Functions
Limits and Continuity

MA1505

Mathematics I

Chapter 1

Functions

Outline

1. Definition of Function.

2. Domain and Range

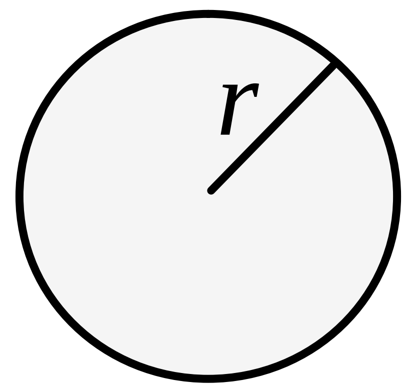
3. Composition

1. Functions

It is common that the value of one variable depends on the value of another.

For example, the area ***A*** of a circle depends on the radius ***r*** of the circle.

$$A = pr^2, \quad r \geq 0$$



When $r = 2$, $A = p(2)^2 = 4p$.

When $r = 3$, $A = p(3)^2 = 9p$.

So the value of A depends on the value of r.

Each value of r gives *exactly one* value of A.

Definition (Function)

If a variable *y* depends on a variable *x* in such a way that each value of *x* determines **exactly one** value of *y*, then we say that *y* is a function of *x*.

Since the area ***A*** of a circle depends on the radius ***r*** of the circle, and each value of ***r*** gives exactly one value of ***A***, we have ***A*** is a function of ***r***.

$$A = pr^2, \quad r \geq 0$$

Example $y = \pm\sqrt{x}$ is NOT a function

When $x = 9$, $y = \pm\sqrt{9} = \pm 3$.

So one value of x gives *more than one* value of y.

Thus, $y = \pm\sqrt{x}$ is NOT a function

PAUSE and THINK

Right or **Wrong** ???

Let $y = \sqrt{x}$

When $x = 9$,

we have $y = \sqrt{9} = \pm 3.$

What is the diffenence ???

When $x = 9$,

find the value(s) of y.

(A) $y = \sqrt{x}$

(B) $y = -\sqrt{x}$

(C) $y^2 = x$

When y is a function of x, we refer to x as the *independent variable* and y the *dependent variable*.

$$A = \pi r^2, \quad r \geq 0$$

r : *independent variable*

A : *dependent variable*

Many years ago, the Swiss mathematician Euler invented the symbol

$$y = f(x)$$

to denote the statement that 'y is a function of x'.

$$\boxed{y = f(x)}$$

dependent variable ***independent*** variable

$$\boxed{y \text{ is a } \textbf{\textit{function}} \text{ of } x}$$

Since the Area A of a circle is a function of the radius r, we can write

$$A(r) = p\,r^2, \quad r \geq 0$$

PAUSE and THINK

$$A(r) = p\,r^2, \qquad r \geq 0$$

Question : What is the value of $A(-2)$?

Can we say $A(-2) = p\,(-2)^2 = 4p$?

$$A(r) = p\,r^2, \quad r \geq 0$$

In the function $A = p\,r^2$, we have a constraint on r.

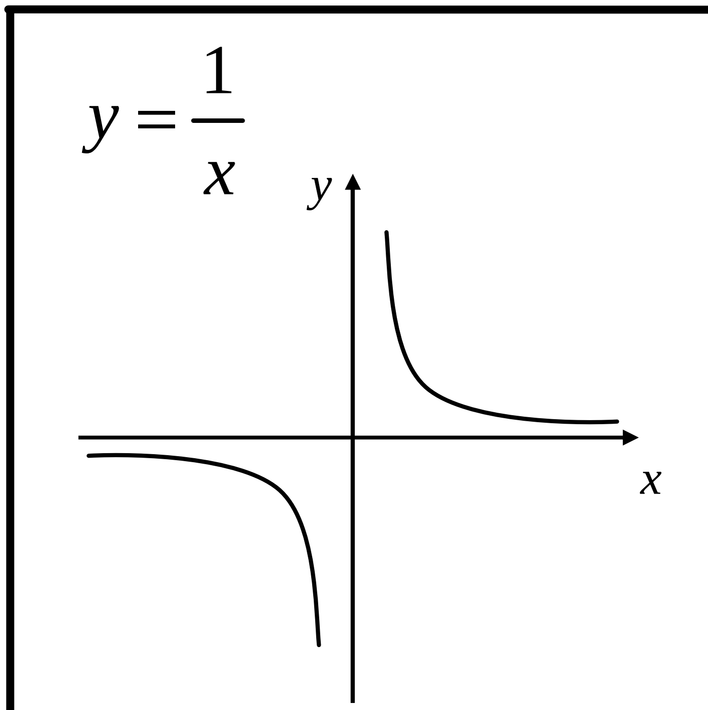
We required $r \geq 0$ since the radius of a circle is positive.

If we look at the function $y = \dfrac{1}{x}$, we require that $x \neq 0$

since y is undefined when $x = 0$.

So we see that if y is a function of x,

there may be constraints on x.



Definition (Domain)

If y is a function of x, the set of values that the variable x is allowed to take is called the domain of the function.

For the function $f(x) = \dfrac{1}{x}$, the domain is the set of all real numbers excluding $x = 0$.

The domain of a function $f(x)$ is usually denoted by D.

PAUSE and THINK

The function $A(r) = p\,r^2$ has domain $r \geq 0$.

Question : What is the value of $A(-2)$?

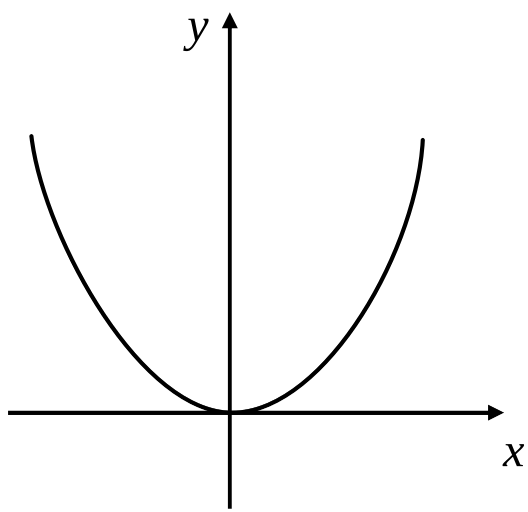
Can we say $A(-2) = p\,(-2)^2 = 4p$?

Definition (Range)

If y is a function of x, the set of values that the variable y can take (when x takes values in the domain) is called the range of the function.

The range of a function $f(x)$ is usually denoted by R.

$y = x^2$



$\boxed{D = \mathbb{R}}$

$\boxed{R = \mathbb{R}_{\geq 0} = [0, \infty)}$

For the function $y = x^2$, since $x^2 \geq 0$, the range of y is the set of all positive real numbers including 0.

In this chapter, we are only concerned with real values (or real numbers).

The set of real numbers is denoted by $\mathbb{R}$

Symbolically, we write

$$f: \; D \to \mathbb{R}$$

to denote f is a real-valued function with domain D.

Interval Notation

Let a and b be two real numbers with $a < b$.

Then the interval notation refers to the following :

$[a,b] = \{x \in \mathbb{R} \mid a \le x \le b\}$ (closed interval from a to b)

$(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$ (open interval from a to b)

$[a,b) = \{x \in \mathbb{R} \mid a \le x < b\}$

Interval Notation

$$(a,b]=\{x\in\mathbb{R}\,|\,a<x\le b\}$$

$$[a,\infty)=\{x\in\mathbb{R}\,|\,x\ge a\}$$

$$(a,\infty)=\{x\in\mathbb{R}\,|\,x>a\}$$

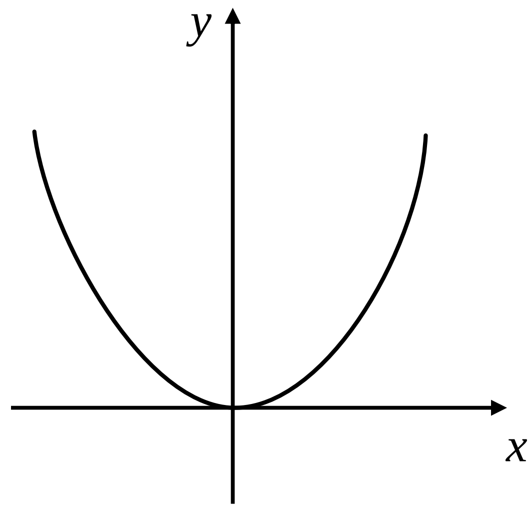
Interval Notation

$$(-\infty, a] = \{x \in \mathbb{R} \mid x \le a\}$$

$$(-\infty, a) = \{x \in \mathbb{R} \mid x < a\}$$

$$(-\infty, \infty) = \mathbb{R}$$

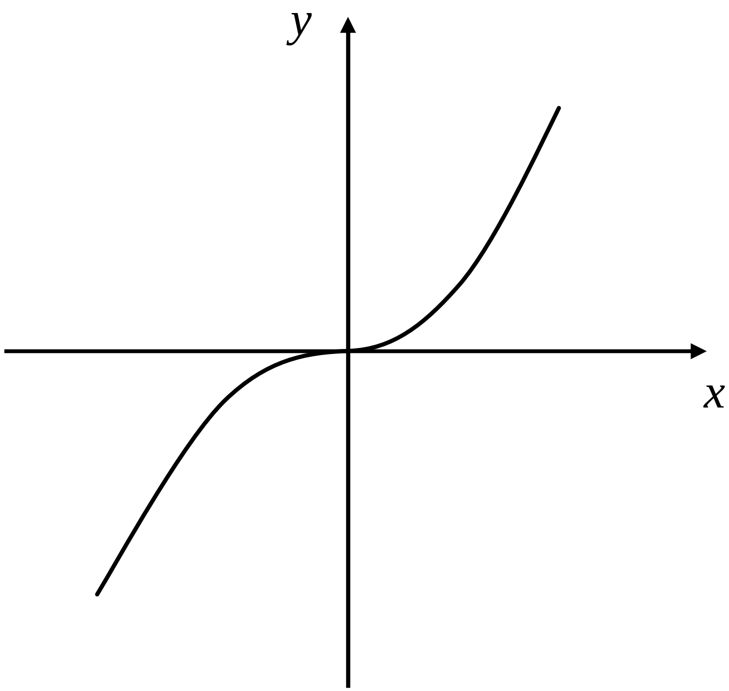
$y = x^2$



$$\boxed{D = \mathbb{R}}$$

$$\boxed{R = \mathbb{R}_{\geq 0} = [0, \infty)}$$

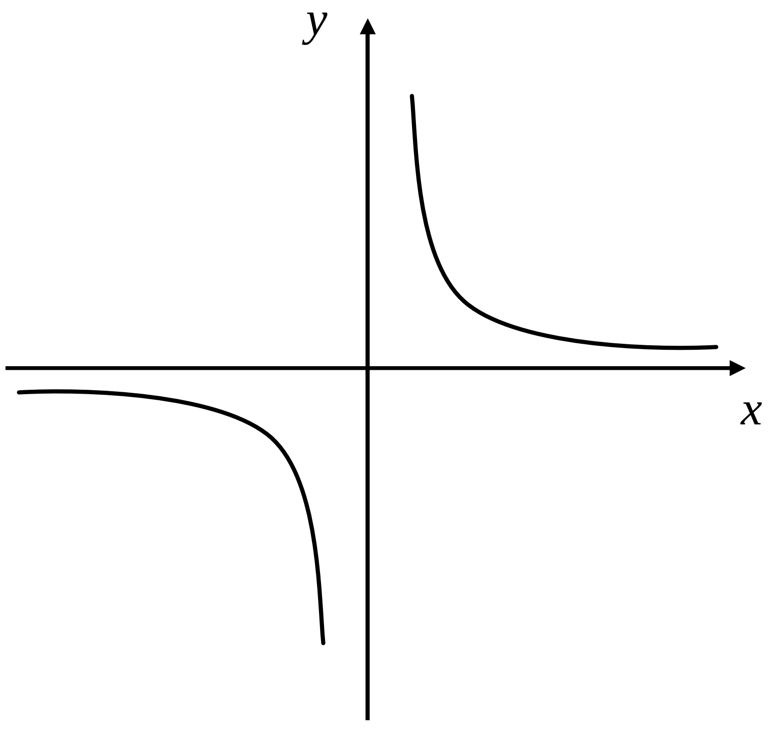
$$y = x^3$$



$$\boxed{D = \mathbb{R}}$$

$$\boxed{R = \mathbb{R}}$$

$$y = \frac{1}{x}$$

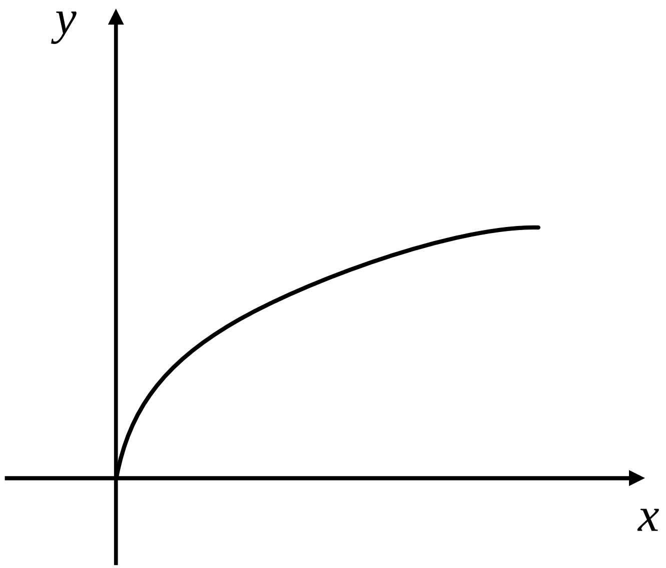


$$D = \mathbb{R} - \{0\}$$

$$R = \mathbb{R} - \{0\}$$

$y = \sqrt{x}$

$\sqrt{\text{Positive number}}$



$\boxed{D = \mathbb{R}_{\geq 0} = [0, \infty)}$

$\boxed{R = \mathbb{R}_{\geq 0} = [0, \infty)}$

PAUSE and THINK

What is the domain and Range?

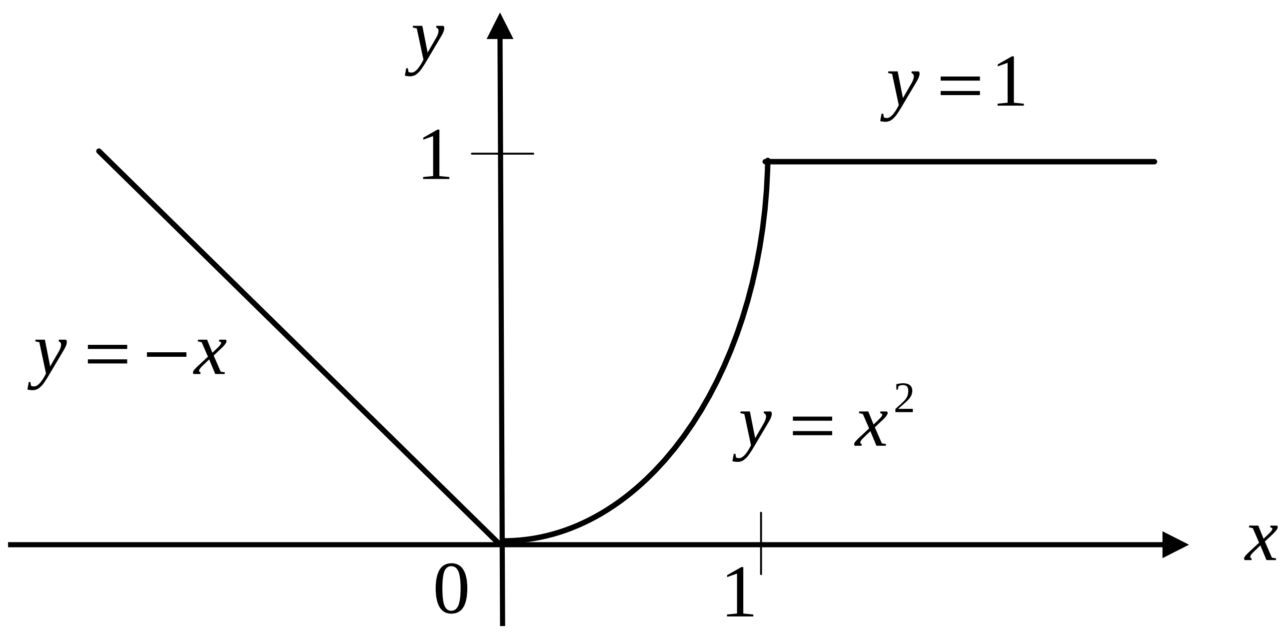
$$y = \sqrt{-x}$$

Functions can be defined in pieces.

<u>Example (a)</u>

$$f : \mathbb{R} \to \mathbb{R}$$

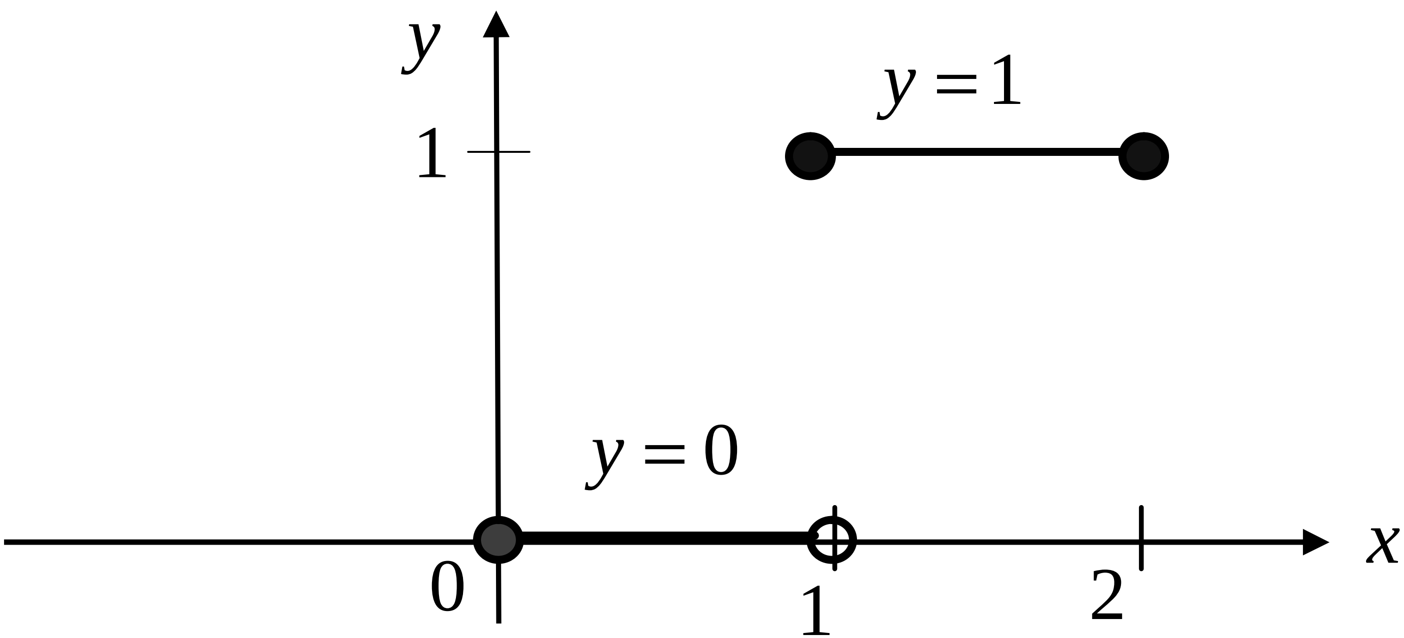
given by $$f(x) = \begin{cases} -x & x < 0 \\ x^2 & 0 \leq x \leq 1 \\ 1 & x > 1. \end{cases}$$



Examples

(b) $f : [0,2] \to \mathbb{R}$

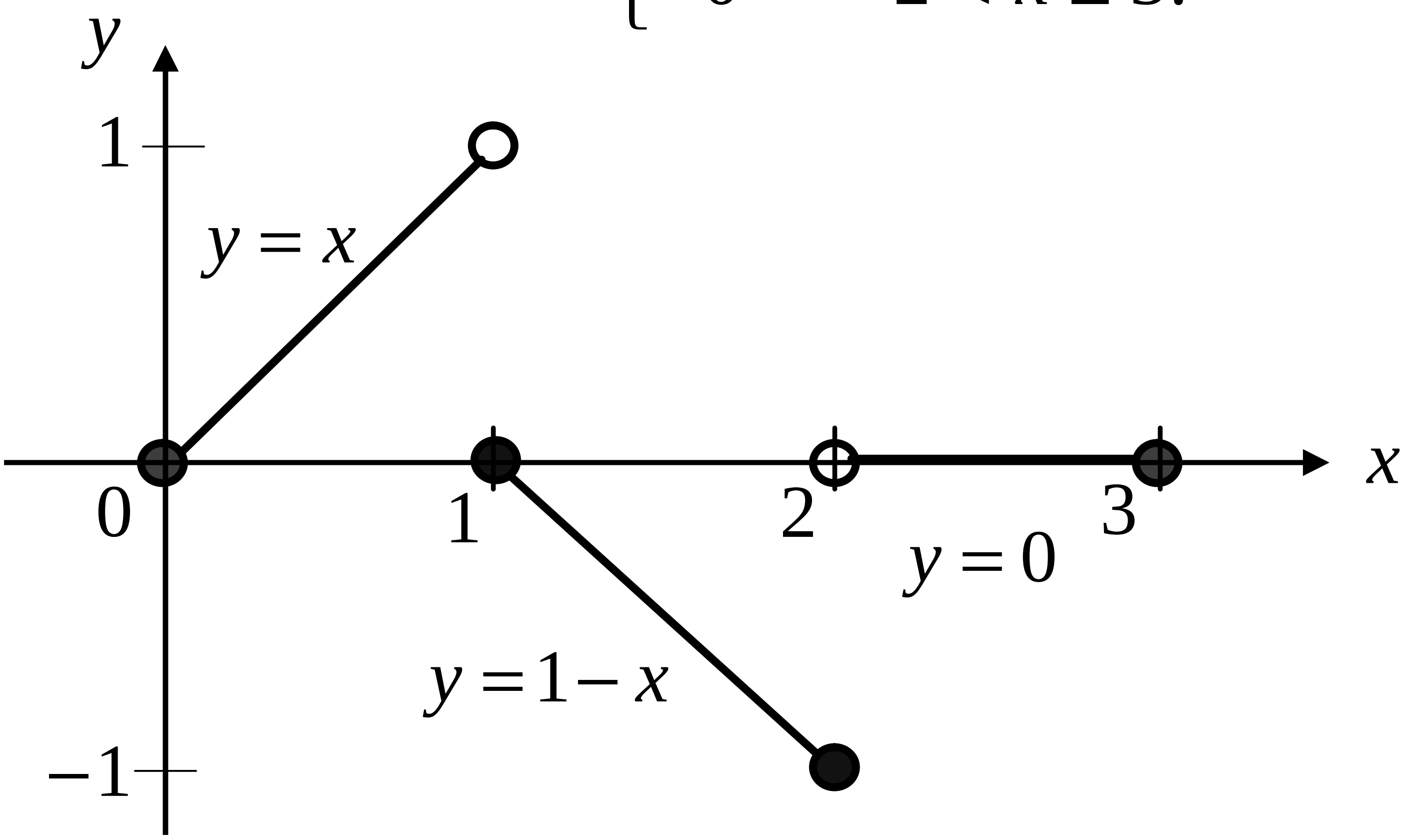
given by $f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & 1 \le x \le 2. \end{cases}$



Examples

(c) $f : [0,3] \to \mathbb{R}$

given by $f(x) = \begin{cases} x & 0 \le x < 1 \\ 1 - x & 1 \le x \le 2 \\ 0 & 2 < x \le 3. \end{cases}$



Let f and g be two functions.

$(f+g)(x) = f(x)+g(x)$. (the sum of f and g)

$(f-g)(x) = f(x)-g(x)$. (the difference of f and g)

$(fg)(x) = f(x)g(x)$. (the product of f and g)

$(f/g)(x) = f(x)/g(x)$ (the quotient of f by g)
where $g(x) \neq 0$.

Examples

$$f(x) = \sin x \qquad g(x) = \cos x$$

$$(f+g)(x) = \sin x + \cos x. \qquad \text{(the sum of } f \text{ and } g\text{)}$$

$$(f-g)(x) = \sin x - \cos x. \qquad \text{(the difference of } f \text{ and } g\text{)}$$

$$(fg)(x) = \sin x \cos x. \qquad \text{(the product of } f \text{ and } g\text{)}$$

$$(f/g)(x) = \frac{\sin x}{\cos x}. \qquad \text{(the quotient of } f \text{ by } g\text{)}$$

Composition

Let f and g be two functions with domains D and D' respectively. We define

$(f \circ g)(x) = f(g(x))$ called f composed with g (or f circle g) with domain consists of all x values in D' for which the values $g(x)$ are in D.

Example

Let $f(x) = x - 7$ with domain $\mathbb{R}$

and $g(x) = x^2$ with domain $\mathbb{R}$.

$$
\begin{aligned}
(f \circ g)(2) &= f(g(2)) \\
&= f(4) \\
&= 4 - 7 \\
&= -3
\end{aligned}
$$

$$
\begin{aligned}
g(2) &= 2^2 \\
&= 4
\end{aligned}
$$

$$
\begin{aligned}
(g \circ f)(2) &= g(f(2)) \\
&= g(-5) \\
&= (-5)^2 \\
&= 25
\end{aligned}
$$

$$
\begin{aligned}
f(2) &= 2 - 7 \\
&= -5
\end{aligned}
$$

(Note: $(f \circ g)(2) \neq (g \circ f)(2)$)

Example

Let $f:\mathbb{R}\to\mathbb{R}$ $\quad f(x)=\sin(x)$

Let $g:[-1,\infty)\to\mathbb{R}$ $\quad g(x)=\sqrt{1+x}$

$$(f\circ g)(x)=f(g(x))$$

$$=f(\sqrt{1+x})$$

$$=\sin(\sqrt{1+x})$$

Replace $g(x)$ by $\sqrt{1+x}$

$f(x)=\sin(x)$

Relace all x by $\boxed{}$

$f(\boxed{\sqrt{1+x}})=\sin(\boxed{\sqrt{1+x}})$

Example

Let $f: \mathbb{R} \to \mathbb{R}$ $\qquad f(x) = \sin(x)$

Let $g: [-1, \infty) \to \mathbb{R}$ $\qquad g(x) = \sqrt{1+x}$

$$
\begin{aligned}
(f \circ g)(x) &= f(g(x)) \\
&= f(\sqrt{1+x}) \\
&= \sin(\sqrt{1+x})
\end{aligned}
$$

Domain of $f \circ g$

$=$ Domain of g

$= [-1, \infty)$.

What is the domain of $f \circ g$?

To find $(f \circ g)(x)$, we perform $g(x)$ first followed by $f(g(x))$.
Thus, we need to start with x values from the domain of g.
After that to compute $f(g(x))$, we require the value of $g(x)$
to be in the domain of f.

Example

Let $f:\mathbb{R}\to\mathbb{R}$ $\qquad f(x)=\sin(x)$

Let $g:[-1,\infty)\to\mathbb{R}$ $\qquad g(x)=\sqrt{1+x}$

$$
\begin{aligned}
(g\circ f)(x) &= g(f(x)) \\
&= g(\sin(x)) \\
&= \sqrt{1+\sin x}
\end{aligned}
$$

Replace $f(x)$ by $\sin(x)$

$$g(x)=\sqrt{1+x}$$

Relace all x by $\boxed{}$

$$g\left(\boxed{\sin(x)}\right)=\sqrt{1+\left(\boxed{\sin(x)}\right)}$$

Example

Let $f : \mathbb{R} \to \mathbb{R}$ $\quad f(x) = \sin(x)$ $\quad -1 \le \sin(x) \le 1$

Let $g : [-1, \infty) \to \mathbb{R}$ $\quad g(x) = \sqrt{1+x}$ $\quad \sqrt{\text{Positive number}}$

$$
\begin{aligned}
(g \circ f)(x) &= g(f(x)) \\
&= g(\sin(x)) \\
&= \sqrt{1 + \sin x}
\end{aligned}
$$

What is the domain of $g \circ f$?

$$
\begin{aligned}
&\text{Domain of } g \circ f \\
&= \text{Domain of } f \\
&= \mathbb{R}.
\end{aligned}
$$

To find $(g \circ f)(x)$, we perform $f(x)$ first followed by $g(f(x))$.

Thus, we need to start with x values from the domain of f.

After that to compute $g(f(x))$, we require the value of $f(x)$ to be in the domain of g.

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