Outline

Twelvefold Way

Distributing n balls into k boxes

- balls could be identical or distinct
- boxes could be identical or distinct
- three subcases
- due to Gian-Carlo Rota

Identical Balls, Identical Boxes

How many ways are there to distribute n identical balls to k identical boxes,

1. with no restriction?
2. if each box has to get at least one ball?
3. if each box has to get at most one ball?

Since boxes are identical, we can order them into non-increasing order. This becomes an integer partition problem.

1) Since some boxes may be empty, $\# = \sum_{j=1}^{k} p(n,j)$

2) $p(n,k)$

3) 0 if $n > k$, otherwise since the boxes are identical, it does not matter which box has 0 or 1 balls. $\# = 1$ if $n \leq k$.

Distinct Balls, Identical Boxes

How many ways are there to distribute n distinct balls to k identical boxes,

1. with no restriction?
2. if each box has to get at least one ball?
3. if each box has to get at most one ball?

Since boxes are identical, this becomes a set partition problem.

1) We can use any # of boxes, hence $\# = \displaystyle\sum_{j=1}^{k} \left\{ {n \atop j} \right\}$

2) $\left\{ {n \atop k} \right\}$
3) 0 if $n > k$, otherwise since the boxes are identical, it does not matter which box contains which ball. $\# = 1$ if $n \leq k$.

Identical Balls, Distinct Boxes

How many ways are there to distribute n identical balls to k distinct boxes,

1. with no restriction?
2. if each box has to get at least one ball?
3. if each box has to get at most one ball?

Since boxes are distinct, $x_1, \ldots, x_k$, this becomes a composition problem.

1) Weak composition $\# = \binom{n+k-1}{k-1}$

2) Composition $\# = \binom{n-1}{k-1}$

3) 0 if $n > k$, choose n boxes to put one ball each $\# = \binom{k}{n}$.

Distinct Balls, Distinct Boxes

How many ways are there to distribute n distinct balls to k distinct boxes,

1. with no restriction?
2. if each box has to get at least one ball?
3. if each box has to get at most one ball?

1) Balls 1 to n, each has k choices. $\# = k \times k \times \ldots k = k^n$
2) Partition $[n]$ into k sets and order them. $\# = k!\left\{ {n \atop k} \right\}$
3) 0 if $n > k$, otherwise, k choices for ball 1, $k-1$ choices for ball 2, etc. $\# = (k)_{\underline{n}}$.

Remark

How many ways are there to distribute n balls to k boxes,

1. with no restriction?
2. if each box has to get at least one ball?
3. if each box has to get at most one ball?

This can be viewed as a mapping $f : [n] \to [k]$

1. with no restriction on f
2. with f surjective (onto)
3. with f injective (1-to-1)

plus some equivalence relation of $[n]$ and $[k]$.

Partitions

Theorem 8.1

$$p(n+k,k) = \sum_{j=1}^{k} p(n,j).$$

Proof: Let λ be a partition of n into k parts and consider the mapping f that removes 1 from each part. $f(\lambda)$ is now a partition of $n-k$ into j parts where $j \leq k$. f is bijective since it is invertible for a fixed k. Hence $p(n,k) = \sum_{j=1}^{k} p(n-k,j)$. Now replace $n \mapsto n+k$. $\square$

This gives a simpler expression for # ways to distribute n identical balls into k identical boxes.

Example 8.2

The university approved funding for SOC (School of Combinatorics) to purchase 8 projectors and hire 8 tutors. How many ways are there to distribute the 8 projectors to 10 tutorial groups? How many ways are there to distribute the 8 tutors to 10 tutorial groups?

Assumptions:
1) Tutorial groups and tutors are distinct but not projectors.
2) Each group require not more than one projector and not more than one tutor.
$\binom{10}{8}$ ways to distribute projectors.
$(10)_{\underline{8}}$ ways to distribute tutors.

Example 8.3

The university approved funding for SOC to hire 8 tutors to conduct 4 question and answer sessions. Students are free to join any of the 4 sessions. How many ways are there to allocate the tutors?

Assumptions:
1) Tutors are distinct, sessions may be distinct (held at different times) or identical (from perspective of tutors, they only care about which colleagues they are working with.)
2) At least one tutor per session.
$4!\left\{ {8 \atop 4} \right\} = 40824$ ways to distribute tutors to distinct sessions.
$\left\{ {8 \atop 4} \right\} = 1701$ ways to distribute tutors to identical sessions.

Example 8.4

The university approved funding for SOC to purchase 100 chairs for 5 tutorial rooms. How many ways are there to allocate the chairs?

Assumptions:
1) Chairs are identical. Since rooms are at different locations, they should be distinct. But sometimes administrators do not really care about the specific rooms. They are happy to know that the rooms have chairs and can be used.
2) No restriction on chairs per room
$\binom{104}{4} = 4598126$ ways to distribute chairs to distinct rooms.
$\sum_{j=1}^{5} p(100, j) = 46262$ ways to distribute chairs to identical rooms.

Example 8.5

10 teams took part in a round robin soccer tournament, where each team played exactly one game against every other team. A team scores 2 points for a win, 1 point for a draw and 0 points if they lose.
How many different outcomes are possible

- *for a particular team L*
- *for team L based on total # wins, draws and losses*
- *for team L based on total # of points*

• 3^9 since each match is distinct with outcomes: win, lose or draw
• Matches are identical, to be distributed to 3 distinct boxes, i.e $\binom{9+2}{2} = 55$
• Every integer from 0 to 18 is possible.

Example 8.5 cont'd

How many different outcomes are possible for the tournament?
How many different outcomes are possible based on total # wins, draws and losses?

- Map the teams to [10]
- $\binom{10}{2} = 45$ distinct matches. Order the teams that are playing by (i, j) with $i < j$. There are 3^{45} possible outcomes because i can win, draw or lose.
- Next consider the matches as identical
- Each match can either be decisive or drawn. A decisive map results in one win and one loss. A drawn match results in one draw. (Could also be two draws depending on how you want to view it.)
- Weak composition of 45 into 2 parts, i.e. # wins and # draws
- $\binom{46}{1} = 46$ outcomes.

Outline

Pigeonhole Principle

Example 9.1

If there are 10 pigeons resting in 9 pigeonholes, then one of the pigeonholes must contain at least 2 pigeons.

- also known as Dirichlet's Drawer Principle
- not a result about distributions
- existence problems

Pigeonhole Principle

Example 9.2

If I have 24 hours of classes in a week, there must be at least a day with five or more hours of classes.
If I wish to arrange my classes so that I have a free day. Then there would be at least one day where I would have at least six hours of classes.

Example 9.3

Among any 13 people, at least two of whom have their birthdays in the same month.

Pigeonhole Principle

Theorem 9.4 (Pigeonhole Principle)

Let $A_1, A_2, \ldots, A_k$ be finite sets that are pairwise disjoint. If

$$| A_1 \cup A_2 \cup \cdots \cup A_k | > kr,$$

then there exists at least one index i such that $|A_i| > r$.

Proof: (By contradiction.)
Suppose not, then for every i, we have $|A_i| \leq r$. By the addition principle,

$$| A_1 \cup A_2 \cup \cdots \cup A_k | = \sum_{i=1}^{k} |A_i| \leq kr,$$

contradicting the original assumption.$\square$

Pigeonhole Principle

Theorem 9.5 (Pigeonhole Principle (Alternate Formulation))

If m pigeons are placed into k pigeonholes, then at least one pigeonhole will contain more than $\lfloor \frac{m-1}{k} \rfloor$ pigeons.

Pigeonhole Principle

Theorem 9.6 (Pigeonhole Principle (Alternate Formulation 2))

If m pigeons are placed into k pigeonholes, then at least one pigeonhole will contain less than $\lceil \frac{m+1}{k} \rceil$ pigeons.

Proof: Suppose every pigeonhole has $\lceil \frac{m+1}{k} \rceil$ or more pigeons. Since total number of pigeons is m, we have

$$m \geq k \times \lceil \frac{m+1}{k} \rceil > m,$$

which is a contradiction. $\square$

Example 9.7

A company has 15 web servers and 10 internet ports. We wish to connect each server to some of the ports, in such a way that we use as few connections as possible but always ensuring each server will have a port to access. A port can be used by one server at any one time and at most 10 web servers will require a port at any one time.
How many connections are needed?

- Answer $= 60$. If $m < 60$ then $\lceil \frac{m+1}{10} \rceil \leq 6$.
- At least one port has 5 or less connections. Call it port A.
- There are at least 10 servers not connected to port A. So if all 10 require a port, there are only 9 ports left which are insufficient.

Pigeonhole Principle

Example 9.8

Given any 10 distinct points within a square of sides 3 units. There are two points that are at a distance of not more than $\sqrt{2}$ apart.

- Divide the square into 9 squares of sides 1 unit.
- By the pigeonhole principle, there must be at least 2 points lying inside the same square.
- What happens if a point lies on the perimeter of the smaller squares?
- Finally check that any two points inside a square of sides 1 unit, cannot be more than $\sqrt{2}$ apart.

Pigeonhole Principle

Example 9.9

There are at least 6 people in Singapore who were born in the same hour of the same day of the same year.

- 5.076 million people in Singapore in 2010 (Dept of Stats)
- Assume that no one is more than 115 years old
- $k = 115 \times 366 \times 24 = 1,010,160$
- A_i be the set of people born in the i-th hour
- $5 \times k = 5,050,800$
- There exist j such that $|A_j| > 5$.

Pigeonhole Principle

Example 9.10

Given any 5 distinct integers, there are at least 3 whose sum is divisible by 3.

• Write the 5 integers a_i as $a_i = 3q_i + r_i$ where $0 \leq r_i \leq 2$.
• Distribute the integers into three boxes according to the remainder r_i.
• If any box has more than 3 integers, then any 3 would have sum divisible by 3.
• Otherwise, there are no empty boxes. So pick an integer from each box and the sum will be divisible by 3.

Example 9.11

Consider the sequence $a_i = 2^i - 1$, i.e. 1, 3, 7, 15, 31, ... Let q be any odd integer, then there exists some element a_k such that q divides a_k.

- Consider $\{a_1, a_2, \ldots a_q\}$. If q divides one of them, done.
- Else consider remainders modulo q
- Write $a_i = d_i q + r_i$ where $0 < r_i < q$, $d_i \geq 0$.
- $\# \ r_i = q$ but $q - 1$ possible values
- There exists $m > n$ such that $r_m = r_n$
- $a_m - a_n = (d_m - d_n)q = (2^m - 1) - (2^n - 1)$
- Therefore $q(d_m - d_n) = 2^n(2^{m-n} - 1) = 2^n a_{m-n}$
- Since q is odd, $q \not| \ 2^n \implies q \mid a_{m-n}$.

Example 9.12

Any subset of $n+1$ distinct integers from $[2n]$ contains
a) at least one pair whose sum is $2n+1$;
b) at least one pair who difference is n.

- Set $B_i = \{i, 2n+1-i\}$, then $[2n] = \bigcup_{i=1}^{n} B_i$
- By the pigeonhole principle, at least one B_i contains 2 integers from any subset of $n+1$ integers
- the sum of these 2 is exactly $2n+1$
- Set $C_i = \{i, n+i\}$, then $[2n] = \bigcup_{i=1}^{n} C_i$
- By the pigeonhole principle, at least one C_i contains 2 integers from any subset of $n+1$ integers
- the difference of these 2 is exactly n

Sequences and Subsequences

Definition 9.13

Let a_n be a sequence, then a subsequence is any

$$a_{i_1}, a_{i_2}, \dots, a_{i_k}, \qquad \textit{such that } 1 \leq i_1 < i_2 < \cdots < i_k.$$

Example 9.14

The sequence $3, 1, 4, 1, 5, 9, 2, 6, 5, 3$ *has the following possible subsequences*

- *3, 1, 4*
- *4, 9, 2, 6*
- *3, 4, 5, 9 is an increasing subsequence of length 4*
- *9, 6, 5, 3 is the longest decreasing subsequence*

Sequences and Subsequences

Example 9.15

Can you find an increasing or decreasing subsequence of length 4 in the following sequence?

$$3, 2, 1, 6, 5, 4, 9, 8, 7.$$

Example 9.16

Can you find an increasing or decreasing subsequence of length 5 in the following sequence?

$$2, 7, 1, 8, 28, 18, 284, 5, 90, 4, 5, 23, 53, 60, 287, 47, 13.$$

2, 7, 8, 28, 284, 287

Sequences and Subsequences

Theorem 9.17 (Erdös and Szekeres)

Given a sequence of n^2+1 distinct integers, either there is an increasing subsequence of $n+1$ terms or a decreasing subsequence of $n+1$ terms.

Proof: Let $a_1, \ldots, a_{n^2+1}$ be the given sequence and define t_i to be the number of terms in the longest increasing subsequence beginning at a_i.
If any $t_i \geq n+1$, we are done. Hence assume $1 \leq t_i \leq n$. We have n^2+1 pigeons (t_i) to be placed into n pigeonholes (1 to n). Hence there are at least

$$\lfloor \frac{(n^2+1)-1}{n} \rfloor + 1 = n+1,$$

a_{i_j} with the same value of t_i.

Sequences and Subsequences

Proof of 9.17 cont'd:
We claim that these $(n+1)$ a_{i_j} forms a decreasing sequence. Suppose not, there exists $a_{i_k} \leq a_{i_l}$ for some $k < l$. Since a_{i_j} are all distinct, $a_{i_k} < a_{i_l}$.
Now a_{i_k} has an increasing sequence of length t_i. But since $a_{i_k} < a_{i_l}$, a_{i_k} has an increasing sequence of length at least $t_i + 1$. This contradicts the fact that a_{i_k} has the longest increasing subsequence of t_i. $\square$

$\mathbb{Q}$ is dense in $\mathbb{R}$

Example 9.18

Let r be any positive irrational number. Then there exists a positive integer n such that the distance between nr and the nearest integer is less than 10^{-10}.

- Map any positive real number to the interval $[0, 1)$ by discarding the integer part.
- Divide $[0, 1)$ into 10^{10} intervals of equal parts.
- Consider $r, 2r, 3r, \ldots, (10^{10} + 1)r$.
- By the pigeonhole principle, there exist jr and kr that are mapped into the same interval.
- Hence $|f(jr) - f(kr)| < 10^{-10} \implies |(j - k)r - m| < 10^{-10}$ for some $m \in \mathbb{Z}$

Example 9.19

10 teams took part in a round robin soccer tournament, where each team played exactly one game against every other team. A team scores 2 points for a win, 1 point for a draw and 0 points if they lose.
Show that there were two teams with the same score if more than 70% of the games ended in a draw.

- Each team can get any total score from 0 to 18
- 19 distinct total to be distributed to 10 teams
- $0.7 \times \binom{10}{2} = 31.5$, so at most 13 games had a winner.

Solution to Example 9.19

Show that there were two teams with the same score if more than 70% of the games ended in a draw.

Proof by contradiction:

- Map score $x \mapsto x - 9$
- Equivalent to W (1 pt), D(0 pt), L (-1 pt)
- Suppose all 10 teams had distinct scores, at least 5 teams had either all positive or all negative scores
- WLOG assume 5 teams has positive total score
- Total points $\geq 1 + 2 + 3 + 4 + 5 = 15$.
- Contradicting the fact that at most 13 games had a winner.

Example 9.20

Let $X \subset [100]$, $|X| = 10$. Prove that it is possible to find two disjoint subsets A and B of X such that $\sum_{a \in A} a = \sum_{b \in B} b$.

- For example, if $X = \{2, 7, 15, 19, 23, 50, 56, 60, 66, 100\}$,
- we can take $A = \{19, 50\}$ and $B = \{2, 7, 60\}$.
- Or $A = \{23, 50\}$ and $B = \{7, 66\}$.

Solution to Example 9.20

Let $X \subset [100]$, $|X| = 10$. Prove that it is possible to find two disjoint proper subsets A and B of X such that $\sum_{a \in A} a = \sum_{b \in B} b$.

- # of nonempty proper subsets of $X = 2^{10} - 2 = 1022$
- $1 \leq \sum_{a \in A} a \leq 92 + 93 + \ldots + 100 = 864$.
- There exists two proper subsets C and D of X which satisfies $\sum_{c \in C} c = \sum_{d \in D} d$.
- C and D need not be distinct, let $A = C - D$ and $B = D - C$.

Ramsey Numbers

Example 9.21

Prove that among six persons, either there are 3 persons who are mutual acquaintances or there are 3 persons who are mutually not acquainted with each other.

- Fix a person A. By the pigeonhole principle, among the 5 other people A either knows at least 3 of them or A do not known at least 3 of them.
- Suppose A knows B, C and D.
- If B, C and D are mutually not acquainted with each other, we are done.
- Otherwise, at least two of them are acquainted, say B and C, hence A, B and C are mutual acquaintances.
- A similar argument holds if A do not know at least 3 of the remaining 5.

Ramsey Numbers

Example 9.22

Six points are in general position in space (no three in a line and no four in a plane.) The fifteen line segments joining the points in pairs are drawn and painted either red or blue. Prove that there is a triangle with all its sides of the same colour.

- Bijectively equivalent to Example 9.21
- Map each point to a person
- If two persons are acquainted paint the segment red, if they are not acquainted paint it blue.

Ramsey Numbers

Definition 9.23

$R(p, q)$ is defined to be the least number of people such that, either there are p persons who are mutual acquaintances or there are q persons who are mutually not acquainted with each other. These are known as Ramsey Numbers.

- $R(p, q) = R(q, p)$
- $R(1, q) = 1$
- $R(2, q) = q$.

Ramsey Numbers

Theorem 9.24

$$R(3,3) = 6.$$

- We know $R(3,3) \leq 6$ by Example 9.21
- To show $R(3,3) \neq 5$, consider 5 points A, B, C, D, E
- Colour AB, BC, CD, DE and EA blue and the remaining segments red.
- None of the $\binom{5}{3}$ triangles has sides of the same colour.

Ramsey Numbers

Theorem 9.25

$$R(p,q) \le R(p-1,q) + R(p,q-1),$$
$$R(p,q) \le \binom{p+q-2}{p-1}, \qquad \textit{for } p,q \ge 2$$

- Paper by Greenwood and Gleason 1955
  http://cms.math.ca/cjm/v7/cjm1955v07.0001-0007.pdf
- Exact value of $R(3,10)$ and $R(4,6)$ is still open
- Dynamic survey by Radziszowski
  http://www.combinatorics.org/Surveys/index.html