Chapter 1. Functions: Limits and Continuity

1.1 Functions

It is common that the values of one variable depend on the values of another. E.g. the area A of a region on the plane enclosed by a circle depends on the radius r of the circle ($A = \pi r^2, \;\; r > 0$.) Many years ago, the Swiss mathematician Euler invented the symbol $y = f(x)$ to denote the statement that "y is a function of x".

A function represents a rule that assigns a *unique* value y to each value x .

We refer to x as the *independent variable* and y the *dependent* variable.

One can also think of a function as an input-output system/process: input the value x and output the value $y = f(x)$. (This becomes particularly useful when we combine or composite functions together.)

Unless otherwise stated, in this course, we are only concerned with real values (or real numbers). The whole collection of real numbers is denoted by $\mathbb{R}$. So for our functions, the values of x and y belong to $\mathbb{R}$.

1.1.1 Domain and Range

There may be constraints on the possible values of x, e.g. $y = 1/x$, where we require that $x \neq 0$. The collection D in which x takes values is called the *domain* of the function f.

Symbolically, we write

$$\begin{aligned} f:\ D &\longrightarrow \mathbb{R} \\ x &\longmapsto y = f(x). \end{aligned}$$

The $\mathbb{R}$ appearing on the right side of the arrow in the above notation is called the *codomain* of f. It indicates that the output values of this function are real numbers.

On the other hand, the *actual* collection of y values, where $y = f(x)$ and x is allowed to take the values in D, is known as the *range* of f (denoted by R.)

E.g. the range of the function $f = 1/x$ is given by $R = \mathbb{R} - \{0\}$.

1.1.2 Example

The function $A : \mathbb{R}_{\geq 0} \to \mathbb{R};\ \ r \mapsto \pi r^2$ (also written as $A(r) = \pi r^2$) gives the area of a circle as a function of its radius; e.g., $A(2) = 4\pi$. It is clear that the range of this function is the set of all nonnegative numbers $\mathbb{R}_{\geq 0}$.

What about $A(-2) =?$. This is not defined since -2 lies outside the domain of A.

1.1.3 Interval Notation

For the next few examples, we shall use the *interval notation*. Let a and b be two real numbers with $a < b$. Then the interval notation refers to the following:

$[a, b] = \{x \in \mathbb{R} |\ a \leq x \leq b\}$

(this set is called the *closed* interval from a to b)

$(a, b) = \{x \in \mathbb{R} |\ a < x < b\}$

(this set is called the *open* interval from a to b)

$[a, b) = \{x \in \mathbb{R} |\ a \leq x < b\}$

$(a, b] = \{x \in \mathbb{R} |\ a < x \leq b\}$

$[a, \infty) = \{x \in \mathbb{R} |\ x \geq a\}$

$(a, \infty) = \{x \in \mathbb{R} |\ x > a\}$

$(-\infty, a] = \{x \in \mathbb{R} |\ x \leq a\}$

$(-\infty, a) = \{x \in \mathbb{R} |\ x < a\}$

$(-\infty, \infty) = \mathbb{R}.$

1.1.4 Examples

Functions can also be defined in pieces. Here are some examples:

(a) $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} -x & x < 0 \\ x^2 & 0 \leq x \leq 1 \\ 1 & x > 1. \end{cases}$$

(b) $f : [0, 2] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & 1 \leq x \leq 2. \end{cases}$$

(c) $f : [0, 3] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x & 0 \leq x < 1 \\ 1 - x & 1 \leq x \leq 2 \\ 0 & 2 < x \leq 3. \end{cases}$$

1.2 Graphs

1.2.1 Definition

The *graph* of a function f consists of all the points in the xy-plane whose coordinates are of the form $(x, f(x))$, that is,

$$\text{graph of } f = \{(x, y) \in \mathbb{R}^2 | \; y = f(x)\}.$$

1.2.2 Examples

(i) $f : \mathbb{R} \to \mathbb{R}$; $f(x) = x^2$. The graph is given in Figure 1 below.

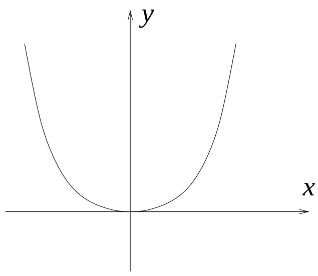


Figure 1

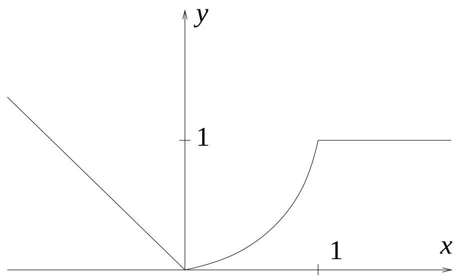


Figure 2

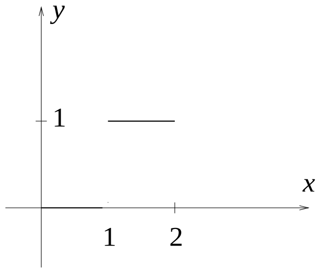


Figure 3

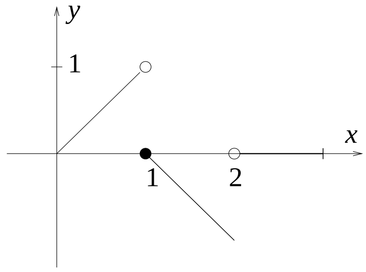


Figure 4

(ii) Figures 2 to 4 above are the graphs of the three functions given in Example 1.1.4 (functions defined in pieces).

1.3 Operations on Functions

1.3.1 Arithmetical operations

Let f and g be two functions.

(i) The functions $(f \pm g)(x) = f(x) \pm g(x)$, called the sum or difference of f and g.

(ii) The function $(fg)(x) = f(x)g(x)$, called the product of f and g.

(iii) The function $(f/g)(x) = f(x)/g(x)$, called the quotient of f by g, is defined where $g(x) \neq 0$;

1.3.2 Composition

Let $f : D \to \mathbb{R}$ and $g : D' \to \mathbb{R}$ be two (real) functions with domains D and D' respectively.

The function

$$(f \circ g)(x) = f(g(x)),$$

called f *composed with* g or f *circle* g, is defined on the subset of D' for which the values $g(x)$ (i.e. the range of g) are in D.

1.3.3 Example

Let $f(x) = x - 7$ and $g(x) = x^2$ (defined on all of $\mathbb{R}$). Then

$$\begin{aligned}(f \circ g)(2) &= f(g(2)) = f(4) = -3, \quad \text{and} \\ (g \circ f)(2) &= g(f(2)) = g(-5) = 25.\end{aligned}$$

Note that in general $f \circ g \neq g \circ f$.

1.3.4 Example

Given the following two functions.

$$f : \mathbb{R} \to \mathbb{R} \quad f(x) = \sin(x)$$

$$g : [-1, \infty) \to \mathbb{R} \quad g(x) = \sqrt{1 + x}$$

Then

$$(g \circ f) : \mathbb{R} \to \mathbb{R} \quad (g \circ f)(x) = \sqrt{1 + \sin(x)}$$

$$(f \circ g) : [-1, \infty) \to \mathbb{R} \quad (f \circ g)(x) = \sin(\sqrt{1 + x})$$

Again notice that $g \circ f \neq f \circ g$. Even their domains are different.

1.4 Limits

In this section we are interested in the behaviour of f as x gets closer and closer to a.

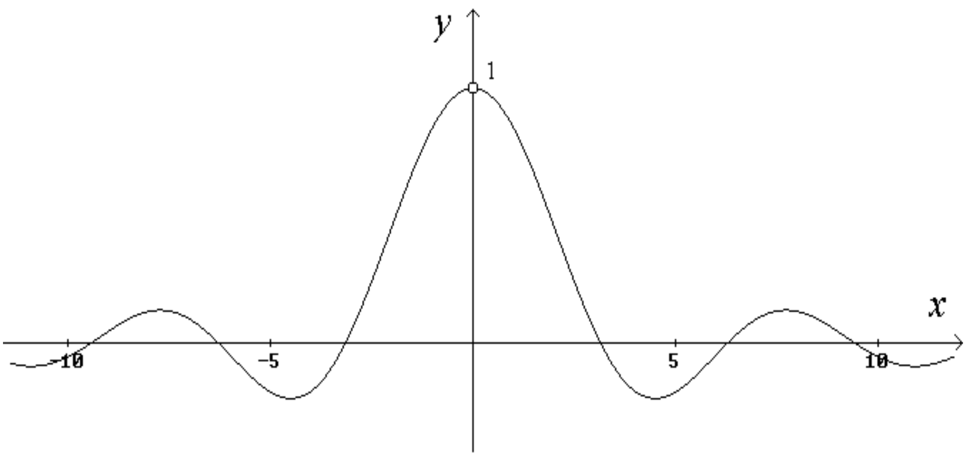
1.4.1 Example

Let $D = \{x \in \mathbb{R} : x \neq 0\}$ and we consider the function $f : D \to \mathbb{R}$ given by $f(x) = \dfrac{\sin(x)}{x}$. ($x$ is in radian.) Describe its behaviour as x tends to 0.

Clearly when $x = 0$, $\dfrac{\sin(0)}{0} = \dfrac{0}{0}$ does not make sense. It is defined everywhere except at 0 and thus it makes sense to ask how it behaves as it is evaluated at arguments which are closer and closer to 0.

If we plot the graph of $f(x)$, we see that as x gets closer and closer to 0 from either sides (and not reaching 0 itself), $f(x)$ approaches 1. In this case, we say that "the limit of f as x tends to 0 is equal to 1". We use the following notation:

$$\lim_{x \to 0} f(x) = 1.$$



1.4.2 Informal Definition

Let $f(x)$ be defined on an open interval I containing x_0, except possibly at x_0 itself. If $f(x)$ gets arbitrary close to L when x is sufficiently close to x_0, then we say that the limit of $f(x)$ as x tends to x_0 is the number L and we write

$$\lim_{x \to x_0} f(x) = L.$$

1.4.3 Rules of Limits

Suppose $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = L'$, then the following statements are easy to verify:

(i) $\lim_{x \to a} (f \pm g)(x) = L \pm L'$;

(ii) $\lim_{x \to a} (fg)(x) = LL'$;

(iii) $\lim_{x \to a} \frac{f}{g}(x) = \frac{L}{L'}$ provided $L' \neq 0$;

(iv) $\lim_{x \to a} kf(x) = kL$ for any real number k.

1.4.4 Example.

Consider how to evaluate

$$\lim_{x \to -1} \frac{\sqrt{x^2 + 8} - 3}{x + 1}.$$

We cannot substitute $x = -1$, and the numerator and denominator have no obvious common factors.

The idea is to multiple the numerator and denominator by the *conjugate expression* $\sqrt{x^2+8}+3$:

$$\frac{\sqrt{x^2+8}-3}{x+1} = \frac{(\sqrt{x^2+8}-3)(\sqrt{x^2+8}+3)}{(x+1)(\sqrt{x^2+8}+3)}$$

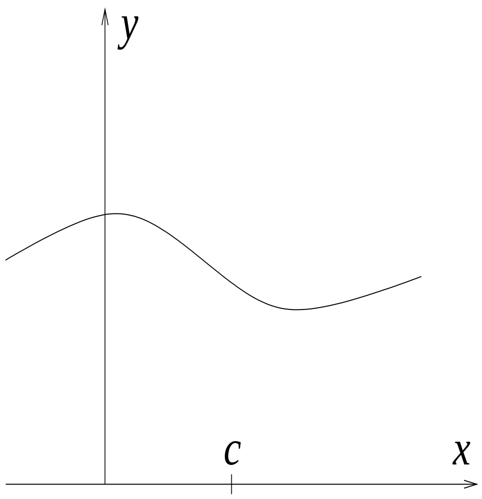
$$= \frac{(x+1)(x-1)}{(x+1)(\sqrt{x^2+8}+3)} = \frac{x-1}{(\sqrt{x^2+8}+3)}\,.$$

The cancelation occurs **when** $x \neq -1$. Hence

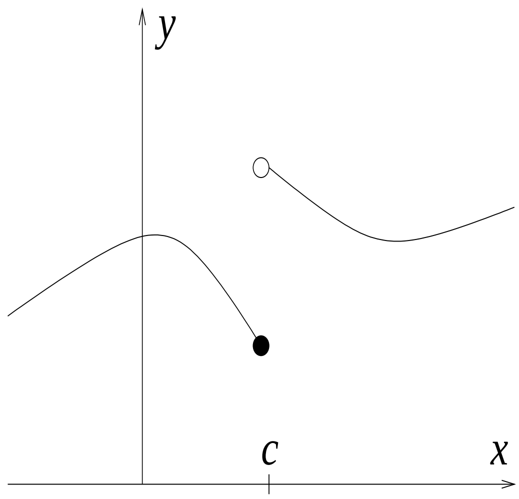
$$\lim_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1} = \lim_{x\to -1} \frac{x-1}{(\sqrt{x^2+8}+3)} = -\frac{1}{3}\,.$$

1.5 Continuity

Continuity is an intuitive concept. Intuitively, a function is continuous if we can draw its graph "in one stroke", or "without lifting up the pen from the paper".



Continuous at c



Not continuous at c

Here is the precise definition.

1.5.1 Definition

A function $f(x)$ is *continuous*

(i) at an interior point c of its domain if $\lim_{x \to c} f(x) = f(c)$;

(ii) at a left endpoint a of its domain if $\lim_{x \to a^+} f(x) = f(a)$;

(iii) at a right endpoint b of its domain if $\lim_{x \to b^-} f(x) = f(b)$.

1.5.2 The continuity test

According to the definition, to test whether a function f is continuous at a point p, we need to do the following 3 things:

(i) check that p is in the domain of f (that is, $f(p)$ is defined),

(ii) check that $\lim_{x \to p} f(x)$ exists (or the appropriate one sided limit if p is an end-point),

(iii) make sure that $\lim_{x \to p} f(x)$ (or the appropriate one sided limit) is equal to $f(p)$.

1.5.3 Example

$f(x) = 1/x$ is continuous at every x except $x = 0$ where it is not defined.

1.5.4 Example

All polynomials are continuous at every point in $\mathbb{R}$ as we have seen in Example **??** that for a polynomial $p(x)$,

$\lim_{x \to a} p(x) = p(a)$ for every a. Also, recall that for a rational function (quotient of 2 polynomials) $p(x)/q(x)$ where p and q are polynomials,

$$\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$$

as long as $q(a) \neq 0$.

This implies that all rational functions are continuous at every point such that $q(x) \neq 0$.

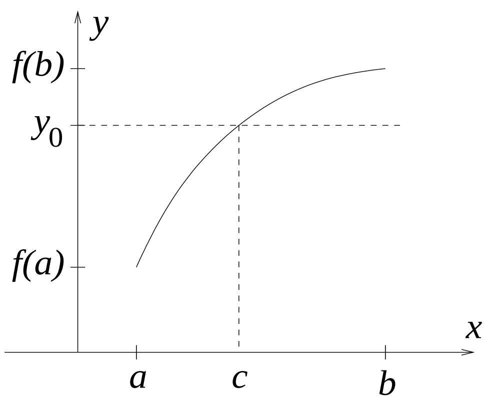
1.5.5 Continuous functions

A function $f : D \to \mathbb{R}$ is called a *continuous* function if f is continuous at ALL points in D.

If the domain D of a continuous function is an interval, then the graph of f is continuous.

1.5.6 The Intermediate Value Theorem (IVT)

If a function $f(x)$ is continuous in a closed interval $[a, b]$, then for any value y_0 in the range $f(a) \leq y_0 \leq f(b)$, we can always find a point $c \in [a, b]$ such that $f(c) = y_0$.



In particular, suppose f is a continuous function on the closed interval $[a, b]$ and $f(a) < 0 < f(b)$. Then the equation $f(x) = 0$ has a solution in $[a, b]$.

1.5.7 Example

Show that the polynomial $x^{11} + 3x - 1$ has a real root between 0 and 1.

Solution. Let $f(x) = x^{11} + 3x - 1$.

This is a continuous function on $\mathbb{R}$. So it is continuous on the closed interval $[0, 1]$.

Now $f(0) = -1$ and $f(1) = 3$.

Since $f(0) < 0 < f(1)$, by IVT, there is a point y_0 in $[0, 1]$ such that $f(y_0) = 0$. So y_0 is a real root of the polynomial between 0 and 1.