CHAPTER 2. OSCILLATIONS

2.1. THE HARMONIC OSCILLATOR

Consider the pendulum shown.

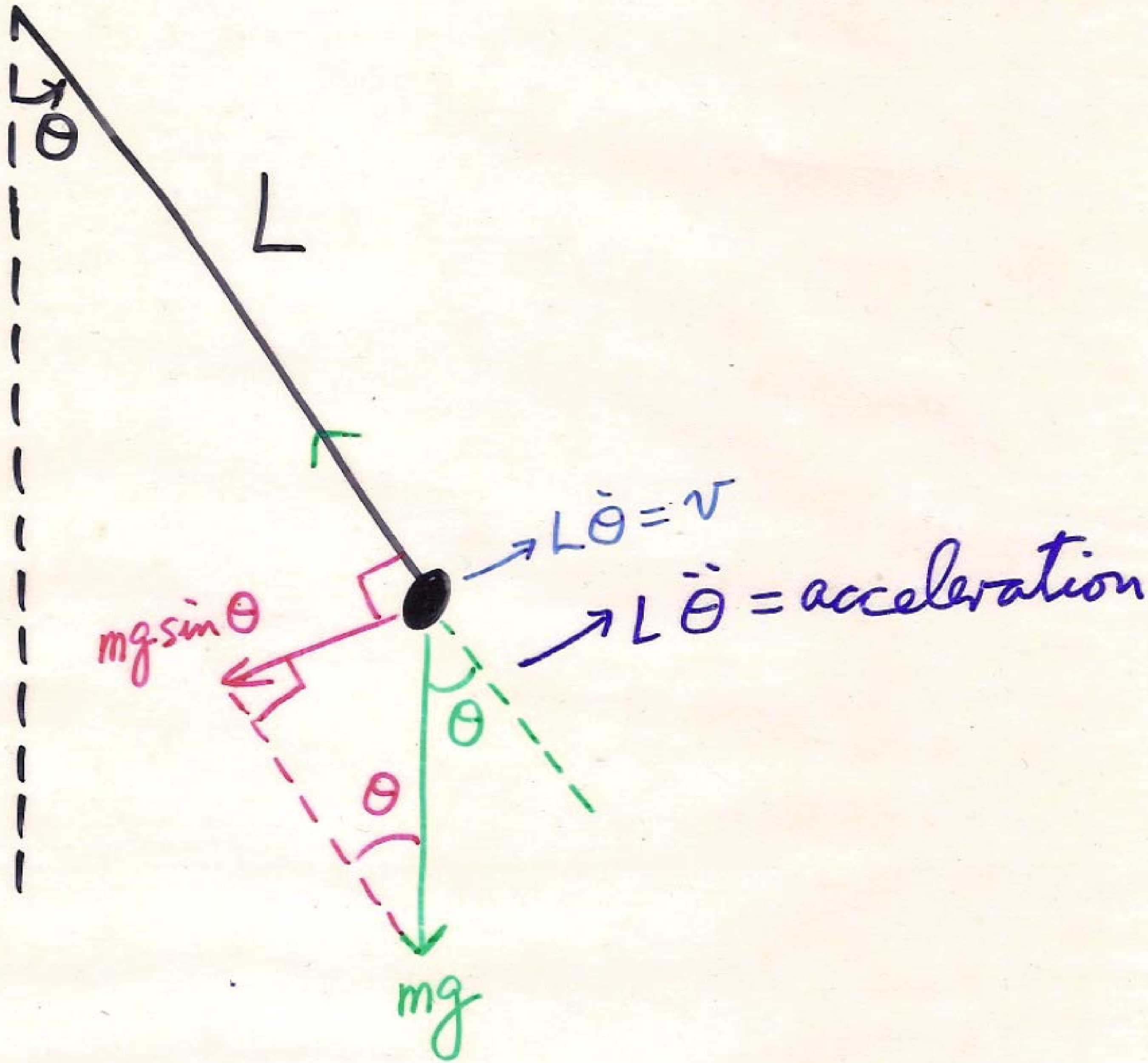
The small object, mass m,

at the end of the pendulum,

is moving on a circle of radius

L, so the component of its velocity

tangential to the circle is $L\dot{\theta}$



θ
L
$L\dot{\theta} = v$
$L\ddot{\theta}$ = acceleration
$mg\sin\theta$
θ
θ
mg

$$mL\ddot{\theta} = -mg\sin\theta$$

The pendulum equation:

$$\ddot{\theta} = -\frac{g}{L}\sin\theta$$

$\because$ g and L are both +ve,

we can define $\omega = \sqrt{\frac{g}{L}}$

$$\boxed{\ddot{\theta} = -\omega^2 \sin\theta} \quad \text{-----} (1)$$

Observe: $\theta \equiv 0$ and $\theta \equiv \pi$ are solutions of 1.

Proof. $\theta \equiv 0 \Rightarrow$ L.H.S. $= \ddot{\theta} = 0$

R.H.S. $= -\omega^2 \sin 0 = 0$

$\theta \equiv \pi$ is similar.

$\theta \equiv 0$ is called an <u>equilibrium solution</u> of ①.

This means that if you set $\theta = 0$ initially, then θ will remain at 0 and the pendulum will not move — which of course we know is correct.

Similarly for $\theta \equiv \pi$.

IN THEORY, if you set the pendulum EXACTLY at $\theta = \pi$, then it will remain in that position forever. IN REALITY, of course, it won't! Because the slightest puff of air will knock it over! So this equilibrium is very different from the one at $\theta = 0$. This is a very important distinction!

Equilibrium is said to be STABLE if a SMALL push away from equilibrium REMAINS small. If the small push tends to grow large, then the equilibrium is UNSTABLE. Obviously this is important for engineers! Especially you want vibrations of structures, engines, etc to remain small.

Let's look at $\theta = \pi$.

In a small neighborhood of π:

$$\frac{\sin\theta - \sin\pi}{\theta - \pi} \simeq \left[\frac{d}{d\theta}\sin\theta\right]_{\theta=\pi}$$

$$= \left[\cos\theta\right]_{\theta=\pi}$$

$$= -1$$

$$\therefore \quad \sin\theta \simeq -(\theta - \pi)$$

$$① \Rightarrow \ddot{\theta} = -\omega^2 \sin\theta$$

$$\approx -\omega^2 \left[-(\theta - \pi) \right]$$

$$= \omega^2 (\theta - \pi)$$

Let $\phi = \theta - \pi$

$\therefore \ddot{\phi} = \ddot{\theta}$

$\therefore$ 1 can be approximated by the linear equation

$$\ddot{\phi} = \omega^2 \phi \quad \text{----} \quad (2)$$

Characteristic equation is

$$\lambda^2 = \omega^2$$

$$\therefore \lambda = \pm \omega$$

$$\therefore \phi = Ae^{\omega t} + Be^{-\omega t}$$

i.e. $\theta = \phi + \pi$

$$= Ae^{\omega t} + Be^{-\omega t} + \pi$$

As you know, the exponential function grows very quickly; so even if θ is close to π initially, it won't stay near to it very long! Very soon, θ will arrive either at $\theta = 0$ or 2π, far away from $\theta = \pi$. The equilibrium is UNSTABLE!

How long does it take for things to get out of control? That is determined by $\sqrt{g/L}$ or rather $\sqrt{L/g}$, which has units of TIME.

Since $\theta = Ae^{\omega t} + Be^{-\omega t} + \pi$

$$= Ae^{\sqrt{\frac{g}{L}}t} + Be^{-\sqrt{\frac{g}{L}}t} + \pi$$

$\therefore |\theta|$ is large when $\sqrt{\frac{g}{L}}t$ is large

(Assume that $A \neq 0$)

Observe that for $\sqrt{\frac{g}{L}}t$ to be large,

t has to be large if L is large

i.e. it takes longer to fall over if

L is large.

EXAMPLE:

An eccentric professor likes to balance pendula near their unstable equilibrium point. In a given performance, the pendulum is initially slightly away from that point, and is initially at rest. The prof's skill is such that he can stop the pendulum from falling provided that the angular deviation from the vertical angle does not double. If the shortest pendulum for which he can perform this trick is 9.8 centimetres long, estimate the speed of his reflexes.

Solution: The problem is saying that the angle ϕ [the deviation from the vertical] is initially very small, and its initial rate of change is zero. So $\phi(0) = \epsilon$ [some very small number] and $\dot{\phi}(0) = 0$.

Recall: $\phi = Ae^{\omega t} + Be^{-\omega t}$

$\therefore \dot{\phi} = A\omega e^{\omega t} - B\omega e^{-\omega t}$

$\phi(0) = \varepsilon \Rightarrow \varepsilon = A + B$

$\dot{\phi}(0) = 0 \Rightarrow 0 = A\omega - B\omega \Rightarrow 0 = A - B$

$\therefore A = \frac{1}{2}\varepsilon, \quad B = \frac{1}{2}\varepsilon$

i.e. $\phi = \frac{1}{2}\varepsilon e^{\omega t} + \frac{1}{2}\varepsilon e^{-\omega t}$

$$= \varepsilon \frac{e^{\omega t} + e^{-\omega t}}{2}$$

$$= \varepsilon \cosh \omega t$$

now $L = 9.8 \text{ cm} = 0.098 \text{ m}$

$g = 9.8 \text{ m/sec}^2$

$$\therefore \quad \omega = \sqrt{\frac{g}{L}} = \sqrt{\frac{9.8}{0.098}} = 10/\text{sec}$$

$$\therefore \quad \underline{\phi = \varepsilon \cosh 10t}$$

Suppose ϕ doubles at $t = \tau$,

i.e. $\phi(\tau) = 2\varepsilon$

$\therefore \quad 2\varepsilon = \varepsilon \cosh 10\tau$

$\therefore \quad \tau = \frac{1}{10}\cosh^{-1}(2)$

$\simeq \underline{\underline{0.132 \text{ sec.}}}$

Now what about $\theta = 0$?

In a small neighborhood of 0:

For $|\theta|$ small, we have

$$\frac{\sin\theta - \sin 0}{\theta - 0} \approx \left[\frac{d}{d\theta}\sin\theta\right]_{\theta=0}$$

$$\therefore \quad \frac{\sin\theta}{\theta} \approx 1$$

$$\therefore \quad \sin\theta \approx \theta$$

$$① \Rightarrow \ddot{\theta} = -\omega^2 \sin\theta \approx -\omega^2\theta$$

i.e. (1) can be approximated by the linear equation

$$\ddot{\theta} = -\omega^2 \theta \quad \cdots\cdots (3)$$

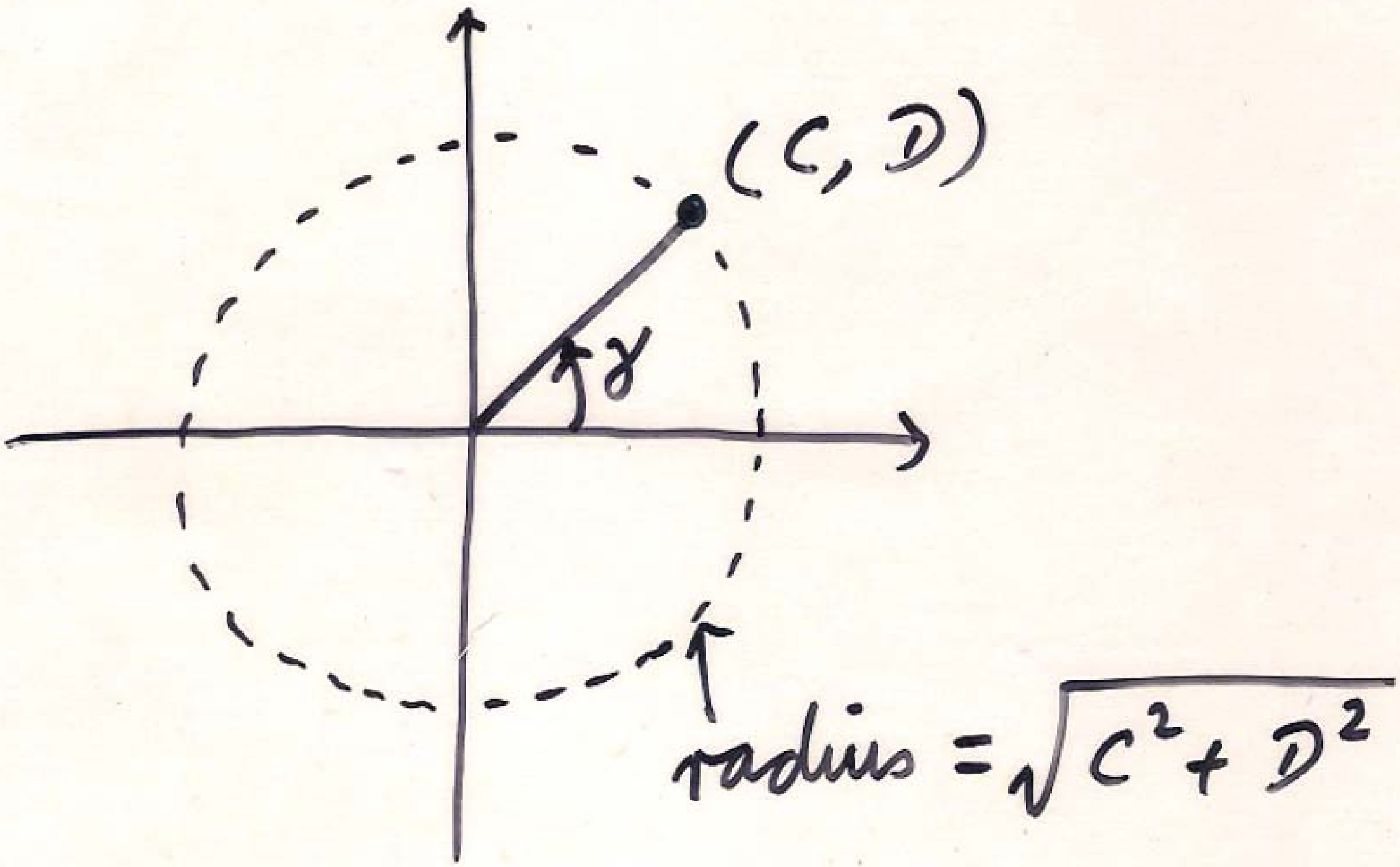
Characteristic equation:

$$\lambda^2 = -\omega^2$$

$$\therefore \quad \lambda = \pm i\omega$$

$$\therefore \quad \theta = C\cos\omega t + D\sin\omega t$$

Define the angle γ by



$$\therefore \quad C = (\cos\gamma)\sqrt{C^2+D^2}$$

$$D = (\sin\gamma)\sqrt{C^2+D^2}$$

$$\therefore \quad \theta = \{\cos\omega t\cos\gamma + \sin\omega t\sin\gamma\}\sqrt{C^2+D^2}$$

$$= \{\cos(\omega t - \gamma)\}\sqrt{C^2+D^2}$$

Let $A = \sqrt{C^2+D^2}$

$$\underline{\underline{\theta = A\cos(\omega t - \gamma)}}$$

A is the amplitude.

Note : $|\theta| = |A\cos(\omega t - \gamma)| \leq |A| = A$

and $\max \theta = A$

$\min \theta = -A$.

Note : The motion is periodic and this is called a simple harmonic motion.

Let T = one period.

$\therefore$ when t goes from 0 to T,

$\omega t - \gamma$ increases by 2π

i.e. $\{\omega T - \gamma\} - \{\omega(0) - \gamma\} = 2\pi$

$$\therefore \quad \omega T = 2\pi$$

$$\therefore \quad \underline{\underline{T = \frac{2\pi}{\omega}}}$$

General case:

$$\ddot{x} = f(x) \quad \text{-------} \quad (*)$$

where f is differentiable and $f(0) = 0$.

Note 1: $X \equiv 0$ is an <u>equilibrium solution</u>.

Proof: Substitute $x \equiv 0$,

$$\text{L.H.S. of } (*) = 0$$

$$\text{R.H.S. of } (*) = f(0) = 0$$

$\therefore X \equiv 0$ is a solution of $(*)$.

Note 2: For $|x|$ small:

$$\frac{f(x)-f(0)}{x-0} \approx f'(0)$$

$$\therefore \quad f(x) \approx \{f'(0)\}\, x$$

$$\therefore \ (*) \approx \quad \ddot{x} = f'(0)\, x \quad \text{-----} \quad (4)$$

Characteristic equation of (4) is

$$\lambda^2 = f'(0)$$

Case 1: $f'(0) > 0$

$$\therefore \lambda = \pm\sqrt{f'(0)}$$

$$\therefore x = c_1 e^{\sqrt{f'(0)}\,t} + c_2 e^{-\sqrt{f'(0)}\,t}$$

exponential growth

$\therefore x \equiv 0$ is an <u>unstable equilibrium</u>.

Case 2: $f'(0) < 0$

$$\therefore \lambda = \pm i\sqrt{-f'(0)}$$

$$\therefore x = C\cos\sqrt{-f'(0)}\,t + D\sin\sqrt{-f'(0)}\,t$$

$$= A\cos\left(\sqrt{-f'(0)}\,t - \gamma\right)$$

Simple harmonic motion with

$$\text{period} = \frac{2\pi}{\sqrt{-f'(0)}}$$

$x \equiv 0$ is a <u>stable equilibrium</u>.

Summary

For $\ddot{x} = f(x)$

and $f(c) = 0$

(i) $f'(c) > 0 \Rightarrow x \equiv c$ is an unstable equilibrium.

(ii) $f'(c) < 0 \Rightarrow x \equiv c$ is a stable equilibrium.

The motion is a S.H.M. with period $= \dfrac{2\pi}{\sqrt{-f'(c)}}$

2.2. OSCILLATOR PHASE PLANE.

Phase plane diagram:

Consider a S.H.M. $\ddot{x} = -\omega^2 x$ ---- (5)

From the calculation in the simple pendulum, we have

$$x = A\cos(\omega t - \delta)$$

$$\therefore \quad x = A\cos(\delta - \omega t)$$

Let $y = \dot{x} = -[A\sin(\delta - \omega t)](-\omega)$

$$= \omega A \sin(\delta - \omega t)$$

$$\therefore \begin{cases} x = A\cos(\delta - \omega t) \\ y = \omega A \sin(\delta - \omega t) \end{cases}$$

We can think of this as a parametric equation of a curve on the x-y plane (called the <u>phase plane</u> for ⑤).

Let $\psi = \delta - \omega t$

$$\begin{cases} \dfrac{x}{A} = \cos\psi \\ \\ \dfrac{y}{\omega A} = \sin\psi \end{cases}$$

$$\therefore \quad \frac{x^2}{A^2} + \frac{y^2}{\omega^2 A^2} = 1$$

This curve is an ellipse.

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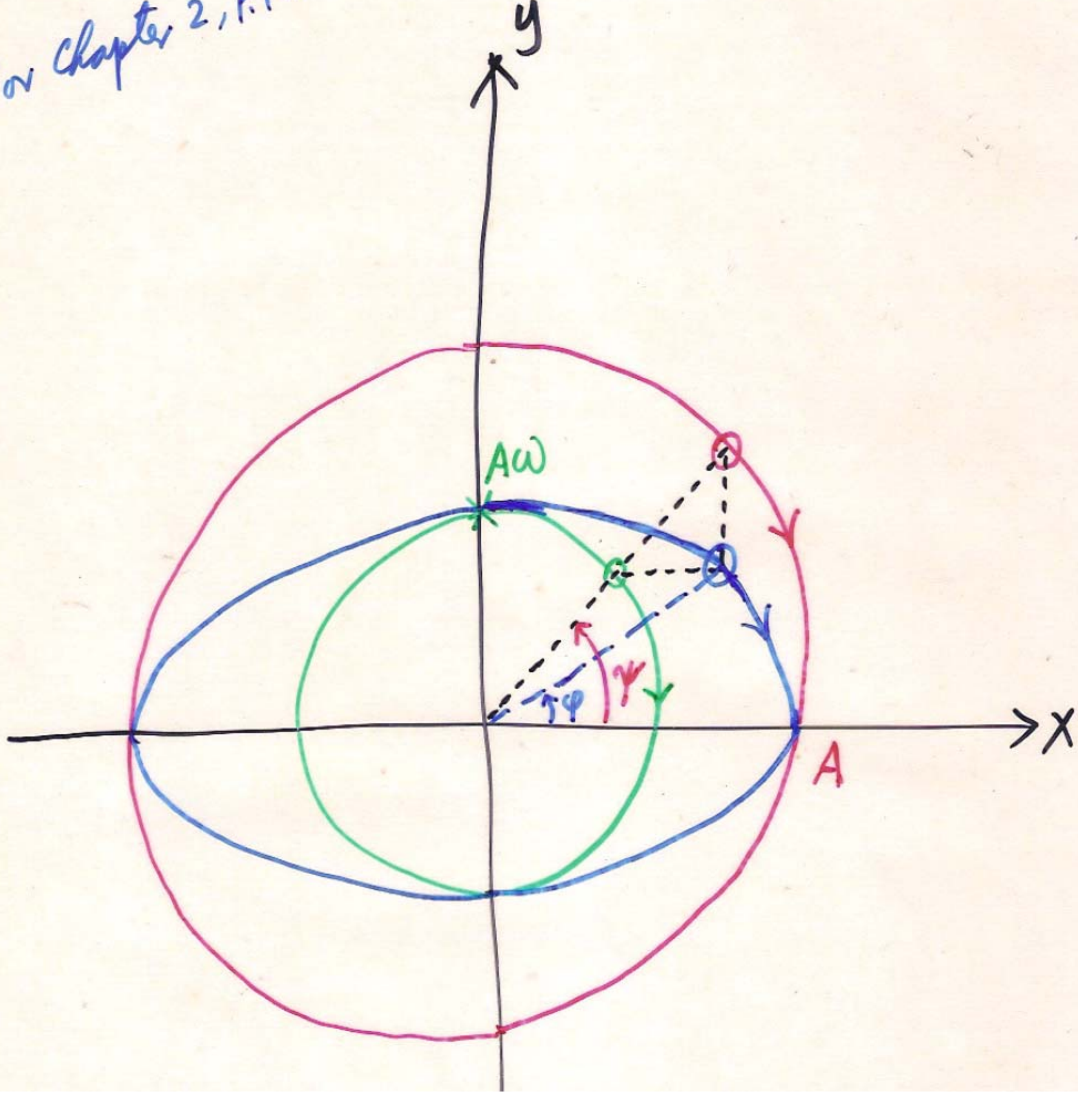
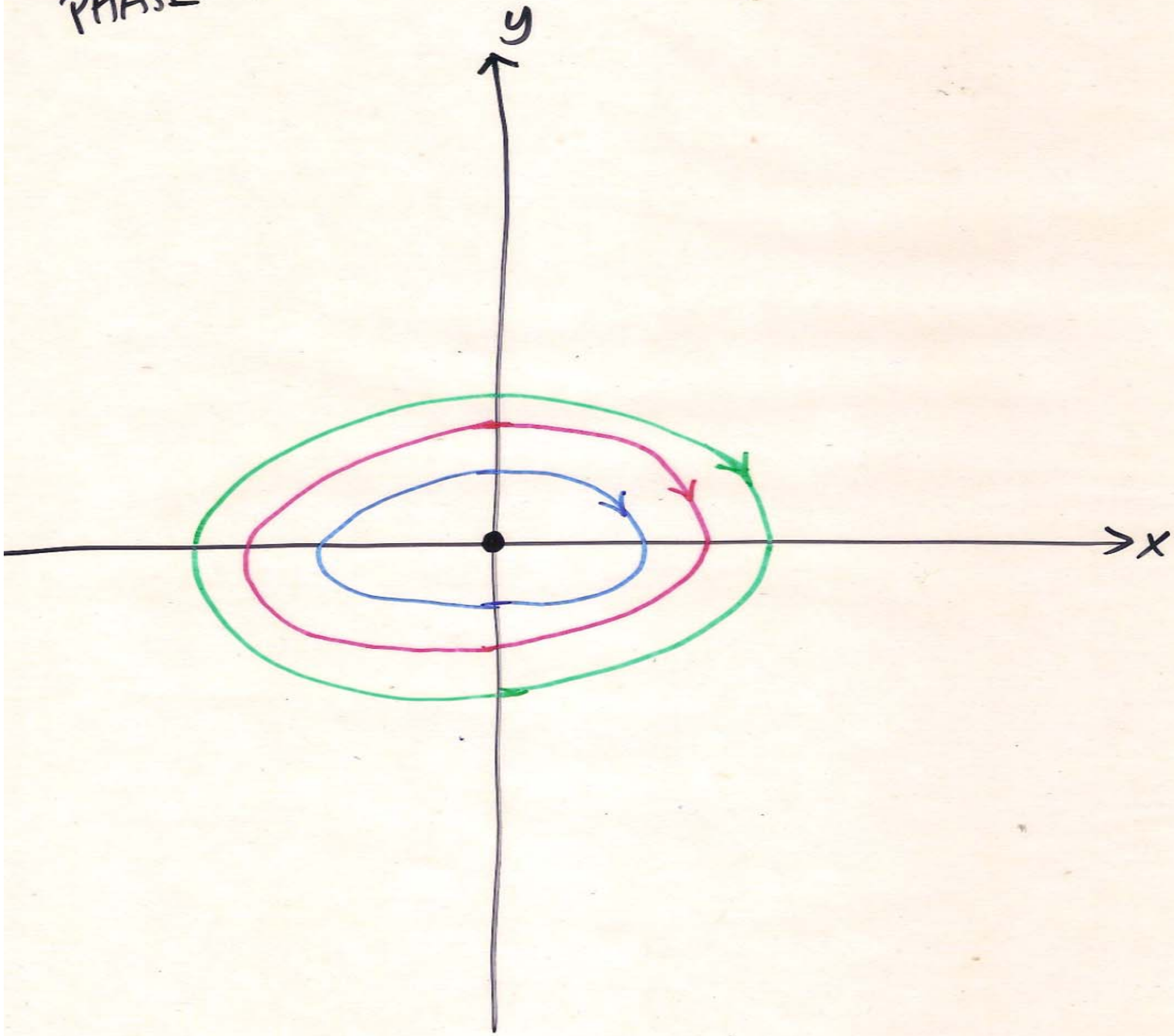


$\psi = \delta - \omega t$

$\therefore \psi \downarrow \ (\because \omega > 0)$

$\Rightarrow \varphi \downarrow$ (see diagram above)

PHASE DIAGRAM of $\ddot{x} = -\omega^2 x$



Origin = stable equilibrium

Example of an unstable case:

$$\begin{cases} \ddot{x} = \omega^2 x \\ x(0) = \alpha \\ \dot{x}(0) = 0 \end{cases}$$

Characteristic equation is

$$\lambda^2 = \omega^2$$

$$\therefore \lambda = \pm \omega$$

$$\therefore x = c_1 e^{\omega t} + c_2 e^{-\omega t}$$

$$\dot{x} = \omega c_1 e^{\omega t} - \omega c_2 e^{-\omega t}$$

$$x(0)=\alpha \Rightarrow \alpha = C_1 + C_2$$

$$\dot{x}(0)=0 \Rightarrow 0 = \omega C_1 - \omega C_2 \Rightarrow C_1 = C_2$$

$$\therefore \quad C_1 = C_2 = \frac{\alpha}{2}$$

$$\therefore \quad x = \frac{\alpha}{2}\left(e^{\omega t} + e^{-\omega t}\right)$$

i.e. $x = \alpha \cosh \omega t$

Let $y = \dot{x}$

$\therefore \quad y = \dot{x} = \alpha\omega \sinh \omega t$

$$\therefore \quad \begin{cases} \dfrac{x}{\alpha} = \cosh \omega t \\[2em] \dfrac{y}{\alpha\omega} = \sinh \omega t \end{cases} \qquad \text{for } \alpha \neq 0.$$

$$\therefore \quad \frac{x^2}{\alpha^2} - \frac{y^2}{\alpha^2\omega^2} = 1$$

a hyperbola.

Note: at $t = 0$, we have $x = \alpha$, $y = 0$

$\therefore$ the curve starts at $(\alpha, 0)$.

$$\begin{cases} x = \alpha \cosh \omega t \\ y = \alpha\omega \sinh \omega t \end{cases}$$

Note that $\omega > 0$ and $t \geqq 0$

$\therefore \cosh \omega t > 0$ and $\sinh \omega t \geqq 0$.

The curve starts at time $t = 0$

i.e. the starting point is $(\alpha, 0)$.

Case 1: $\alpha > 0$

$t > 0 \Rightarrow x > 0$ and $y > 0$

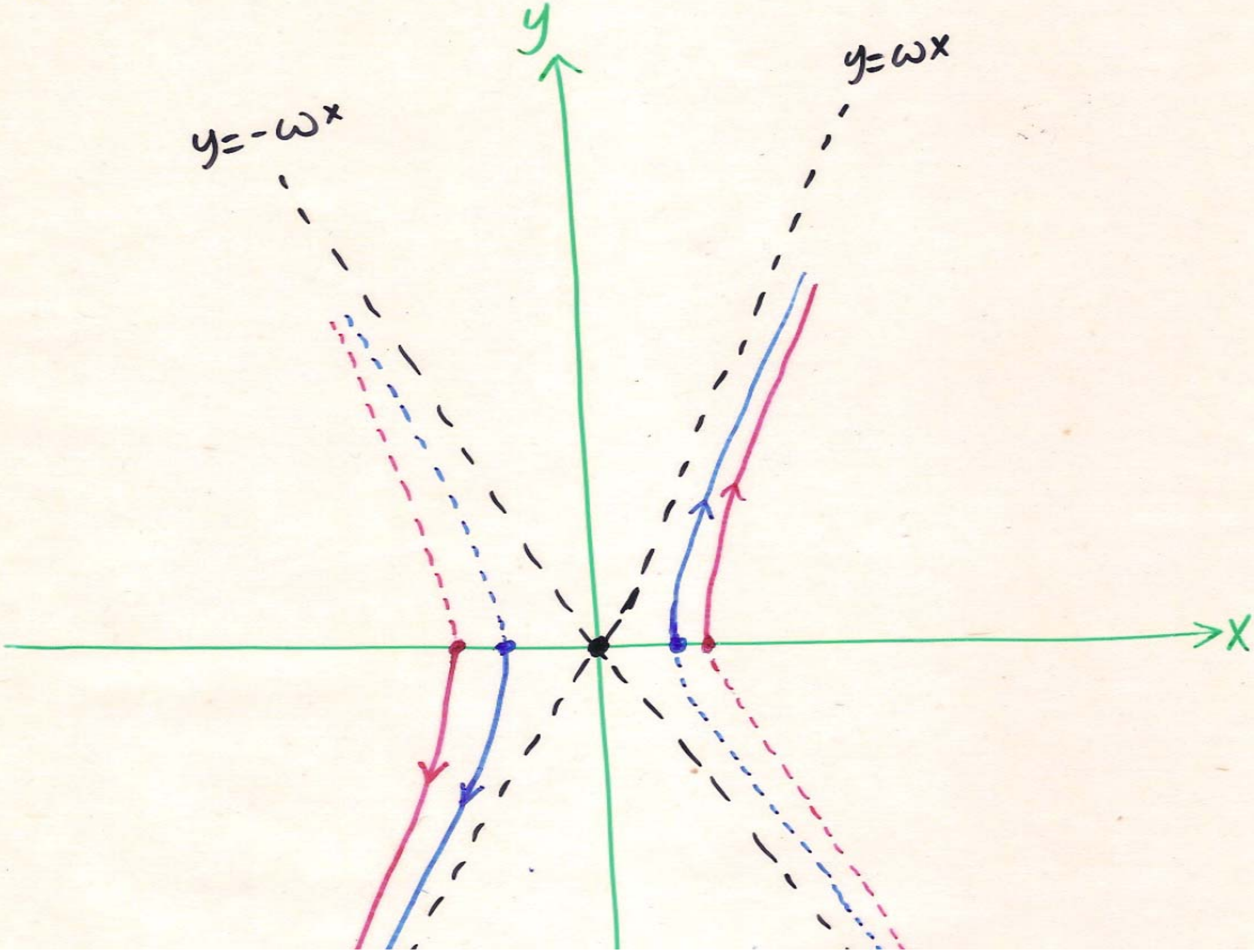
$\therefore (x, y)$ is in the first quadrant.

Case 2: $\alpha < 0$

$t > 0 \Rightarrow x < 0$ and $y < 0$

$\therefore (x, y)$ is in the 3rd quadrant.

Phase diagram of $\ddot{x} = \omega^2 x$, $x(0)=\alpha$, $\dot{x}(0)=0$



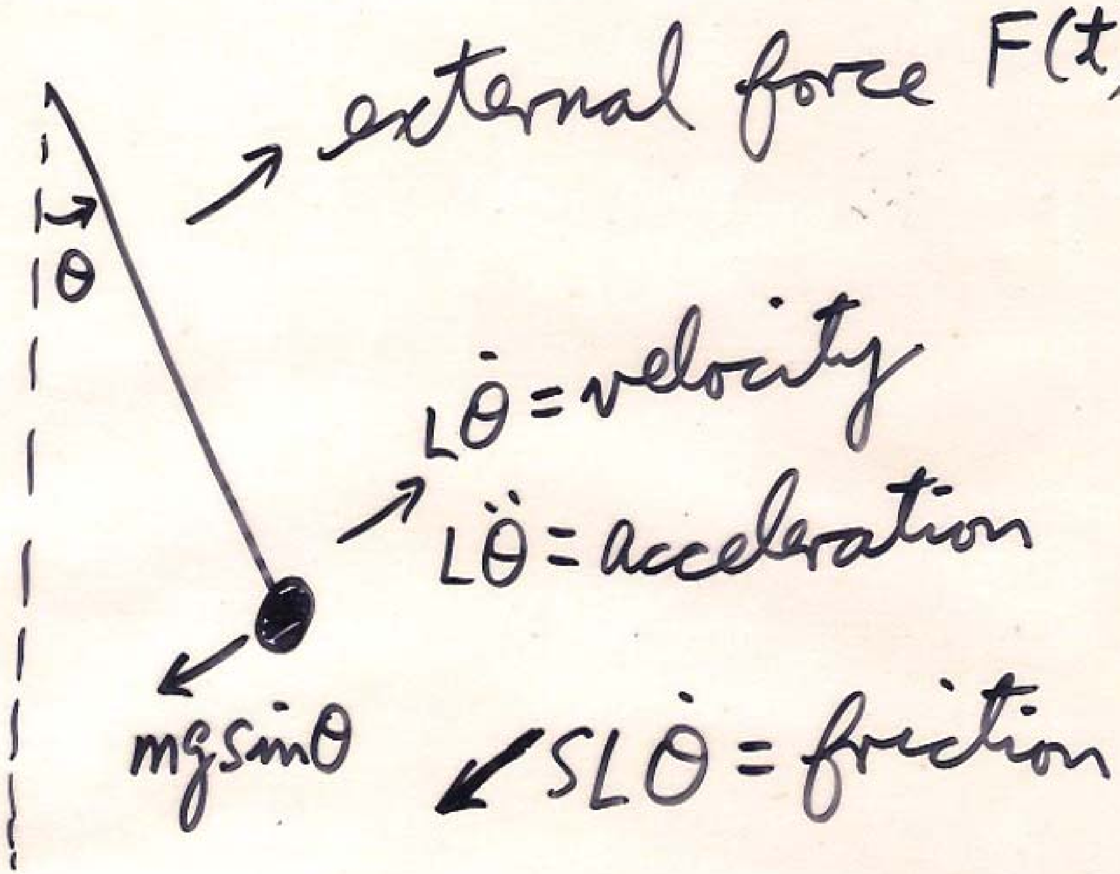
2.3. DAMPED, FORCED OSCILLATORS.

When an object moves fairly slowly through air, the RESISTANCE DUE TO FRICTION is approximately proportional to its speed, and of course in the OPPOSITE DIRECTION. So in the case of the pendulum, where the speed of the object is $L\dot{\theta}$, the DAMPING FORCE is

$$-SL\dot{\theta}$$

where S is some positive constant.

We can also attach a motor to the pendulum, that is, an external force $F(t)$ which may depend on time.



$$mL\ddot{\theta} = -mg\sin\theta - SL\dot{\theta} + F(t)$$

$$m\ddot{\theta} + s\dot{\theta} + \frac{mg}{L}\sin\theta = \frac{1}{L}F(t)$$

when $|\theta|$ is small,

$$\therefore \sin\theta \approx \theta$$

$$\therefore \quad m\ddot{\theta} + s\dot{\theta} + \frac{mg}{L}\theta = \frac{1}{L}F(t)$$

The case $F(t) \equiv 0$ is called <u>Damped Harmonic Motion</u>.

The case $F(t) \not\equiv 0$ is called

<u>Forced Damped Harmonic Motion</u>.

2.4. MODELS OF ELECTRICAL CIRCUITS.

For an electrical circuit, voltage drops can occur in

3 ways:

[a] Across a RESISTOR:

$$V = RI$$

where R is a constant called the RESISTANCE, $I(t)$ is the current, and $V(t)$ is the voltage drop. Resistors try to stop currents.

[b] Across an INDUCTOR:

$$V(t) = L\frac{dI}{dt}$$

where L is a constant called the INDUCTANCE.

Inductors try to stop CHANGES of currents.

[c] Across a CAPACITOR:

$$V(t) = \frac{1}{C} \int I(t) dt$$

where C is a constant called the CAPACITANCE.

Capacitors try to stop currents from building up an accumulation of charge.

So in a circuit like the one shown, the source of voltage has to supply

$$V(t) = RI + L\dot{I} + \tfrac{1}{c}\int^{t} I \, dt.$$

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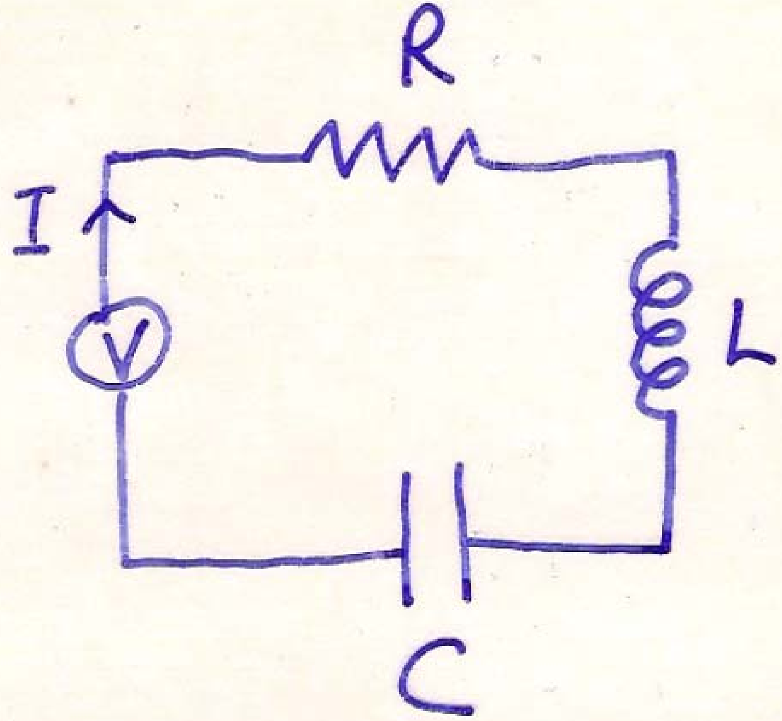


$$V = IR + L\dot{I} + \frac{1}{C}\int_0^t I\,dt$$

Let $Q = \int_0^t I\,dt$ = the charge on the capacitor.

$\therefore \dot{Q} = I$

and $\ddot{Q} = \dot{I}$

$\therefore L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = V$

But this is exactly the same equation as in forced, damped, harmonic motion!

We see that inductance is like mass, resistance is like FRICTION, the voltage is like external force, etc.

2.5. DAMPED, UNFORCED OSCILLATORS.

Damped Harmonic Motion

$$m\ddot{\theta} + s\dot{\theta} + \frac{mg}{L}\theta = 0 \quad \cdots\cdots (*)$$

The characteristic equation is

$$m\lambda^2 + s\lambda + \frac{mg}{L} = 0 \; ; \quad m, s, g, L \text{ all } +ve$$

i.e. $\lambda^2 + \frac{s}{m}\lambda + \frac{g}{L} = 0 \; - - - - \; ①$

Quadratic in λ, so 3 cases: both roots real, both complex, both equal.

[a] BOTH REAL : OVERDAMPING

Example, $\ddot{x} + 3\dot{x} + 2x = 0$

$$\lambda^2 + 3\lambda + 2 = 0 \to \lambda = -1, -2,$$

general solution $B_1 e^{-t} + B_2 e^{-2t}$. The motion very rapidly dies away to zero. Obviously we have too much friction.

[b] BOTH COMPLEX: UNDERDAMPING

Example, $\ddot{x} + 4\dot{x} + 13x = 0$

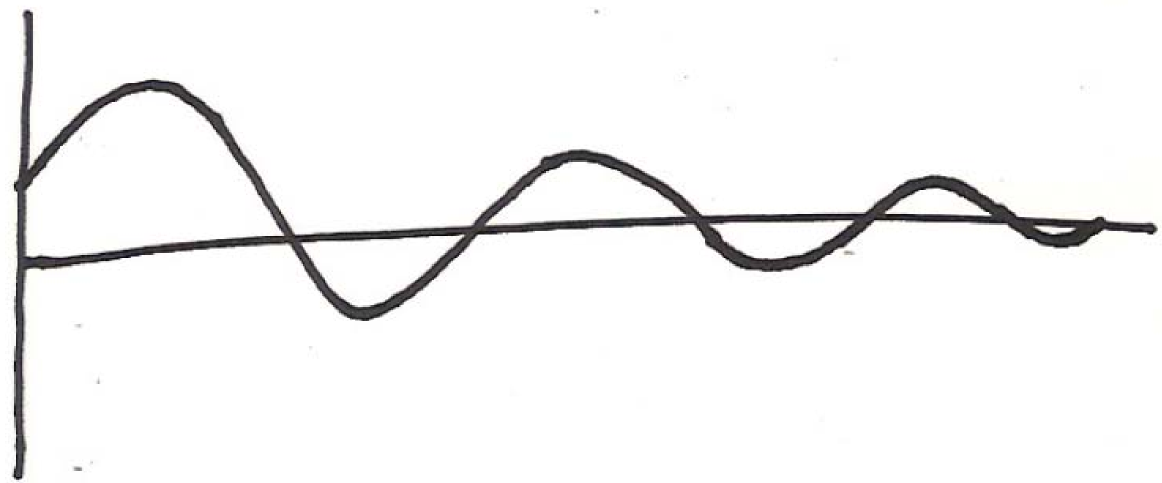
$$\lambda^2 + 4\lambda + 13 = 0 \rightarrow \lambda = -2 \pm 3i$$

general solution $B_1 e^{-2t} \cos(3t) + B_2 e^{-2t} \sin(3t)$, which

can be written as

$$x = Ae^{-2t} \cos(3t - \delta).$$

The graph is obtained by "multiplying together" the graphs of e^{-2t} and $A\cos(3t-\delta)$:



This is like a simple harmonic oscillator such that the amplitude is a function of time.

In general if we have an equation for an unforced, damped harmonic oscillator, such that

$$m\ddot{x} + b\dot{x} + kx = 0,$$

$$m\lambda^2 + b\lambda + k = 0,$$

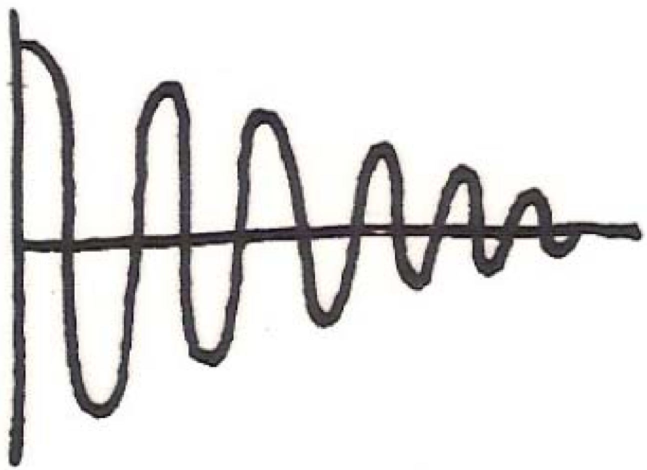
then if the solutions for λ are COMPLEX, we say that the system is UNDERDAMPED. We get

$$x(t) = Ae^{\frac{-bt}{2m}} \cos(\beta t - \delta)$$

where $\beta = \frac{1}{2m}\sqrt{4mk - b^2}$. You can think of this as

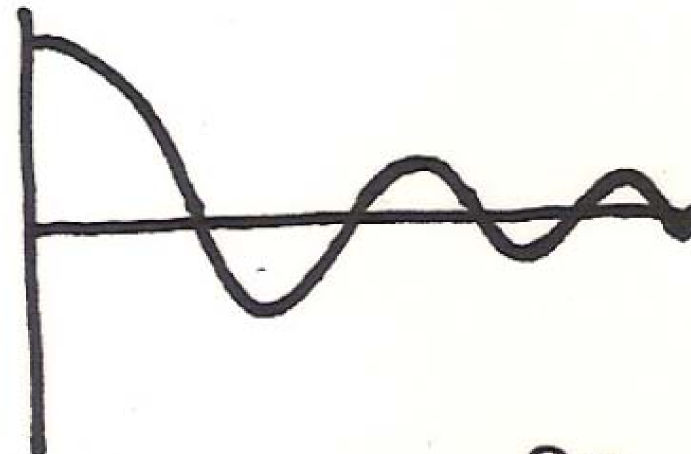
"SHM with frequency β and amplitude $Ae^{\frac{-bt}{2m}}$."

Here β is often called the QUASI-FREQUENCY and $\frac{2\pi}{\beta}$ is the QUASI-PERIOD. Notice that in this problem, UNLIKE in true SHM, there are actually TWO independent time scales: $\frac{2\pi}{\beta}$ has units of time but so does $\frac{2m}{b}$. This second time scale tells you how quickly the amplitude dies out. [They are INDEPENDENT because given m and b you can work out $\frac{2m}{b}$ but not $\frac{2\pi}{\beta}$ [since you have not been given k.]



LARGE $\frac{2m}{b}$

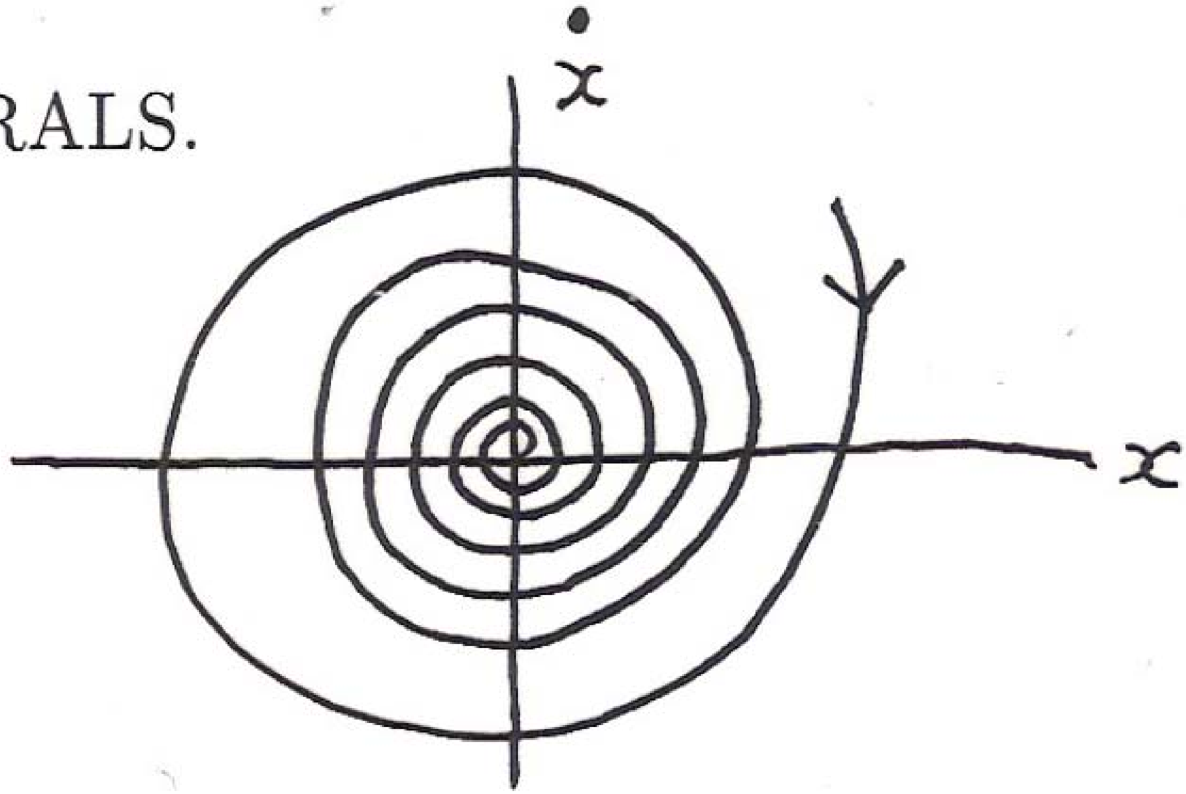
SMALL $\frac{2\pi}{\beta}$



SMALL $\frac{2m}{b}$

LARGE $\frac{2\pi}{\beta}$

Remember that the phase plane diagram for SHM was a set of ellipses. For underdamped harmonic motion, it must be like that, but now the point $(x, \dot{x}) \to (0, 0)$ [Check that $x, \dot{x}$ always $\to 0$ as $t \to \infty$.] So we get SPIRALS.

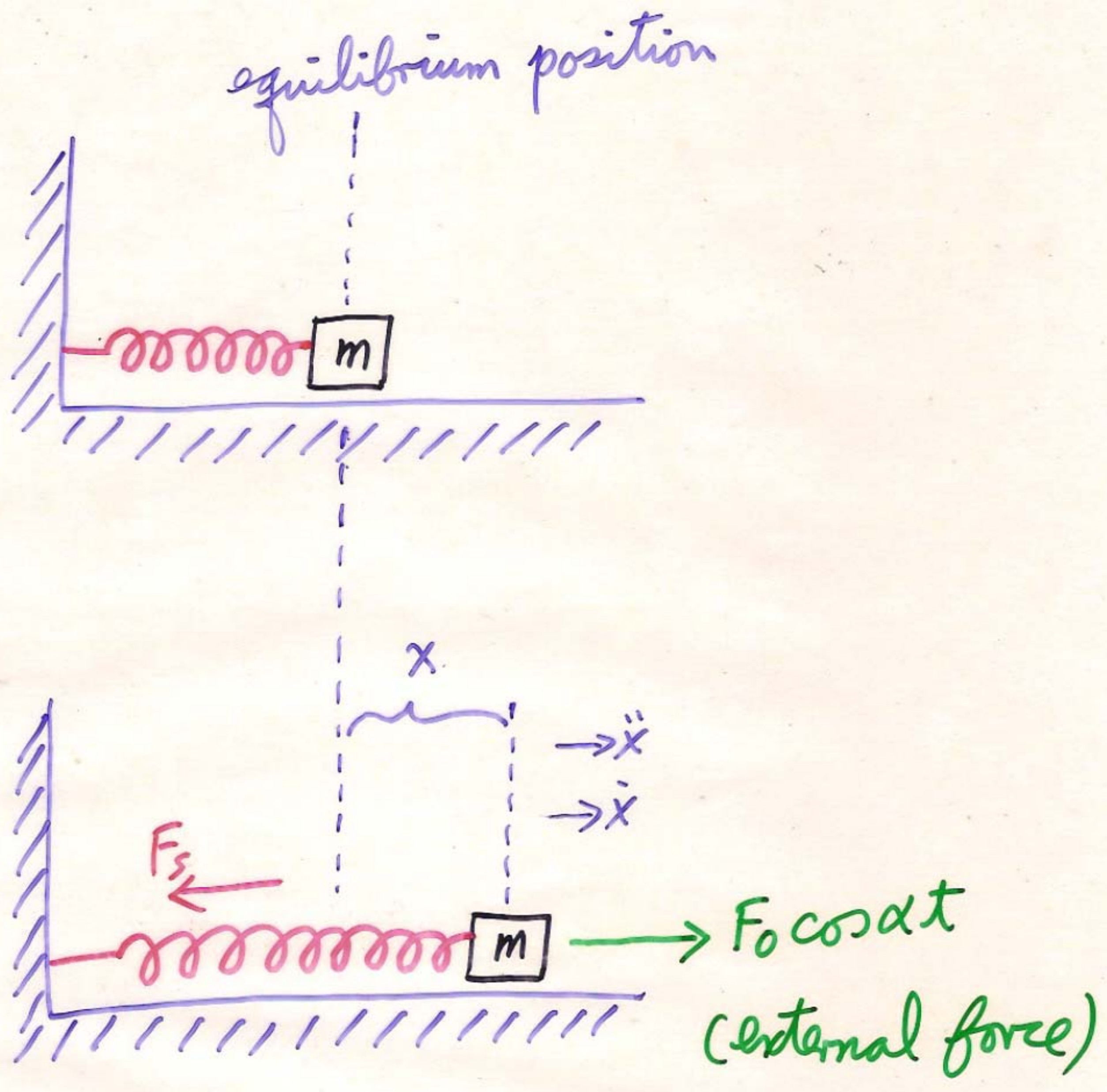


2.6. FORCED OSCILLATIONS

Suppose you have a mass m which can move in a horizontal line. It is attached to the end of a spring which exerts a force

$$F_{\text{spring}} = -kx$$

where x is the extension of the spring and k is a constant (called the spring constant). This is Hooke's Law. Now we attach an external MOTOR to the mass m. This motor exerts a force $F_0 \cos(\alpha t)$, where F_0 is the amplitude of the external force and α is the frequency.


equilibrium position
m
x
$\rightarrow \ddot{x}$
$\rightarrow \dot{x}$
F_s
m
$F_0 \cos \alpha t$
(external force)

Force due to spring $= F_s$

$= kx$ (Hooke's Law)

where k is called the <u>spring constant</u>.

$$m\ddot{x} = F_0 \cos \alpha t - F_s$$

$$\therefore m\ddot{x} + F_s = F_0 \cos \alpha t$$

$\therefore m\ddot{x} + kx = F_0 \cos \alpha t$

Here, we have ignored friction.

i.e. We have forced, undamped motion.

The equation for this case is

$$\underline{m\ddot{x} + kx = F_0 \cos \alpha t}$$

where $m > 0$, $k > 0$, $F_0 \geq 0$, $\alpha \geq 0$.

Case 1: No external force.

i.e. $F_0 = 0$.

$$m\ddot{x} + kx = 0$$

$$\ddot{x} + \omega^2 x = 0 \quad \text{where } \omega = \sqrt{\frac{k}{m}}.$$

called the natural frequency

$$\lambda^2 + \omega^2 = 0$$

$$\lambda = \pm i\omega$$

$$x = C\cos\omega t + D\sin\omega t$$
$$= A\cos(\omega t - \delta)$$

Case 2 : External force $\neq 0$

i.e. $F_0 \neq 0$.

$$m\ddot{x} + kx = F_0 \cos\alpha t$$

$$\ddot{x} + \omega^2 x = \frac{F_0}{m}\cos\alpha t \quad \text{-----} \ (1)$$

Let $\ddot{y} + \omega^2 y = \frac{F_0}{m}\sin\alpha t \quad \text{----} \ (2)$

$$① + i② \Rightarrow \underbrace{(x+iy)}_{z}'' + \omega^2 \underbrace{(x+iy)}_{z} = \frac{F_0}{m} e^{i\alpha t}$$

$$\Rightarrow z'' + \omega^2 z = \frac{F_0}{m} e^{i\alpha t} \quad \text{----} ③$$

Case 2.1 : $\alpha \neq \omega$.

$$\text{Try } z = Ce^{i\alpha t}$$

$$\therefore z' = i\alpha C e^{i\alpha t}$$

$$z'' = -\alpha^2 C e^{i\alpha t}$$

$$\therefore ③ \Rightarrow -\alpha^2 C + \omega^2 C = \frac{F_0}{m}$$

$$\Rightarrow C = \frac{F_0}{m(\omega^2 - \alpha^2)}$$

$\therefore z = \dfrac{F_0/m}{\omega^2 - \alpha^2} e^{i\alpha t}$ solves 3

$\therefore x = \mathrm{Re}(z) = \dfrac{F_0/m}{\omega^2 - \alpha^2} \cos \alpha t$ solves 1

$\therefore$ General Solution of 1 is

$$x = A \cos(\omega t - \delta) + \frac{F_0/m}{\omega^2 - \alpha^2} \cos \alpha t$$

$$\therefore\ \dot{x} = -\omega A \sin(\omega t - \delta) - \frac{F_0/m}{\omega^2 - \alpha^2}\alpha \sin \alpha t$$

Suppose we are given the initial conditions: $x(0) = \dot{x}(0) = 0$.

$$\therefore \begin{cases} 0 = A\cos\delta + \dfrac{F_0/m}{\omega^2-\alpha^2} & \text{---- (i)} \\[2ex] 0 = \omega A \sin\delta & \text{----- (ii)} \end{cases}$$

$\because F_0 \neq 0$, $\therefore A \neq 0$ by (i)

$\therefore \sin\delta = 0$ by (ii)

We choose $\delta = 0$.

(If we choose $\delta = \pi$, we will get back the same value for x on the next page.)

$$\therefore (\text{i}) \Rightarrow A = -\frac{F_0/m}{\omega^2 - \alpha^2}$$

$$\therefore x = -\frac{F_0/m}{\omega^2 - \alpha^2}\cos(\omega t - 0)$$

$$+ \frac{F_0/m}{\omega^2 - \alpha^2}\cos \alpha t$$

$$= \frac{F_0/m}{\omega^2 - \alpha^2}(\cos \alpha t - \cos \omega t)$$

Recall:

$$\cos(u-v) = \cos u \cos v + \sin u \sin v$$

$$\cos(u+v) = \cos u \cos v - \sin u \sin v$$

$$(-) \Rightarrow \cos(u-v) - \cos(u+v) = 2 \sin u \sin v$$

let $A = u - v$

$B = u + v$

$$\therefore u = \frac{A+B}{2}, \quad v = \frac{B-A}{2} = -\frac{A-B}{2}$$

$$\Rightarrow \cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$\therefore\ x = -\frac{2F_0/m}{\omega^2-\alpha^2}\sin\frac{\alpha+\omega}{2}t\,\sin\frac{\alpha-\omega}{2}t$$

i.e. $$x = \left\{\frac{2F_0/m}{\alpha^2-\omega^2}\sin\frac{\alpha-\omega}{2}t\right\}\left\{\sin\frac{\alpha+\omega}{2}t\right\}$$

Observe: Suppose α is near to ω.

For $\sin \frac{\alpha-\omega}{2} t$,

$$\text{frequency} = \frac{\alpha-\omega}{2} \quad \text{very small}$$

$$\text{period} = \frac{2\pi}{\frac{\alpha-\omega}{2}} = \frac{4\pi}{\alpha-\omega} \quad \text{very large}$$

For $\sin \frac{\alpha+\omega}{2} t$,

frequency $= \frac{\alpha+\omega}{2}$ very large

compare with $\frac{\alpha-\omega}{2}$

period $= \frac{2\pi}{\frac{\alpha+\omega}{2}} = \frac{4\pi}{\alpha+\omega}$ very small

compare with $\frac{4\pi}{\alpha-\omega}$

Let $A(t) = \dfrac{2F_0/m}{\alpha^2 - \omega^2} \sin \dfrac{\alpha - \omega}{2} t$

Then $A(t)$ has very small frequency and very large period.

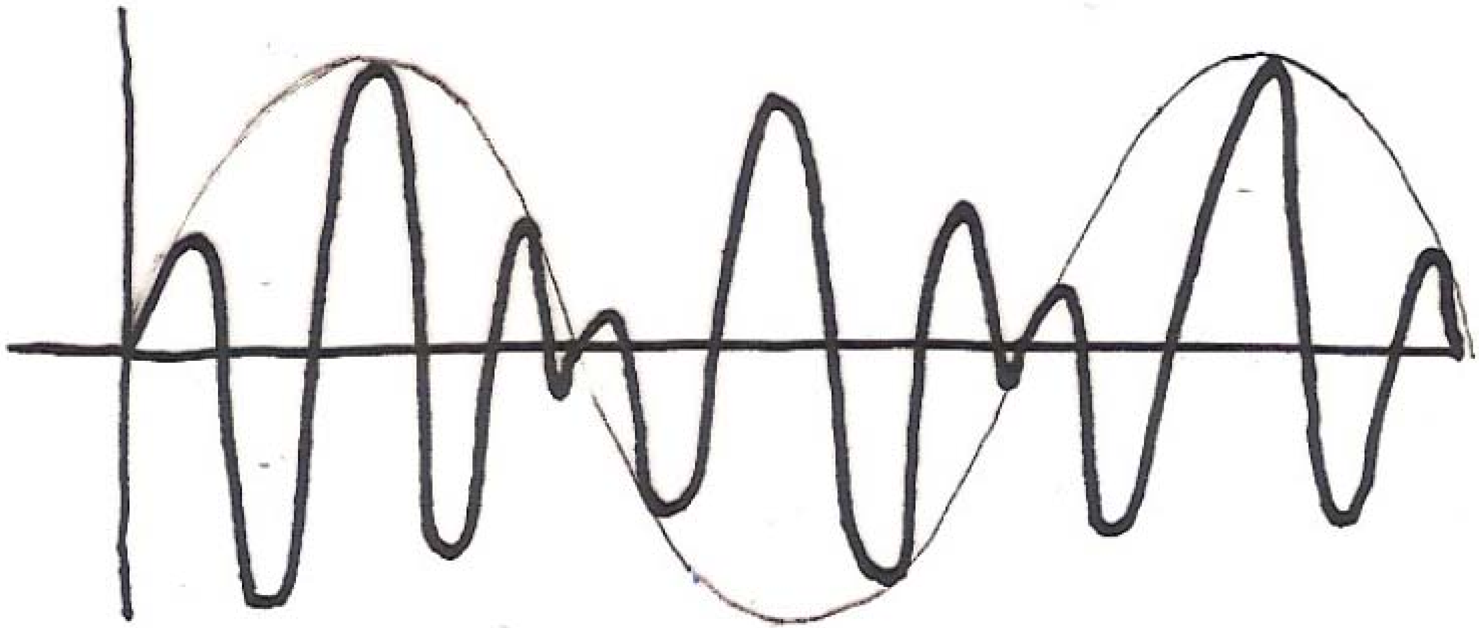
Now $x = A(t) \sin \frac{\alpha + \omega}{2} t$

has very large frequency and very small period and oscillates within $A(t)$. ($\because |x| \leq |A(t)|$)

The frequency of $A(t)$, $\frac{\alpha-\omega}{2}$ is called the <u>beat frequency</u>.

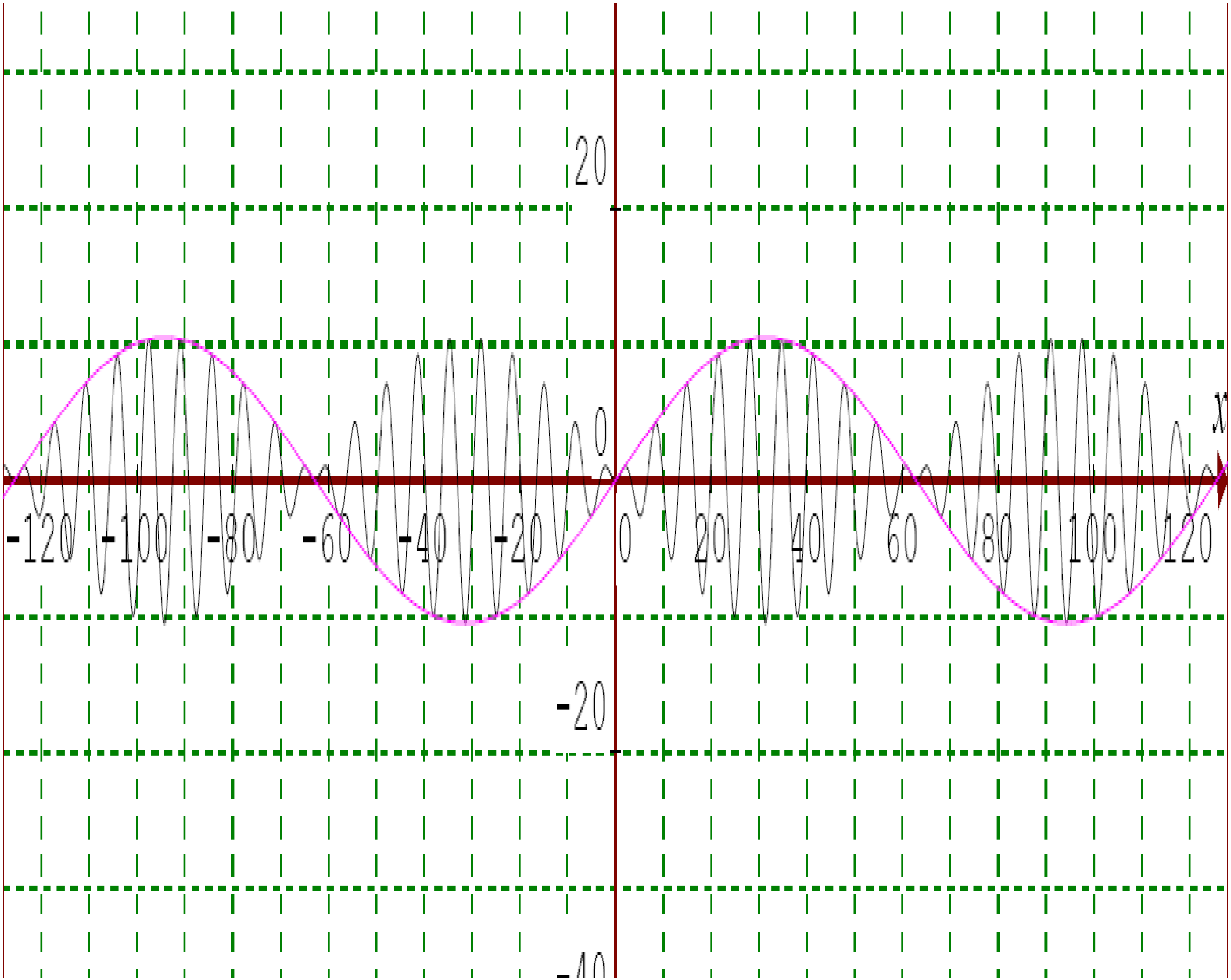
We have a high frequency curve $X(t)$, moving within a low frequency boundary curve $A(t)$.

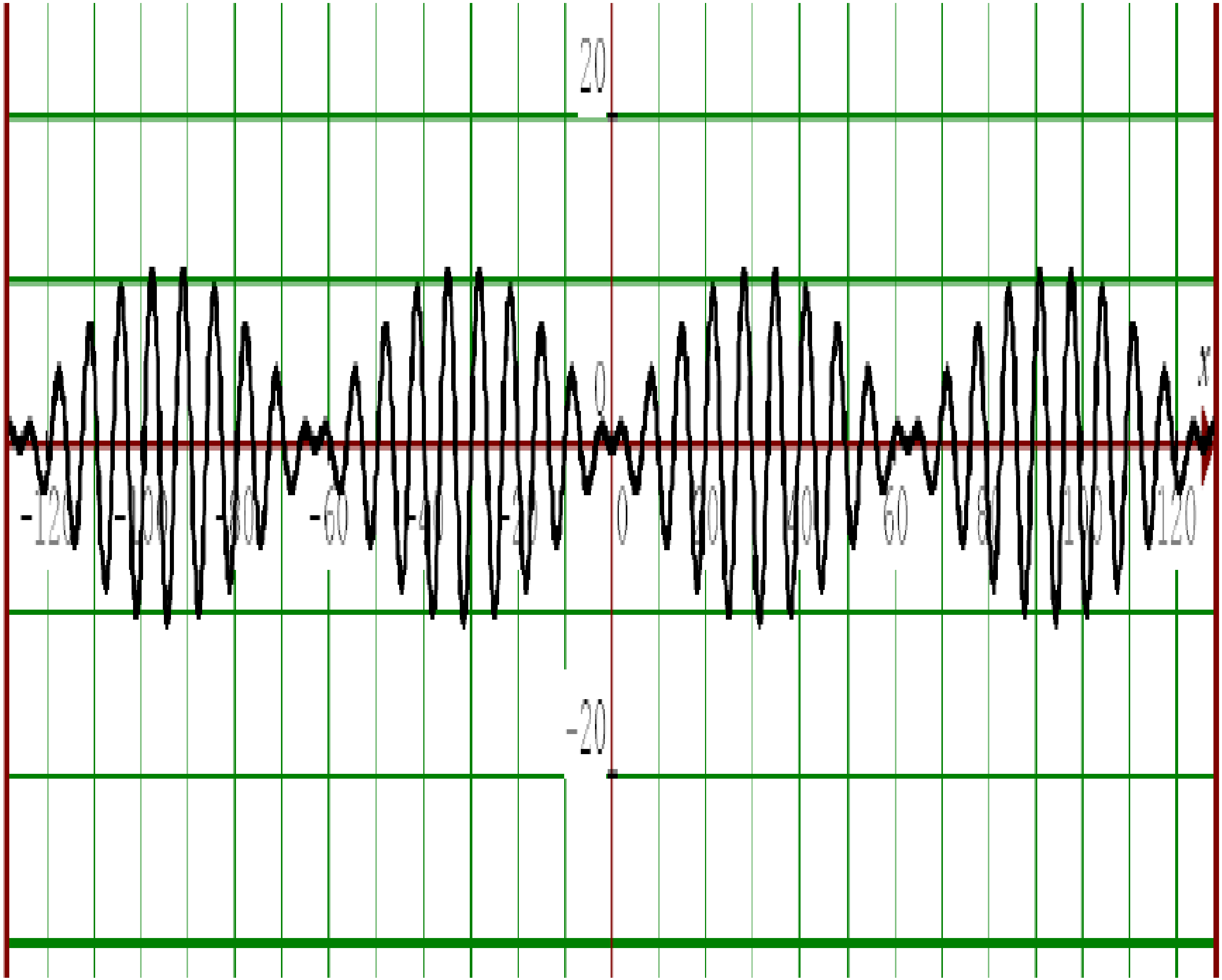
We say that the system has a <u>beat</u>.



Observe: $\left|\dfrac{2F_0/m}{\alpha^2-\omega^2}\right|$ is very large

when α is near ω.


20
0
x
-120
-100
-80
-60
-40
-20
0
20
40
60
80
100
120
-20
-40


20
0
x
-120
-100
-80
-60
-40
-20
0
20
40
60
80
100
120
-20

Case 2.2 : $\alpha=\omega$.

We can use the usual undetermined coefficient method to find x.
It turns out that the answer is the same as letting $\alpha \to \omega$ in case 2.1.

From 2-1, we have

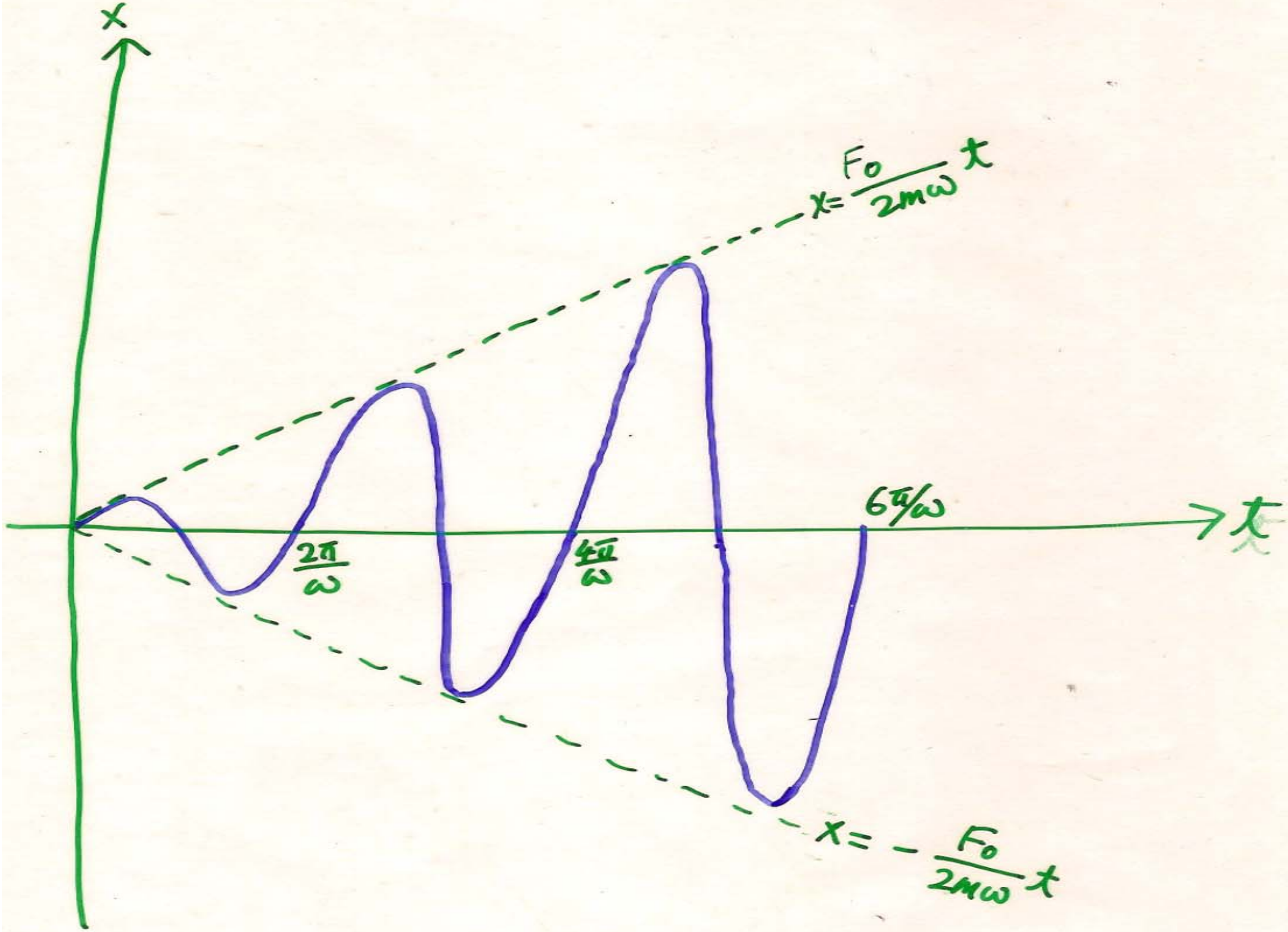
$$X=\left\{\frac{2F_0/m}{\alpha^2-\omega^2}\sin\frac{\alpha-\omega}{2}t\right\}\sin\frac{\alpha+\omega}{2}t$$

$$\therefore\quad X=\frac{F_0/m}{\alpha+\omega}\left(\frac{\sin\frac{\alpha-\omega}{2}t}{\frac{\alpha-\omega}{2}t}\right)t\sin\frac{\alpha+\omega}{2}t$$

$\therefore\ \alpha\to\omega \Rightarrow$

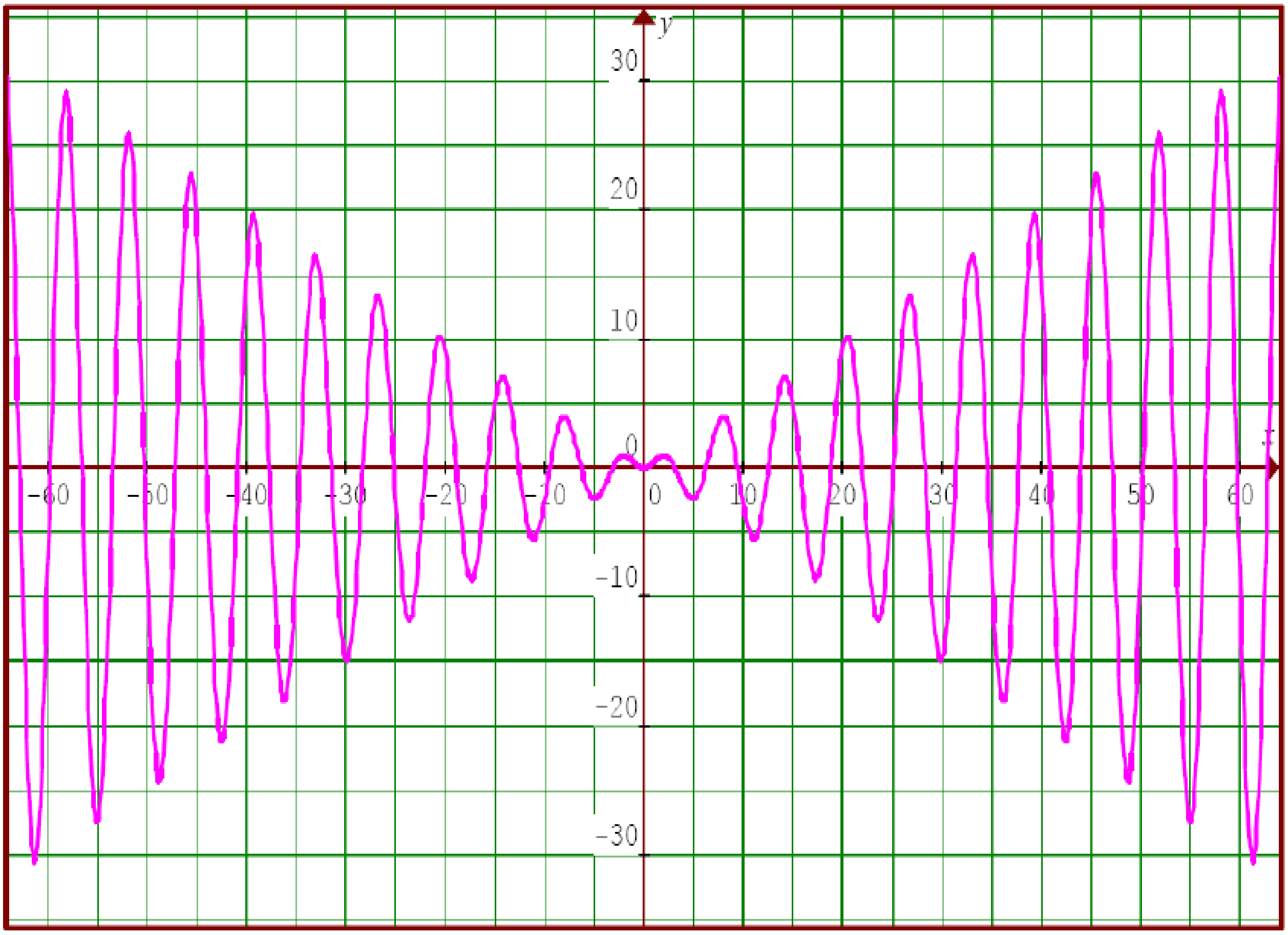
$$x=\frac{F_0}{2\omega m}t\sin\omega t$$

Observe: X oscillates between the envelopes $X = \dfrac{\pm F_0}{2m\omega} t$.



x
t
$x = \frac{F_0}{2m\omega} t$
$x = -\frac{F_0}{2m\omega} t$
$\frac{2\pi}{\omega}$
$\frac{4\pi}{\omega}$
$6\pi/\omega$

Resonance



y
30
20
10
0
−10
−20
−30
x
−60
−50
−40
−30
−20
−10
0
10
20
30
40
50
60

Damped, Forced Oscillations (i.e. $b \neq 0$, $F_0 \neq 0$)

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos\alpha t \quad \text{----} \; (1)$$

Let $$m\ddot{y} + b\dot{y} + ky = F_0 \sin\alpha t \quad \text{----} \; (2)$$

$$(1) + i(2) \Rightarrow m(\underbrace{x+iy}_{z})'' + b(x+iy)' + k(x+iy)$$

$$= F_0 e^{i\alpha t}$$

$$\Rightarrow mz'' + bz' + kz = F_0 e^{i\alpha t} \quad \text{----} \; (3)$$

and $x = \mathrm{Re}(z)$.

Try $z = Ce^{i\alpha t}$

$$z' = i\alpha Ce^{i\alpha t}$$

$$z'' = -\alpha^2 Ce^{i\alpha t}$$

$$\therefore (3) \Rightarrow -m\alpha^2 c e^{i\alpha t} + ib\alpha c e^{i\alpha t}$$
$$+ k c e^{i\alpha t} = F_0 e^{i\alpha t}$$

$$\Rightarrow \quad c = \frac{F_0}{(k - m\alpha^2) + ib\alpha}$$

$$= \frac{F_0 \{(k - m\alpha^2) - ib\alpha\}}{(k - m\alpha^2)^2 + b^2\alpha^2}$$

$$\therefore z = Ce^{i\alpha t}$$

$$= \frac{F_0}{(k-m\alpha^2)^2 + b^2\alpha^2}\{(k-m\alpha^2) - ib\alpha\}\{\cos\alpha t + i\sin\alpha t\}$$

$$\therefore x = Re(z)$$

$$= \frac{F_0}{(k-m\alpha^2)^2 + b^2\alpha^2}\{(k-m\alpha^2)\cos\alpha t + b\alpha\sin\alpha t\}$$

Let x_h = general solution of the Damped, Unforced case

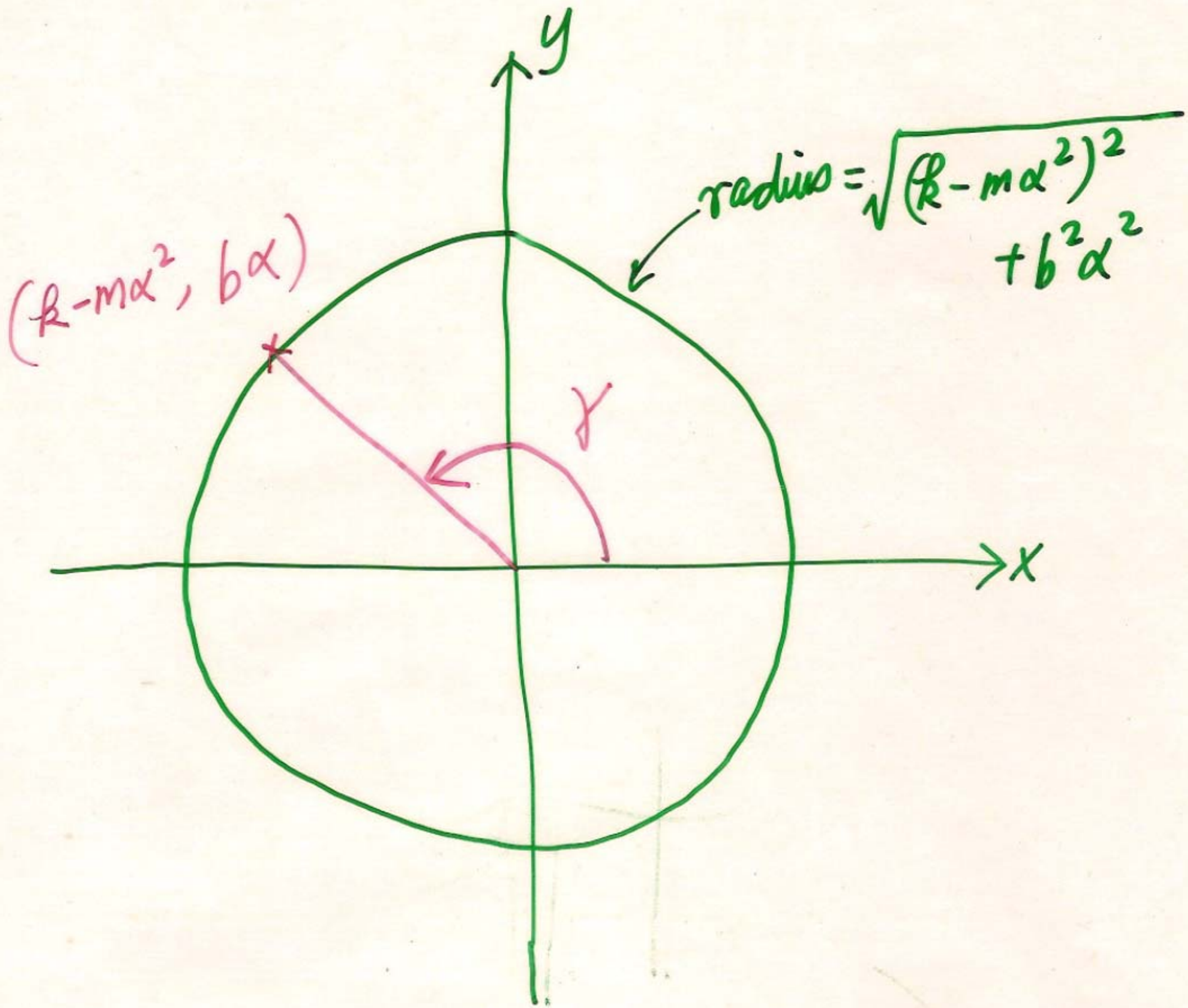
Then the general solution of ① is

$$x = x_h + \frac{F_0}{(k-m\alpha^2)^2+b^2\alpha^2}\left\{(k-m\alpha^2)\cos\alpha t + b\alpha\sin\alpha t\right\}$$

We know from the Damped, Unforced case that $x_h \longrightarrow 0$ very fast (exponential with negative power)

x_h is called the <u>transient</u>.

So after a short time, when X_h is very small, X will be dominated by the expression involving F_0, this is called the <u>steady-state</u>.


y
x
radius $= \sqrt{(k - m\alpha^2)^2 + b^2\alpha^2}$
$(k - m\alpha^2, b\alpha)$
γ

after some time

$$x \approx \frac{F_0}{\sqrt{(k-m\alpha^2)^2+b^2\alpha^2}}(\cos\gamma\cos\alpha t+\sin\gamma\sin\alpha t)$$

$$= \frac{F_0\cos(\alpha t-\gamma)}{\sqrt{(k-m\alpha^2)^2+b^2\alpha^2}}$$

$$= \frac{(F_0/m)\cos(\alpha t-\gamma)}{\sqrt{\left(\frac{k}{m}-\alpha^2\right)^2+\frac{b^2}{m^2}\alpha^2}}$$

Recall: for the undamped, unforced case, $m\ddot{x} + kx = 0$, we let $\omega = \sqrt{\frac{k}{m}}$ (called the natural frequency)

$$\therefore\ x \approx \frac{F_0/m}{\sqrt{(\omega^2 - \alpha^2)^2 + \frac{b^2}{m^2}\alpha^2}} \cos(\alpha t - \gamma)$$

Amplitude: $A(\alpha) = \dfrac{F_0/m}{\sqrt{(\omega^2-\alpha^2)^2 + \frac{b^2}{m^2}\alpha^2}}$

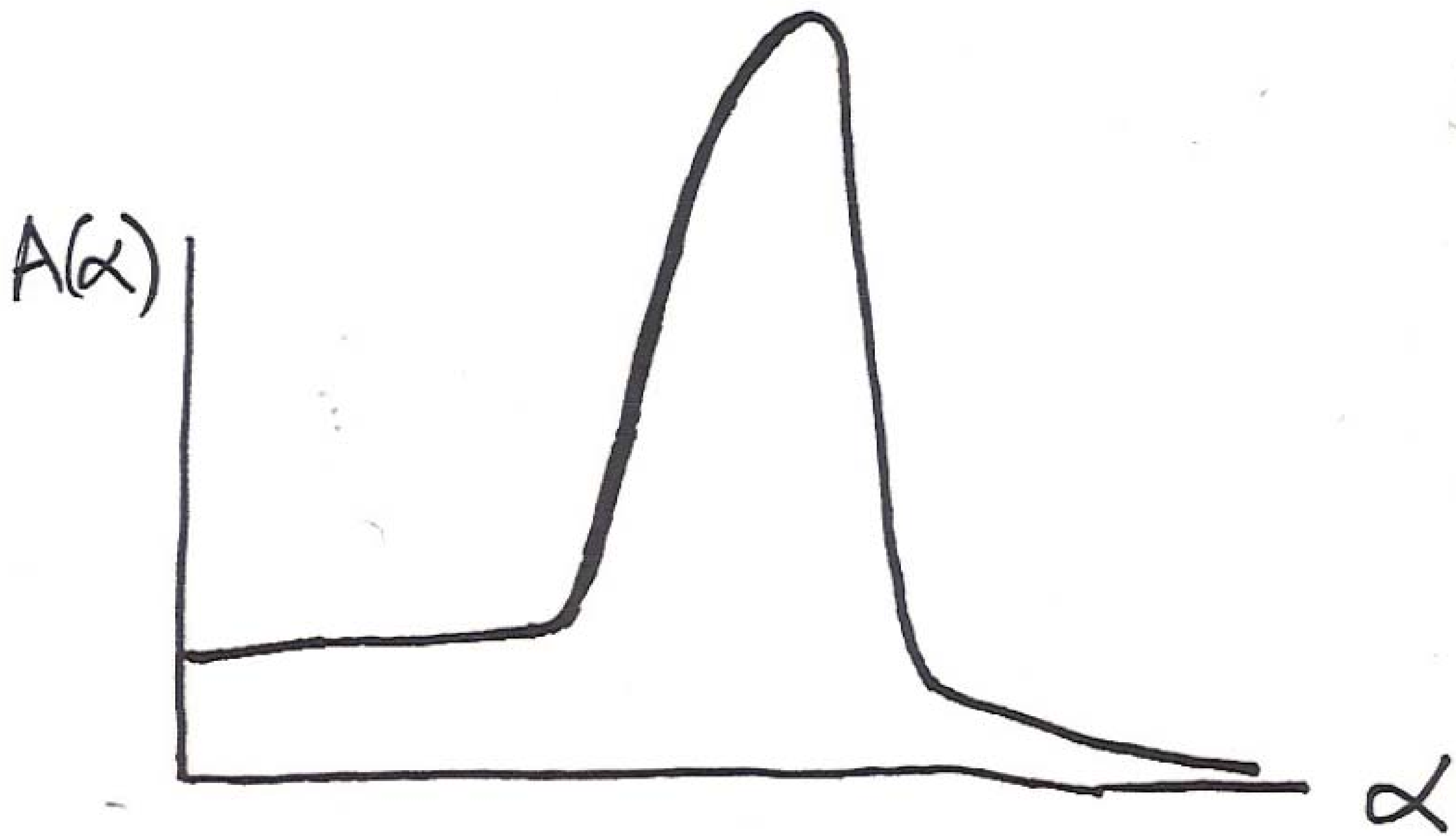
frequency $= \alpha$

period $= \dfrac{2\pi}{\alpha}$.

So the system eventually settles down into a steady oscillation, BUT AT FREQUENCY α, NOT ω! Also, the AMPLITUDE of this oscillation is a FUNCTION OF α,

$$A(\alpha) = \frac{F_0/m}{\sqrt{(\omega^2 - \alpha^2)^2 + \frac{b^2}{m^2}\alpha^2}}$$

The graph of this is called the AMPLITUDE RESPONSE CURVE. Depending on the values of the parameters, this curve might have a SHARP MAXIMUM, meaning that the system will suddenly respond strongly if α is chosen to be the value that gives that maximum. This is something to be avoided in some cases [things might break] but welcomed in others [you want your mobile phone to ignore all frequencies but one].



A(α)
α

2.7. CONSERVATION.

Recall: The equation for Simple Harmonic Motion is

$$m\ddot{x} = -kx$$

Observe: $\ddot{x} = \dfrac{d\dot{x}}{dt} = \dfrac{dx}{dt} \cdot \dfrac{d\dot{x}}{dx} = \dot{x}\dfrac{d\dot{x}}{dx} = \dfrac{d}{dx}\left(\tfrac{1}{2}\dot{x}^2\right)$

$\therefore \; m\dfrac{d}{dx}\left(\tfrac{1}{2}\dot{x}^2\right) = -kx$

$\therefore \; \tfrac{1}{2}m\dot{x}^2 = -\tfrac{1}{2}kx^2 + E$ ← constant of integration

i.e. $\underbrace{\tfrac{1}{2}m\dot{x}^2}_{\text{kinetic energy}} + \underbrace{\tfrac{1}{2}kx^2}_{\text{potential energy}} = \underbrace{E}_{\text{total energy}}$

$\therefore$ the total energy is constant
this property is called the
<u>conservation of energy</u>.

When there is friction, but no external force, the equation is

$$m\ddot{x} + b\dot{x} + kx = 0$$

$$\therefore \quad m\frac{d}{dx}\left(\frac{1}{2}\dot{x}^2\right) + kx = -b\dot{x}$$

$$\therefore \quad \frac{d}{dx}\left(\frac{1}{2}m\dot{x}^2\right) + \frac{d}{dx}\left(\frac{1}{2}kx^2\right) = -b\dot{x}$$

$$\therefore \quad \frac{d}{dx}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right) = -b\dot{x}$$

Let $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$ = total energy

So in this case, the total energy is not constant, but satisfies

$$\frac{dE}{dx} = -b\dot{x}$$

$$\therefore \frac{dE}{dt} = \frac{dE}{dx} \cdot \frac{dx}{dt}$$

$$= (-b\dot{x})(\dot{x})$$

$$= -b\dot{x}^2 < 0$$

$\therefore$ E is a decreasing function which is what we would expect.

Now let $y = \dot{x}$

$\therefore \quad \frac{1}{2} k x^2 + \frac{1}{2} m y^2 = E$ and $E \downarrow$

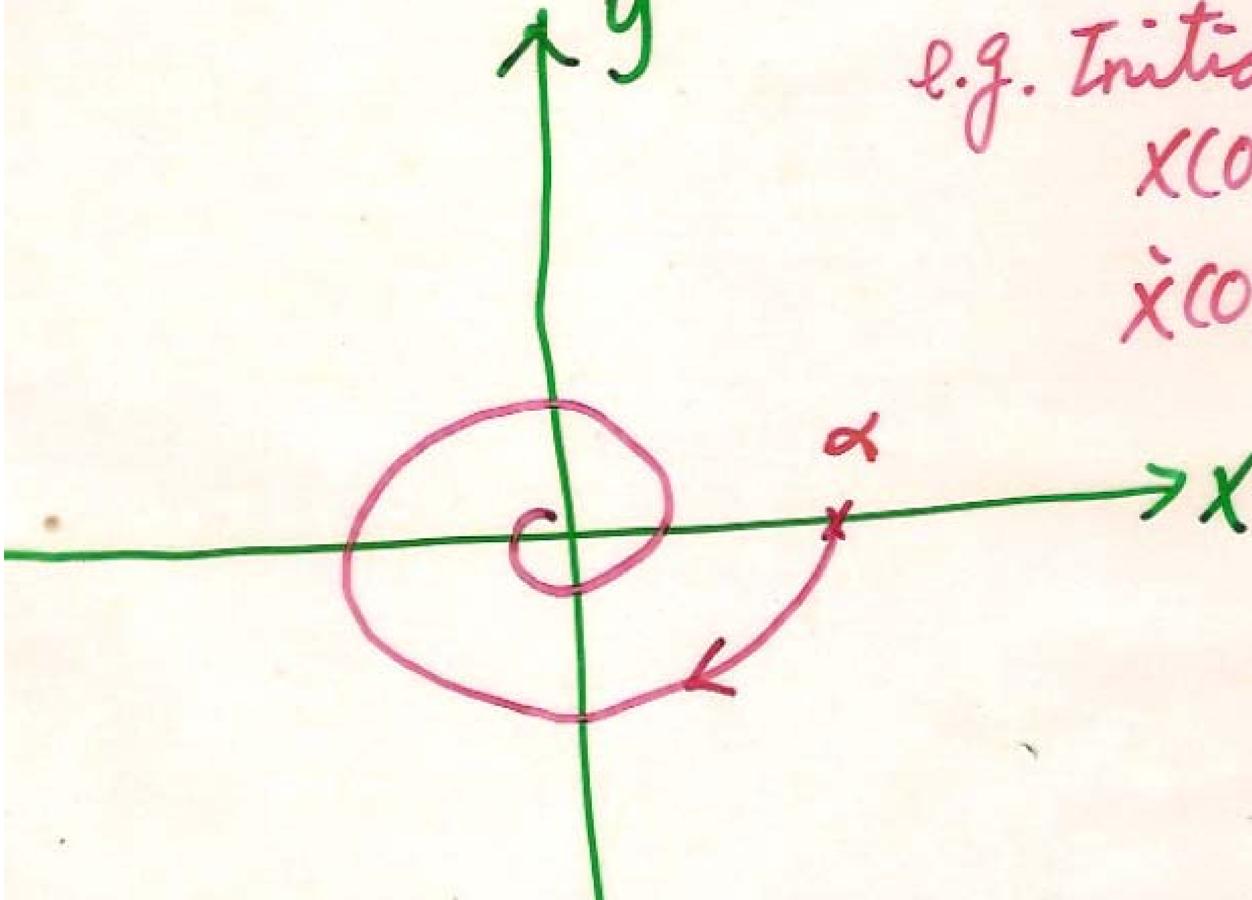
$$\therefore \quad \frac{x^2}{m} + \frac{y^2}{k} = \frac{2E}{km}$$

$$\therefore \quad \frac{x^2}{(\sqrt{m})^2} + \frac{y^2}{(\sqrt{k})^2} = \frac{2E}{km}$$

$\therefore$ the graph is an "elliptical" spiral which spirals to the origin.

In the overdamping case, as we have seen before, there is not enough time for the motion to oscillate and so the graph just goes to the origin without spiraling.

In the underdamping case, as we have seen before, there is enough time for the motion to oscillate before damping puts it to a complete stop. There will be some spiraling:



e.g. Initial conditions

$$x(0) = \alpha$$

$$\dot{x}(0) = 0$$

For a general one dimensional dynamics problem of the form

$$m\ddot{x} = F$$

the kinetic energy, as usual is defined to be

$$\frac{1}{2}m\dot{x}^2.$$

We define the potential energy $V(x)$

by $$V(x) = -\int_0^x F(y)\,dy$$

where we assume that the force F

is a function of position only

(i.e. independant of time t).

$$\therefore \frac{dV}{dx} = -F(x)$$

$$\therefore m\ddot{x} = -\frac{dV}{dx}$$

$$m\ddot{x} + \frac{dV}{dx} = 0$$

$$m\frac{d}{dx}\left(\frac{1}{2}\dot{x}^2\right) + \frac{dV}{dx} = 0$$

$$\frac{d}{dx}\left\{\frac{1}{2}m\dot{x}^2+V\right\}=0$$

$$\Rightarrow \frac{1}{2}m\dot{x}^2+V=E=\text{constant}$$

$\Rightarrow$ Conservation of energy

Observe: In the one dimensional case,

Conservation of energy

$$\Rightarrow \frac{1}{2} m \dot{x}^2 + V(x) = E$$

Let $y = \dot{x}$

$\therefore \frac{1}{2} m y^2 + V(x) = E$

This gives the equation of the phase-plane trajectory of this equation $m\ddot{x} = -\frac{dV}{dx}$.

for example, take

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = E,$$

and define $y = \dot{x}$, so

$$\left(\frac{1}{2}k\right)x^2 + \left(\frac{1}{2}m\right)y^2 = E,$$

and you should recognise this as an ellipse in the xy plane. Similarly in the case of UNSTABLE MO-

TION,

$$\ddot{x} = +\omega^2 x$$

we have

$$\frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2 = E$$

which is the equation of a HYPERBOLA in the $(x,\ \dot{x})$ plane.