Limits

MA1505

Mathematics I

Chapter 1

Outline

1. Definition of Limits.
2. Basic Results on Limits
3. One-sided Limits

Limits

In the concept of ***limits***, we are interested in the behaviour of the values of $f(x)$ when x get closer and closer to some number a.

Important Remark

When we consider limits, the value of $f(x)$ when $x = a$ is not important. In fact, $f(a)$ need not be defined.

Example 1.

Let $f(x) = \dfrac{\sin x}{x}$.

Note : $f(0) = \dfrac{\sin 0}{0} = \dfrac{0}{0}$. So $f(0)$ is not defined.

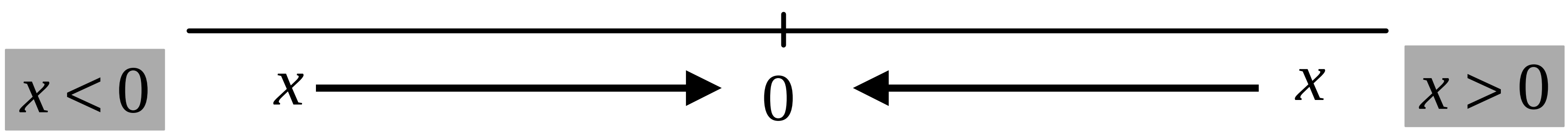
Domain of $f = \{x \in \mathbb{R} : x \neq 0\}$.

We shall now look at the behaviour of the values of $f(x)$ when x is close to 0.

Let $f(x) = \dfrac{\sin x}{x}$. Domain of $f = \{x \in \mathbb{R} : x \neq 0\}$.

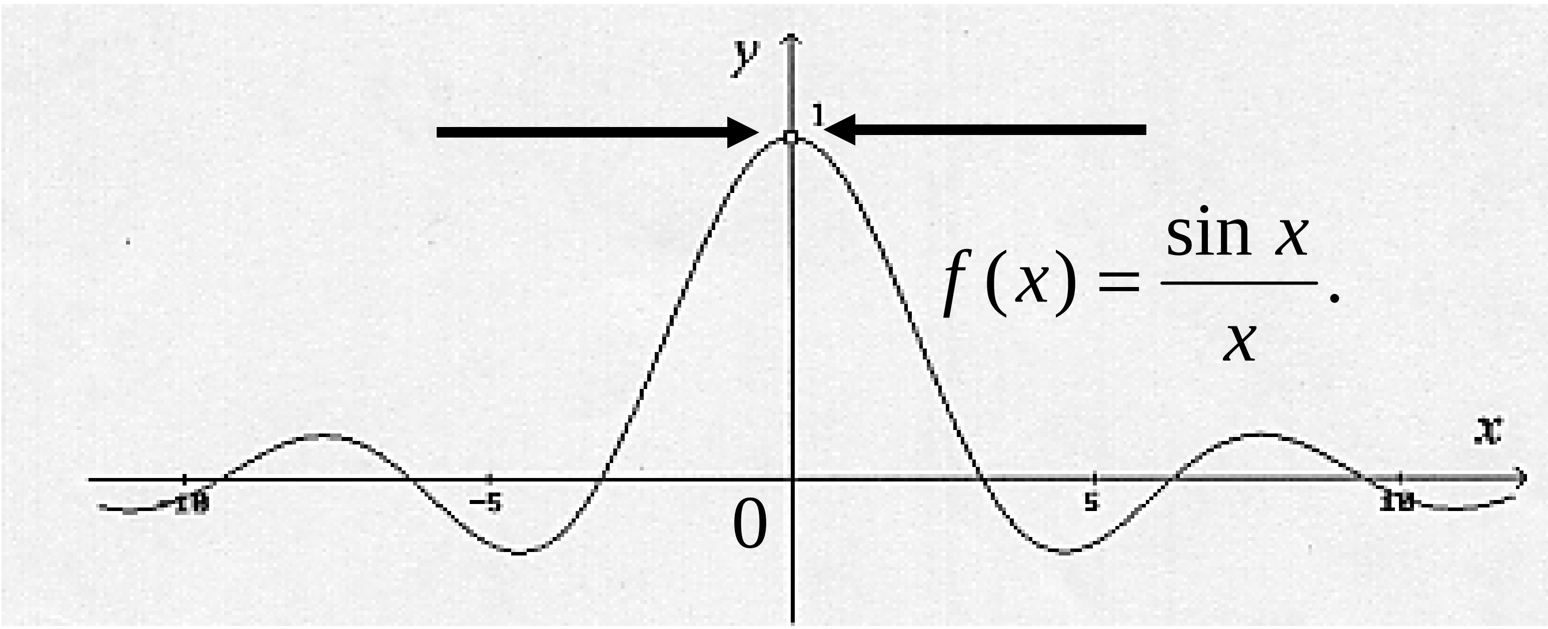
x	$\dfrac{\sin x}{x}$
0.01	0.999983333
0.001	0.999999833
0.0001	0.999999998

x	$\dfrac{\sin x}{x}$
-0.01	0.999983333
-0.001	0.999999833
-0.0001	0.999999998



So we see that when x get closer and closer to 0,

the value of $\dfrac{\sin x}{x}$ approaches 1.



If we plot the graph of $f(x) = \dfrac{\sin x}{x}$, we can also see from the graph that when x get closer and closer to 0 from either side, the value of $\dfrac{\sin x}{x}$ approaches 1.

So we know that the value of $f(x) = \dfrac{\sin x}{x}$ approaches 1 when x get closer and closer to 0.

In this case, we say that

"the limit of $f(x)$ as x tends to 0 is equal to 1."

We also use the following notation :

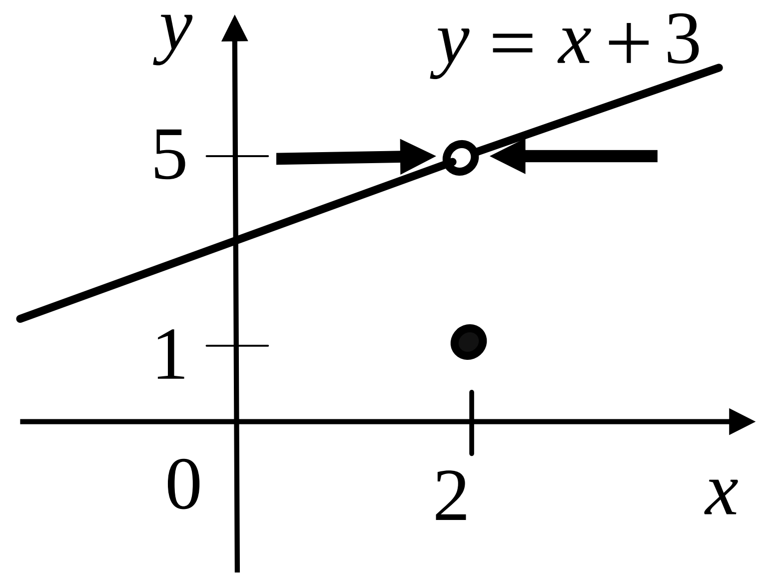
$$\lim_{x \to 0} f(x) = 1.$$

Note : in this case, $\lim_{x \to 0} f(x) = 1$ but $f(0)$ is undefined.

Example 2.

Let $f(x) = \begin{cases} x+3 & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

Describe the behaviour of $f(x)$ as x tends to 2.



From the graph of $f(x)$, we see that when x gets closer and closer to 2 (but not equal to 2), the value of $f(x)$ approaches 5.

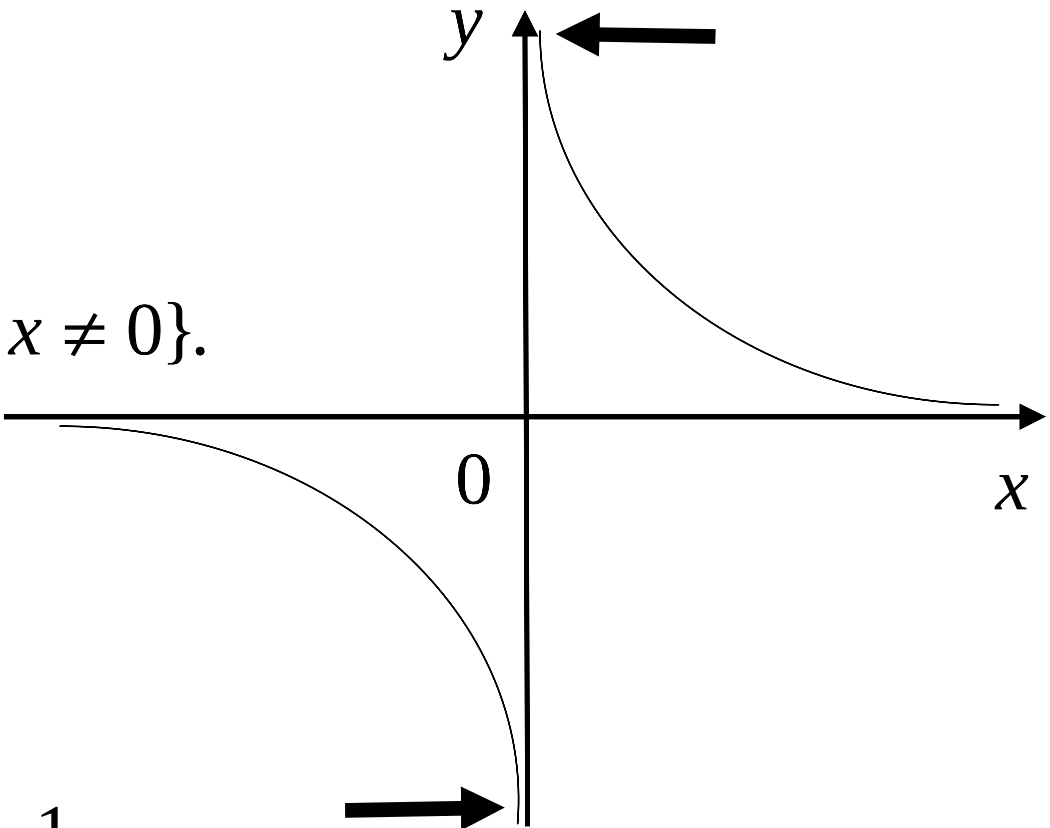
In notation we have: $\lim_{x \to 2} f(x) = 5.$

Note: the value of $f(2) = 1$ does not have an effect on the limit. Also $\lim_{x \to 2} f(x) \neq f(2)$.

Example 3.

Let $f(x) = \dfrac{1}{x}$.

Domain of $f = \{x \in \mathbb{R} : x \neq 0\}$.



From the graph of $f(x) = \dfrac{1}{x}$, we see that when x gets closer and closer to 0, the value of $f(x)$ does not approach to any value.

In this case, we say that $\lim_{x \to 0} f(x)$ does not exist.

Remarks

1. $\lim_{x \to a} f(x)$ may not exist.

2. If the limit exists, it is not affected by the number $f(a)$.

3. $\lim_{x \to a} f(x)$ need not be equal to $f(a)$.

In Example 2, we have $\lim_{x \to 2} f(x) = 5$ but $f(2) = 1$, so $\lim_{x \to 2} f(x) \neq f(2)$.

Remarks

1. $\lim_{x \to a} f(x)$ may not exist.

2. If the limit exists, it is not affected by the number $f(a)$.

3. $\lim_{x \to a} f(x)$ need not be equal to $f(a)$.

4. $f(a)$ need not be defined.

In Example 1, we have $f(x) = \dfrac{\sin x}{x}$

$\lim_{x \to 0} f(x) = 1$ but $f(0)$ is not defined.

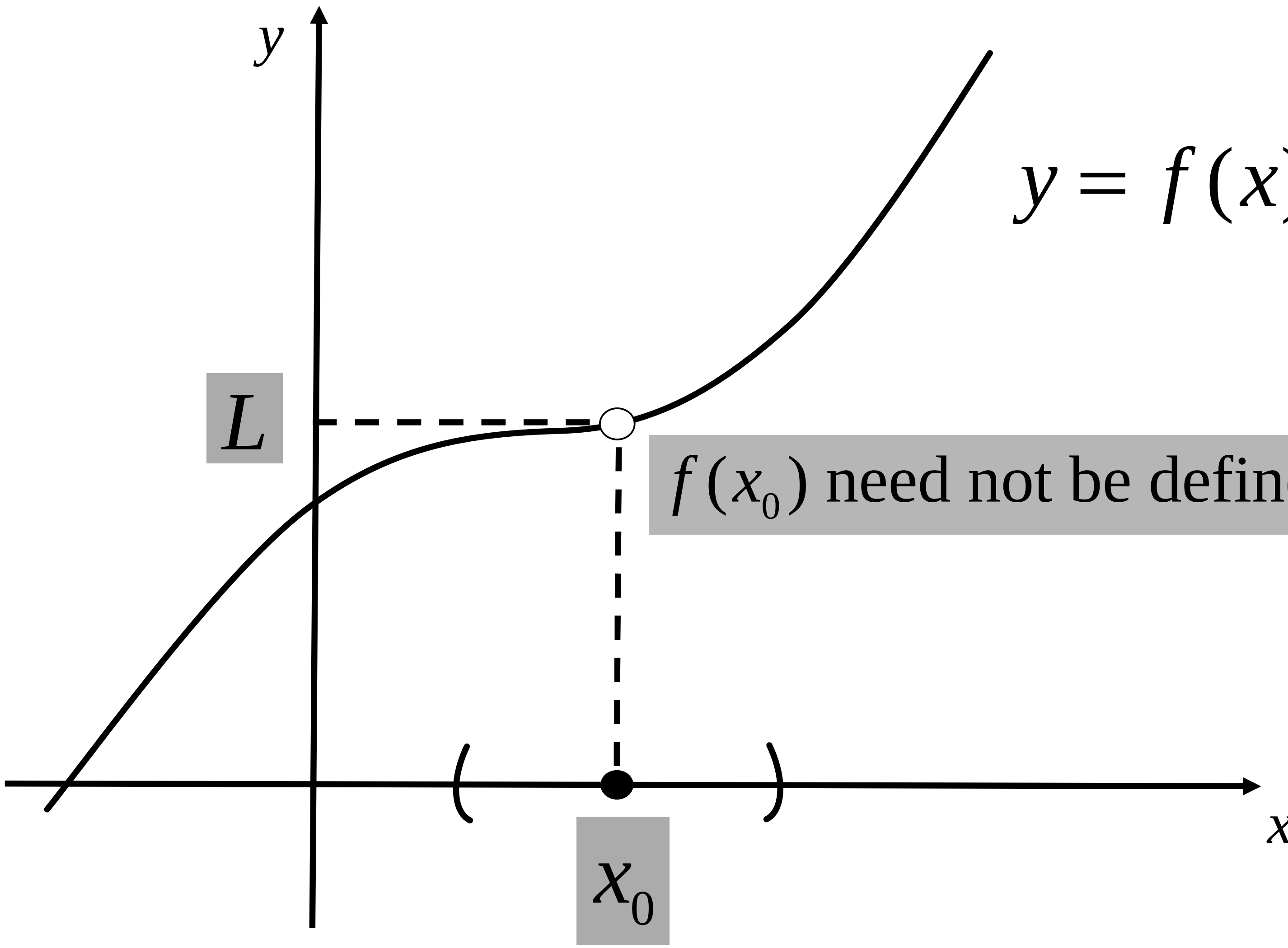
Limits (An Informal Definition).

Let $f(x)$ be defined on an open interval containing x_0, except possibly at x_0 itself. If $f(x)$ gets arbitrary close to L when x is sufficiently close to x_0, then we say that the limit of $f(x)$ as x tends to x_0 is the number L and we write

$$\lim_{x \to x_0} f(x) = L,$$

which is read "the limit of $f(x)$ as x approaches x_0 is L."

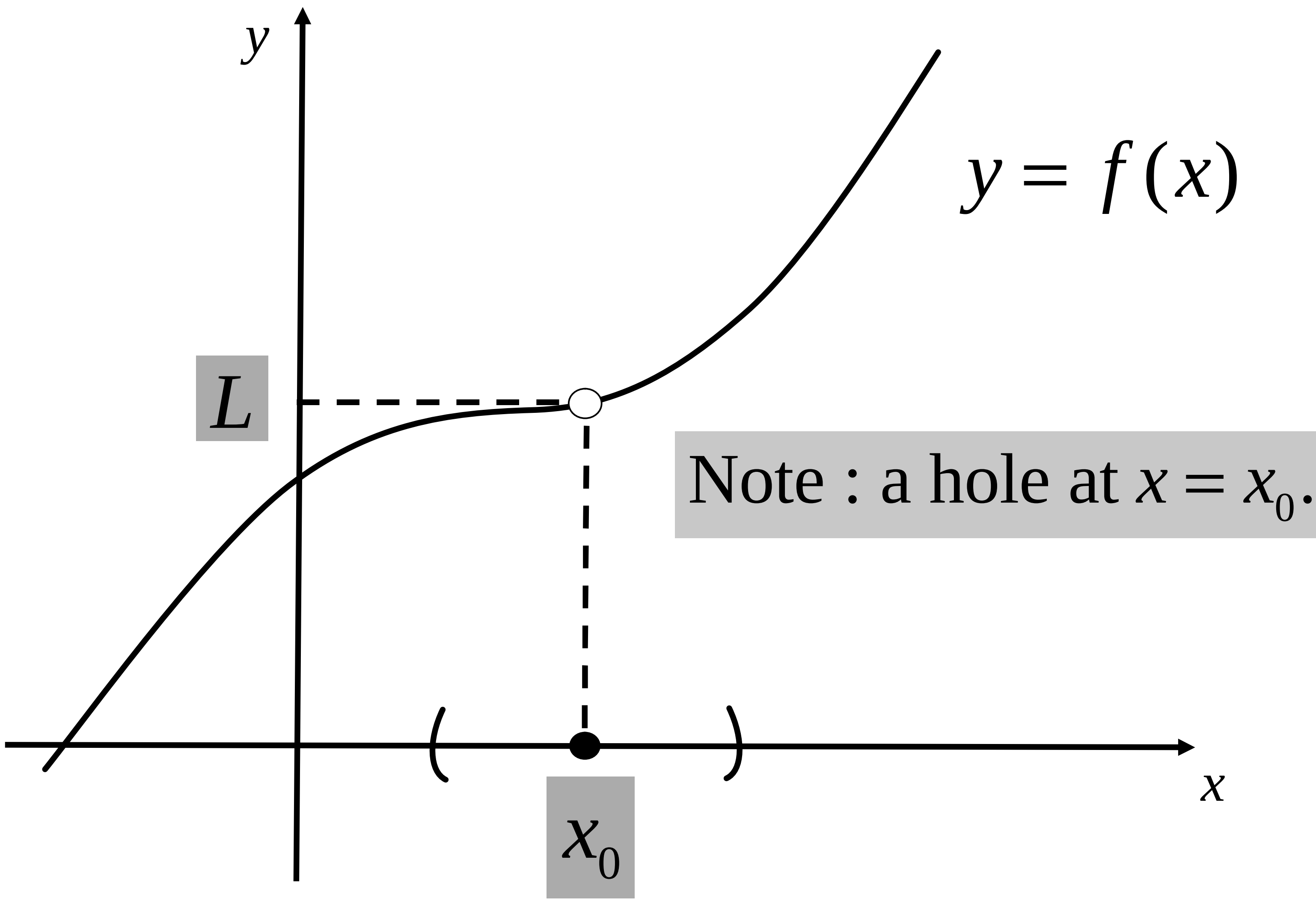
$$\lim_{x\to x_0} f(x) = L$$



What happen at $x = x_0$ is not important.

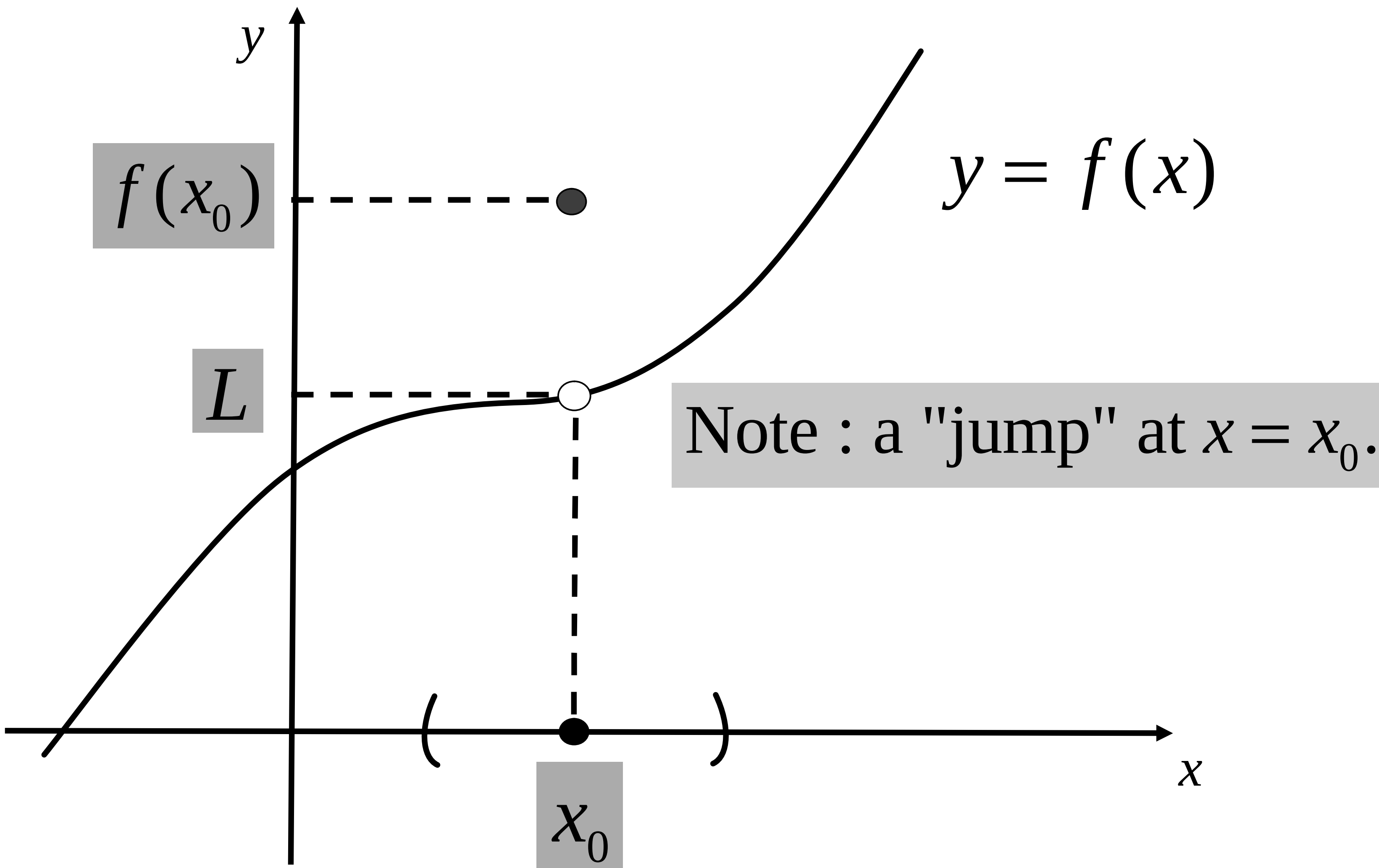
$$\lim_{x \to x_0} f(x) = L$$

Case (1). $f(x_0)$ not defined.



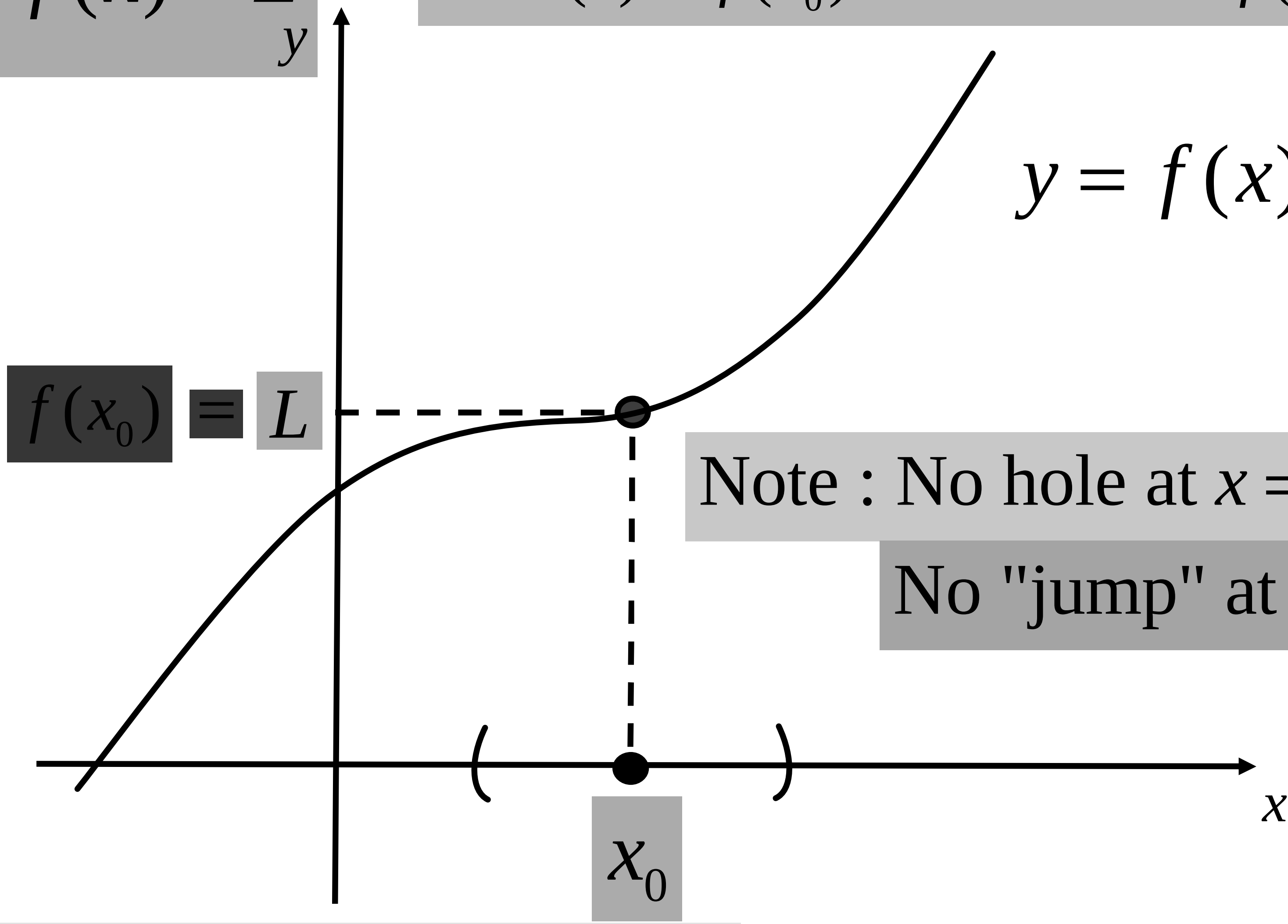
$$\lim_{x \to x_0} f(x) = L$$

Case (2). $f(x_0)$ is defined and $f(x_0) \neq$ L.



$\lim_{x \to x_0} f(x) = L$

Case (3). $f(x_0)$ is defined and $f(x_0) = \text{L}$.



Note : No hole at $x = x_0$.

No "jump" at $x = x_0$.

Note : $\lim_{x \to x_0} f(x) = L = f(x_0)$

We say that $f(x)$ is continuous at $x = x_0$.

Rules of Limits

Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = L'$. Then

(i) $\lim_{x\to a}(f+g)(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L + L'$;

(ii) $\lim_{x\to a}(f-g)(x) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L - L'$;

(iii) $\lim_{x\to a}(fg)(x) = \lim_{x\to a} f(x) \lim_{x\to a} g(x) = LL'$;

(iv) $\lim_{x\to a}(\frac{f}{g})(x) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{L'}$ provided $L' \neq 0$;

(v) $\lim_{x\to a} kf(x) = k\lim_{x\to a} f(x) = kL$ for any real number k.

Suppose $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = L'$. Then

(iv) $\lim_{x \to a} (\frac{f}{g})(x) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{L'}$ provided $L' \neq 0$;

Remarks

1. If $L \neq 0$ and $L' = 0$, then $\lim_{x \to a} \frac{f}{g}(x)$ does not exist.

2. If $L = L' = 0$, then $\lim_{x \to a} \frac{f}{g}(x)$ may or may not exist.

We will be dealing with such limits problems in the next chapter (L'Hospital Rule).

Example. Evaluate $\lim_{x \to -1} \frac{\sqrt{x^2+8}-3}{x+1}$.

When cannot substitute x by -1 into the function since the denomintor would be zero.

Example. Evaluate $\lim_{x \to -1} \frac{\sqrt{x^2+8}-3}{x+1}$.

In order to find the limit, we multiply the numerator and denominator by the *conjugate* expression $\sqrt{x^2+8}+3$.

$$\frac{\sqrt{x^2+8}-3}{x+1} \times \frac{\sqrt{x^2+8}+3}{\sqrt{x^2+8}+3} = \frac{(x-1)(x+1)}{(x+1)(\sqrt{x^2+8}+3)}$$

$$= \frac{(x-1)}{\sqrt{x^2+8}+3)}$$

$$(A-B)(A+B) = A^2 - B^2$$

$$(\sqrt{x^2+8}-3)(\sqrt{x^2+8}+3) = (x^2+8)-9$$

$$= x^2-1$$

$$= (x-1)(x+1)$$

Example. $\boxed{\text{Evaluate } \lim_{x \to -1} \frac{\sqrt{x^2+8}-3}{x+1}.}$

In order to find the limit, we multiply the numerator and denominator by the *conjugate* expression $\sqrt{x^2+8}+3$.

$$\frac{\sqrt{x^2+8}-3}{x+1} \times \boxed{\frac{\sqrt{x^2+8}+3}{\sqrt{x^2+8}+3}} = \frac{(x-1)(x+1)}{(x+1)(\sqrt{x^2+8}+3)}$$

$$= \frac{(x-1)}{\sqrt{x^2+8}+3)}$$

Therefore $\displaystyle \lim_{x \to -1} \frac{\sqrt{x^2+8}-3}{x+1} = \lim_{x \to -1} \frac{x-1}{\sqrt{x^2+8}+3}.$

$$= \frac{-1-1}{\sqrt{(-1)^2+8}+3)} = -\frac{1}{3}$$

Pause and Think !!!

Evaluate $\lim\limits_{x \to 6} \dfrac{4-\sqrt{x+10}}{x^2-36}$.

$\text{Ans}: -\dfrac{1}{96}$

Question :

How to decide when $\lim_{x \to a} f(x)$ exists ?

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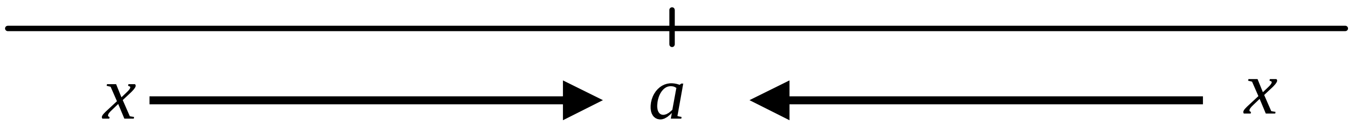
How to decide when $\lim_{x \to a} f(x)$ exists ?

<u>Theorem (to decide when limit exists)</u>

$\lim_{x \to a} f(x)$ exists if and only if $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ both exist and are equal.

One-sided Limits

When x tends to a, x can be tending to a from the left side or right side.



$x < a$ from left side $\quad$ $x > a$ from right side

If we consider x tending to a from one side, we obtain the *one-sided* limits.

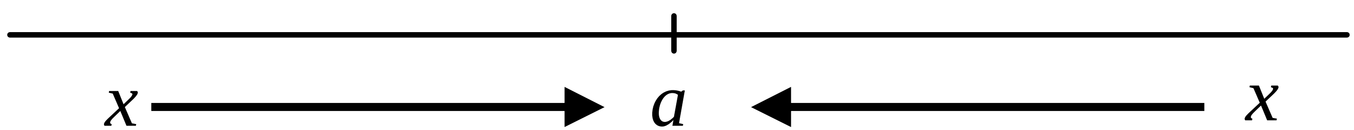
If $x \to a$ and $x > a$, we write $x \to a^+$ and call the limit the *right-hand* limits.

If $x \to a$ and $x < a$, we write $x \to a^-$ and call the limit the *left-hand* limits.

Limits (Definition).

Let $f(x)$ be defined on an open interval containing x_0, except possibly at a itself. If $f(x)$ gets arbitrary close to L when x is sufficiently close to a, then we say that the limit of $f(x)$ as x tends to a is the number L and we write

$$\lim_{x\to a} f(x) = L.$$



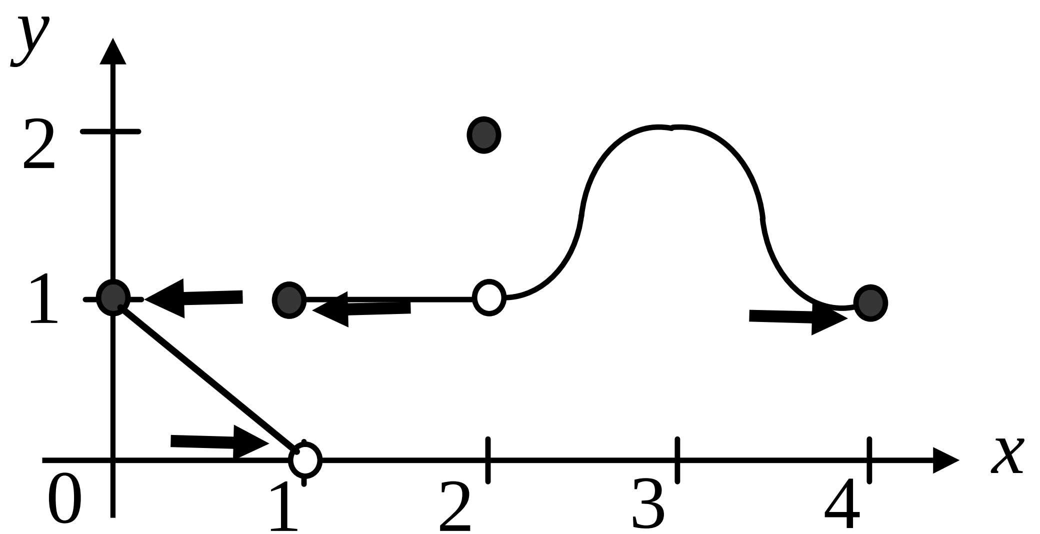
$x < a$ from left side $\qquad$ $x > a$ from right side

Theorem (to decide when limit exists)

$\lim_{x\to a} f(x)$ exists if and only if $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ both exist and are equal.

Example.

The graph of the function $f:[0,4]\to\mathbb{R}$ is given below :



(i) $\lim_{x\to 0^+} f(x)=1$

$\lim_{x\to 4^-} f(x)=1$

(ii) $\lim_{x\to 1^-} f(x)=0$

$\lim_{x\to 1^+} f(x)=1$

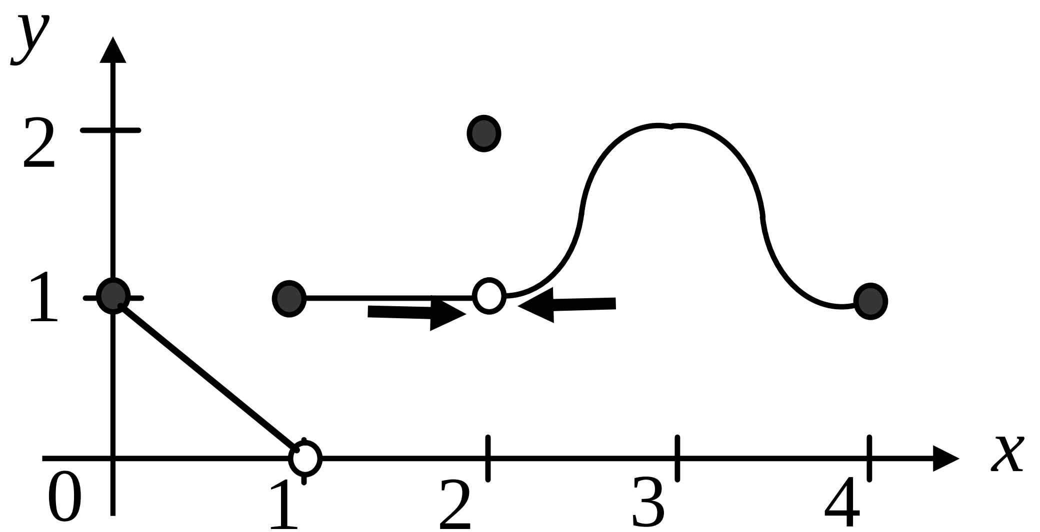
Since $\lim_{x\to 1^-} f(x)\neq \lim_{x\to 1^+} f(x)$, we conclude that

$\lim_{x\to 1} f(x)$ does not exist.

Recall that $\lim_{x\to a} f(x)$ exists if and only if $\lim_{x\to a^+} f(x)=\lim_{x\to a^-} f(x)$.

The graph of the function $f : [0, 4] \to \mathbb{R}$ is given below :

(iii) $\lim_{x \to 2^-} f(x) = 1$



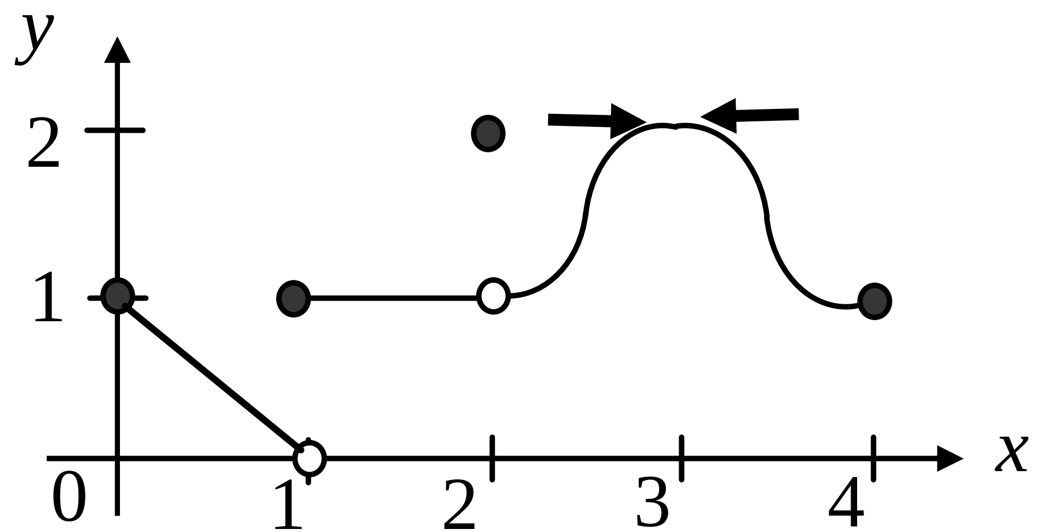
$\lim_{x \to 2^+} f(x) = 1$

Since $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = 1$, we conclude that

$\lim_{x \to 2} f(x) = 1.$

Recall that $\lim_{x \to a} f(x)$ exists if and only if $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$.

The graph of the function $f:[0,4] \to \mathbb{R}$ is given below :



(iv) $\lim\limits_{x \to 3^-} f(x) = 2$

$\lim\limits_{x \to 3^+} f(x) = 2$

Since $\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^+} f(x) = 2$, we conclude that

$\lim\limits_{x \to 3} f(x) = 2.$

Recall that $\lim\limits_{x \to a} f(x)$ exists if and only if $\lim\limits_{x \to a^+} f(x) = \lim\limits_{x \to a^-} f(x)$.

Limits involving infinity

Sometimes, we would like to talk about the behaviour of a function as the x value gets larger and larger without bound.

The notation for this is $\lim_{x \to \infty} f(x)$, and is read

"the limit of $f(x)$ as x tends to infinity."

The symbol $x \to \infty$ means that x gets larger and larger without bound.

Note that ∞ is NOT a real number, it is a symbol which is used in the situation where a quantity increases without bound. Do not write $x = \infty$.

Limits involving infinity

Similarly, we can talk about the behaviour of $f(x)$ as x decreases without bound.

The notation for this is $\lim_{x \to -\infty} f(x)$, and is read

"the limit of $f(x)$ as x tends to negative infinity."

The symbol $x \to -\infty$ means that x decreases without bound.

Example.

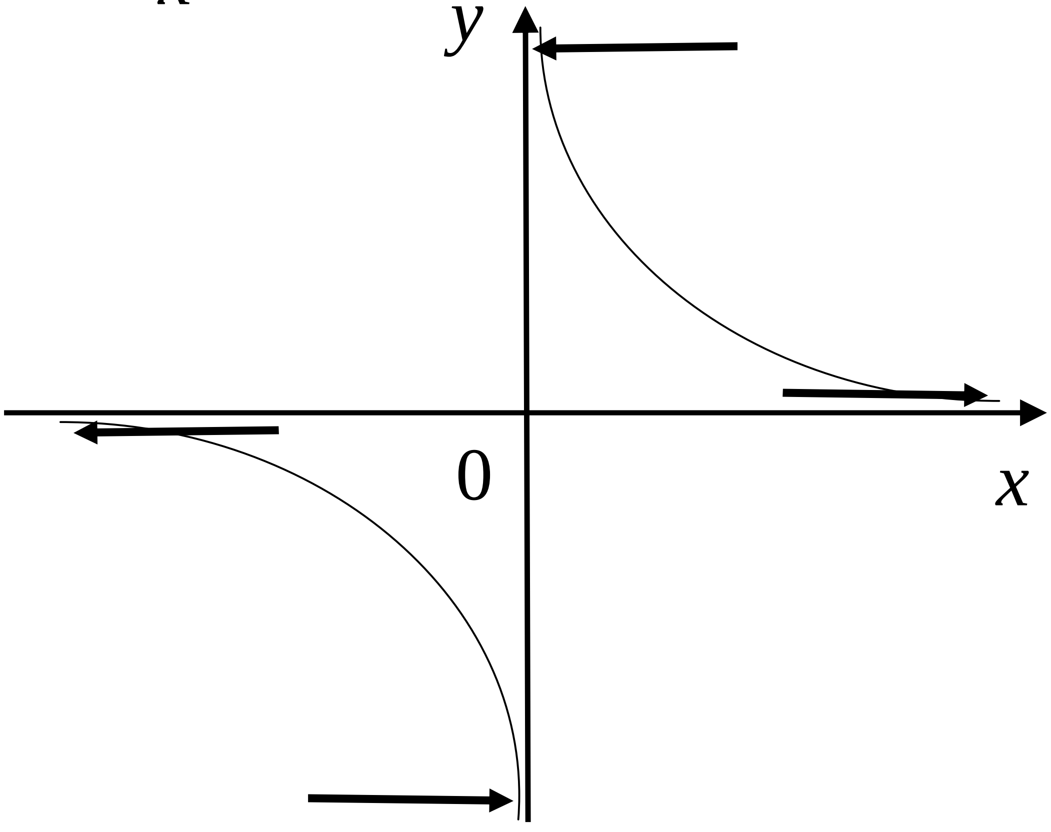
Consider the function $f(x)=\dfrac{1}{x}$ where $x \neq 0$.

(i) $\lim\limits_{x\to\infty} f(x) = 0$

(ii) $\lim\limits_{x\to-\infty} f(x) = 0$

(iii) $\lim\limits_{x\to0^+} f(x) = \infty$

(iv) $\lim\limits_{x\to0^-} f(x) = -\infty$



Some Important Remarks.

The following are a few common situations where $\lim\limits_{x \to a} f(x)$ does not exist.

(i) There is a "jump" in the graph of $f(x)$ at $x = a$ so that

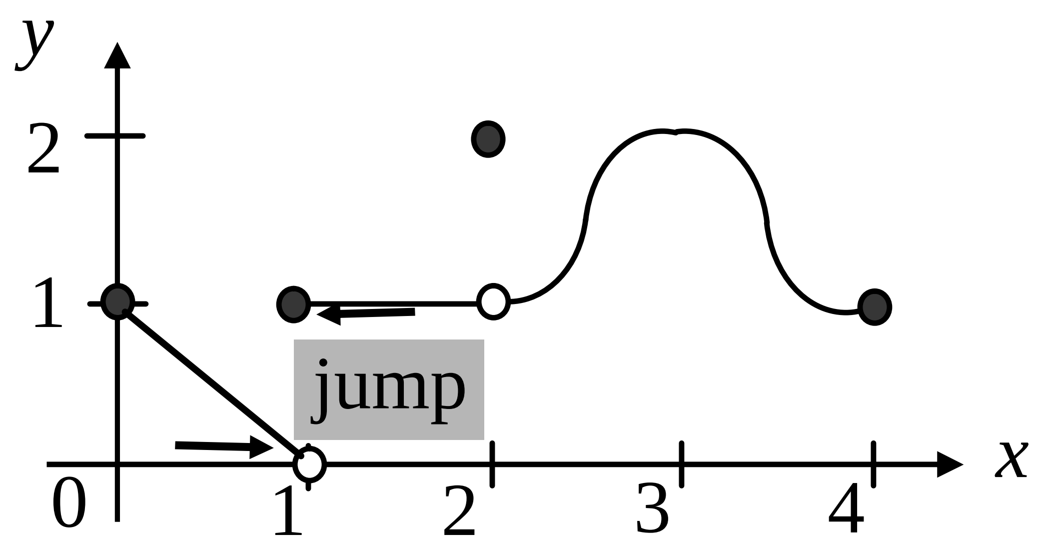
$$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x).$$

$$\lim_{x \to 1^-} f(x) = 0$$

$$\lim_{x \to 1^+} f(x) = 1$$

So $\lim\limits_{x \to 1^-} f(x) \neq \lim\limits_{x \to 1^+} f(x)$.

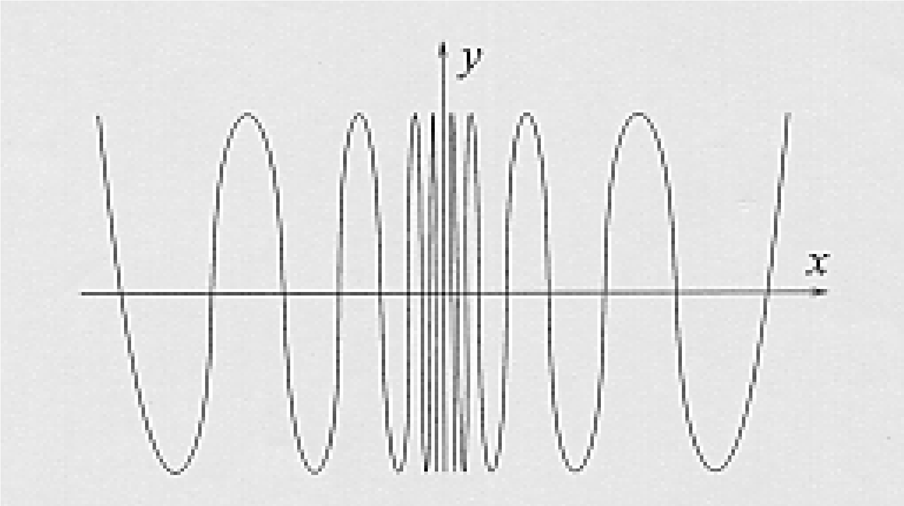
Thus $\lim\limits_{x \to 1} f(x)$ does not exist.



Some Important Remarks.

(ii) The graph of $f(x)$ *fluctuates* when x approaches a.

The graph of $f(x) = \sin(\frac{1}{x})$ is given below.



$$-1 \le \sin(\frac{1}{x}) \le 1$$

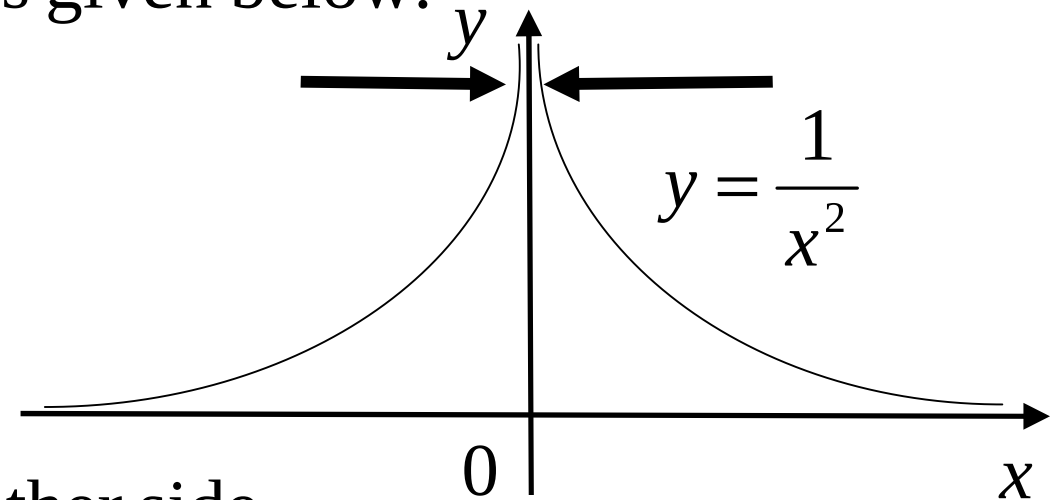
Note that the graph of $f(x)$ fluctuates when x approaches 0.

Thus $\lim_{x \to 0} f(x)$ does not exist.

Some Important Remarks.

(iii) The function $f(x)$ simply increases without bound as x approaches a. In this case, we write $\lim_{x \to a} f(x) = \infty$.

The graph of $f(x) = \dfrac{1}{x^2}$ is given below.



Note that as $x \to 0$ from either side, the value of $f(x)$ increases without bound.

Thus $\lim_{x \to 0} \dfrac{1}{x^2} = \infty$.

Some Important Remarks.

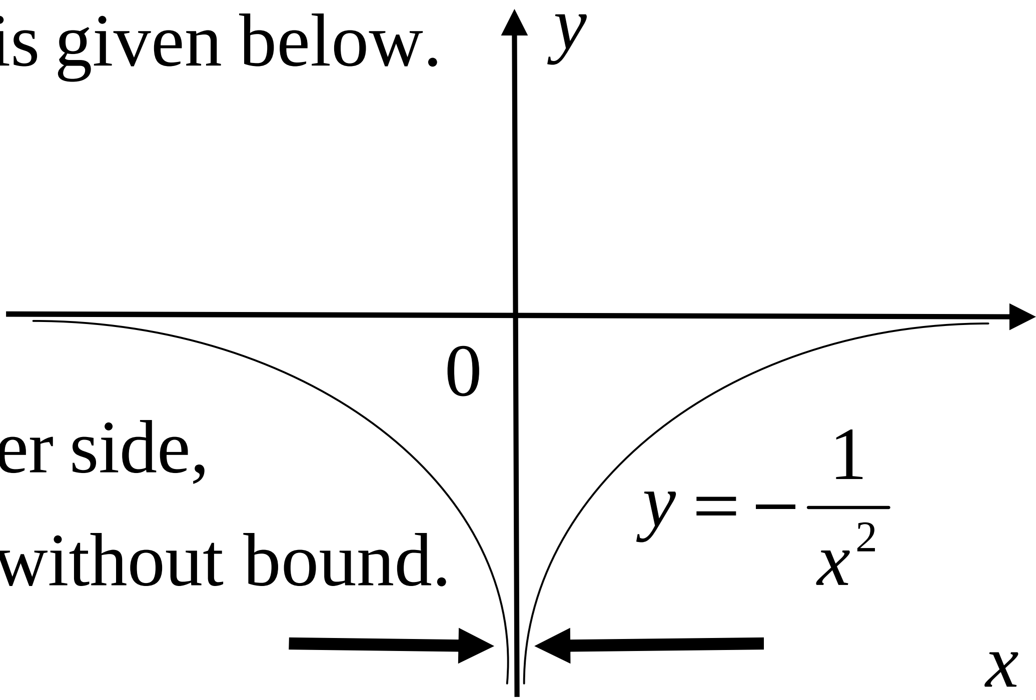
(iii) The function $f(x)$ simply increases without bound as x approaches a. In this case, we write $\lim_{x \to a} f(x) = \infty$.

Note that $\lim_{x \to a} f(x) = \infty$ means that the limit of f as x approaches a DOES NOT EXIST and it does not exist because of the following specific reason: f increases without bound (and thus does not approach any value) as x tends to a.

Some Important Remarks.

(iv) The function $f(x)$ decreases without bound as x approaches a. In this case, we write $\lim_{x \to a} f(x) = -\infty$.

The graph of $f(x) = -\dfrac{1}{x^2}$ is given below.

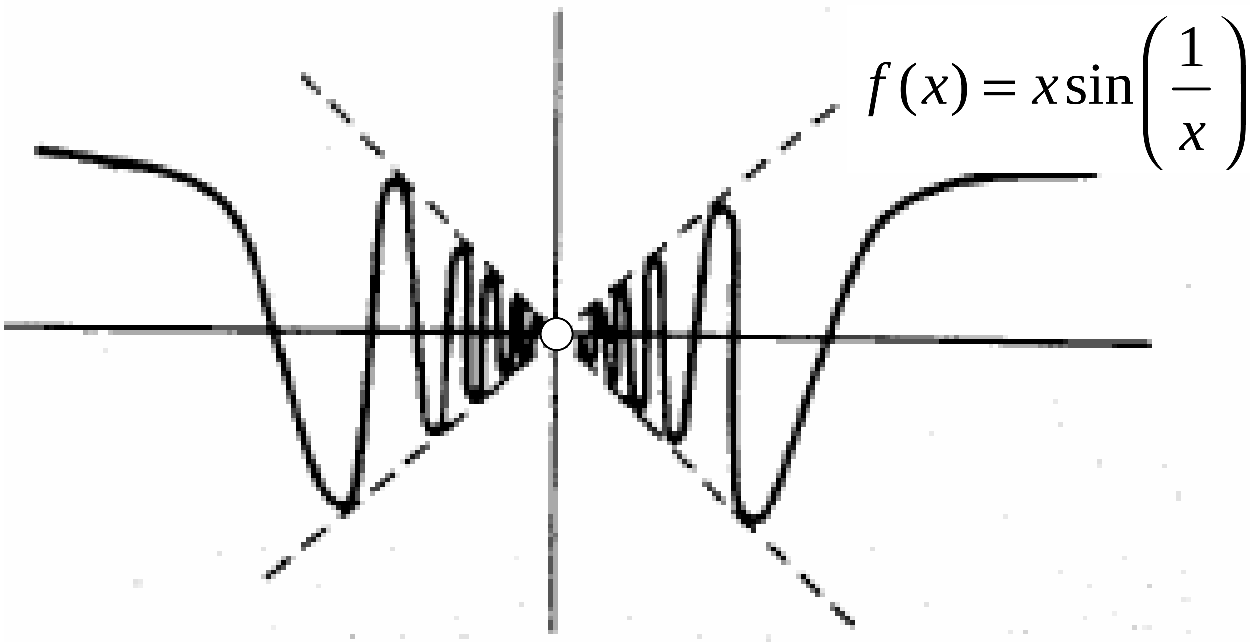


Note that as $x \to 0$ from either side, the value of $f(x)$ decreases without bound.

Thus $\lim_{x \to 0} -\dfrac{1}{x^2} = -\infty$.

Pause and Think !!!

- $f(x) = x\sin(\frac{1}{x}) \quad (x \neq 0)$

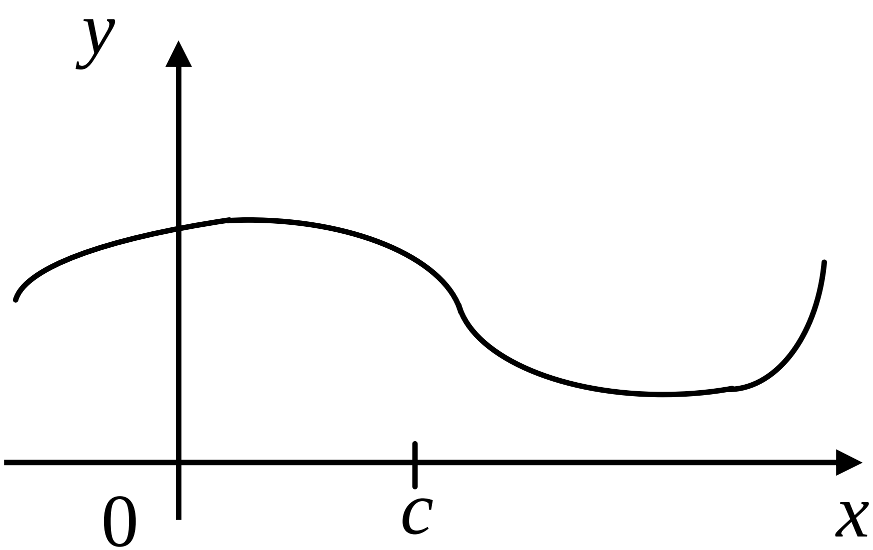


$$-1 \leq \sin(\frac{1}{x}) \leq 1$$

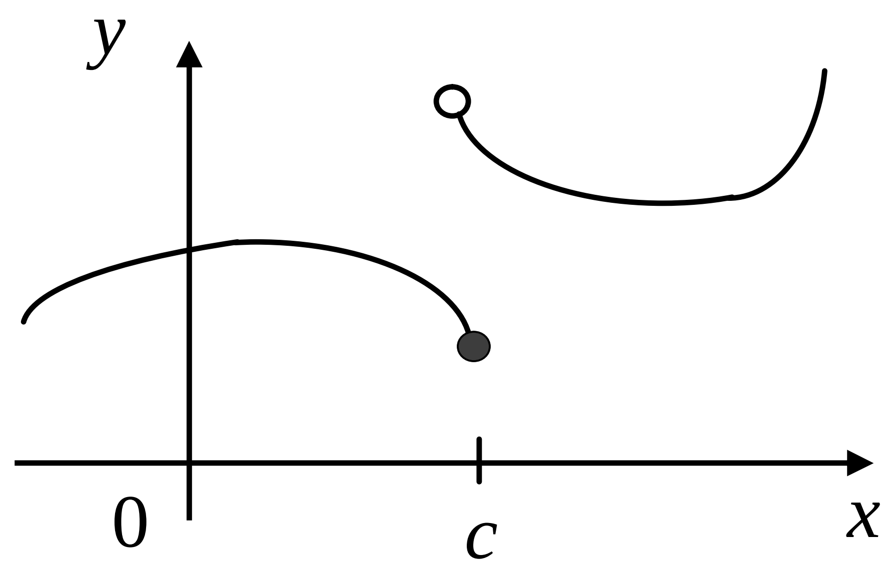
The value of $\lim_{x \to 0} f(x) =$?

Continuity

Intuitively, a function is continuous if we can draw its graph "in one stroke", or "without lifting up the pen from the paper".



Continuous at $x = c$.



NOT Continuous at $x = c$.

A function $f(x)$ is *continuous*

at a point c if $\lim_{x \to c} f(x) = f(c)$.

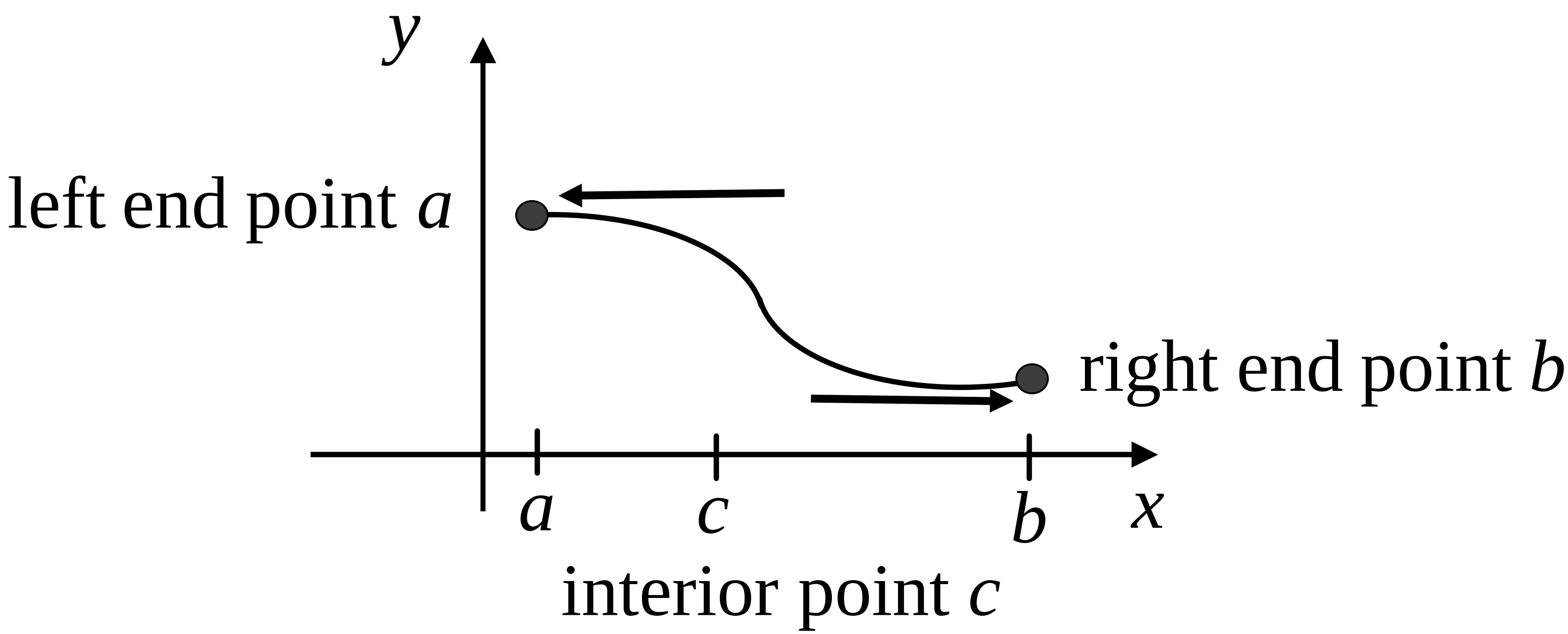
Definition (Continuity).

A function $f(x)$ is *continuous*

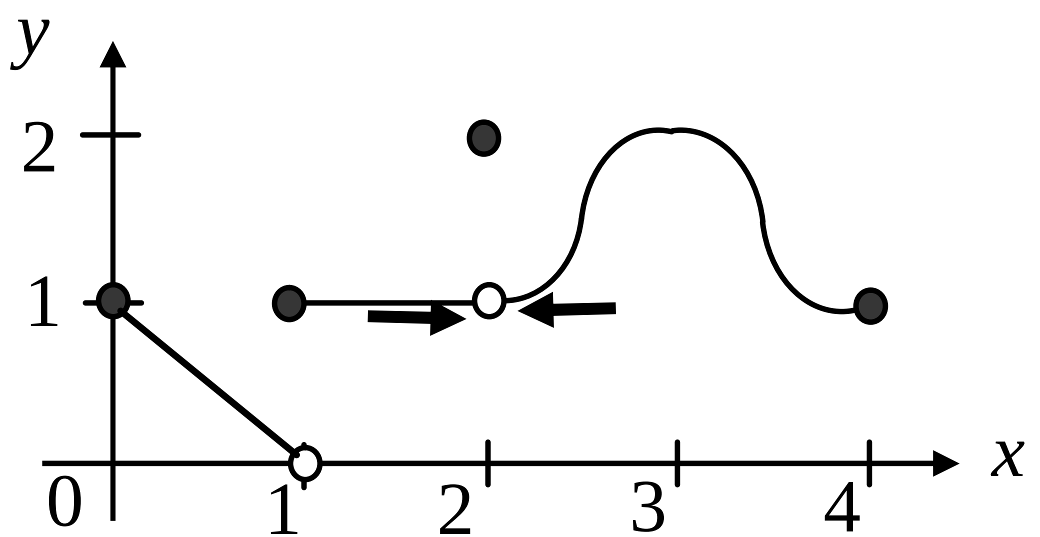
(i) at an interior point c if $\lim_{x \to c} f(x) = f(c)$.

(ii) at a left end point a if $\lim_{x \to a^+} f(x) = f(a)$.

(iii) at a right end point b if $\lim_{x \to b^-} f(x) = f(b)$.



The graph of the function $f:[0,4] \to \mathbb{R}$ is given below :



(iii) $\lim_{x \to 2^-} f(x) = 1$

$\lim_{x \to 2^+} f(x) = 1$

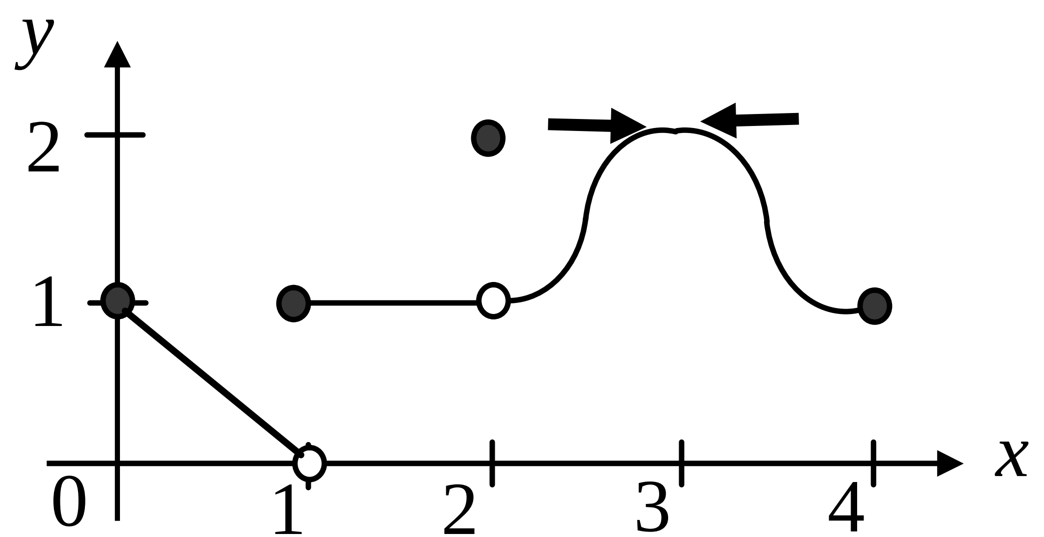
Since $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = 1$, we conclude that

$\lim_{x \to 2} f(x) = 1.$

Note that $\lim_{x \to 2} f(x) = 1 \neq f(2) = 2.$

Therefore $f(x)$ is not continuous at $x = 2.$

The graph of the function $f:[0,4]\to\mathbb{R}$ is given below :



(iv) $\lim_{x\to 3^-} f(x)=2$

$\lim_{x\to 3^+} f(x)=2$

Since $\lim_{x\to 3^-} f(x)=\lim_{x\to 3^+} f(x)=2$, we conclude that

$\lim_{x\to 3} f(x)=2.$

Note that $\lim_{x\to 3} f(x)=2 \;=\; f(3)=2.$

Therefore $f(x)$ is continuous at $x=3.$

The Continuity Test.

To test whether a function f is continuous at a point p,

we need to checking the following 3 things :

(1) check that p is in the domain of f ($f(p)$ is defined),

(2) check that $\lim_{x \to p} f(x)$ exists (or $\lim_{x \to p^+} f(x)$ or $\lim_{x \to p^-} f(x)$).

(3) check that $\lim_{x \to p} f(x)$ (or $\lim_{x \to p^+} f(x)$ or $\lim_{x \to p^-} f(x)$)

is equal to $f(p)$.

A function $f(x)$ is *continuous*

(i) at an interior point p if $\lim_{x \to p} f(x) = f(p)$.

(ii) at a left end point p if $\lim_{x \to p^+} f(x) = f(p)$.

(iii) at a right end point p if $\lim_{x \to p^-} f(x) = f(p)$.

The Continuity Test.

To test whether a function f is continuous at a point p,

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(1) check that p is in the domain of f ($f(p)$ is defined),

(2) check that $\lim_{x\to p} f(x)$ exists (or $\lim_{x\to p^+} f(x)$ or $\lim_{x\to p^-} f(x)$).

(3) check that $\lim_{x\to p} f(x)$ (or $\lim_{x\to p^+} f(x)$ or $\lim_{x\to p^-} f(x)$)

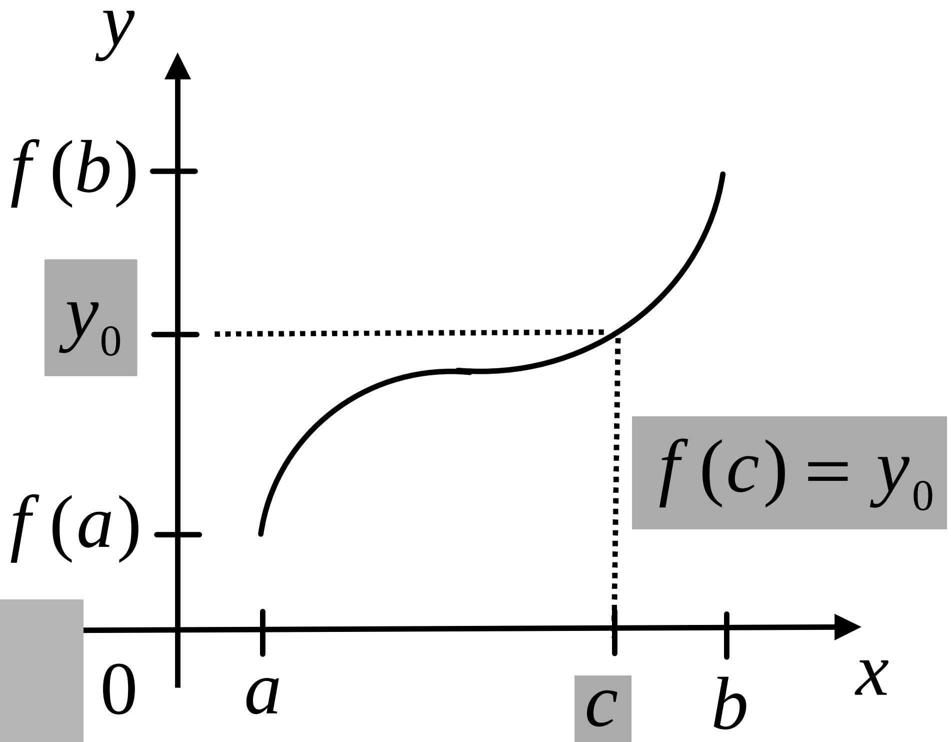
is equal to $f(p)$.

If (1), (2) or (3) fails to hold,

then f is not continuous at $x = p$.

Intermediate Value Theorem.

A function $f(x)$ which is continuous on a closed interval $[a,b]$, takes on every value between $f(a)$ and $f(b)$, that is, if $f(a) \leq y_0 \leq f(b)$, then there exists $c \in [a,b]$ such that $f(c) = y_0$.

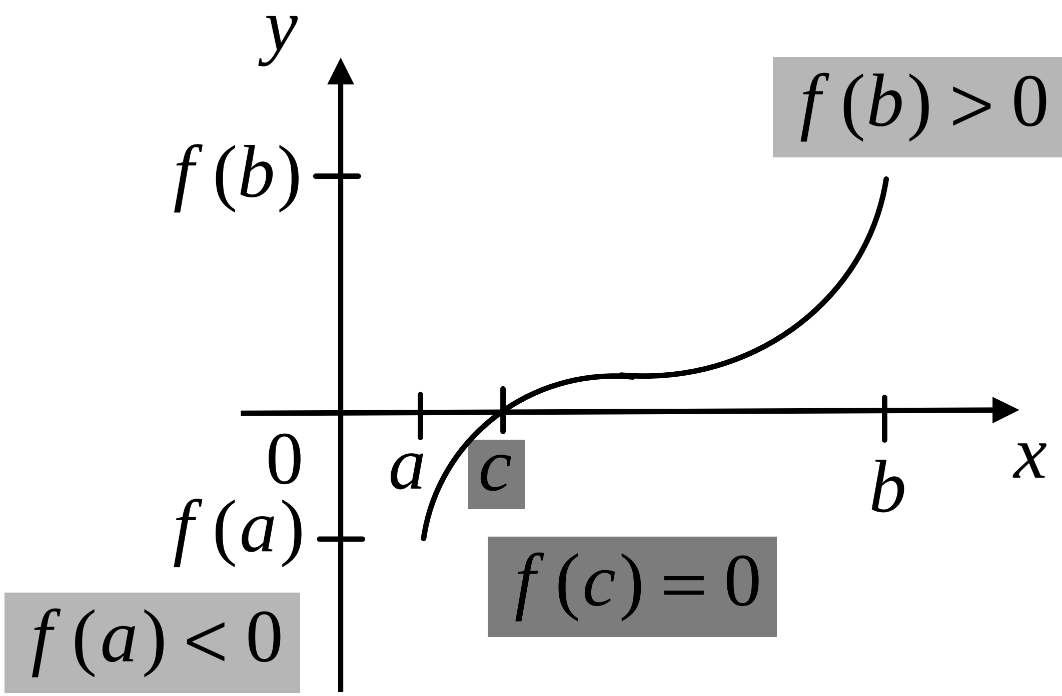


Sometime we write the Intermediate Value Theorem as IVT for short.

The following theorem is an immediate consequence of the Intermediate Value Theorem by setting $y_0 = 0$.

Theorem.

Let $f(x)$ be a continuous function on a closed interval $[a, b]$. Suppose $f(a) < 0$ and $f(b) > 0$ (or $f(a) > 0$ and $f(b) < 0$). Then the equation $f(x) = 0$ has a solution in $[a, b]$.



Example.

Use the IVT to show that

$$f(x) = x^{11} + 3x - 1$$

has a root between 0 and 1.

Note: $f(x) = x^{11} + 3x - 1$ is a continuous function on $[0,1]$.

$$f(0) = -1 < 0. \qquad f(1) = 1 + 3 - 1 = 3 > 0.$$

By the IVT, the equation $f(x) = 0$ has a solution in $[0,1]$.

Hence, $f(x) = x^{11} + 3x - 1$ has a real root between 0 and 1.

End