Summary: 2nd Order Linear D.E.

$$y'' + p(x)y' + q(x)y = F(x)$$

$$\boxed{\begin{aligned} y &= y_h + y_p \\ &= \underbrace{c_1 y_1 + c_2 y_2}_{} + y_p \end{aligned}}$$

If *p* and *q* are constants, use charac. equation $\lambda^2 + a\lambda + b = 0$

Method of undetermined coeff

Variation of parameters

Method of Variation of Parameters

Standard form $y'' + p(x)y' + q(x)y = r(x)$

$$y_p = uy_1 + vy_2$$

$$u = -\int \frac{y_2 r}{y_1 y_2' - y_1' y_2}\,dx,$$

$$v = \int \frac{y_1 r}{y_1 y_2' - y_1' y_2}\,dx.$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Abel's Theorem (for Wronskian)

$$y'' + p(x)y' + q(x)y = 0$$

$$y_h = Ay_1 + By_2$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = Ce^{-\int p(x)dx}$$

Not useful practically since we don't know what is the value of the constant ***C***

Abel's Theorem (for Wronskian): Example

$$\ddot{Q} + 100\dot{Q} + 50000Q = 0$$

$$Q_h = C_1 e^{-50t} \cos mt + C_2 e^{-50t} \sin mt$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = Ce^{-\int 100 dt} = Ce^{-100t}$$

If your Wronskian looks complicated, you might have made a mistake.

Method of undetermined coeff

$$y'' + ay' + by = F(x)$$

$F(x) = \ldots$	Try
$ax^k + bx^{k-1} + \ldots$	$Ax^k + Bx^{k-1} + \ldots$
$(ax^k + \ldots)e^{mx}$	$u(x)e^{mx}$ or $(Ax^k + \ldots)e^{mx}$
$(ax^k + \ldots)\cos(mx)$	$u(x)e^{imx}$ Then take real part or $u(x)\cos(mx) + v(x)\sin(mx)$

Method of undetermined coeff (cont'd)

$$y'' + ay' + by = F(x)$$

Pay attention to your $\boldsymbol{y_h}$

If your $\boldsymbol{y_p}$ coincides with your $\boldsymbol{y_h}$,

Multiply $\boldsymbol{x}$ to $\boldsymbol{y_p}$.

(Multiply higher powers of $\boldsymbol{x}$ if necessary)

Paying attention to hom. sol: Example 1

$$y'' + 2y' = 8x$$

Try $y_p = Ax + B$

$$\lambda^2 + 2\lambda = 0 \Rightarrow y_h = C_1 e^{-2x} + C_2$$

Try $y_p = x(Ax + B) = Ax^2 + Bx$

Ans: $y_p = 2x^2 - 2x$

Paying attention to hom. sol: Example 2

$$y'' + y' - 2y = -9e^{-2x}$$

Try $y_p = u(x)e^{-2x}$ or Ae^{-2x}

$$\lambda^2 + \lambda - 2 = 0 \Rightarrow y_h = C_1 e^{x} + C_2 e^{-2x}$$

Try $y_p = x(Ae^{-2x})$

$$y_p'' + y_p' - 2y_p = -3Ae^{-2x}$$

Example 3

$$y'' - 4y' + 3y = x + e^x$$

What should you try?

“Superposition” for non-hom.

$$y'' + p(x)y' + q(x)y = F_1(x) + F_2(x)$$

$$\boxed{y = y_h + y_{p_1} + y_{p_2}}$$

is a solution to

$$y'' + p(x)y' + q(x)y = 0$$

$$y'' + p(x)y' + q(x)y = F_1(x)$$

$$y'' + p(x)y' + q(x)y = F_2(x)$$

“Superposition” for non-hom. : Example

$$y'' - 4y' + 3y = x + e^x$$

$$\boxed{y \;=\; y_h + y_{p_1} + y_{p_2}}$$

$$y_h'' - 4y_h' + 3y_h = 0 \;\Rightarrow\; y_h = C_1 e^{3x} + C_2 e^x$$

$$y_p'' - 4y_p' + 3y_p = x \;\Rightarrow\; y_p = A_p x + B_p$$

$$y_q'' - 4y_q' + 3y_q = e^x \;\Rightarrow\; y_q = A_q x e^x$$

$$y = C_1 e^{3x} + C_2 e^x + \frac{3x+4}{9} - \frac{x}{2} e^x$$

Practice: Guess your y_p

$$y'' + 4y' = 1 \qquad y_p = At$$

$$y'' + 4y' = t \qquad y_p = t(At + B)$$

$$y'' + y' = 6\sin 2t \qquad y_p = A\cos 2t + B\sin 2t$$

$$y'' + 4y' + 4y = te^{-t} \qquad y_p = (At + B)e^{-t}$$

$$y'' + 4y' + 4y = te^{-2t} \qquad y_p = t^2(At + B)e^{-2t}$$

Practice: Guess your y_p

$$y'' + 4y = 12\cos^2 t$$

$$y_p = At\cos 2t + Bt\sin 2t + C$$

$$y'' - 3y' + 2y = e^t \sin t$$

$$y_p = e^t(A\cos t + B\sin t)$$

$$y'' - 3y' - 4y = 3e^{2x} + 2\sin x - 8e^x \cos 2x$$

$$\begin{aligned} y_p &= Ae^{2x} + B\cos x + C\sin x \\ &\quad + e^x(D\cos 2x + E\sin 2x) \end{aligned}$$