Sinusoidal Response (EXAMPLE)

Find the steady-state response of $H(s) = \dfrac{20}{s+1}$ to the input $x(t) = \sin(10t)u(t)$.

Method A: Using Laplace transform

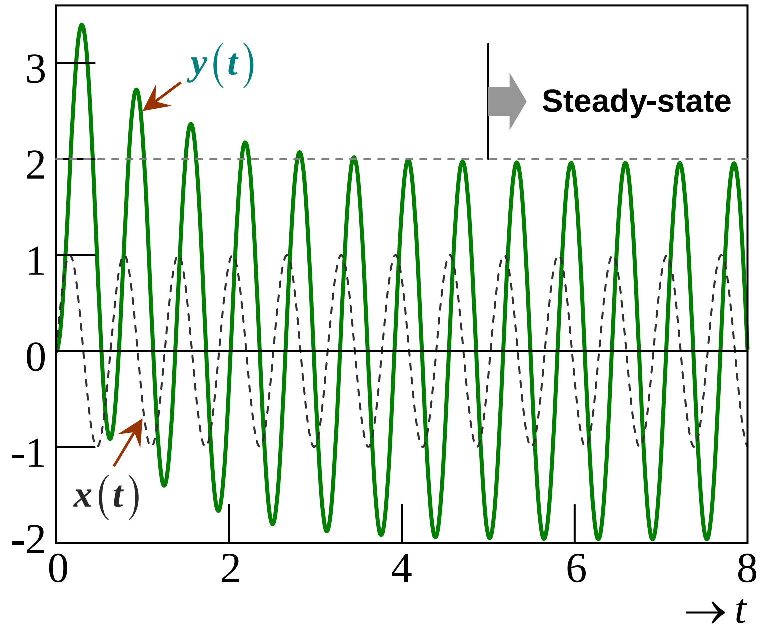
$$X(s) = \frac{10}{s^2+100}$$

$$Y(s) = X(s)H(s) = \frac{200}{(s+1)(s^2+100)}$$

$$= \frac{200}{101}\left[\frac{1}{s+1}\right] + \frac{20}{101}\left[\frac{10}{s^2+100}\right] - \frac{200}{101}\left[\frac{s}{s^2+100}\right]$$

$$y(t) = \left[\frac{200}{101}\exp(-t) + \frac{20}{101}\left[\sin(10t) - 10\cos(10t)\right]\right]u(t)$$

$$= \left[\underbrace{\frac{200}{101}\exp(-t)}_{y_{tr}(t)} + \underbrace{\frac{20}{(101)^{1/2}}\sin\left(10t - \tan^{-1}(10)\right)}_{y_{ss}(t)}\right]u(t)$$



$$\boldsymbol{y_{ss}(t) = \frac{20}{10.05}\sin(10t - 1.47)}$$

Method B: Using $y_{ss}(t) = M_\omega \sin(\omega t + \phi_\omega) = |H(j\omega)| \sin(\omega t + \angle H(j\omega))$

$$H(j\omega) = \frac{20}{j\omega + 1} \quad \cdots\cdots \quad \begin{cases} |H(j\omega)| = M_\omega = \dfrac{20}{\left(\omega^2+1\right)^{1/2}} \\ \angle H(j\omega) = \phi_\omega = -\tan^{-1}(\omega) \end{cases}$$

$$x(t) = \sin(10t)u(t); \quad \omega = 10$$

$$H(j10) = \frac{20}{j10 + 1} \quad \cdots\cdots \quad \begin{cases} |H(j10)| = M_{10} = \dfrac{20}{(101)^{1/2}} = \dfrac{20}{10.05} \\ \angle H(j10) = \phi_{10} = -\tan^{-1}(10) = -1.47 \end{cases}$$

$$\boldsymbol{y_{ss}(t)} = |H(j10)| \sin(10t + \angle H(j10)) = \mathbf{\frac{20}{10.05} \sin(10t - 1.47)}$$

This is a easier way to calculate the **steady-state response of a system to a <u>sinusoidal input</u>** compared to **Method A** if the transient response is not required.

Method C: Using Fourier transform

$$x(t)=\sin(10t)=\frac{1}{j2}\left[\exp(j10t)-\exp(-j10t)\right] \leftarrow \begin{cases} \textit{Note that we have removed } u(t) \textit{ because we assume} \\ \textit{that } x(t)=\sin(10t) \textit{ has started since } t=-\infty \end{cases}$$

$$X(\omega)=\sin(10t)=\frac{1}{j2}\left[2\pi\delta(\omega-10)-2\pi\delta(\omega+10)\right] \quad \text{and} \quad H(\omega)=\frac{20}{j\omega+1}$$

$$Y(\omega)=X(\omega)H(\omega)=\frac{1}{j2}\left[2\pi\delta(\omega-10)-2\pi\delta(\omega+10)\right]\frac{20}{j\omega+1}$$

$$=\frac{1}{j2}\left[\frac{20}{1+j10}2\pi\delta(\omega-10)-\frac{20}{1-j10}2\pi\delta(\omega+10)\right]$$

$$y(t)=\Im^{-1}\{Y(\omega)\}=\frac{1}{j2}\left[\frac{20}{1+j10}\cdot\Im^{-1}\{2\pi\delta(\omega-10)\}-\frac{20}{1-j10}\cdot\Im^{-1}\{2\pi\delta(\omega+10)\}\right]$$

$$=\frac{1}{j2}\left[\frac{20}{1+j10}\exp(j10t)-\frac{20}{1-j10}\exp(-j10t)\right]$$

$$\cdots\cdots \textit{ with } \frac{1}{1\pm j10}=\frac{1}{10.05}\exp(\mp j1.47)$$

$$=\frac{20}{10.05}\frac{1}{j2}\cdot\left[\exp\left(j(10t-1.47)\right)-\exp\left(-j(10t-1.47)\right)\right]=\mathbf{\frac{20}{10.05}\sin(10\mathit{t}-1.47)}$$

This is another easier way to calculate the **steady-state response of a system to a** **sinusoidal input** compared to **Method A** if the transient response is not required.