MA1506

Mathematics II

Chapter 6

Linear Transformation

Linearity

$$\frac{d}{dx}(af + bg) = a\frac{df}{dx} + b\frac{dg}{dx}$$

$$\int (af + bg) = a \int f + b \int g$$

$$L(af + bg) = aL(f) + bL(g)$$

$f(x) = x$	$f(x) = x^2$	$f(x) = \sin x$
Linear	Nonlinear	Nonlinear

6.1 What is a Linear Transformation

Transformations are mappings (rules) that send vectors to vectors

Linear transformation further satisfies

$$T(c\vec{u}) = cT(\vec{u})$$

$$T(\vec{u} + \vec{v}) = T\vec{u} + T\vec{v}$$

6.1 Example

Identity transformation is linear

$$\boxed{I\vec{u} = \vec{u}}$$

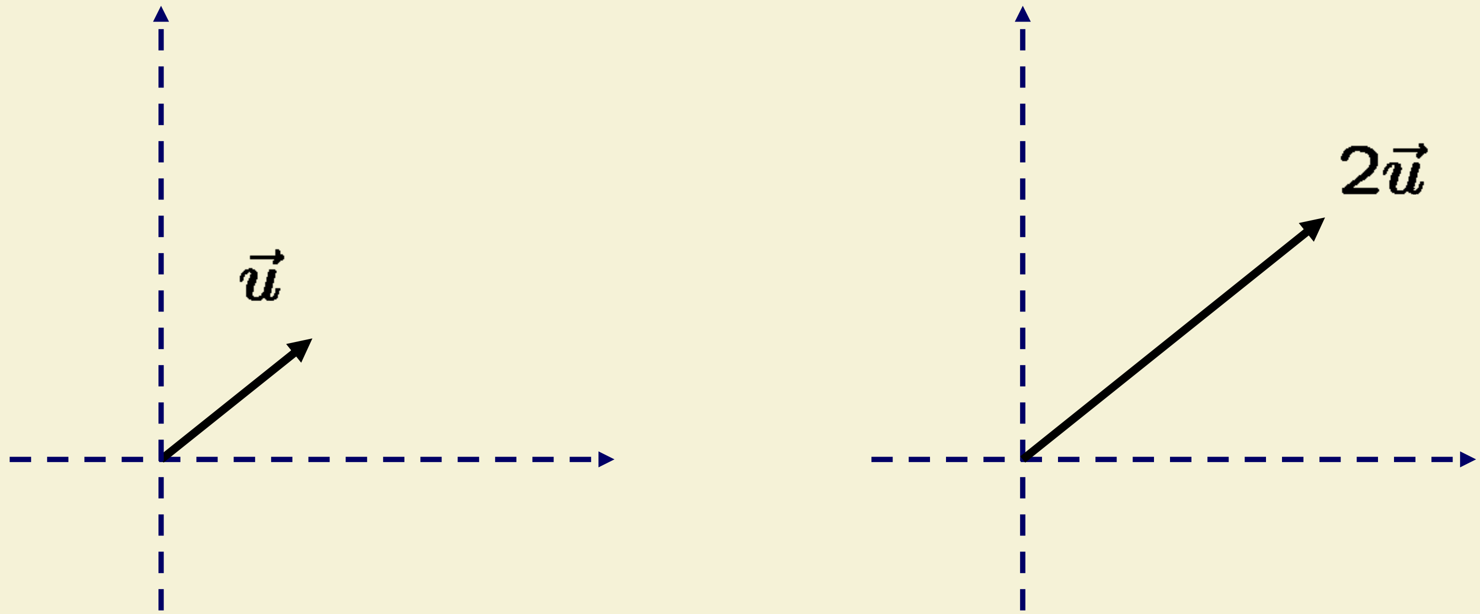
Check

$$I(c\vec{u}) = c\vec{u} = cI(\vec{u})$$

$$I(\vec{u} + \vec{v}) = \vec{u} + \vec{v} = I\vec{u} + I\vec{v}$$

6.1 Example

Let ***D*** be the transformation $\boxed{D\vec{u} = 2\vec{u}}$



Check $$D(c\vec{u}) = 2c\vec{u} = c(2\vec{u}) = cD\vec{u}$$

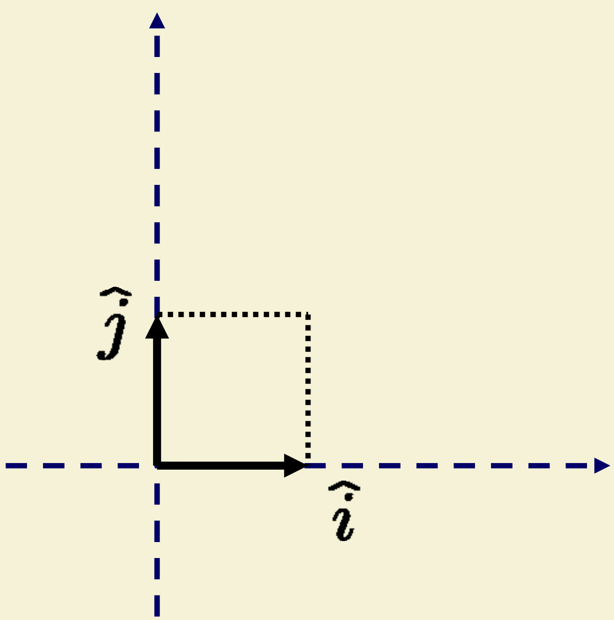
$$D(\vec{u} + \vec{v}) = 2(\vec{u} + \vec{v}) = 2\vec{u} + 2\vec{v} = D\vec{u} + D\vec{v}$$

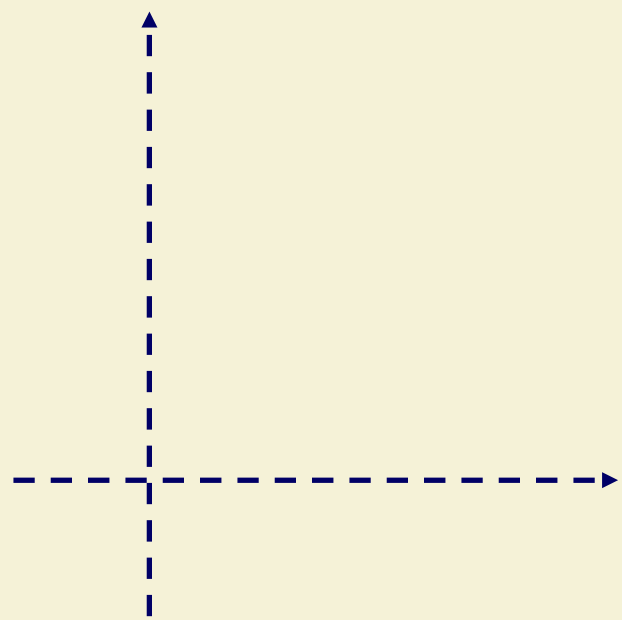
6.2 Basic Box in 2-D

Every vector can be defined as $a\hat{i} + b\hat{j}$

$$T(a\hat{i} + b\hat{j}) = aT\hat{i} + bT\hat{j}$$

just need to know

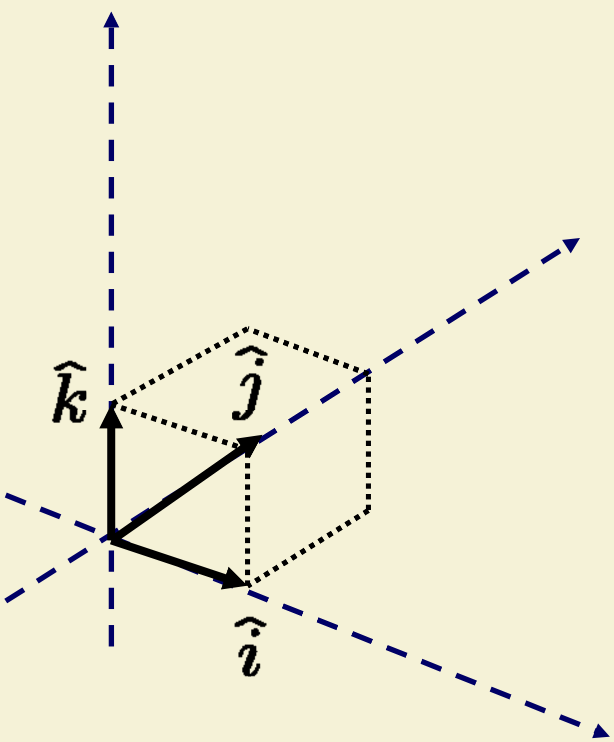


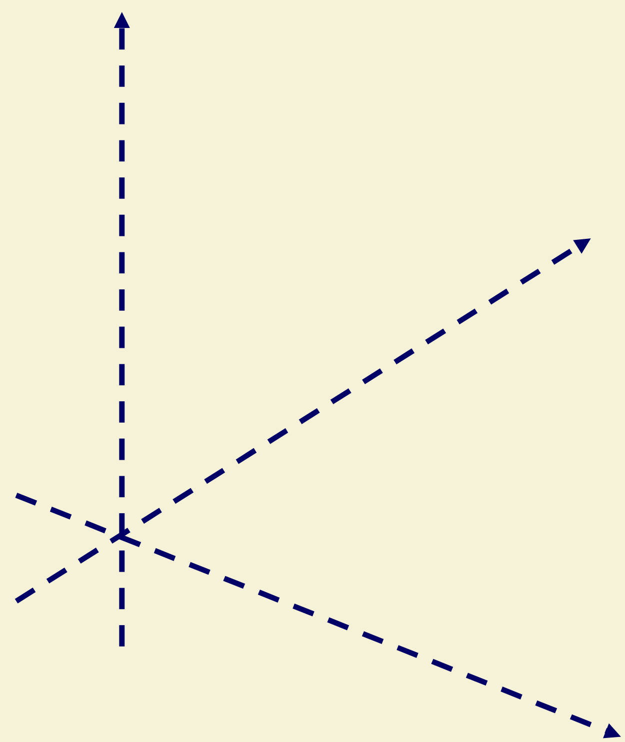


6.2 Basic Box in 3-D

Every vector can be defined as $a\hat{i} + b\hat{j} + c\hat{k}$

$$T(a\hat{i} + b\hat{j} + c\hat{k}) = aT\hat{i} + bT\hat{j} + cT\hat{k}$$

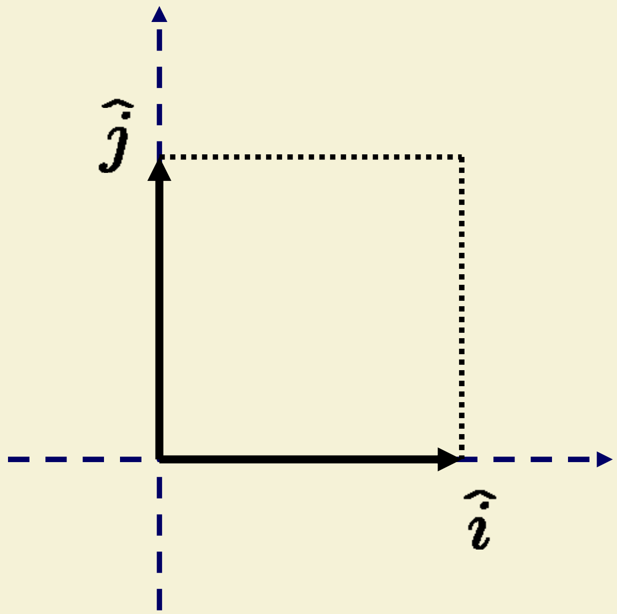


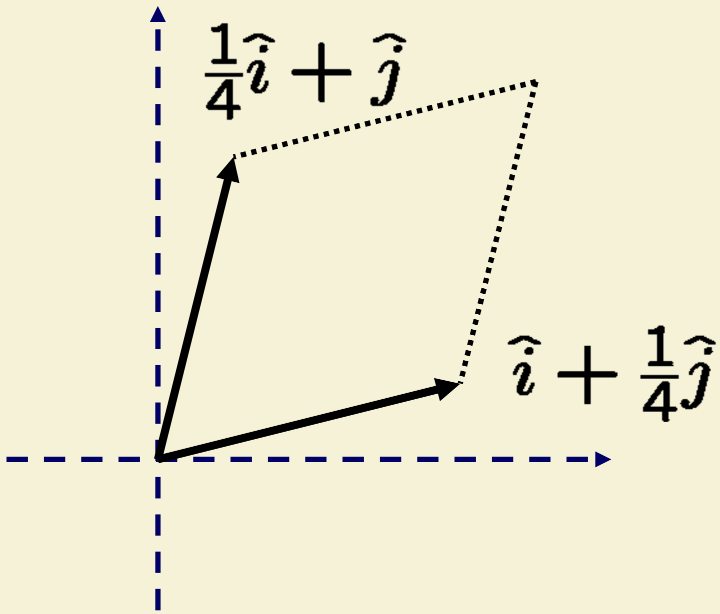


Example $\quad T(\hat{i}) = \hat{i} + \frac{1}{4}\hat{j},\ T(\hat{j}) = \frac{1}{4}\hat{i} + \hat{j}$

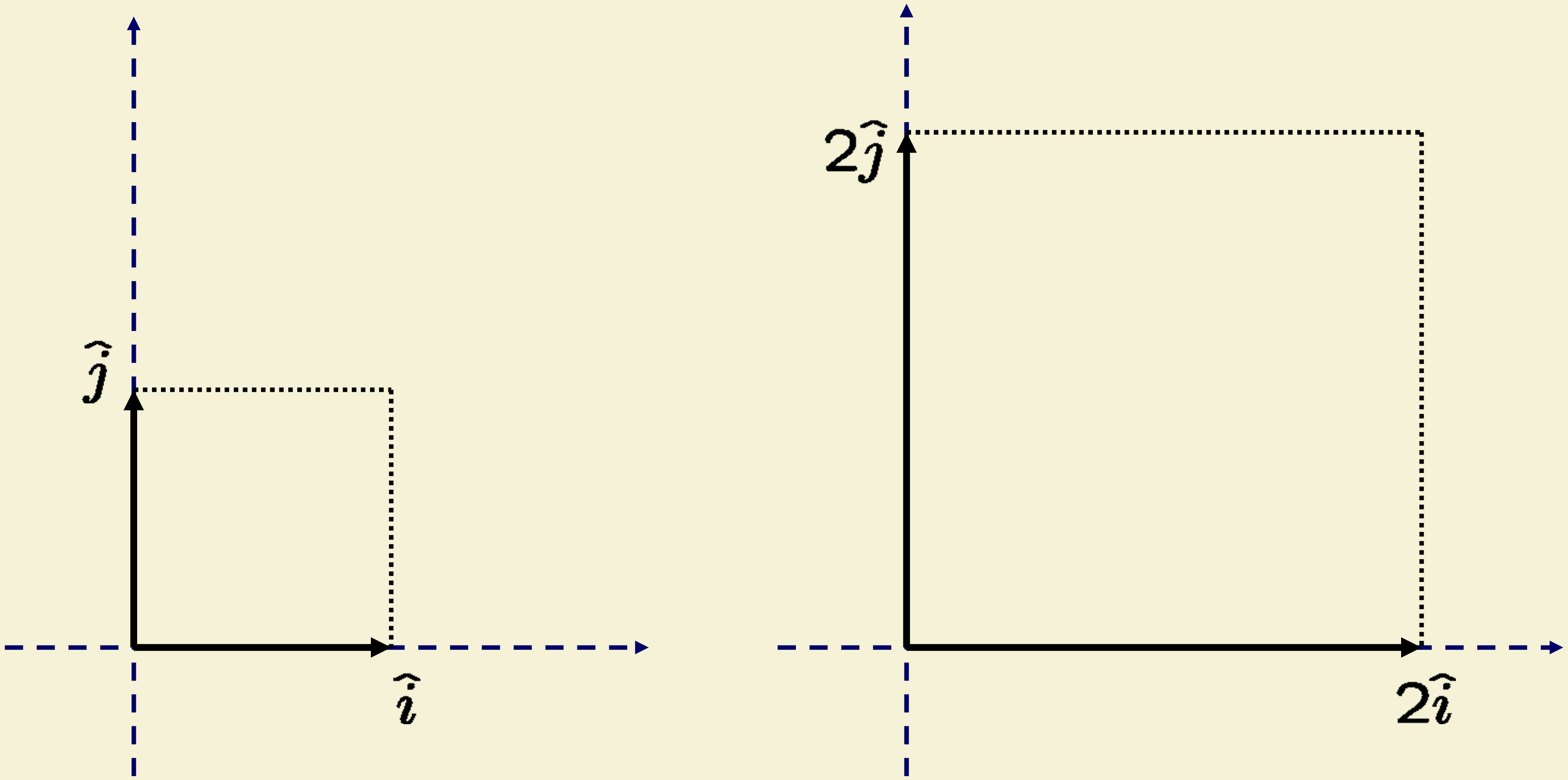
$$T(2\hat{i} + 3\hat{j}) = 2T\hat{i} + 3T\hat{j}$$

$$= 2\left(\hat{i} + \frac{1}{4}\hat{j}\right) + 3\left(\frac{1}{4}\hat{i} + \hat{j}\right) = \frac{11}{4}\hat{i} + \frac{7}{2}\hat{j}$$





Example $D\vec{u} = 2\vec{u}$



6.2 Matrices

$$\left.\begin{aligned} T\hat{i} &= a\hat{i} + c\hat{j} = \begin{bmatrix} a \\ c \end{bmatrix} \\ T\hat{j} &= b\hat{i} + d\hat{j} = \begin{bmatrix} b \\ d \end{bmatrix} \end{aligned}\right\} \quad \text{All necessary info about } \boldsymbol{T}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$ = Matrix of ***T*** relative to $\hat{i}, \hat{j}$

Example

Identity transformation $I\vec{u} = \vec{u}$

$$I\hat{i} = \hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad I\hat{j} = \hat{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Matrix of ***I*** = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

also known as identity matrix

Example

transformation $D\vec{u} = 2\vec{u}$

$$D\hat{i} = 2\hat{i} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad D\hat{j} = 2\hat{j} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Matrix of ***D*** = $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Example

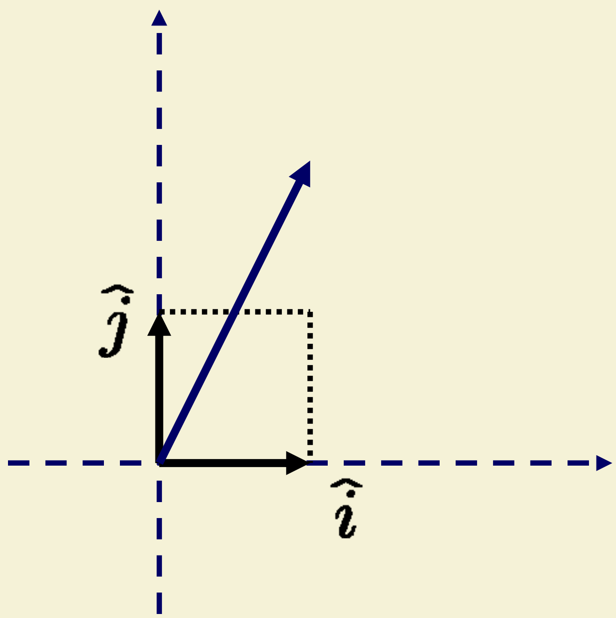
transformation $T(\hat{i}) = \hat{i} + \frac{1}{4}\hat{j},\ T(\hat{j}) = \frac{1}{4}\hat{i} + \hat{j}$

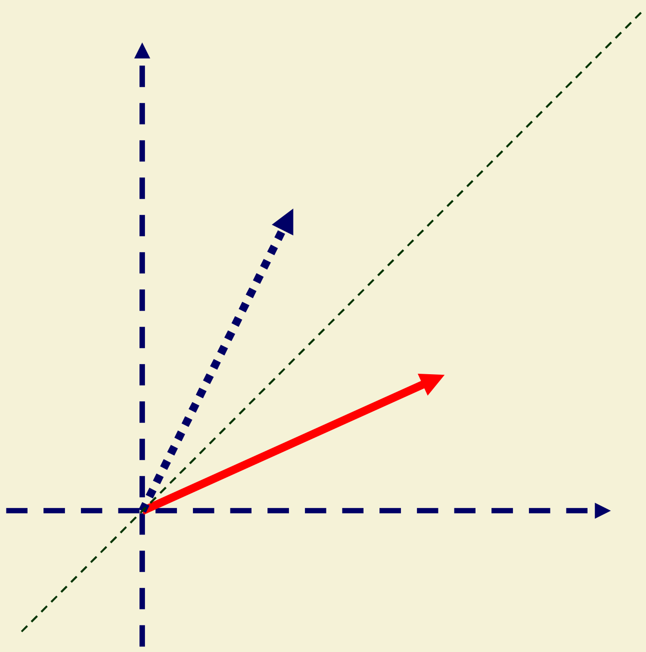
Matrix of ***T*** = $\begin{bmatrix} 1 & \frac{1}{4} \\ \frac{1}{4} & 1 \end{bmatrix}$

Example

transformation $T(\hat{i}) = \hat{j},\ T(\hat{j}) = \hat{i}$

Matrix of ***T*** = $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ reflection





Example

$$T\widehat{i} = \widehat{i} + 4\widehat{j} + 7\widehat{k},$$
$$T\widehat{j} = 2\widehat{i} + 5\widehat{j} + 8\widehat{k},$$
$$T\widehat{k} = 3\widehat{i} + 6\widehat{j} + 9\widehat{k}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \text{ relative to } \widehat{i}, \widehat{j}, \widehat{k}$$

Example

$$T\hat{i} = \hat{i} + \hat{j} + 2\hat{k}$$
$$T\hat{j} = \hat{i} - 3\hat{k}$$

T : 2-D vectors $\longrightarrow$ 3-D vectors

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 2 & -3 \end{bmatrix}$$

Example

$$T\widehat{i} = 2\widehat{i},$$
$$T\widehat{j} = \widehat{i} + \widehat{j},$$
$$T\widehat{k} = \widehat{i} - \widehat{j}$$

$\boldsymbol{T}$: 3-D vectors $\longrightarrow$ 2-D vectors

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

Dimension

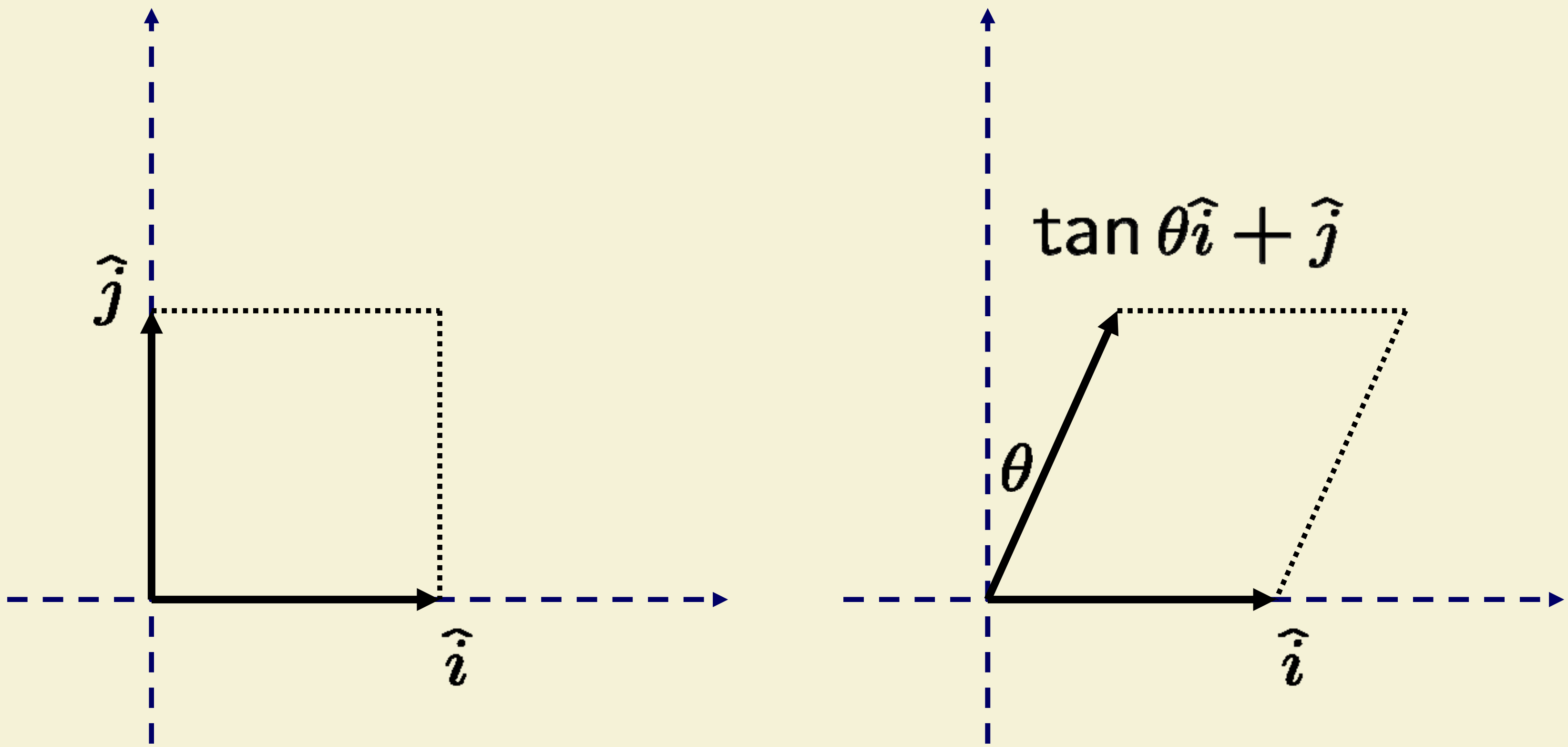
$\boldsymbol{T}$ is 2 dimensional if

$\boldsymbol{T}$: 2-D vectors $\longrightarrow$ 2-D vectors

$\boldsymbol{T}$ is 3 dimensional if

$\boldsymbol{T}$: 3-D vectors $\longrightarrow$ 3-D vectors

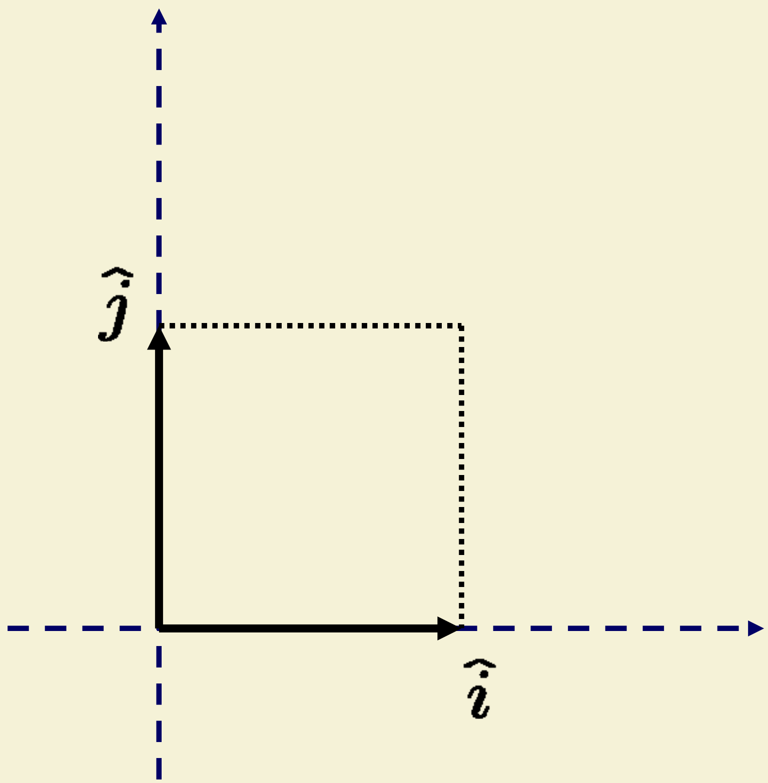
Shear parallel to x-axis

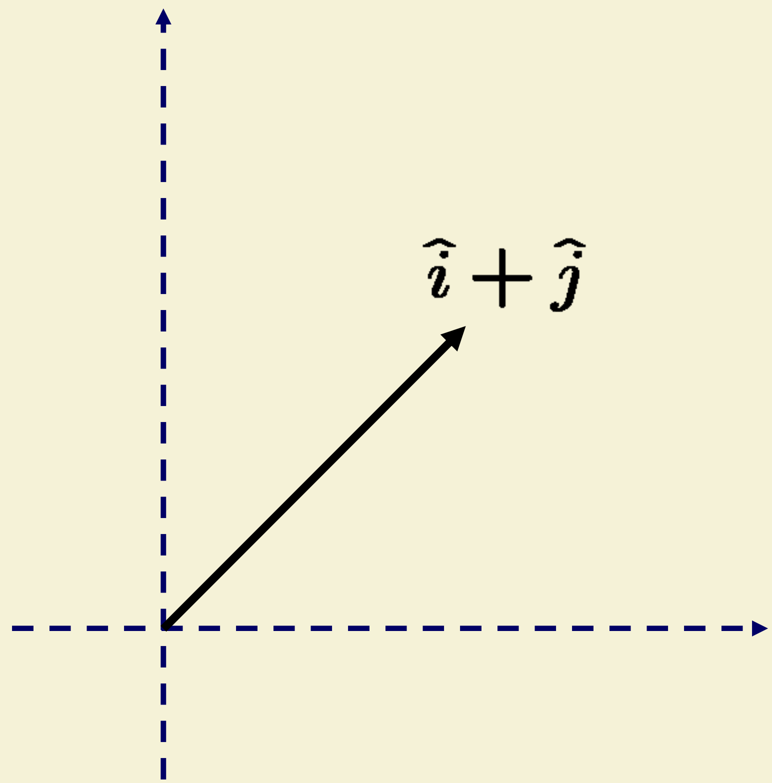


$$S\hat{i}=\hat{i}$$
$$S\hat{j}=\tan\theta\hat{i}+\hat{j}$$

$$\Rightarrow \begin{bmatrix} 1 & \tan\theta \\ 0 & 1 \end{bmatrix}$$

Example $Ti = \hat{i} + \hat{j}$, $T\hat{j} = \hat{i} + \hat{j}$





$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Basic box has zero volume

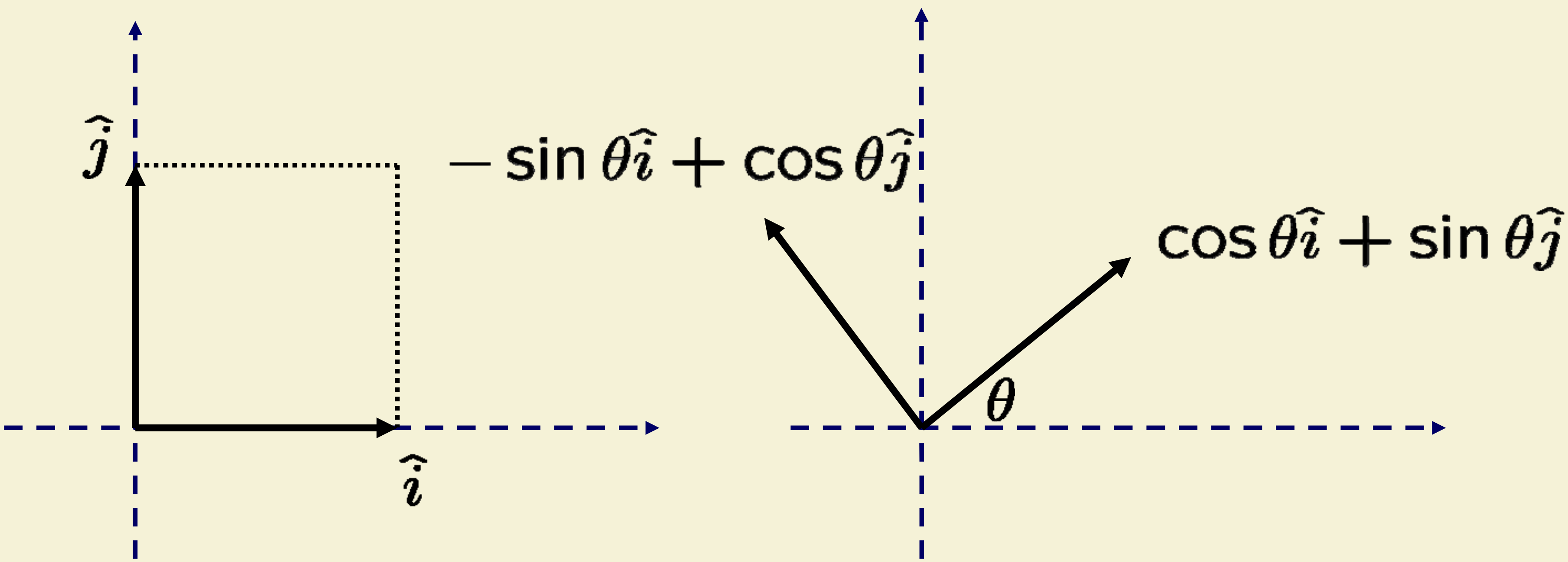
Rotation

Rotation (anti-clockwise) through angle θ

$$R\hat{i} = \cos\theta\hat{i} + \sin\theta\hat{j}$$

$$R\hat{j} = -\sin\theta i + \cos\theta\hat{j}$$

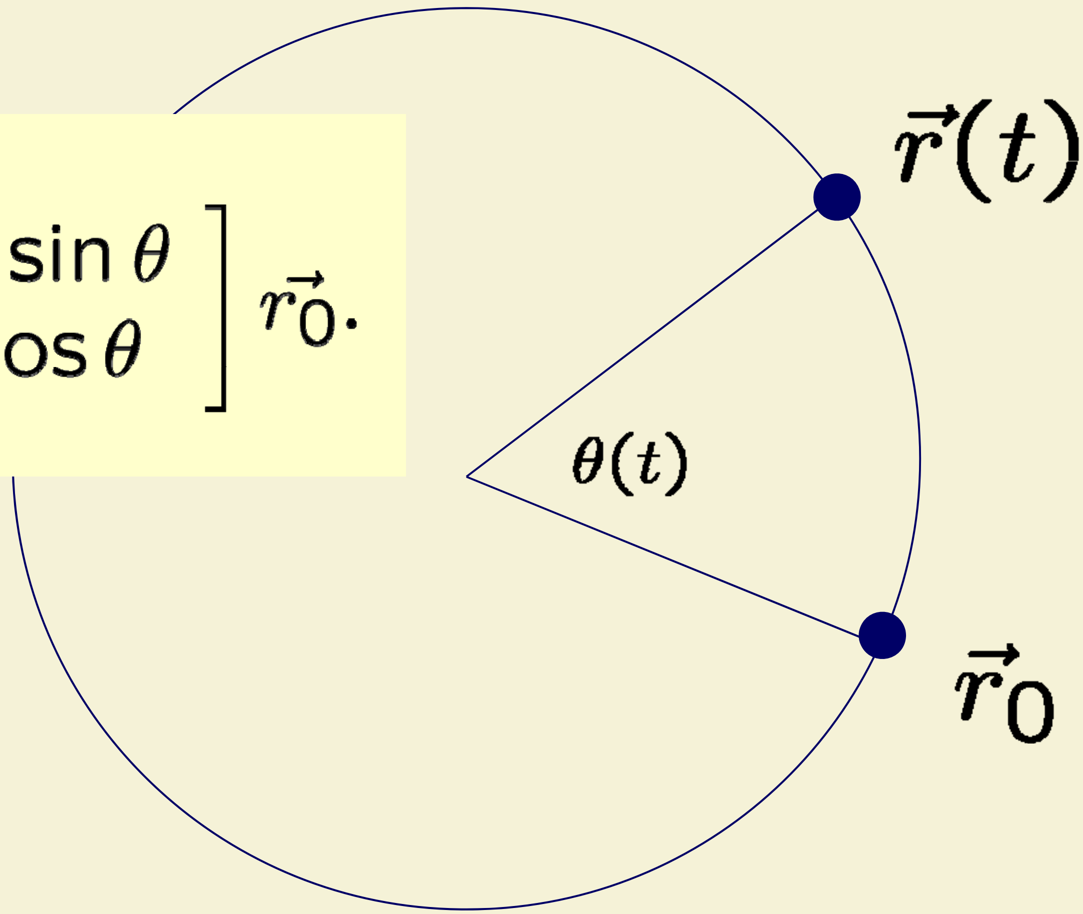
$$\Rightarrow \quad R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



Application

Suppose an object is moving in a circle at constant angular speed ω . What is its acceleration?

$$\vec{r}(t) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \vec{r_0}.$$



Application

Suppose an object is moving in a circle at constant angular speed ω . What is its acceleration?

$$\vec{r}(t) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \vec{r_0}$$

By chain rule

$$\frac{d\vec{r}}{dt} = \dot{\theta} \begin{bmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix} \vec{r_0}$$

$$\boxed{\dot{\theta} = \omega}$$

$$\frac{d^2\vec{r}}{dt^2} = \begin{bmatrix} -\cos\theta & \sin\theta \\ -\sin\theta & -\cos\theta \end{bmatrix} \omega^2 \vec{r_0} = -\omega^2 \vec{r}(t)$$

Summary

Transformations are mappings (rules) that send vectors to vectors

Linear transformation further satisfies

$$T(c\vec{u}) = cT(\vec{u})$$

$$T(\vec{u} + \vec{v}) = T\vec{u} + T\vec{v}$$

Linear transformation can be associated with a matrix with relative to $\hat{i}, \hat{j}, \hat{k}$

Recall composition of functions

$$f(x) = \sin(x),\ g(x) = x^2$$

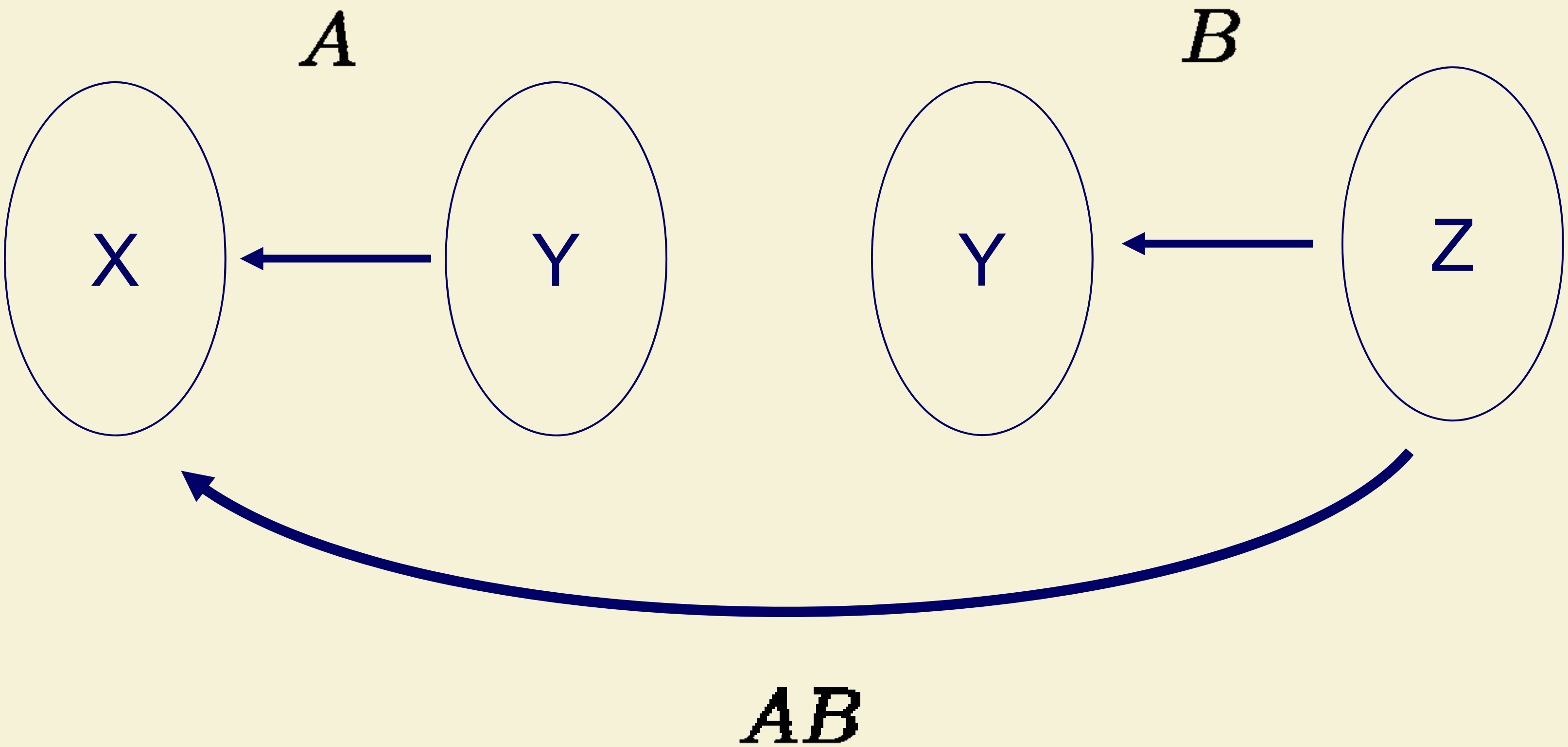
$f \circ g$ do *g* first , then do *f*

$$f \circ g(x) = \sin(x^2) \qquad g \circ f(x) = \sin^2(x)$$

For linear transformations

$\boldsymbol{AB}$ do *B* first then do *A*

Composition

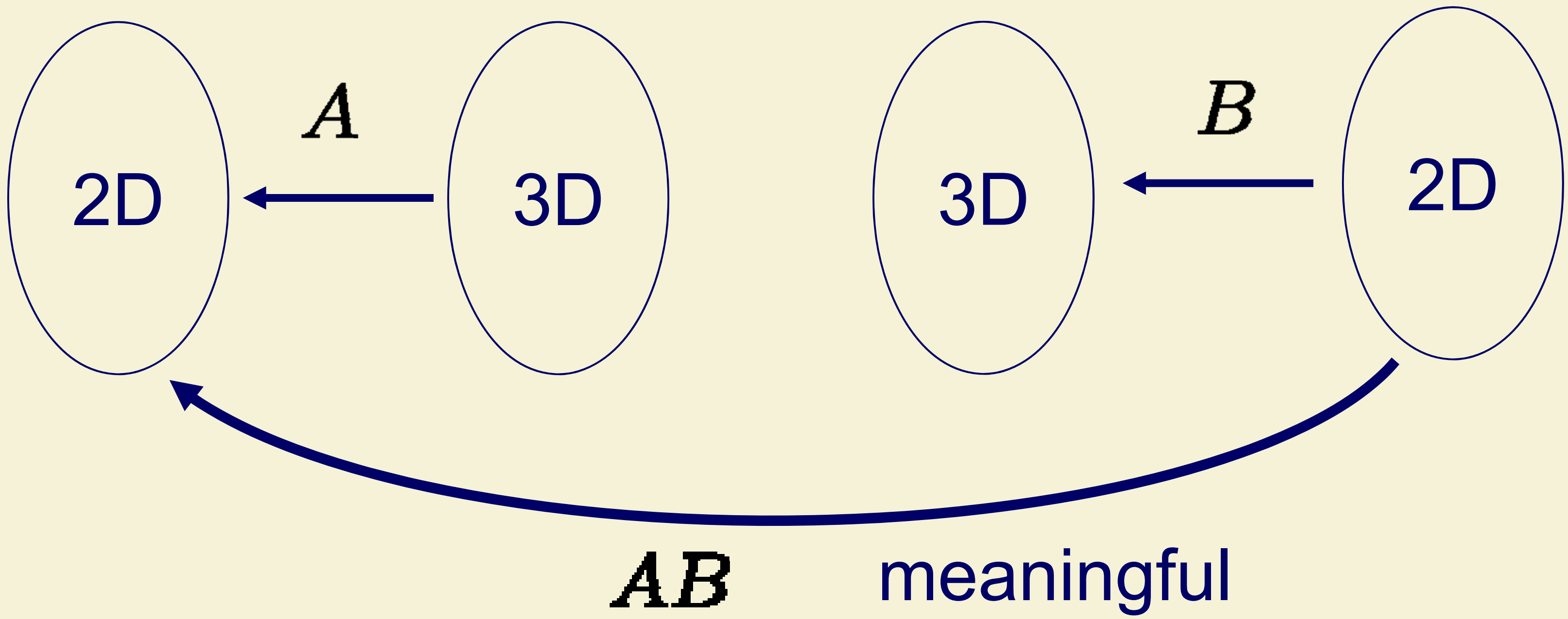


Note: composition must be well defined

Composition

AB not meaningful

Example



$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \qquad \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

Fact: composition = matrix multiplication

$$A = (a_{ij}) \qquad\qquad B = (b_{ij})$$

If *AB* is meaningful, the matrix of *AB*

= matrix product of (a_{ij}) and (b_{ij})

$$AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

Example

S: shear 45 degrees parallel to x axis $S(\theta) = \begin{bmatrix} 1 & \tan\theta \\ 0 & 1 \end{bmatrix}$

R: rotate 90 degrees anticlockwise $R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Compare *RS* versus *SR* ?

$$R(90^\circ)S(45^\circ) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

$$S(45^\circ)R(90^\circ) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

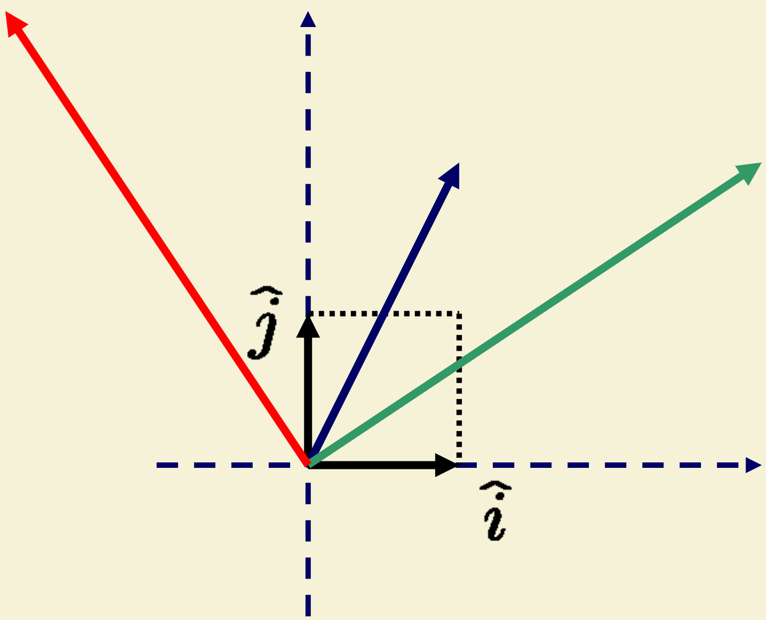
$$\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

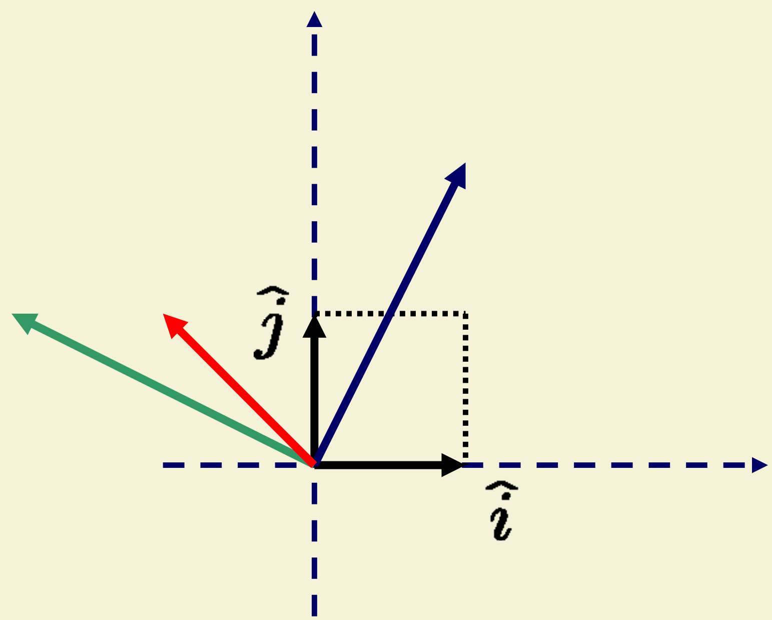
Example

S: shear 45 degrees parallel to x axis

R: rotate 90 degrees anticlockwise

$$RS\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}-2\\3\end{bmatrix} \qquad SR\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}-1\\1\end{bmatrix}$$





Example

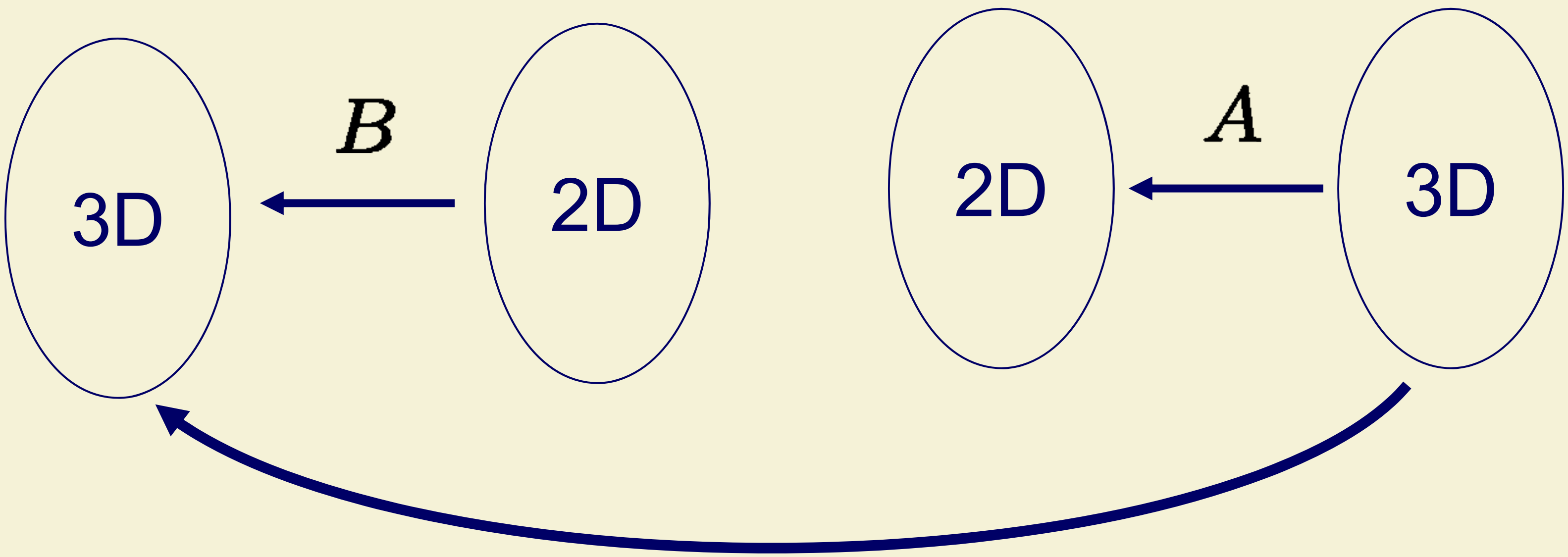
$$\begin{bmatrix} 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \qquad\qquad \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{tikzcd} \text{2D} & \text{3D} \arrow[l, "A"'] & \text{3D} & \text{2D} \arrow[l, "B"'] \arrow[lll, bend left, "AB"] \end{tikzcd}$$

$$AB = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$$

Example

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$



$$BA = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$$

Composing two shears

S: shear θ degrees parallel to x axis

S: shear ϕ degrees parallel to x axis

$$S(\phi)S(\theta) = \begin{bmatrix} 1 & \tan\phi \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \tan\theta \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \tan\phi + \tan\theta \\ 0 & 1 \end{bmatrix}.$$

Still a shear but note that $\tan(\phi + \theta) \neq \tan\phi + \tan\theta$

Example: Rotation in 3D

Rotate 90 degrees (anticlockwise) about z-axis

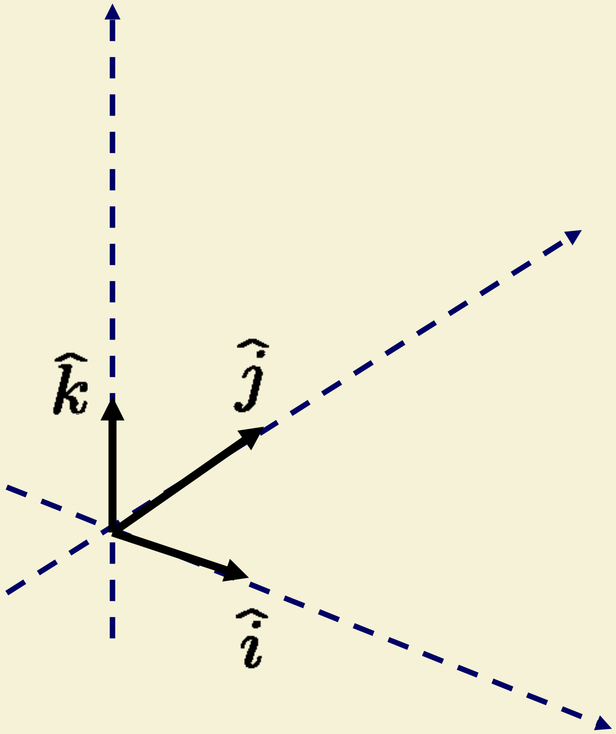
Rotate 90 degrees (anticlockwise) about x-axis

$$\widehat{i} \longrightarrow \widehat{j}$$
$$\widehat{j} \longrightarrow -\widehat{i}$$
$$\widehat{k} \longrightarrow \widehat{k}$$

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

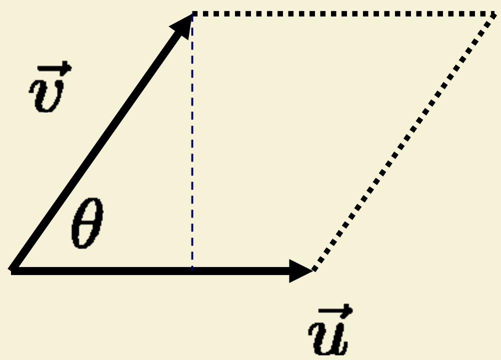
$$\widehat{i} \longrightarrow \widehat{i}$$
$$\widehat{j} \longrightarrow \widehat{k}$$
$$\widehat{k} \longrightarrow -\widehat{j}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

6.4 Area and Volume



Area of parallelogram = base x height

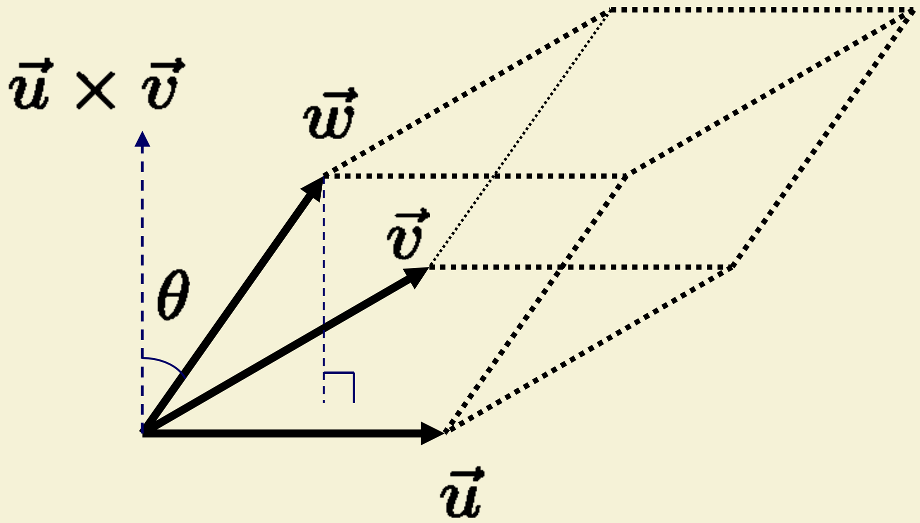
$$= |\vec{u}| \times |\vec{v}| \sin\theta$$

normal multiplication

$$= |\vec{u} \times \vec{v}|$$

cross product

6.4 Area and Volume

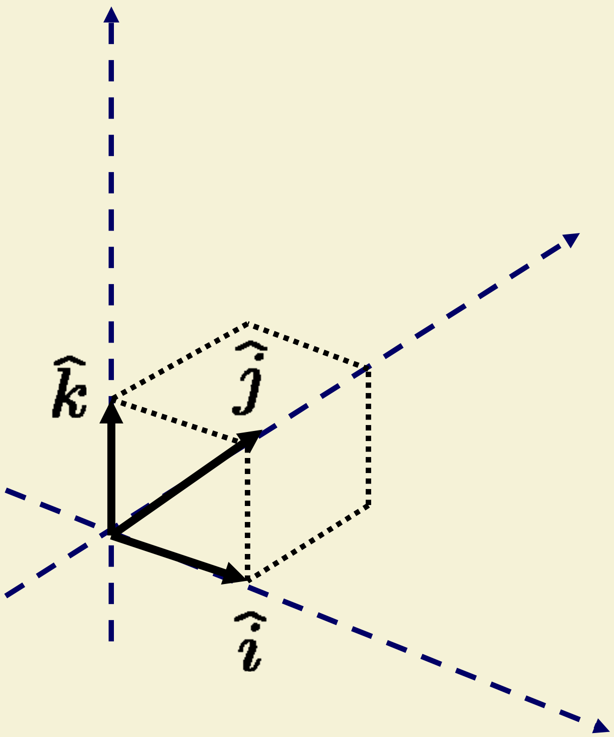


vol of parallelepiped = base area x height

$$= |\vec{u} \times \vec{v}| \times \left(|\vec{w}| \sin\left(\frac{\pi}{2} - \theta\right)\right)$$

$$= |\vec{u} \times \vec{v}||\vec{w}| \cos\theta$$

$$= |(\vec{u} \times \vec{v}) \cdot \vec{w}|$$



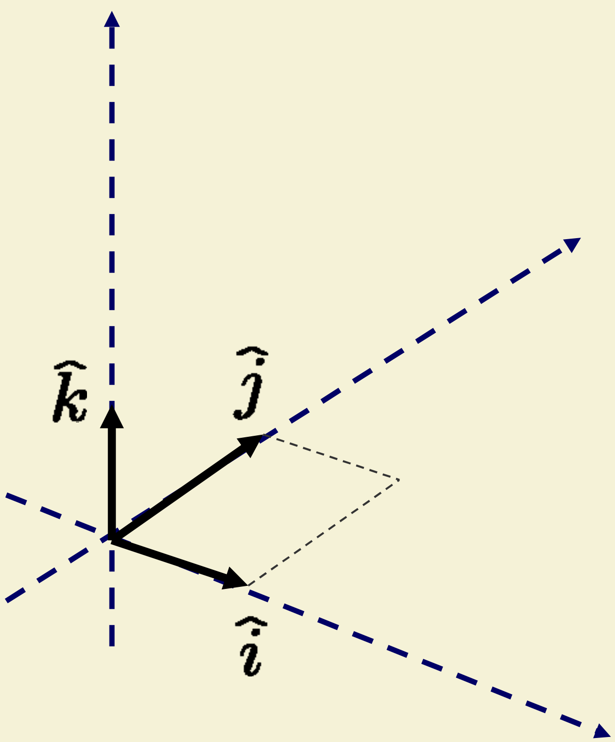
basic box in 3D

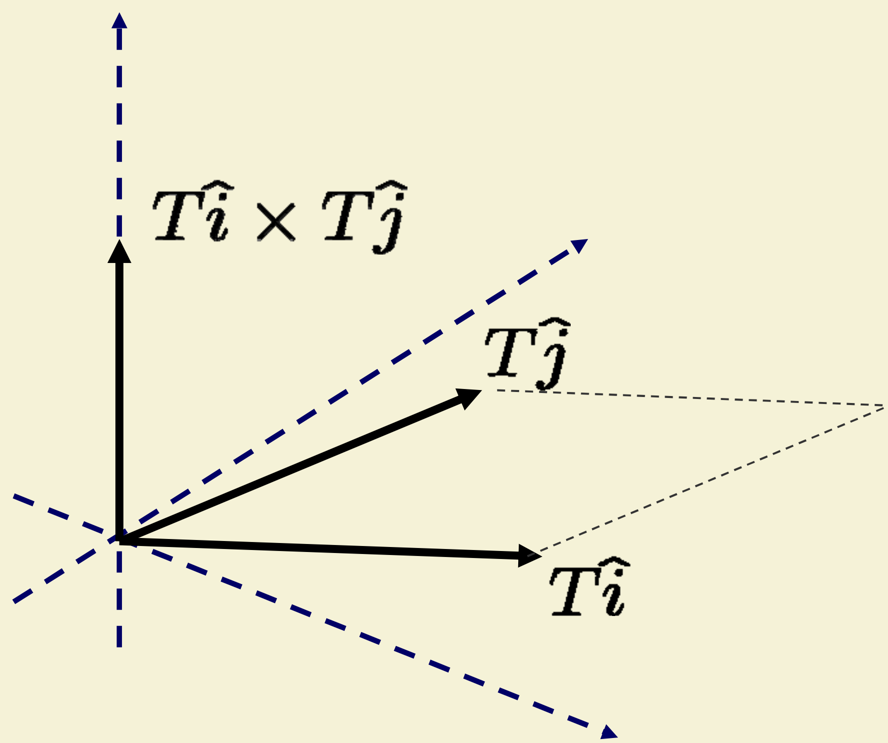
$$|(\hat{i} \times \hat{j}) \cdot \hat{k}| = |\hat{k} \cdot \hat{k}| = 1$$

6.4 Determinant (2D)

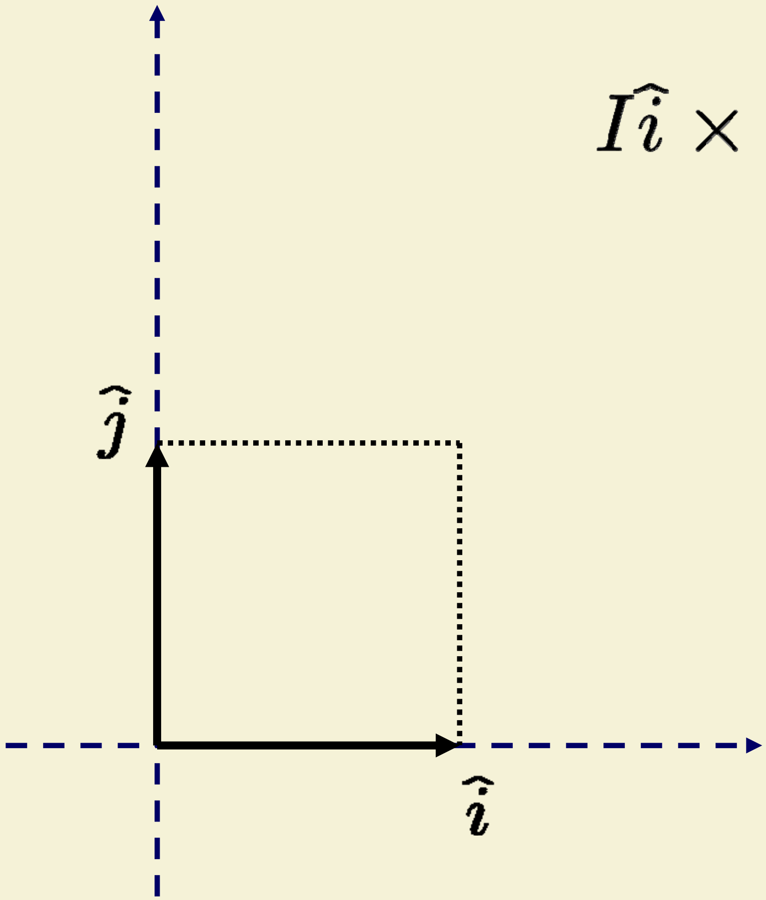
$$T : \hat{i} \longrightarrow T\hat{i}$$
$$T : \hat{j} \longrightarrow T\hat{j}$$

$$(T\hat{i}) \times (T\hat{j}) = \det(T)\hat{k}$$





Example: Identity *I*



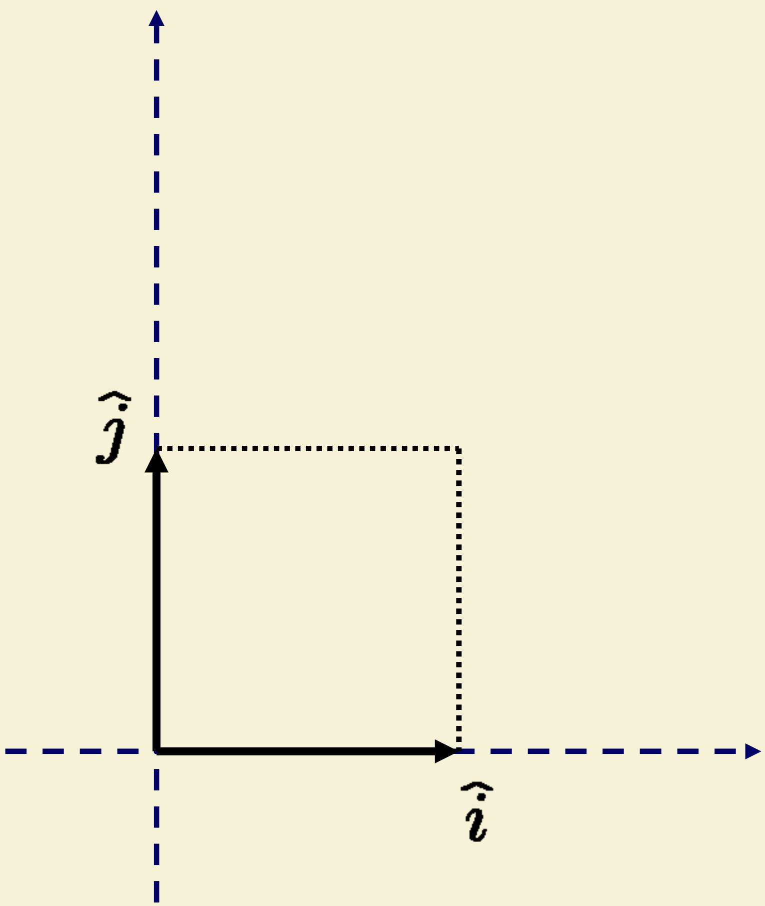
$$I\hat{i} \times I\hat{j} = \hat{i} \times \hat{j} = \hat{k} = 1\hat{k}$$

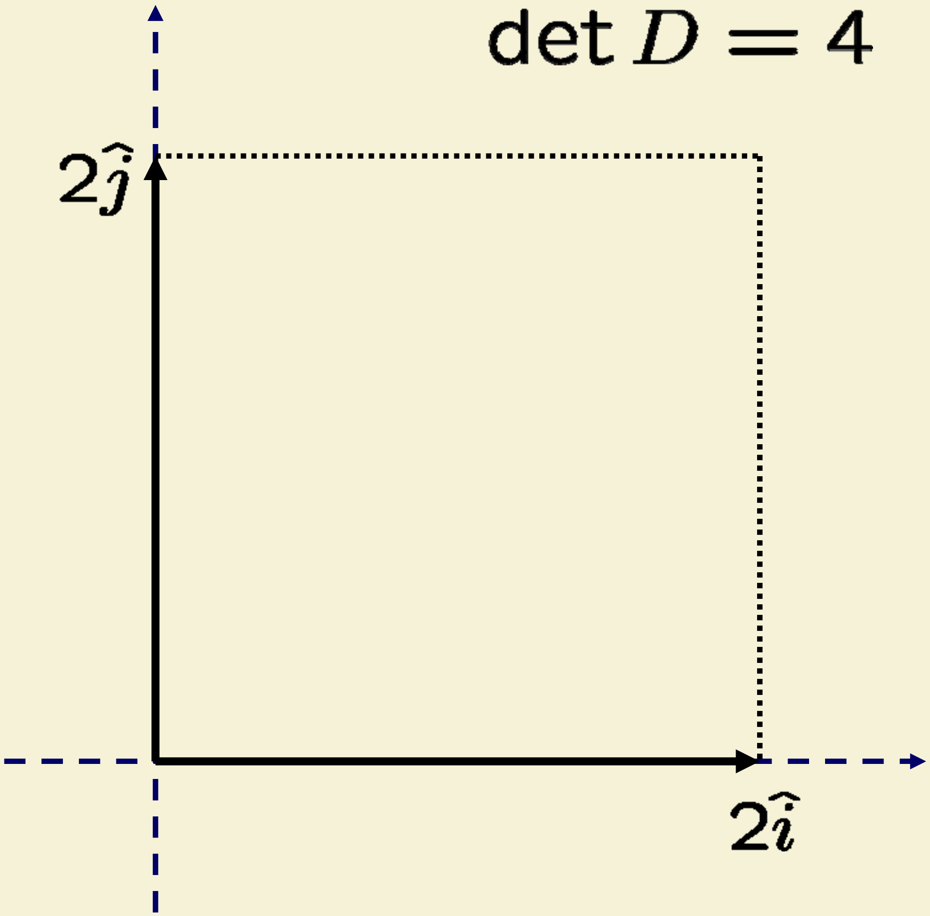
$$\det(I) = 1$$

Example $\quad D\vec{u} = 2\vec{u}$

$$D\hat{i} \times D\hat{j} = 2\hat{i} \times 2\hat{j} = 4\hat{i} \times \hat{j} = 4\hat{k}$$

$\det D = 4$



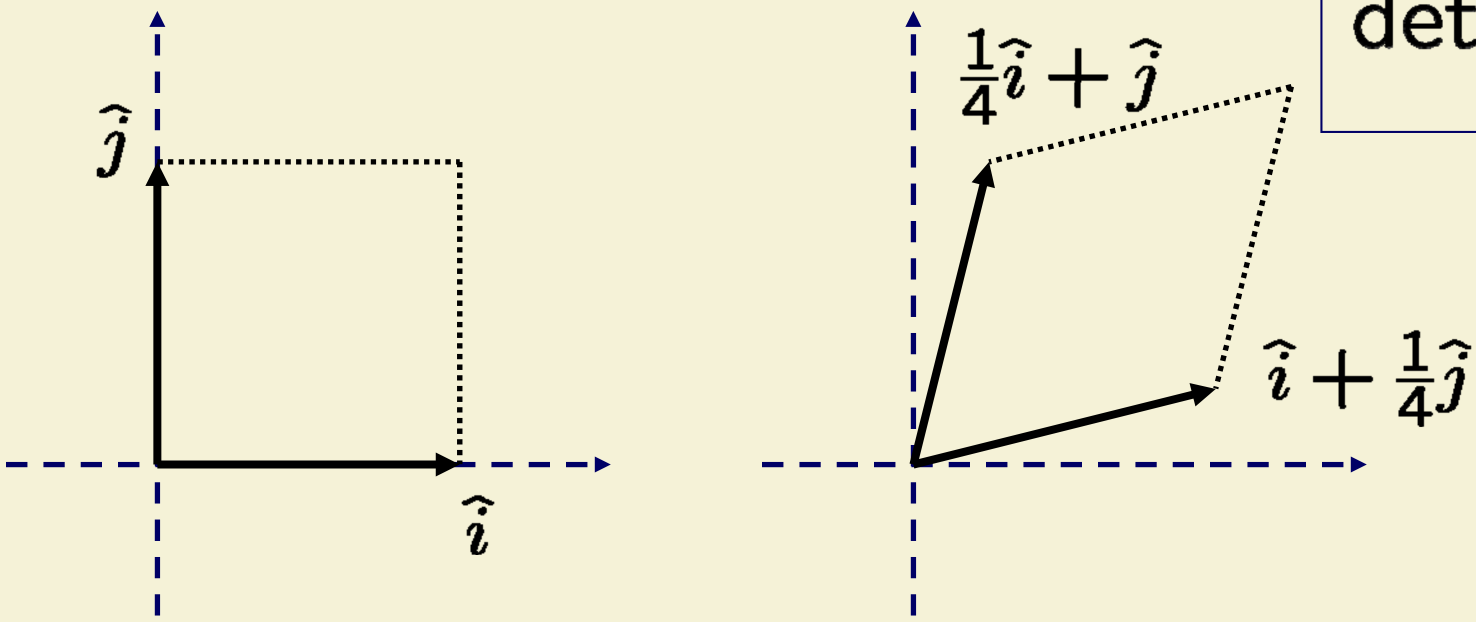


Example $T(\hat{i}) = \hat{i} + \frac{1}{4}\hat{j},\ T(\hat{j}) = \frac{1}{4}\hat{i} + \hat{j}$

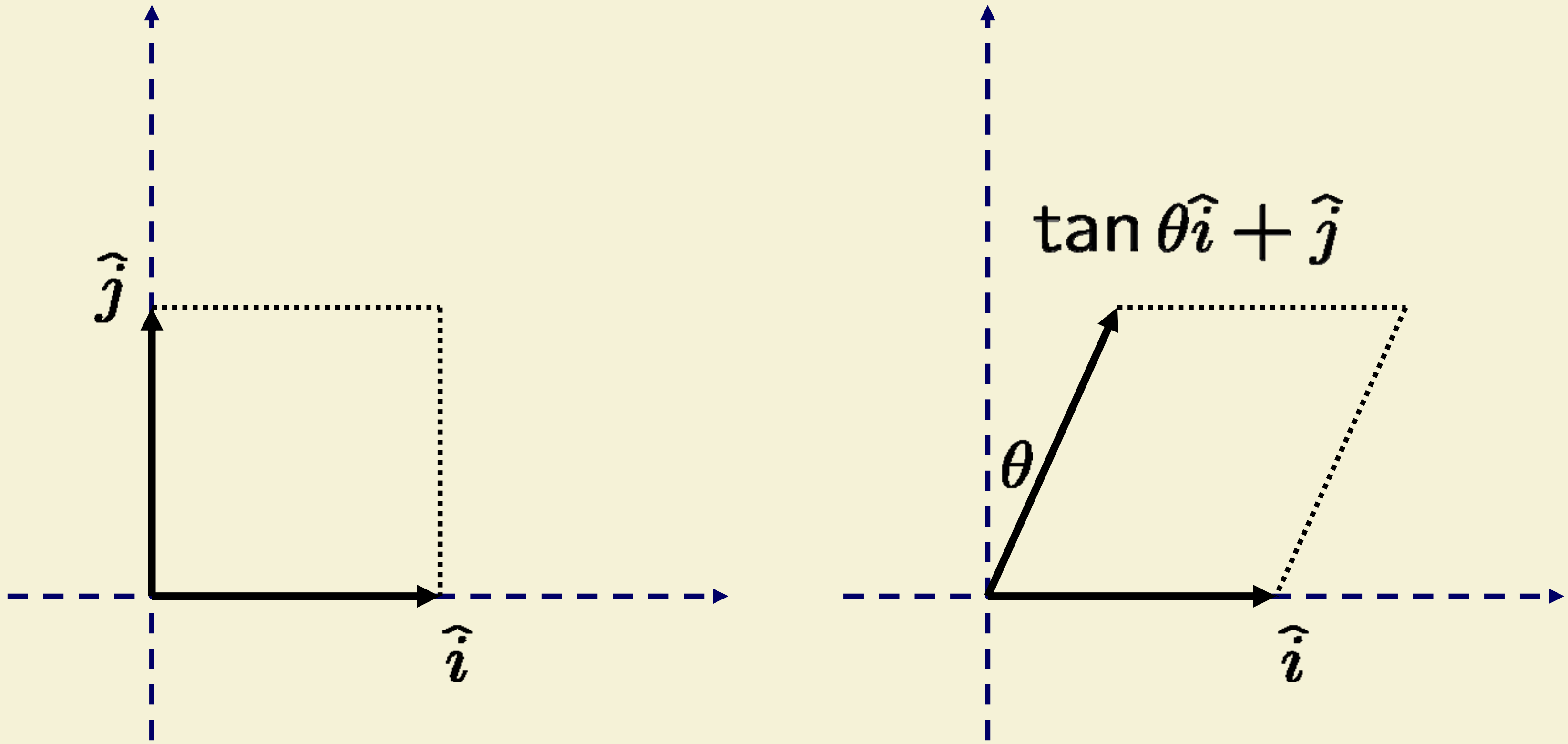
$$T\hat{i} \times T\hat{j} \;=\; \left(\hat{i} + \frac{1}{4}\hat{j}\right) \times \left(\frac{1}{4}\hat{i} + \hat{j}\right)$$

$$= \hat{i} \times \hat{j} + \frac{1}{16}\hat{j} \times \hat{i} \;=\; \frac{15}{16}\hat{k}$$

$$\boxed{\det T = \frac{15}{16}}$$

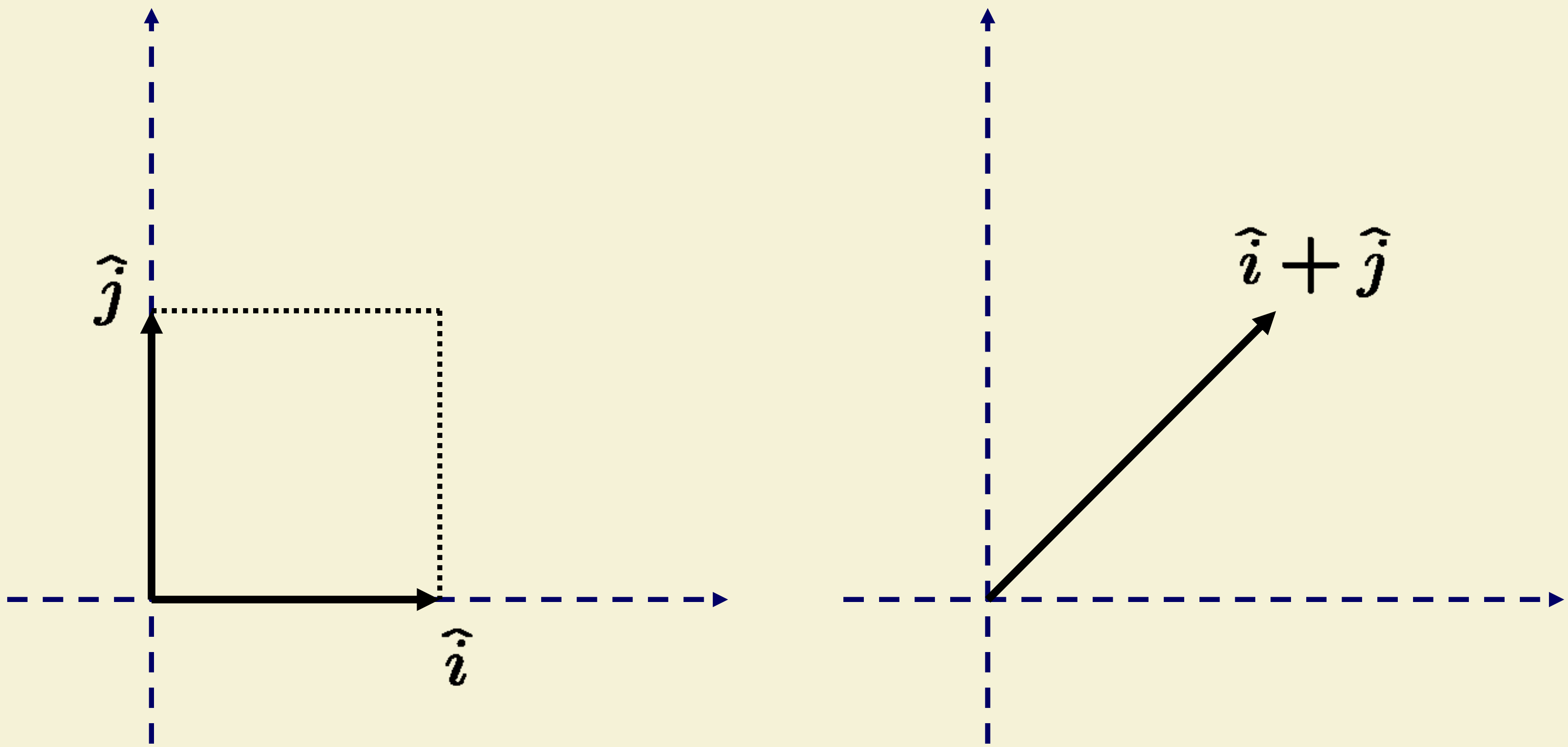


Shear $\qquad S\hat{i} = \hat{i} \qquad S\hat{j} = \tan\theta\,\hat{i} + \hat{j}$



$$S\hat{i} \times S\hat{j} = \hat{k} \quad \Rightarrow \quad \det(S) = 1$$

Example $Ti = \widehat{i} + \widehat{j},\ T\widehat{j} = \widehat{i} + \widehat{j}$



$$T\widehat{i} \times T\widehat{j} = (\widehat{i} + \widehat{j}) \times (\widehat{i} + \widehat{j}) = 0$$

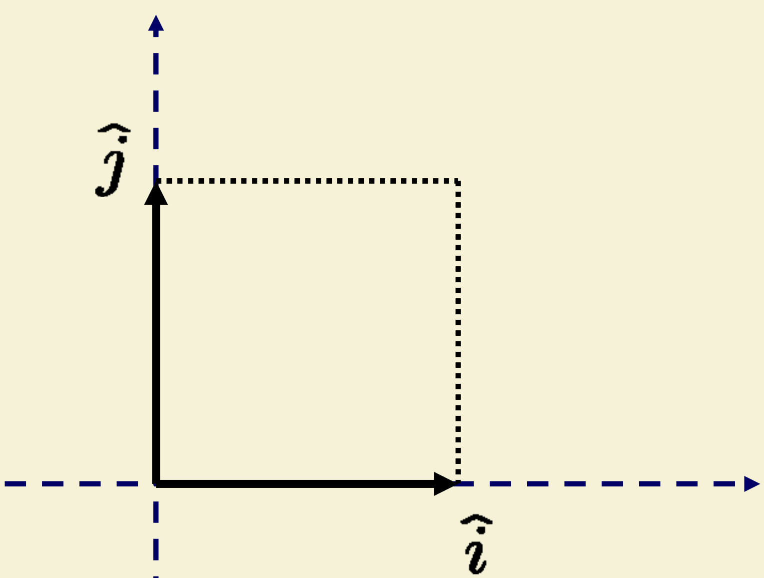
Determinant =0

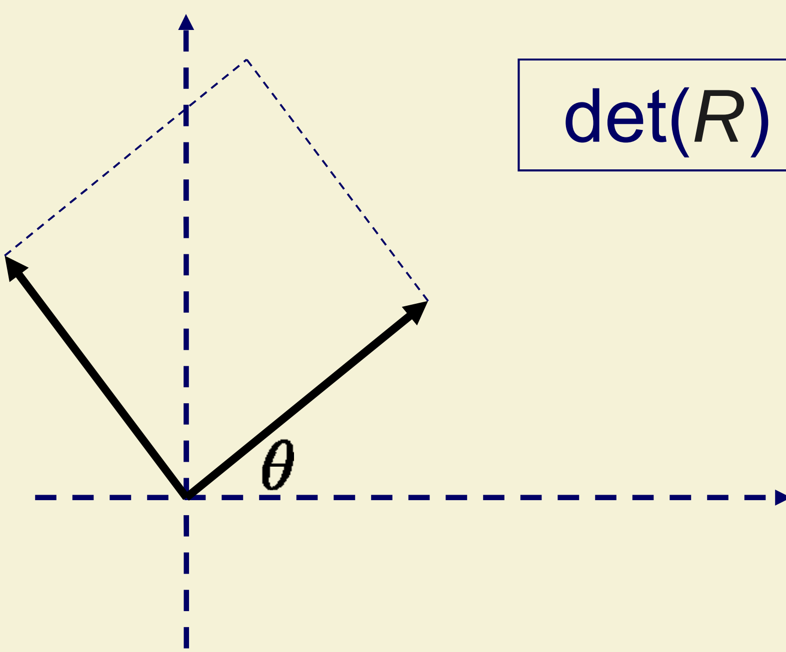
Example Rotation (anti-clockwise) through θ

$$R\hat{i} = \cos\theta\hat{i} + \sin\theta\hat{j}$$

$$R\hat{j} = -\sin\theta i + \cos\theta\hat{j}$$

$$\begin{aligned} R\hat{i} \times R\hat{j} &= (\cos\theta\hat{i} + \sin\theta\hat{j}) \times (-\sin\theta\hat{i} + \cos\theta\hat{j}) \\ &= (\cos^2\theta - (-\sin^2\theta))\hat{k} = \hat{k}. \end{aligned}$$





$\det(R) = 1$

Summary

$$|T\hat{\imath} \times T\hat{\jmath}| = |\det T\ \hat{k}| = |\det T|$$

$$\boxed{\frac{\text{Final area of basic box}}{\text{Initial area of basic box}} = \frac{|\det T|}{1} = |\det T|.}$$

det(T) = amount by which areas are changed by T

shears, rotation, reflection : |det =1|

det =0 means basic box is flattened to zero volume

Formula for 2D determinant

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

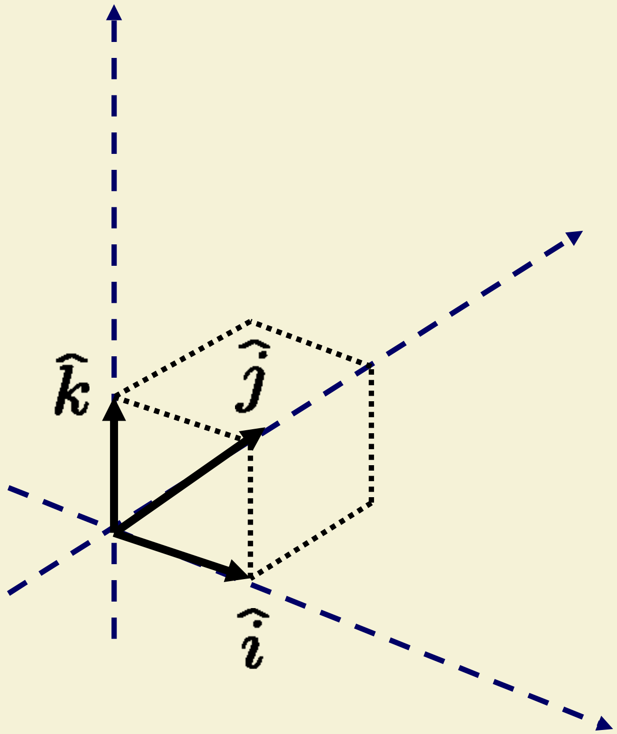
$$M\hat{\imath} = a\hat{\imath} + c\hat{\jmath}$$
$$M\hat{\jmath} = b\hat{\imath} + d\hat{\jmath}$$

$$M\hat{\imath} \times M\hat{\jmath} = \left(a\hat{\imath} + c\hat{\jmath}\right) \times \left(b\hat{\imath} + d\hat{\jmath}\right)$$

$$= (ad - bc)\hat{k}$$

$$\boxed{\begin{vmatrix} a & b \\ c & d \end{vmatrix} := \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc}$$

Extend to 3D determinant



Volume of basic box:

$$|(\hat{i} \times \hat{j}) \cdot \hat{k}| = |\hat{k} \cdot \hat{k}| = 1$$

Volume of new basic box:

$$\left|\left(T\hat{i} \times T\hat{j}\right) \cdot T\hat{k}\right|$$

$$|\det T| = \frac{\text{Final volume of basic box}}{\text{Initial volume of basic box}}$$

Formula for 3D determinant

$$\left|\left(T\hat{i} \times T\hat{j}\right) \cdot T\hat{k}\right|$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Cofactor expansion

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Formula for 3D determinant

Cofactor expansion can be done about any row or column

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= -a_{21} \begin{vmatrix} * & a_{12} & a_{13} \\ * & * & * \\ * & a_{32} & a_{33} \end{vmatrix} + a_{22} \begin{vmatrix} a_{11} & * & a_{13} \\ * & * & * \\ a_{31} & * & a_{33} \end{vmatrix} - a_{23} \begin{vmatrix} a_{11} & a_{12} & * \\ * & * & * \\ a_{31} & a_{32} & * \end{vmatrix}$$

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Examples

$$\begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 2 & 0 & 0 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix} - (-1)\begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} + 0\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix}$$

$$= 0 + 2 + 0 = 2$$

$$= -1\begin{vmatrix} -1 & 0 \\ 0 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} - (-1)\begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix}$$

$$= 0 + 0 + 2 = 2$$

Remark: Formula for determinant

Cofactor expansion can be used for any

n x n determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

$$\begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}$$

Important Properties of Determinants

$$\boxed{\det(ST) = \det(TS) = \det S \det T}$$

but $ST \neq TS$

$$\boxed{\det M^T = \det M}$$

size of ***M***

$$\boxed{\det(cM) = c^n \det M}$$

Usually, $\det(A + B) \neq \det A + \det B$

Example: Orthogonal Matrices

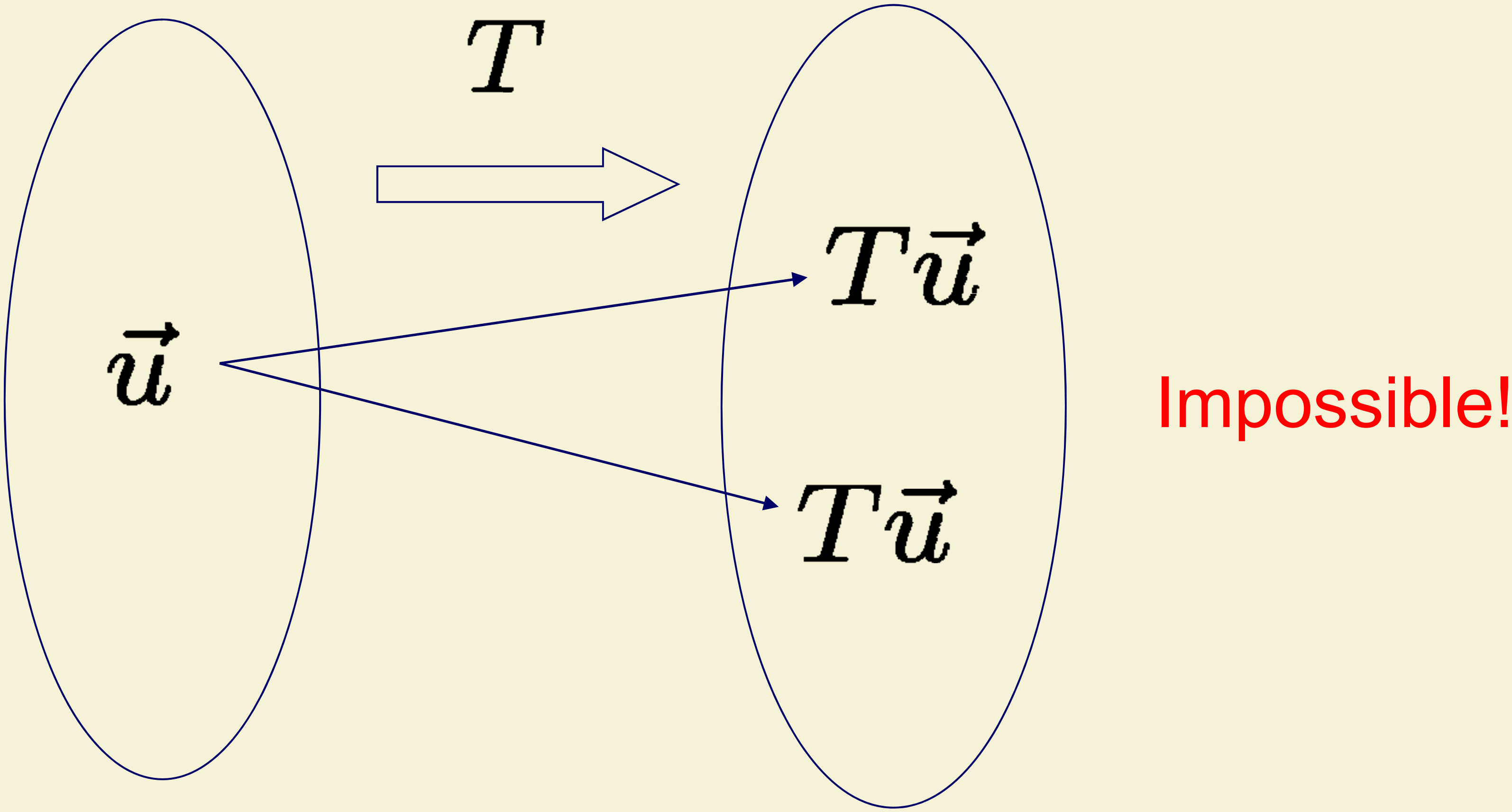
$$MM^T = I$$

$$\begin{aligned} \det(MM^T) &= \det(M)\det(M^T) \\ &= \det(M) \times \det(M) \\ &= (\det M)^2 \end{aligned}$$

$$\boxed{\det M = \pm 1}$$

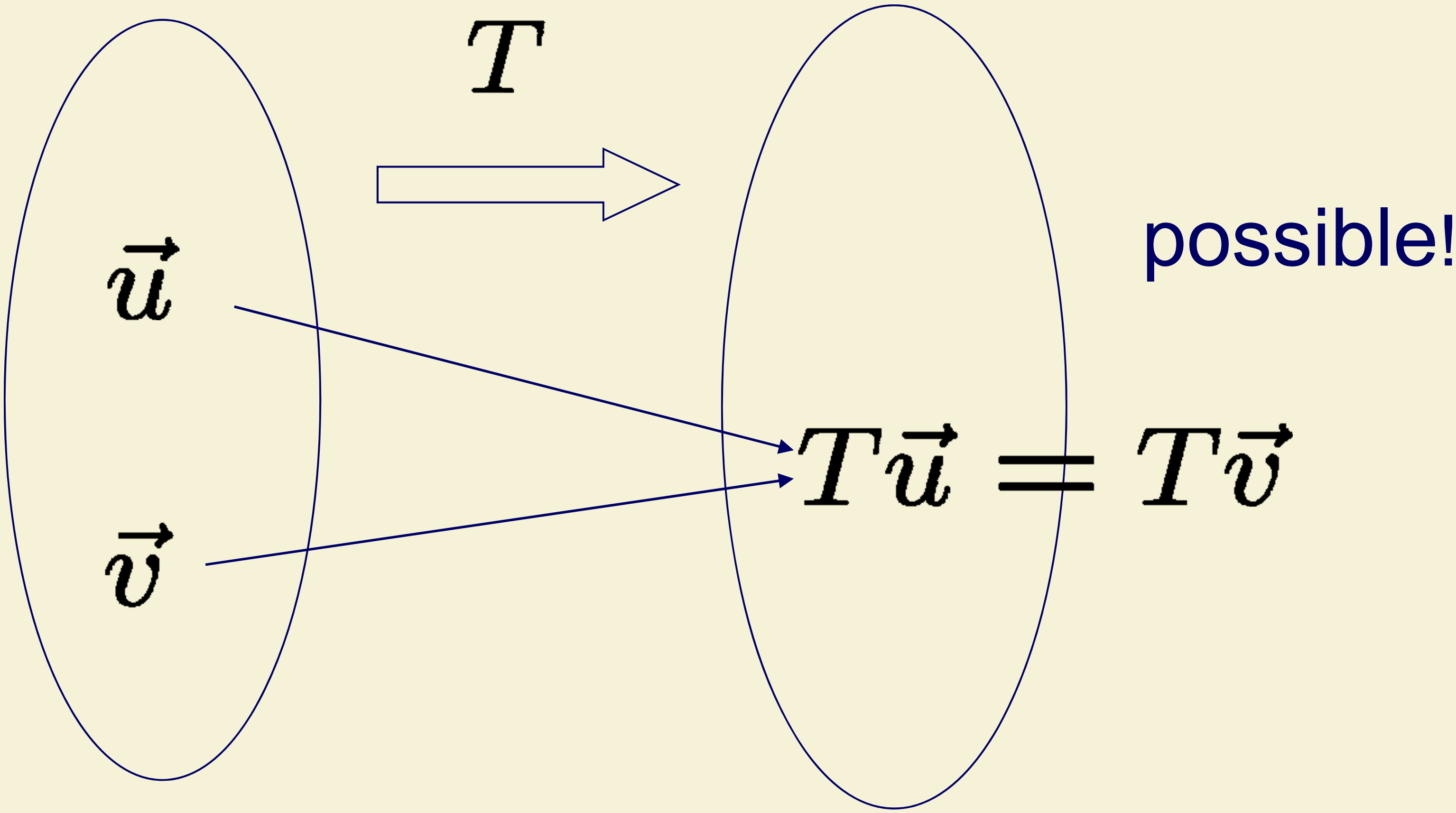
$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

6.5 Well Defined Mapping



T sends each vector to a unique vector

6.5 Well Defined but not 1 to 1



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

6.5 Not 1 to 1

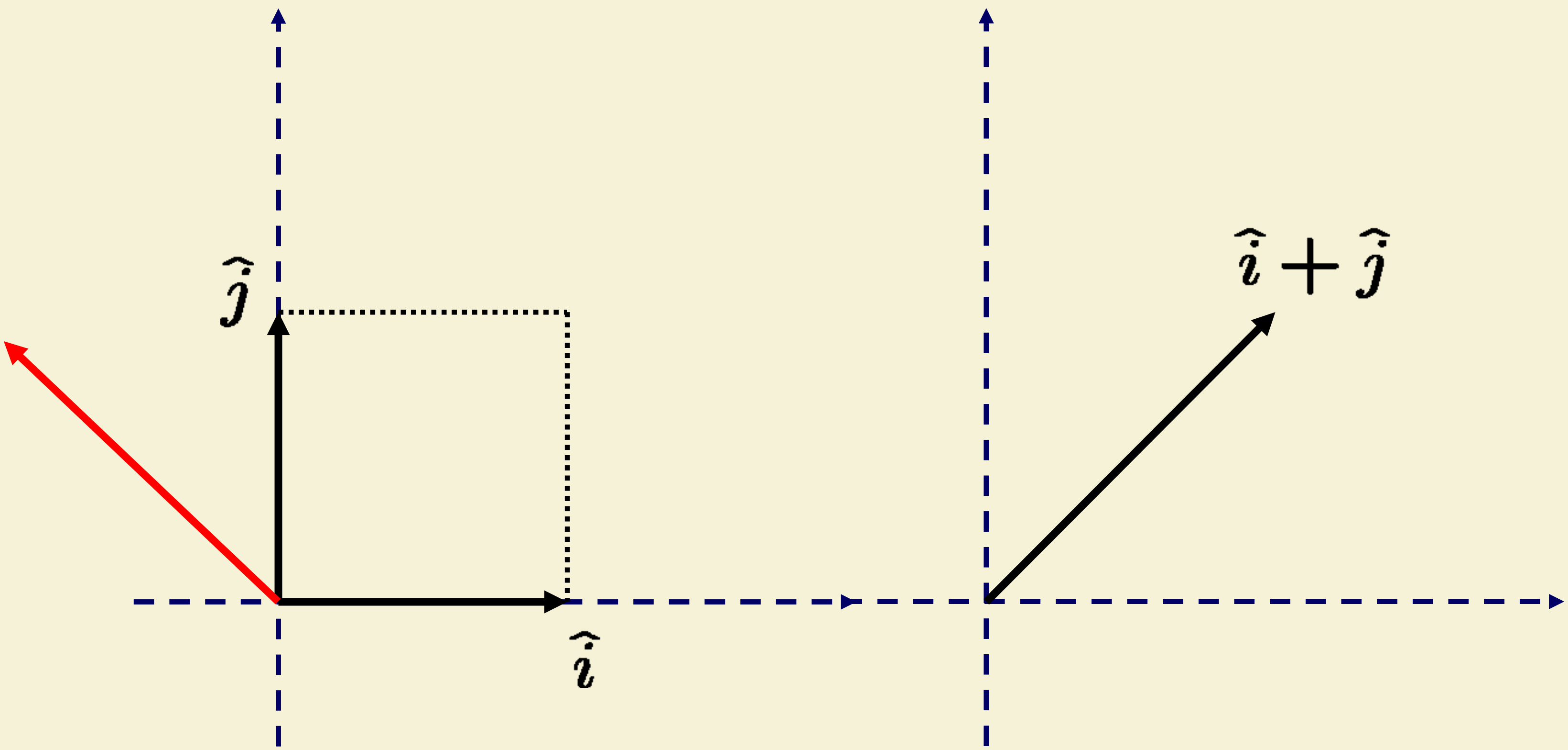
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\hat{j}, \hat{k}$ are destroyed (mapped to $\vec{0}$)

$$\boxed{\begin{gathered} \vec{u} \neq \vec{v} \\ T\vec{u} = T\vec{v} \end{gathered}} \Rightarrow \boxed{T(\vec{u} - \vec{v}) = \vec{0}}$$

Not 1 to 1 means ***T*** maps something to $\vec{0}$

Example $Ti = \widehat{i} + \widehat{j},\ T\widehat{j} = \widehat{i} + \widehat{j}$



$T(-\widehat{i} + \widehat{j}) = \vec{0}$

Determinant =0

Singular Transformations

1. Maps two different vectors to one vector

$$\vec{u} \neq \vec{v} \qquad\qquad T\vec{u} = T\vec{v}$$

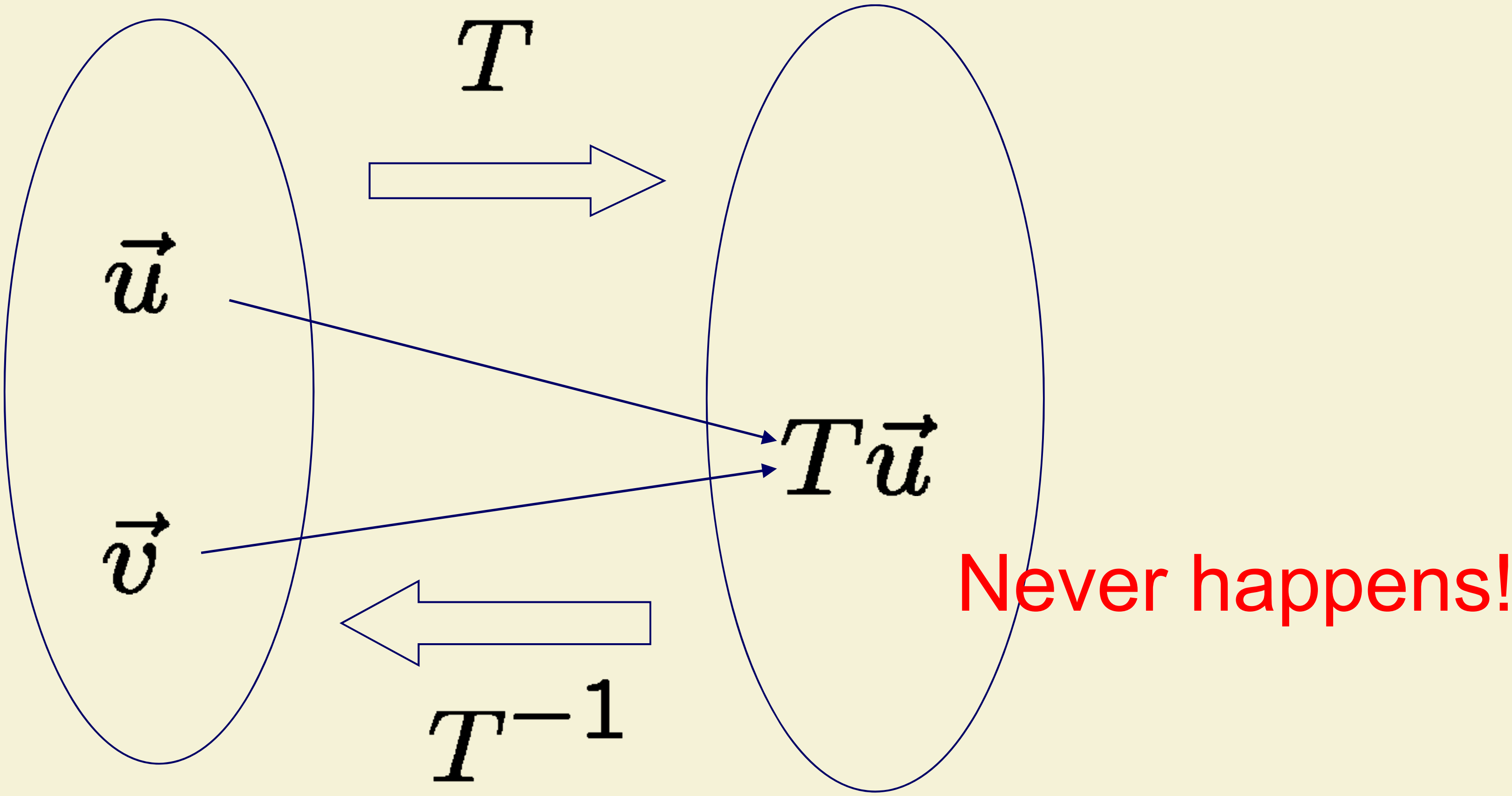
2. Destroys all of the vectors in at least one direction

$$T(\vec{u} - \vec{v}) = T\vec{w} = \vec{0}$$

3. Loses all info associated with those directions

4. $\boxed{\det(T) = 0}$ (basic box has 0 area/volume)

Non–singular (1 to 1 mapping)



Inverse mapping exist

$$\boxed{T^{-1} \text{ exist if and only if } \det T \neq 0}$$

Examples: Singular Transformation

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} k \\ -k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Also, $\det\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = 0$

Examples: Non Singular Transformation

Suppose $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ maps two vectors to the same vector

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -b \\ a \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

$$\Rightarrow a = x,\ b = y$$

Faster way: $\det\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right) = 1$

Non–singular (1 to 1 mapping)

By definition $T^{-1} : T\vec{u} \mapsto \vec{u}$

$T^{-1}T\vec{u} = \vec{u} = I\vec{u}$ for every vector $\vec{u}$

$$\boxed{T^{-1}T = I}$$

Remark

$A\vec{u} = B\vec{u}$ does not mean ***A* = *B***

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Non–singular (1 to 1 mapping)

$$\boxed{T^{-1}T = I}$$

To find T^{-1}

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & -a \\ d & -c \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Formula for 2 x 2 Inverse

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Determinant must be non zero

Recall: 5.4 Leontief Model of Manufacturing

$$(I - T)\vec{u} = \vec{c}$$

Find ***S*** such that ***S(I-T) = I***

$$\Longrightarrow \quad \vec{u} = S\vec{c}$$

$$\begin{bmatrix} 0.7 & -0.4 \\ -0.5 & 0.7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 150 \\ 100 \end{bmatrix}$$

$$S = \frac{1}{29} \begin{bmatrix} 70 & 40 \\ 50 & 70 \end{bmatrix}$$

Does the job

Formula for n x n Inverse (Cofactor Expansion)

- Work out the matrix of cofactor of each entry
- Take transpose
- Divide by determinant

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\Rightarrow \quad M^{-1} = \frac{1}{2} \begin{bmatrix} \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \\ -\begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \end{bmatrix}^T$$

Formula for n x n Inverse (Cofactor Expansion)

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$M^{-1} = \frac{1}{2} \begin{bmatrix} \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \\ -\begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \end{bmatrix}^T$$

$$= \frac{1}{2} \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Row operations technique to find inverse

Refer to Textbook Section 3.3 for more details

3 possible row operations on a matrix

- cR_1 (multiply a constant to a row)
- $R_1 \leftrightarrow R_2$ (switch two rows)
- $R_1 + cR_2 \rightarrow R_1$ (add a multiple of another row)

To find inverse of $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$

$\mathbf{A}^{-1}$

Start with $[\ \mathbf{A} \mid \mathbf{I}\ ]$ simplify till $[\ \mathbf{I} \mid \mathbf{B}\ ]$

Example

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right) \xrightarrow{R_3 + 2R_2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right) \xrightarrow{-R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right) \xrightarrow[R_2 + 3R_3]{R_1 - 3R_3}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right) \xrightarrow{R_1 - 2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

Summary

A non singular transformation T satisfies det $T \neq 0$

T has a unique inverse T^{-1}

$$\boxed{T^{-1}T = I = TT^{-1}}$$

$$\boxed{(AB)^{-1} = B^{-1}A^{-1}}$$

Proof

$$\boxed{(AB)^{-1} = B^{-1}A^{-1}}$$

$$(B^{-1}A^{-1})AB = B^{-1}(A^{-1}A)B$$

$$= B^{-1}IB = B^{-1}B = I$$

Application: Solving Linear Systems

To solve

$$\begin{aligned} x + 2y + 3z &= 1 \\ 4x + 5y + 6z &= 2 \\ 7x + 8y + 9z &= 4. \end{aligned}$$

Write as

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

Find the inverse

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

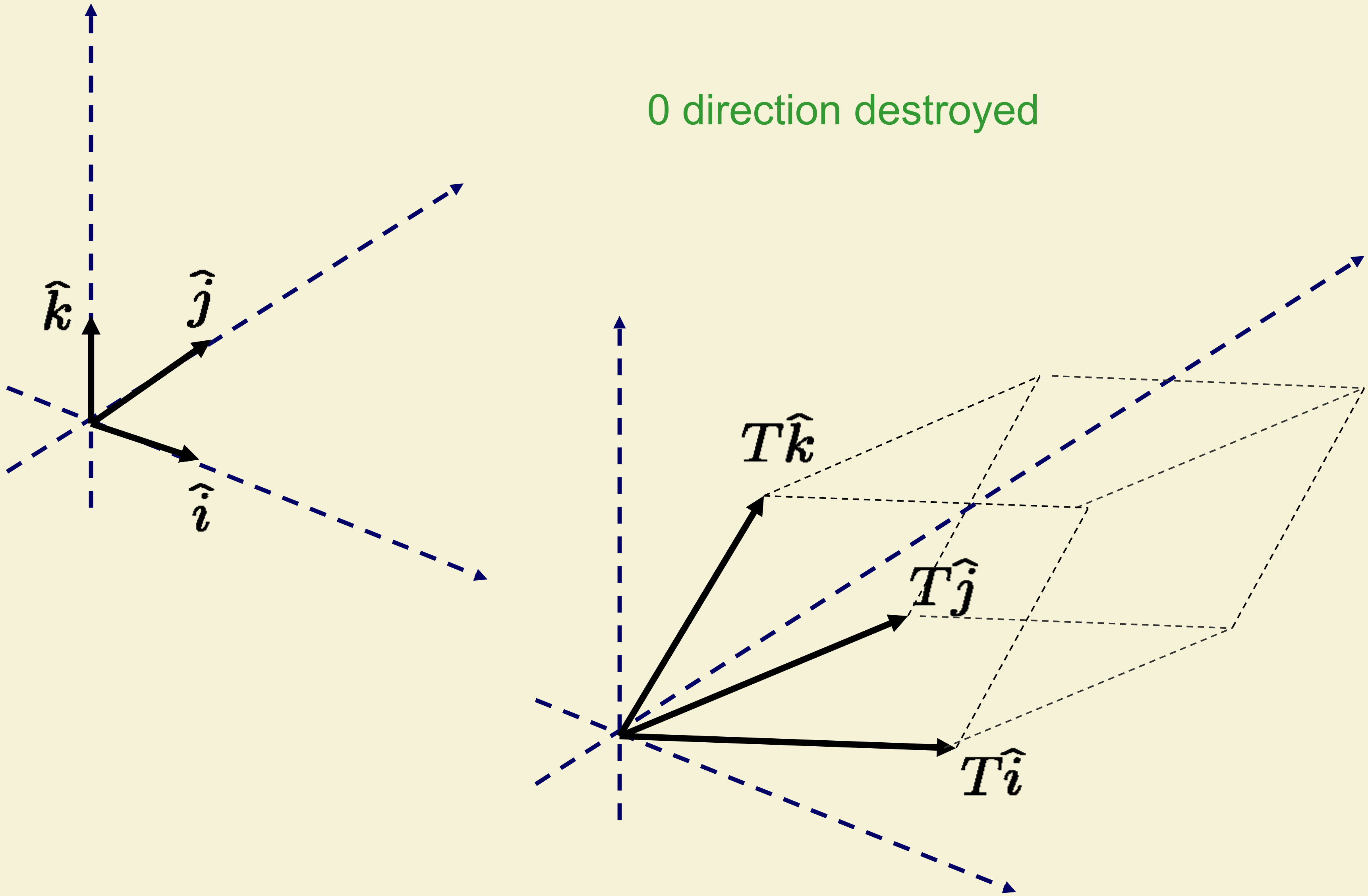
Does not exist

Application: Solving Linear Systems

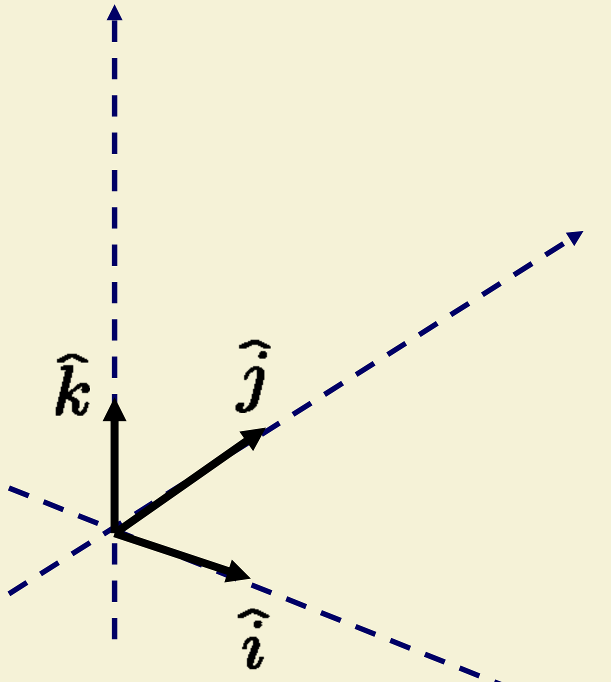
Check $\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 0$

Or $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Rank 3: det (volume) non zero

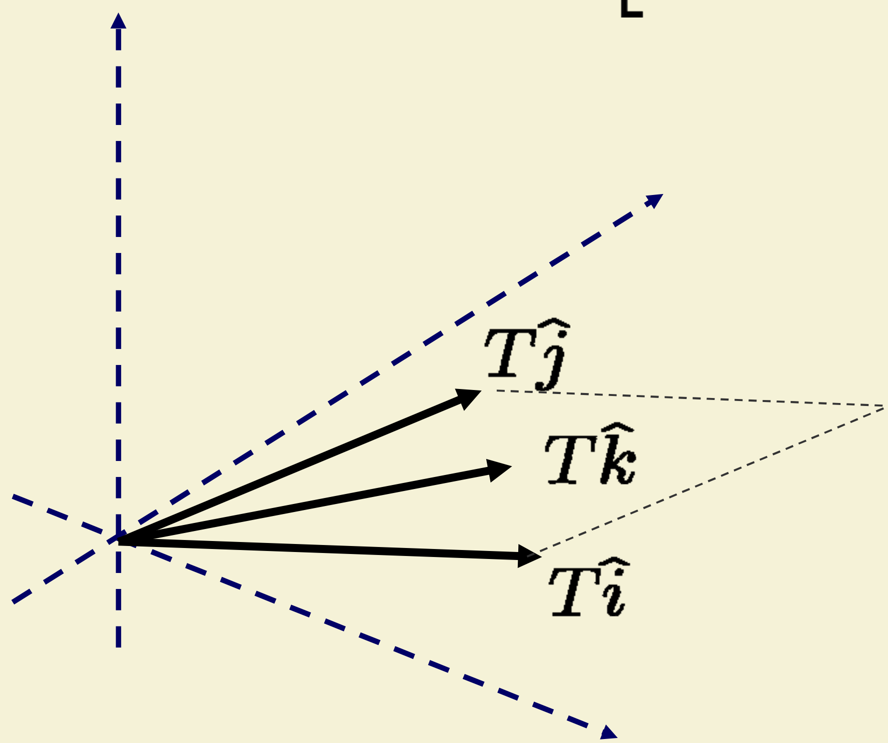


Rank 2: det (volume) = 0

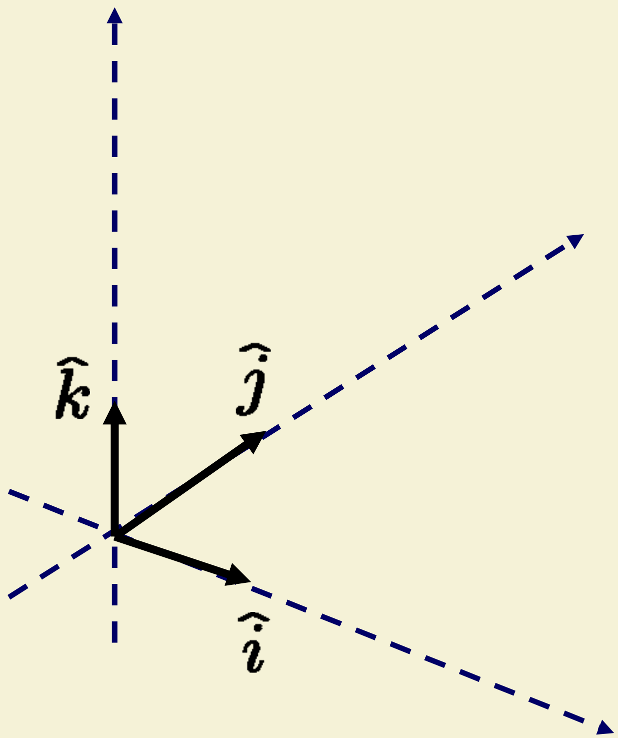


1 direction destroyed

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

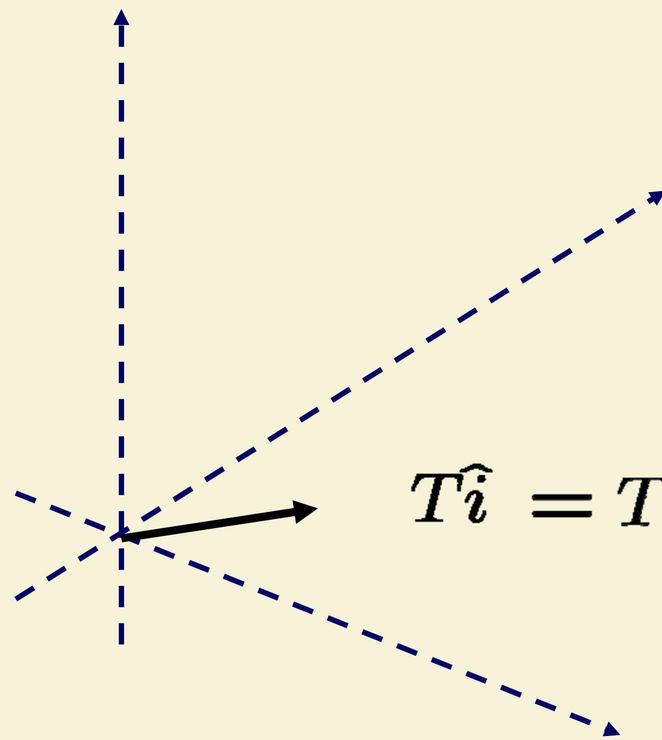


Rank 1: det (volume) = 0



2 directions destroyed

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$



$$T\hat{i} = T\hat{j} = T\hat{k}$$

Remark: Rank

A 3-D transformation can have

- Rank 3 — i.e invertible, non-zero determinant
- Rank 2 — Maps to 2-D space

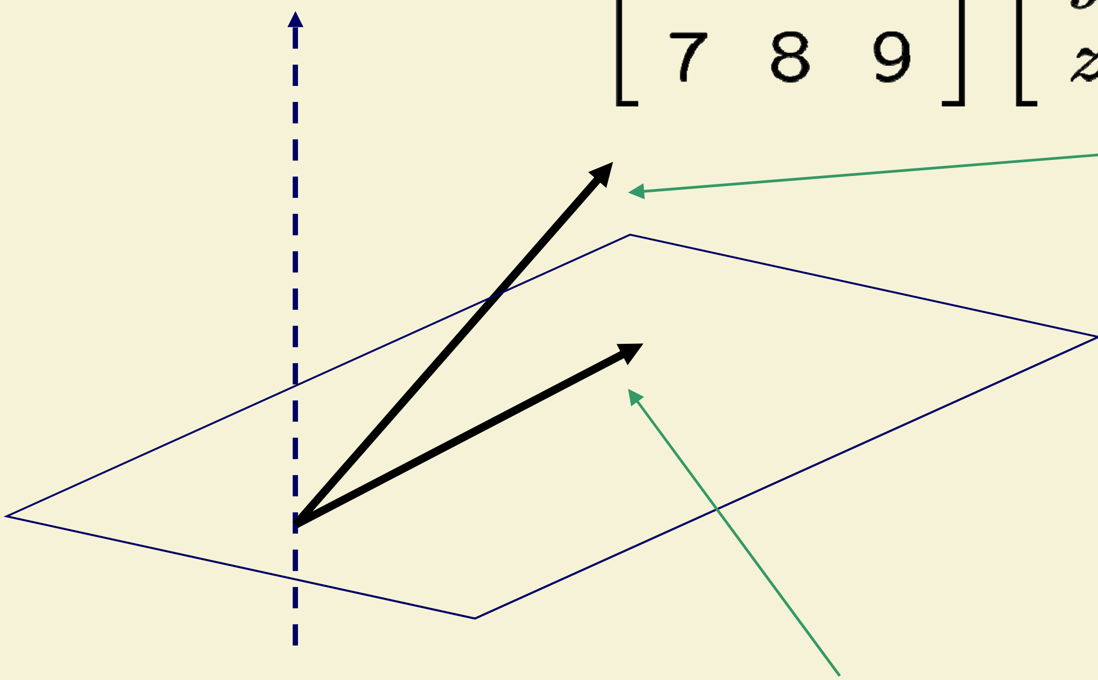
$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

- Rank 1 — Maps to 1-D space

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank 2: det (volume) = 0

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$



has no solution

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

has infinitely many solutions

Why infinitely many solutions? $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} -1/3 \\ 2/3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \left(\begin{bmatrix} -1/3 \\ 2/3 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Application: Solving Linear Systems

Any linear system can be written as

$$\boxed{M\vec{r} = \vec{a}}$$

If $\boldsymbol{M}$ is square and det $\boldsymbol{M} \neq 0$,

$$\boxed{\vec{r} = M^{-1}\vec{a}}$$

If det $\boldsymbol{M} = 0$, there could be no solutions or infinitely many solutions

Application: Solving Linear Systems

Any linear system can be written as

$$\boxed{M\vec{r} = \vec{a}}$$

If ***M*** is not square, ***M***$^{-1}$ does not make sense

The system can still be solved by using row operations on $[M|\vec{a}]$