CHAPTER 4

THE LAPLACE TRANSFORM

$$
\begin{tikzcd}
\text{O.D.E.} \arrow[r, "\text{Laplace}"] \arrow[d, dashed] & \text{Algebraic equation} \arrow[d, "\text{easily solved}"] \\
\text{right ans.} & \text{ans} \arrow[l, "\text{Inverse Laplace}"]
\end{tikzcd}
$$

Definition. Let f be a function defined for all $t \geq 0$. The Laplace transform of f is the function $F(s)$ defined by

$$F(s) = L(f) = \int_0^\infty e^{-st} f(t) dt, \qquad (1)$$

provided the improper integral on the right exists.

Recall: $\int_0^\infty e^{-st} f(t)\,dt$ is convergent

means that $\lim_{b\to\infty} \int_0^b e^{-st} f(t)\,dt$ is

finite.

The original function $f(t)$ in (1) is called the inverse transform or inverse of $F(s)$ and is denoted by $L^{-1}(F)$; i.e.,

$$f(t) = L^{-1}(F(s)).$$

Notation: Original functions are denoted by lower case letters and their Laplace transforms by the same letters in capitals. Thus $F(s) = L(f(t))$, $Y(s) = L(y(t))$ etc.

Property L is linear.

i.e. $L(\alpha f + \beta g) = \alpha L f + \beta L g$.

Proof $L(\alpha f + \beta g)(s)$

$$= \int_0^\infty e^{-st} (\alpha f + \beta g)(t)\, dt$$

$$= \alpha \int_0^\infty e^{-st} f(t)\, dt + \beta \int_0^\infty e^{-st} g(t)\, dt$$

$$= \alpha L f(s) + \beta L g(s). \quad //$$

Property. L^{-1} is linear.

Let $L^{-1}F = f$, $L^{-1}G = g$

$\therefore\ Lf = F$ and $Lg = G$

$$\because\ L(\alpha f + \beta g) = \alpha Lf + \beta Lg$$
$$= \alpha F + \beta G$$
$$\therefore\ L^{-1}(\alpha F + \beta G) = \alpha f + \beta g$$
$$= \alpha L^{-1}F + \beta L^{-1}G. \ //$$

Example

To find $\mathcal{L}(e^{at})$ when a = real constant.

Solution

Let $\mathcal{L}(e^{at}) = F$

Then $F(s) = \int_0^\infty e^{-st} e^{at}\,dt$

$$= \int_0^\infty e^{(a-s)t}\,dt$$

Case 1: $s < a$

$$F(s) = \frac{1}{a-s} e^{(a-s)t} \Big|_0^{\infty} = \infty$$

Case 2: $s = a$

$$F(s) = \int_0^{\infty} dt = t \Big|_0^{\infty} = \infty$$

Case 3: $s > a$

$$F(s) = \frac{1}{a-s} e^{(a-s)t} \Big|_0^\infty$$

$$= \frac{1}{a-s} \left\{ e^{-\infty} - e^0 \right\} = \frac{1}{s-a}$$

$$\boxed{L(e^{at}) = \frac{1}{s-a} \quad \text{for } s > a}$$

A similar calculation shows that the same formula

$$L(e^{at}) = \frac{1}{s-a}$$

holds when a is complex and

s > Re(a)

Example

From $\mathcal{L}(e^{at})(s) = \dfrac{1}{s-a}$, $s > \mathrm{Re}(a)$.

Put $a=0 \implies \boxed{\mathcal{L}(1)(s) = \dfrac{1}{s}\ , \ s>0}$

Now we can find $L(k)(s)$ for any constant k.

$$L(k)(s) = k\,L(1)(s) = \frac{k}{s}, \quad s > 0.$$

Put $a = i\alpha \Rightarrow L(e^{it\alpha})(s) = \dfrac{1}{s - i\alpha}, \quad s > 0$

$$\therefore L(\cos\alpha t + i\sin\alpha t)(s) = \frac{1}{s - i\alpha} = \frac{s + i\alpha}{s^2 + \alpha^2}$$

$$\| $$

$$L(\cos\alpha t)(s) + i\,L(\sin\alpha t)(s) = \frac{s}{s^2+\alpha^2} + i\,\frac{\alpha}{s^2+\alpha^2}$$

$$\Rightarrow \boxed{L(\cos\alpha t)(s) = \frac{s}{s^2+\alpha^2}, \; s > 0}$$

$$\boxed{L(\sin\alpha t)(s) = \frac{\alpha}{s^2+\alpha^2}, \; s > 0}$$

Next

$$\mathcal{L}(\cosh \alpha t)(s) = \mathcal{L}\left(\tfrac{1}{2}e^{\alpha t} + \tfrac{1}{2}e^{-\alpha t}\right)(s)$$

$$= \frac{1}{2}\frac{1}{s-\alpha} + \frac{1}{2}\frac{1}{s-(-\alpha)}, \quad s > |\alpha|$$

$$= \frac{1}{2}\frac{s+\alpha+s-\alpha}{s^2-\alpha^2} = \frac{s}{s^2-\alpha^2}$$

$$L(\sinh \alpha t)(s) = L\left(\frac{1}{2}e^{\alpha t} - \frac{1}{2}e^{-\alpha t}\right)(s)$$

$$= \frac{\alpha}{s^2-\alpha^2}, \quad s > |\alpha|$$

Example 4. If $F(s) = \frac{3}{s} + \frac{5}{s-7}$, find $f(t) = L^{-1}(F)$.

Recall: $L(e^{at}) = \frac{1}{s-a}$

$\therefore \quad L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

$$\therefore L^{-1}\left(\frac{3}{s}+\frac{5}{s-7}\right)$$

$$= 3L^{-1}\left(\frac{1}{s-0}\right)+5L^{-1}\left(\frac{1}{s-7}\right)$$

$$= 3e^{0t}+5e^{7t}$$

$$= 3+5e^{7t}$$

Example 6. To show that $L(t^n) = \frac{n!}{s^{n+1}}$, $n = 0, 1, 2, \ldots$.

Proof By Induction on n.

step 1: $n=0$

$L(t^0)(s) = L(1)(s) = \frac{1}{s}$ is true.

Step 2: Assume

$$L(t^k)(s) = \frac{k!}{s^{k+1}}$$

$$L(t^{k+1})(s) = \int_0^\infty e^{-st} t^{k+1} dt$$

$$= -\frac{1}{s}\int_0^\infty t^{k+1} d(e^{-st})$$

$$= -\frac{1}{s}\left\{ t^{k+1} e^{-st} \Big|_0^\infty \right.$$

$$\left. - \int_0^\infty e^{-st} (k+1) t^k \, dt \right\}$$

$$= \frac{k+1}{s} \int_0^\infty e^{-st} t^k \, dt$$

$$\therefore L(t^{k+1})(s) = \frac{k+1}{s} L(t^k)(s)$$

$$= \frac{k+1}{s} \frac{k!}{s^{k+1}}$$

$$= \frac{(k+1)!}{s^{k+2}}$$

$\therefore$ The result is true by Induction.

Example

$$\because\ L(t^n)(s) = \frac{n!}{s^{n+1}}$$

$$\therefore\ L^{-1}\left(\frac{n!}{s^{n+1}}\right)(t) = t^n$$

$$\therefore\ n!\, L^{-1}\left(\frac{1}{s^{n+1}}\right) = t^n$$

$$\therefore\ L^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{1}{n!} t^n$$

$$L(e^{at}) = \frac{1}{s-a} \qquad L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$$

$$L(\cos at) = \frac{s}{s^2+a^2} \qquad L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$$

$$L(\sin at) = \frac{a}{s^2+a^2} \qquad L^{-1}\left(\frac{a}{s^2+a^2}\right) = \sin at$$

$$L(\cosh at) = \frac{s}{s^2-a^2} \qquad L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$$

$$L(\sinh at) = \frac{a}{s^2-a^2} \qquad L^{-1}\left(\frac{a}{s^2-a^2}\right) = \sinh at$$

$$L(t^n) = \frac{n!}{s^{n+1}} \qquad L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!}$$

Example 7. Given $F(s) = \frac{2s+5}{s^2+9}$, find $L^{-1}(F(s))$.

Solution

$$L^{-1}\left(\frac{2s+5}{s^2+9}\right)$$

$$= 2L^{-1}\left(\frac{s}{s^2+9}\right) + 5L^{-1}\left(\frac{1}{s^2+9}\right)$$

$$= 2L^{-1}\left(\frac{s}{s^2+3^2}\right) + \frac{5}{3}L^{-1}\left(\frac{3}{s^2+3^2}\right)$$

$$= 2\cos 3t + \frac{5}{3}\sin 3t$$

Example

To find $L^{-1}\left(\frac{-2s-11}{s^2-3s-10}\right)$

Solution

We can factorize the denominator

$$s^2-3s-10=(s+2)(s-5)$$

Next we do a partial fraction

$$\frac{-2s-11}{s^2-3s-10} = \frac{A}{s+2} + \frac{B}{s-5}$$

$$\therefore -2s-11 = A(s-5) + B(s+2)$$

$$s=-2 \Rightarrow A=1$$

$$s=5 \Rightarrow B=-3$$

$$L^{-1}\left(\frac{-2s-11}{s^2-3s-10}\right) = L^{-1}\left(\frac{1}{s+2} + \frac{-3}{s-5}\right)$$

$$= L^{-1}\left(\frac{1}{s+2}\right) - 3L^{-1}\left(\frac{1}{s-5}\right)$$

$$= L^{-1}\left(\frac{1}{s-(-2)}\right) - 3L^{-1}\left(\frac{1}{s-5}\right)$$

$$= \underline{\underline{e^{-2t} - 3e^{5t}}}$$

Theorem

$$L(f^{(n)}) = s^n L(f) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

$$= s^n L(f) - \sum_{k=0}^{n-1} s^{n-1-k} f^{(k)}(0)$$

for $n = 1, 2, 3, \ldots$

<u>Proof</u> By Induction.

Step 1 : $n=1$.

$$L(f') = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= \int_0^\infty e^{-st}\, d(f(t))$$

$$= e^{-st} f(t) \Big|_0^{\infty} + s \int_0^{\infty} e^{-st} f(t)\, dt$$

$$= -f(0) + s\, L(f)$$

Here we have used $\lim_{t \to \infty} e^{-st} f(t) = 0$

which is true if $L(f)$ exists, i.e. $\int_0^{\infty} e^{-st} f(t)\, dt$ converges.

Step 2: Assume the result is true for n.

$$L(f^{(n+1)}) = L(f^{(n)\prime})$$

$$= sL(f^{(n)}) - f^{(n)}(0)$$

$\left(\begin{array}{l}\text{Use } n=1 \text{ case apply to} \\ g = f^{(n)}\end{array}\right)$.

$$= s\{s^n L(f) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)\}$$

$$- f^{(n)}(0)$$

($\because$ we assume the result holds for n)

$$= s^{n+1} L(f) - s^n f(0) - \cdots$$

$$- s f^{(n-1)}(0) - f^{(n)}(0)$$

$\therefore$ the result is true for $n+1$. //

Example

$$\boxed{L(f') = sL(f) - f(0)}$$

Example

$$\boxed{L(f'') = s^2 L(f) - sf(0) - f'(0)}$$

Example 8

To find $L(\sin^2 t)$

Solution

$$L(\sin^2 t) = L\left(\frac{1-\cos 2t}{2}\right)$$

$$= \frac{1}{2} L(1) - \frac{1}{2} L(\cos 2t)$$

$$= \frac{1}{2s} - \frac{1}{2} \frac{s}{s^2+2^2}$$

$$= \frac{1}{2} \left\{ \frac{s^2+2^2-s^2}{s(s^2+2^2)} \right\}$$

$$= \frac{2}{s(s^2+4)}$$

Example 8 (Second solution: as in the notes)

To find $L(\sin^2 t)$

Solution

Let $f(t) = \sin^2 t$

$\therefore \ f'(t) = 2\sin t \cos t$

$= \sin 2t$

and $f(0) = \sin^2 0 = 0$

Now we use

$$L(f') = sL(f) - f(0)$$

$$\therefore L(\sin 2t) = sL(\sin^2 t) - 0$$

$$\therefore \quad sL(\sin^2 t) = \frac{2}{s^2 + 2^2}$$

$$\therefore \quad L(\sin^2 t) = \frac{2}{s(s^2+4)}$$

Example 9

Find $L(t \sin \alpha t)$

Solution

We start with

$$\frac{\alpha}{s^2+\alpha^2} = L(\sin \alpha t) = \int_0^\infty e^{-st} \sin \alpha t \, dt$$

Differentiate with respect to s :

$$-\frac{\alpha}{(s^2+\alpha^2)^2}(2s) = \frac{d}{ds}\int_0^\infty e^{-st}\sin\alpha t\, dt$$

$$= \int_0^\infty \frac{\partial}{\partial s} e^{-st}\sin\alpha t\, dt$$

$$= \int_0^\infty e^{-st}(-t)\sin \alpha t \, dt$$

$$= -\int_0^\infty e^{-st}\, t \sin \alpha t \, dt$$

$$= -L(t \sin \alpha t)$$

$$\therefore L(t \sin \alpha t) = \underline{\underline{\frac{2\alpha s}{(s^2+\alpha^2)^2}}}$$

Note: By differentiating with respect to s n times, we can find $L(t^n \sin \alpha t)$ in a similar way.

Example 9 (Second solution: as in the notes)

Find $L(t\sin\alpha t)$

Solution

Let $f(t) = t\sin\alpha t$

$\therefore \quad f(0) = 0$

and $f'(t) = \sin\alpha t + \alpha t \cos\alpha t$

$\therefore f'(0) = 0$

and $f''(t) = 2\alpha\cos\alpha t - \alpha^2 t \sin\alpha t$

Now we use

$$L(f''(t)) = s^2 L(f) - s f(0) - f'(0)$$

$$\therefore s^2 L(t\sin\alpha t) = 2\alpha L(\cos\alpha t) - \alpha^2 L(t\sin\alpha t)$$

$$\therefore (s^2+\alpha^2) L(t\sin\alpha t) = 2\alpha \frac{s}{s^2+\alpha^2}$$

$$\therefore L(t\sin\alpha t) = \frac{2\alpha s}{(s^2+\alpha^2)^2}$$

Solution of Initial value problems

Example 10

Solve $\begin{cases} y'' + y = e^{2t} \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$

Solution

Step 1: Take the Laplace transform of both sides of the DE.

Let $Y = L(y)$

$$L(y'' + y) = L(e^{2t})$$

$$L(y'') + L(y) = L(e^{2t})$$

$$s^2 L(y) - s\,y(0) - y'(0) + L(y) = \frac{1}{s-2}$$

$$s^2 Y - 1 + Y = \frac{1}{s-2}$$

Step 2: Solve for Y.

$$(s^2+1)\,Y = \frac{1}{s-2} + 1 = \frac{s-1}{s-2}$$

$$\therefore \quad Y = \frac{s-1}{(s-2)(s^2+1)}$$

Step 3: $y = L^{-1}(Y)$

Let $\dfrac{s-1}{(s-2)(s^2+1)} = \dfrac{A}{s-2} + \dfrac{Bs+C}{s^2+1}$

$\therefore\ s-1 = A(s^2+1) + (Bs+C)(s-2)$

$s=2 \Rightarrow A = \frac{1}{5}$

Compare $s^2 \Rightarrow 0 = A + B \Rightarrow B = -\frac{1}{5}$

Compare $s \Rightarrow 1 = -2B + C \Rightarrow C = 1 + 2B = \frac{3}{5}$

$$\therefore\ y = L^{-1}\left\{ \frac{1/5}{s-2} + \frac{-\frac{1}{5}s + \frac{3}{5}}{s^2+1} \right\}$$

$$= \frac{1}{5}L^{-1}\left(\frac{1}{s-2}\right) - \frac{1}{5}L^{-1}\left(\frac{s}{s^2+1}\right) + \frac{3}{5}L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= \frac{1}{5}e^{2t} - \frac{1}{5}\cos t + \frac{3}{5}\sin t$$

Theorem

$$L\left(\int_0^t f(u)\,du\right) = \frac{1}{s} L(f(t))$$

Proof

Let $g(t) = \int_0^t f(u)\,du$

$\therefore \quad g'(t) = f(t)$ (Fundamental Theorem of calculus)

and $g(0) = \int_0^0 f(u)\,du = 0.$

$$\because \; L(g'(t)) = sL(g(t)) - g(0)$$

$$\therefore \; L(f(t)) = sL\left(\int_0^t f(u)\,du\right)$$

$$\therefore \; L\left(\int_0^t f(u)\,du\right) = \frac{1}{s} L(f(t)) \quad //$$

Example 11

Find $L^{-1}\left(\frac{1}{s^2(s^2+\omega^2)}\right)$

Solution

Let $\frac{1}{s^2(s^2+\omega^2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+\omega^2}$

$$1 = As(s^2+\omega^2) + B(s^2+\omega^2) + s^2(Cs+D)$$

$$s=0 \Rightarrow 1 = B\omega^2 \Rightarrow B = \frac{1}{\omega^2}$$

$$s=i\omega \Rightarrow 1 = -\omega^2(Ci\omega+D)$$

$$= -D\omega^2 - iC\omega^3$$

$$\Rightarrow D = -\frac{1}{\omega^2}, \quad C = 0$$

Compare coefficient of $s \Rightarrow 0 = A\omega^2 \Rightarrow A = 0$

$$\therefore \quad \frac{1}{s^2(s^2+\omega^2)} = \frac{1/\omega^2}{s^2} + \frac{-1/\omega^2}{s^2+\omega^2}$$

$$\therefore \; L^{-1}\left(\frac{1}{s^2(s^2+\omega^2)}\right)$$

$$= \frac{1}{\omega^2} L^{-1}\left(\frac{1}{s^2}\right) - \frac{1}{\omega^2} L^{-1}\left(\frac{1}{s^2+\omega^2}\right)$$

$$= \frac{1}{\omega^2} t - \frac{1}{\omega^3} L^{-1}\left(\frac{\omega}{s^2+\omega^2}\right)$$

$$= \frac{1}{\omega^2} t - \frac{1}{\omega^3} \sin \omega t$$

Example 11 (Second solution: as in the notes)

Find $L^{-1}\left(\dfrac{1}{s^2(s^2+\omega^2)}\right)$

Solution

We start with

$$L(\sin \omega t) = \frac{\omega}{s^2+\omega^2}$$

$$\therefore\ L\left(\frac{1}{\omega}\sin\omega t\right) = \frac{1}{s^2+\omega^2}$$

By Theorem 5,

$$L\left(\int_0^t \frac{1}{\omega}\sin\omega u\,du\right) = \frac{1}{s}L\left(\frac{1}{\omega}\sin\omega t\right)$$

$$= \frac{1}{s(s^2+\omega^2)}$$

$$\therefore L\left(-\frac{1}{\omega^2}\cos\omega u \Big|_0^t \right) = \frac{1}{s(s^2+\omega^2)}$$

$$\therefore L\left(\frac{1-\cos\omega t}{\omega^2} \right) = \frac{1}{s(s^2+\omega^2)}$$

Using Theorem 5 again, we have

$$L\left(\int_0^t \frac{1-\cos\omega u}{\omega^2}\,du\right) = \frac{1}{s} L\left(\frac{1-\cos\omega t}{\omega^2}\right)$$

$$= \frac{1}{s^2(s^2+\omega^2)}$$

$$\therefore \mathcal{L}^{-1}\left(\frac{1}{s^2(s^2+\omega^2)}\right)$$

$$= \int_0^t \frac{1-\cos\omega u}{\omega^2}\,du$$

$$= \frac{1}{\omega^2}\left[u - \frac{1}{\omega}\sin\omega u\right]_0^t$$

$$= \frac{1}{\omega^2}\left(t - \frac{1}{\omega}\sin\omega t\right)$$

Theorem (s-Shifting)

If $L(f(t)) = F(s)$,

then $L(e^{ct} f(t)) = F(s-c)$.

Proof

$$L(e^{ct} f(t)) = \int_0^\infty e^{-st} e^{ct} f(t)\,dt$$

$$= \int_0^\infty e^{-(s-c)t} f(t)\,dt \quad \cdots\cdots (1)$$

Next, we have

$$F(s) = L(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$$

$$\therefore F(s-c) = \int_0^\infty e^{-(s-c)t} f(t)\,dt$$

$$= L(e^{ct} f(t)) \quad (\text{by } 1)$$

//

Example

(i) $L(t^n) = \dfrac{n!}{s^{n+1}} \Rightarrow \boxed{L(e^{ct}t^n) = \dfrac{n!}{(s-c)^{n+1}}}$

$$\Rightarrow L^{-1}\left\{\frac{n!}{(s-c)^{n+1}}\right\} = e^{ct}t^n$$

$$\Rightarrow \boxed{L^{-1}\left\{\frac{1}{(s-c)^n}\right\} = \frac{e^{ct}t^{n-1}}{(n-1)!}}$$

(ii) $L(\cos\omega t) = \dfrac{s}{s^2+\omega^2} \Rightarrow \boxed{L(e^{ct}\cos\omega t) = \dfrac{s-c}{(s-c)^2+\omega^2}}$

$$\Rightarrow \boxed{L^{-1}\left(\frac{s-c}{(s-c)^2+\omega^2}\right) = e^{ct}\cos\omega t}$$

(iii) $L(\sin \omega t) = \dfrac{\omega}{s^2+\omega^2} \Rightarrow \boxed{L(e^{ct} \sin \omega t) = \dfrac{\omega}{(s-c)^2+\omega^2}}$

$$\Rightarrow \boxed{L^{-1}\left(\frac{\omega}{(s-c)^2+\omega^2}\right) = e^{ct} \sin \omega t}$$

Example 12. Solve $y'' + 2y' + 5y = 0$,

$y(0) = 2$, $y'(0) = -4$.

Solution

Set $L(y) = Y$.

We have $L(y'' + 2y' + 5y) = 0$

$$\therefore\ s^2 Y - s\,y(0) - y'(0) + 2\{sY - y(0)\} + 5Y = 0$$

$$\therefore \quad (s^2+2s+5)Y - 2s = 0$$

$$\therefore \quad Y = \frac{2s}{s^2+2s+5} = \frac{2\{(s+1)-1\}}{(s+1)^2+2^2}$$

$$= \frac{2(s+1)}{(s+1)^2+2^2} - \frac{2}{(s+1)^2+2^2}$$

$$\therefore\ y(t) = L^{-1}(Y) = 2L^{-1}\left(\frac{(s+1)}{(s+1)^2+2^2}\right) - L^{-1}\left(\frac{2}{(s+1)^2+2^2}\right)$$

$$= 2e^{-t}\cos 2t - e^{-t}\sin 2t$$

Example 13. Solve $y'' - 2y' + y = e^t + t$, $y(0) = 1$, $y'(0) = 0$.

Solution

Set $L(y) = Y$.

We have

$$L(y'' - 2y' + y) = L(e^t + t)$$

$$\therefore\ s^2 Y - s y(0) - y'(0)$$

$$-2\{sY - y(0)\} + Y = \frac{1}{s-1} + \frac{1}{s^2}$$

$$\therefore\ (s^2 - 2s + 1)Y - s + 2 = \frac{1}{s-1} + \frac{1}{s^2}$$

$$\therefore \quad Y = \frac{s-2}{(s-1)^2} + \frac{1}{(s-1)^3} + \frac{1}{s^2(s-1)^2}$$

$$= \frac{(s-1)-1}{(s-1)^2} + \frac{1}{(s-1)^3} + \frac{1}{s^2(s-1)^2}$$

$$= \frac{1}{s-1} - \frac{1}{(s-1)^2} + \frac{1}{(s-1)^3}$$

$$+ \frac{1}{s^2(s-1)^2}$$

Let $$\frac{1}{s^2(s-1)^2} \equiv \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{(s-1)^2}$$

$$\therefore\ 1 \equiv As(s-1)^2 + B(s-1)^2 + Cs^2(s-1) + Ds^2$$

$$s=0 \Rightarrow B=1$$

$$s=1 \Rightarrow D=1$$

Compare coefficients:

s^3 : $0 = A + C$

s : $0 = A - 2B \Rightarrow A = 2$

$\therefore C = -2$

$$\therefore \quad Y = \frac{1}{s-1} - \frac{1}{(s-1)^2} + \frac{1}{(s-1)^3}$$

$$+ \frac{2}{s} + \frac{1}{s^2} - \frac{2}{s-1} + \frac{1}{(s-1)^2}$$

$$= \frac{2}{s} + \frac{1}{s^2} - \frac{1}{s-1} + \frac{1}{(s-1)^3}$$

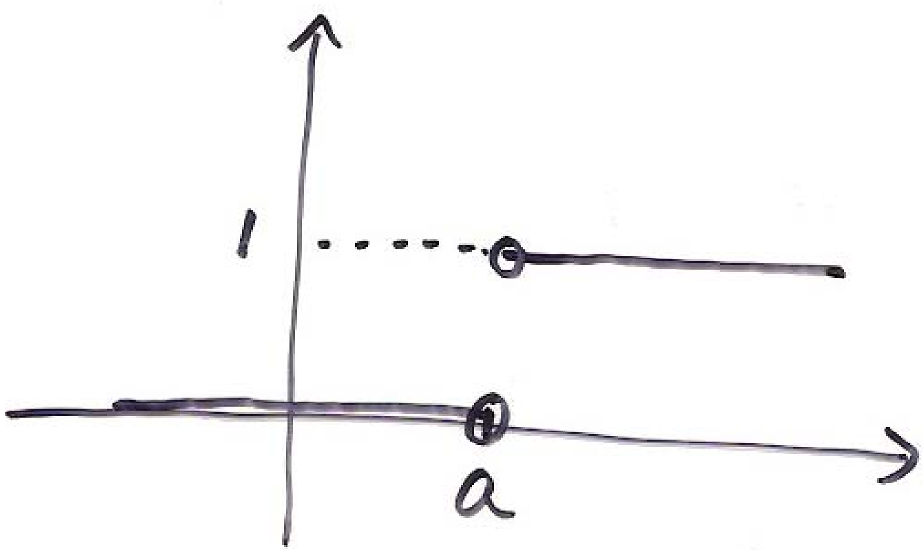
$$\therefore \quad y(t) = \underline{\underline{2 + t - e^t + \frac{1}{2}t^2 e^t}}$$

Unit Step (Heaviside) function

Definition.

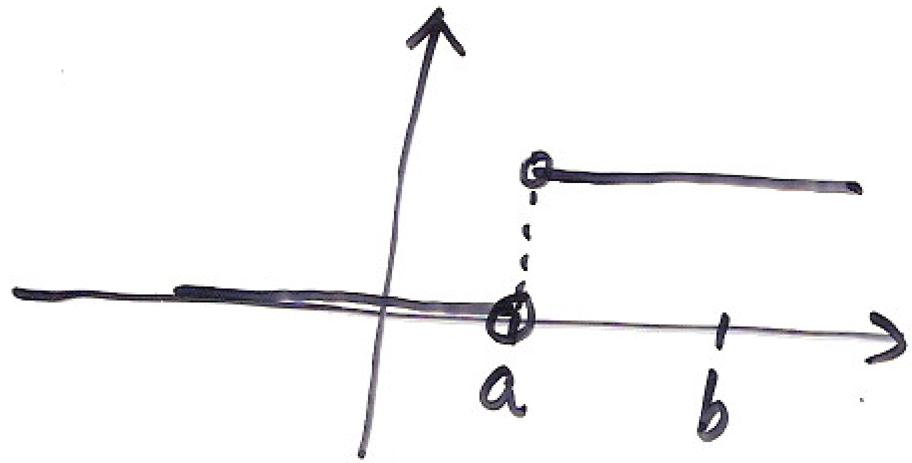
$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$$

Graph of $u(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$

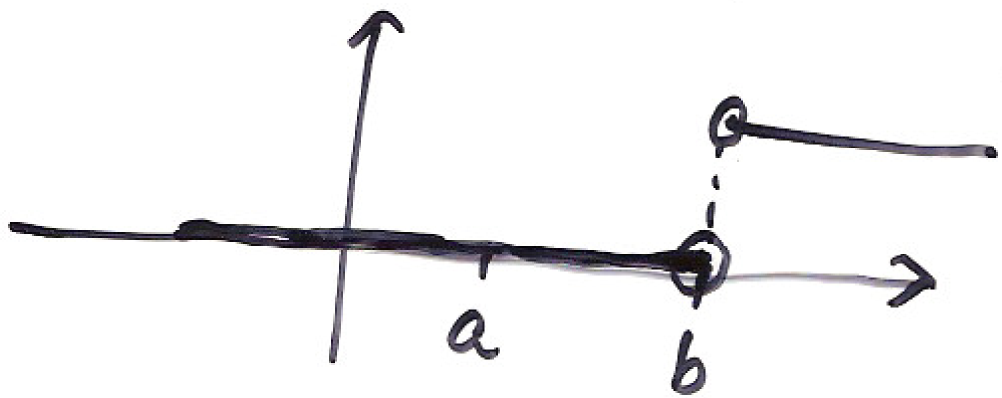


Let $0 < a < b$

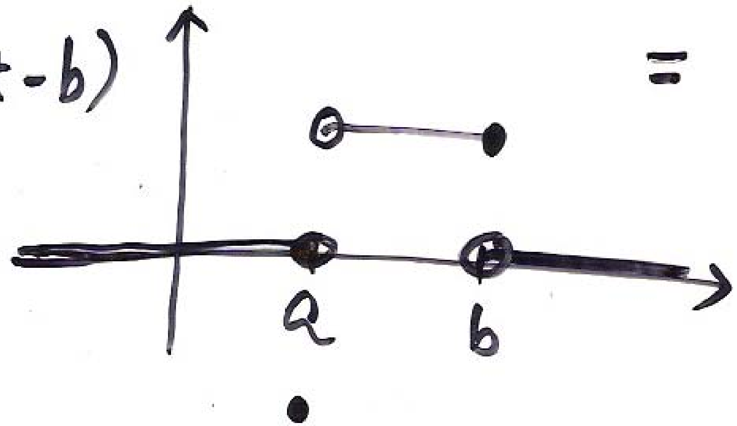
$u(t-a)$



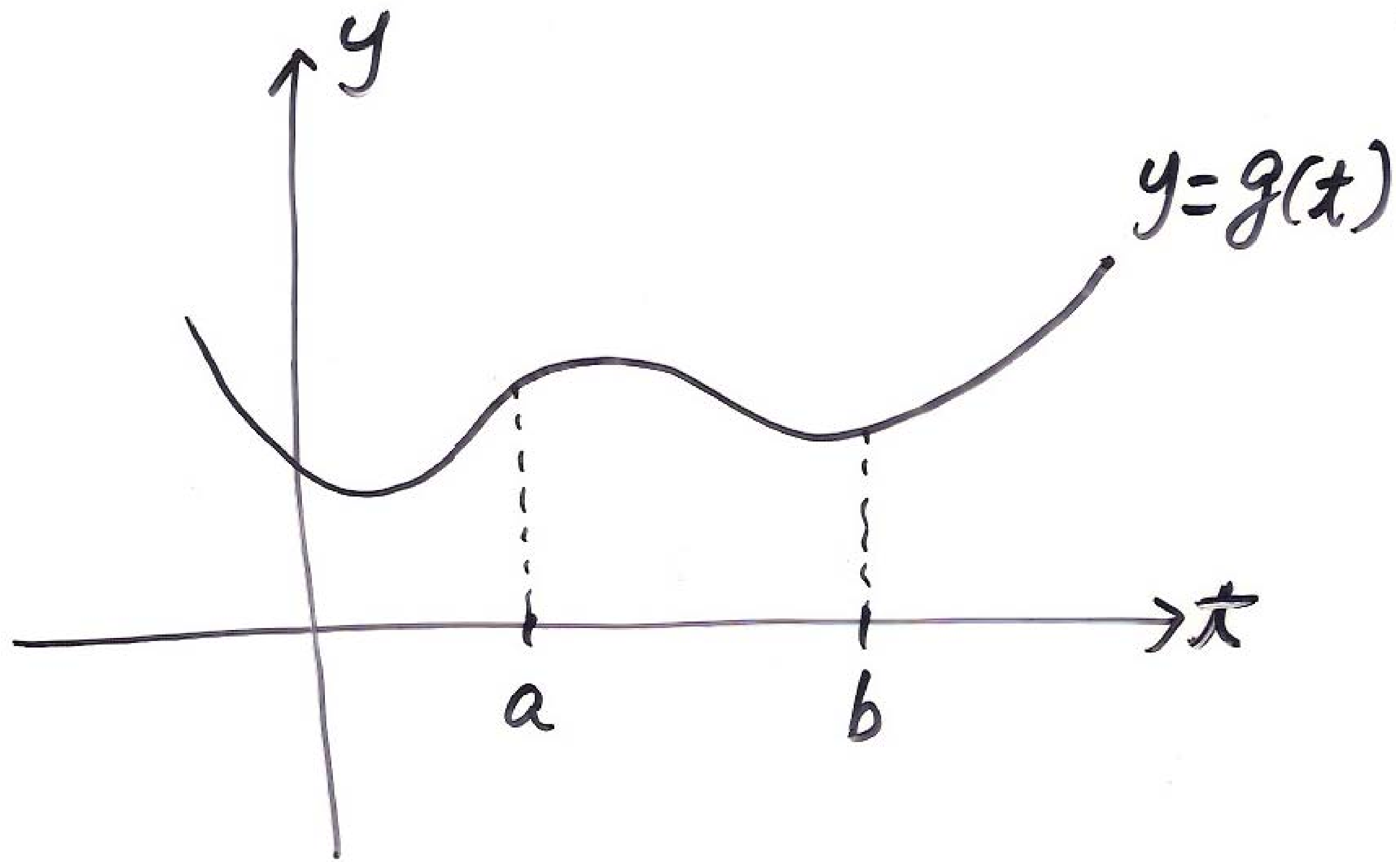
$u(t-b)$



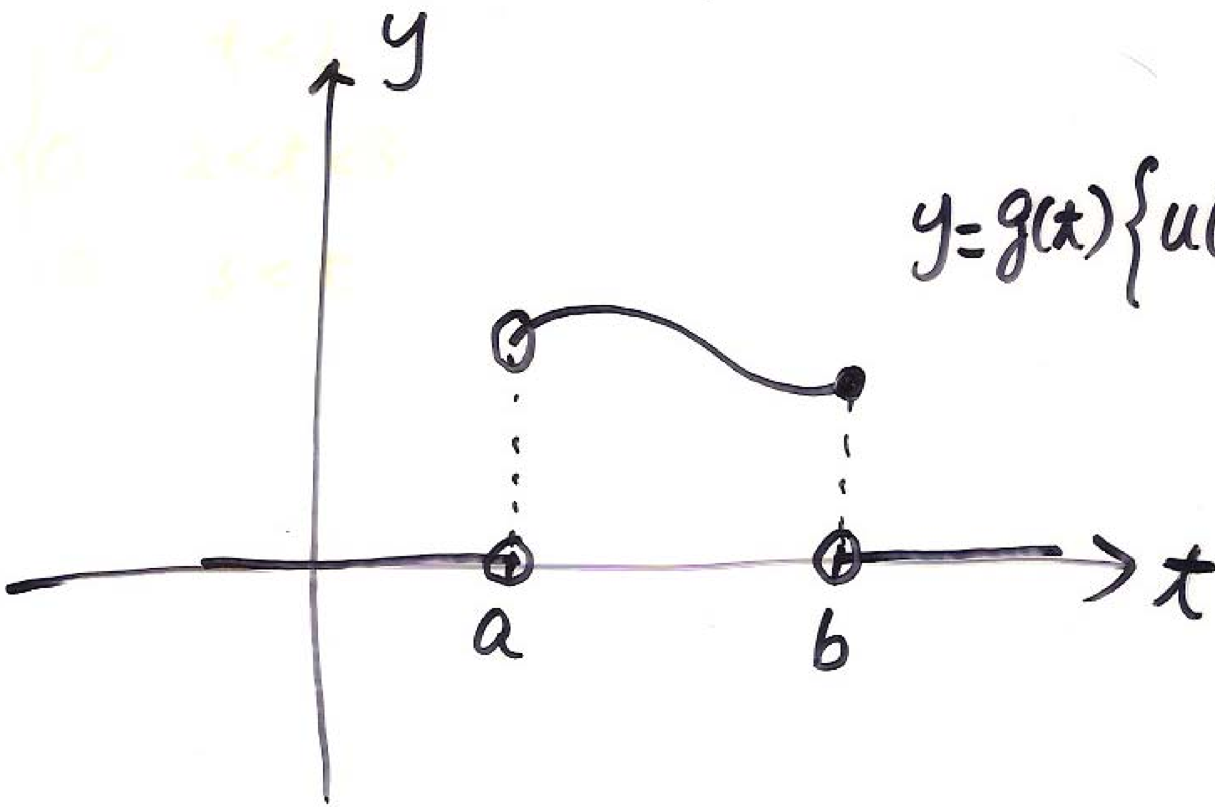
$$u(t-a)-u(t-b)$$



$$= \begin{cases} 0 & t < a \\ 1 & a < t < b \\ 0 & t > b \end{cases}$$



y
y = g(t)
t
a
b



$$y = g(t)\{u(t-a) - u(t-b)\}$$

$$g(t)\{u(t-a)-u(t-b)\} = \begin{cases} 0 & t<a \\ g(t) & a<t<b \\ 0 & b<t \end{cases}$$

Example 15. Express

$$f(t) = \begin{cases} t, & 0 < t < 1 \\ 2 - t, & 1 < t < 2 \\ 0, & 2 < t < 3 \\ 1, & t > 3 \end{cases}$$

$$f(t) = \begin{cases} t & 0<t<1 \\ 2-t & 1<t<2 \\ 0 & 2<t<3 \\ 1 & 3<t \end{cases}$$

$$= \begin{cases} 0 & t<0 \\ t & 0<t<1 \\ 0 & 1<t \end{cases} + \begin{cases} 0 & t<1 \\ 2-t & 1<t<2 \\ 0 & 2<t \end{cases}$$

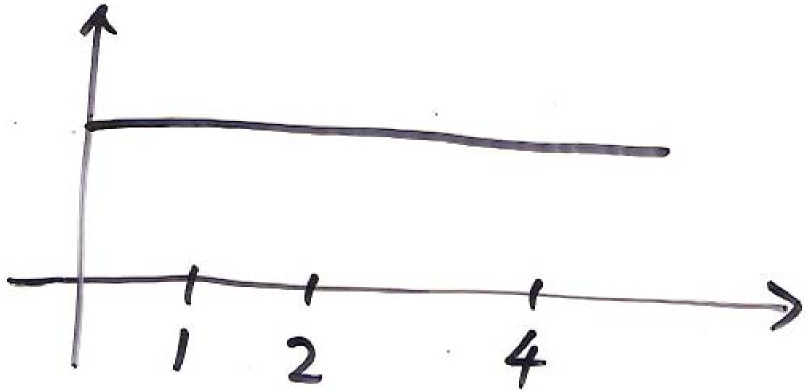
$$+ \begin{cases} 0 & t<2 \\ 0 & 2<t<3 \\ 0 & 3<t \end{cases} + \begin{cases} 0 & t<3 \\ 1 & 3<t \end{cases}$$

$$
\begin{aligned}
&= t\{u(t) - u(t-1)\} \\
&\quad + (2-t)\{u(t-1) - u(t-2)\} \\
&\quad + u(t-3)
\end{aligned}
$$

Example 16. Sketch

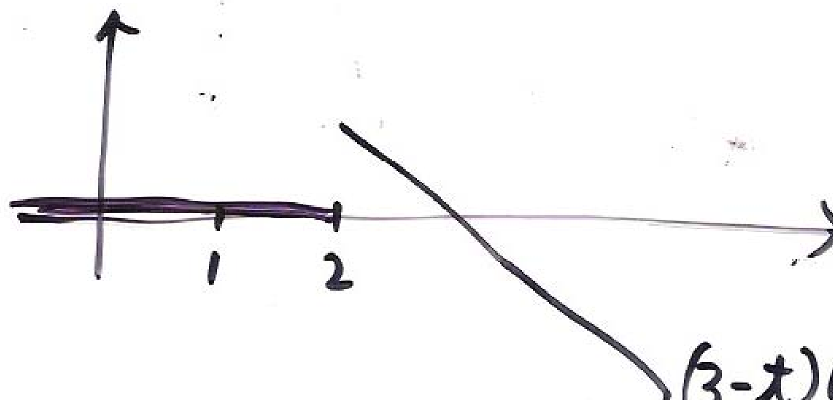
$$g(t) = 2u(t) + tu(t-1) + (3-t)u(t-2)$$
$$-3u(t-4), \quad t > 0.$$

Solution

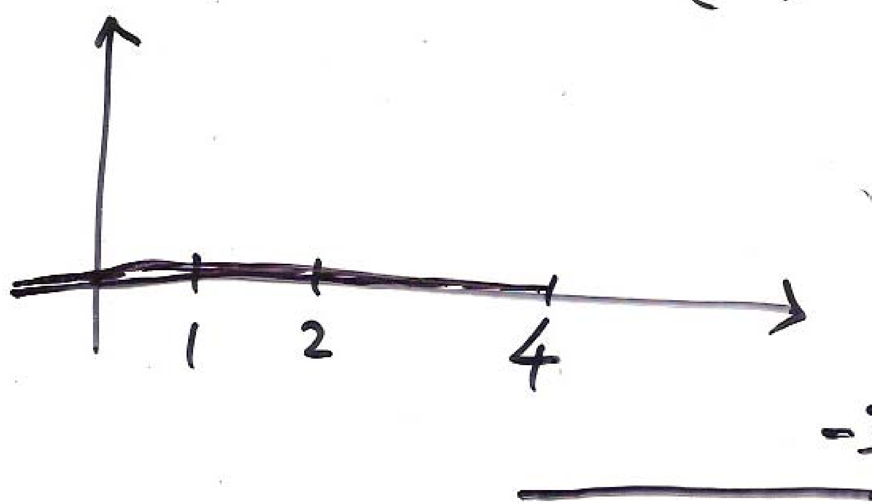


$2u(t) = 2, \; t > 0$

$tu(t-1) = t, \; t > 1$



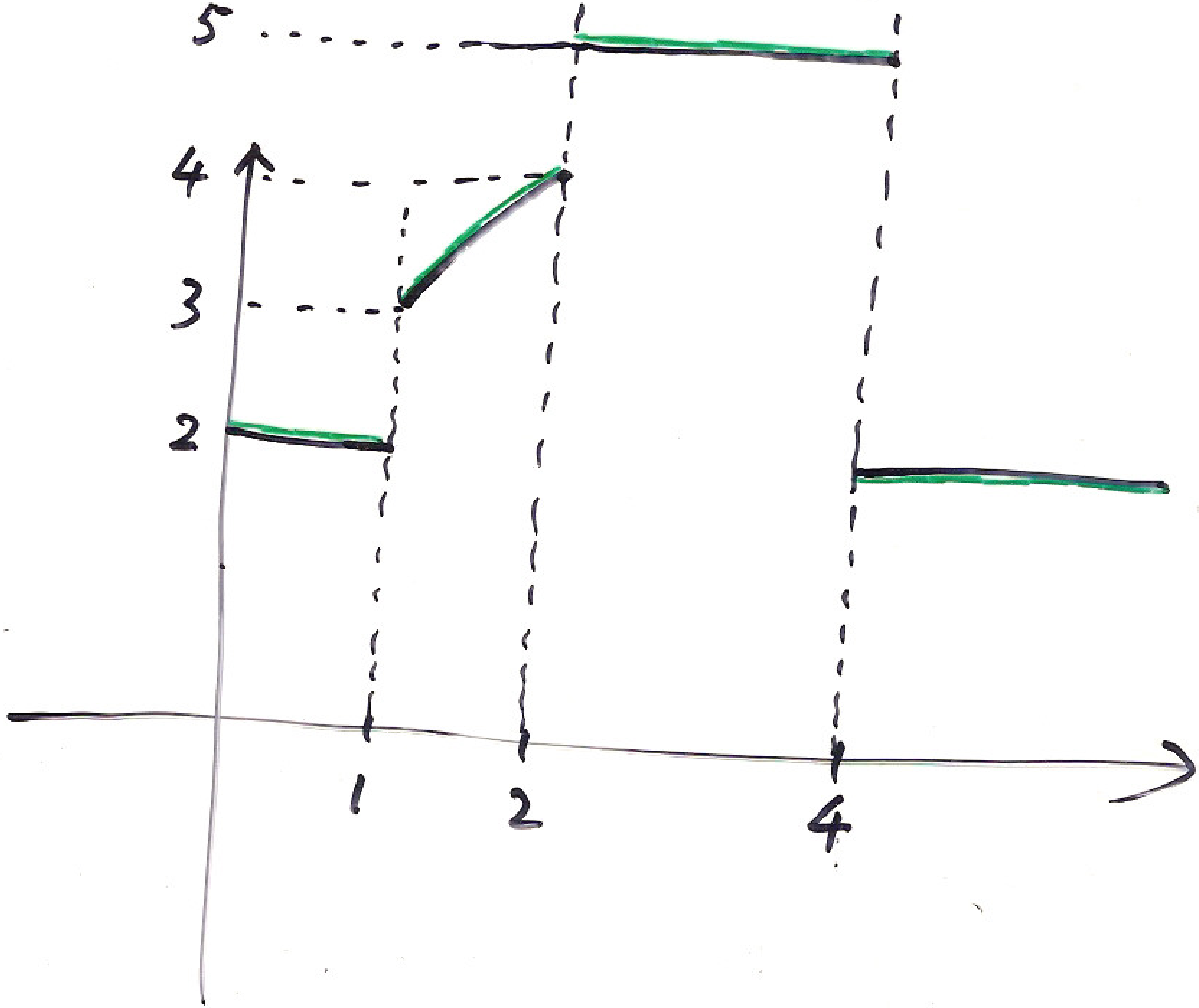
$(3-t)u(t-2) = 3-t, \quad t>2$



$-3u(t-4) = -3, \quad t>4$

$$\therefore g(t) = \begin{cases} 2 & 0 < t < 1 \\ 2+t & 1 < t < 2 \\ 2+t+(3-t) & 2 < t < 4 \\ 2+t+(3-t)-3 & 4 < t \end{cases}$$

$$= \begin{cases} 2 & 0 < t < 1 \\ 2+t & 1 < t < 2 \\ 5 & 2 < t < 4 \\ 2 & 4 < t \end{cases}$$


5
4
3
2
1
2
4

Theorem (t-Shifting)

If $L(f(t)) = F(s)$, then

$$\boxed{L\{f(t-a)u(t-a)\} = e^{-as}F(s)}$$

i.e. $$\boxed{L^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)}$$

Proof

$$\mathcal{L}\{f(t-a)u(t-a)\}$$

$$= \int_0^\infty e^{-st} f(t-a)u(t-a)\,dt$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt \quad \left(\because \begin{array}{l} u(t-a)=0 \\ \text{for } t<a \end{array}\right)$$

$$= \int_a^\infty e^{-s(t-a+a)} f(t-a)\, d(t-a)$$

$$= \int_a^\infty e^{-s(t-a)} e^{-sa} f(t-a)\, d(t-a)$$

$$= e^{-sa} \int_0^\infty e^{-su} f(u)\, du \quad (\text{let } u = t-a)$$

$$= e^{-sa} L(f(u)) = e^{-sa} F(s) \quad //$$

Example 17

Put $f(t-a) \equiv 1$.

Then $L(u(t-a)) = e^{-as} L(1)$

$$= \frac{e^{-as}}{s}$$

$$\boxed{L(u(t-a)) = \frac{e^{-as}}{s}}$$

$$\boxed{L^{-1}\left(\frac{e^{-as}}{s}\right) = u(t-a)}$$

Example 18. Compute $L(t^2 u(t-1))$.

Solution

$$L\{t^2 u(t-1)\}$$
$$= L\{((t-1)+1)^2 u(t-1)\}$$
$$= L\{(t-1)^2 u(t-1) + 2(t-1)u(t-1) + u(t-1)\}$$

$$= e^{-s}\frac{2}{s^3} + 2e^{-s}\frac{1}{s^2} + \frac{e^{-s}}{s}$$

$$= e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right)$$

Example 19. Compute $L((e^t + 1)u(t-2))$.

Solution

$$L\{(e^t+1)u(t-2)\}$$

$$= L\{(e^{(t-2)+2}+1)u(t-2)\}$$

$$= L\{e^2 e^{(t-2)}u(t-2) + u(t-2)\}$$

Recall $\mathcal{L}(e^{t}) = \frac{1}{s-1}$

$$\therefore \mathcal{L}(e^{t-2}u(t-2)) = e^{-2s}\frac{1}{s-1}$$

$$\therefore \mathcal{L}\{(e^{t}+1)u(t-2)\}$$

$$= e^{2}\frac{e^{-2s}}{s-1} + \frac{e^{-2s}}{s}$$

$$= e^{-2s}\left\{\frac{e^{2}}{s-1} + \frac{1}{s}\right\}$$

The next and final problem in this section is rather complicated, but it really just involves assembling a lot of small bits and pieces. Notice that

NONE of the methods we learned in earlier chapters would have allowed us to solve this problem, so it should convince you that the Laplace Transform is really useful!

Example 20. Solve the initial value problem

$$y'' + 3y' + 2y = g(t), \quad y(0) = 0, \quad y'(0) = 1,$$

with

$$g(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & t > 1 \end{cases}.$$

Solution

First rewrite $g(t)$ as

$$g(t) = u(t) - u(t-1).$$

$$= 1 - u(t-1).$$

$\therefore L(y''+3y'+2y) = L\{1-u(t-1)\}$

Putting $Y(s) = L(y)$, we have

$$s^2Y - sy(0) - y'(0) + 3\{sY - y(0)\} + 2Y = \frac{1}{s} - \frac{e^{-s}}{s}$$

$$\therefore\ s^2Y - 1 + 3sY + 2Y = \frac{1}{s} - \frac{e^{-s}}{s}$$

$$\therefore\ (s^2+3s+2)Y = \frac{1}{s} + 1 - \frac{e^{-s}}{s}$$

$$= \frac{s+1}{s} - \frac{e^{-s}}{s}$$

$$\therefore\ Y = \frac{s+1}{s(s^2+3s+2)} - \frac{e^{-s}}{s(s^2+3s+2)}$$

$$\frac{s+1}{s(s^2+3s+2)} = \frac{s+1}{s(s+1)(s+2)}$$

$$= \frac{1}{s(s+2)}$$

$$= \frac{1}{2}\left\{\frac{1}{s} - \frac{1}{s+2}\right\}$$

$$\therefore L^{-1}\left(\frac{s+1}{s(s^2+3s+2)}\right) = \frac{1}{2}(1-e^{-2t}).$$

$$\frac{1}{s(s^2+3s+2)} = \frac{1}{s(s+1)(s+2)}$$

$$\equiv \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$\therefore\ 1 \equiv A(s+1)(s+2) + Bs(s+2) + Cs(s+1)$$

$$s=0 \Rightarrow A = \frac{1}{2}$$

$$s=-1 \Rightarrow B = -1$$

$$s=-2 \Rightarrow C = \frac{1}{2}$$

$$\therefore L^{-1}\left(\frac{1}{s(s^2+3s+2)}\right) = L^{-1}\left(\frac{1/2}{s} + \frac{-1}{s+1} + \frac{1/2}{s+2}\right)$$

$$= \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$$

$$\therefore L^{-1}\left\{\frac{e^{-s}}{s(s^2+3s+2)}\right\}$$

$$= \left\{\frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}\right\} u(t-1)$$

(use t-shifting law)

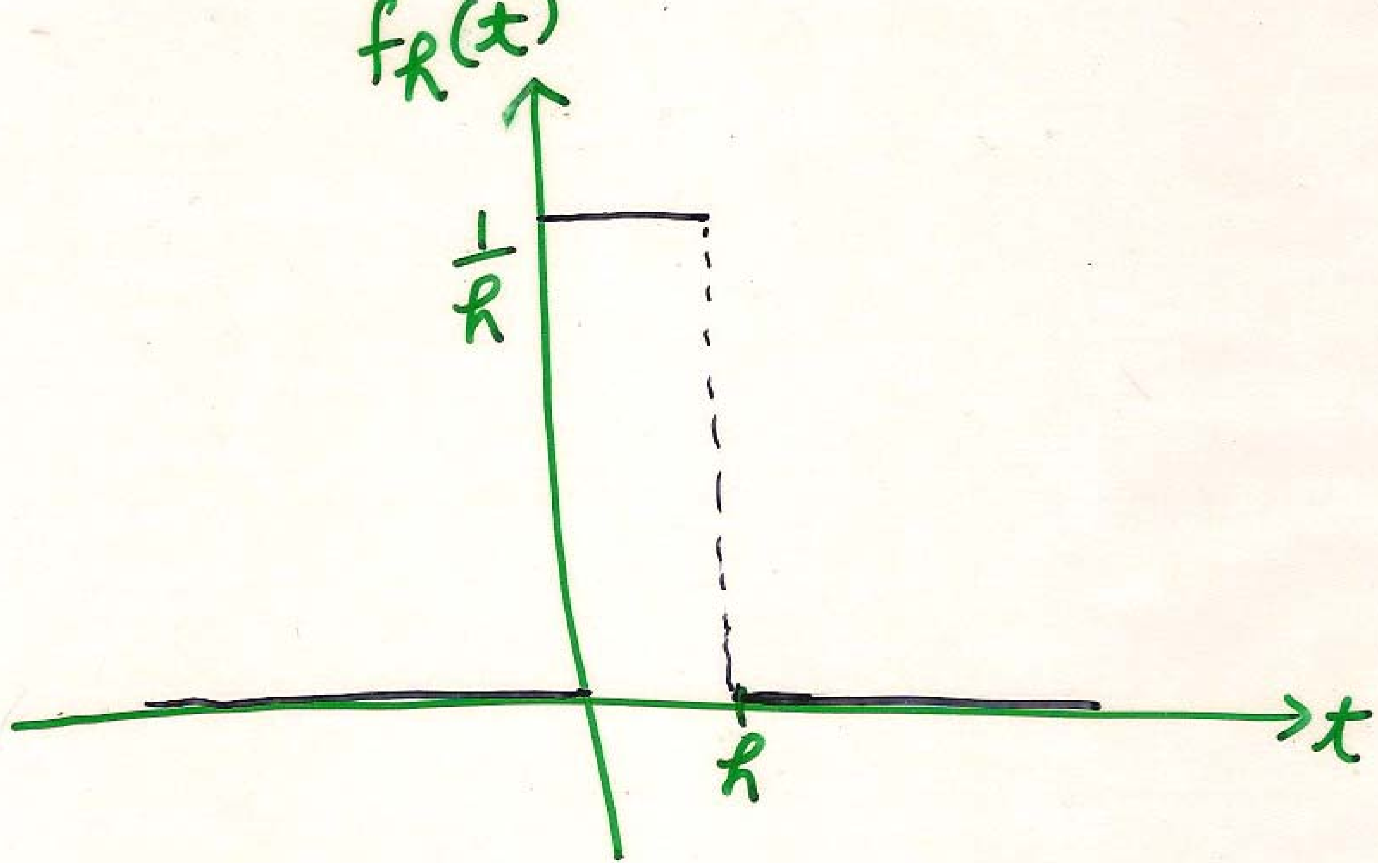
$$\therefore\ y(t) = \frac{1}{2}(1 - e^{-2t})$$

$$- \left\{ \frac{1}{2} - e^{-(t-1)} + \frac{1}{2} e^{-2(t-1)} \right\} u(t-1)$$

THE DIRAC DELTA FUNCTION

For each $h > 0$, define

$$f_h(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{h} & 0 \leq t \leq h \\ 0 & h < t \end{cases}$$



Observe: $\int_0^\infty f_h(t)\,dt = \left(\frac{1}{h}\right)(h) = 1$.

Thus for example $f_{10^{-100}}(t)$ is a function with maximum value 10^{100} and yet the area under its graph is still 1. The graph is an extremely tall but sharp and narrow spike next to $t = 0$.

Define $\delta(t) = \lim_{h \to 0} f_h(t)$.

Let g be a continuous function.

Then $\int_0^\infty f_h(t)\,g(t)\,dt$

$= \frac{1}{h}\int_0^h g(t)\,dt$

$$= \frac{1}{h} g(c)(h-0) \quad \text{where } 0 \le c \le h$$

(Using first Mean Value Theorem for integrals)

$$= g(c)$$

Letting $h \to 0$

$$\Rightarrow \int_0^\infty \delta(t) g(t)\, dt = g(0).$$

Similarly, we define for any a,

$$f_R(t-a) = \begin{cases} 0 & t < a \\ \frac{1}{R} & a \le t \le a+R \\ 0 & a+R < t \end{cases}$$

$$\delta(t-a) = \lim_{R \to 0} f_R(t-a)$$

Then for any continuous function g, we have

$$\int_0^\infty \delta(t-a)\, g(t)\, dt = g(a).$$

Example

$$L(\delta(t-a)) = \int_0^\infty e^{-st}\delta(t-a)\,dt$$

$$= e^{-st}\Big|_{t=a}$$

$$= e^{-as}$$

$$\therefore \boxed{L(\delta(t-a) = e^{-as}}$$

$$\boxed{L^{-1}(e^{-as}) = \delta(t-a)}$$

Let $a = 0$.

$$\boxed{L(\delta(t)) = 1}$$

$$\boxed{L^{-1}(1) = \delta(t)}$$

EXAMPLE: INJECTIONS!

Suppose that a doctor injects, almost instantly, 100 mg of morphine into a patient. He does it again 24 hours later. Suppose that the HALF-LIFE of morphine in the patient's body is 18 hours. Find the amount of morphine in the patient at any time.

Suppose c mg of medicine is injected into a person at time $t=a$.

We try to model this mathematically.

We observe that although the injection is done at $t=a$, the whole process still needs a bit of time, say h seconds, to complete. So actually the medicine enters the blood during the time interval a to $a+h$.

Since c mg enters the blood during h sec.,

the average rate is $c/h = c f_h(t-a)$

Now $\lim_{h \to 0} f_h(t-a) = \delta(t-a)$

$\therefore \quad c/h \simeq c\,\delta(t-a)$ when h is small

Therefore, we can use $c\,\delta(t-a)$ to model

the rate of this injection.

Solution: Half-life refers to the exponential function e^{-kt}. "Half-life 18 hours" $= 0.75$ days means $\frac{1}{2} = e^{-k\times 0.75}$, that is, k $= \dfrac{\ell n(2)}{0.75} = 0.924$. So without the injections,

$$\frac{dy}{dt} = -ky, \quad k = 0.924. \quad y(0)=0.$$

Next with two injections at $t=0$ and $t=1$ at 100 mg each time, we have

$$\frac{dy}{dt} = -(0.924)y + 100\,\delta(t) + 100\,\delta(t-1).$$

By the way, notice that we have to think of the delta function as something which itself has UNITS. In this case you should think of the delta function as something that has units of [1/time], so that this equation has consistent units. In many problems, it is actually very helpful to work out what units the delta function has — it can have different units in different problems! This is particularly useful in physics problems where something gets hit suddenly and gains some momentum instantly — you can use this in some of the tutorial problems.

Let $L(y) = Y$.

$$\therefore \; sY - y(0) = -0.924Y + 100 + 100e^{-s}$$

$$\therefore \; (s + 0.924)Y = 100 + 100e^{-s}$$

$$\therefore \; Y = \frac{100}{s+0.924} + 100\frac{e^{-s}}{s+0.924}$$

$$\therefore y = 100\, L^{-1}\left(\frac{1}{s+0.924}\right)$$

$$+100\, L^{-1}\left(\frac{1}{s+0.924}e^{-s}\right)$$

$$\because L^{-1}\left(\frac{1}{s+0.924}\right) = e^{-0.924t}$$

$$\therefore L^{-1}\left(\frac{1}{s+0.924}e^{-s}\right) = e^{-0.924(t-1)}\, u(t-1)$$

(by the t-shifting theorem)

$$\therefore\ y = 100e^{-0.924t} + 100e^{-0.924(t-1)}u(t-1)$$

$$= 100e^{-0.924t} + 100e^{0.924}e^{-0.924t}u(t-1)$$

$$= \begin{cases} 100e^{-0.924t}, & 0<t<1 \\ 100e^{-0.924t}(1+e^{0.924}), & t>1 \end{cases}$$

EXAMPLE: PARAMETER RECONSTRUCTION!

Sometimes it happens, in Engineering applications, that you have a system whose nature you understand [for example, you may know that it is a damped harmonic oscillator] but you don't know the values of the parameters [the spring constant, the mass, the friction coefficient]. In such a case, what you can do is to "poke" the system with a sudden, sharp force, and watch how it behaves. Then you can **reconstruct the parameters of the system** as follows. Suppose for example that you have a damped harmonic oscillator — say, a spring with a mass attached — which is initially at rest, that is, x(0) = $\dot{x}(0) = 0$. and you poke it with a unit impulse [change of momentum = 1 in MKS units] at time t = 1; in other words, the applied force is just $\delta(t-1)$.

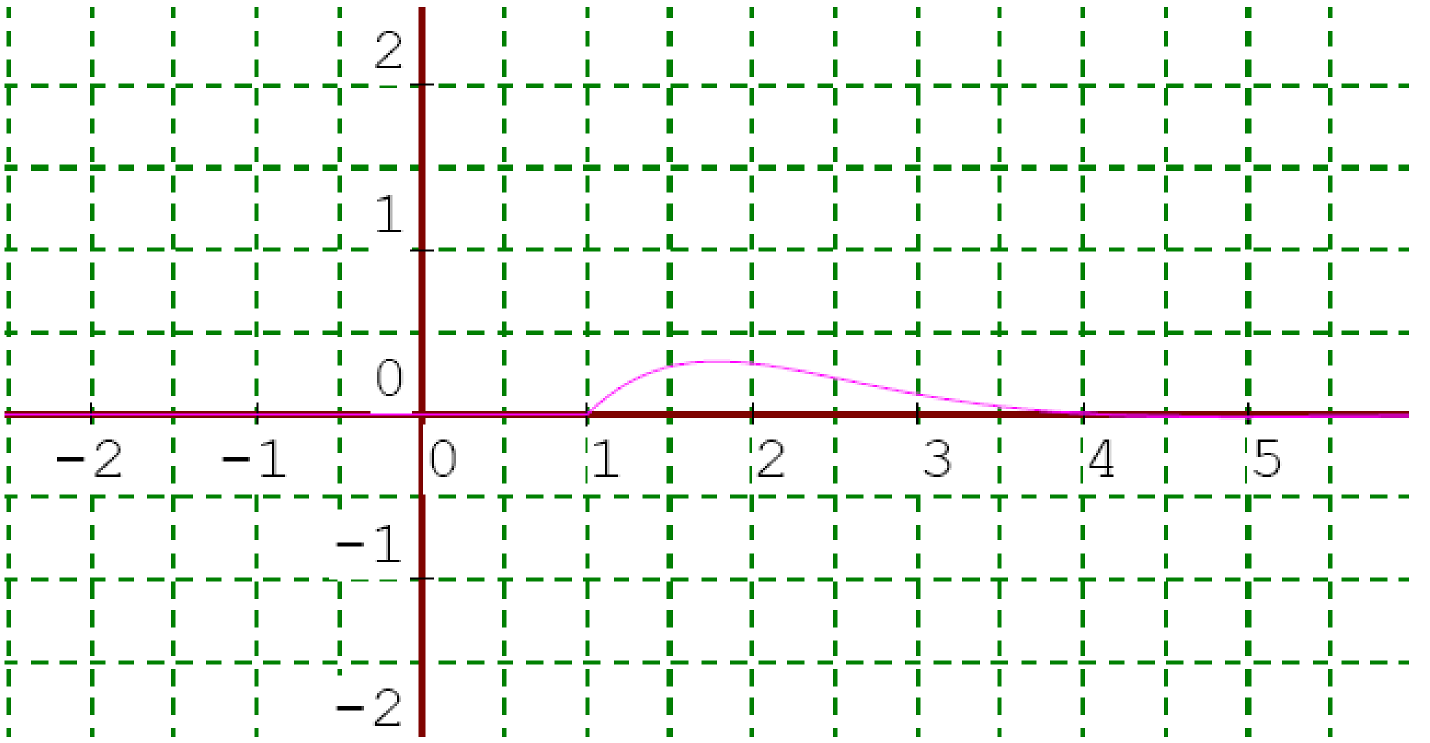
You observe that the displacement of the mass is

$$x(t) = u(t-1)e^{-(t-1)}sin(t-1).$$

[You can use Graphmatica to look at the graph by putting in

$$y = exp(-(x-1)) * sin(x-1) * step(x-1);$$

note that this oscillator is underdamped, despite the shape of the graph!] Question: what are the values of the mass, the spring constant, the frictional coefficient?



2
1
0
−1
−2
−2
−1
0
1
2
3
4
5

Solution: Newton's Second Law in this case says:

$$M\ddot{x} = -kx - b\dot{x} + \delta(t-1).$$

With the given initial data, taking the Laplace transform gives

$$Ms^2 X(s) = -kX(s) - bsX(s) + e^{-s},$$

and so

$$X(s) = \frac{e^{-s}}{Ms^2 + bs + k}.$$

But the Laplace transform of the given response function is [using both t-shifting and s-shifting!]

$$X(s) = \frac{e^{-s}}{(s+1)^2 + 1} = \frac{e^{-s}}{s^2 + 2s + 2},$$

so by inspection we see that M = 1, b = 2, k = 2 in MKS units, and we have successfully reconstructed the parameters of this system, just by hitting it with a delta function impulse! Similar ideas work for electrical circuits etc etc etc. Useful idea.

If you look at the graph of the solution you will see that the derivative jumps suddenly at $t = 1$; that is, there is a sharp corner there. This is typical in problems involving delta-function impulses, basically because such impulses cause the momentum, and therefore the velocity, to change suddenly.

A Short Table of Laplace Transforms

$f(t)$	$F(s) = \mathcal{L}(f(t)) = \int_0^\infty f(t)\, e^{-st} dt$
$t^n \,;\, n = 0, 1, 2, 3, \ldots$	$\dfrac{n!}{s^{n+1}}$
e^{at}	$\dfrac{1}{s - a}$
$\sin at$	$\dfrac{a}{s^2 + a^2}$

$\cos at$	$\frac{s}{s^2+a^2}$
$U(t-a)\ ;\ a>0$	$\frac{e^{-as}}{s}$
$e^{at} f(t)$	$F(s-a)$
$f(t-a)\ U(t-a)\ ;\ a>0$	$e^{-as}\ F(s)$
$\frac{d}{dt}\{f(t)\}$	$sF(s)-f(0)$

$\frac{d^2}{dt^2}\{f(t)\}$	$s^2 F(s) - s f(0) - f'(0)$
$\frac{d^n}{dt^n}\{f(t)\}$; $n = 1, 2, 3, \ldots$	$s^n F(s) - s^{n-1} f(0) - \sum_{r=1}^{n-1} s^{n-r-1} f^{(r)}(0)$
$\delta(t-a)$	e^{-as}