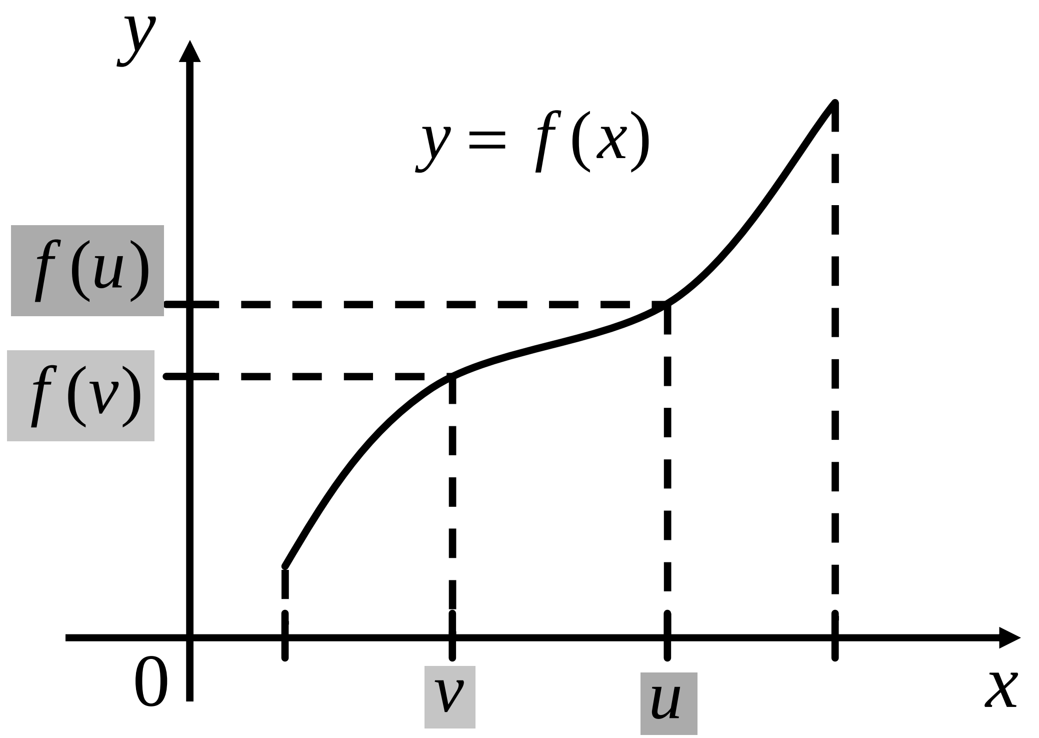
Increasing functions

Let f be a function defined on an interval I.



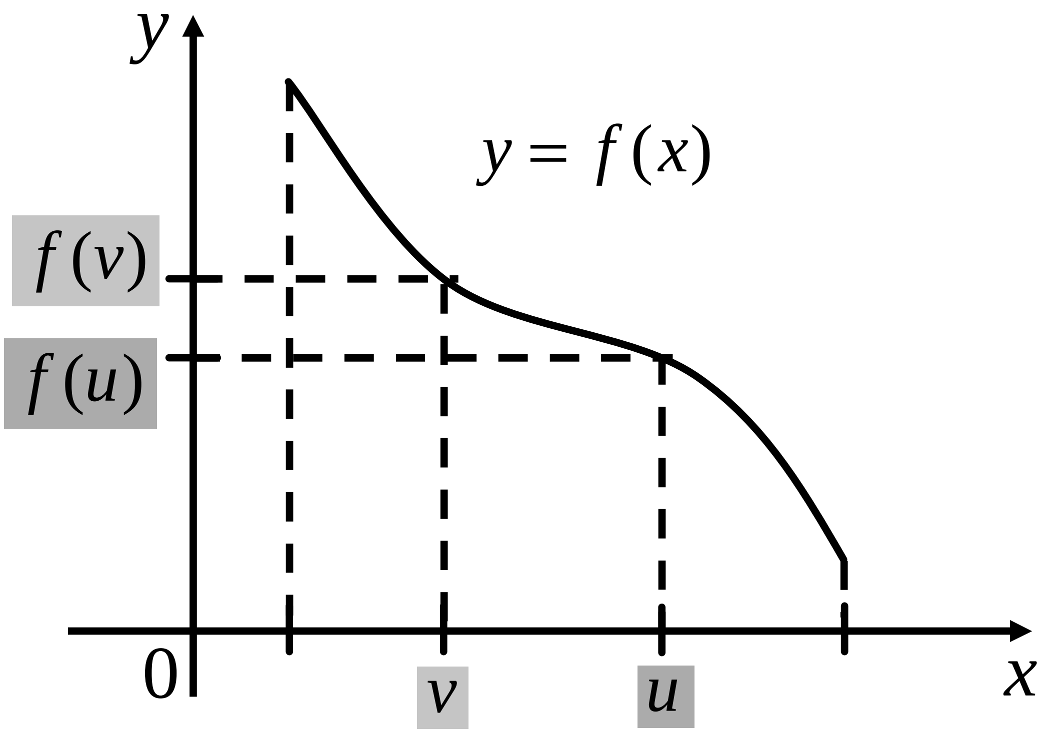
f is ***increasing*** on I if $u > v \Rightarrow f(u) > f(v)$.

Bigger x value, bigger $f(x)$ value

y increases as x increases

Decreasing functions

Let f be a function defined on an interval I.



f is ***decreasing*** on I if $u > v \Rightarrow f(u) < f(v)$.

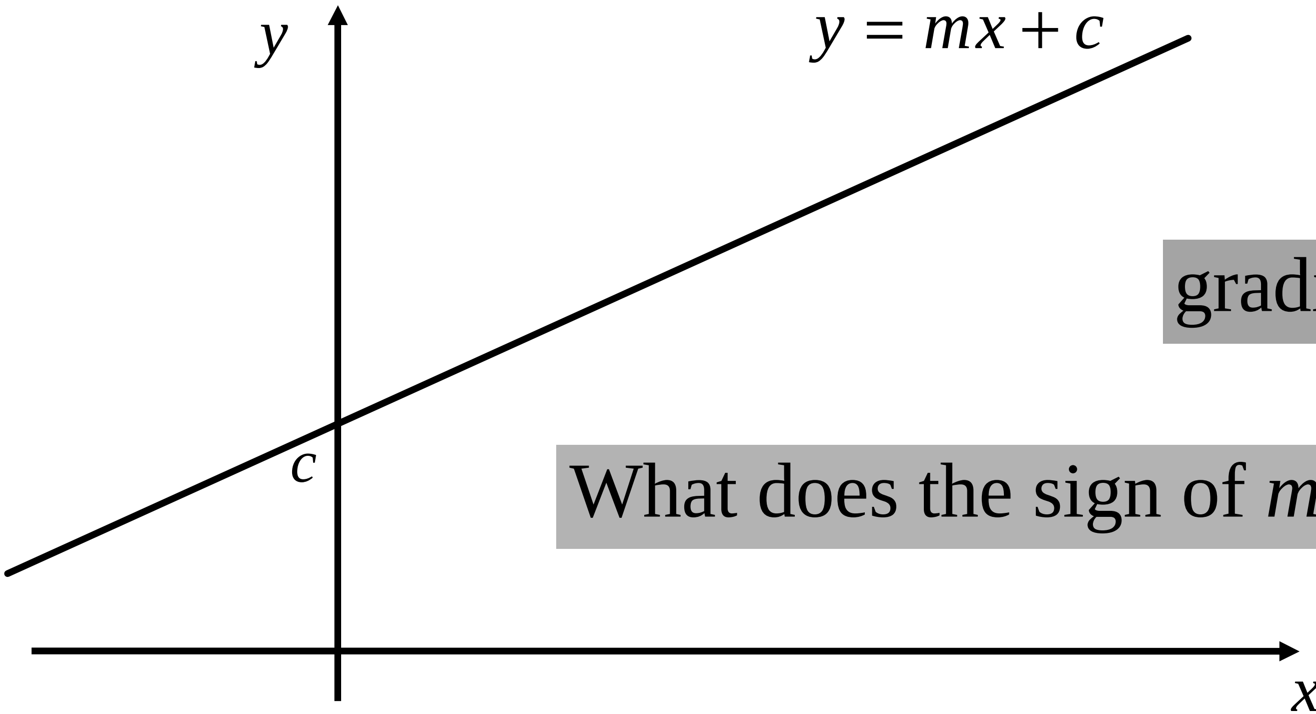
Bigger x value, smaller $f(x)$ value

y decreases as x increases

Question:

How to check a function $f(x)$ is increasing / decreasing ??

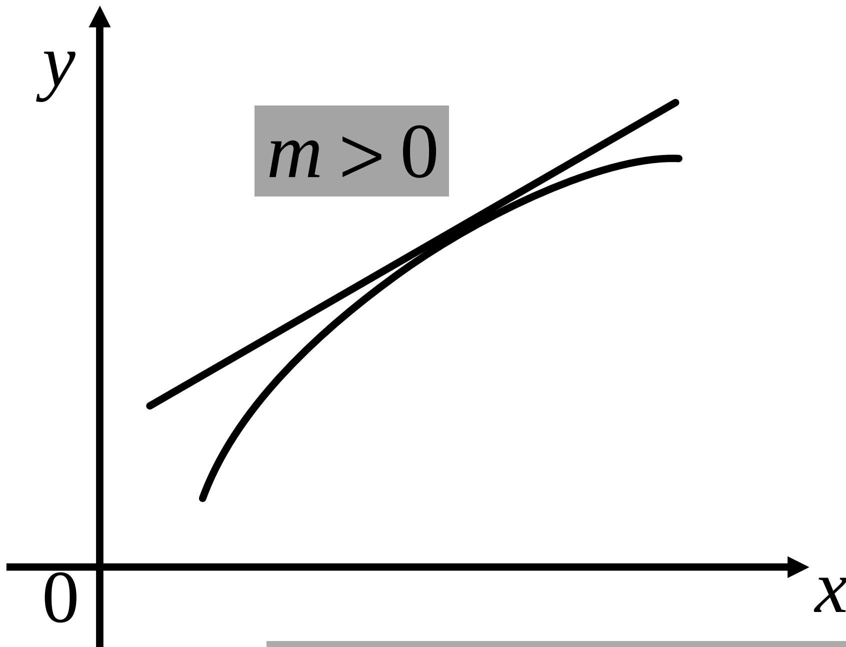
Pause and Think !!!

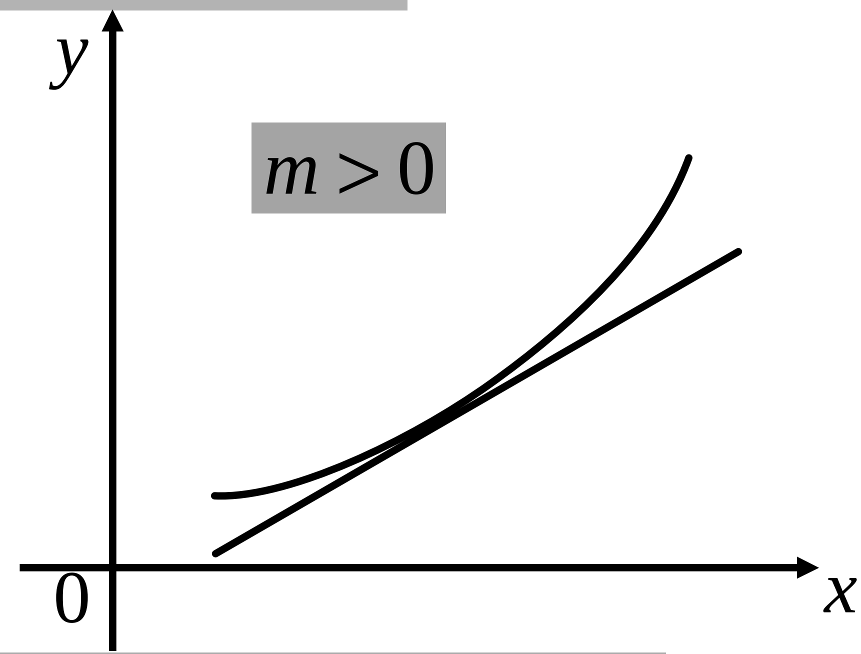


gradient $= m$

What does the sign of m tell you?

What does the sign of $\frac{dy}{dx}$ tell you?





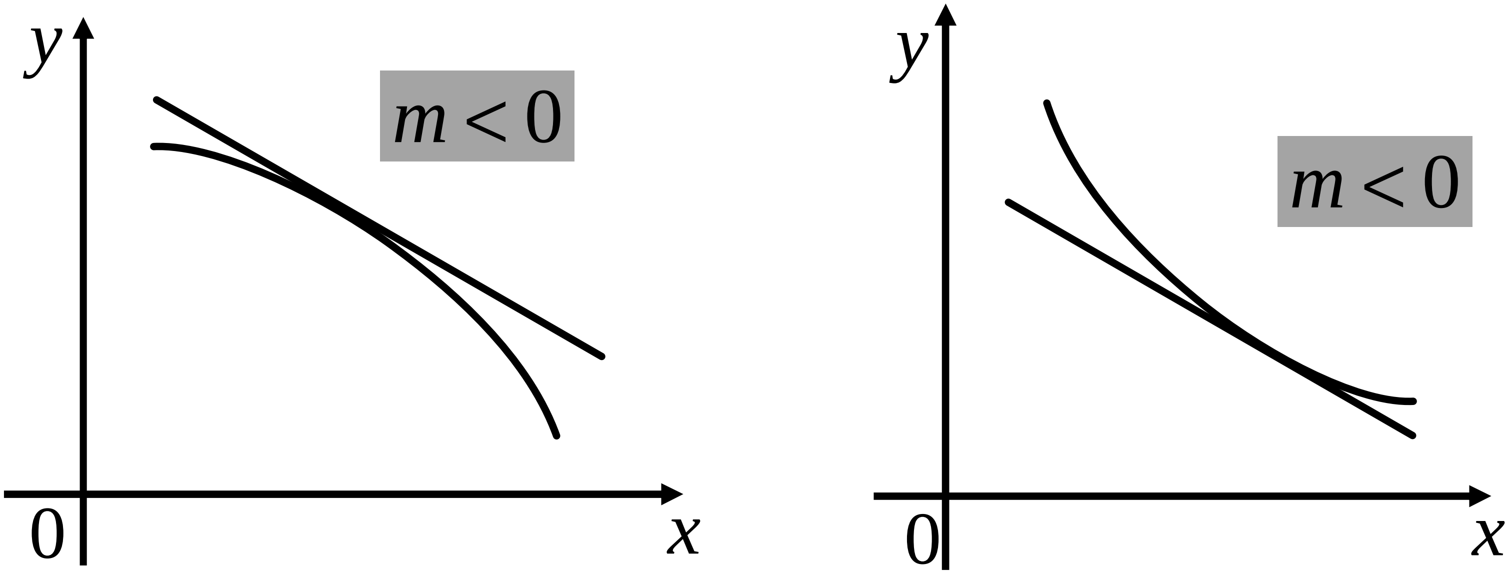
For both graphs, y is increasing.

For both graphs, $\frac{dy}{dx} > 0$.

$\therefore$ y increases if $\frac{dy}{dx} > 0$.

$\therefore$ $f(x)$ increases if $f'(x) > 0$.

What does the sign of $\frac{dy}{dx}$ tell you?



For both graphs, y is decreasing.

For both graphs, $\frac{dy}{dx} < 0$.

$\therefore$ y decreases if $\frac{dy}{dx} < 0$.

$\therefore$ $f(x)$ decreases if $f'(x) < 0$.

Test for Increasing / Decreasing function

$f'(x) > 0$ for all values of x in I,

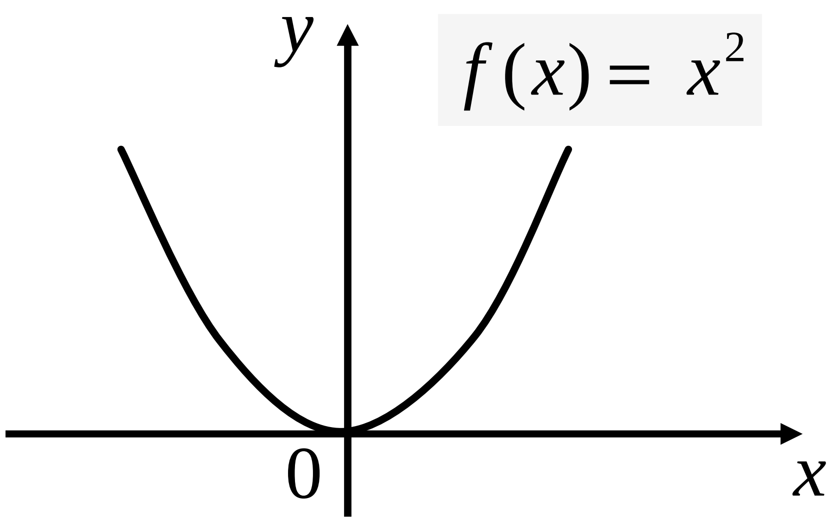
then f is ***increasing*** on I.

$f'(x) < 0$ for all values of x in I,

then f is ***decreasing*** on I.

$f'(x) > 0$ for all values of x in I, then f is ***increasing*** on I.

$f'(x) < 0$ for all values of x in I, then f is ***decreasing*** on I.



$f(x) = x^2$

$f'(x) = 2x$

For $x > 0$, $f'(x) = 2x > 0$, $f(x)$ is increasing

For $x < 0$, $f'(x) = 2x < 0$, $f(x)$ is decreasing

$f'(x) > 0$ for all values of x in I, then f is ***increasing*** on I.

$f'(x) < 0$ for all values of x in I, then f is ***decreasing*** on I.

$$f(x) = \frac{2}{3}x^3 + x^2 + 2x + 1$$

$$f'(x) = 2x^2 + 2x + 2$$

$$= 2\left((x + \tfrac{1}{2})^2 + \tfrac{3}{4}\right) \qquad \text{(Completing square)}$$

For all x, $f'(x) > 0$, $f(x)$ is increasing

Example Let $f(x) = x^3(x-1)^2$.

$$\text{Then } f'(x) = x^3(2)(x-1) + 3x^2(x-1)^2$$

$$= x^2(x-1)(5x-3)$$

Set $f'(x) = 0$, we have $x = 0,\ 1$ or $\dfrac{3}{5}$.

		0		$\frac{3}{5}$		1	
$f'(x)$	(+)(-)(-)		(+)(-)(-)		(+)(-)(+)		(+)(+)(+)
$f(x)$	$\nearrow$		$\nearrow$		$\searrow$		$\nearrow$

Show that $\ln(1+x) < x$ for all $x > 0$.

It is the same as showing $\ln(1+x) - x < 0$ for all $x > 0$.

Let $f(x) = \ln(1+x) - x$.

Then $f'(x) = \dfrac{1}{1+x} - 1$.

Note that : Since $x > 0$, we have $1 + x > 1$.

Thus, $f'(x) = \dfrac{1}{1+x} - 1 < 0$ for all $x > 0$.

Hence, $f(x)$ is decreasing on $[0, \infty)$.

Show that $\ln(1+x) < x$ for all $x > 0$.

It is the same as showing $\ln(1+x) - x < 0$ for all $x > 0$.

Let $f(x) = \ln(1+x) - x$.

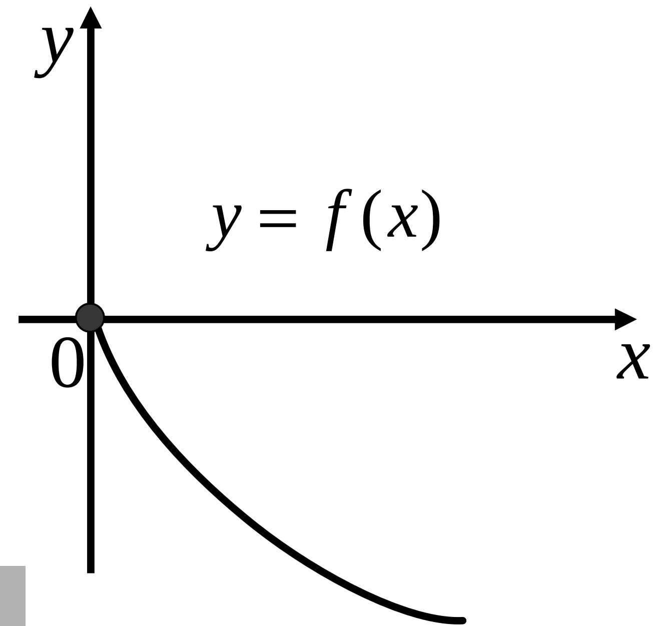
Note : $f(0) = \ln(1+0) - 0$
$= 0$

$f(x)$ is decreasing on $[0, \infty)$.

When $x = 0$, $f(0) = 0$.

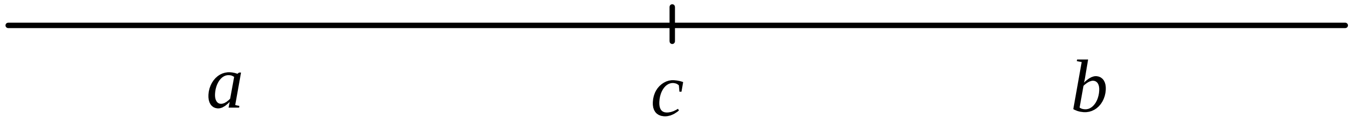
Thus, for $x < 0$, $f(x) < 0$.

Hence, $\ln(1+x) - x < 0$ for all $x > 0$.



Derivative Test

First Derivative Test for Local Extremes



Suppose that $c \in (a, b)$ is a ***critical point*** of f. If

(i) $f'(x) > 0$ for $x \in (a, c)$ and $f'(x) < 0$ for $x \in (c, b)$,

then $f(c)$ is a ***local maximum***.

(ii) $f'(x) < 0$ for $x \in (a, c)$ and $f'(x) > 0$ for $x \in (c, b)$,

then $f(c)$ is a ***local minimum***.

Note: $f'(x)$ changes sign

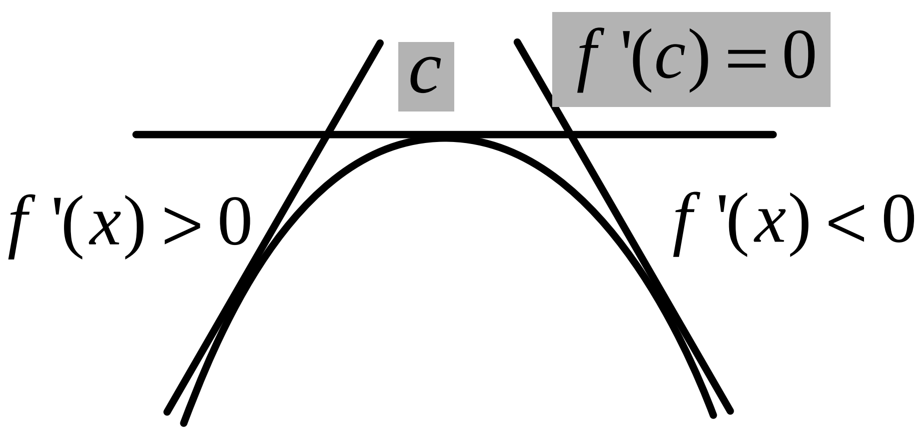
The test is applicable whether $f'(c)$ exists or not.

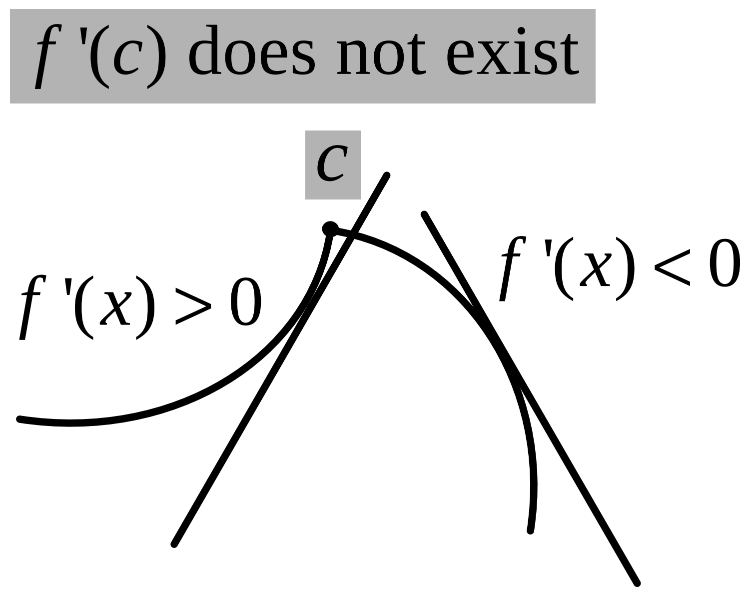
First Derivative Test for Local Extremes

Suppose that $c \in (a,b)$ is a ***critical point*** of f. If

(i) $f'(x) > 0$ for $x \in (a,c)$ and $f'(x) < 0$ for $x \in (c,b)$,

then $f(c)$ is a ***local maximum***.



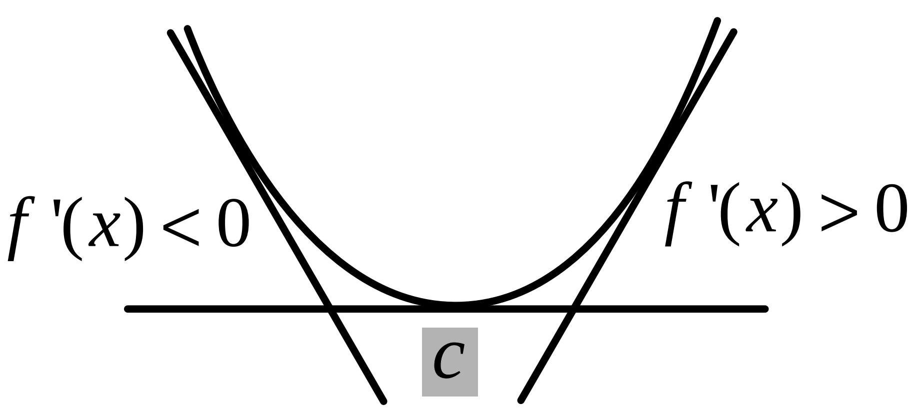


The test is applicable whether $f'(c)$ exists or not.

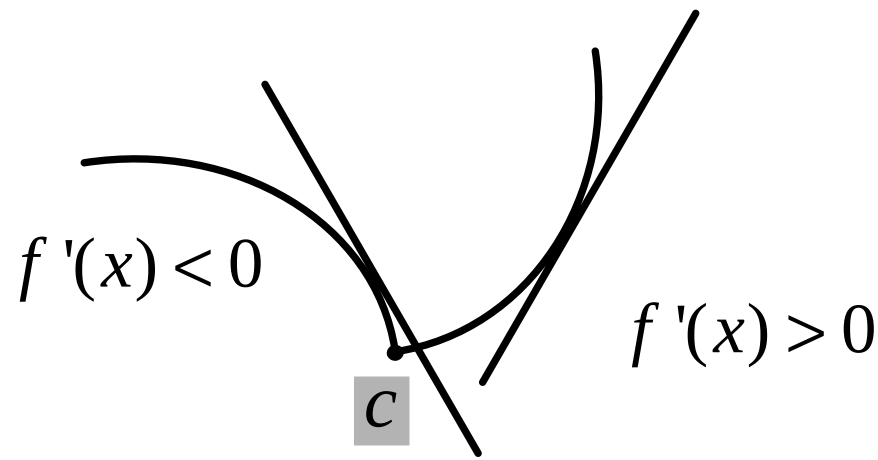
First Derivative Test for Local Extremes

Suppose that $c \in (a,b)$ is a ***critical point*** of f. If

(ii) $f'(x) < 0$ for $x \in (a,c)$ and $f'(x) > 0$ for $x \in (c,b)$, then $f(c)$ is a ***local minimum***.



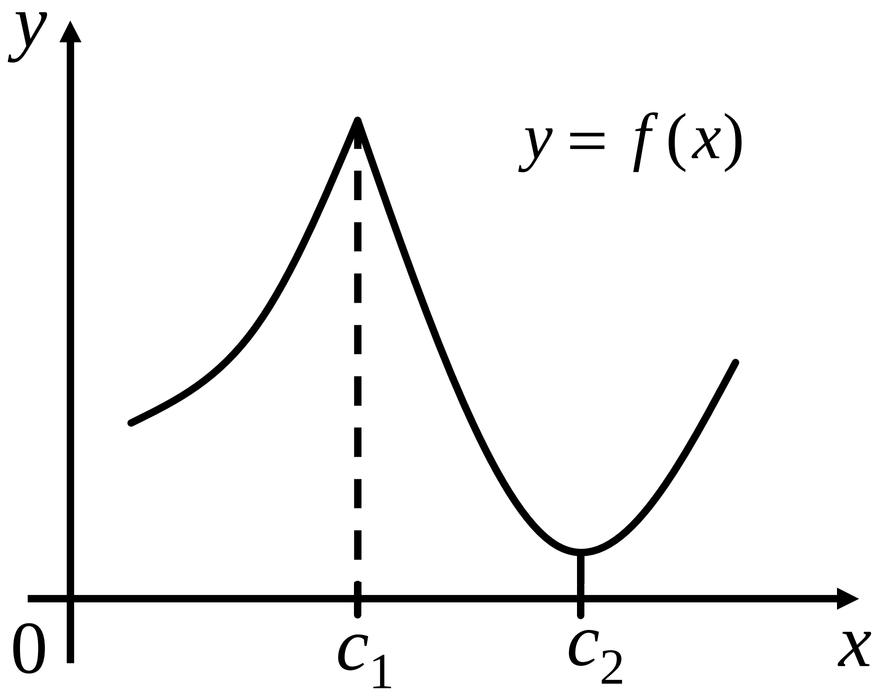
$f'(c) = 0$



$f'(c)$ does not exist

The test is applicable whether $f'(c)$ exists or not.

First Derivative Test



f has a local maxmium at $x = c_1$ and a local minimum at $x = c_2$.

Example

$$f(x)=\begin{cases} x^2-4x+9, & x\le 3 \\ 6-\sqrt{x-3}, & x>3 \end{cases}$$

$$f'(x)=\begin{cases} 2(x-2) & , \ x<3 \\ -\dfrac{1}{2\sqrt{x-3}}, & x>3 \end{cases}$$

For $f'(3)=\lim\limits_{x\to 3}\left(\dfrac{f(x)-f(3)}{x-3}\right)$ to exist,

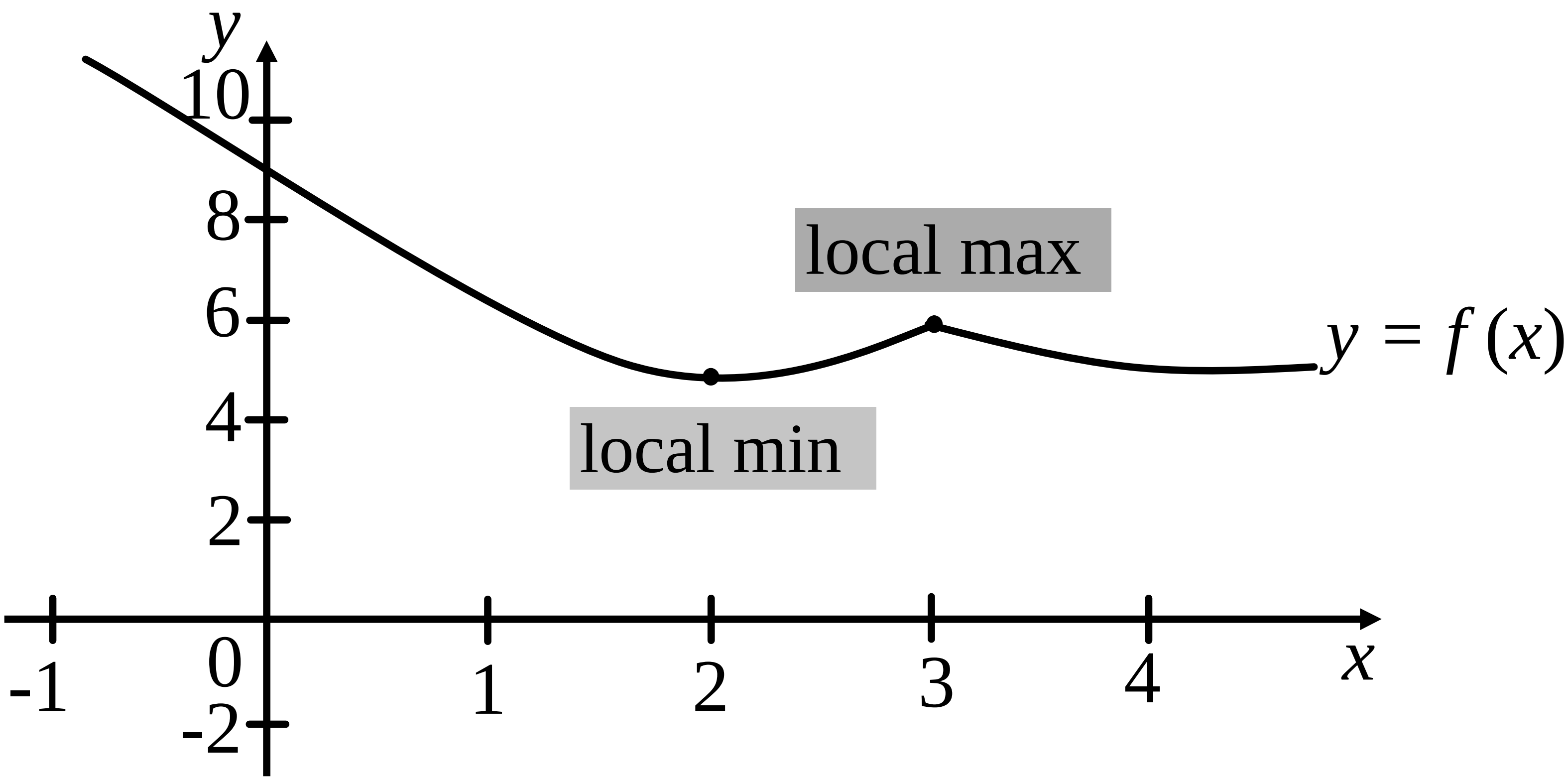
$$\lim_{x\to 3^-}\left(\frac{f(x)-f(3)}{x-3}\right)=\lim_{x\to 3^+}\left(\frac{f(x)-f(3)}{x-3}\right)$$

It can be checked that $f'(3)$ does not exist.

Note that $f'(2)=0$

Thus, $x=2$ and $x=3$ are two ***critical points***.

$$f(x)=\begin{cases} x^2-4x+9, & x\le 3 \\ 6-\sqrt{x-3}, & x>3 \end{cases}$$



By ***First Derivative Test***, f has a ***local minimum*** at $x = 2$ & a ***local maximum*** at $x = 3$.

x	2^-	2^+
$f'(x)$	negative	positive
curve	$\searrow$	$\nearrow$

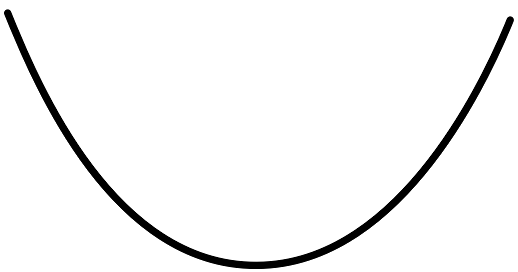
local min at $x = 2$

x	3^-	3^+
$f'(x)$	positive	negative
curve	$\nearrow$	$\searrow$

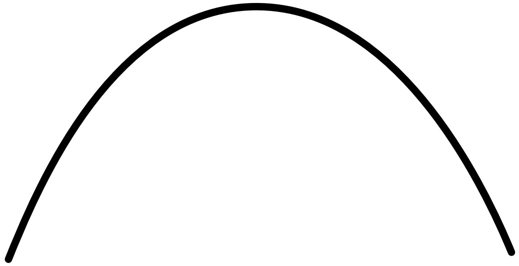
local max at $x = 3$

Concavity

straight line



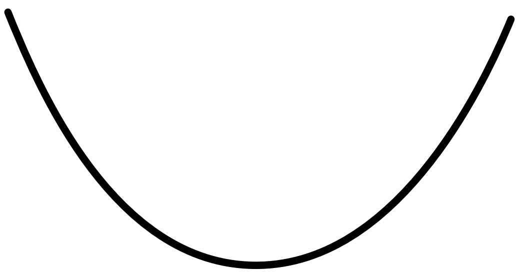
concave up



concave down

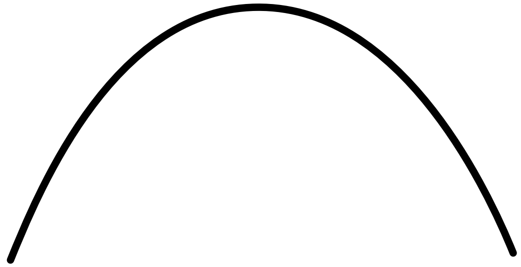
Concavity

Concave Up



$$f''(x) > 0$$

Concave Down



$$f''(x) < 0$$

Increasing / Decreasing

$f'(x)$ tells you how $f(x)$ changes with x.

If $f'(x) > 0$, then $f(x)$ increases.

If $f'(x) < 0$, then $f(x)$ decreases.

Similarly, $f''(x)$ tells you how $f'(x)$ changes with x.

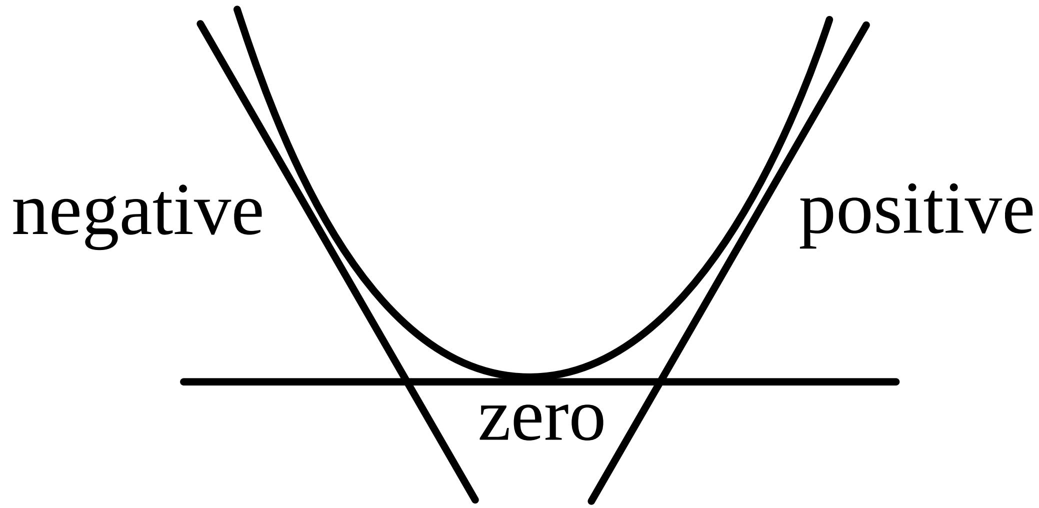
If $f''(x) > 0$, then $f'(x)$ increases.

If $f''(x) < 0$, then $f'(x)$ decreases.

$$\text{If } f''(x) > 0, \text{ then } f'(x) \text{ increases.}$$

$$\text{If } f''(x) < 0, \text{ then } f'(x) \text{ decreases.}$$

Concavity - Concave Up



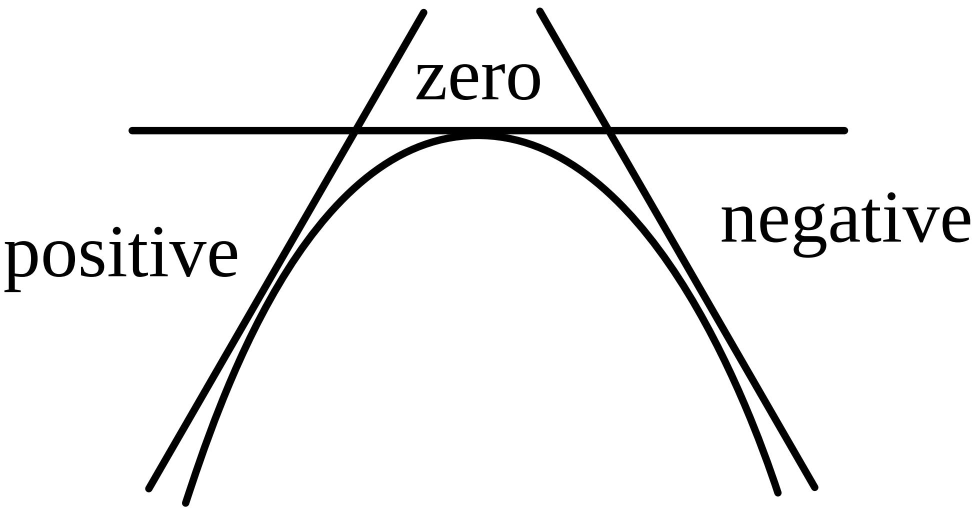
Note that we have increasing gradient, i.e., increasing $f'(x)$.

$$\text{Thus,} \quad f''(x) > 0$$

If $f''(x) > 0,$ then $f'(x)$ increases.

If $f''(x) < 0,$ then $f'(x)$ decreases.

Concavity - Concave Down



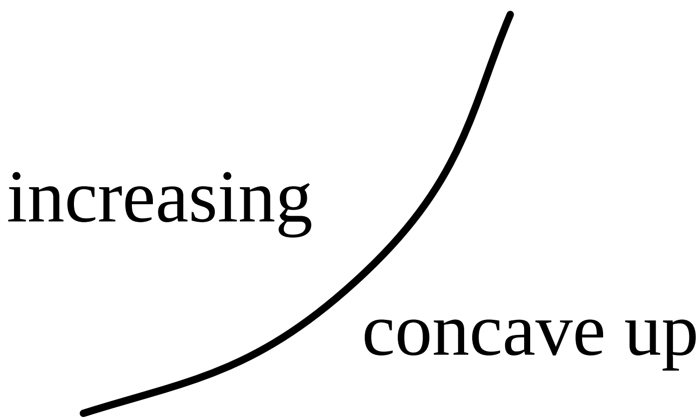
Note that we have decreasing gradient, i.e., decreasing $f'(x)$.

Thus, $f''(x) < 0$

Concavity

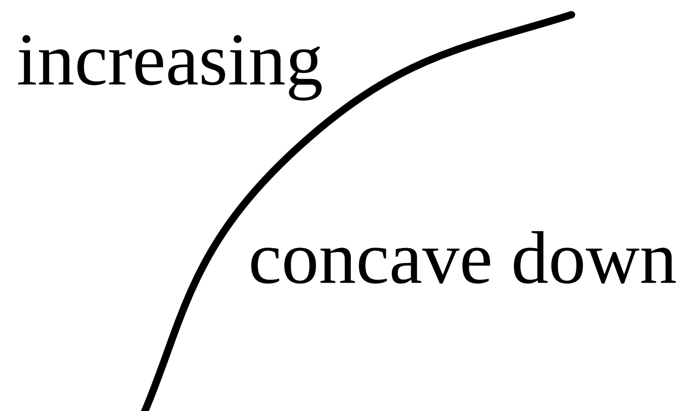
$f'(x)$ determines if f is increasing or decreasing.

$f''(x)$ determines if f is concave up or down.



$$f'(x) > 0$$

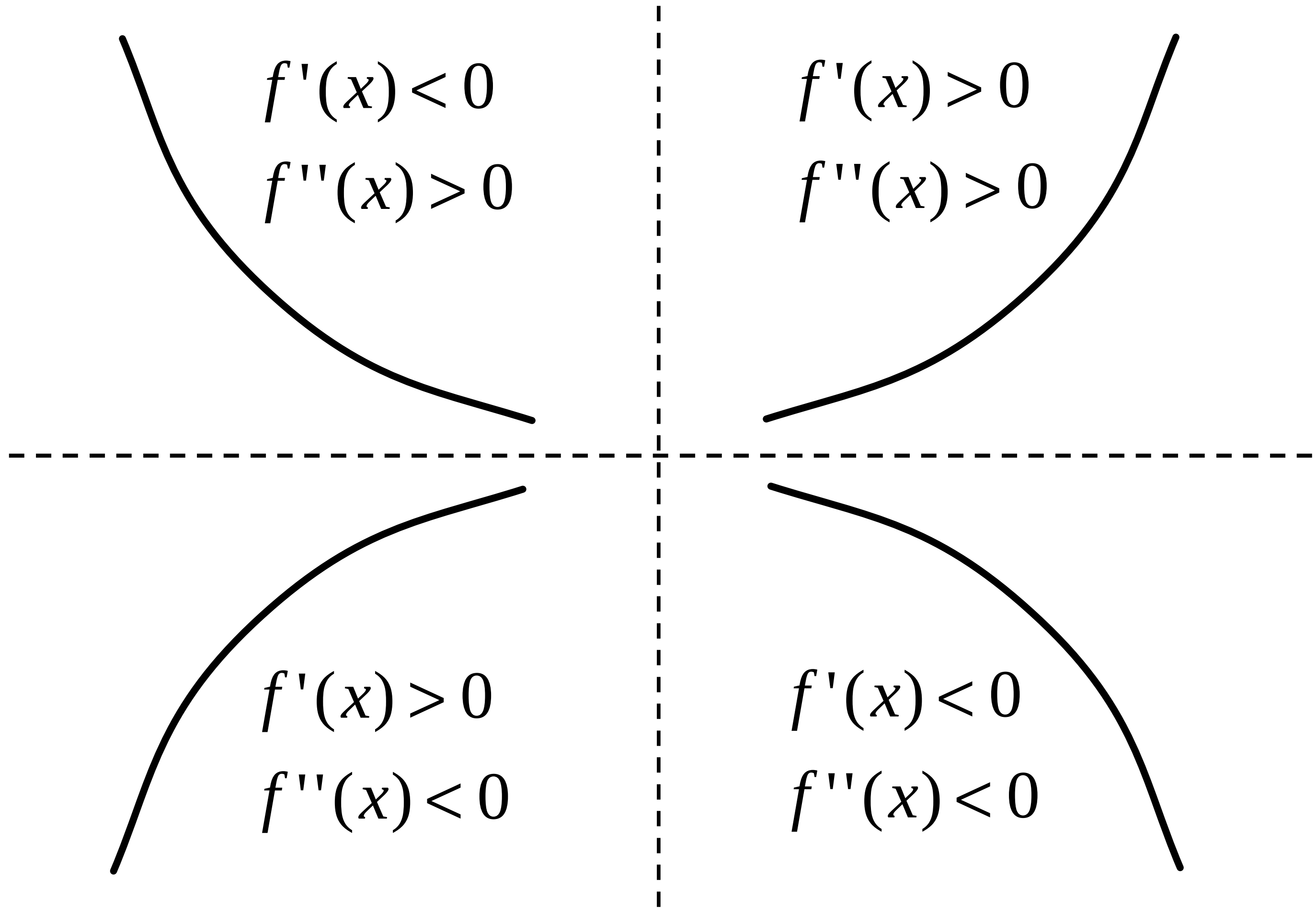
$$f''(x) > 0$$



$$f'(x) > 0$$

$$f''(x) < 0$$

Concavity



Concavity Test

$f''(x) > 0$ concave up

$f''(x) < 0$ concave down

Concavity - Example

Let $y = f(x) = x^3$

Then $f'(x) = 3x^2 \geq 0$ for all values of x

$$f''(x) = 6x$$

$f''(x) = 6x > 0$ for $x > 0$

(Concave up)

$f''(x) = 6x < 0$ for $x < 0$

(Concave down)