MA1506 TUTORIAL 4 SOLUTIONS

Question 1

(i) $\ddot{x} = \cosh(x)$. An equilibrium solution of an ODE is just a solution that is identically constant. That is not possible here because the cosh function never vanishes. So there is no equilibrium for this ODE.
(ii) $\ddot{x} = \cos(x)$. Equilibria are at $x = \pi/2$, $3\pi/2$, etc etc. Taylor expansion at $\pi/2$ is

$$\cos(x) = \cos'(\pi/2)[x - \pi/2] + ...$$

where we drop the higher order terms since we wish to consider a small perturbation. If we define $y = x - \pi/2$ then we have

$$\ddot{y} = -y$$

so we have simple harmonic motion [ie, stable equilibrium] with angular frequency [approximately] 1.

Taylor expansion at $3\pi/2$ is

$$\cos(x) = \cos'(3\pi/2)[x - 3\pi/2] + ...$$

where we drop the higher order terms since we wish to consider a small perturbation. If we define $y = x - 3\pi/2$ then we have

$$\ddot{y} \approx +y$$

so this equilibrium is unstable. The other equilibria are like these two; they alternate as we consider larger and smaller equilibrium values of x.
(iii) $\ddot{x} = \tan(\sin(x))$. Equilibria are at 0, π, 2π etc etc etc. Taylor expansion at 0 is

$$\tan(\sin(x)) = \cos(0)\sec^2(\sin(0))[x - 0] + ...$$

where we drop the higher order terms since we wish to consider a small perturbation. We have

$$\ddot{x} \approx +x$$

so we have an unstable equilibrium.

Taylor expansion at π is

$$\tan(\sin(x)) = \cos(\pi)\sec^2(\sin(\pi))[x - \pi] + ...$$

where we drop the higher order terms since we wish to consider a small perturbation. If we define $y = x - \pi$ then we have

$$\ddot{y} \approx -y$$

so we have simple harmonic motion [ie, stable equilibrium] with angular frequency approximately 1. The other equilibria are like these two; they alternate as we consider larger and smaller equilibrium values of x.

Question 2

The equation governing such a circuit is

$$\ddot{\mathrm{Q}} \;+\; \mathrm{R}\dot{\mathrm{Q}}/\mathrm{L} \;+\; \mathrm{Q}/\mathrm{LC} \;=\; \mathrm{V}/\mathrm{L},$$

where Q is the charge on the capacitor, R is the resistance, L is the inductance, C is the capacitance, and V is the applied voltage. So here we have

$$\frac{\mathrm{d}^2\mathrm{Q}}{\mathrm{dt}^2} + 100\frac{\mathrm{dQ}}{\mathrm{dt}} + 50000\mathrm{Q} = 4000\,\cos 100\mathrm{t}.$$

We can solve this in the usual way. The roots of the quadratic equation turn out to be $50 \pm 50\ \mathrm{i}\ \sqrt{19}$, and using the method of undetermined coefficients you will find that a particular solution is

$$\frac{16}{170}\cos 100\mathrm{t} + \frac{4}{170}\sin 100\mathrm{t}$$

so the general solution of the equation is

$$\mathrm{Q} = \mathrm{c}_1\mathrm{e}^{-50\mathrm{t}}\cos 50\sqrt{19}\mathrm{t} \;+\; \mathrm{c}_2\mathrm{e}^{-50\mathrm{t}}\sin 50\sqrt{19}\mathrm{t} + \frac{16}{170}\cos 100\mathrm{t} + \frac{4}{170}\sin 100\mathrm{t}.$$

We are told that $\mathrm{Q}(0) = 0$ and

$$\frac{\mathrm{dQ}}{\mathrm{dt}}\{\mathrm{t} = 0\} = 0.$$

The first condition gives

$$0 = \mathrm{q}(0) = \mathrm{c}_1(1) + \mathrm{c}_2(0) + \frac{16}{170}$$

but in order to use the second condition we need to differentiate first:

$$\begin{aligned}\frac{dQ}{dt} \quad &= \quad -0.0941\,(-50\mathrm{e}^{-50\mathrm{t}}\cos 50\sqrt{19}\,\mathrm{t} \;-\; 50\sqrt{19}\mathrm{e}^{-50\mathrm{t}}\sin 50\sqrt{19}\,\mathrm{t}) \\ &\quad +\mathrm{c}_2(-50\mathrm{e}^{-50\mathrm{t}}\sin 50\sqrt{19}\,\mathrm{t} + 50\sqrt{19}\mathrm{e}^{-50\mathrm{t}}\cos 50\sqrt{19}\,\mathrm{t}) - \frac{160}{17}\sin 100\mathrm{t} + \frac{40}{17}\cos 100\mathrm{t}\end{aligned}$$

So we get

$$0 = \frac{\mathrm{dQ}}{\mathrm{dt}}\{\mathrm{t} = 0\} = -0.0941\,(-50) + \mathrm{c}_2(50\sqrt{19}) + \frac{40}{17}.$$

Solving these two simultaneous equations for c_1 and c_2, we substitute them back into the formula we found for dQ/dt [since that is the current] and we find

$$-\,2.35\,\mathrm{e}^{-50\mathrm{t}}\cos(50\sqrt{19}\mathrm{t}) + 22.13\,\mathrm{e}^{-50\mathrm{t}}\sin(50\sqrt{19}\mathrm{t}) + 2.35\cos(100\mathrm{t}) \;-\; 9.41\sin(100\mathrm{t})$$

Question 3

The amplitude, as a function of the input frequency α, is given in Chapter 2 by

$$A(\alpha) = \frac{F_0/m}{\sqrt{(\omega^2 - \alpha^2)^2 + \frac{b^2}{m^2}\alpha^2}}$$

Here F_0, m, and ω are to be regarded as fixed constants which determine the nature of the particular system. For sufficiently small values of the friction constant b, the shape of the graph of this function is as follows: it begins with a value of $F_0/m\omega$ at $\alpha = 0$, then it rises to a local maximum [this is the resonance situation] and then decreases monotonically towards zero. If b is too large, however, the function simply decreases monotonically from $F_0/m\omega$ — there is no resonance. In that case the maximum amplitude is just $F_0/m\omega$ at $\alpha = 0$.

Differentiating A with respect to α and setting the derivative equal to zero we get

$$4\alpha[\alpha^2 - \omega^2] + 2b^2\alpha/m^2 = 0.$$

Simplifying this we get

$$\alpha^2 = \omega^2 - \frac{b^2}{2m^2}.$$

Of course the left side cannot be negative, so if $b \geq \sqrt{2}m\omega$ then there is no resonance; this is the situation described above; in that case the maximum amplitude is at $\alpha = 0$ and is given by $F_0/m\omega$. Otherwise the maximal value of the amplitude is obtained by substituting this value of α into $A(\alpha)$. The result, after some simple algebra, is

$$A_{Resonance} = \frac{F_0/b\omega}{\sqrt{1 - (b^2/4m^2\omega^2)}}.$$

If $b^2/m^2\omega^2$ is negligible then this is approximately $F_0/b\omega$. That is, the resonance amplitude grows without limit as b becomes smaller.

All of these results are reflected in the graphs: there is no resonance when $b^2 = 2$, but resonance is present in all the other cases, and the maximum becomes steadily larger and sharper as b decreases.

Question 4

When the ship is at rest, the part of it which is under sea level has a volume of Ad [that is, the area of the base times the height]. Therefore, this is the volume of seawater that has been pushed aside by the ship. If the density of seawater is ρ, then the mass of seawater pushed aside is ρAd, and its weight is ρAdg. This upward force exactly balances the weight of the ship, so we have

$$\rho\,\mathrm{A\,d\,g} = \mathrm{M\,g}.$$

Thus

$$\mathrm{d} = \mathrm{M}/\rho\,\mathrm{A}.$$

Now if the ship is moving and the distance from sea level to the bottom of the ship is d + x, where x is a function of time, we have to use Force = mass $\times$ acceleration. Taking the downwards direction to be positive, we find that the buoyancy force is now $-\rho\mathrm{A}(\mathrm{d} + \mathrm{x})\mathrm{g}$, so we have

$$\mathrm{M\ddot{x}} = \mathrm{Mg} - \rho\,\mathrm{A}\,(\mathrm{d} + \mathrm{x})\mathrm{g},$$

which, using our formula for d, is just

$$\mathrm{\ddot{x}} = -\frac{\rho\,\mathrm{A\,g}}{\mathrm{M}}\,\mathrm{x}$$

This represents simple harmonic motion with angular frequency $\sqrt{\rho\,\mathrm{A\,g/M}}$, as claimed. The ship will bob up and down at this frequency. Note the inverse dependence on M, which is to be expected, but also that the frequency increases if A is large, which is not so obvious.

Taking into account the friction and the force exerted by the waves, Force = mass $\times$ acceleration gives

$$\mathrm{M\ddot{x}} \;=\; \mathrm{Mg} \;-\; \rho\,\mathrm{A}\,(\mathrm{d}+\mathrm{x})\mathrm{g} \;-\mathrm{b}\,\dot{\mathrm{x}} \;+\; \mathrm{F_0}\cos(\alpha\,\mathrm{t})$$

or

$$\mathrm{M\ddot{x}} \;+\mathrm{b}\,\dot{\mathrm{x}} \;+\; \rho\,\mathrm{A}\,\mathrm{g}\,\mathrm{x} \;=\; \mathrm{F_0}\cos(\alpha\,\mathrm{t}).$$

This is exactly the equation studied in the notes, except that k is replaced by ρAg. We assume that b is small, so that, in the absence of waves, the ship will undergo damped harmonic motion. [The ship is sailing in seawater, not honey....]

So after the transient terms [the solution of the homogeneous equation, which decay exponentially and so can be neglected] die out, we will have [see page 34 of the notes]

$$\mathrm{x(t)} = \frac{\frac{1}{\mathrm{M}}\mathrm{F_0}\cos(\alpha\mathrm{t}-\gamma)}{\sqrt{(\omega^2-\alpha^2)^2+\frac{\mathrm{b}^2}{\mathrm{M}^2}\alpha^2}},$$

where γ is a constant and where ω denotes $\sqrt{\rho\,\mathrm{A}\,\mathrm{g}/\mathrm{M}}$. So eventually the ship bobs up and down at the same frequency as the waves, but the amplitude of x is given by

$$\mathrm{A}(\alpha) = \frac{\mathrm{F_0/M}}{\sqrt{(\omega^2-\alpha^2)^2+\frac{\mathrm{b}^2}{\mathrm{M}^2}\alpha^2}}$$

Notice that this can be very large even if $\mathrm{F_0}$ is quite small. So there is a danger that even if the ship is safe for most values of α, it might sink if α takes a particular value. [The ship will probably sink if the maximum possible value of x exceeds H, because that means that the deck of the ship will be under water!].

To see how large $\mathrm{A}(\alpha)$ can be, look at the expression inside the square root in the denominator and regard it as a function of β, defined to be α^2. This expression is

$$\mathrm{f}(\beta) \;=\; \beta^2 \;+\; [\frac{\mathrm{b}^2}{\mathrm{M}^2}-2\omega^2]\beta \;+\; \omega^4,$$

which is a quadratic with minimum at $\omega^2 \;-\; b^2/2M^2$; we may assume that b is so small that this is positive. [If b is not so small, then actually the ship is in no danger — left to you as an exercise.] Thus the most dangerous value of α is just

$$\alpha_{\mathrm{danger}} \;=\; \sqrt{\omega^2 \;-\; (\mathrm{b}^2/2\mathrm{M}^2)}$$

and simple algebra shows that [remembering the definition of ω]

$$\mathrm{A}(\alpha_{\mathrm{danger}}) \;=\; 2\mathrm{MF_0/b}\sqrt{4\rho\mathrm{MAg} \;-\; \mathrm{b}^2}$$

So to design a ship, we need a good estimate of the largest possible value of $\mathrm{F_0}$ [see http://en.wikipedia.org/wiki/Rogue_wave], we need to measure b and ρ, and then we should choose A and M in such a way that

$$2\mathrm{MF_0/b}\sqrt{4\rho\mathrm{MAg} \;-\; \mathrm{b}^2} \;<\; \mathrm{H},$$

because then, even in the worst possible case, the ship will never go so far down that the sea comes over the deck.