Question 1. The thing to remember here is that there are six possible states in which the cunning Miss Tan can find herself: she can have \$0, \$1, \$2....up to \$5. She can never have more than that because Ah Huat won't allow it. If at any point she has exactly \$0, then the probability that she will still have \$0 after a round of the game is 1, because by then she has been booted out into the street. So the first column of the matrix is $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. If at any time she has \$1, then there is a probability of 1/2 that she will have \$0 after the next round, and a probability 1/2 that she will have \$2; all other probabilities are zero. So the next column is $\begin{pmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. Similarly for \$2,3,4; in each case she can either win \$1 or lose it, each with probability 1/2, so there are exactly two non-zero entries in each of those columns [the 3rd, 4th, and 5th]. Finally, if at any point she finds herself with \$5, then the probability that she will have \$5 next time is 1, because again she will soon find herself outside in the gutter if she does that; her gambling career is over. So the final column is $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. Hence the full Markov matrix in this case is just

$$\begin{pmatrix} 1 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1 \end{pmatrix}.$$

The fifth power of this matrix can be computed by the wims website. We only care about its fourth column, because that contains all of the probabilities given that Miss TAL started with \$3. That fourth column is $\begin{pmatrix} 0.21875 \\ 0 \\ 0.25 \\ 0 \\ 0.15625 \\ 0.375 \end{pmatrix}$. The probability that she is thrown out into the street with zero dollars is the top entry, 0.21875; the probability that she gets kicked out, sore but triumphant, with the vast sum of \$5 in her pocket is the last entry, 0.375.

Note that there is no chance that she has either \$1 or \$3 at this point; that is interesting, though irrelevant. You might like to compute higher powers of this matrix. One finds, of course, that it becomes more and more likely that TAL will be thrown out one way or the other [that is, the sum of the first and last entries approaches 1]. Note that Ah Huat is, as always, the loser here, in that TAL is significantly more likely to exit with a profit than with a loss. [The ratio of probabilities is $0.375/0.21875 = 1.7143$.]

Question 2. According to Chapter 5, we just have to work out

$$[I-T]^{-1}\begin{pmatrix}140\\20\\2\end{pmatrix}$$

where T is the technology matrix $\begin{bmatrix}0.30 & 0.00 & 0.00\\0.10 & 0.20 & 0.20\\0.05 & 0.01 & 0.02\end{bmatrix}$. So the answers are given by

$$\begin{bmatrix}1.43 & 0.00 & 0.00\\0.20 & 1.25 & 0.26\\0.07 & 0.01 & 1.02\end{bmatrix}\begin{pmatrix}140\\20\\2\end{pmatrix}=\begin{pmatrix}200.2\\53.52\\12.04\end{pmatrix}.$$

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Question 3.

$$\begin{aligned}&\begin{pmatrix}1\\0\end{pmatrix},2\\&\begin{pmatrix}1\\-1\end{pmatrix},0,\begin{pmatrix}1\\1\end{pmatrix},2\\&\begin{pmatrix}1\\1/2\end{pmatrix},3,\begin{pmatrix}1\\-1/2\end{pmatrix},-1\\&\begin{pmatrix}1\\-i\end{pmatrix},2-i,\begin{pmatrix}1\\i\end{pmatrix},2+i\\&\begin{pmatrix}1\\0\end{pmatrix},0,\begin{pmatrix}0\\1\end{pmatrix},1\end{aligned}$$

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Question 4.

$$\begin{vmatrix}2-\lambda & 1\\0 & 2-\lambda\end{vmatrix}=0\rightarrow(2-\lambda)^2=0$$

$\rightarrow \lambda = 2$ (only one eigenvalue)

$$\begin{bmatrix}2-\lambda & 1\\0 & 2-\lambda\end{bmatrix}\begin{bmatrix}1\\\alpha\end{bmatrix}=\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}1\\\alpha\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$\rightarrow \alpha=0\rightarrow\begin{bmatrix}1\\0\end{bmatrix}$$

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$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \to (1-\lambda)^2 = 1$$

$$\to \lambda = 0, 2$$

$\lambda = 0$

$$\to \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \alpha \end{bmatrix} = 0 \to \alpha = -1 \to \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\lambda = 2$

$$\to \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ \alpha \end{bmatrix} = 0 \to \alpha = 1 \to \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

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$$\begin{vmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{vmatrix} = 0 \to (1-\lambda)^2 = 4$$

$\to \lambda = 3, -1$

$\lambda = 3 \to$

$$\left| \begin{matrix} -2 & 4 \\ 1 & -2 \end{matrix} \right] \begin{bmatrix} 1 \\ \alpha \end{bmatrix} = 0 \to \alpha = \tfrac{1}{2} \to \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$$

$\lambda = -1 \to$

$$\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ \alpha \end{bmatrix} = 0 \to \alpha = -1/2 \to \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}$$

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$$\begin{bmatrix} 2-\lambda & 1 \\ -1 & 2-\lambda \end{bmatrix} = 0 \to (2-\lambda)^2 = -1$$

$\to \lambda = 2-i, 2+i$

$\lambda = 2-i \to$

$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \begin{bmatrix} 1 \\ \alpha \end{bmatrix} = 0 \to \alpha = -i \to \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$\lambda = 2+i$

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} 1 \\ \alpha \end{bmatrix} = 0 \to \alpha = i \to \begin{bmatrix} 1 \\ i \end{bmatrix}$$

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$$\begin{vmatrix} -\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0 \to \lambda(\lambda-1) = 0 \to$$

$\lambda = 0, 1$

$\lambda = 0$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \alpha \end{bmatrix} = 0 \rightarrow \alpha = 0 \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\lambda = 1$

$$\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ 1 \end{bmatrix} = 0 \rightarrow \alpha = 0 \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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Question 5. A 2×2 stochastic matrix has the form $S = \begin{pmatrix} a & b \\ 1-a & 1-b \end{pmatrix}$. For this to have an eigenvalue equal to 1, we must have

$$det \begin{pmatrix} a-1 & b \\ 1-a & -b \end{pmatrix} = 0,$$

which is obviously true. It is important because in studying Markov processes, we need to take large powers of S. If both eigenvalues were less than 1 in absolute value, then $S^n$ would tend to zero as n becomes large; if one or both were greater than 1 in absolute value, not all of the components of $S^n$ could remain bounded. Neither possibility makes sense, since the entries have to be interpreted as probabilities.

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Question 6. Eigenvectors are

$$\begin{pmatrix} 1 \\ -\frac{1}{16}(3\sqrt{33}-19) \\ -\frac{1}{8}(3\sqrt{33}-11) \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1}{16}(3\sqrt{33}+19) \\ \frac{1}{8}(3\sqrt{33}+11) \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Any vector with $T\vec{u} = 0$ is an eigenvector. There is only one eigenvector here with eigenvalue zero (the last one). So the rank is 2.

To show that $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ does not lie in that plane, just work out the triple product

$$\text{(first eigenvector)} \times \text{(second eigenvector)} \cdot \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

You will find that the answer is 2.154211... which is not zero as it would have to be if $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ lay in that plane.

If you repeat this process (for example, get WIMS to work out the determinant) you will find that $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ does lie in that plane i.e.

$$\text{(first eigenvector)} \times \text{(second eigenvector)} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 0.$$

There will be infinitely many solutions because if $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is a solution, so is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} + r \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ where $r$ is any number.