NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 EXAMINATION 2007/2008

MA2214 Combinatorial Analysis

April/May 2008 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains a total of **FIVE (5)** questions and comprises **THREE (3)** printed pages.

2. Answer **ALL** questions. The marks for each question are indicated at the beginning of the question.

3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Attempt ALL questions. Each question carries 20 marks.

Question 1 [20 marks]

(a) From 1 to 8000, find the number of integers which are divisible by neither 14 nor 21 but divisible by either 4 or 6.

(b) Find the number of ways of arranging 25 people to sit around 4 identical round tables such that each table must be seated by at least 5 persons.

Question 2 [20 marks]

(a) Find the number of 6-element sets $\{a, b, c, d, e, f\}$ where a, b, c, d, e, f are integers chosen from 1 to 100 such that $b > a + 3$, $c > b + 1$, $d > c + 2$, $e > d + 4$, $f > e + 5$.

(b) Observe that in each of the five integers, namely, 11, 222, 3003, 31131 and 132231, any digit which occurs in an integer occurs at least twice in the integer. For each integer $n > 1$, let a_n denote the number of such n-digit integers (where the leading digit cannot be zero) comprising some or all of the four digits, namely, 0, 1, 2 and 3.

(i) Find a suitable generating function for a_n.

(ii) Hence find an expression of a_n in terms of n.

Question 3 [20 marks]

(a) Find the number of permutations of all the 11 letters of the word COMMUTATORS which contain

(i) none of the three words, namely, COM, MAT and TOUR;

(ii) exactly one of the three words as mentioned above;

(iii) exactly two of the three words as mentioned above.

(b) Eight boys want to participate in a game with their parents in an annual carnival of their school. The game requires that these 24 people are to be divided into 8 groups of 3 each, such that each group comprises a boy, a male parent and a female parent, and no boy should be in the same group with ANY of his parents. Find the number of ways of grouping these 24 people if

(i) there is no further requirement;

(ii) one particular couple do not want to be in the same group.

Question 4 [20 marks]

(a) For each positive integer n, let a_n denote the number of ways of paving a rectangular wall of 2 metres by n metres, using some or all of the three kinds of tiles, namely, 1m $\times$ 1m tiles, 1m $\times$ 2m tiles and 2m $\times$ 2m tiles.

(i) Find a recurrence relation for a_n with the necessary initial conditions.

(ii) Evaluate a_8.

(b) For each positive integer n, let a_n denote the number of n-digit integers (where the leading digit cannot be zero) formed by the digits 0, 1, 2, 3 and 4 which contain neither a block of 11 nor a block of 23.

(i) Find a recurrence relation for a_n with the necessary initial conditions.

(ii) Evaluate a_8.

Question 5 [20 marks]

(a) For each positive integer n, let a_n denote the number of ways of distributing n distinct objects into 7 distinct boxes labelled from 1 through 7 such that the total number of objects in boxes 1 and 2 is even, the total number of objects in boxes 3 and 4 is odd and the total number of objects in boxes 5, 6 and 7 is at least 2.

(i) Find a suitable generating function for a_n.

(ii) Evaluate a_8.

(b) For each positive integer n, let a_n denote the number of non-negative integral solutions to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = n$$

subject to the conditions that $x_1 + x_2 + x_3 + x_4$ is even and $x_5 + x_6 + x_7 + x_8$ is odd.

(i) Find a suitable generating function for a_n.

(ii) Evaluate a_9.

END OF PAPER