MA1506

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CHAPTER 1

DIFFERENTIAL EQUATIONS

1.1 Introduction

A *differential* equation is an equation that contains one or more derivatives of a differentiable function. [In this course we deal only with ordinary DEs, NOT partial DEs.]

The *order* of a d.e. is the order of the equation's highest order derivative; and a d.e. is *linear* if it can be put in the form

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y^{(1)}(x) + a_0 y(x) = F,$$

where $a_i,\ 0 \le i \le n$, and F are all functions of x.

For example, $y' = 5y$ and $xy' - \sin x = 0$ are first order linear d.e.; $(y''')^2 + (y'')^5 - y' = e^x$ is third order, nonlinear.

We observe that in general, a d.e. has many solutions, e.g. $y = \sin x + c$, c an arbitrary constant, is a solution of $y' = \cos x$.

Such solutions containing arbitrary constants are called *general solution* of a given d.e.. Any solution obtained from the general solution by giving specific values to the arbitrary constants is called a *particular solution* of that d.e. e.g. $y = \sin x + 1$ is a particular solution of $y' = \cos x$.

Basically, differential equations are solved using integration, and it is clear that there will be as many integrations as the order of the DE. Therefore, THE GENERAL SOLUTION OF AN nth-ORDER DE WILL HAVE n ARBITRARY CONSTANTS.

1.2 Separable equations

A first order d.e. is *separable* if it can be written in the form $M(x) - N(y)y' = 0$ or equivalently, $M(x)dx = N(y)dy$. When we write the d.e. in this form, we say that we have *separated the variables*, because everything involving x is on one side, and everything involving y is on the other.

We can solve such a d.e. by integrating w.r.t. x:

$$\int M(x)dx = \int N(y)dy + c.$$

Example 1. Solve $y' = (1 + y^2)e^x$.

Solution. We separate the variables to obtain

$$e^x dx = \frac{1}{1 + y^2} dy.$$

Integrating w.r.t. x gives

$$e^x = \tan^{-1} y + c,$$

or

$$\tan^{-1} y = e^x - c,$$

or

$$y = \tan(e^x - c).$$

Example 2. Experiments show that a radioactive substance decomposes at a rate proportional to the amount present. Starting with 2 mg at certain time, say $t = 0$, what can be said about the amount available at a later time?

Example 2

Let y = amount of substance at time t.

Then

$$\begin{cases} \dfrac{dy}{dt} = ky \\ \\ y(0) = 2 \end{cases}$$

$$\therefore \quad \frac{dy}{y} = k\,dt$$

$$\therefore \quad \int \frac{dy}{y} = \int k\,dt$$

$$\ln|y| = kt + C$$

$$|y| = e^{c} e^{kt}$$

$$y = A e^{kt}$$

$$y(0) = 2 \Rightarrow 2 = A e^{k(0)} = A$$

$$\therefore \quad \boxed{y = 2e^{kt}}$$

Example 3. A copper ball is heated to 100°C. At $t = 0$ it is placed in water which is maintained at 30°C. At the end of 3 mins the temperature of the ball is reduced to 70°C. Find the time at which the temperature of the ball is 31°C.

Physical information: Experiments show that the rate of change dT/dt of the temperature T of the ball w.r.t. time is proportional to the difference between T and the temp T_0 of the surrounding medium. Also, heat flows so rapidly in copper that at any time the temperature is practically the same at all points of the ball.

Example 3

Let T = temperature of the ball at time t.

Then

$$\begin{cases} \dfrac{dT}{dt} = k(T-30) \\ T(0) = 100 \\ T(3) = 70 \end{cases}$$

$$\therefore \quad \frac{dT}{T-30} = k\,dt$$

$$\ln|T-30| = kt + C$$

$$T - 30 = Ae^{kt}$$

$$T(0) = 100 \Rightarrow 100 - 30 = Ae^{k(0)}$$

$$\Rightarrow 70 = A$$

$\therefore \quad T = 30 + 70e^{kt}$

$T(3) = 70 \Rightarrow \quad 70 = 30 + 70e^{3k}$

$\Rightarrow \quad 4 = 7e^{3k}$

$\Rightarrow \quad k = \frac{1}{3}\left(\ln \frac{4}{7}\right) = \underline{\frac{\ln 4 - \ln 7}{3}}$

$$\therefore \quad T = 30 + 70e^{(\ln 4 - \ln 7)t/3}$$

$$\therefore \quad T = 31 \Rightarrow 1 = 70e^{(\ln 4 - \ln 7)t/3}$$

$$\Rightarrow \frac{(\ln 4 - \ln 7)}{3}t = \ln \frac{1}{70}$$

$$= -\ln 70$$

$$\Rightarrow t = \frac{3 \ln 70}{\ln 7 - \ln 4}$$

$$\approx \underline{\underline{22.78 \text{ min.}}}$$

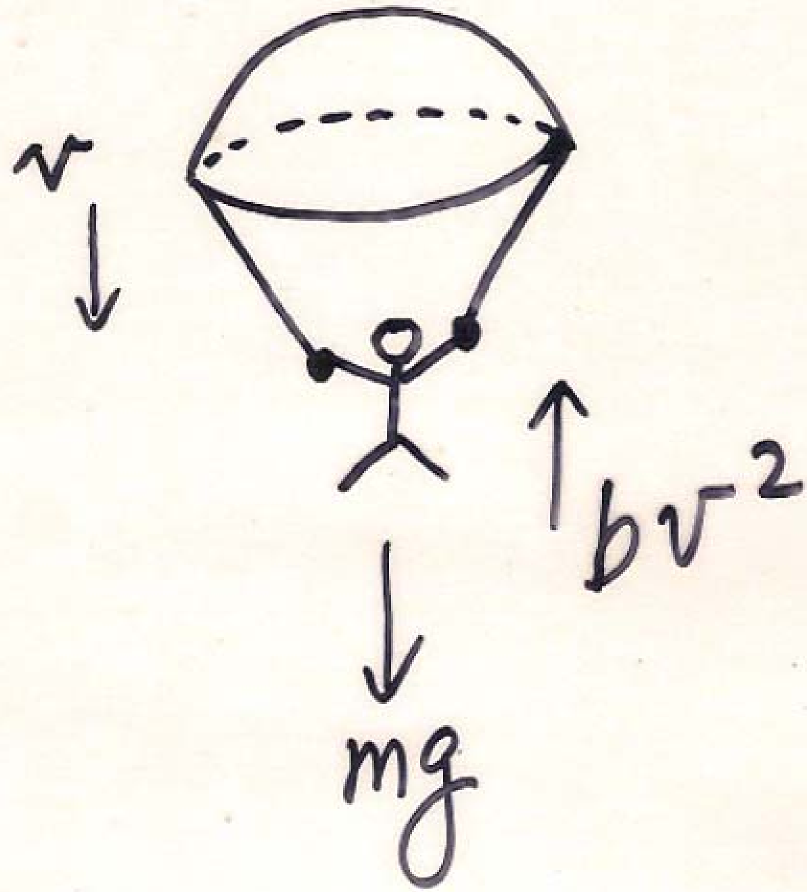
Example 4. Suppose that a sky diver falls from rest toward the earth and the parachute opens at an instant $t = 0$, when sky diver's speed is $v(0) = v_0 = 10\,\mathrm{m/s}$. Find the speed of the sky diver at any later time t.

Physical assumptions and laws:

weight of the man + equipment = 712N,

air resistance = bv^2, where $b = 30\,\mathrm{kg/m}$.

Example 4



Newton's second law

$$\Rightarrow m\frac{dv}{dt} = mg - bv^2$$

$$\frac{dv}{dt} = g - \frac{b}{m}v^2$$

$$\frac{dv}{dt} = -\frac{b}{m}\left(v^2 - \frac{mg}{b}\right)$$

Define $k = \sqrt{\frac{mg}{b}}$ (called the terminal velocity)

$$\therefore \frac{dv}{dt} = -\frac{b}{m}(v^2 - k^2)$$

$$\frac{dv}{v^2 - k^2} = -\frac{b}{m}dt$$

$$\frac{1}{2k}\left(\frac{1}{v-k} - \frac{1}{v+k}\right)dv = -\frac{b}{m}dt$$

$$\left(\frac{1}{v-k}-\frac{1}{v+k}\right)dv=-\frac{2kb}{m}dt$$

$$\ln|v-k|-\ln|v+k|=-\frac{2kb}{m}t+C$$

$$\ln\left|\frac{v-k}{v+k}\right|=-\frac{2kb}{m}t+C$$

$$\frac{v-k}{v+k}=Ae^{-\frac{2kb}{m}t}$$

$$v - k = (v + k) A e^{-\frac{2kb}{m}t}$$

$$v\left(1 - Ae^{-\frac{2kb}{m}t}\right) = k\left(1 + Ae^{-\frac{2kb}{m}t}\right)$$

$$v = \left(\frac{1 + Ae^{-\frac{2kb}{m}t}}{1 - Ae^{-\frac{2kb}{m}t}}\right)k$$

Note: $\lim_{t \to \infty} v = k$

$= \text{terminal velocity}.$

In example 4, we have

$$v(0) = 10 \ m/s$$

$$mg = 712 \ N$$

$$b = 30 \ kg/m$$

$$\therefore\ k = \sqrt{\frac{mg}{b}}$$

$$= \sqrt{\frac{712}{30}} \approx 4.87 \text{ m/s.}$$

$$10 = v(0) = \left(\frac{1+A}{1-A}\right)k$$

$$\Rightarrow A \approx 0.345$$

$$\therefore\ v = \left(\frac{1 + 0.345e^{-4.02t}}{1 - 0.345e^{-4.02t}}\right)(4.87)$$

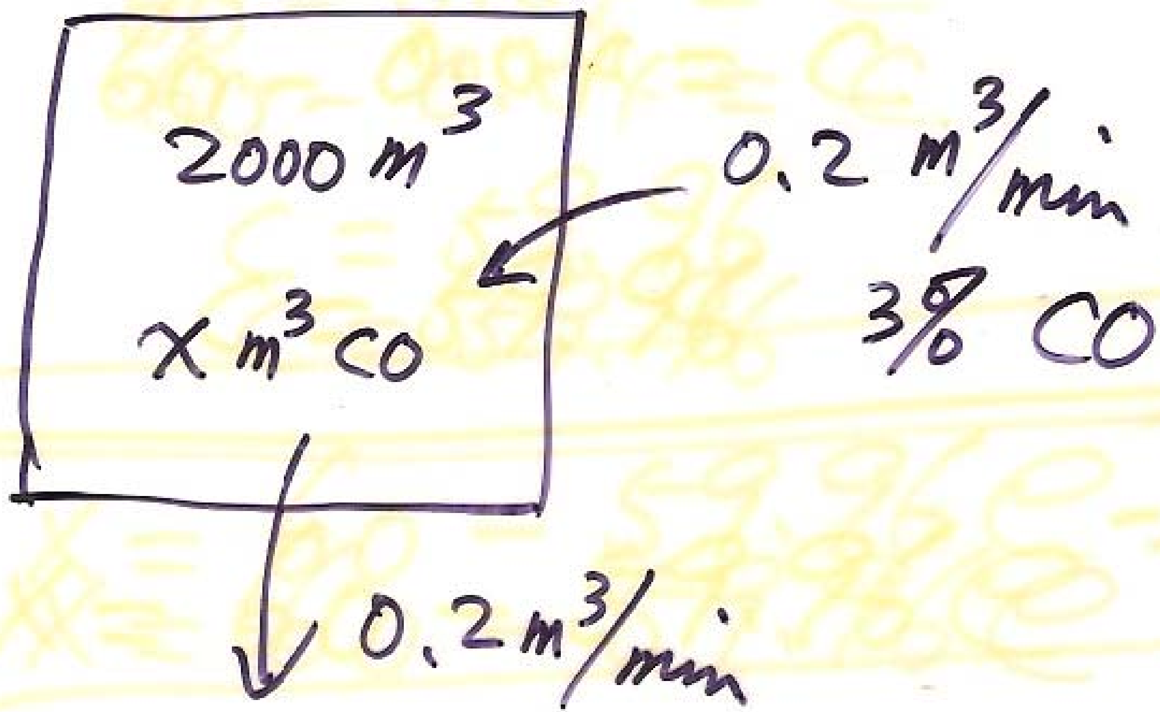
Example 5. A conference room with volume 2000m^3 contains air with 0.002% CO. At time $t = 0$ the ventilation system starts blowing in air which contains 3% CO (by volume). If the ventilation system blows in (and extracts) air at a rate of $0.2\text{m}^3/\text{min}$, how long will it take for the air in the room to contain 0.015% CO?

Solution. Let $x\,\mathrm{m}^3$ of CO be in the room at time t. Then

$t=0$

$$\boxed{\begin{array}{c} 2000\,m^3 \\ 0.002\%\ CO \end{array}}$$

t



at $t+\Delta t$, we have $x+\Delta x$ m^3 CO

$$\Rightarrow \Delta x = (0.2 \times 3\%)\Delta t$$

$$- \left(0.2 \times \frac{x}{2000}\right)\Delta t$$

$$\frac{\Delta x}{\Delta t} = 0.2 \times 0.03 - 0.2 \frac{x}{2000}$$

$\Delta t \to 0$ $\qquad \frac{dx}{dt} = 0.006 - 0.0001x$

$$= 0.0001(60 - x)$$

$$\frac{dx}{60-x} = 0.0001\,dt$$

$$-\ln|60-x| = 0.0001t + c_1$$

$$60 - x = ce^{-0.0001t}$$

$t=0,\ x = 2000 \times 0.002\% = 0.04$

$60 - 0.04 = C$

$C = 59.96$

$$\boxed{x = 60 - 59.96e^{-0.0001t}}$$

When $x = 0.015\% \times 2000\ m^3$

$$= 0.00015 \times 2000 = 0.3$$

$$\Rightarrow 0.3 = 60 - 59.96\, e^{-0.0001t}$$

$$\Rightarrow t = \frac{-1}{0.0001} \ln \frac{59.7}{59.96}$$

$$\approx \underline{\underline{43.5 \text{ mins}}}$$

e.g. 2 $$\frac{dy}{dt} = ky$$

e.g. 3 $$\frac{dT}{dt} = k(T - 30)$$

e.g. 4 $$\frac{dv}{dt} = -\frac{b}{m}(v^2 - k^2)$$

e.g. 5 $\quad \dfrac{dx}{dt} = 0.0001\,(60 - x)$

Autonomous equations i.e. t does not appear on the R.H.S.

Reduction to separable form

Certain first order d.e. are not separable but can be made separable by a simple change of variable. This holds for equations of the form

$$y' = g(\frac{y}{x}) \tag{1}$$

where g is any function of $\frac{y}{x}$. We set $\frac{y}{x} = v$, then $y = vx$ and $y' = v+xv'$. Thus (1) becomes $v + xv' = g(v)$, which is separable. Namely, $\frac{dv}{g(v) - v} = \frac{dx}{x}$. We can now solve for v, hence obtain y.

Example 6.

(a) Solve $2xyy' - y^2 + x^2 = 0$. $[x^2 + y^2 = cx]$

Example 6 (a)

$$2xyy' - y^2 + x^2 = 0$$

$$\Rightarrow \quad y' = \frac{-x^2 + y^2}{2xy} = \frac{-1 + \left(\frac{y}{x}\right)^2}{2\left(\frac{y}{x}\right)}$$

Let $v = \dfrac{y}{x} \implies y = xv$

$$\therefore y' = v + xv'$$

$$\therefore v + xv' = \frac{-1+v^2}{2v}$$

$$xv' = \frac{-1+v^2}{2v} - v = \frac{-1-v^2}{2v}$$

$$\frac{2v\,dv}{1+v^2} = -\frac{dx}{x}$$

$$\ln|1+v^2| = -\ln|x| + C$$

$$1+v^2 = A_1 e^{-\ln|x|} = A_1 e^{\ln\frac{1}{|x|}} = \frac{A_1}{|x|}$$

$$1+\frac{y^2}{x^2} = \frac{A}{x}$$

$$\underline{\underline{x^2+y^2 = Ax}}$$

(b) Solve the initial value problem $y' = \dfrac{y}{x} + \dfrac{2x^3 \cos x^2}{y}$, $y(\sqrt{\pi}\,) = 0$. $[y = x\sqrt{}(2 \sin x^2)]$

Example 6 (b)

$$\begin{cases} y' = \dfrac{y}{x} + \dfrac{2x^3 \cos x^2}{y} \\ y(\sqrt{\pi}) = 0 \end{cases}$$

Let $v = \dfrac{y}{x} \Rightarrow y = xv$

$\Rightarrow y' = v + xv'$

$$\therefore \quad v + xv' = v + \frac{2x^2\cos x^2}{v}$$

$$\therefore \quad x\frac{dv}{dx} = \frac{2x^2\cos x^2}{v}$$

$$v\,dv = 2x\cos x^2\,dx$$

$$\frac{1}{2}v^2 = \sin x^2 + C$$

$$y^2 = 2x^2(\sin x^2 + C)$$

$$y(\sqrt{\pi}) = 0 \Rightarrow 0 = 2\pi(\sin\pi + C)$$

$$\Rightarrow C = 0$$

$$\therefore \underline{\underline{y^2 = 2x^2 \sin x^2}}$$

A d.e. of the form $y' = f(ax + by + c)$, where f is continuous and $b \neq 0$ (if $b = 0$, the equation is separable) can be solved by setting $u = ax + by + c$.

Example 7. $(2x-4y+5)y'+x-2y+3=0.$

Set $x-2y=u$, we have

$$(2u+5)\frac{1}{2}(1-u')+u+3=0,$$

$$(2u+5)u'=4u+11.$$

Separating variables and integrating :

$$\left(1 - \frac{1}{4u+11}\right) du = 2dx.$$

Thus $\quad u - \frac{1}{4}\ell n|4u+11| = 2x + c_1,$

or $\quad 4x + 8y + \ell n|4x - 8y + 11| = c.$

1.3 Linear First Order ODEs

A d.e. which can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{1}$$

where P and Q are functions of x, is called a linear first order d.e. Relation (1) above is the standard form of such a d.e.

Method to solve

$$y' + Py = Q \quad \text{----} \; (1)$$

Step 1: multiply (1) by R which will be determined later.

$$\therefore \; Ry' + RPy = RQ \quad \text{----} \; (2)$$

Step 2: Set the L.H.S. of 2 equal to $(Ry)'$.

$$\therefore \ (Ry)' = Ry' + RPy$$

$$\Rightarrow R'y + Ry' = Ry' + RPy$$

$$\Rightarrow R'y = RPy$$

$$\Rightarrow \quad R' = RP$$

$$\Rightarrow \quad \frac{dR}{dx} = RP$$

$$\therefore \quad \frac{dR}{R} = P\,dx$$

$$\therefore \quad \ln|R| = \int P\,dx + C$$

$$\therefore \quad R = Ae^{\int P\,dx}$$

Any choice of $A \neq 0$ will do, so we take $A = 1$ for simplicity

$$\boxed{R = e^{\int P\,dx}} \quad \text{-----} \; (3)$$

R is called an <u>integrating factor</u> of (1).

Step 3 : With R given by 3,

2 $\Rightarrow (Ry)' = RQ$

$\Rightarrow d(Ry) = RQ\,dx$

$\Rightarrow Ry = \int RQ\,dx$

$$\Rightarrow \boxed{y = \frac{1}{R}\int RQ\,dx} \quad \text{----} \; 4$$

Summary

To solve $y' + Py = Q$:

First: find $R = e^{\int P\,dx}$.

Second: Write down the answer

$$y = \frac{1}{R}\int RQ\,dx$$

Example 8. Solve

(i) $xy' - 3y = x^2,\ x > 0.$

(ii) $y' - y = e^{2x}.$

Example 8 (i)

$$xy' - 3y = x^2, \quad x > 0$$

$$\therefore \quad y' - \frac{3}{x}y = x$$

i.e. $P = -\frac{3}{x}, \quad Q = x.$

$$\therefore R = e^{\int P dx} = e^{\int -\frac{3}{x} dx}$$

$$= e^{-3\ln x} = e^{\ln \frac{1}{x^3}} = \frac{1}{x^3}$$

$$\therefore \quad y = \frac{1}{R}\int RQ\,dx$$

$$= \frac{1}{1/x^3}\int \frac{1}{x^3}\,x\,dx$$

$$= x^3 \int \frac{1}{x^2}\,dx$$

$$= x^3\left(-\frac{1}{x} + C\right)$$

$$\therefore \quad \underline{\underline{y = -x^2 + Cx^3}}$$

Example 8 (ii)

$$y' - y = e^{2x}$$

$$\therefore P = -1, \quad Q = e^{2x}$$

$$\therefore R = e^{\int P dx} = e^{\int - dx} = e^{-x}$$

$$\therefore \quad y = \frac{1}{R}\int RQ\,dx$$

$$= \frac{1}{e^{-x}}\int e^{-x}e^{2x}\,dx$$

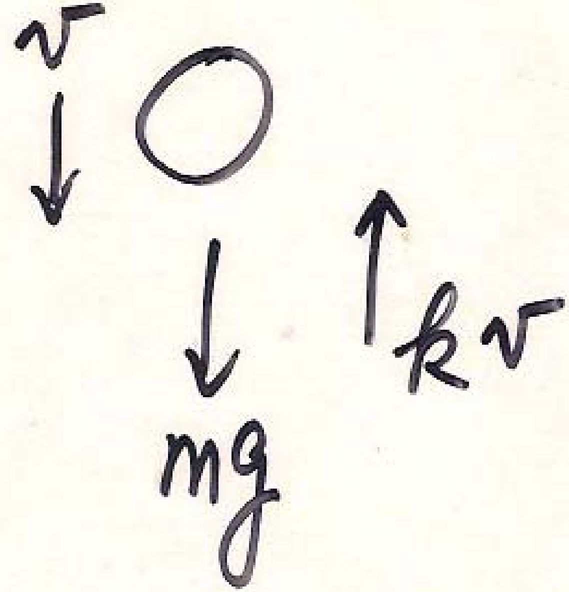
$$= e^{x}\int e^{x}\,dx$$

$$= e^{x}(e^{x}+c)$$

$$\therefore \quad \underline{\underline{y = e^{2x} + ce^{x}}}$$

Example 9. Consider an object of mass m dropped from rest in a medium that offers a resistance proportional to the magnitude of the instantaneous velocity of the object. The goal is to find the position $x(t)$ and velocity $v(t)$ at any time t.

Example 9



$$\begin{cases} m\dfrac{dv}{dt} = mg - kv \\ v(0) = 0 \\ x(0) = 0 \end{cases}$$

where v = velocity at time t.

x = vertical distance from starting point at time t.

$$\therefore \quad \frac{dv}{dt} + \frac{k}{m}v = g$$

$$P = \frac{k}{m}, \quad Q = g$$

$$R = e^{\int P\,dt} = e^{\int \frac{k}{m}dt} = e^{\frac{k}{m}t}$$

$$v = \frac{1}{R}\int RQ\,dt$$

$$= \frac{1}{e^{\frac{k}{m}t}}\int e^{\frac{k}{m}t} g\,dt$$

$$= e^{-\frac{k}{m}t}\left(\frac{mg}{k}e^{\frac{k}{m}t} + c_1\right)$$

$$= \frac{mg}{k} + c_1 e^{-\frac{k}{m}t}$$

$$v(0) = 0 \Rightarrow 0 = \frac{mg}{k} + C_1$$

$$\Rightarrow C_1 = -\frac{mg}{k}$$

$$\therefore \quad v = \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right)$$

$$\frac{dx}{dt} = \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right)$$

$$dx = \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right) dt$$

$$x = \frac{mg}{k}\left(t + \frac{m}{k} e^{-\frac{k}{m}t}\right) + C_2$$

$$x(0)=0 \Rightarrow 0=\frac{m^2 g}{k^2}+C_2$$

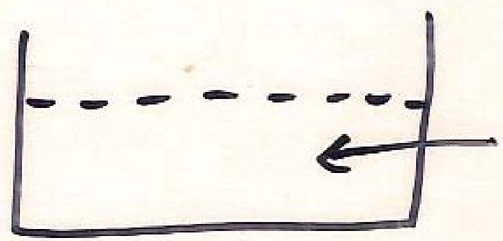
$$\Rightarrow C_2=-\frac{m^2 g}{k^2}$$

$$\therefore \quad x=\frac{mg}{k}t+\frac{m^2 g}{k^2}\left(e^{-\frac{k}{m}t}-1\right)$$

Example 10. At time $t = 0$ a tank contains 20 lbs of salt dissolved in 100 gal of water. Assume that water containing 0.25 lb of salt per gallon is entering the tank at a rate of 3 gal/min and the well stirred solution is leaving the tank at the same rate. Find the amount of salt at any time t.

Example 10

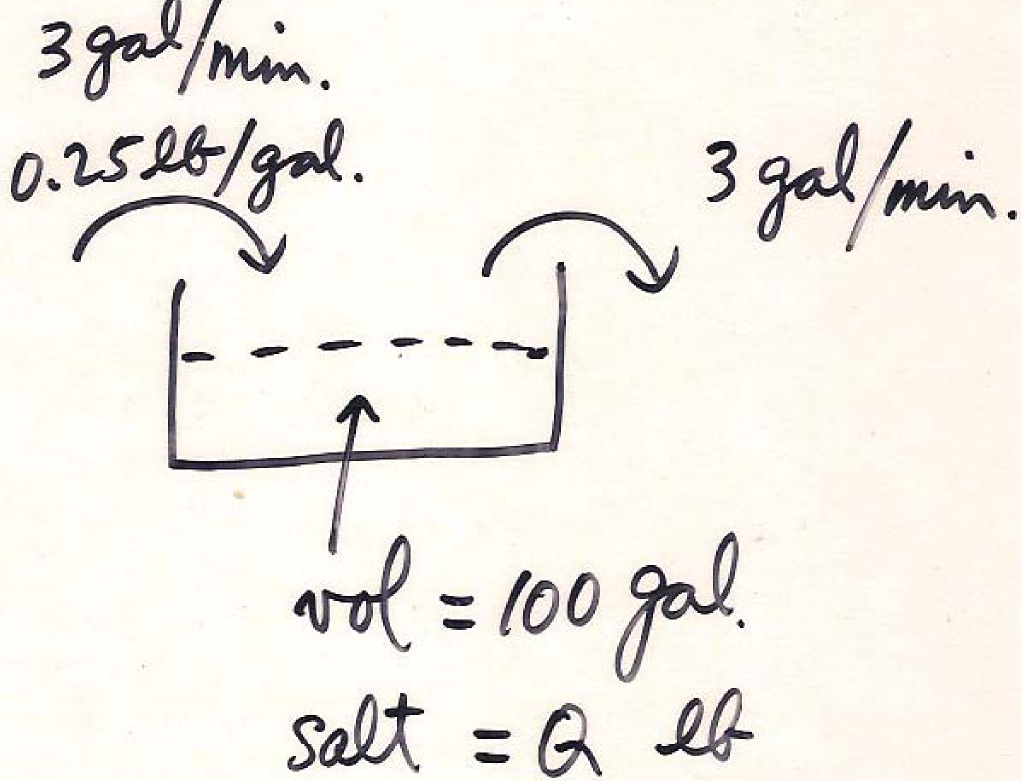
$t = 0$



vol = 100 gal.

salt = 20 lb.

t



Note: density of salt solution at t

is $\dfrac{Q}{100}$ lb/gal.

Suppose at $t+\Delta t$, there is $Q+\Delta Q$ lb of salt.

$$\Delta Q = (\text{salt input}) - (\text{salt output})$$
$$= 3 \times 0.25 \times \Delta t - 3 \times \frac{Q}{100} \times \Delta t$$

$$\frac{\Delta Q}{\Delta t} = 0.75 - \frac{3Q}{100}$$

$$\Delta t \to 0 \Rightarrow \frac{dQ}{dt} = 0.75 - 0.03Q$$

$$\frac{dQ}{dt} + 0.03Q = 0.75$$

$$R = e^{\int 0.03\,dt} = e^{0.03t}$$

$$Q = \frac{1}{R}\int R(0.75)\,dt$$

$$= e^{-0.03t}\int 0.75\,e^{0.03t}\,dt$$

$$= e^{-0.03t}\left(25e^{0.03t} + c\right)$$

$$Q(0) = 20 \Rightarrow 20 = 25 + C$$

$$\Rightarrow C = -5$$

$$\therefore \underline{\underline{Q = 25 - 5e^{-0.03t}}}$$

Note that

$\lim_{t\to\infty} Q(t) = 25$. Thus after sufficiently long time, the salt concentration remains constant at 25 lbs/100 gal.

Example 11. In Example 2 in Section 1.2, we saw that radioactive substances typically decay at a rate proportional to the amount present. Sometimes the product of a radioactive decay is itself a radioactive substance which in turn de-

decays (at a different rate). An interesting example of this is provided by *Uranium-Thorium dating*, which is a method used by palaeontologists to determine how old certain fossils [especially ancient corals] are. Corals filter the sea-water in which they live. Sea-water contains a tiny amount of a certain kind of Uranium [Uranium 234] and the corals absorb this into their bodies. Uranium 234 decays, with a half-life of 245000 years, into Thorium 230, which itself decays with a half-life of 75000 years. Thorium is not found in sea-water; so when the coral dies, it has a certain amount of Uranium in it but no Thorium [because the lifetime of a coral polyp is negligible compared with 245000 years]. It is possible to measure the ratio of the amounts of Uranium and Thorium in any given sample.

From this ratio we want to work out the age of the sample [the time when it died]. This is important if we want to know whether global warming is causing corals to die now. [Maybe they die off regularly over long periods of time and the current deaths have nothing to do with global warming.]

Let $U(t)$ be the amount of Uranium in a particular sample of ancient coral and let $T(t)$ be the amount of Thorium. Because each decay of one Uranium atom produces one Thorium atom, Thorium atoms are being born at exactly the same rate at which Uranium atoms die: so we have

$$U \xrightarrow{k_U U} T \longrightarrow k_T T$$

$$k_U \neq k_T$$

both k_U and k_T are +ve.

$$\begin{cases} \dfrac{d\upsilon}{dt} = -k_\upsilon \upsilon & \text{-----} \quad (4) \\ \dfrac{dT}{dt} = k_\upsilon \upsilon - k_T T & \text{----} \quad (5) \\ \upsilon(0) = \upsilon_0 \\ T(0) = 0 \end{cases}$$

④ $\Rightarrow \dfrac{dv}{v} = -k_v \, dt$

$\Rightarrow v = v_0 e^{-k_v t}$ ...... ⑥

$\therefore$ half-life of U is 245000 years

$$\therefore \quad \frac{U_0}{2} = U_0 e^{-k_U(245000)}$$

$$\therefore \quad -\ln 2 = -k_U(245000)$$

$$\therefore \quad k_U = \frac{\ln 2}{245000}$$

In a similar way, we find that

$$k_T = \frac{\ln 2}{75000}$$

5 and 6

$$\Rightarrow \quad \frac{dT}{dt} + k_T T = k_v v_0 e^{-k_v t}$$

Integrating factor is

$$R = e^{\int k_T \, dt} = e^{k_T t}$$

$$\therefore \quad T = \frac{1}{e^{k_T t}} \int e^{k_T t} k_v v_0 e^{-k_v t} \, dt$$

$$= e^{-k_T t} \left\{ \frac{k_v v_0}{k_T - k_v} e^{(k_T - k_v) t} + c \right\}$$

$$T(0) = 0 \Rightarrow 0 = \frac{k_U U_0}{k_T - k_U} + C$$

$$\therefore C = -\frac{k_U U_0}{k_T - k_U}$$

$$\therefore T = \frac{k_U U_0}{k_T - k_U}\left(e^{-k_U t} - e^{-k_T t}\right) \quad \cdots\cdots (7)$$

$$\frac{⑦}{⑥} \Rightarrow \frac{T}{U} = \frac{k_U}{k_T - k_U}\left(1 - e^{-(k_T - k_U)t}\right)$$

Note:

1). $\because k_T > k_U$

$\therefore e^{(k_T - k_U)t} > 1$, when $t > 0$

$\therefore e^{-(k_T - k_U)t} < 1$ when $t > 0$

$\therefore t > 0 \Rightarrow \frac{T}{U}$ is +ve.

2). Observe that $\lim_{t \to \infty} T = \lim_{t \to \infty} U = 0$,

but $\lim_{t \to \infty} \frac{T}{U} = \frac{k_U}{k_T - k_U} \neq 0$.

3). By measuring $\frac{I}{U}$ at the present time, we can calculate t which gives the age of the sample.

Reduction to linear form

Certain nonlinear d.e.s can be reduced to a linear form. The most important class of such equations are the Bernoulli equations of the form $y'+p(x)y = q(x)y^n$ where n is any real number.

Bernoulli equations

$$\frac{dy}{dx} + Py = Qy^n, \qquad n \neq 0, 1.$$

Let $\boxed{z = y^{1-n}}$

$$\frac{dz}{dx} = (1-n)\, y^{-n} \frac{dy}{dx} = \frac{1-n}{y^n} \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{y^n}{1-n} \frac{dz}{dx}$$

$$\frac{y^n}{1-n} \frac{dz}{dx} + Py = Qy^n$$

$$\frac{dz}{dx} + (1-n) P y^{1-n} = Q(1-n)$$

$$\underbrace{\frac{dz}{dx} + (1-n) P z = Q(1-n)}_{\text{linear in } z}$$

Examples. To solve

(i) $y' - Ay = -By^2$, A, B constants.

$$y' - Ay = -By^2$$

Let $z = y^{1-2} = y^{-1}$

$\therefore \quad z' = -y^{-2}\, y'$

$\therefore \quad -y^2 z' - Ay = -By^2$

$$z' + A y^{-1} = B$$

$$\therefore \quad z' + A z = B$$

$$R = e^{\int A dx} = e^{Ax}$$

$$z = e^{-Ax} \int e^{Ax} B \, dx$$

$$= e^{-Ax} \left(\frac{B}{A} e^{Ax} + c \right)$$

$$\therefore \frac{1}{y} = \frac{B}{A} + c e^{-Ax}$$

$$\therefore y = \frac{1}{\frac{B}{A} + C e^{-Ax}}$$

(ii) $y' + y = x^2y^2$. $[y(Ae^x + x^2 + 2x + 2) = 1]$

$$y' + y = x^2y^2$$

Let $z = y^{1-2} = y^{-1}$

$$\therefore z' = -y^{-2}y'$$

$$\therefore -y^2z' + y = x^2y^2$$

$$z' - y^{-1} = -x^2$$

$$z' - z = -x^2$$

$$R = e^{\int -dx} = e^{-x}$$

$$z = e^{x} \int e^{-x}(-x^2)\,dx$$

$$= e^{x} \left\{ \int x^2\, d(e^{-x}) \right\}$$

$$= e^{x}\left\{ x^{2}e^{-x} - \int 2xe^{-x}\,dx \right\}$$

$$= e^{x}\left\{ x^{2}e^{-x} - \int (-2x)\,d(e^{-x}) \right\}$$

$$= e^{x}\left\{ x^{2}e^{-x} + 2xe^{-x} - 2\int e^{-x}\,dx \right\}$$

$$= e^{x}\left\{ x^{2}e^{-x} + 2xe^{-x} + 2e^{-x} + c \right\}$$

$$\therefore \frac{1}{y} = ce^x + x^2 + 2x + 2$$

$$\therefore y = \frac{1}{ce^x + x^2 + 2x + 2}$$

1.4 Second order linear differential equations

The general form of a second order linear d.e. is

$$y'' + p(x)y' + q(x)y = F(x). \qquad (1)$$

If $F(x) \equiv 0$, the linear d.e. is called *homogeneous* otherwise it is called *nonhomogeneous.*

The d.e.

$$y'' + 4y = e^{-x} \sin x$$

is a nonhomogeneous linear second order d.e.;

$(1-x^2)y'' - 2xy' + 6y = 0$ or in the above standard form $y'' - \dfrac{2x}{1-x^2}y' + \dfrac{6}{1-x^2}y = 0$ is homogeneous; whereas $x(y''y+(y')^2)+2y'y = 0$ and $y'' = \sqrt{1+(y')^2}$ are nonlinear. Note that (1) is *linear* in the sense that it is linear in y and its derivatives.

A *solution* of a second order d.e. on some interval I is a function $y = h(x)$ with derivatives $y' = h'(x)$ and $y'' = h''(x)$ satisfying the d.e. for all x in I.

Homogeneous d.e.s

The general solutions of homogeneous equations can be found with the help of the **Superposition** or **linearity** principle, which is contained in the following theorem.

Theorem. For a homogeneous linear d.e.

$$y'' + p(x)y' + q(x)y = 0, \qquad (2)$$

any linear combination of two solutions on an open interval I is also a solution on I. In particular for such an equation, sums and constant multiples of solutions are again solutions.

Proof. Let y_1 and y_2 be solution of (2) on I.

Then $y_1'' + py_1' + qy_1 = 0$ and $y_2'' + py_2' + qy_2 = 0$.

Substituting $y = c_1y_1 + c_2y_2$ in the left of (2),

we get

$$(c_1y_1 + c_2y_2)'' + p(c_1y_1 + c_2y_2)' + q(c_1y_1 + c_2y_2)$$

$$= c_1y_1'' + c_2y_2'' + pc_1y_1' + pc_2y_2' + qc_1y_1 + qc_2y_2$$

$$= c_1(y_1'' + py_1' + qy_1) + c_2(y_2'' + py_2' + qy_2)$$

$$= c_1 \cdot 0 + c_2 \cdot 0 = 0.$$

Thus $c_1y_1 + c_2y_2$ is also a solution of (2).

Caution

The above result does not hold for nonhomogeneous or nonlinear d.e.s. For example, $y = 1 + \cos x$ and $y = 1 + \sin x$ are solutions of the nonhomogeneous linear d.e. $y'' + y = 1$, but $2(1 + \cos x)$ and $2 + \cos x + \sin x$ are not its solutions. Similarly, $y = 1$ and $y = x^2$ are solutions of the nonlinear d.e. $yy'' - xy' = 0$. But $-x^2$ and $x^2 + 1$ are not its solutions.

Example 12. Solve the initial value problem

$y'' - y = 0$, $y(0) = 5$, $y'(0) = 3$.

Solution. It is easy to see that e^x and e^{-x} are solutions of $y'' - y = 0$. Thus $y = c_1 e^x + c_2 e^{-x}$ is also a solution. From $y(0) = 5$ we get $c_1 + c_2 = 5$, and $y'(0) = 3$ gives $c_1 - c_2 = 3$. Solving, $c_1 = 4$, $c_2 = 1$. The required solution is $y = 4e^x + e^{-x}$.

General solution of homogeneous linear second order d.e.

Let $y_1(x)$ and $y_2(x)$ be defined on some interval I. Then y_1 and y_2 are said to be *linearly dependent* on I if one of them is a CONSTANT MULTIPLE OF THE OTHER ONE. Otherwise they are LINEARLY INDEPENDENT.

A *general solution* of $y'' + py' + qy = 0$ on an open interval I is $y = c_1 y_1 + c_2 y_2$, where y_1 and y_2 are linearly independent solutions of the d.e. and c_1, c_2 are arbitrary constants.

A *particular solution* of the d.e. on I is obtained if specific values are assigned to c_1 and c_2.

For example, $y_1 = \cos x$ and $y_2 = \sin x$ are linearly independent solutions of $y'' + y = 0$. A general solution is $y = c_1 \cos x + c_2 \sin x$.

A particular solution is, for example, $y = 2\cos x + \sin x$, (which satisfies $y(0) = 2$ and $y'(0) = 1$).

Homogeneous d.e. with constant coefficients

Consider

$$y'' + ay' + by = 0, \;\; a, \;\; b \;\; \text{constants.} \qquad (1)$$

Recall that a first order linear d.e. $y' + ky = 0$, k constant, has $y = e^{-kx}$ as a solution. We now try the function $y = e^{\lambda x}$ as a solution of (1). Substituting $y = e^{\lambda x}$ in (1) we obtain $(\lambda^2 + a\lambda + b)e^{\lambda x} = 0$, which implies that $e^{\lambda x}$ is a solution if λ is a solution of

$$\lambda^2 + a\lambda + b = 0. \tag{2}$$

This equation is called the *characteristic* equation (or *auxiliary* equation) of (1).

The roots of (2) are

$$\begin{aligned}\lambda_1 &= \tfrac{1}{2}(-a + \sqrt{a^2 - 4b}\,), \\ \lambda_2 &= \tfrac{1}{2}(-a - \sqrt{a^2 - 4b}\,).\end{aligned}$$

We obtain $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ as solutions of (1).

Depending on the sign of $a^2 - 4b$, equation (2) will have

Case 1: two real roots if $a^2 - 4b > 0$,

Case 2: a real double root (i.e. $\lambda_1 = \lambda_2$) if $a^2 - 4b = 0$,

Case 3: complex conjugate roots if $a^2 - 4b < 0$.

Case 1. (2) has two distinct real roots λ_1 and λ_2.

In this case $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ are linearly independent solutions of (1) on any interval. The corresponding general solution of (1) is $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$.

Example 13. Solve $y'' + y' - 2y = 0$, with $y(0) = 4$, $y'(0) = -5$.

$$\begin{cases} y'' + y' - 2y = 0 \\ y(0) = 4 \\ y'(0) = -5 \end{cases}$$

The characteristic equation is

$$\lambda^2 + \lambda - 2 = 0$$

$$\therefore (\lambda + 2)(\lambda - 1) = 0$$

$$\therefore \lambda = -2, \ 1.$$

General solution is

$$y = c_1 e^{-2x} + c_2 e^{x}.$$

$$\therefore \; y' = -2c_1 e^{-2x} + c_2 e^{x}$$

$$y(0) = 4 \Rightarrow 4 = C_1 + C_2$$

$$y'(0) = -5 \Rightarrow -5 = -2C_1 + C_2$$

$$\therefore C_1 = 3, \quad C_2 = 1$$

The answer is

$$\underline{\underline{y = 3e^{-2x} + e^{x}}}$$

Case 2. (2) has a real double root $\lambda_1\,(=\lambda_2)$.

This occurs when $a^2-4b=0$, and $\lambda_1=\lambda_2=-\frac{a}{2}$, from which we get one solution $y_1=e^{-\frac{a}{2}x}$.

To find a second solution y_2, we try $y_2=xe^{-\frac{ax}{2}}$.

This does work: $y_2' = e^{-\frac{ax}{2}} - \frac{a}{2}xe^{-\frac{ax}{2}}$, and

$$y_2'' = -\frac{a}{2}e^{-\frac{ax}{2}} - \frac{a}{2}e^{-\frac{ax}{2}} + \frac{a^2}{4}xe^{-\frac{ax}{2}}$$

$$= \left(-a + \frac{a^2}{4}x\right)e^{-\frac{ax}{2}}$$

So

$$y_2'' + ay_2' + by_2 = \left[-a + \frac{a^2}{4}x + a - \frac{a^2}{2}x + bx\right]e^{-\frac{ax}{2}}$$

$$= \left[b - \frac{a^2}{4}\right]xe^{-\frac{ax}{2}} = 0$$

because $b = \frac{a^2}{4}$.

Thus in this case when $a^2 - 4b = 0$, a linearly independent pair of solutions of $y''+ay'+by = 0$ on any interval is $e^{-\frac{ax}{2}}$, $xe^{-\frac{ax}{2}}$. The corresponding general solution is $y = (c_1 + c_2x)e^{-\frac{ax}{2}}$.

Example 14.

(i) Solve $y'' + 8y' + 16y = 0$.

$$y'' + 8y' + 16y = 0$$

$\therefore \lambda^2 + 8\lambda + 16 = 0$

$\therefore (\lambda + 4)^2 = 0$

$\therefore \lambda = -4$ is a double root.

$\therefore y = c_1 e^{-4x} + c_2 x e^{-4x}$

(ii) Solve the initial problem $y'' - 4y' + 4y = 0$,

$y(0) = 3$, $y'(0) = 1$.

$\lambda^2 - 4\lambda + 4 = 0 \Rightarrow (\lambda - 2)^2 = 0 \Rightarrow \lambda = 2$.

$\therefore \; y = c_1 e^{2x} + c_2 x e^{2x}$

$$y' = 2c_1 e^{2x} + c_2 e^{2x} + 2c_2 x e^{2x}$$

$$y(0) = 3 \Rightarrow 3 = c_1 + 0 \Rightarrow c_1 = 3$$

$$y'(0) = 1 \Rightarrow 1 = 2c_1 + c_2 \Rightarrow 1 = 6 + c_2$$

$$\Rightarrow c_2 = -5$$

$$\therefore \underline{\underline{y = 3e^{2x} - 5xe^{2x}}}$$

Case 3. (2) has two complex roots λ_1, λ_2.

Say $\lambda_1 = \alpha + \beta i$

$\lambda_2 = \alpha - \beta i$

$$\therefore \; y_1 = e^{\lambda_1 x} = e^{\alpha x + i\beta x} = e^{\alpha x}(\cos\beta x + i\sin\beta x)$$

and $y_2 = e^{\alpha x}(\cos\beta x - i\sin\beta x)$

are solutions of $(*)$.

$\therefore\ y = \frac{1}{2}(y_1 + y_2) = e^{\alpha x} \cos \beta x$

and $y = \frac{1}{2i}(y_1 - y_2) = e^{\alpha x} \sin \beta x$

are also solutions of (*) and they are not proportional.

General solution is

$$y = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x$$

Example 15.

(i) Solve $y'' + 2y' + 5y = 0$.

$$y'' + 2y' + 5y = 0$$

$$\lambda^2 + 2\lambda + 5 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$$

$$\therefore \underline{\underline{y = c_1 e^{-x} \cos 2x + c_2 e^{-x} \sin 2x}}$$

(ii) Solve $y'' + 2y' + 5y = 0$, $y(0) = 1$, $y'(0) = 5$.

From (i), $y = C_1 e^{-x} \cos 2x + C_2 e^{-x} \sin 2x$

$$y' = -C_1 e^{-x} \cos 2x - 2C_1 e^{-x} \sin 2x$$

$$- C_2 e^{-x} \sin 2x + 2C_2 e^{-x} \cos 2x$$

$$y(0)=1 \Rightarrow 1=c_1$$

$$y'(0)=5 \Rightarrow 5=-c_1+2c_2=-1+2c_2$$

$$\Rightarrow c_2=3$$

$$\therefore \underline{\underline{y=e^{-x}\cos 2x+3e^{-x}\sin 2x}}$$

Nonhomogeneous equations

We consider

$$y'' + p(x)y' + q(x)y = r(x), \; r(x) \not\equiv 0. \tag{1}$$

The corresponding homogeneous equation is

$$y'' + p(x)y' + q(x)y = 0. \tag{2}$$

Let y_1 and y_2 be any two solutions of (1). Then

$$y_1'' + p(x)y_1' + q(x)y_1 = r(x), \quad (3)$$

and

$$y_2'' + p(x)y_2' + q(x)y_2 = r(x). \quad (4)$$

Subtracting (4) from (3):

$$y_1'' - y_2'' + p(x)(y_1' - y_2') + q(x)(y_1 - y_2) = r(x) - r(x) = 0.$$

Thus $(y_1 - y_2)'' + p(x)(y_1 - y_2)' + q(x)(y_1 - y_2) = 0$, i.e. $y_1 - y_2$ is a solution of (2).

On the other hand, if y_0 is a solution of (2) and y_1 a solution of (1), then clearly $y_1 + y_0$ is again a solution of (1). This suggests the following definition:

Definition. A *general solution* of the non-homogeneous d.e. (1) is of the form

$$y(x) = y_h(x) + y_p(x), \tag{5}$$

where $y_h(x) = c_1 y_1(x) + c_2 y_2(x)$ is a general solution of the homogeneous d.e. (2) and $y_p(x)$ is any solution of (1) containing no arbitrary constants.

Thus to solve (1), we have to solve the homogeneous equation (2) and find a (particular) solution of (1). The sum of these two is what we want.

Determination of $y_p(x)$

(I) Method of undetermined coefficients

This method applies to equations of the form $y'' + ay' + by = r(x)$, where a and b are constants, and $r(x)$ is a polynomial, exponential function, sine or cosine, or sums or products of such functions.

We denote $y'' + ay' + by$ by $L(y)(x)$, where a and b may be complex numbers. We'll make use of the Principle of super-position:

If $y_1(x)$ is a solution of $L(y)(x) = g_1(x)$ and $y_2(x)$ is a solution of $L(y)(x) = g_2(x)$, then for any constants c_1 and c_2, $y = c_1y_1(x)+c_2y_2(x)$ is a solution of $L(y)(x) = c_1g_1(x)+c_2g_2(x)$. Using this principle, the problem is reduced to finding a particular solution of $L(y)(x) = p(x)e^{kx}$, where $p(x)$ is a polynomial in x and k is a real or complex constant. Method of undetermined coefficients is adequately described by considering the following three cases, which we illustrate with examples.

1. Polynomial case

In this case $k = 0$. The method begins with "try a polynomial with unknown coefficients".

Example 16.

$$y'' - 4y' + y = x^2 + x + 2.$$

$$y'' - 4y' + y = x^2 + x + 2 \text{ ---- } (1)$$

<u>Step 1</u> Look at $y'' - 4y' + y = 0$ ---- (2)

$$\lambda^2 - 4\lambda + 1 = 0$$

$$\lambda = \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm \sqrt{12}}{2}$$

$$= 2 \pm \sqrt{3}$$

gen. sol. of (2) is

$$y = c_1 e^{(2+\sqrt{3})x} + c_2 e^{(2-\sqrt{3})x}$$

Step 2. Try $y = Ax^2 + Bx + C$

$$y' = 2Ax + B$$

$$y'' = 2A$$

$\therefore$ 1 $\Rightarrow$ $2A - 4(2Ax + B) + (Ax^2 + Bx + C)$

$= x^2 + x + 2$

$\therefore Ax^2 + (-8A + B)x + (2A - 4B + C)$
$= x^2 + x + 2$

Compare coefficients

$\Rightarrow A = 1, \; B = 9, \; C = 36$

Particular sol. of ① is

$y = x^2 + 9x + 36$

Step 3: General Solution of (1) is

$$y = c_1 e^{(2+\sqrt{3})x} + c_2 e^{(2-\sqrt{3})x} + x^2 + 9x + 36$$

Example 17. $y'' - 2y = 2x^3$.

Step 1

$$y'' - 2y = 0 \quad \text{----} \quad (1)$$

$$\lambda^2 - 2 = 0$$

$$\lambda = \pm\sqrt{2}$$

Step 2 Try $y = Ax^3 + Bx^2 + Cx + D$

$$\therefore y' = 3Ax^2 + 2Bx + C$$

$$y'' = 6Ax + 2B$$

$$\therefore y'' - 2y = -2Ax^3 - 2Bx^2 + (6A - 2C)x + (2B - 2D)$$

Compare coefficients

$$\begin{cases} -2A = 2 \\ -2B = 0 \\ 6A - 2C = 0 \\ 2B - 2D = 0 \end{cases}$$

$$\therefore A = -1,\ B = 0,\ C = -3,\ D = 0$$

$$\therefore y = -x^3 - 3x$$

Step 3 General solution of 1 is

$$y = c_1 e^{\sqrt{2}x} + c_2 e^{-\sqrt{2}x} - x^3 - 3x$$

2. **Exponential case**

Here k is real but not zero. The method begins with "put $y = ue^{kx}$, where $u = u(x)$." This substitution will remove e^{kx} from the equation and reduce the problem to the polynomial case 1 (above).

Example 18.

$$y'' - 4y' + 2y = 2x^3e^{2x}. \qquad (1)$$

Step 1 $\quad y'' - 4y' + 2y = 0 \quad \text{----} \quad (2)$

$$\lambda^2 - 4\lambda + 2 = 0$$

$$\lambda = \frac{4 \pm \sqrt{16-8}}{2} = \frac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}$$

gen. sol. of ② is

$$y = c_1 e^{(2+\sqrt{2})x} + c_2 e^{(2-\sqrt{2})x}$$

Step 2 Try $y = ue^{2x}$

where u is a function of x

to be determined.

$$y' = u'e^{2x} + 2ue^{2x}$$

$$y'' = u''e^{2x} + 2u'e^{2x} + 2u'e^{2x} + 4ue^{2x}$$

$$= u''e^{2x} + 4u'e^{2x} + 4ue^{2x}$$

(1) $\Rightarrow$ $u''e^{2x} + 4u'e^{2x} + 4ue^{2x}$
$- 4u'e^{2x} - 8ue^{2x}$
$+ 2ue^{2x} = 2x^3e^{2x}$

$\Rightarrow$ $u'' - 2u = 2x^3$ ---- (3)

Using the result of example 17, $u = -x^3 - 3x$.

Thus a particular solution is $y_p(x) = (-x^3 - 3x)e^{2x}$.

Step 3 General Solution of 1 is

$$\boxed{y = c_1 e^{(2+\sqrt{2})x} + c_2 e^{(2-\sqrt{2})x} + (-x^3 - 3x)e^{2x}}$$

Example 19.

$$y'' - 4y' + 4y = 20x^3e^{2x}.$$

$$y'' - 4y' + 4y = 20x^3e^{2x} \quad \text{----} \quad 1$$

Step 1 Solve $y'' - 4y' + 4y = 0$ - - - - (2)

$$\lambda^2 - 4\lambda + 4 = 0$$

$$(\lambda - 2)^2 = 0$$

$\lambda = 2$ is a double root.

Gen. sol. of (2) is

$$y = c_1 e^{2x} + c_2 x e^{2x}$$

Step 2. Try $y = (\underbrace{Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F}_{\overset{\text{let}}{=} u})e^{2x}$

$$y' = u'e^{2x} + 2ue^{2x}$$

$$y'' = u''e^{2x} + 4u'e^{2x} + 4ue^{2x}$$

$$1 \Rightarrow u''e^{2x} + 4u'e^{2x} + 4ue^{2x}$$
$$- 4u'e^{2x} - 8ue^{2x}$$
$$+ 4ue^{2x} = 20x^3e^{2x}$$

$$\Rightarrow u''e^{2x} = 20x^3 e^{2x}$$

$$\Rightarrow u'' = 20x^3 \quad \text{-----} \; 3$$

$$u = Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F$$

$$u' = 5Ax^4 + 4Bx^3 + 3Cx^2 + 2Dx + E$$

$$u'' = 20Ax^3 + 12Bx^2 + 6Cx + 2D$$

Compare coefficients:

$$x^3: \quad 20A = 20 \Rightarrow A = 1$$

$$x^2: \quad 12B = 0 \Rightarrow B = 0$$

$$x: \quad 6C = 0 \Rightarrow C = 0$$

$$\text{constant}: \quad 2D = 0 \Rightarrow D = 0$$

$\therefore u = x^5 + Ex + F$

Put $E = F = 0$

$\therefore u = x^5$

Step 3. Gen. sol. of ① is

$$y = c_1 e^{2x} + c_2 x e^{2x} + x^5 e^{2x}$$

Another way to do step 2 (as in the notes)

From the previous calculation,

we have

$$u'' = 20x^3 \quad \text{- - - - ③}$$

Instead of comparing coefficients, we integrate ③ two times:

$$u' = 5x^4 + c_1$$

$$u = x^5 + c_1 x + c_2$$

Any choice of c_1, c_2 will give a particular solution.

Say we take $c_1 = c_2 = 0$

$$\therefore \quad u = x^5$$

as before.

3. **Trigonometric case**

Example 20. Solve

$$y'' + 4y = 16x \sin 2x. \tag{1}$$

To solve

$$y'' + 4y = 16x\sin 2x \quad \text{-----} \quad 1$$

Let $\quad v'' + 4v = 16x\cos 2x \quad \text{----} \quad 2$

$$② + i① \Rightarrow (v+iy)'' + 4(v+iy)$$

$$= 16x(\cos 2x + i\sin 2x)$$

$$\Rightarrow z'' + 4z = 16xe^{i2x} \quad \text{----} ③$$

where $z = v + iy$

$\therefore \; y = \text{Im}(z)$

Step 1 Solve the homogeneous equation

$$y'' + 4y = 0. \quad \text{-------} \quad (4)$$

The characteristic equation is

$$\lambda^2 + 4 = 0$$

$$\therefore \lambda = \pm 2i$$

(Here, we observe that e^{i2x} is a solution of ④. Also note that $2i$ is a simple root (i.e. not a double root).

$\therefore$ In the next step, when we try $z = ue^{i2x}$, we have to use a degree 2 polynomial for u.

$\uparrow$

$(= \{\text{degree of } (16x)\} + 1 = 1 + 1 = 2)$).

Step 2

Try $z = ue^{i2x}$

where $u = Ax^2 + Bx + C$.

$\therefore z' = u'e^{i2x} + 2iue^{i2x}$

$z'' = u''e^{i2x} + 4iu'e^{i2x} - 4ue^{i2x}$

$$\therefore \text{③} \Rightarrow u'' e^{i2x} + 4i u' e^{i2x} = 16 x e^{i2x}$$

$$\Rightarrow u'' + 4i u' = 16x$$

$$\Rightarrow 2A + 4i(2Ax + B) = 16x$$

Compare coefficients:

x : $8Ai = 16$

$$\therefore A = \frac{16}{8i} = -2i$$

constant: $2A + 4iB = 0$

$$\therefore B = -\frac{2A}{4i} = 1$$

C can be any number and we take $C=0$ for convenience.

$\therefore\ u = -2ix^2 + x$

$$\therefore z = u e^{i2x}$$

$$= (-2ix^2 + x)(\cos 2x + i \sin 2x)$$

$$= Re(z) + i\{-2x^2 \cos 2x + x \sin 2x\}$$

$$\therefore y = Im(z)$$

$$= -2x^2 \cos 2x + x \sin 2x.$$

Step 3

General solution of ① is

$$y = c_1 \cos 2x + c_2 \sin 2x - 2x^2 \cos 2x + x \sin 2x$$

Example 21 To solve

$$y'' + 2y' + 5y = 16xe^{-x}\cos 2x \quad \text{------} \quad (*)$$

Step 1 $y'' + 2y' + 5y = 0$ - - - - - - - - ①

$\lambda^2 + 2\lambda + 5 = 0$

$\lambda = (-2 \pm \sqrt{4-20})/2 = -1 \pm 2i$

General solution of ① is

$y = C_1 e^{-x} \cos 2x + C_2 e^{-x} \sin 2x$.

Step 2 $y'' + 2y' + 5y = 16xe^{-x}\cos 2x$ ------- (2)

Let $iv'' + 2iv' + 5iv = 16xe^{-x}\, i\sin 2x$ ---- (3)

Let $z = y + iv \Rightarrow y = \text{Re}(z)$

(2) + (3) $\Rightarrow z'' + 2z' + 5z = 16xe^{-x}(\cos 2x + i\sin 2x)$

$$= 16xe^{-x}e^{i2x}$$

$$\Rightarrow z'' + 2z' + 5z = 16xe^{-x+2ix} \quad \text{-----} \ (4)$$

Try $z = ue^{-x+2ix}$ in (4)

$$z' = u'e^{-x+2ix} + (-1+2i)ue^{-x+2ix}$$

$$z'' = u''e^{-x+2ix} + 2(-1+2i)u'e^{-x+2ix} + (-1+2i)^2 ue^{-x+2ix}$$

$$④ \Rightarrow u'' + 2(-1+2i)u' + (-1+2i)^2 u$$

$$+ 2u' \qquad + 2(-1+2i)u$$

$$+ 5u = 16x$$

$$\Rightarrow u'' + 4iu' = 16x \quad ---- ⑤$$

Try $u = ax^2 + bx + c$ (where a, b, c are complex constants) in 5

$u' = 2ax + b$

$u'' = 2a$

(Note: We try a quadratic expression for u because we observe that $e^{-x}\cos 2x$ is a solution of the homogeneous equation 1)

5 $\Rightarrow 2a + 8aix + 4bi = 16x$

$$\Rightarrow \begin{cases} 8ai = 16 \\ 2a + 4bi = 0 \end{cases} \Rightarrow \begin{cases} a = -2i \\ b = 1 \end{cases}$$

$\therefore\ u = -2ix^2 + x$

$$\therefore\ z = u e^{-x+2ix} = (-2ix^2+x)e^{-x}(\cos 2x + i\sin 2x)$$

$$= e^{-x}\{(x\cos 2x + 2x^2\sin 2x)$$

$$+ i(-2x^2\cos 2x + x\sin 2x)\}$$

$$\therefore\ y = \mathrm{Re}(z)$$

$$= e^{-x}(x\cos 2x + 2x^2\sin 2x)$$

Step 3

General solution of (*) is

$$y = c_1 e^{-x} \cos 2x + c_2 e^{-x} \sin 2x$$
$$+ x \cos 2x \, e^{-x} + 2x^2 \sin 2x \, e^{-x}$$

(II) Method of variation of parameters

Variation of parameters

$$y'' + py' + qy = r \quad \text{------} \quad (1)$$

Suppose y_1, y_2 are linearly independent solution of

$$y'' + py' + qy = 0 \quad \text{------} \quad (2)$$

We try $y = uy_1 + vy_2$ in 1

$$y' = u'y_1 + uy_1' + v'y_2 + vy_2'$$

Impose the condition

$$u'y_1 + v'y_2 = 0 \quad \text{-----} \quad 3$$

$$\therefore y' = uy_1' + vy_2'$$

$$\therefore\ y'' = u'y_1' + uy_1'' + v'y_2' + vy_2''$$

$$py' = \qquad pu y_1' \qquad + pvy_2'$$

$$qy = \qquad qu y_1 \qquad + qvy_2$$

① & ② $\Rightarrow$

$$u'y_1' + v'y_2' = r \quad ----- \quad ④$$

$$u'y_1 + v'y_2 = 0 \text{ ------ } (3)$$

$$u'y_1' + v'y_2' = r \text{ ------ } (4)$$

(3) & (4) $\Rightarrow$

$$u' = \frac{\begin{vmatrix} 0 & y_2 \\ r & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-y_2 r}{y_1 y_2' - y_1' y_2}$$

$$v' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & r \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1 r}{y_1 y_2' - y_1' y_2}$$

Integrating

$$
\begin{aligned}
u &= -\int \frac{y_2 r}{y_1 y_2' - y_1' y_2}\, dx, \\
v &= \int \frac{y_1 r}{y_1 y_2' - y_1' y_2}\, dx.
\end{aligned}
$$

We obtain now y_p and hence a general solution of (1).

Note. The term $y_1y_2'-y_1'y_2$ may be viewed as the determinant $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. It's called the Wronskian of y_1 and y_2.

Caution. When applying the above procedure to solve (1), make sure that the given d.e. is in standard form (1) where the coefficient of y'' is 1.

Example 22. Solve $y'' + y = \tan x$, $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

$$y'' + y = \tan x \quad \text{---} \quad (*)$$

Step 1. $y'' + y = 0 \quad \text{---} \quad (1)$

$$\lambda^2 + 1 = 0$$

$$\lambda = \pm i$$

$y_1 = \cos x, \quad y_2 = \sin x$

are lin. indep. sol. of (1)

Step 2 Try $y = u\cos x + v\sin x$ in (*)

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$u = -\int \sin x \tan x \, dx$$

$$v = \int \cos x \tan x \, dx$$

$$u = -\int \frac{\sin^2 x}{\cos x}\,dx$$

$$= -\int \frac{1-\cos^2 x}{\cos x}\,dx$$

$$= \int (\cos x - \sec x)\,dx$$

$$= \sin x - \ln|\sec x + \tan x|$$

$$v = \int \cos x \tan x \, dx$$

$$= \int \sin x \, dx$$

$$= -\cos x$$

$$y = u y_1 + v y_2$$

$$= (\sin x - \ln|\sec x + \tan x|)\cos x$$

$$+ (-\cos x)\sin x$$

$$= -\cos x \ln|\sec x + \tan x|$$

Step 3 Gen. sol. of (*) is

$$y = C_1 \cos x + C_2 \sin x - \cos x \ln|\sec x + \tan x|$$

Example 23. Solve $y'' - y = e^{-x} \sin e^{-x} + \cos e^{-x}$.

$$y'' - y = e^{-x}\sin(e^{-x}) + \cos(e^{-x}) \quad \cdots (1)$$

<u>Step 1</u> Solve $y'' - y = 0 \quad \cdots\cdots (2)$

Let $\lambda^2 - 1 = 0$

$\therefore \lambda = \pm 1$

general solution of (2) is

$$y = c_1 e^{x} + c_2 e^{-x}.$$

Step 2 Try $y = ue^{x} + ve^{-x}$

$$\begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix} = -1-1 = -2$$

$$\therefore u' = -\frac{e^{-x}\left(e^{-x}\sin e^{-x} + \cos e^{-x}\right)}{-2}$$

$$= \frac{1}{2}\left(e^{-2x}\sin e^{-x} + e^{-x}\cos e^{-x}\right)$$

$$u = \frac{1}{2}\int e^{-2x}\sin e^{-x}\,dx + \frac{1}{2}\int e^{-x}\cos e^{-x}\,dx$$

$$\int e^{-2x}\sin e^{-x}\,dx = +\int e^{-x}\,d(\cos e^{-x})$$

$$= e^{-x}\cos e^{-x} - \int \cos e^{-x}\,d(e^{-x})$$

$$= e^{-x}\cos e^{-x} - \sin e^{-x}$$

$$\int e^{-x}\cos e^{-x}\,dx = -\int \cos e^{-x}\,d(e^{-x})$$

$$= -\sin e^{-x}$$

$$\therefore u = \frac{1}{2}\left(e^{-x}\cos e^{-x} - 2\sin e^{-x}\right)$$

Next

$$v' = \frac{e^{x}(e^{-x}\sin e^{-x} + \cos e^{-x})}{-2}$$

$$= -\frac{1}{2}(\sin e^{-x} + e^{x}\cos e^{-x})$$

$$\therefore\ v = -\frac{1}{2}\left(\int \sin e^{-x}\,dx + \int e^{x}\cos e^{-x}\,dx\right)$$

$$\int \sin e^{-x}\,dx = \int e^{x}\,d(\cos e^{-x})$$

$$= e^{x}\cos e^{-x} - \int e^{x}\cos e^{-x}\,dx$$

$$\therefore \quad v = -\tfrac{1}{2}e^{x}\cos e^{-x}$$

$$\therefore\ y = ue^{x} + ve^{-x}$$

$$= \frac{1}{2}\left(\cos e^{-x} - 2e^{x}\sin e^{-x}\right)$$

$$-\frac{1}{2}\cos e^{-x}$$

$$= -e^{x}\sin e^{-x}$$

Step 3

Ans: $y = -e^{x} \sin e^{-x} + c_1 e^{x} + c_2 e^{-x}$