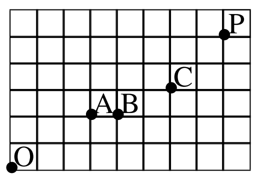
NATIONAL UNIVERSITY OF SINGAPORE
DEPARTMENT OF MATHEMATICS
MA2214 COMBINATORIAL ANALYSIS

TUTORIAL 3 **SEMESTER II, AY 2010/2011**

1. (a) Prove the identity
$$\sum_{k=0}^{m} (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m}.$$
   (b) Find a closed formula for the sum
$$\sum_{k=0}^{n} \binom{2n+1}{k}.$$

2. (a) Let $n \geq 2$ and define the set
$$T = \{(x,y,z) \mid x,y,z \in [n+1], x < z \text{ and } y < z\}.$$
   By counting $|T|$ in two ways show that
$$\sum_{k=1}^{n} k^2 = \binom{n+1}{2} + 2\binom{n+1}{3}.$$
   (b) Extend the previous result to find a formula for $\sum_{k=1}^{n} k^3$.

3. Consider the following street network.



Find the number of shortest paths from O to P that

(i) passes through A;
(ii) passes through street AB;
(iii) passes through A and C;
(iv) can be taken if street AB is closed.

4. Find the number of 7-digit (proper) integers where each digit that appears must appear at least 3 times.

5. Find the coefficient of x^{29} in the expansion of
$$(1+x^5+x^7+x^9)^{1000}.$$

6. A ternary sequence of length n is a sequence of n letters concatenated together, where there are 3 choices (X, Y or Z) for each letter. For example a ternary sequence of length 4 could look like $XXYZ$ or $ZXZZ$.

   Find an expression for the number of ternary sequences of length n with exactly k Xs.

7. (Challenging problem for students who know some linear algebra.)
   Define the matrix P_n as a square matrix of order n with (i,j)-th entries given by

$$p_{ij} = \binom{i+j-2}{i-1}.$$

   Examples:

$$P_1 = \begin{pmatrix} 1 \end{pmatrix}, P_2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, P_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}, P_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{pmatrix}.$$

   Prove that $\det(P_n) = 1$.

Answers

3. (i) $\binom{5}{2}\binom{8}{3}$; (ii) $\binom{5}{2}\binom{7}{3}$; (iii) $\binom{5}{2}\binom{4}{1}\binom{4}{2}$;
(iv) $\binom{13}{5} - \binom{5}{2}\binom{7}{3}$

4. 2844

5. 123754368753000