NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

$\mathbb{MA}$ 1505 Mathematics I
Tutorial 6

1. Imagine you are visiting a country in the winter season. Let $T(x, y) = 36 - \frac{1}{5}[x^2 + (y - 5)^2]$ be the temperature at location (x, y) in a 10ft $\times$ 10ft hotel room with a heater on at night. One corner of the room is at $(0, 0)$ and the opposite corner is at $(10, 10)$.

   (i) What is the domain of the temperature function?

   (ii) Where is the likely location of the heater?

   (iii) Suppose you like to sleep within the temperature range of 20°C to 25°C. Where would you put your bed?

   (iv) Determine the locations in the room where the temperature is lowest.

   Ans. (ii) $(0, 5)$; (iv) $(10, 0)$ and $(10, 10)$.

2. In an electric circuit, the voltage of V volts (V), current of I amperes (A), and resistance of R ohms (Ω) are governed by Ohm's Law $V = I \times R$.

   (i) If the resistance is fixed at 15 Ω, how fast is the current increasing with respect to voltage?

   (ii) If the voltage is fixed at 120 V, how fast is the current increasing with respect to resistance at the instant when resistance is 20 Ω?

   (iii) If the resistance is slowly increasing as the resistor heats up, how is the current changing at the moment when $R = 400\Omega$, $I = 0.08$A, $dV/dt = -0.01$ V/s and $dR/dt = 0.03$ Ω/s?

   Ans. (i) ≈ 0.0667 A/V; (ii) *decreasing* at 0.3 A/Ω; (iii) decreasing at 3.1×10^{-5} A/s

3. Find the directional derivative of $f(x, y) = xe^{2y-x}$ at $P(-2, -1)$ in the direction

   (i) $\dfrac{1}{\sqrt{2}}\mathbf{i} + \dfrac{1}{\sqrt{2}}\mathbf{j}$; (ii) $3\mathbf{i} + 4\mathbf{j}$;

   Find the direction that gives the *largest possible* directional derivative of f at P.

   Ans. (i) $-\sqrt{2}/2$; (ii) $-7/5$; $f_x(-2, -1)\mathbf{i} + f_y(-2, -1)\mathbf{j} = 3\mathbf{i} - 4\mathbf{j}$

4. Let $f(x, y, z) = \sin(xyz)$ and $P = (\frac{1}{2}, \frac{1}{3}, \pi)$.

   (i) Find the rate of change of f at P in the direction $\mathbf{u} = \dfrac{1}{\sqrt{3}}\mathbf{i} - \dfrac{1}{\sqrt{3}}\mathbf{j} + \dfrac{1}{\sqrt{3}}\mathbf{k}$.

   (ii) Suppose P moves 0.1 unit along $\mathbf{u}$ in part (i). How much will the value of f have changed?

   Ans. (i) $\frac{1}{12}(1 - \pi)$; (ii) decreases by ≈ 0.01785.

5. Find the local maximum and minimum values and saddle points (if any) of each of the following functions.

   (i) $f(x, y) = \ln(x^2 y) - xy - 2x$, where $x > 0,\ y > 0$

   (ii) $g(x, y) = xy(1 - x - y)$

   (iii) $h(x, y) = x^2 + y^2 + x^{-2}y^{-2}$, where $x \neq 0,\ y \neq 0$

   Ans. (i) $f(1/2, 2) = -\ln 2 - 2$ is a local maximum, (ii) $(0, 0)$, $(1, 0)$, $(0, 1)$ are saddle points, $g(1/3, 1/3) = 1/27$ is a local maximum, (iii) $h(\pm 1, \pm 1) = h(\pm 1, \mp 1) = 3$ are local minima.

6. Let $u = u(x, y)$ be a twice differentiable function of x and y. If u satisfies $u > 0$ and

$$u\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u}{\partial x}\frac{\partial u}{\partial y},$$

   prove that $\dfrac{\partial(\ln u)}{\partial y}$ is a function of one variable y only.