mathbf{v}\| = \sqrt{x_0^2 + y_0^2 + z_0^2}$$

$$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$$

$$\|\mathbf{r}'(t)\| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$$

How do we compute $\int_C f(x,y)\,ds$?

Let C: $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, where $a \le t \le b$.



The arc length of C is given by

$$s = \int_a^b \|\mathbf{r}'(t)\|\ dt$$

$$= \int_a^b \sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2}\ dt$$

Thus, $s(t) = \int_a^t \|\mathbf{r}'(u)\|\ du$ and $\dfrac{ds}{dt} = \|\mathbf{r}'(t)\|$

Note: $ds = \|\mathbf{r}'(t)\|\ dt$

Hence, $$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\,\|\mathbf{r}'(t)\|\ dt$$

$$= \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

Steps to find line integral for scalar function, $\int_C f(x,y)\,ds$:

(1) Find the vector function for C.

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j},\quad a \le t \le b.$$

(2) Substitution: $f(x(t), y(t))$.

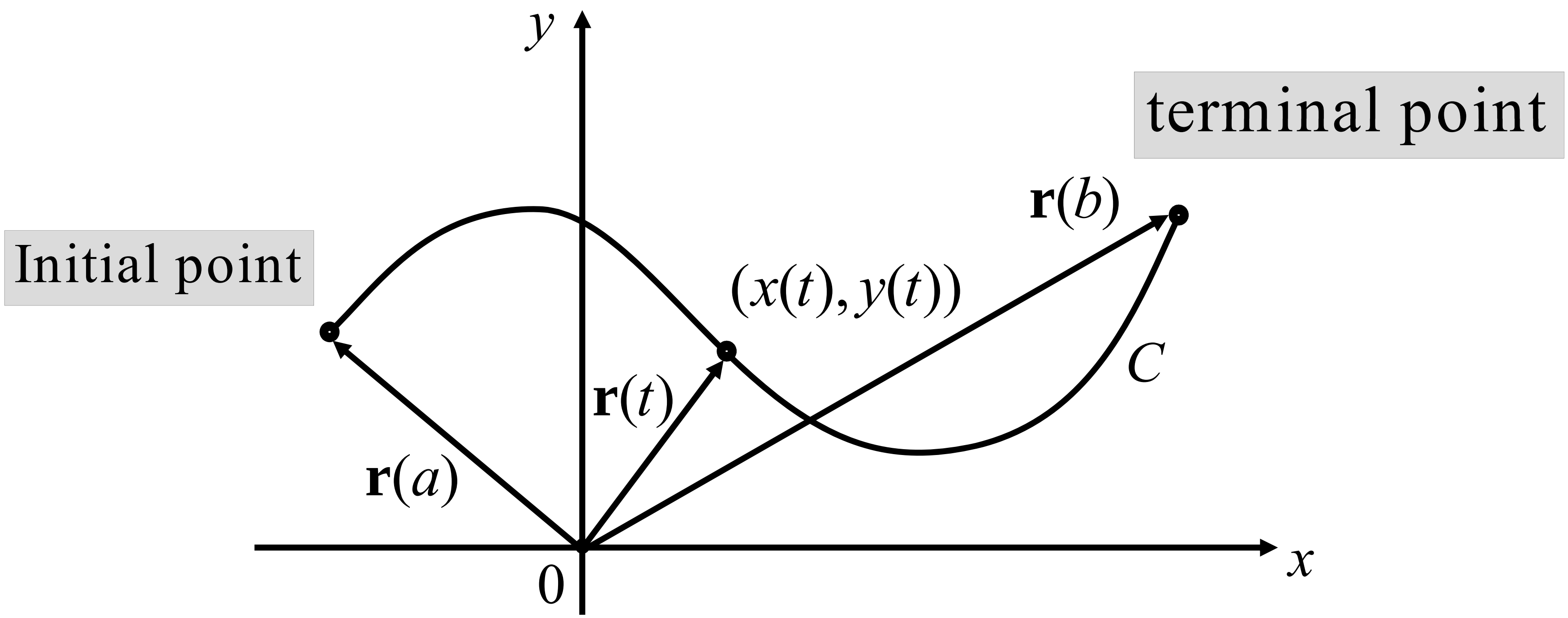
(3) Find derivative of $\mathbf{r}(t)$:

$$\mathbf{r}'(t) = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}$$

(4) Compute $\int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$

Vector function for C

Let $C: \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, where $a \le t \le b$.

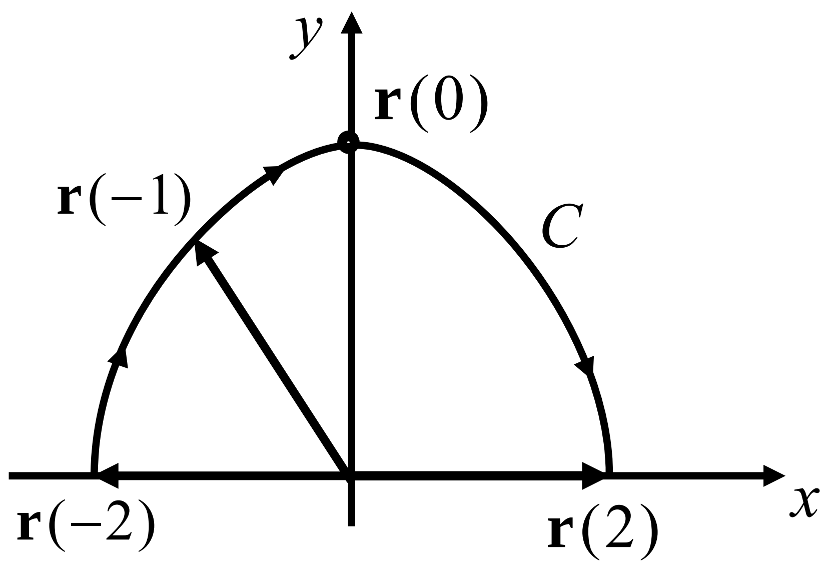


Take note of the direction as t increases from a to b.

Example

Vector function for C

Let $C: \mathbf{r}(t) = t\mathbf{i} + (4 - t^2)\mathbf{j}$, where $-2 \le t \le 2$.



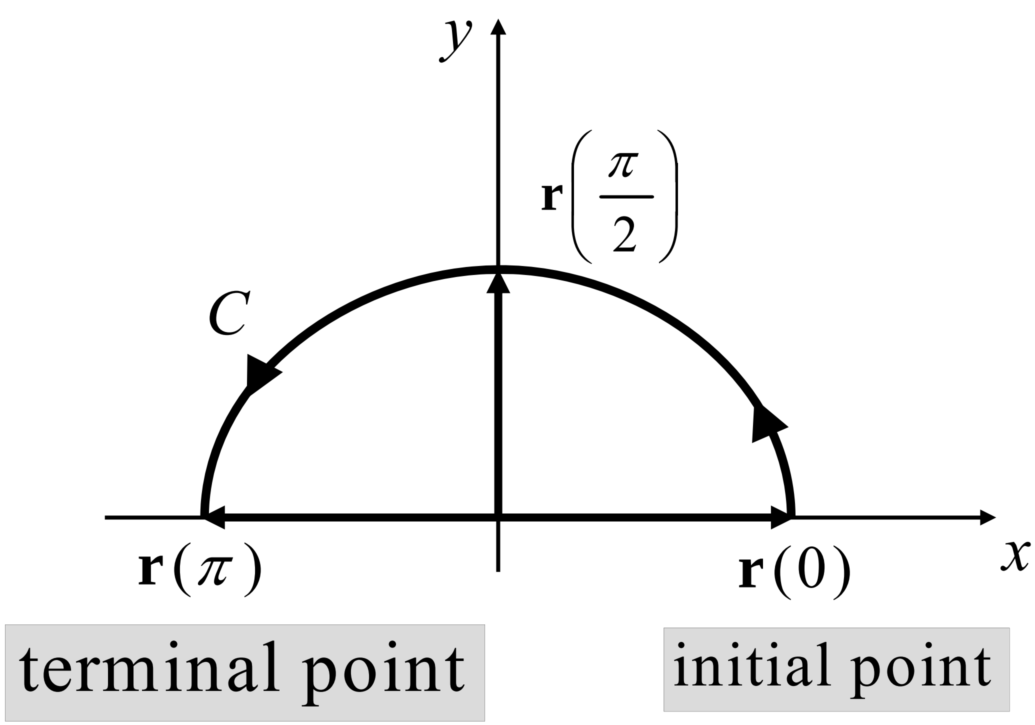
initial point

terminal point

Take note of the direction as t increases from -2 to 2.

Example (Semi-circle)

Let $C: \ \mathbf{r}(t) = \cos t \ \mathbf{i} + \sin t \ \mathbf{j}$, where $0 \le t \le \pi$.



Take note of the direction as t increases from -2 to 2.

Example (circle)

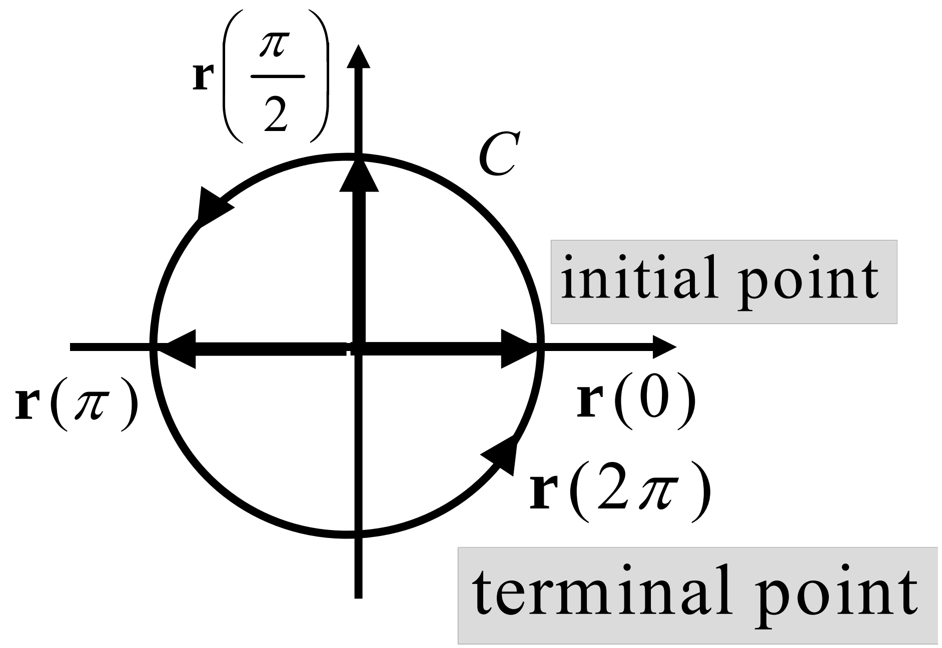
Let $C: \ \mathbf{r}(t) = \cos t \ \mathbf{i} + \sin t \ \mathbf{j}, \ \text{ where } \ 0 \le t \le 2\pi.$

$$\mathbf{r}(0) = \cos 0 \ \mathbf{i} + \sin 0 \ \mathbf{j} = \mathbf{i}$$

$$\mathbf{r}(\frac{\pi}{2}) = \cos\frac{\pi}{2} \ \mathbf{i} + \sin\frac{\pi}{2} \ \mathbf{j} = \mathbf{j}$$

$$\mathbf{r}(\pi) = \cos\pi \ \mathbf{i} + \sin\pi \ \mathbf{j} = -\mathbf{i}$$

$$\mathbf{r}(2\pi) = \cos 2\pi \ \mathbf{i} + \sin 2\pi \ \mathbf{j} = \mathbf{i}$$



Take note of the direction as t increases from 0 to 2π.

Evaluate

$$\int_C (2y + x^2 y)\, ds,$$

where C is the upper half of the unit circle centered at the origin.

Let C : $\mathbf{r}(t) = \cos t\ \mathbf{i} + \sin t\ \mathbf{j}$, where $0 \le t \le \pi$.

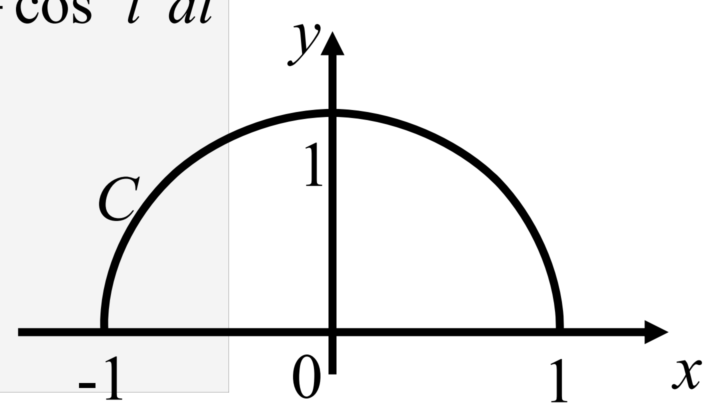
$x(t) = \cos t$ and $y(t) = \sin t$

$\mathbf{r}'(t) = -\sin t\ \mathbf{i} + \cos t\ \mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{\sin^2 t + \cos^2 t}$

$$\int_C f(x,y)\, ds = \int_a^b f(x(t), y(t))\, \|\mathbf{r}'(t)\|\ dt$$
$$= \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$\int_C (2y + x^2 y)\, ds = \int_0^\pi (2\sin t + \cos^2 t \sin t)\sqrt{\sin^2 t + \cos^2 t}\ dt$$
$$= \int_0^\pi (2\sin t + \cos^2 t \sin t)\ dt$$
$$= \left[-2\cos t - \frac{1}{3}\cos^3 t \right]_0^\pi = \frac{14}{3}$$



Evaluation of Line Integral

For line integral of a function $f(x, y, z)$ along a space curve

$$C: \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k},$$

we have the similar definitions:

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Evaluate $\int_C xy \sin z \, ds$ where C is the circular helix

$$\mathbf{r}(t) = \cos t \, \mathbf{i} + \sin t \, \mathbf{j} + t\mathbf{k}, \ t \in \left[0, \frac{\pi}{2}\right].$$

$$x(t) = \cos t, \quad y(t) = \sin t \ \text{ and } \ z(t) = t$$

$$\mathbf{r}'(t) = \sin t \, \mathbf{i} + \cos t \, \mathbf{j} + \mathbf{k}$$

$$\int_C f(x, y, z) \, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt$$

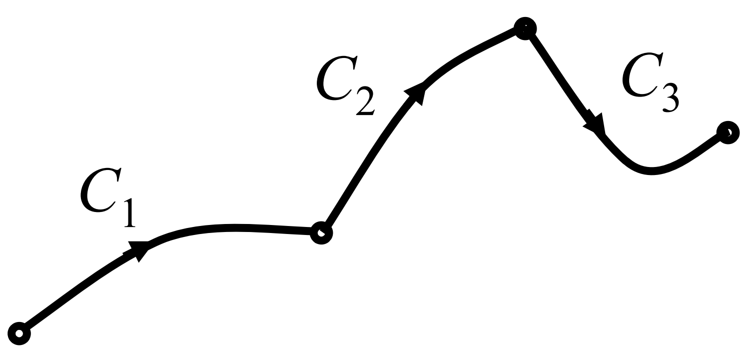
$$\begin{aligned}
\int_C xy \sin z \, ds &= \int_0^{\pi/2} (\cos t)(\sin t)(\sin t) \sqrt{\sin^2 t + \cos^2 t + 1} \, dt \\
&= \sqrt{2} \int_0^{\pi/2} \cos t \sin^2 t \, dt \\
&= \frac{\sqrt{2}}{3} \left[\sin^3 t\right]_0^{\pi/2} = \frac{\sqrt{2}}{3}
\end{aligned}$$

Piecewise Smooth Curves

We denote the union of a finite number of (smooth) curves $C_1, C_2, \cdots, C_n$ by

$$C = C_1 + C_2 + \cdots + C_n.$$

We say C is a ***piecewise - smooth*** curve.



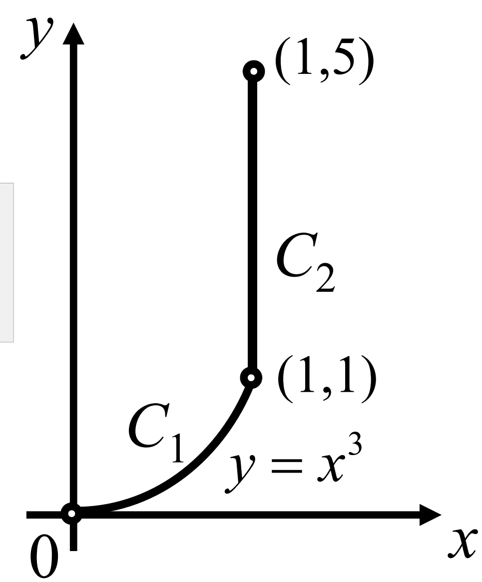
Then the line integral f along C is defined to be

$$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \cdots + \int_{C_n} f(x,y)\,ds.$$

Piecewise Smooth Curves - Example

Evaluate $\int_C 9y\,ds$, where C consists of the arc C_1 of the cubic $y = x^3$ from $(0,0)$ to $(1,1)$ followed by the vertical line segment C_2 from $(1,1)$ to $(1,5)$.

$$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds$$



To find $\int_{C_1} f(x,y)\, ds$

C_1: $y = x^3$ from $(0,0)$ to $(1,1)$ can be parameterized by

$$\begin{cases} x = t \\ y = t^3 \end{cases}, \quad 0 \le t \le 1.$$

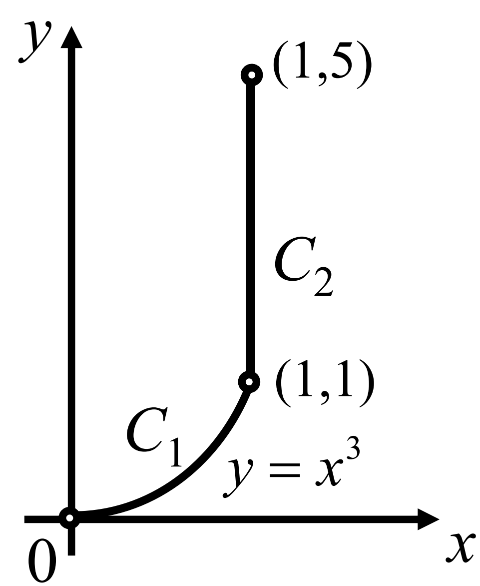
Hence, $\mathbf{r}_1(t) = t\mathbf{i} + t^3\mathbf{j}$ and $\mathbf{r}_1'(t) = \mathbf{i} + 3t^2\mathbf{j}$.

$$\int_a^b f(x(t), y(t))\, \|\mathbf{r}'(t)\|\, dt$$

$$\int_{C_1} 9y\, ds = \int_0^1 9t^3\sqrt{1+9t^4}\, dt$$

$$= \frac{1}{6}\left[(1+9t^4)^{3/2}\right]_0^1$$

$$= \frac{1}{6}(10\sqrt{10} - 1).$$



To find $\int_{C_2} f(x,y)\,ds$

C_2 : $x=1$ from $(1,1)$ to $(1,5)$ can be parameterized by

$$\begin{cases} x=1 \\ y=t \end{cases}, \quad 1 \le t \le 5.$$

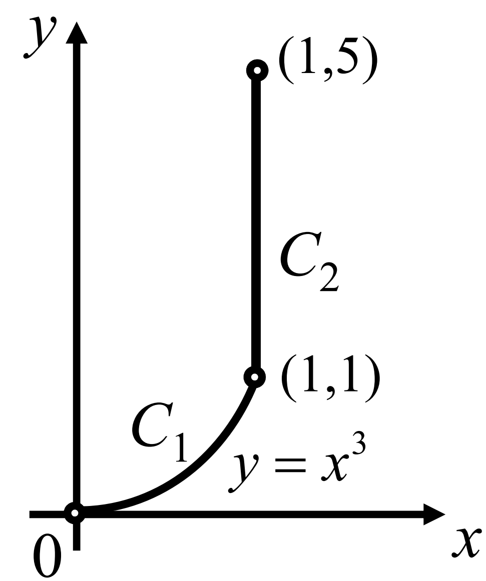
Hence, $\mathbf{r}_2(t)=\mathbf{i}+t\mathbf{j}$ and $\mathbf{r}_2'(t)=\mathbf{j}$.

$$\int_a^b f(x(t),y(t))\,\|\mathbf{r}'(t)\|\,dt$$

$$\int_{C_2} 9y\,ds = \int_1^5 9t\,dt$$

$$= \frac{9}{2}\left[t^2\right]_1^5$$

$$=108.$$



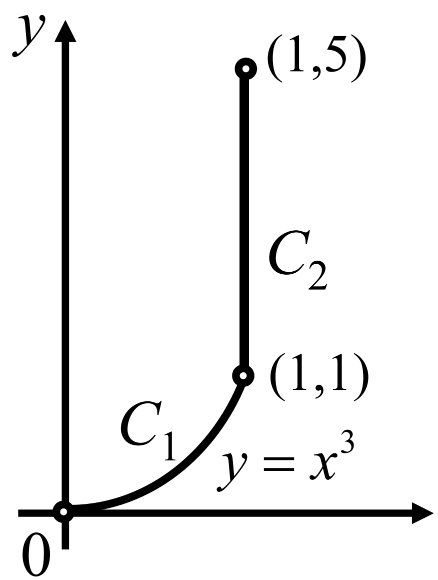
Piecewise Smooth Curves - Example

Evaluate $\int_C 9y\,ds$, where C consists of the arc C_1 of the cubic $y = x^3$ from (0,0) to (1,1) followed by the vertical line segment C_2 from (1,1) to (1,5).

$$\int_{C_1} 9y\,ds = \frac{1}{6}(10\sqrt{10}-1)$$

$$\int_{C_2} 9y\,ds = 108$$

$$\begin{aligned}\int_C 9y\,ds &= \int_{C_1} 9y\,ds + \int_{C_2} 9y\,ds \\ &= \frac{1}{6}(10\sqrt{10}-1)+108 \\ &= \frac{1}{6}(10\sqrt{10}+647).\end{aligned}$$



Line Integrals of Vector Fields

Let $C:\ \mathbf{r}(t),\ \ a \le t \le b$ and $\mathbf{F}:$ vector field on C.

The line integral of $\mathbf{F}$ along C is

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\, dt$$

where $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$,

$\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$ and $\mathbf{F}(\mathbf{r}(t)) = \mathbf{F}(x(t), y(t), z(t))$.

For line integral of a function $f(x,y,z)$ along a space curve

$$C:\ \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k},$$

we have:

$$\int_C f(x,y,z)\, ds = \int_a^b f(x(t), y(t), z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Line Integrals (Vector Fields)

Line Integrals of Vector Fields $\int_C \mathbf{F} \cdot d\mathbf{r}$

1. Find the vector equatin of C:

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \quad a \le t \le b$$

2. Substitution: $\mathbf{F}(x(t), y(t), z(t))$

3. Find derivative of $\mathbf{r}$(t): $\mathbf{r}'(t) = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$

4. Compute $\int_a^b \mathbf{F}(x(t), y(t), z(t)) \cdot \mathbf{r}'(t)\, dt$

integration of single variable function in t

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F}(x, y, z) = x\mathbf{i} + xy\mathbf{j} + xyz\mathbf{k}$$

and C is the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}, \quad t \in [0, 2]$.

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}$$

$$\int_c \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

$$x(t) = t, \quad y(t) = t^2 \quad \text{and} \quad z(t) = t^3$$

$$\begin{aligned}\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) &= (t\mathbf{i} + t \cdot t^2\mathbf{j} + t \cdot t^2 \cdot t^3\mathbf{k}) \cdot (\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}) \\ &= t + 2t^4 + 3t^8\end{aligned}$$

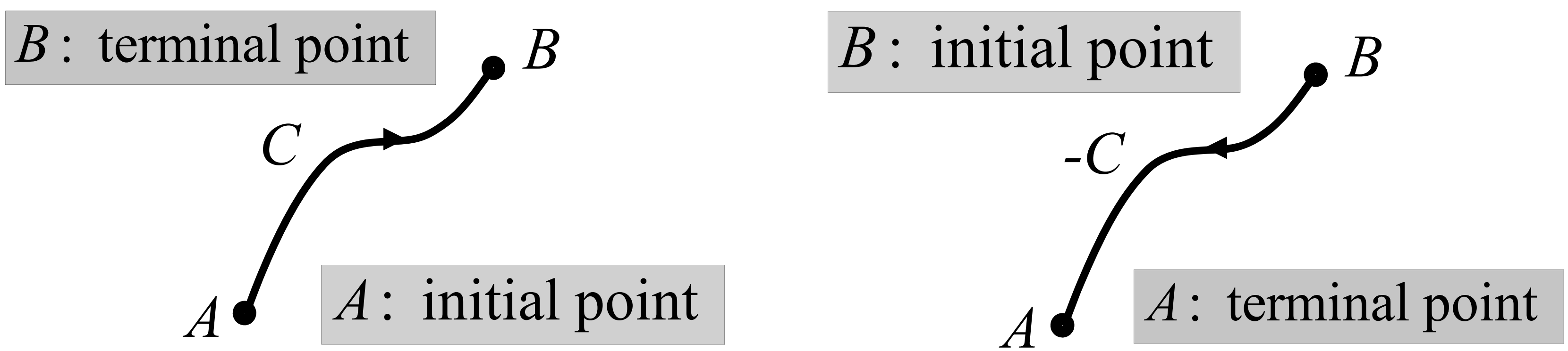
Thus,

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^2 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\ &= \int_0^2 t + 2t^4 + 3t^8\, dt \\ &= \frac{2782}{15}.\end{aligned}$$

Orientation of Curves

The vector equation of a curve C determines an orientation (direction) of C.

The orientation of a curve C is specified by the initial point, terminal point and a "direction".



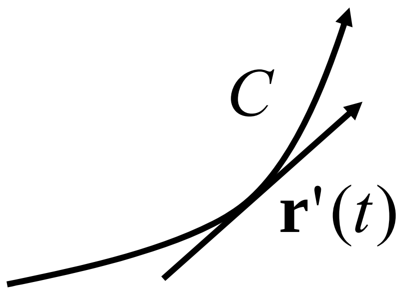
The same curve with the opposite direction of C is denoted by $-C$.

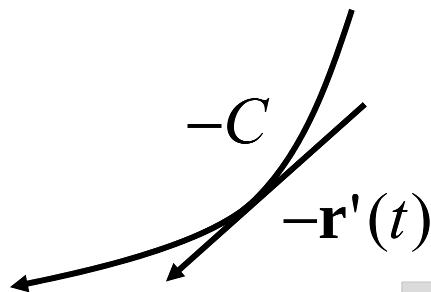
The orientation of C is determined by the vector equation

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, \quad t \in [a,b]$$

where $\mathbf{r}(a)$: initial point and $\mathbf{r}(b)$: terminal point

Orientation of Curve





$\mathbf{r}'(t)$ changes in sign in $-C$.

$$\int_C f(x,y,z)\,ds = \int_a^b f(\mathbf{r}(t))\,\|\mathbf{r}'(t)\|\,dt$$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt$$

Line Integrals of Scalar Functions

$$\int_C f(x,y)\,ds = \int_{-C} f(x,y)\,ds$$

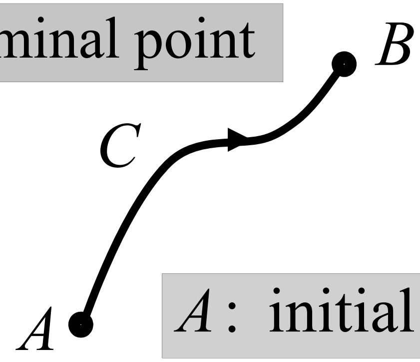
arc length is always positive

Line Integrals of Vector Fields

$$\int_{-C} \mathbf{F}\cdot d\mathbf{r} = -\int_C \mathbf{F}\cdot d\mathbf{r}$$

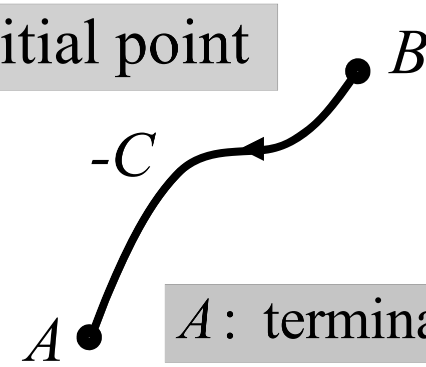
The orientation of a curve C is specified by the initial point, terminal point and a "direction".

$B:$ terminal point



$A:$ initial point

$B:$ initial point



$A:$ terminal point

The same curve with the opposite direction of C is denoted by $-C$.

The orientation of C is determined by the vector equation

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, \quad t \in [a,b]$$

where $\mathbf{r}(a):$ initial point A and $\mathbf{r}(b):$ terminal point B

PAUSE and THINK !!

If we know the vector equation of C, how can we find the vector equation for $-C$???

Line Integrals in Component Form

Suppose $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$.

$$C : \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, \quad t \in [a, b].$$

We may write the line integral as

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\ dx + Q\ dy.$$

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\
&= \int_a^b [P(\mathbf{r}(t))\mathbf{i} + Q(\mathbf{r}(t))\mathbf{j}] \cdot \left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} \right) dt \\
&= \int_a^b \left[P(\mathbf{r}(t))\frac{dx}{dt} + Q(\mathbf{r}(t))\frac{dy}{dt} \right] dt \\
&= \int_C P\, dx + Q\, dy.
\end{aligned}$$

Line Integrals in Component Form

Similarly, for three variable vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, we can write the line integral as

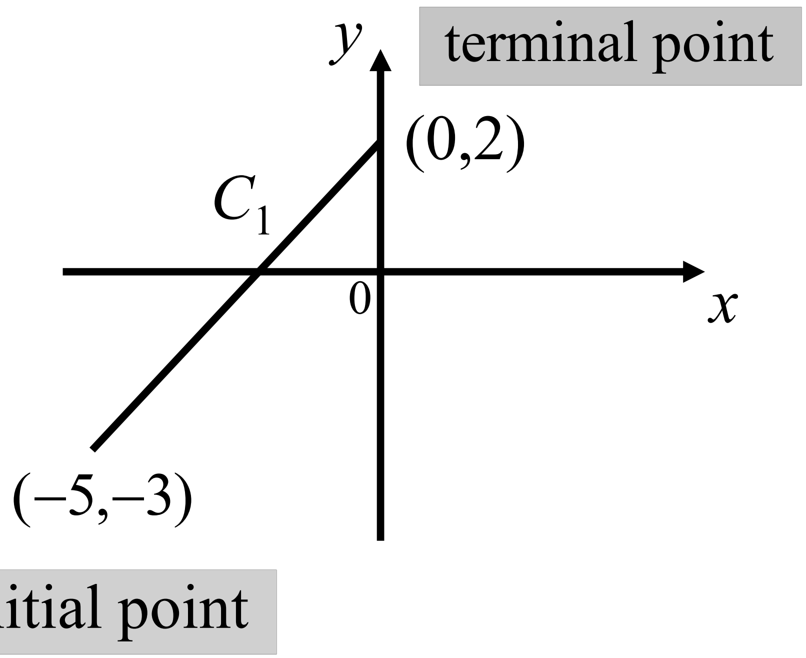
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\ dx + Q\ dy + R\ dz.$$

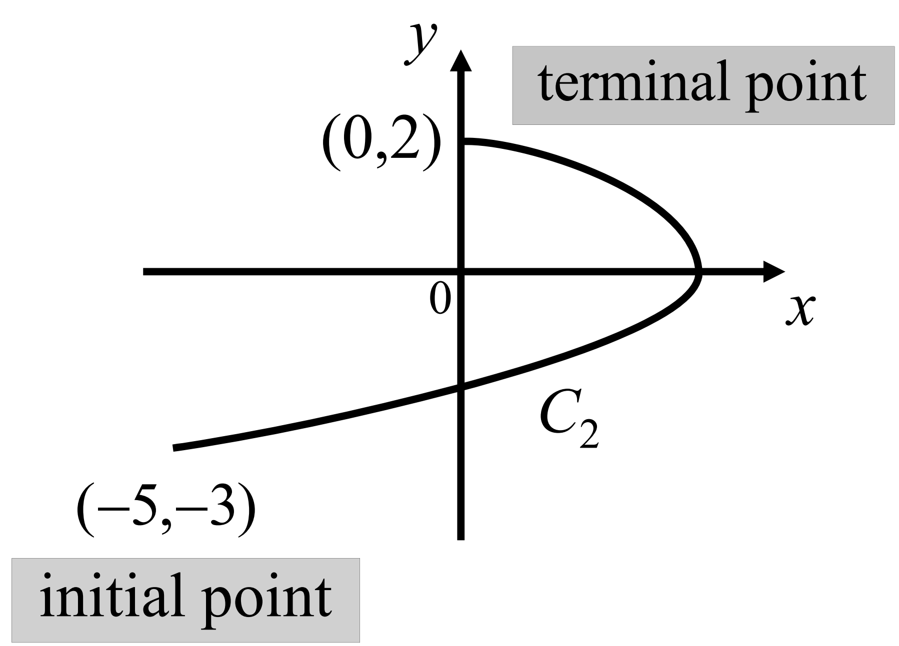
Example

Evaluate the line integral $\int_C y^2\, dx + x\, dy$, where

(a) $C = C_1$ is the line segment from $(-5,-3)$ to $(0,2)$.

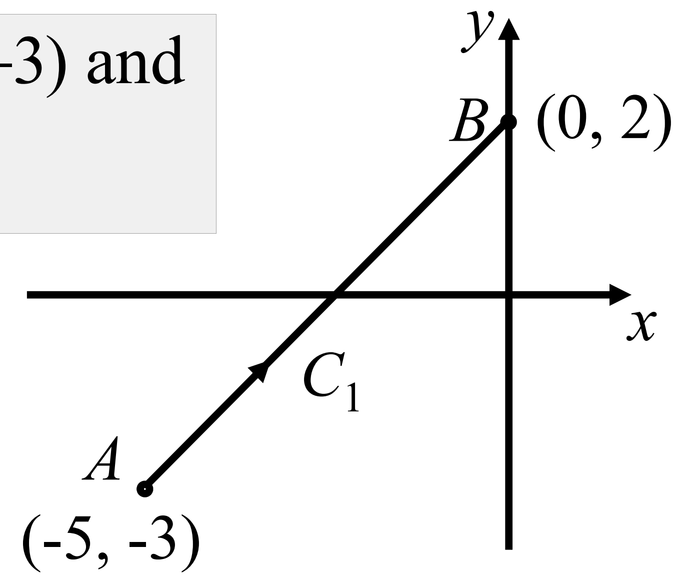
(b) $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$.





(a) $C = C_1$ is the line segment from $(-5,-3)$ to $(0,2)$.

C_1 is a line passing through the point $(-5,-3)$ and parallel to the vector $\overrightarrow{AB}$.



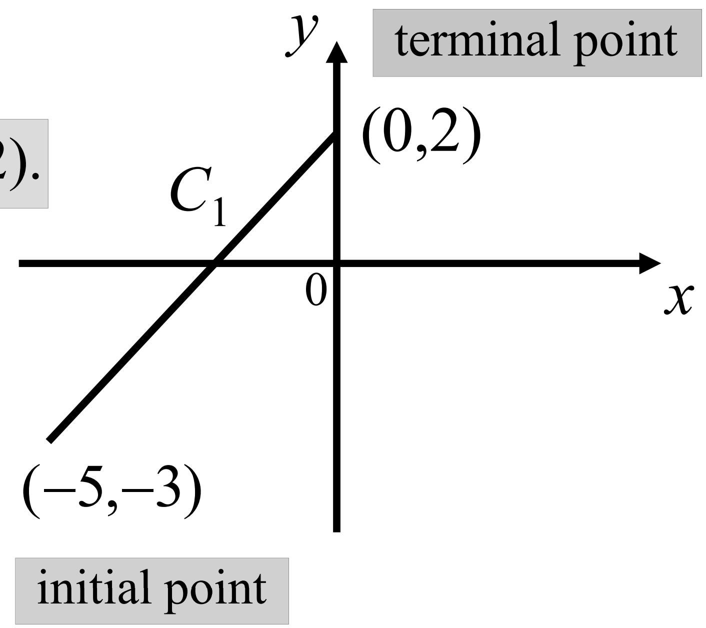
$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$
$$= (0\mathbf{i} + 2\mathbf{j}) - (-5\mathbf{i} - 3\mathbf{j})$$
$$= 5\mathbf{i} + 5\mathbf{j}$$

Vector function of C_1 :

$$\mathbf{r}(t) = \overrightarrow{OA} + t\,\overrightarrow{AB}$$
$$= (-5\mathbf{i} - 3\mathbf{j}) + t(5\mathbf{i} + 5\mathbf{j})$$
$$= (5t-5)\mathbf{i} + (5t-3)\mathbf{j}\ ,\ \ 0 \le t \le 1$$

Evaluate the line integral $\int_C y^2\, dx + x\, dy$, where

(a) $C = C_1$ is the line segment from $(-5,-3)$ to $(0,2)$.



The vector function of C_1 is given by

$\mathbf{r}(t) = (5t-5)\mathbf{i} + (5t-3)\mathbf{j}$ with $0 \le t \le 1$.

$$x = 5t-5 \quad \text{and} \quad y = 5t-3$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \left[P(\mathbf{r}(t))\frac{dx}{dt} + Q(\mathbf{r}(t))\frac{dy}{dt} \right] dt$$
$$= \int_C P\, dx + Q\, dy.$$

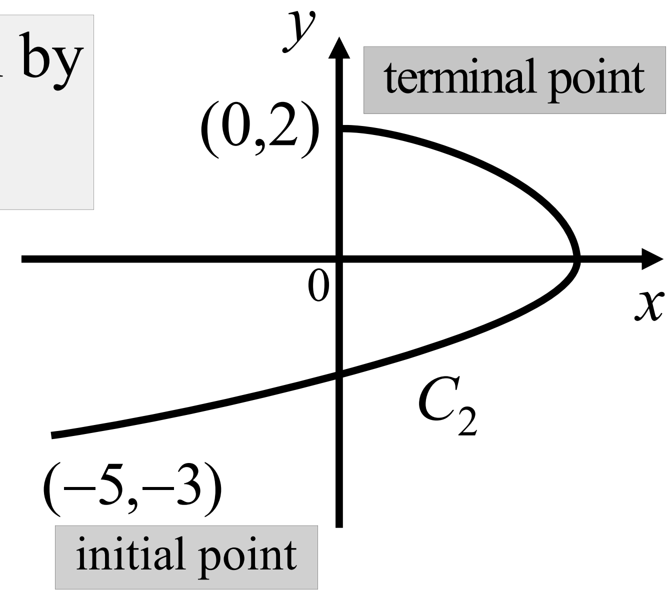
$$\int_{C_1} y^2\, dx + x\, dy = \int_0^1 (5t-3)^2 \frac{dx}{dt}\, dt + \int_0^1 (5t-5)\frac{dy}{dt}\, dt$$
$$= \int_0^1 (5t-3)^2 (5)\, dt + \int_0^1 (5t-5)(5)\, dt$$
$$= -\frac{5}{6}.$$

Evaluate the line integral $\int_C y^2\, dx + x\, dy$, where

(b) $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$.

Set $y = t$, we have the vector function C_2 given by

$\mathbf{r}(t) = (4 - t^2)\mathbf{i} + t\mathbf{j}$ with $-3 \le t \le 2$.



$x = 4 - t^2$ and $y = t$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \left[P(\mathbf{r}(t)) \frac{dx}{dt} + Q(\mathbf{r}(t)) \frac{dy}{dt} \right] dt$$
$$= \int_C P\, dx + Q\, dy.$$

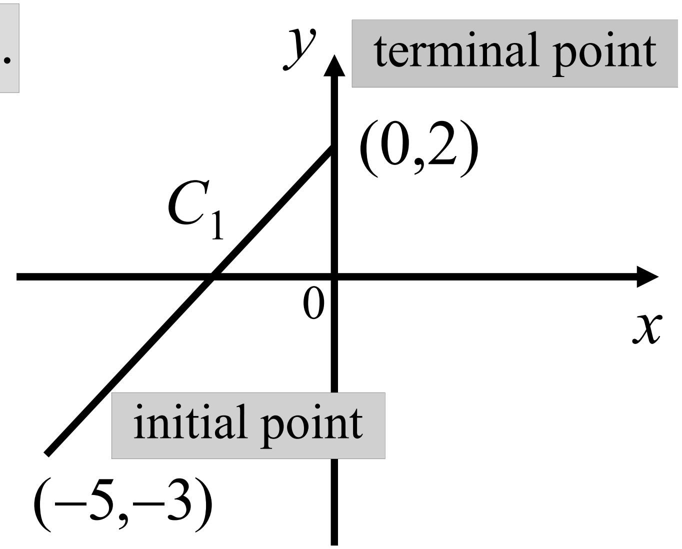
$$\int_{C_2} y^2\, dx + x\, dy = \int_{-3}^{2} t^2 \frac{dx}{dt} dt + \int_{-3}^{2} (4 - t^2) \frac{dy}{dt} dt$$
$$= \int_{-3}^{2} t^2(-2t)\, dt + \int_{-3}^{2} (4 - t^2)(1)\, dt$$
$$= \frac{245}{6}.$$

(a) $C = C_1$ is the line segment from $(-5,-3)$ to $(0,2)$.

$$\int_{C_1} y^2\,dx + x\,dy = \int_0^1 (5t-3)^2 \frac{dx}{dt}\,dt + \int_0^1 (5t-5)\frac{dy}{dt}\,dt$$

$$= \int_0^1 (5t-3)^2(5)dt + \int_0^1 (5t-5)(5)dt$$

$$= -\frac{5}{6}.$$

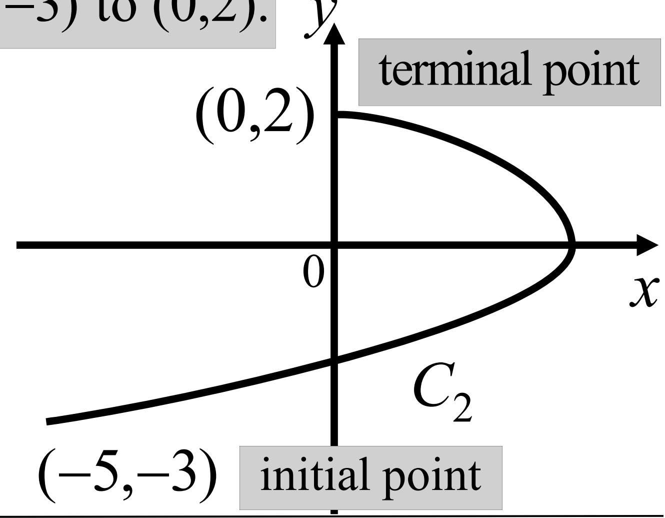


(b) $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$.

$$\int_{C_2} y^2\,dx + x\,dy = \int_{-3}^2 t^2 \frac{dx}{dt}\,dt + \int_{-3}^2 (4-t^2)\frac{dy}{dt}\,dt$$

$$= \int_{-3}^2 t^2(-2t)\,dt + \int_{-3}^2 (4-t^2)(1)\,dt$$

$$= \frac{245}{6}.$$



Note $\int_{C_1} y^2\,dx + x\,dy \neq \int_{C_2} y^2\,dx + x\,dy$ since $-\frac{5}{6} \neq \frac{245}{6}$.

Evaluating an integral on different curve path will result in different answer!

Fundamental Theorem of Calculus

$$\frac{d}{dx}\int_a^x F(t)\,dt = F(x)$$

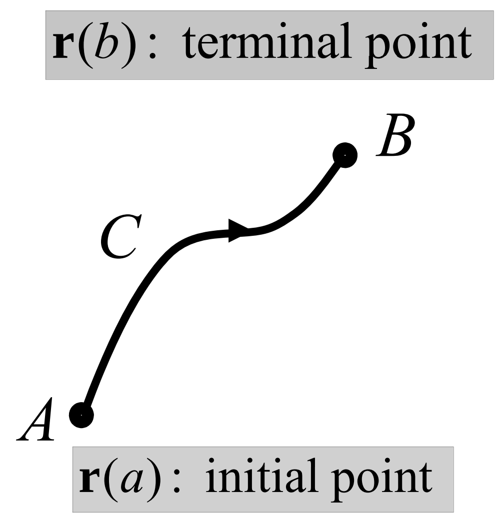
$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Generalization

The Fundamental Theorem of Line Integrals

Let C be a smooth curve given by $\mathbf{r}(t), t \in [a,b]$.

Let f be a (scalar) function of 2 or 3 variables.



$$\boxed{\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))}$$

This line integral only depends on the initial and terminal points of the curve, and not the path of the curve.

The ***gradient field*** of $f(x, y)$:

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$$

$\mathbf{F}$ is ***conservative*** if $\mathbf{F} = \nabla f$ for some f

(f is called a ***potential*** function for $\mathbf{F}$).

Not all vector fields are conservative.

Such fields do rise frequently in physics;

e.g., ***gravitational field***, ***electric field***, etc.

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$. $\quad$ $\mathbf{F}$ is conservative $\Leftrightarrow \dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$

$\mathbf{F}$ is ***conservative*** if $\mathbf{F} = \nabla f$ for some f (f is called a ***potential*** function for $\mathbf{F}$).

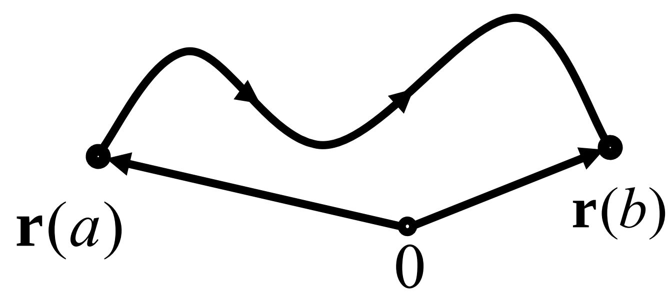
Fundamental Theorem for Line Integrals

$$\mathbf{F} = \nabla f$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r}$$

$$= f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

$C : \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j},\ a \le t \le b$



F is conservative $\rightarrow \int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path

$\int_C \mathbf{F} \cdot d\mathbf{r}$ only depends on the initial and terminal points of the curve, and not the path of the curve.

Consequences of Conservative Fields

If $\mathbf{F}$ is a ***conservative*** vector field, then $\int_C \mathbf{F} \cdot d\mathbf{r}$ is ***independent of path***, i.e.,

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r}$$

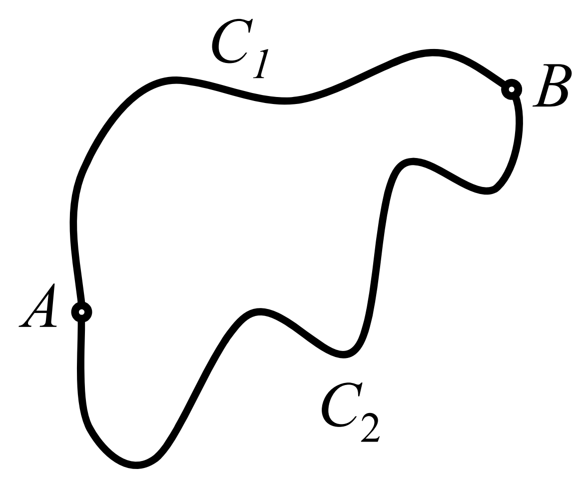
for any 2 paths C_1 and C_2 that have the same initial and terminal points.

Fundamental Theorem for Line Integrals

$$\mathbf{F} = \nabla f$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r}$$

$$= f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$



Example

Find the work done by the (earth) gravitational field given by

$$\mathbf{G} = \frac{-m_1 m_2 Kx}{(x^2+y^2+z^2)^{3/2}}\mathbf{i} + \frac{-m_1 m_2 Ky}{(x^2+y^2+z^2)^{3/2}}\mathbf{j} + \frac{-m_1 m_2 Kz}{(x^2+y^2+z^2)^{3/2}}\mathbf{k}$$

in moving a particle of mass m from the point $(3,4,12)$ to the point $(1,0,0)$ along a curve C.

By our earlier example, $\mathbf{G} = \nabla g$, with $g(x,y,z) = \dfrac{m_1 m_2 K}{\sqrt{(x^2+y^2+z^2)}}$,

where M is the mass of the earth and K the gravitational constant.

$$\begin{aligned} W &\equiv \int_C \mathbf{G}\cdot d\mathbf{r} \\ &= \int_C \nabla g \cdot d\mathbf{r} = g(1,0,0) - g(3,4,12) \\ &= \frac{12}{13} mMK. \end{aligned}$$