Question:

How to derive the ESD and PSD of a periodic signal as defined on Pg 3.4 & 3.6 in Chapter 3 of the lecture notes?

Answer:

Derivation of ESD:

$$E_x(f) = \left|X(f)\right|^2 \quad \leftarrow \quad \text{Definition of ESD}$$

$$= \left[\sum_{k=-\infty}^{\infty} X_k \delta\left(f - \frac{k}{T_p}\right)\right]\left[\sum_{l=-\infty}^{\infty} X_l^* \delta\left(f - \frac{l}{T_p}\right)\right] = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} X_k X_l^* \underbrace{\delta\left(f - \frac{k}{T_p}\right)\delta\left(f - \frac{l}{T_p}\right)}_{=0 \ if \ k \neq l}$$

$$= \sum_{k=-\infty}^{\infty} \left|X_k\right|^2 \delta^2\left(f - \frac{k}{T_p}\right) = \sum_{k=-\infty}^{\infty} \left(\infty \cdot \left|X_k\right|^2\right)\delta\left(f - \frac{k}{T_p}\right) \quad \cdots\cdots \ because \ \ \delta^2(\bullet) = \infty \cdot \delta(\bullet)$$

Derivation of PSD:

$$P_x(f) = \lim_{T\to\infty} \frac{1}{2T}\left|X_T(f)\right|^2 \quad \leftarrow \quad \text{Definition of PSD}$$

$\cdots\cdots$ *let* $2T$ *spans* $(2N+1)$ *periods of* $x_p(t)$ *i.e.* $2T = (2N+1)T_p$ *with* $-(N+0.5)T_p < t < (N+0.5)T_p$,
so $T \to \infty$ *is equivalent to* $N \to \infty$

$$= \lim_{N\to\infty} \frac{1}{(2N+1)T_p}\left|\int_{-(N+0.5)T_p}^{(N+0.5)T_p} x_p(t)\exp(-j2\pi ft)\,dt\right|^2$$

$$= \lim_{N\to\infty} \frac{1}{(2N+1)T_p}\left|\sum_{n=-N}^{N}\int_{-\infty}^{\infty} x_p(t)\,\text{rect}\left(\frac{t-nT_p}{T_p}\right)\exp(-j2\pi ft)\,dt\right|^2$$

$$= \lim_{N\to\infty} \frac{1}{(2N+1)T_p}\left|\sum_{n=-N}^{N}\int_{-\infty}^{\infty} \underbrace{\sum_{k=-\infty}^{\infty} X_k \exp\left(j2\pi\frac{k}{T_p}t\right)}_{x_p(t)}\text{rect}\left(\frac{t-nT_p}{T_p}\right)\exp(-j2\pi ft)\,dt\right|^2$$

$$= \lim_{N\to\infty} \frac{1}{(2N+1)T_p}\left|\sum_{n=-N}^{N}\sum_{k=-\infty}^{\infty} X_k \int_{-\infty}^{\infty} \exp\left(-j2\pi\left(f - \frac{k}{T_p}\right)t\right)\text{rect}\left(\frac{t-nT_p}{T_p}\right)dt\right|^2$$

$$= \lim_{N\to\infty} \frac{1}{(2N+1)T_p}\left|\sum_{k=-\infty}^{\infty} X_k T_p \text{sinc}\left(fT_p - k\right)\sum_{n=-N}^{N}\exp\left(-j2\pi nT_p f\right)\right|^2$$

$$= \lim_{N\to\infty} \frac{1}{(2N+1)T_p}\left[\begin{array}{l}\left[\displaystyle\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} X_k X_l^* T_p^2 \text{sinc}\left(fT_p - k\right)\text{sinc}\left(fT_p - l\right)\sum_{n=-N}^{N}\exp\left(-j2\pi nT_p f\right)\right] \\ \times \displaystyle\sum_{m=-N}^{N}\exp\left(j2\pi mT_p f\right)\end{array}\right]$$

$$\cdots\cdots \quad but \quad \lim_{N\to\infty}\sum_{m=-N}^{N}\exp\left(j2\pi mT_p f\right) = \sum_{m=-\infty}^{\infty}\exp\left(j2\pi mT_p f\right) = \frac{1}{T_p}\sum_{m=-\infty}^{\infty}\delta\left(f - \frac{m}{T_p}\right)$$

$$= \lim_{N\to\infty}\frac{1}{\cancel{(2N+1)}\,\cancel{T_p}}\sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} X_k X_l^* \cancel{T_p^2}\,\underbrace{\text{sinc}(m-k)}_{=\begin{cases}1; & k=m\\ 0; & k\neq m\end{cases}}\underbrace{\text{sinc}(m-l)}_{=\begin{cases}1; & l=m\\ 0; & l\neq m\end{cases}}\cancel{\underbrace{\sum_{n=-N}^{N}\overbrace{\exp(-j2\pi nm)}^{=1}}_{=2N+1}}\frac{1}{\cancel{T_p}}\delta\left(f - \frac{m}{T_p}\right)$$

$$= \sum_{m=-\infty}^{\infty}\left|X_m\right|^2\delta\left(f - \frac{m}{T_p}\right)$$
