

# Chapter 10. Surface Integrals

## 10.1 Parametric Surfaces

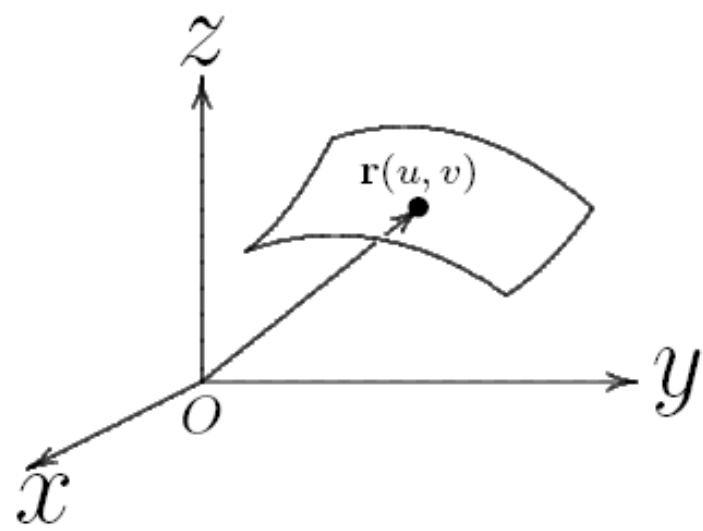
A **parametric representation** of a surface is given by the two-variable vector function

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k} \quad (1)$$

where  $u$  and  $v$  are two independent parameters.

The collection of points with position vectors (1) form a surface in the  $xyz$ -space.

The equations  $x = x(u, v), y = y(u, v), z = z(u, v)$  are called the **parametric equations** of the surface.



### 10.1.1 Example (Planes)

For a general plane  $ax + by + cz = d$ , we can let two of the three components be  $u$  and  $v$  and obtain the remaining component in terms of  $u$  and  $v$  using the above equation.

E.g.  $3x + 2y - 4z = 6$ : Let  $x(u, v) = u$ ,  $y(u, v) = v$ .

Then  $z(u, v) = \frac{1}{4}(3x + 2y - 6)$ . So the parametric representation of this plane is

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + \left(\frac{1}{4}(3u + 2v - 6)\right) \mathbf{k}.$$

If one variable is absent from the equation, we let the missing component be  $u$  or  $v$ .

E.g.  $2y + x = 7$ : Let  $z(u, v) = u$ . Then  $y(u, v) = v$

and  $x(u, v) = 7 - 2v$ .

$$\mathbf{r}(u, v) = (7 - 2v)\mathbf{i} + v\mathbf{j} + u\mathbf{k}.$$

If two variables are absent from the equation, we let the two missing components be  $u$  or  $v$ .

E.g. The  $xy$ -plane is given by

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + 0\mathbf{k}.$$



### 10.1.2 Example (Surfaces of the form $z = f(x, y)$ )

A natural parametric representation of  $S$  is

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + f(u, v)\mathbf{k}$$

E.g. The paraboloid  $z = x^2 + y^2$ .

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + (u^2 + v^2)\mathbf{k}.$$

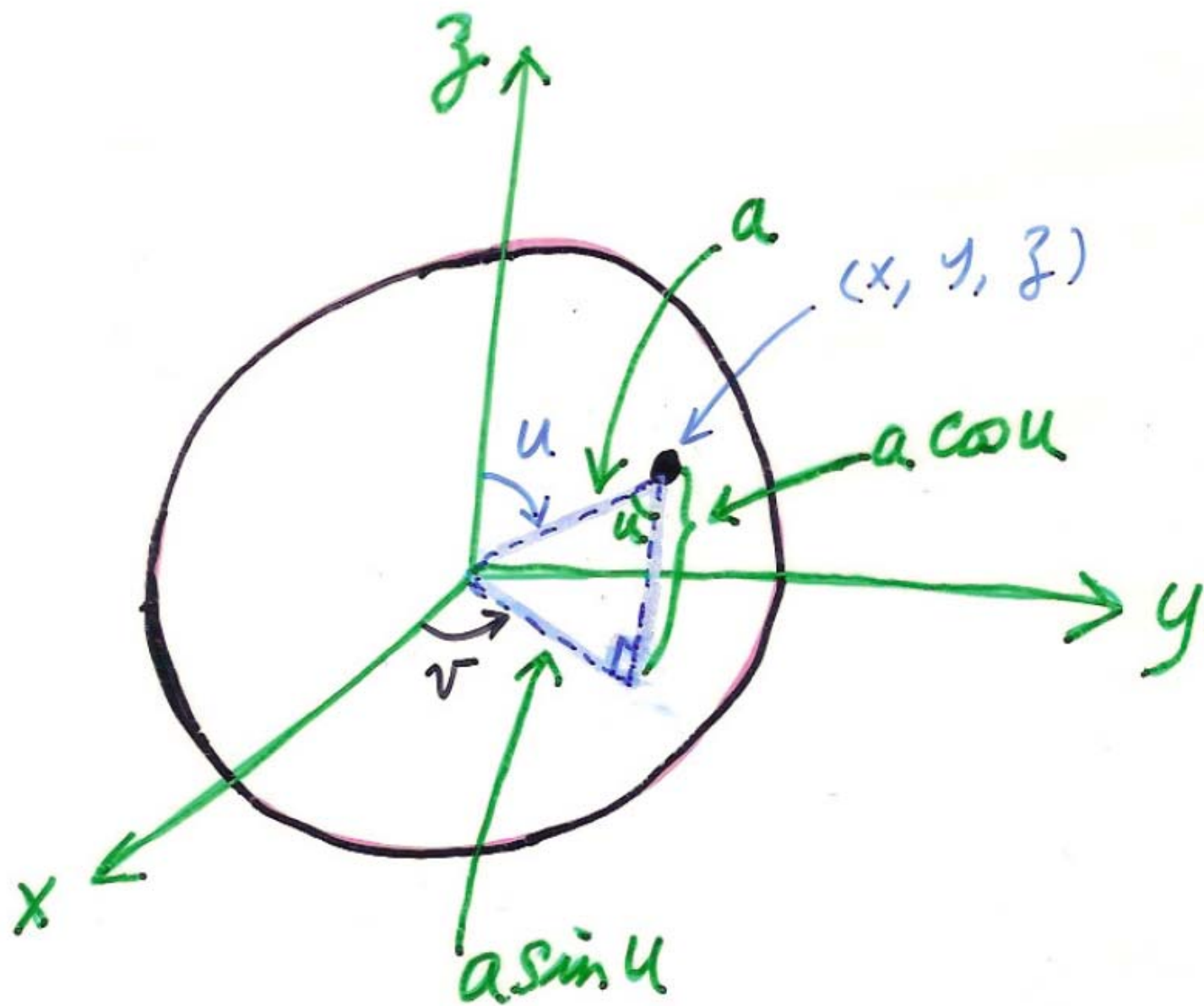
E.g. The upper cone  $z = \sqrt{x^2 + y^2}$ .

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + \sqrt{u^2 + v^2}\mathbf{k}.$$

### 10.1.3 Example (Spheres )

We have a standard parametric representation for a sphere  $x^2 + y^2 + z^2 = a^2$  of radius  $a$  centered at the origin:

$$\mathbf{r}(u, v) = (a \sin u \cos v)\mathbf{i} + (a \sin u \sin v)\mathbf{j} + (a \cos u)\mathbf{k}.$$



$$\begin{cases} x = a \sin u \cos v \\ y = a \sin u \sin v \\ z = a \cos u \end{cases}$$

$$0 \leq u \leq \pi$$

$$0 \leq v \leq 2\pi$$

$$\vec{r}(u, v) = a \sin u \cos v \vec{i} + a \sin u \sin v \vec{j} + a \cos u \vec{k}$$

E.g. When  $0 \leq u \leq \pi$ ,  $0 \leq v \leq 2\pi$ , the representation gives the full sphere.

When  $0 \leq u \leq \pi/2$ ,  $0 \leq v \leq 2\pi$ , the representation gives the upper hemisphere.

e.g.  $\frac{\pi}{2} \leq u \leq \pi, \quad 0 \leq v \leq 2\pi$

gives the lower hemi-sphere.

e.g.  $0 \leq u \leq \frac{\pi}{2}, \quad 0 \leq v \leq \frac{\pi}{2}$

gives the part of the sphere  
in the first octant.

### 10.1.4 Example (Circular cylinders)

We have a standard parametric representation for circular cylinder  $x^2 + y^2 = a^2$  about the  $z$ -axis:

$$\mathbf{r}(u, v) = (a \cos u)\mathbf{i} + (a \sin u)\mathbf{j} + v\mathbf{k}.$$



Here  $u$  measures the angle from the positive  $x$ -axis (about the  $z$ -axis) while  $v$  measures the height from the  $xy$ -plane along the cylinder.

Similarly, for  $x^2 + z^2 = a^2$  and  $y^2 + z^2 = a^2$  (cylinders about  $y$ - and  $x$ -axes resp.), we have respectively

$$\mathbf{r}(u, v) = (a \cos u)\mathbf{i} + v\mathbf{j} + (a \sin u)\mathbf{k}$$

and

$$\mathbf{r}(u, v) = v\mathbf{i} + (a \cos u)\mathbf{j} + (a \sin u)\mathbf{k}.$$

### 10.1.5 Tangent planes and normal vectors

Let  $S$  be a surface given by the parametric representation

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k} \quad (2)$$

We shall find the equation of the tangent plane to  $S$  at a point  $P_0$  with position vector  $\mathbf{r}_0 = \mathbf{r}(u_0, v_0)$ .

Let us fix  $v = v_0$  in (2) above.

Then the vector equation

$$\mathbf{r}(u, v_0) = x(u, v_0)\mathbf{i} + y(u, v_0)\mathbf{j} + z(u, v_0)\mathbf{k}$$

represents a space curve  $C_1$  on  $S$  passing through the point  $P_0$ .

The tangent vector to  $C_1$  at  $P_0$  is given by  $\frac{d}{du}\mathbf{r}(u, v_0) \big|_{u=u_0}$ ,

which is simply

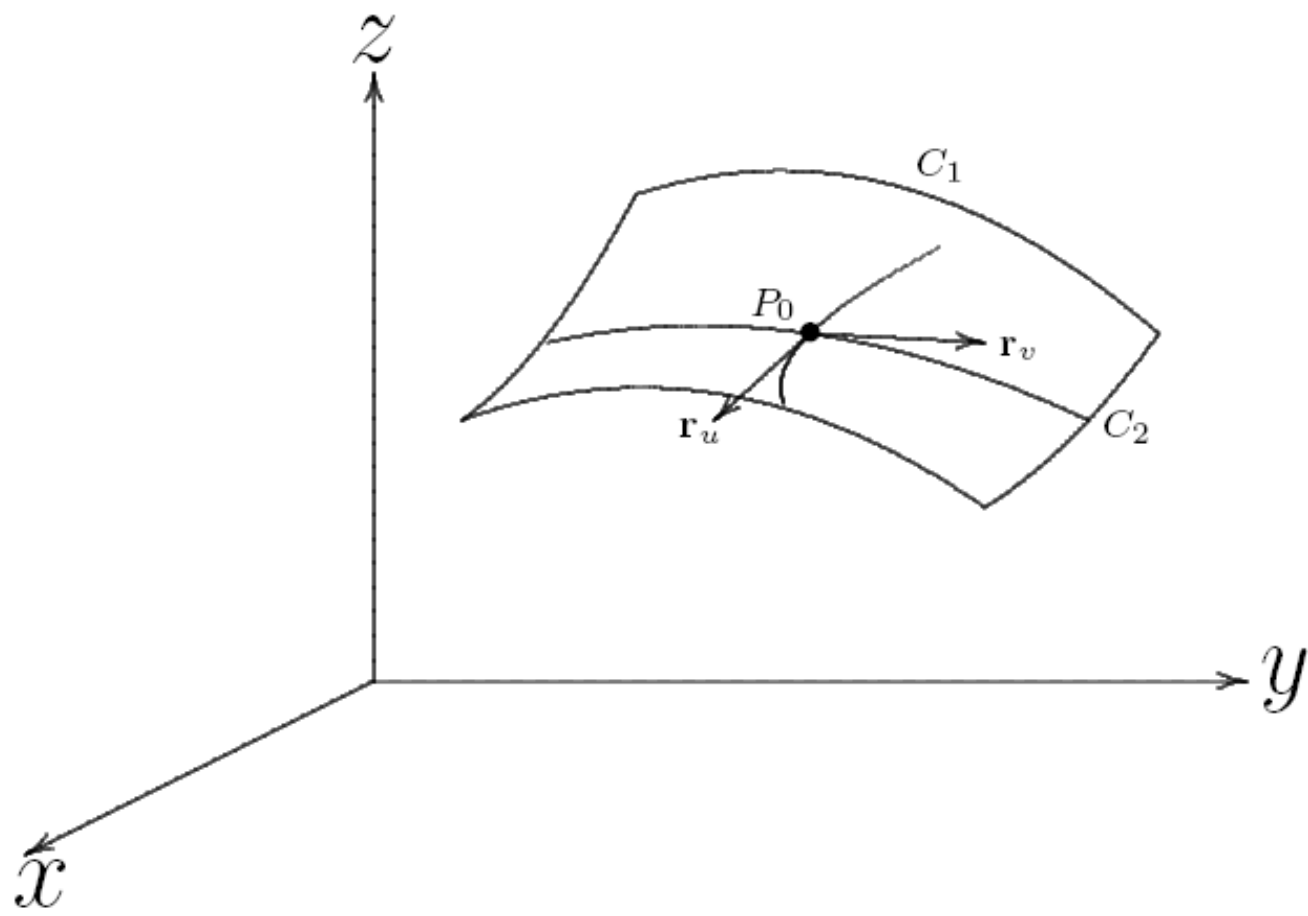
$$\mathbf{r}_u \equiv \frac{\partial x}{\partial u}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\mathbf{k}.$$

Similarly, if we fix  $u = u_0$  in (2), we get another space curve  $C_2$  with vector equation

$$\mathbf{r}(u_0, v) = x(u_0, v)\mathbf{i} + y(u_0, v)\mathbf{j} + z(u_0, v)\mathbf{k}.$$

The tangent vector to  $C_2$  at  $P_0$  is given by

$$\mathbf{r}_v \equiv \frac{\partial x}{\partial v}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\mathbf{k}.$$



Both vectors  $\mathbf{r}_u$  and  $\mathbf{r}_v$  lie in the tangent plane to  $S$  at  $P_0$ . Therefore, the cross product  $\mathbf{r}_u \times \mathbf{r}_v$ , assuming it is nonzero, provides a normal vector to the tangent plane to  $S$  at  $P_0$ . Therefore,  $(\mathbf{r} - \mathbf{r}_0) \cdot (\mathbf{r}_u \times \mathbf{r}_v) = 0$  is the equation of the tangent plane.



### 10.1.6 **Example**

Find the equation of the tangent plane to the surface  
with parametric representation

$$\mathbf{r}(u, v) = u\mathbf{i} + v^2\mathbf{j} + (u^2 - v)\mathbf{k}$$

at the point  $(1, 4, -1)$ .

## Solution:

To find the tangent plane to

$$S: \vec{r}(u,v) = u\vec{i} + v^2\vec{j} + (u^2 - v)\vec{k}$$

at the point  $(1, 4, -1)$ .

$$\vec{r}_u = \vec{i} + 0\vec{j} + 2u\vec{k}$$

$$\vec{r}_v = 0\vec{i} + 2v\vec{j} - \vec{k}$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2u \\ 0 & 2v & -1 \end{vmatrix} = -4uv\vec{i} + \vec{j} + 2v\vec{k}$$

at  $(1, 4, -1)$ , we have  $u=1$ ,  $v=2$

$$\therefore \vec{r}_u \times \vec{r}_v = -8\vec{i} + \vec{j} + 4\vec{k}$$

$\therefore$  Equation of Tangent plane is

$$\begin{aligned} -8x + y + 4z &= (1, 4, -1) \cdot (-8, 1, 4) \\ &= -8 + 4 - 4 = -8 \end{aligned}$$

i.e.

$$\underline{\underline{-8x + y + 4z + 8 = 0}}$$

### 10.1.7 Example

If  $S$  has Cartesian equation  $z = f(x, y)$ . Then a parametric representation of  $S$  is

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + f(u, v)\mathbf{k}.$$

Thus,  $\mathbf{r}_u = \mathbf{i} + 0\mathbf{j} + f_u\mathbf{k}$  and  $\mathbf{r}_v = 0\mathbf{i} + \mathbf{j} + f_v\mathbf{k}$ .

So the normal vector  $\mathbf{r}_u \times \mathbf{r}_v = -f_u\mathbf{i} - f_v\mathbf{j} + 1\mathbf{k}$ .

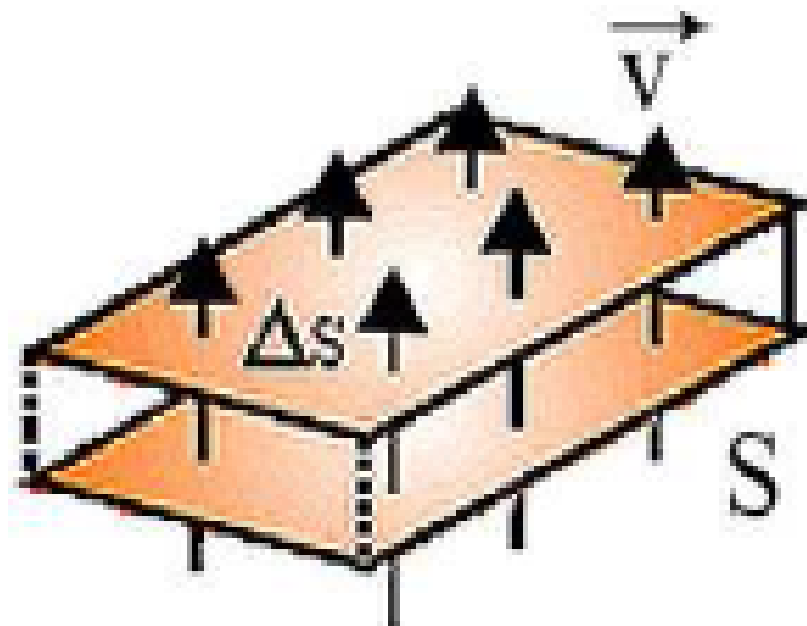
## 10.2 Surface Integrals

Similar to line integrals, surface integrals involve integration over some (bounded) surfaces. Suppose  $S$  is a surface and imagine a fluid with velocity  $\mathbf{v}$  flows through  $S$ . We wish to calculate the total volume of fluid flowing out of  $S$  per unit time.

Case (i): The fluid velocity is constant over flat surface  $S$  and its direction is perpendicular to  $S$ . Then the volume flow rate is given by distance traveled per unit time multiplied with the area of  $S$ :

$$w = \|\mathbf{v}\|\Delta s.$$

distance  
travelled  
per unit time



(i)

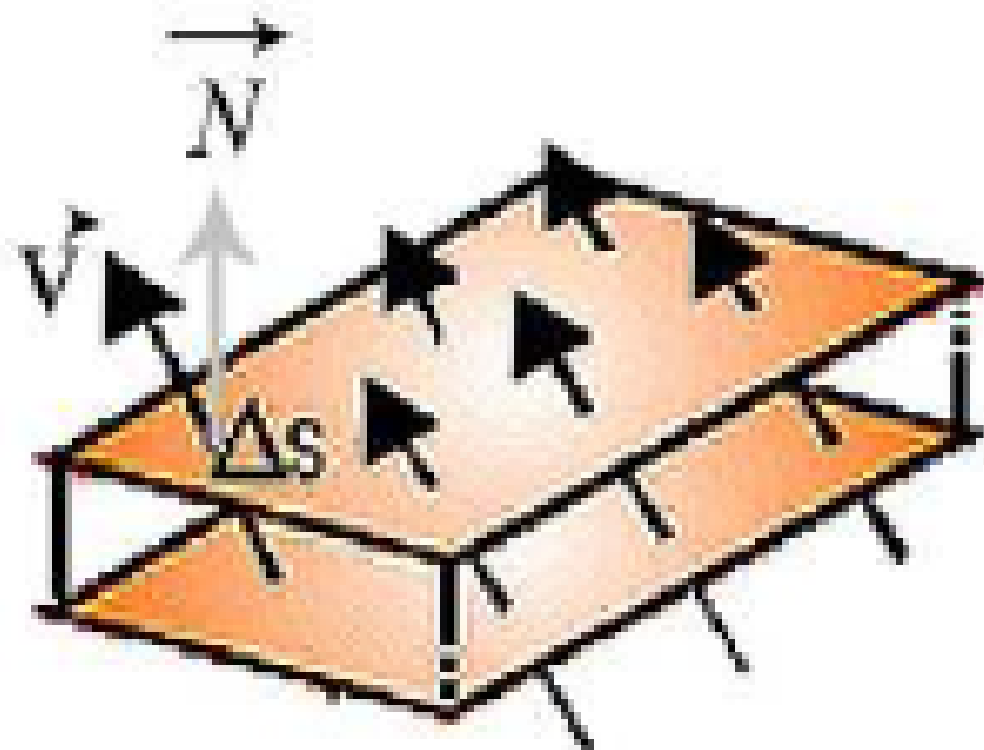


Case (ii): The fluid velocity is constant over flat surface  $S$  but its direction is not perpendicular to  $S$ .

Then the volume flow rate is given by

$$w = \mathbf{v} \cdot \mathbf{N} \Delta s$$

where  $\mathbf{N}$  is the unit normal vector to  $S$ .



(ii)

Case (iii): The fluid velocity is changing over curved surface  $S$ . We can divide up the surface into small segments and then sum the volume flow rate of the individual segments to get the total flow rate. In a particular segment, we have

$$w_i \approx \mathbf{v}(x_i, y_i, z_i) \cdot \mathbf{N}_i \Delta s_i.$$

So the total flow rate is approximately

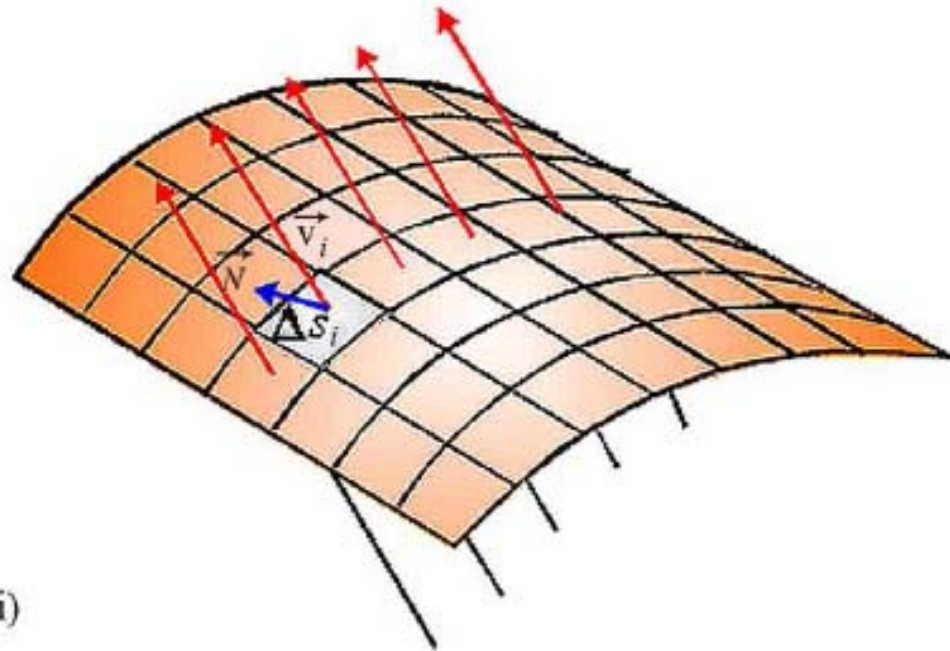
$$w \approx \sum_1^n \mathbf{v}(x_i, y_i, z_i) \cdot \mathbf{N}_i \Delta s_i \quad (3)$$

If we let  $n$  goes to infinity, the RHS of (3) becomes an integral

$$\iint_S \mathbf{v}(x, y, z) \cdot \mathbf{N} ds$$

which represents the actual total volume flow rate.

This integral is called a surface integral of the vector field  $\mathbf{v}$ .



General case (velocity is changing  
on a curved surface)

There are two types of surface integrals, one for scalar functions and the other for vector fields.

### 10.2.1 Surface integrals of scalar functions

Let  $f(x, y, z)$  be a function defined on a (bounded) surface  $S$ . Then for the parametric representation  $\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$  of  $S$ , the corresponding set of ordered pairs  $(u, v)$  come from a bounded domain  $D$ .

The **surface integral** of a scalar function  $f$  over  $S$  is

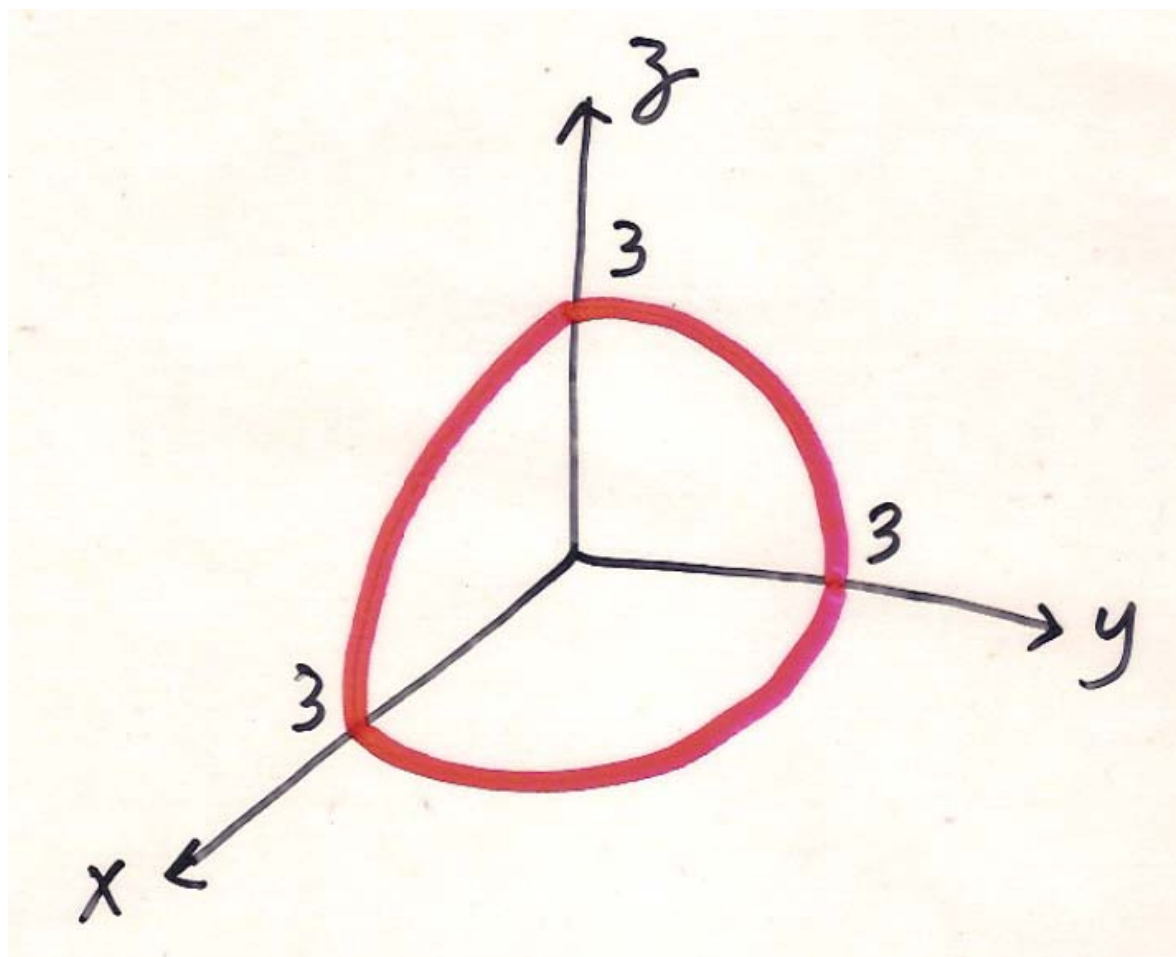
$$\iint_S f(x, y, z) dS = \iint_D f(\mathbf{r}(u, v)) \|\mathbf{r}_u \times \mathbf{r}_v\| dA.$$

The RHS of the above equation is a double integral over a domain  $D$ . Usually,  $D$  can be described by giving the ranges of  $u$  and  $v$ .



## 10.2.2 Example

Evaluate  $\iint_S (xz + yz) \, dS$ , where  $S$  is part of the sphere  $x^2 + y^2 + z^2 = 9$  in the first octant.



**Solution:** A parametric representation of the sphere is given by (see Example 10.1.3)

$$\mathbf{r}(u, v) = 3 \sin u \cos v \mathbf{i} + 3 \sin u \sin v \mathbf{j} + 3 \cos u \mathbf{k}.$$

To represent the first octant, the domain  $D$  is given by  $0 \leq u \leq \pi/2$  and  $0 \leq v \leq \pi/2$ .

$$\begin{aligned}
 \mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 \cos u \cos v & 3 \cos u \sin v & -3 \sin u \\ -3 \sin u \sin v & 3 \sin u \cos v & 0 \end{vmatrix} \\
 &= 9 \sin^2 u \cos v \mathbf{i} + 9 \sin^2 u \sin v \mathbf{j} + 9 \sin u \cos u \mathbf{k}.
 \end{aligned}$$

Therefore,  $\|\mathbf{r}_u \times \mathbf{r}_v\| = 9 \sin u$ .

The surface integral is given by

$$\begin{aligned}& \iint_S (xz + yz) \, dS \\&= \iint_D (9 \sin u \cos u \cos v + 9 \sin u \cos u \sin v) \|\mathbf{r}_u \times \mathbf{r}_v\| \, dA \\&= \int_0^{\pi/2} \int_0^{\pi/2} 81 \sin^2 u \cos u (\cos v + \sin v) \, du \, dv \\&= 81 \int_0^{\pi/2} \sin^2 u \cos u \, du \int_0^{\pi/2} (\cos v + \sin v) \, dv \\&= 81 \left( \left[ \frac{1}{3} \sin^3 u \right]_0^{\pi/2} \right) (2) = 54.\end{aligned}$$

Some additional details:

$$\int_0^{\frac{\pi}{2}} \sin^2 u \cos u \, du = \int_0^{\frac{\pi}{2}} \sin^2 u \, d(\sin u)$$

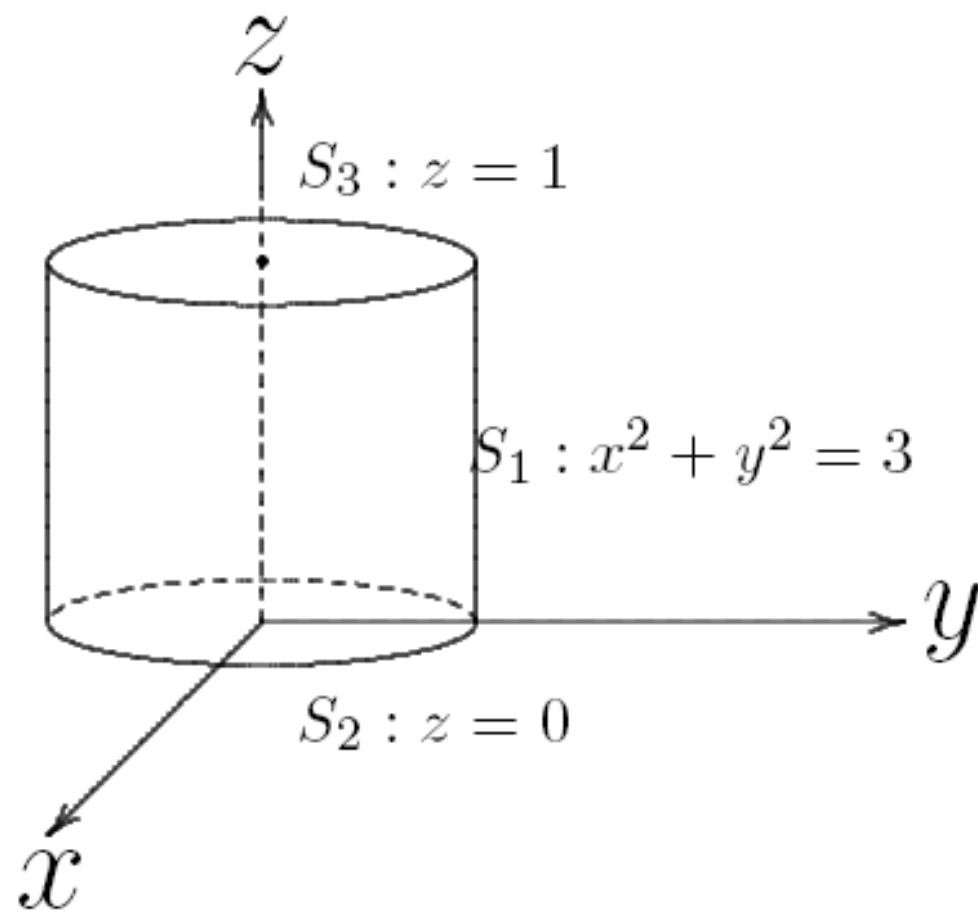
$$= \frac{1}{3} [\sin^3 u]_0^{\frac{\pi}{2}} = \frac{1}{3}$$

$$\int_0^{\frac{\pi}{2}} (\cos v + \sin v) \, dv$$

$$= [\sin v - \cos v]_0^{\frac{\pi}{2}} = 1 - (-1) = 2$$

### 10.2.3 Example

Evaluate  $\iint_S z \, dS$ , where  $S$  is the closed surface bounded laterally by  $S_1$ : the cylinder  $x^2 + y^2 = 3$ ; bounded below by  $S_2$ : the  $xy$ -plane and bounded on top by  $S_3$ : the horizontal plane  $z = 1$ .





**Solution:** The surface integral is the sum of three surface integrals:

$$\iint_S z \, dS = \iint_{S_1} z \, dS + \iint_{S_2} z \, dS + \iint_{S_3} z \, dS.$$

The surface  $S_1$  is part of a circular cylinder. By

Example 10.1.4, it has a parametric representation

$$\mathbf{r}(u, v) = \sqrt{3} \cos u \mathbf{i} + \sqrt{3} \sin u \mathbf{j} + v \mathbf{k}.$$

Thus,  $\mathbf{r}_u \times \mathbf{r}_v = \sqrt{3} \cos u \mathbf{i} + \sqrt{3} \sin u \mathbf{j} + 0 \mathbf{k}$

and  $\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{3}$ .

Since  $S_1$  is a full cylinder, the range of  $u$  is given by

$$0 \leq u \leq 2\pi.$$

Moreover,  $S_1$  is bounded above by the plane  $z = 1$

and below by  $z = 0$ , so the range of  $v$  is given by

$$0 \leq v \leq 1.$$

Therefore,

$$\begin{aligned}\iint_{S_1} z \, dS &= \iint_D v \|\mathbf{r}_u \times \mathbf{r}_v\| \, dA \\ &= \int_0^{2\pi} \int_0^1 \sqrt{3}v \, dv du = \int_0^{2\pi} \frac{\sqrt{3}}{2} du \\ &= \sqrt{3}\pi.\end{aligned}$$

$S_2$  is on the  $xy$ -plane, so we have  $z = 0$ . Thus the integrand of  $\iint_{S_2} z \, dS$  is zero so that the integral has value zero. Therefore,

$$\iint_{S_2} z \, dS = 0.$$

The surface  $S_3$  is on the horizontal plane  $z = 1$ . Thus

$$\iint_{S_3} z \, dS = \iint_{S_3} dS = \text{area of } S_3 = \pi(\sqrt{3})^2 = 3\pi.$$

Consequently,

$$\iint_S z \, dS = \iint_{S_1} z \, dS + \iint_{S_2} z \, dS + \iint_{S_3} z \, dS = (3 + \sqrt{3})\pi.$$

## 10.2.4 Surface integrals of vector fields

Let  $\mathbf{F}$  be a continuous vector field defined on a surface  $S$  with a unit normal vector  $\mathbf{n}$ . We have seen at the beginning of this section that the **surface integral** of  $\mathbf{F}$  over  $S$  is  $\iint_S \mathbf{F} \cdot \mathbf{n} \, dS$ . We usually simplify the notation as

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

This integral is also called the **flux** of  $\mathbf{F}$  over  $S$  as it is related to the volume flow rate of fluid.

If  $S$  is given by the parametric representation  $\mathbf{r} = \mathbf{r}(u, v)$  with domain  $D$ ,

then

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_S \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|} dS \\
&= \iint_D \left[ \mathbf{F}(\mathbf{r}(u, v)) \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|} \right] \|\mathbf{r}_u \times \mathbf{r}_v\| dA \\
&= \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA.
\end{aligned}$$

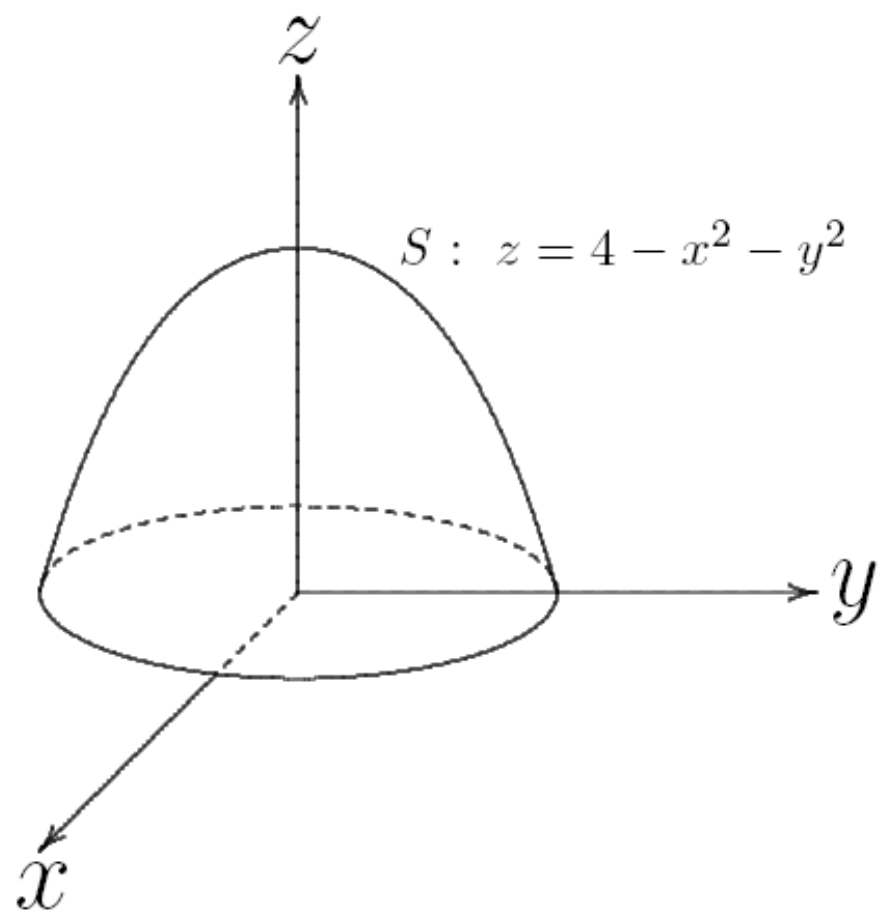
Therefore,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA$$



### 10.2.5 Example

Evaluate  $\iint_S \mathbf{F} \cdot d\mathbf{S}$ , where  $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + xy\mathbf{k}$ , and  $S$  is the part of the paraboloid  $z = 4 - x^2 - y^2$  above the  $xy$ -plane.



**Solution:** Since  $S$  has Cartesian equation  $z = 4 - x^2 - y^2$ , the parametric representation is

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + (4 - u^2 - v^2)\mathbf{k}.$$

The region  $D$  is then the projection onto the  $xy$ -plane, which is the disk of radius 2.

We have  $\mathbf{r}_u = \mathbf{i} + 0\mathbf{j} - 2u\mathbf{k}$ ,  $\mathbf{r}_v = 0\mathbf{i} + \mathbf{j} - 2v\mathbf{k}$  and

$$\mathbf{r}_u \times \mathbf{r}_v = 2u\mathbf{i} + 2v\mathbf{j} + \mathbf{k}.$$

Therefore,

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA \\ &= \iint_D (u\mathbf{i} + v\mathbf{j} + uv\mathbf{k}) \cdot (2u\mathbf{i} + 2v\mathbf{j} + \mathbf{k}) dA \\ &= \iint_D (2u^2 + 2v^2 + uv) dA \end{aligned}$$

Change to Polar Coordinates

$$\iint_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^2 \left( 2r^2 + \frac{r^2 \sin 2\theta}{2} \right) r dr d\theta$$

$$= \int_0^{2\pi} \left[ \frac{1}{2} r^4 + \frac{1}{8} r^4 \sin 2\theta \right]_{r=0}^{r=2} d\theta$$

$$= \int_0^{2\pi} (8 + 2\sin 2\theta) d\theta$$

$$= [8\theta - \cos 2\theta]_0^{2\pi}$$

$$= 16\pi$$

$$\underline{\underline{\quad}}$$

### 10.2.6 Example

Let  $\mathbf{F}(x, y, z) = y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$ . Evaluate  $\iint_S \mathbf{F} \cdot d\mathbf{S}$ ,  
where  $S$  is the sphere  $x^2 + y^2 + z^2 = 1$ .

**Solution:** A parametric representation of the unit sphere is given by

$$\mathbf{r}(u, v) = \sin u \cos v \mathbf{i} + \sin u \sin v \mathbf{j} + \cos u \mathbf{k},$$

with  $D$  given by  $0 \leq u \leq \pi$  and  $0 \leq v \leq 2\pi$ .



We have

$$\mathbf{r}_u \times \mathbf{r}_v = \sin^2 u \cos v \mathbf{i} + \sin^2 u \sin v \mathbf{j} + \sin u \cos u \mathbf{k}$$

and

$$\mathbf{F}(\mathbf{r}(u, v)) = \sin u \sin v \mathbf{i} + \sin u \cos v \mathbf{j} + \cos u \mathbf{k}.$$

Thus,

$$\mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) = 2 \sin^3 u \sin v \cos v + \sin u \cos^2 u.$$

Therefore,

$$\begin{aligned}& \iint_S \mathbf{F} \cdot d\mathbf{S} \\&= \int_0^{2\pi} \int_0^\pi (2 \sin^3 u \sin v \cos v + \sin u \cos^2 u) \, du dv \\&= \int_0^\pi \sin^3 u \, du \int_0^{2\pi} \sin 2v \, dv + \int_0^\pi \sin u \cos^2 u \, du \int_0^{2\pi} dv \\&= 4\pi/3.\end{aligned}$$

Some additional details:

$$\int_0^{2\pi} \sin 2v \, dv = -\frac{1}{2} [\cos 2v]_0^{2\pi} = 0$$

$$\int_0^{\pi} \sin u \cos^2 u \, du = -\int_0^{\pi} \cos^2 u \, d(\cos u)$$

$$= -\frac{1}{3} [\cos^3 u]_0^{\pi} = \frac{2}{3}$$

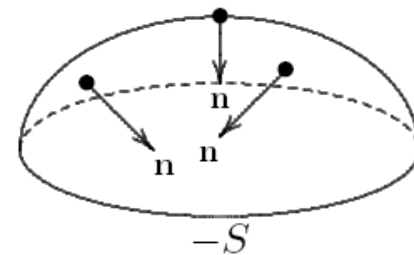
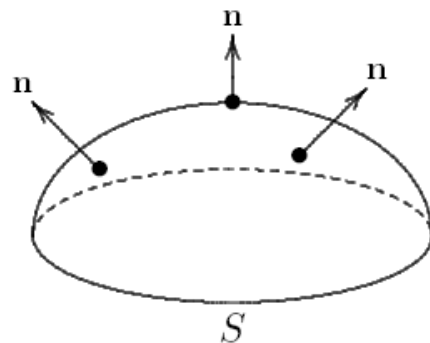
## 10.2.7 Orientation of surfaces

Note that, in the above example, if we switch the order of  $u$  and  $v$ , then

$$\mathbf{r}_v \times \mathbf{r}_u = -\mathbf{r}_u \times \mathbf{r}_v$$

and the surface integral will be evaluated to  $-4\pi/3$ .

Therefore, for surface integral of a vector field, the value depends on the choice of the normal vector, which is known as the **orientation** of the surface.



If  $S$  is a surface given in parametric form by  $\mathbf{r} = \mathbf{r}(u, v)$ , then the normal vector  $\mathbf{r}_u \times \mathbf{r}_v$  automatically supply an orientation to  $S$ .

The opposite orientation is given by  $\mathbf{r}_v \times \mathbf{r}_u = -\mathbf{r}_u \times \mathbf{r}_v$  and the corresponding oriented surface is denoted by  $-S$ . Then

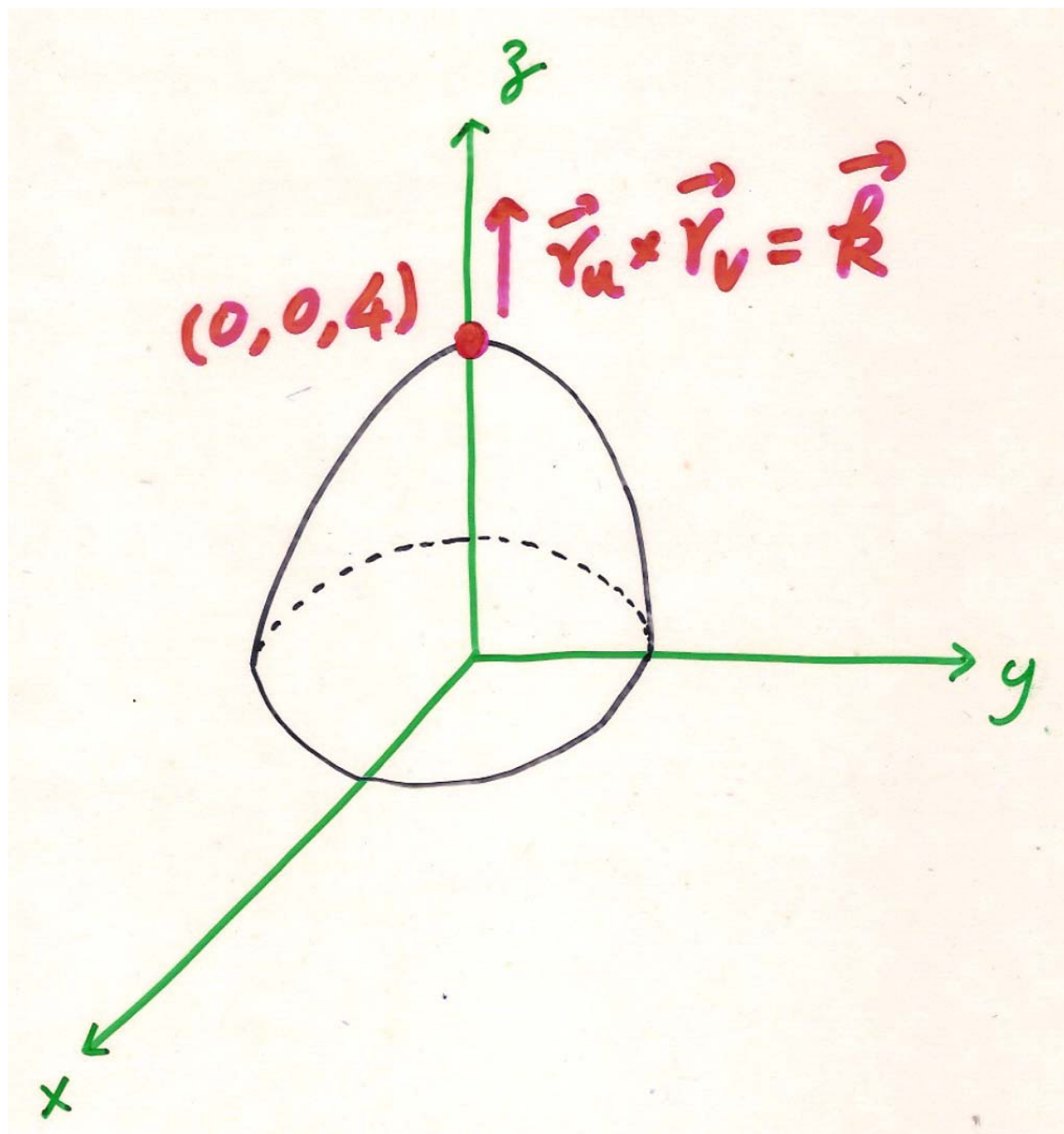
$$\iint_{-S} \mathbf{F} \cdot d\mathbf{S} = - \iint_S \mathbf{F} \cdot d\mathbf{S}.$$

## 10.2.8 Example

In example 10.2.5, the normal vector we used is  $\mathbf{r}_u \times \mathbf{r}_v = 2u\mathbf{i} + 2v\mathbf{j} + \mathbf{k}$ . Consider the point  $(0, 0, 4)$  on the paraboloid. This point corresponds to  $u = 0, v = 0$ .

At this point,  $\mathbf{r}_u \times \mathbf{r}_v = \mathbf{k}$ , which is pointing “upwards”. Hence the orientation of the paraboloid we used in this example is given by the **upward normal vector**.





## 10.2.9 Example

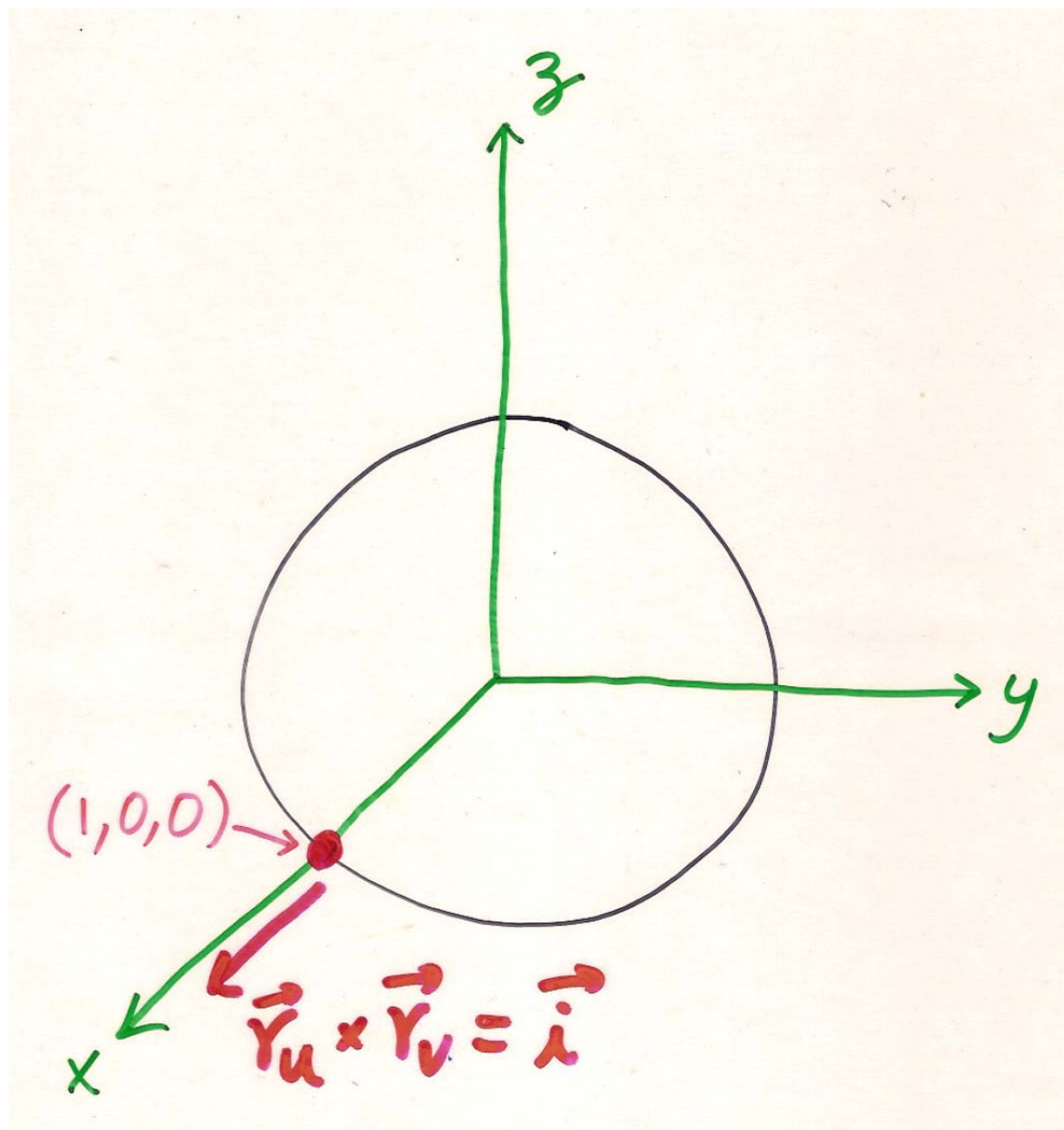
In Example 10.2.6, the normal vector we used is

$$\mathbf{r}_u \times \mathbf{r}_v = \sin^2 u \cos v \mathbf{i} + \sin^2 u \sin v \mathbf{j} + \sin u \cos u \mathbf{k}.$$

Consider the point  $(1, 0, 0)$  on the sphere.

This point corresponds to  $u = \pi/2, v = 0$ .

At this point,  $\mathbf{r}_u \times \mathbf{r}_v = \mathbf{i}$ , which is pointing “outwards” away from the sphere. Hence the orientation of the sphere we used in this example is given by the **outward normal vector**.



## 10.3 Curl and Divergence

In this section, we introduce two operators on vector fields which will be used in the subsequent sections.

### 10.3.1 Curl

Let  $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$  be a vector field in the  $xyz$ -space. The **curl** of  $\mathbf{F}$  is defined by

$$\operatorname{curl} \mathbf{F} = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$$

is a vector field.

## 10.3.2 Divergence

Let  $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$  be a vector field in the  $xyz$ -space. The **divergence** of  $\mathbf{F}$  defined by

$$\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

is a scalar function.

### 10.3.3 Del operator

The curl and divergence operators can be expressed in terms of the **del operator**:

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$$

Then



(i) taking the cross product of  $\nabla$  with a vector field

$$\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k},$$

$$\begin{aligned}\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\ &= \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}\end{aligned}$$

So

$$\nabla \times \mathbf{F} = \text{curl } \mathbf{F}$$

(ii) taking the dot product of  $\nabla$  with a vector field

$$\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k},$$

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \left( \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \right) \cdot (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \\ &= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\end{aligned}$$

So  $\nabla \cdot \mathbf{F} = \operatorname{div} \mathbf{F}$ .

### 10.3.4 Example

Let  $\mathbf{F}(x, y, z) = x^2yz\mathbf{i} + xy^2z\mathbf{j} + xyz^2\mathbf{k}$ .

$$\begin{aligned}\operatorname{curl} \mathbf{F} &= \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & xy^2z & xyz^2 \end{vmatrix} \\ &= (xz^2 - xy^2)\mathbf{i} + (x^2y - yz^2)\mathbf{j} + (y^2z - x^2z)\mathbf{k}.\end{aligned}$$

$$\begin{aligned}\operatorname{div} \mathbf{F} &= \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(xy^2z) + \frac{\partial}{\partial z}(xyz^2) \\ &= 6xyz.\end{aligned}$$

For a scalar function  $f$

$$\nabla f = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k}$$

is the gradient of  $f$ .

### 10.3.5 Example

Show that  $\text{curl } (\nabla f) = \mathbf{0}$ .

**Solution:**

$$\begin{aligned}\text{curl } (\nabla f) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \\ &= \left( \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right) \mathbf{i} + \left( \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \right) \mathbf{j} + \left( \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) \mathbf{k} \\ &= \mathbf{0}\end{aligned}$$

since  $f_{xy} = f_{yx}$  etc.

### 10.3.6 Curl and conservative fields

Let  $\mathbf{F}$  be a vector field in the  $xyz$ -space.

If  $\text{curl } \mathbf{F} = \mathbf{0}$ , then  $\mathbf{F}$  is a conservative field.

The converse is also true.

Note: Let  $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$

$$\nabla \times \vec{F} = \vec{0}$$

$$\Leftrightarrow \begin{cases} \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z} \\ \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \\ \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \end{cases}$$

$\Leftrightarrow \vec{F}$  is conservative.



### 10.3.7 Example

Find the curl of the velocity vector fields defined by

(a)  $\mathbf{F}_1 = x\mathbf{i} + y\mathbf{j}$ , (b)  $\mathbf{F}_2 = -y\mathbf{i} + x\mathbf{j}$ , (c)  $\mathbf{F}_3 = \cos y\mathbf{i} + \sin x\mathbf{j}$ .

**Solution:**

(a)  $\text{curl } \mathbf{F}_1 = \mathbf{0}$ , (b)  $\text{curl } \mathbf{F}_2 = 2\mathbf{k}$ , (c)  $\text{curl } \mathbf{F}_3 = (\cos x + \sin y)\mathbf{k}$ .

$$\text{Curl } \vec{F}_3 = \nabla \times \vec{F}_3$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos y & \sin x & 0 \end{vmatrix}$$

$$= (\cos x + \sin y) \vec{k}$$

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### 10.3.8 Example

Find the divergence of the velocity vector fields defined by (a)  $\mathbf{F}_1 = x\mathbf{i} + y\mathbf{j}$ , (b)  $\mathbf{F}_2 = -y\mathbf{i} + x\mathbf{j}$ , (c)  $\mathbf{F}_3 = -x^2\mathbf{i} + y^2\mathbf{j}$ .

**Solution:** (a)  $\operatorname{div} \mathbf{F}_1 = 2$ , (b)  $\operatorname{div} \mathbf{F}_2 = 0$ , (c)  $\operatorname{div} \mathbf{F}_3 = 2(y - x)$ .

$$(c) \quad \vec{F}_3 = -x^2 \vec{i} + y^2 \vec{j}$$

$$\operatorname{div} \vec{F}_3$$

$$= \nabla \cdot \vec{F}_3$$

$$= \frac{\partial}{\partial x}(-x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(0)$$

$$= \underline{\underline{-2x + 2y}}$$

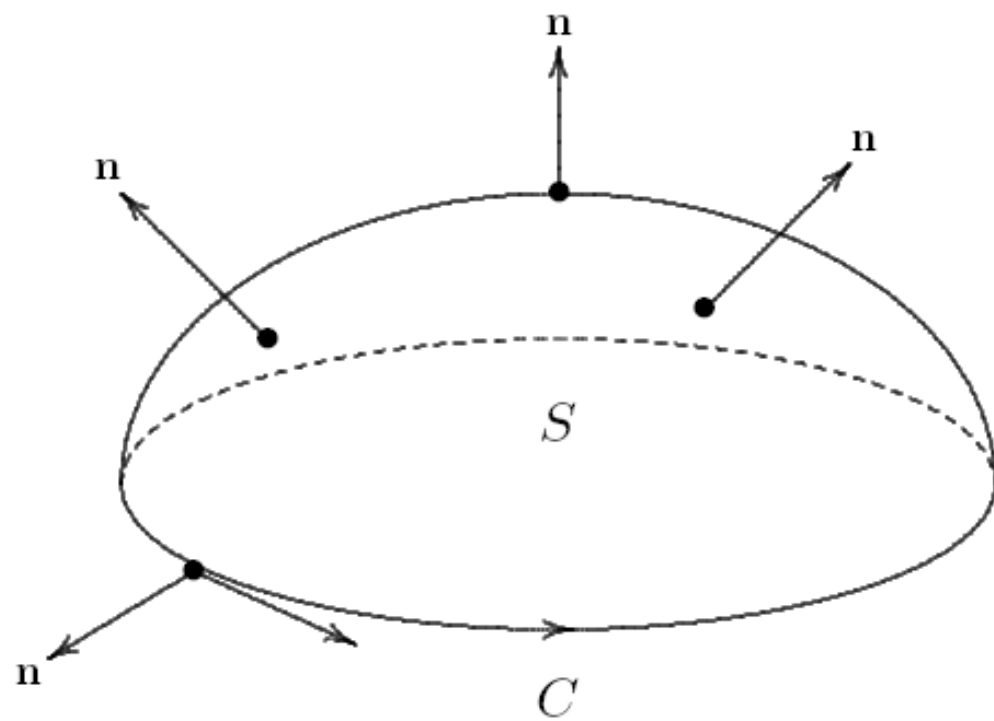
## 10.4 Stokes' Theorem

Let  $S$  be an oriented piecewise-smooth surface that is bounded by a closed, piecewise-smooth boundary curve  $C$ .

Let  $\mathbf{F}$  be a vector field whose components have continuous partial derivatives on  $S$ . Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \cdot d\mathbf{S}.$$

**Note:** In the above equation, the orientation of  $C$  must be consistent with that of  $S$ : when you walk in the direction (orientation) around  $C$  with your head pointing in the direction of the normal vector of  $S$ , the corresponding orientation of  $S$  should be on your left.





Note: Green's Theorem is a special case of Stoke's Theorem.

Consider a closed curve  $C$

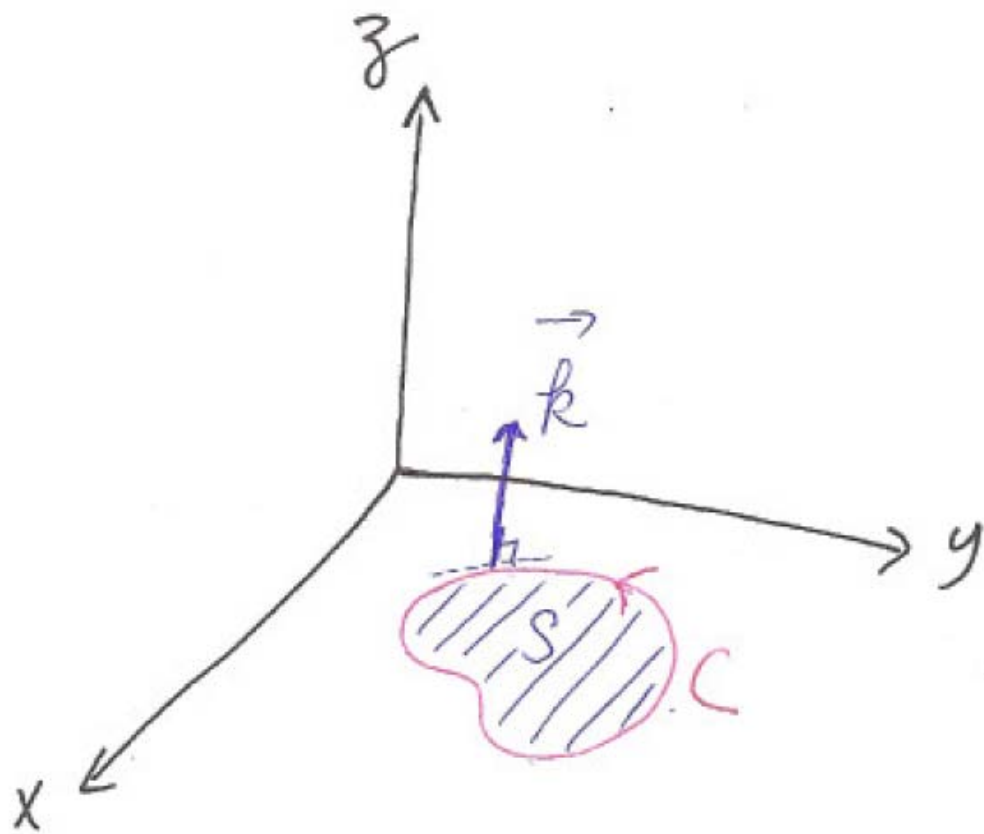
$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}, \quad a \leq t \leq b.$$

on the  $x$ - $y$  plane.

Let  $S$  denote the surface on the  $x$ - $y$  plane bounded by  $C$ .

Choose  $\vec{k}$  to be the orientation of  $S$ .

Choose orientation of  $C$  so that it is compatible with Stoke's Theorem.



$$\text{Let } \vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$$

Now  $S$  has a parametric representation:

$$\vec{r}(u,v) = u\vec{i} + v\vec{j} + 0\vec{k}$$

where  $(u,v) \in S$ .

$$\therefore \vec{r}_u = (1, 0, 0) = \vec{i}, \vec{r}_v = (0, 1, 0) = \vec{j}$$

$$\therefore \vec{r}_u \times \vec{r}_v = \vec{i} \times \vec{j} = \vec{k} \Rightarrow d\vec{S} = \vec{k} dA$$

$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k}$$

$\therefore$  Stoke's Theorem

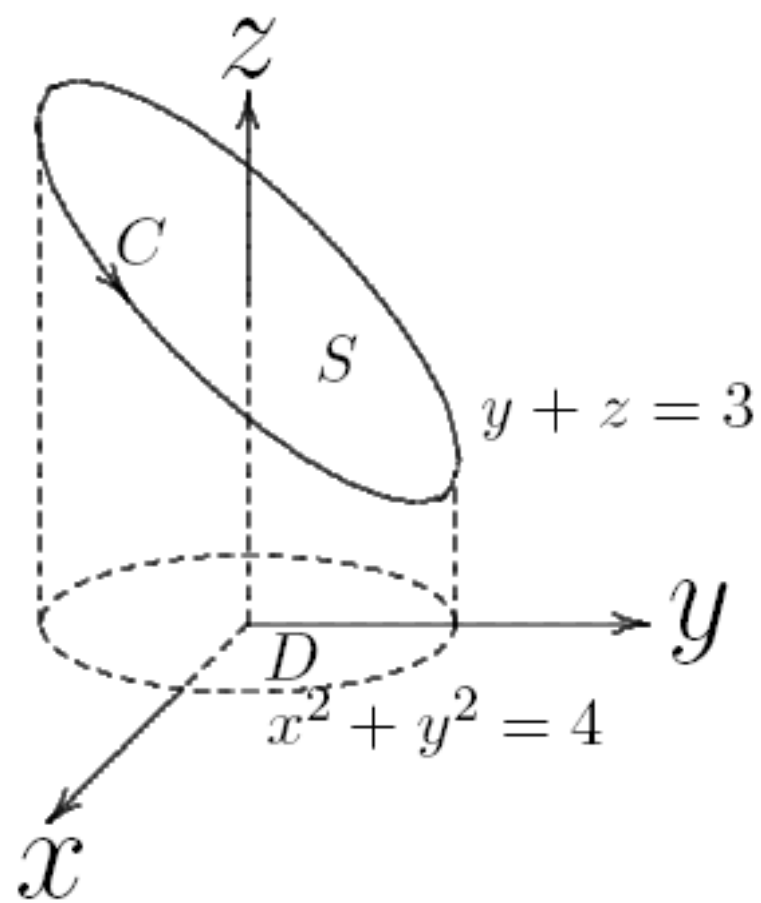
$$\Rightarrow \int_C \vec{F} \cdot d\vec{r} = \iint_S \text{Curl } \vec{F} \cdot d\vec{S}$$

$$\Rightarrow \int_C P dx + Q dy = \iint_S \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

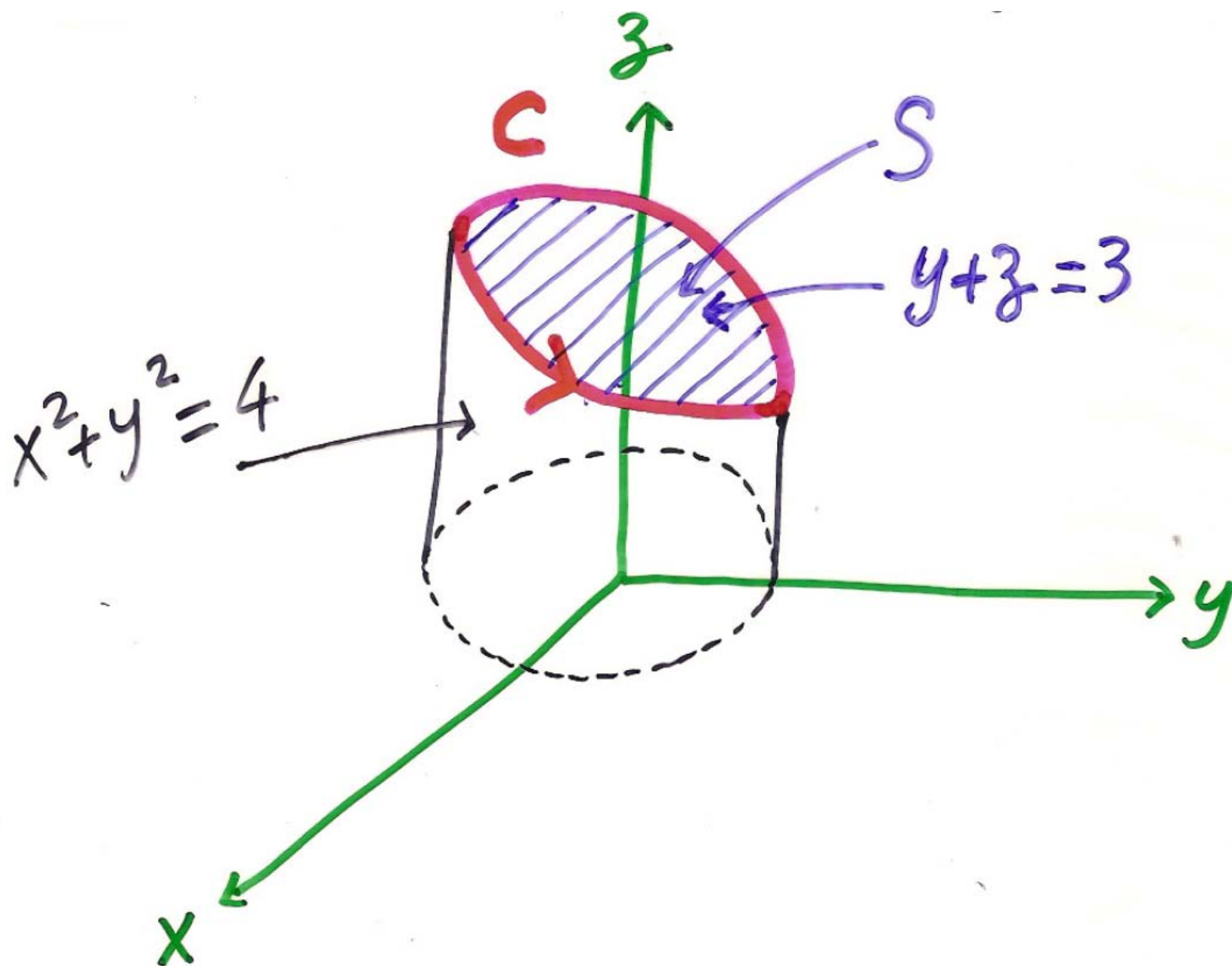
$\Rightarrow$  Green's Theorem //

### 10.4.1 Example

Use Stokes' Theorem to evaluate  $\int_C \mathbf{F} \cdot d\mathbf{r}$ , where  $\mathbf{F} = yz\mathbf{i} - xz\mathbf{j} + xy\mathbf{k}$  and  $C$  is the curve of intersection of the plane  $y + z = 3$  and the cylinder  $x^2 + y^2 = 4$ . ( $C$  is oriented in the counterclockwise sense when viewed from above.)



**Solution:** Let  $S$  be the (bounded) surface enclosed by  $C$  on the plane  $y + z = 3$ . So  $S$  has parametric representation  $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + (3 - v)\mathbf{k}$  and the region  $D$  is the disk of radius 2.





We have  $\mathbf{r}_u \times \mathbf{r}_v = \mathbf{j} + \mathbf{k}$ , which is the upward normal vector of  $S$ . This gives the orientation of  $S$  which agrees with that of  $C$ .

$$\text{Also curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & -xz & xy \end{vmatrix} = 2x\mathbf{i} - 2z\mathbf{k}.$$

By Stokes' Theorem,

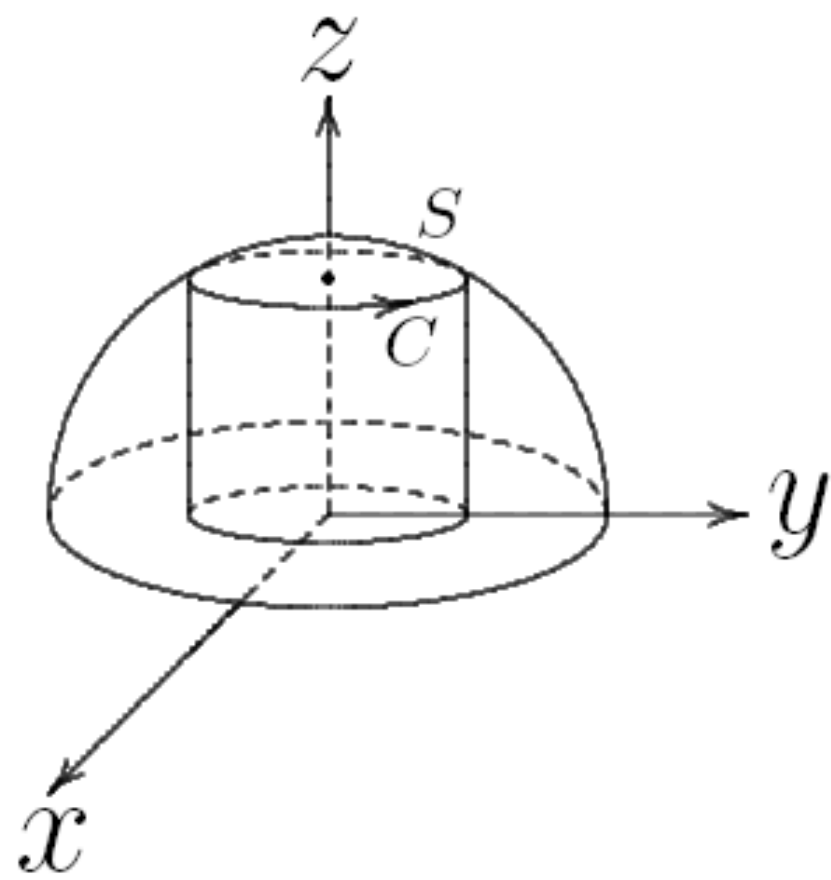
$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} \\
&= \iint_D (2u\mathbf{i} - 2(3-v)\mathbf{k}) \cdot (\mathbf{j} + \mathbf{k}) \, dA \\
&= \iint_D (-6 + 2v) \, dA
\end{aligned}$$

Since  $D$  is the disk of radius 2, we may use polar coordinates:

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \int_0^2 (-6 + 2r \sin \theta) r dr d\theta \\ &= \int_0^{2\pi} \left(-12 + \frac{16}{3} \sin \theta\right) d\theta = -24\pi.\end{aligned}$$

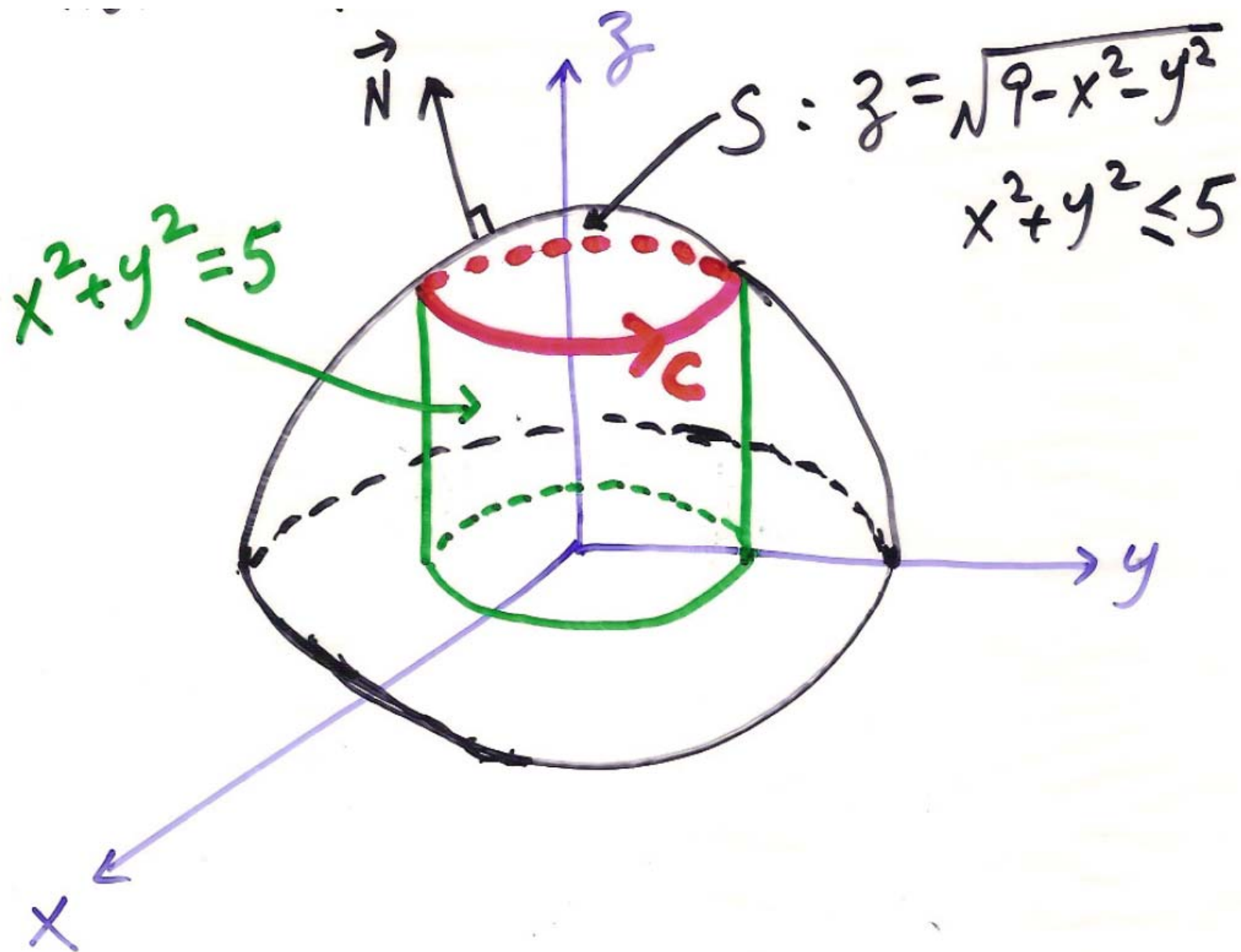
## 10.4.2 Example

Use Stokes' Theorem to compute  $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ , where  $\mathbf{F}(x, y, z) = y^2 z \mathbf{i} + x \mathbf{j} + (x + y) \mathbf{k}$  and  $S$  is the part of the upper hemisphere  $z = \sqrt{9 - x^2 - y^2}$  that lies within the cylinder  $x^2 + y^2 = 5$  and the orientation of  $S$  is given by the upward normal vector.



**Solution:** The boundary  $C$  of  $S$  is given by the intersection of the cylinder  $x^2 + y^2 = 5$  and the upper hemisphere  $z = \sqrt{9 - x^2 - y^2}$ . Solving the two equations, we have  $z = 2$ . So the curve  $C$  has a vector equation given by

$$\mathbf{r}(t) = \sqrt{5} \cos t \mathbf{i} + \sqrt{5} \sin t \mathbf{j} + 2\mathbf{k}.$$



With this vector equation, the curve traverses in anticlockwise direction when viewed from top. This agrees with the given orientation of  $S$ .

Now  $\mathbf{r}'(t) = -\sqrt{5} \sin t \mathbf{i} + \sqrt{5} \cos t \mathbf{j} + 0 \mathbf{k}$  and

$$\mathbf{F}(\mathbf{r}(t)) = 10 \sin^2 t \mathbf{i} + \sqrt{5} \cos t \mathbf{j} + \sqrt{5}(\cos t + \sin t) \mathbf{k}.$$



By Stokes' Theorem,

$$\begin{aligned}\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \int_C \mathbf{F} \cdot d\mathbf{r} \\ &= \int_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt \\ &= \int_0^{2\pi} (-10\sqrt{5} \sin^3 t + 5 \cos^2 t) \, dt\end{aligned}$$

$$= \int_{-\pi}^{\pi} \left( \underbrace{-10\sqrt{5} \sin^3 t}_{\text{odd}} + \underbrace{5 \cos^2 t}_{\text{even}} \right) dt$$

( $\because$  the integrand is  $2\pi$  periodic.

$$\therefore \int_0^{2\pi} = \int_{-\pi}^{\pi} )$$

$$= 10 \int_0^{\pi} \cos^2 t \, dt$$

$$= 10 \int_0^{\pi} \frac{1 + \cos 2x}{2} dx$$

$$= 5 \left[ x + \frac{1}{2} \sin 2x \right]_0^{\pi}$$

$$= \underline{\underline{5\pi}}$$

## 10.5 Divergence Theorem (or Gauss' Theorem)

Let  $E$  be a solid region and let  $S$  be the boundary of  $E$ , given with the **outward orientation**<sup>\*</sup>. Let  $\mathbf{F}$  be a vector field whose component functions have continuous partial derivatives in  $E$ . Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV.$$

\* The outward orientation of the boundary surface of a solid region  $E$  is the one for which the normal vector point outward from  $E$ .

### 10.5.1 Example

Let  $\mathbf{F}(x, y, z) = (x+y)\mathbf{i} + (y+z)\mathbf{j} + (z+x)\mathbf{k}$ . Evaluate

$\iint_S \mathbf{F} \cdot d\mathbf{S}$ , where  $S$  is the sphere  $x^2 + y^2 + z^2 = 1$

with orientation given by the outward normal vector.

**Solution:** By the Divergence Theorem,

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E \operatorname{div} \mathbf{F} \, dV = \iiint_E 3 \, dV \\ &= 3 \times \text{volume of the unit ball} = 4\pi.\end{aligned}$$

### 10.5.2 Example

Evaluate  $\iint_S \mathbf{F} \cdot d\mathbf{S}$ , where

$$\mathbf{F} = x^2\mathbf{i} + (xy + x \cos z)\mathbf{j} + e^{xy}\mathbf{k}$$



and  $S$  is the surface of the cubic region  $E$  bounded by the three coordinate planes  $x = 0, y = 0, z = 0$  and the three planes  $x = 1, y = 1, z = 1$ . The orientation of  $S$  is given by the outward normal vector.

**Solution:** The cubic region  $E$  can be described as

$$E : \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1, \quad 0 \leq z \leq 1$$

By the Divergence Theorem, we have

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E \operatorname{div} \mathbf{F} \, dV \\ &= \iiint_E 3x \, dV = 3 \int_0^1 \int_0^1 \int_0^1 x \, dx dy dz \\ &= \frac{3}{2}. \end{aligned}$$