Supplementary solutions of T3

2(d) Solve

$$y'' + 4y = (\sin x)^2 = \frac{1}{2}(1 - \cos 2x)$$
$$= \frac{1}{2} - \frac{1}{2}\cos 2x$$

Soln: Consider

$$y'' + 4y = \frac{1}{2}$$

and $y'' + 4y = -\frac{1}{2}\cos 2x.$

A particular soln for $y'' + 4y = \frac{1}{2}$ is $\frac{1}{8}$

Next consider $y'' + 4y = -\frac{1}{2}\cos 2x$

First note that a general soln for $y'' + 4y = 0$ is $c\cos 2x + D\sin 2x$

$\cos 2x$ appears in the above soln.

So let

$$y_p = x\,(A\cos 2x + B\sin 2x)$$

for $y'' + 4y = -\frac{1}{2}\cos 2x$

Subst. y_p'' and y_p into above ODE

get $A = 0, \quad B = -\frac{1}{8}$

$$\therefore\ y_p = -\frac{1}{8}\,x\sin 2x$$

$\therefore$ A particular soln for

$$y'' + 4y = \frac{1}{2} - \frac{1}{2}\cos 2x$$

is $\frac{1}{8} - \frac{1}{8}\,x\sin 2x$

4 (a) Prove that

$$\frac{d^2 y(x)}{dx^2} = \frac{d}{dy} \left(y'(x)\right)^2/2$$

Proof

$$\frac{d}{dy}\left(y'(x)\right)^2/2$$

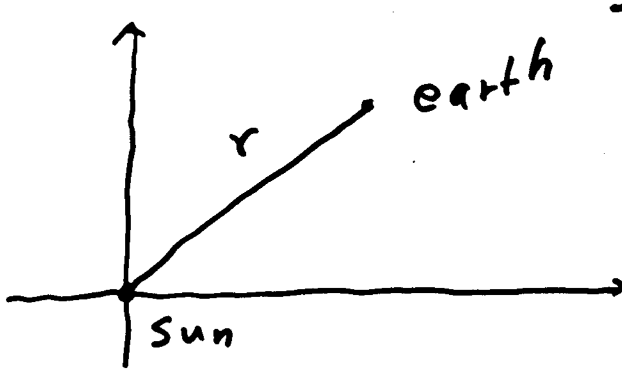
$$= \frac{1}{2}\ 2\ y'(x)\ \frac{d}{dy}\ y'(x)$$

$$= y'(x)\ \frac{d}{dx}\ y'(x)\ \frac{dx}{dy} \quad \text{chain Rule}$$

$$= \frac{dy}{dx}\ \frac{d^2y}{dx^2}\ \frac{dx}{dy}$$

$$= \frac{d^2y}{dx^2}$$

(b)



Suppose the earth were to stop moving. Then the earth would fall towards the sun according to

$$\ddot{r} = -\frac{GM}{r^2}$$

By (a),

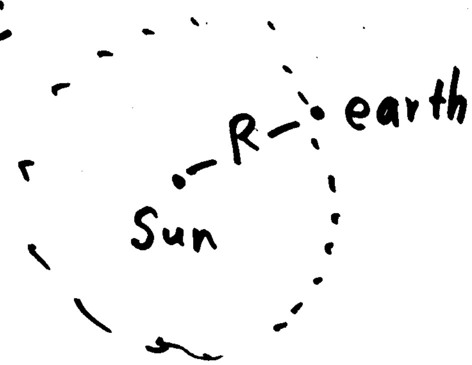
$$\frac{d^2 r}{dt} = -\frac{GM}{r^2}$$

can be written as

$$\frac{d}{dr}\frac{(\dot{r})^2}{2} = -\frac{GM}{r^2}$$

$$d(\dot{r})^2 = -\frac{2GM}{r^2}\,dr$$

$$\therefore (\dot{r})^2 = \frac{2GM}{r} + c$$



Note that at $r = R$

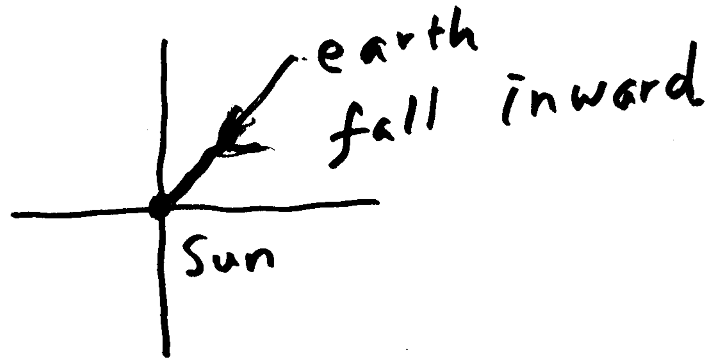
$$\dot{r} = 0$$

$$\therefore c = -\frac{2GM}{R}$$

$$\therefore (\dot{r})^2 = 2GM\left(\frac{1}{r} - \frac{1}{R}\right)$$

$$\dot{r}(t) = \pm\sqrt{2GM}\sqrt{\frac{1}{r}-\frac{1}{R}}$$

We take $-$, since



$$\frac{dr}{dt} = -\sqrt{2GM}\sqrt{\frac{1}{r}-\frac{1}{R}}$$

$$\int dt = \frac{1}{\sqrt{2GM}}\,\frac{-1}{\sqrt{\frac{1}{r}-\frac{1}{R}}}\,dr \, .$$

Let $x = \frac{r}{R}$ $\quad dr = R\,dx$

$$\int dt = \int \frac{R^{\frac{3}{2}}}{\sqrt{2GM}}\,\frac{-dx}{\sqrt{\frac{1}{x}-1}}$$

$$\int \frac{1}{\sqrt{\frac{1}{x} - 1}}\, dx = \int \frac{\sqrt{x}}{\sqrt{1-x}}\, dx$$

$$= \sin^{-1}(\sqrt{x}) - \sqrt{x(1-x)}$$

From online integrator

or

$$\frac{1}{\sqrt{1-x}} = (1-x)^{-\frac{1}{2}}$$

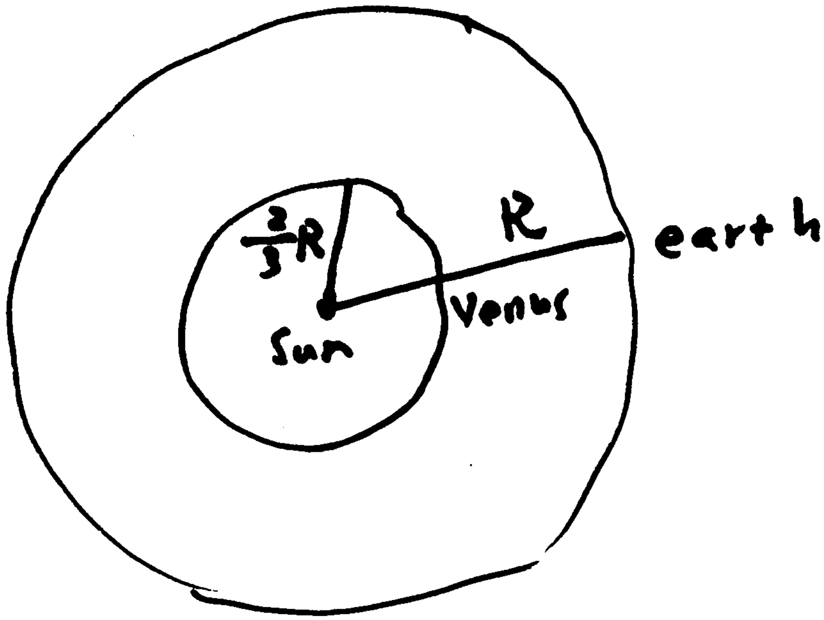
$$= 1 + \frac{1}{2}x + \frac{1}{2}\frac{3}{4}x^2 + \frac{1}{2}\frac{3}{4}\frac{5}{6}x^3 + \cdots$$

for $-1 < x < 1$

$$\int \frac{\sqrt{x}}{\sqrt{1-x}}\, dx = \int \left(\sqrt{x} + \frac{1}{2}x^{\frac{3}{2}} + \frac{1}{2}\frac{3}{4}x^{\frac{5}{2}} + \cdots\right) dx$$

$$\approx \int \left(\sqrt{x} + \frac{1}{2}x^{\frac{3}{2}} + \frac{1}{2}\frac{3}{4}x^{\frac{5}{2}}\right) dx$$

How long reach the orbit of Venus



when $r = R$, $x = \frac{r}{R} = 1$

$r = \frac{2}{3}R$, $x = \frac{2}{3}$

$$t = \int_0^t dt = \frac{-R^{\frac{2}{3}}}{\sqrt{2GM}} \int_1^{\frac{2}{3}} \frac{1}{\sqrt{\frac{1}{x} - 1}}\, dx$$

$\simeq$ 44.74 days