Chapter 3. Probability (B)

January 26, 2011

1 Conditional Probability

Example Why are some mutual fund managers more successful than others? One possible factor is where the manager did his or her MBA. The following table compares mutual fund performance against the ranking of the school where the fund manager earned their MBA.

$$A_1 = \text{Fund manager graduated from a top-20 MBA program}$$

$$A_2 = \text{Fund manager did not graduate from a top-20 MBA program}$$

$$B_1 = \text{Fund outperforms the market}$$

$$B_2 = \text{Fund does not outperform the market}$$

	Mutual fund outperforms the market (i.e. B_1)	Mutual fund doesnt outperform the market (i.e. B_2)
Top 20 MBA program (i.e. A_1)	.11	.29
Not top 20 MBA program (i.e. A_2)	.06	.54

By frequency probability approach

$$\text{P}(A_2 \text{ and } B_1) = .06, \quad \text{P}(A_1 \text{ and } B_2) = .29, \quad \ldots$$

Marginal probabilities ($P(A_i), P(B_i)$) based on the table

	B_1	B_2	P(A_i)
A_1	.11	.29	0.4
A_2	.06	.54	0.6
P(B_i)	0.17	0.83	1.00

the probability of “a fund manager isn’t from a top program”

$$P(A_2) = .06 + .54$$

the probability of “a fund outperforms the market?”

$$P(B_1) = .11 + .06$$

Conditional probability is used to determine how two events are related; that is, we can determine the probability of one event given the occurrence of another event.

Example Randomly select one student in a class.

P(the student is male) $=$

P(the student is male|given the student is taller than the average) $=$

Conditional probabilities are written as P(A | B) and read as "the probability of A given B" (more precisely, the probability of A given that B has occurred) and is defined as (providing P(B)$\neq$ 0):

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

"the probability of B given A " (more precisely, the probability of B given that A has occurred) is defined as (providing P(A)$\neq$ 0):

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Example An experiment consists of rolling an even die once. Let X be the outcome. Let F $= \{$X $=$ 6$\}$, and let E $= \{$X $>$ 4$\}$.

Suppose E has occurred. This leaves only two possible outcomes: 5 and 6. So the probability of F becomes 1/2, (intuitively) making P(F|E) $=$ 1/2.

Note that $E \cap F = F$. So, the above formula/definition gives

$$P(F|E) = \frac{P(F \cap E)}{P(E)} = \frac{1/6}{1/3} = \frac{1}{2};$$

in agreement with the intuitive calculation, indicating that our definition of conditional probability make sense.

Some properties of conditional probability: for any B with P(B)> 0

- $0 \le P(A|B) \le 1$ for any A.

- $P(\Omega|B) = 1$

- $P(E \cup F|B) = P(E|B) + P(F|B) - P(EF|B)$

For the mutual fund example above, what is the probability that a fund will outperform the market given that the manager graduated from a top-20 MBA program?, i.e. what is P($B_1|A_1$) ?

	B_1	B_2	P(A_i)
A_1	0.11	0.29	0.4
A_2	0.06	0.54	0.6
P(B_i)	0.17	0.83	1.00

$$P(B_1|A_1) = \frac{P(B_1 \cap A_1)}{P(A_1)} = \frac{0.11}{0.4} = 27.5\% \quad (\neq P(B_1))$$

Thus, there is a 27.5% chance that that a fund will outperform the market given that the manager graduated from a top-20 MBA program. This chance is bigger than the average chance/probability 17%.

Inverting the order of conditioning Sometimes, it is beneficial to be able to swap the event of interest and the conditioning event when we are computing probabilities.

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$$

Example (inverting conditioning) Suppose we classify the entire female population into 2 classes: healthy controls and cancer patients. If a woman has a positive mammogram result, what is the probability that she has breast cancer? Suppose we obtain medical evidence for a subject in terms of the results of her mammogram (imaging) test: positive or negative mammogram.

$$\text{P(Positive Test)} = 0.107, \qquad \text{P(Cancer)} = 0.1,$$

$$\text{P(Positive test} \mid \text{Cancer)} = 0.8,$$

then we can easily calculate the probability of real interest— the chance that the subject has cancer:

$$\begin{aligned}
P(\text{Cancer}|\text{Positive Test}) &= \frac{P(\text{“Positive Test” and “Cancer”})}{P(\text{Positive Test})} \\
&= \frac{P(\text{Positive Test}|\text{Cancer}) \times P(\text{Cancer})}{P(\text{Positive Test})} \\
&= \frac{0.8 \times 0.1}{0.107}
\end{aligned}$$

One of the objectives of calculating conditional probability is to determine whether two events are related. Now consider the mutual fund again

	B_1	B_2	P(A_i)
Birthplace (W)est	0.068	0.332	0.4
Birthplace (E)ast	0.102	0.498	0.6
P(B_i)	0.17	0.83	1.00

$$P(B_1|E) = \frac{P(B_1 \cap E)}{P(E)} = \frac{0.102}{0.6} = 0.17 = P(B_1)$$

and

$$P(B_1|W) = \frac{P(B_1 \cap W)}{P(W)} = \frac{0.068}{0.4} = 0.17 = P(B_1)$$

Indicating that a manager's performance does not depend on his birthplace.

Independent If A and B are any two events in a sample space Ω, we say that A is **independent** of B if and only if one of the following equations holds

$$P(A|B) = P(A),$$

$$P(B|A) = P(B),$$

$$P(A \cap B) = P(A) \cdot P(B)$$

(Thus, we also say A and B are independent events)

Proof of the equivalence: Assume first that A and B are independent. Then $P(A|B) = P(A)$, and so

$$P(A \cap B) = P(A|B)P(B) = P(A)P(B)$$

Assume next that $P(A \cap B) = P(A)P(B)$. Then

$$P(A|B) = P(A \cap B)/P(B) = P(A)$$

Also,

$$P(B|A) = P(B \cap A)/P(A) = P(B)$$

Therefore, A and B are independent.

Experiments can be physically independent (roll 1 die, then roll another die), or seem physically related and still be independent.

Example George tosses an (even) die and considers any event A of the experiment, and John tosses an (even) die, consider any event B of the experiment. Then A and B are independent.

Example Drawing a card from a deck of 52 cards. Event A = draw an ace; C = draw a heart. are A and C independent?

$$P(A) = \frac{4}{52}, \quad P(C) = \frac{13}{52}$$

and

$$P(A \cap C) = \frac{1}{52} = P(A) \times P(C)$$

They are independent!

Independence vs. disjointness/mutual-exclusiveness

Mutual-exclusiveness and independence are different concepts.

A and B are independent $\Leftrightarrow P(A \cap B) = P(A)P(B)$

Events C and D are disjoint/mutually-exclusive $\Leftrightarrow P(C \cap D) = 0$

Events that are mutually exclusive (disjoint) with nonzero probability cannot be independent.

Example Electrical engineers are considering two alternative systems for sending messages. A message consists of a word that is either a 0 or a 1. The probability, p, that

$$P\,[\text{A transmitted 1 is received as 0}] = p$$

$$P\,[\text{A transmitted 0 is received as 1}\,] = p$$

Scheme A: send a single digit. Scheme B: repeat the selected digit three times in succession, any of 101, 110, 011 , or 111 will mean 1 was sent; otherwise 0 was sent.

(a) Evaluate the probability that a transmitted 1 will be received as a 1 under the three-digit scheme when p = 0.01 , 0.02, or 0.05. For Scheme B:

$$P(\text{transmitted 1 is received correctly})$$

$$= P(111) + P(110) + P(101) + P(011)$$

$$= (1-p)(1-p)(1-p) + (1-p)(1-p)p$$

$$+(1-p)p(1-p) + p(1-p)(1-p)$$

$$= (1-p)(1-p)(1-p) + 3(1-p)(1-p)p$$

p	0.01	0.02	0.05
P[Correct of scheme A]	0.99	0.98	0.95
P[Correct of scheme B]	0.9997	0.9988	0.9928

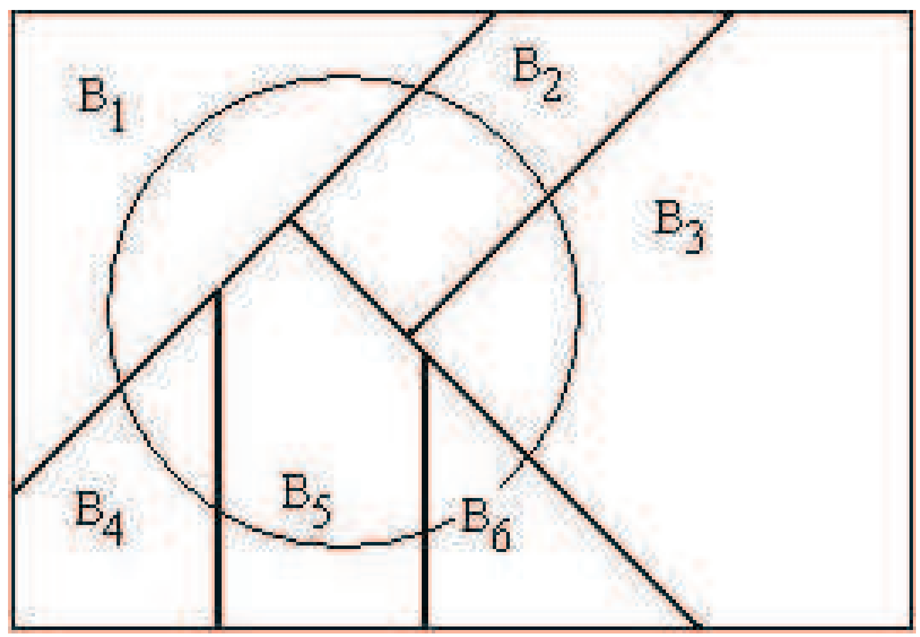
(b) Suppose a message with p=0.05, consisting of the two words, a 1 followed by 0, is to be transmitted using the three-digit scheme. Both of the words 1 and 0 must be received correctly. the probability that the total message is correctly received is

Scheme A: $0.95 \times 0.95 = 0.903$

Scheme B: $0.9928^2 = 0.986.$

2 Rule of total probability

If $B_1, B_2 \ldots., B_n$ are mutually exclusive events (i.e. $B_i \cap B_j = \emptyset$ for any $i \neq j$), and $B_1 \cup B_2 \cup \ldots \cup B_n = \Omega$, we call $B_1, B_2 \ldots., B_n$ a partition of Ω.

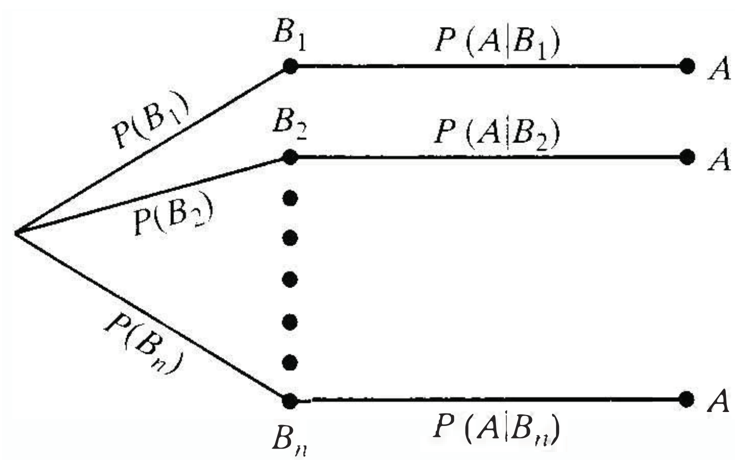


Theorem If $B_1, B_2 ...., B_n$ is a partition of Ω, then

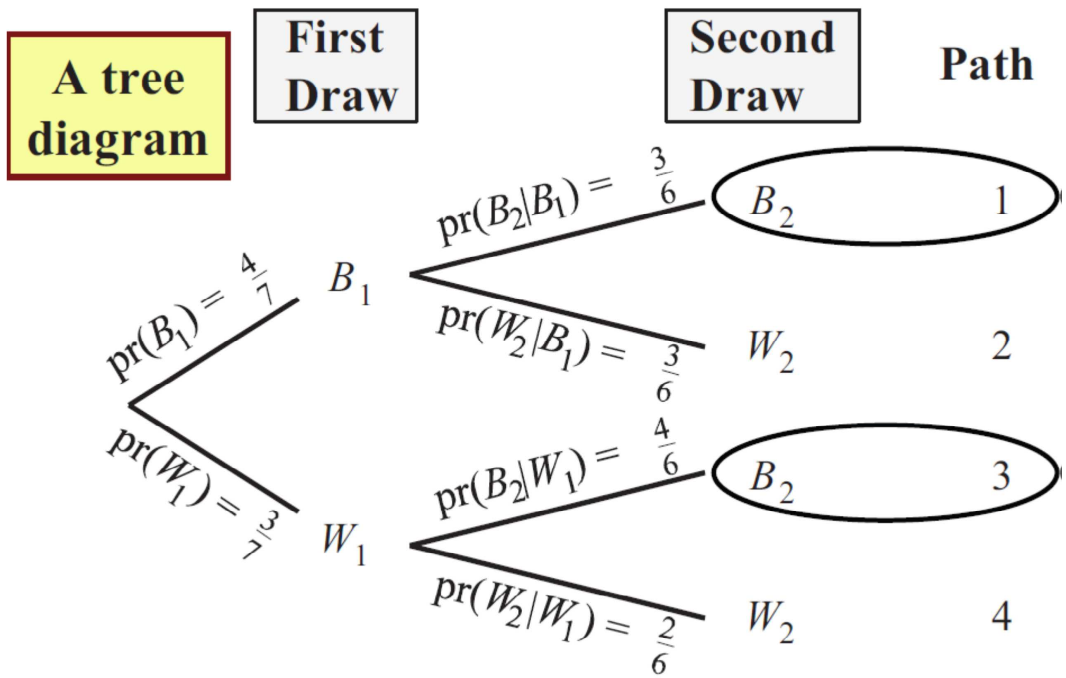
$$P(A) = \sum_{i=1}^{n} P(B_i) \times P(A|B_i)$$

as a special case, $P(A) = P(B) \times P(A|B) + P(B^c) \times P(A|B^c)$.

we can use the following probability tree to show the theorem.



Example Suppose we draw 2 balls randomly, one at a time without replacement from an urn containing 4 black and 3 white balls. What is the probability that the second ball is black?



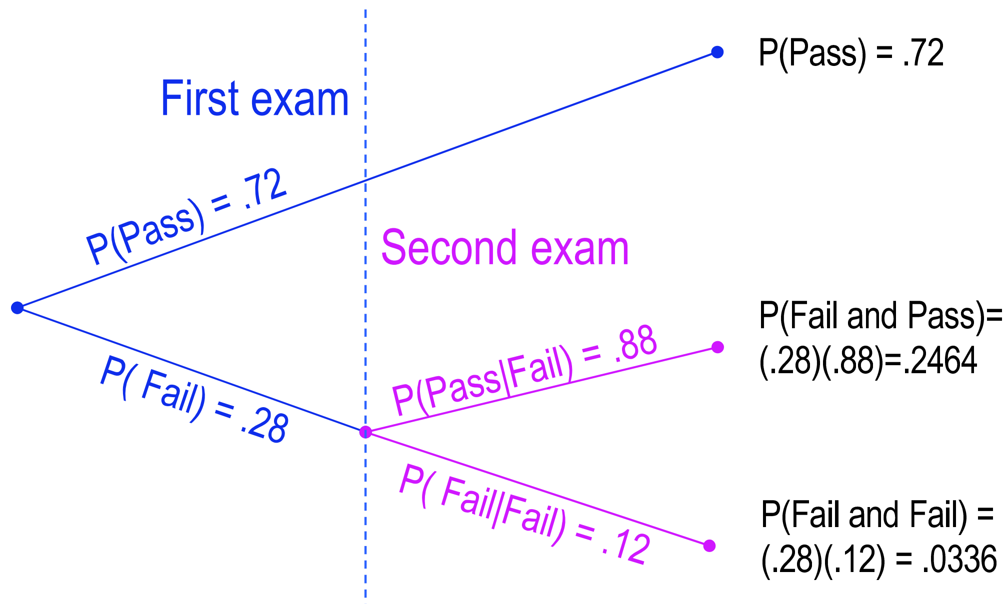
$$
\begin{aligned}
P(\{\text{2nd ball is black}\}) &= P(\{\text{2nd is black }\} \text{ and } \{\text{ 1st is black}\}) \\
&\quad + P(\{\text{2nd is black}\} \text{ and } \{\text{1st is white}\}) \\
&= P(\{\text{2nd is black }\} \mid \{\text{ 1st is black}\}) P(\{\text{1st is black}\}) \\
&\quad + P(\{\text{2nd is black}\} \mid \{\text{1st is white}\}) P(\{\text{1st is white}\}) \\
&= \frac{4}{7} \times \frac{3}{6} + \frac{4}{6} \times \frac{3}{7} = 4/7.
\end{aligned}
$$

Example In a certain county. 60% are Republicans; 30% are Democrats ; 10% are Independents. When those voters were asked about increasing military spending 40% of Republicans opposed it; 65% of the Democrats opposed it; 55% of the Independents opposed it. What is the probability that a randomly selected voter in this county opposes increasing military spending?

$\Omega = \{$voters in the county$\}$; R $= \{$republicans$\}$, P(R) $= 0.6$; D $= \{$democrats$\}$, P(D) $= 0.3$; I $= \{$independents$\}$, P(I) $= 0.1$; O $= \{$voters opposing military spending$\}$; P(O$|$R) $= 0.4$, P(O$|$D) $= 0.65$, P(O$|$I) $= 0.55$. By the total probability theorem:

$$P(O) \;=\; P(O|R)P(R) + P(O|D)P(D) + P(O|I)P(I) = 0.49.$$

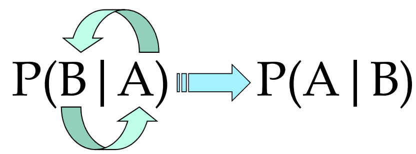
Example Law school grads must pass a bar exam. Suppose pass rate for first-time test takers is 72%. They can re-write if they fail and 88% pass their second attempt [P(pass take 2|fail take 1)]. What is the probability that a randomly grad passes the bar?



3 Bayes Theorem

Bayes' Law is named for Thomas Bayes, an eighteenth century mathematician.

If we know P(B | A), we can apply Bayes' Law to determine P(A | B)



Bayes Theorem. Let $A_1, ..., A_n$ be a partition of Ω. For any event B

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_{i=1}^{n} P(A_i)P(B|A_i)}$$

as a special case,

$$P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A) + P(A^c)P(B|A^c)}$$

Example Suppose there are 2 urns A and B. In A, 2 black balls and 3 white balls. In B, 4 black and 1 white. one ball is randomly drawn from all the balls (equal chance to select A and B, and each chance to select to select all ball).

- What is the probability that the selected ball is a black?

$$P(\text{black}) = P(A)P(\text{black}|A) + P(B)P(\text{black}|B) = 0.5 \times \frac{2}{5} + 0.5 \times \frac{4}{5}$$

- If a black ball is observed, what is the probability that the ball is from urn A?

$$
\begin{aligned}
P(A|\mathsf{black}) &= \frac{P(\mathsf{black}|A)P(A)}{P(\mathsf{black})} = \frac{P(\mathsf{black}|A)P(A)}{P(A)P(\mathsf{black}|A) + P(B)P(\mathsf{black}|B)} \\
&= \frac{1}{3} \\
P(B|\mathsf{black}) &= \frac{P(\mathsf{black}|B)P(B)}{P(\mathsf{black})} = \frac{P(\mathsf{black}|B)P(B)}{P(A)P(\mathsf{black}|A) + P(B)P(\mathsf{black}|B)} \\
&= \frac{2}{3}
\end{aligned}
$$

or

$$
P(B|\mathsf{black}) = 1 - P(A|\mathsf{black}) \ = \ \frac{2}{3}
$$

(we call P(A)=0.5, P(B) = 0.5 the *prior probabilities*, P(A|black) = 1/3 and P(B|black)=2/3 the *posterior* probabilities)

Example Suppose a Laboratory blood test is used as evidence for a disease. Assume P(positive Test$|$Disease) $=$ 0.95, P(positive Test$|$no Disease)$=$0.01 and P(Disease) $=$ 0.005. Find P(Disease$|$positive Test)$=$?

Denote D $=$ the test person has the disease, $D^c = \{$the test person does not have the disease$\}$ and T $= \{$the test result is positive$\}$. Then

$$\begin{aligned} P(D|T) &= \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|D^c)P(D^c)} \\ &= \frac{0.95 \times 0.005}{0.95 \times 0.005 + 0.01 \times 0.995} \\ &= 0.3231293 \end{aligned}$$

Example A clinic has 3 doctors Dr A sees 41% of the patients and sent 5% of his patients for blood test. Dr B sees 32% of the patients and requested blood test on 8% of her patients. Dr C sees the rest and sent 6% of his patient for blood test. What is the probability that a blood est request was sent by Dr A?

Example (Monty Hall problem[1]) Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door. The host, who knows what's behind the doors, opens another door, which has a goat. He then says to you, "Do you want to change your choice to the other door?"

Suppose your first choice is Door 1.

Door 1	Door 2	Door 3	result if switching	result if staying
Car	Goat	Goat	Goat	Car
Goat	Car	Goat	Car	Goat
Goat	Goat	Car	Car	Goat

The probability of winning a car by staying with the initial choice is therefore 1/3, while the probability of winning by switching is 2/3.

[1]This is a well-known problem in probability. You can ignore it if you find it is too difficult.