- **Multiple Pole Case**

If $D(s)$ has multiple roots, i.e., it contains factors of the form $(s+p_n)^r$, we say that $-p_n$ is a ***multiple pole of $\boldsymbol{F(s)}$ with multiplicity $\boldsymbol{r}$***. The expansion of $F(s)$ will consist of terms of the form

$$\frac{\gamma_1}{s+p_n}+\frac{\gamma_2}{(s+p_n)^2}+\cdots+\frac{\gamma_r}{(s+p_n)^r} \tag{6.19}$$

where

$$\gamma_{r-k}=\frac{1}{k!}\frac{d^k}{ds^k}\left[(s+p_n)^r F(s)\right]\Big|_{s=-p_n} \quad ; \qquad k=0,1,\cdots,r-1. \tag{6.20}$$

Example 6.4:

$$F(s)=\frac{s^2+2s+5}{(s+3)(s+5)^2}=\frac{\alpha_1}{(s+3)}+\frac{\gamma_1}{(s+5)}+\frac{\gamma_2}{(s+5)^2}$$

Using $(\mathbf{6.18})$: $\alpha_1=(s+3)F(s)\big|_{s=-3}=2$

Errata

Using $\underbrace{(\mathbf{6.20})}_{\text{with } k=0}$: $\gamma_2=(s+5)^2F(s)\Big|_{s=-5}=-10$

Using $\underbrace{(\mathbf{6.20})}_{\text{with } k=1}$: $\gamma_1=\dfrac{d}{ds}\left[(s+5)^2F(s)\right]\Big|_{s=-5}=\dfrac{d}{ds}\left[\dfrac{s^2+2s+5}{s+3}\right]\Bigg|_{s=-5}=\dfrac{s^2+6s+1}{(s+3)^2}\Bigg|_{s=-5}=-1$

$$\therefore F(s)=\frac{2}{(s+3)}-\frac{1}{(s+5)}-\frac{10}{(s+5)^2} \quad \Rightarrow \quad \left(f(t)=\mathcal{L}^{-1}\{F(s)\}=\left(2e^{-3}-e^{-5}-10te^{-5t}\right)u(t)\right)$$

B. $F(s)$ *is an Improper Rational Function* $(M \geq N)$

If $M \geq N$, we can apply long division to express $F(s)$ in the form

$$F(s)=\frac{N(s)}{D(s)}=Q(s)+\frac{R(s)}{D(s)} \tag{6.21}$$

such that the $\begin{cases}\text{Quotient} \quad : Q(s) \text{ is a polynomial in } s \text{ with degree } (M-N), \\ \text{Remainder} : R(s) \text{ is a polynomial in } s \text{ with degree strictly less than } N.\end{cases}$

Errata

The inverse Laplace transform of $R(s)/D(s)$, which is now a ***proper rational function***, can be computed by first expanding into partial fractions.

The inverse Laplace transform of $Q(s)$ can be computed by using

$$\mathcal{L}^{-1}\left\{s^k\right\}=\frac{d^k}{dt^k}\delta(t); \quad k=0,1,2,\cdots\cdots\cdots. \tag{6.22}$$

Example 6.5:

$$F(s)=\underbrace{\frac{2s^2+10s+10}{(s+1)(s+3)}=2+\frac{2s+4}{(s+1)(s+3)}}_{\text{by long division}}$$

$$\therefore F(s)=2+\frac{1}{s+1}+\frac{1}{s+3} \rightarrow \left(f(t)=\mathcal{L}^{-1}\{F(s)\}=2\delta(t)+e^{-t}u(t)+e^{-3t}u(t)\right)$$

6.4 Relationship between the Fourier Transform and the Laplace Transform

Fourier Transform }: $$\Im\{f(t)\} = \int_{-\infty}^{\infty} f(t)\exp(-j\omega t)\,dt \tag{6.23}$$

Bilateral Laplace Transform }: $$\tilde{F}(s) = \int_{-\infty}^{\infty} f(t)\exp(-st)\,dt \tag{6.24}$$

Unilateral Laplace Transform }: $$F(s) = \int_{0^-}^{\infty} f(t)\exp(-st)\,dt \tag{6.25}$$

Errata

- From (6.23) and (6.24), we see that the Fourier transform is a special case of the **bilateral** Laplace transform in which $\boldsymbol{s} = \boldsymbol{j\omega}$, that is

$$\Im\{f(t)\} = \tilde{F}(s)\Big|_{s=j\omega} \tag{6.26}$$

- Setting $\boldsymbol{s} = \boldsymbol{\sigma} + \boldsymbol{j\omega}$ in (6.24), we have

$$\begin{aligned}\tilde{F}(\sigma + j\omega) &= \int_{-\infty}^{\infty} f(t)\exp\big(-(\sigma + j\omega)t\big)\,dt = \int_{-\infty}^{\infty} \big[f(t)\exp(-\sigma t)\big]\exp(-j\omega t)\,dt \\ &= \Im\{f(t)\exp(-\sigma t)\}\end{aligned} \tag{6.27}$$

which shows that the **bilateral** Laplace transform of $f(t)$ can be viewed as the Fourier transform of $f(t)e^{-\sigma t}$.

- Considering the Laplace transform as a generalization of the Fourier transform where the frequency variable is generalized from $j\omega$ to $s=\sigma+j\omega$, the complex variable $\boldsymbol{s}$ is often referred to as the ***complex frequency***.

Errata

- It should not be automatically assumed that the Fourier transform of $f(t)$ is the Laplace transform with s replaced by $j\omega$. If $\boldsymbol{f(t)}$ **is absolutely integrable, i.e.** $\underbrace{\int_{-\infty}^{\infty}|\boldsymbol{f(t)}|\boldsymbol{dt}<\infty}_{\text{Dirichlet's }4^{\text{th}}\text{-condition}}$ **,** then the Fourier transform of $f(t)$ can be obtained from its bilateral Laplace transform with s replaced by $j\omega$. This is not generally true if $f(t)$ is not absolutely integrable.

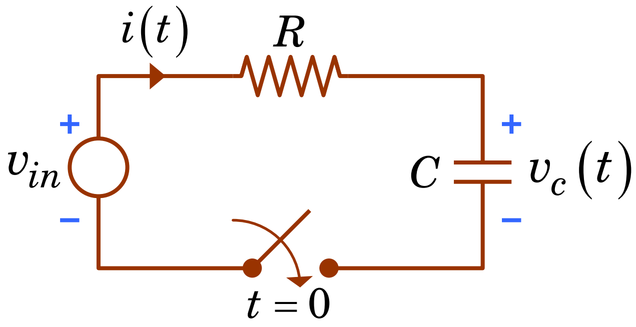
The above relationship between the Fourier Transform and the *Bilateral* Laplace Transform extends fully to *Unilateral* Laplace transform if $\boldsymbol{f(t)}$ is, in addition, a right-sided function.

Example 6.6:

	$f(t)$	$F(s)$	$\Im\{f(t)\}$	Right-sided?	Absolutely Integrable?	$\Im\{f(t)\}=F(j\omega)$
Unit Impulse	$\delta(t)$	1	1	Yes	Yes	Yes
Unit Step	$u(t)$	$\frac{1}{s}$	$\pi\delta(\omega)+\frac{1}{j\omega}$	Yes	No	No
Exponential	$\exp(-\lvert\alpha\rvert t)u(t)$	$\frac{1}{s+\lvert\alpha\rvert}$	$\frac{1}{j\omega+\lvert\alpha\rvert}$	Yes	Yes	Yes

Example 6.7: Series RC Circuit

For the RC circuit shown, find the voltage $v_c(t)$ across the capacitor C.



$$\underbrace{RC\frac{dv_c(t)}{dt}}_{i(t)} + v_c(t) = v_{in}$$

$$sRCV_c(s) - RCv_c(0^-) + V_c(s) = \frac{v_{in}}{s}$$

$$V_c(s) = \frac{RCv_c(0^-)}{sRC+1} + \frac{v_{in}}{s(sRC+1)} = \frac{RCv_c(0^-)}{sRC+1} + \frac{v_{in}}{s} - \frac{v_{in}RC}{sRC+1} \quad (\clubsuit)$$

$$\left.\begin{aligned} v_c(t) &= \mathcal{L}^{-1}\{V_c(s)\} = \mathcal{L}^{-1}\left\{\frac{v_c(0^-)-v_{in}}{s+\frac{1}{RC}}\right\} + \mathcal{L}^{-1}\left\{\frac{v_{in}}{s}\right\} \\ &= \left[v_c(0^-) - v_{in}\right]\exp\left(-\frac{t}{RC}\right) + v_{in} \end{aligned}\right\} \quad (\heartsuit)$$

Errata

From $(\clubsuit)$ and $(\heartsuit)$, we observe that: $\underbrace{\lim_{t\to\infty} v_c(t) = \lim_{s\to 0} sV_c(s)}_{\text{Final Value Theorem}} = v_{in}$

RETURN

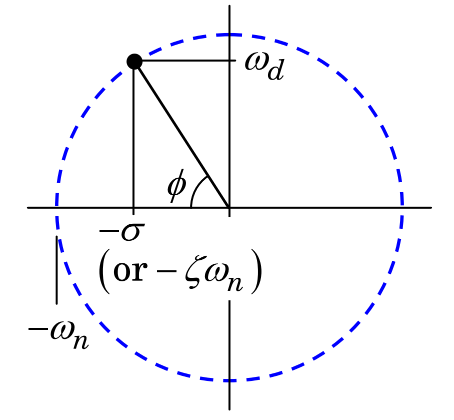
Unilateral Laplace Transform: $X(s)=\int_{0^-}^{\infty} x(t)\exp(-st)\,dt$

ROC added

	$x(t)$	$X(s)$	ROC
Unit Impulse	$\delta(t)$	1	All s
Unit Step	$u(t)$	$1/s$	$\mathrm{Re}[s]>0$
Ramp	$t\,u(t)$	$1/s^2$	$\mathrm{Re}[s]>0$
nth order Ramp	$t^n u(t)$	$\dfrac{n!}{s^{n+1}}$	$\mathrm{Re}[s]>0$

	$x(t)$	$X(s)$	ROC
Exponential	$\exp(-\alpha t)u(t)$	$1/(s+\alpha)$	$\mathrm{Re}[s]>-\mathrm{Re}[\alpha]$
Damped Ramp	$t\exp(-\alpha t)u(t)$	$1/(s+\alpha)^2$	$\mathrm{Re}[s]>-\mathrm{Re}[\alpha]$
Cosine	$\cos(\omega_o t)u(t)$	$s/(s^2+\omega_o^2)$	$\mathrm{Re}[s]>0$
Sine	$\sin(\omega_o t)u(t)$	$\omega_o/(s^2+\omega_o^2)$	$\mathrm{Re}[s]>0$

	$x(t)$	$X(s)$	ROC
Damped Cosine	$\exp(-\alpha t)\cos(\omega_o t)u(t)$	$(s+\alpha)/\left[(s+\alpha)^2+\omega_o^2\right]$	$\mathrm{Re}[s]>-\mathrm{Re}[\alpha]$
Damped Sine	$\exp(-\alpha t)\sin(\omega_o t)u(t)$	$\omega_o/\left[(s+\alpha)^2+\omega_o^2\right]$	$\mathrm{Re}[s]>-\mathrm{Re}[\alpha]$

	$x(t)$	$X(s)$	
Step response of 1st order system	$\left[1-\exp(-at)\right]u(t)$	$\dfrac{a}{s(s+a)}$	
Step response of 2nd order underdamped system	$K\left\{1-\dfrac{\exp(-\omega_n\zeta t)}{\sqrt{1-\zeta^2}}\sin\left[\omega_n\sqrt{1-\zeta^2}\,t+\phi\right]\right\}u(t)$	$\dfrac{K\omega_n^2}{s\left(s^2+2\zeta\omega_n s+\omega_n^2\right)}$	
	$K\left\{1-\dfrac{\sqrt{\sigma^2+\omega_d^2}\,\exp(-\sigma t)}{\omega_d}\sin\left[\omega_d t+\phi\right]\right\}u(t)$	$\dfrac{K\omega_d^2}{s\left[(s+\sigma)^2+\omega_d^2\right]}$	