Lecture 4

Energy storage elements : Inductance and Capacitance

Resistors convert electrical energy into heat and/or light. Whenever current flows through a resistor, electrical energy is dissipated in it (converted to other form of energy).

Unlike resistors, capacitors and inductors can receive, store and return back electrical energy. Capacitance is associated with energy stored in electric field (charge at rest – more appropriately charge separation) and inductance is associated with energy stored in magnetic field (charge in motion / current). Capacitance and inductance are employed in designing transducers to measure other physical quantities. Filters made up of inductance and capacitors are used to remove unwanted noise from measurement signals.

However, these elements do not generate energy and hence are still known as passive elements. These also are linear elements in the sense that current and voltage for these elements hold a linear relationship, albeit of integral-differential nature.

Capacitance

Capacitors are constructed by separating two sheets of conductors by a thin layer of insulating material. The insulating material is called dielectric and it can be air, paper, Mylar, polyester etc.

When a DC voltage source is connected across a capacitor, then electrons get repelled from the negative terminal of the source towards the capacitor, but stop at the insulating material (as they cannot pass through). Similarly, electrons from the other side of the capacitor get attracted to the positive terminal of the voltage source. Thus, the two plates of the capacitor get charged oppositely. An electric field is developed inside the insulating material, from positively charged plate towards the negatively charged plate. A voltage develops across the capacitor which equals the voltage source at steady state.

In ideal capacitors, the charge developed in each plate (charge stored) is proportional to the voltage across it.

$$Q = CV$$

where C, the proportionality constant is called the capacitance. Its unit is farad (F). One farad is equivalent to coulomb per volt. One farad is a large value and practical capacitors have values only a few pico farad (10^{-12}) to a few micro farad (10^{-6}).

Capacitor current in terms of voltage

Current is the rate of flow of charge. When charge is accumulating on the capacitor plates, the rate of this accumulation can be obtained as:

$$i(t) = \frac{dq(t)}{dt} = \frac{d}{dt}\left(Cv(t)\right) = C\frac{dv(t)}{dt}$$

$i(t) = C\dfrac{dv(t)}{dt}$ is i-v relation for a capacitor.

From the equation above, a capacitor current is present only when charge is building up. At steady state, when the voltage is stable, the capacitor current will be zero. Hence, capacitors in DC circuits behave as an open circuit in steady state.

Capacitor voltage in terms of current

Rearranging the above equation we get the rate of change of voltage in terms of the capacitor current.

$$\frac{dv(t)}{dt} = \frac{1}{C}i(t)$$

$$v(t) - v(t_0) = \frac{1}{C}\int_{t_0}^{t} i(t)dt$$

$$v(t) = \frac{q_0}{C} + \frac{1}{C}\int_{t_0}^{t} i(t)dt$$

The initial charge on the capacitor is taken as q_0.

Stored Energy

The power delivered to the capacitor is the product of the capacitor voltage and current.

$$p(t) = v(t)i(t) = v(t)C\frac{dv(t)}{dt} = Cv(t)\frac{dv(t)}{dt}$$

The energy supplied to the capacitor will be integral of the power over time.

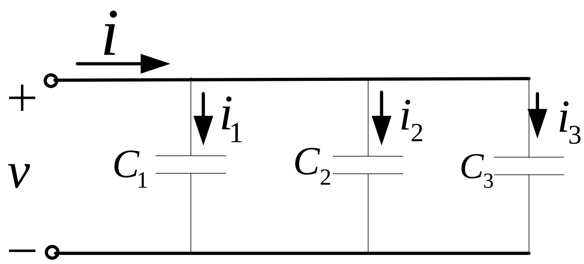
$$e = \int_{t_0}^{t} p(t)dt = \int_{t_0}^{t} Cv(t)\frac{dv}{dt}dt = \int_{v_0}^{v} Cvdv = \frac{1}{2}C\left(v^2 - v_0^2\right)$$

If the capacitor is initially discharged $v_0 = 0$, $e = \dfrac{1}{2}Cv^2$.

Capacitances in series and parallel

Capacitors in parallel add. Capacitors in series combine according to the same rule as for the resistors in parallel.

When capacitors are in parallel, then the voltage across all of them will be same.



$$v_1(t) = v_2(t) = v_3(t) = v(t)$$

The current in each capacitor will be dependent on the value of each capacitor.

$$i_1(t) = C_1 \frac{dv(t)}{dt}$$

$$i_2(t) = C_2 \frac{dv(t)}{dt}$$

$$i_3(t) = C_3 \frac{dv(t)}{dt}$$

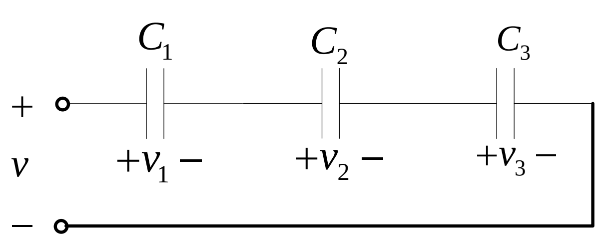
From KCL, the total current entering into all the capacitors will be the sum of individual currents entering into the capacitors.

$$i(t) = i_1(t) + i_2(t) + i_3(t) = C_1 \frac{dv(t)}{dt} + C_2 \frac{dv(t)}{dt} + C_3 \frac{dv(t)}{dt} = \left(C_1 + C_2 + C_3\right) \frac{dv(t)}{dt} = C_{eq} \frac{dv(t)}{dt}$$

The equivalent capacitance of the capacitors in parallel would be sum of individual capacitances:

$$C_{eq} = C_1 + C_2 + C_3$$

When capacitances are in series, the charge in each of them will be identical as charge will be separated into positive and negative between two nearby conductor plates, but cannot move through the insulators.



$$q_1(t) = q_2(t) = q_3(t) = q(t)$$

We can use the relationship between the voltage and charge stored as

$$v_1(t) = \frac{q_1(t)}{C_1} = \frac{q(t)}{C_1}$$

$$v_2(t) = \frac{q_2(t)}{C_2} = \frac{q(t)}{C_2}$$

$$v_3(t) = \frac{q_3(t)}{C_3} = \frac{q(t)}{C_3}$$

From KVL, the total voltage across the capacitors is the sum of individual voltage drops across the capacitors:

$$v_1(t) + v_2(t) + v_3(t) = v(t)$$

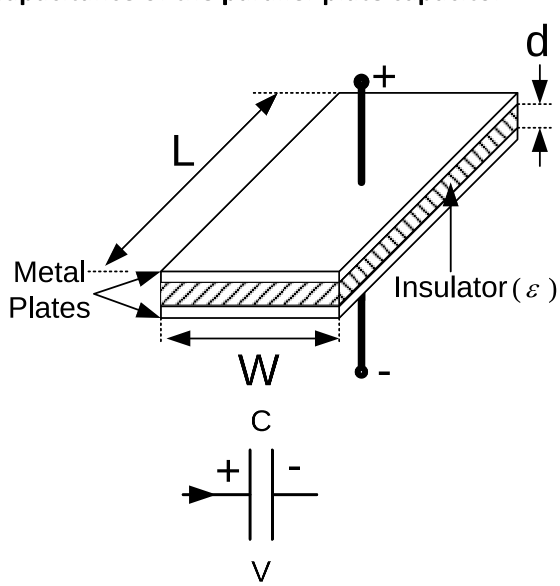
$$\frac{q(t)}{C_1} + \frac{q(t)}{C_2} + \frac{q(t)}{C_3} = q(t)\left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right) = \frac{q(t)}{C_{eq}}$$

The value of the equivalent capacitance will be:

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

It is worth mentioning that the resistances and capacitances follow opposite formulas for series and parallel connection.

Capacitance of the parallel-plate capacitor



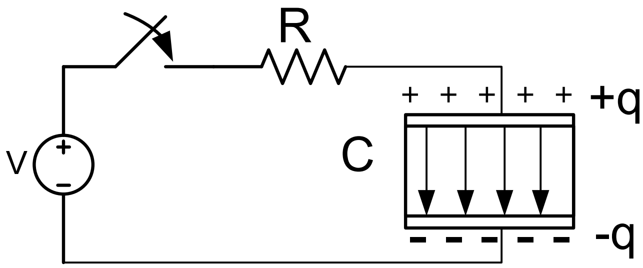


Fig. Parallel plate capacitors

If the two conducting plate of length L and width W are separated by a dielectric material of thickness d, then the capacitance of the capacitor will be

$C = \frac{\varepsilon A}{d}$ where the $A = WxL$ is the area of each plate and $\varepsilon = \varepsilon_r \varepsilon_0$ where $\varepsilon, \varepsilon_r, \varepsilon_0$ are the permittivity of the dielectric material, the relative permittivity and permittivity of free space respectively.

$\varepsilon_0 = 8.85 \times 10^{-12} F/m$

Table 3.1 Relative permittivity of selected materials

Air	1.0
Diamond	5.5
Mica	7.0
Polyester	3.4
Quartz	4.3
Silicon dioxide	3.9

Practical Capacitors

Real capacitors have maximum voltage ratings. Beyond certain value the dielectric will breakdown. There is a trade-off between compact size and voltage rating.

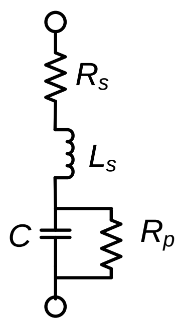
Electrolytic Capacitors

In such capacitors, one of the plates is metallic aluminum or tantalum, the dielectric is an oxide layer on the surface of the metal and the other ‘plate’ is an electrolytic solution. This process results in high capacitance per volume. These capacitors have polarity. If voltage of the opposite polarity is applied, the oxide layer is attacked and the capacitor may fail at high voltage. Hence, bulk capacitor is commonly used in DC voltage systems to filter out voltage ripples from the DC supply. These capacitors cannot be used where voltage polarity reverses.

On the other hand capacitors built with Mylar, polyester, polyethylene can be used in applications where voltage polarity reverses.

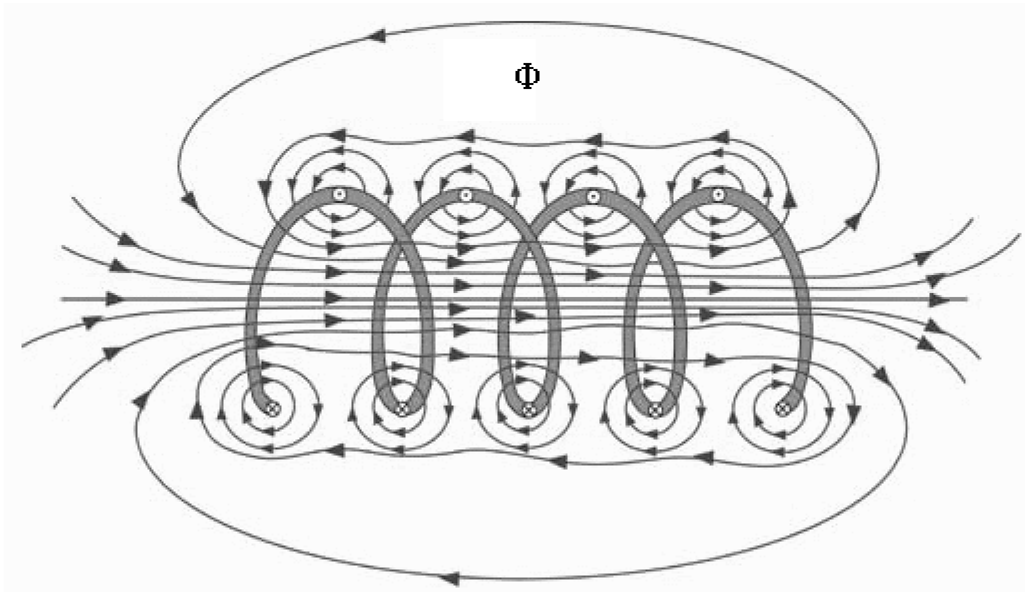
Parasitic Effects

Practical capacitor can be more accurately modeled with a series resistance Rs, a series inductance Ls (both coming from the leads and plates) and a parallel resistor representing the very small leakage current through the dielectric material. Typically, the series resistance and inductance are very small in magnitude whereas the parallel resistance is very large in magnitude. These are called parasitic elements.



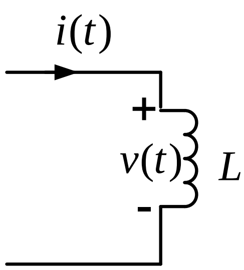
Inductance

An inductor is constructed by coiling a wire around some type of form. When current flows in the coil, a magnetic field is produced, with the magnetic flux linking the coil.



According to Faraday's law, a voltage is induced in a coil when the magnetic field linking it varies with time. For an ideal inductor, the voltage induced is proportional to the rate of change of the current. The proportionality constant is the inductance of the coil.

$v(t) = L\dfrac{di(t)}{dt}$ is the v-i characteristic of an inductor.



Inductance has units of **henries** (H) which is equivalent **volt seconds per ampere**.

Inductor current in terms of voltage

If we know the initial current $i(t_0)$ and the voltage $v(t)$ across an inductance, we can find current at any time $t > t_0$.

Rearrange the voltage equation for inductance,

$$di = \frac{1}{L}v(t)dt$$

Integrating both sides,

$$\int_{i(t_0)}^{i(t)} di = \frac{1}{L}\int_{t_0}^{t} v(t)dt \Rightarrow i(t) = \frac{1}{L}\int_{t_0}^{t} v(t)dt + i(t_0)$$

From the above equation, we can notice that as long as the voltage across the inductance is finite, the current can change by an incremental amount in a time increment. *Hence, current in an inductor must be continuous.*

Energy stored in an inductor

We use power as the product of voltage and current: $p(t) = v(t)i(t)$ and energy stored during a time period as power integrated over time, i.e. $e(t) = \int_{t_0}^{t} p(t)dt$.

$$e(t) = \int_{t_0}^{t} Li(t)\frac{di}{dt}dt = \int_{i_0}^{i(t)} Lidi = \frac{1}{2}L\left(i^2 - i_0^2\right)$$

If the initial current flowing in the inductance were zero, then the energy stored would be

$$e(t) = \frac{1}{2}Li^2(t)$$

This energy is stored in the inductor and is returned back when the current become zero again.

Inductance in series and parallel

When inductances are in series, the same current will be flowing through them (KCL) and the total voltage will be sum of voltages across individual inductances (KVL).

$$i_1(t) = i_2(t) = i_3(t) = i(t)$$
$$v_1(t) + v_2(t) + v_3(t) = v(t)$$
$$L_1\frac{di(t)}{dt} + L_2\frac{di(t)}{dt} + L_3\frac{di(t)}{dt} = L_{eq}\frac{di(t)}{dt}$$
$$L_{eq} = L_1 + L_2 + L_3$$

When inductances are in parallel, the voltage across them will be equal and the total current flowing into the inductors will be sum of currents flowing into individual inductances.

$$i_1(t) + i_2(t) + i_3(t) = i(t)$$
$$\frac{di_1(t)}{dt} + \frac{di_2(t)}{dt} + \frac{di_3(t)}{dt} = \frac{di(t)}{dt}$$
$$\frac{v_1(t)}{L_1} + \frac{v_2(t)}{L_2} + \frac{v_3(t)}{L_3} = \frac{v(t)}{L_{eq}}$$

As all the voltages are equal,

$$v_1 = v_2 = v_3 = v$$

$$v\left(\frac{1}{L_1}+\frac{1}{L_2}+\frac{1}{L_3}\right)=\frac{v}{L_{eq}} \Rightarrow \frac{1}{L_{eq}}=\frac{1}{L_1}+\frac{1}{L_2}+\frac{1}{L_3}$$

Mutual Inductance

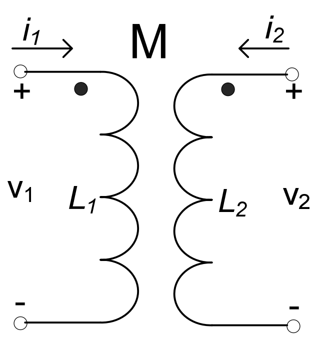
When several coils are wound on the same form, the magnetic flux from one coil links all other coils on the form. Then, a time varying current in one coil induces voltages in the other coils. These coils are said to have a mutual inductance.

The total voltage induced in each coil will be sum of self-induced voltage and the mutually induced voltage:

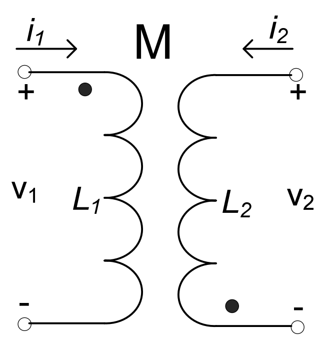
$$v_1(t) = L_1\frac{di_1(t)}{dt} + M\frac{di_2(t)}{dt}$$

$$v_2(t) = L_2\frac{di_2(t)}{dt} + M\frac{di_1(t)}{dt}$$

The voltage and current polarities and directions follow the passive reference convention. The two black dots being on the same end indicates that the flux due to the two coils aid each other.



(a)



(b)

If the dots were at two opposite ends as shown in Fig. (b), then the magnetic fields due to the two fields would oppose each other and voltage equations would change. The sign of the mutual terms in the voltage equation depends on how the currents are referenced with respect to the dots. If both the currents are referenced in to (or both are referenced out of) the dotted terminals, then the mutual term is positive. If one current is referenced into and the other is referenced out of the dotted terminal then the mutual term carries a negative sign as:

$$v_1 = L_1 \frac{di_1}{dt} - M \frac{di_2}{dt}$$

$$v_2 = L_2 \frac{di_2}{dt} - M \frac{di_1}{dt}$$

Comparison of capacitor and inductor

For a capacitor $i_c(t) = C \frac{dv_c(t)}{dt}$

For an inductor $v_L(t) = L \frac{di_L(t)}{dt}$

It can be seen that the roles of voltage and current are reversed in the two elements but both are described by a differential equation of the same form. There is a duality between inductor and capacitor.

We can make important observations from the differential relationship between voltage and current.
For capacitor, the voltage must be continuous as otherwise the current has to be infinity which is not possible.

For inductor, the current must be continuous as otherwise the voltage has to be infinity which is not possible.

These properties of inductor and capacitor give rise to many useful circuits.

Transients

Learning objectives:

1) Understand the meaning of transients.
2) Write differential equations for circuits containing inductors and capacitors.
3) Determine the DC steady-state solution of circuits containing inductors and capacitors.
4) Write differential equation of first-order circuits in standard form, and determine the complete solution of first-order circuits excited by switched DC sources.
5) Write the differential equation of second-order circuits in standard form, and determine the complete solution of second-order excited by switched DC sources.

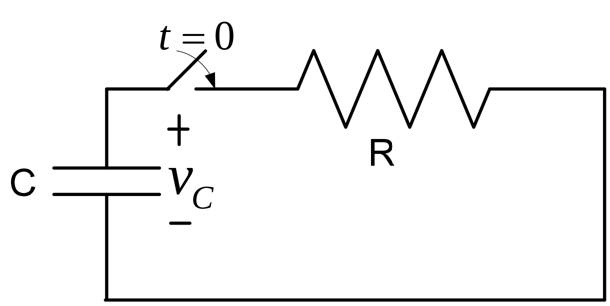
The time-varying voltages and currents resulting from the adding or removing voltage and current source to circuits containing energy storage elements, are called **transients**. Such circuits contain sources, switches, resistances, inductances and/or capacitances. The presence of inductance and capacitance brings integral and differential equations. Thus, the study of transients requires us to solve differential equations.

Circuits with resistors and a single energy storage element (either inductor or capacitor) are said to be first-order circuit. In general, the response of a first-order circuit to a switched DC source will appear in one of the two forms: decaying exponential waveform or rising exponential waveform.

First order RC circuits

We deal with circuits having DC sources, resistances and single capacitance.

Discharge of a capacitor through a resistance



Prior to t=0, the capacitor was charged to an initial voltage V_i. Then at t=0, the switch is closed and the capacitor discharges through the resistance.

According to KCL, the current entering in to the capacitor equals to the current into the resistor:

$$C\frac{dv_c(t)}{dt}+\frac{v_c(t)}{R}=0$$
$$RC\frac{dv_c(t)}{dt}+v_c(t)=0$$
To find the value of the capacitor voltage we need to solve a first order differential equation.

The solution to the first order differential should be a function which has the same form as its first derivative. As the first derivative of an exponential function is also an exponential function, it could be a possible solution.

Let us try the function $v_c(t)=Ke^{st}$.

With this function, the differential equation will be

$RCKse^{st}+Ke^{st}=0$.

Solving for *s*, we obtain $s=\frac{-1}{RC}$ and solution would be $v_c(t)=Ke^{-t/RC}$.

For the complete solution, we need to find out the value of K.

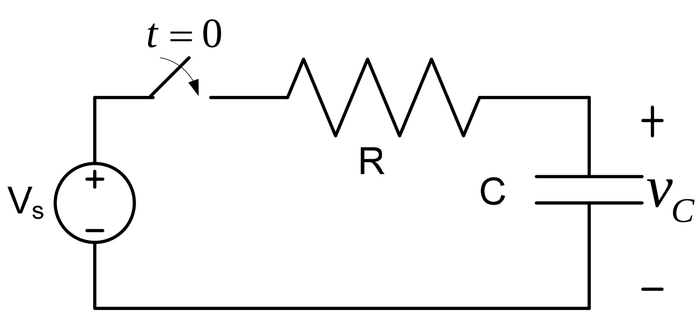
We know that the voltage across the capacitor cannot change instantaneously. Hence, the voltage immediately after switch the closed will be same as the voltage before switch was closed.

$$v_c(t+)=V_i$$
$$Ke^{st}\Big|_{t=0}=K=V_i$$

Final solution will be $v_c(t)=V_ie^{-t/RC}$.

After time interval $\tau=RC$, the capacitor voltage will be at 0.368 times the original voltage. After five times this time interval, the remaining voltage on the capacitor is negligible.

Charging capacitor form a DC source through a resistance



From KVL, starting with the negative polarity of the DC supply and summing the voltage drops, we get

$$-V_s + v_R + v_C = 0$$
$$v_R = iR = RC\frac{dv_c}{dt}$$
$$-V_s + RC\frac{dv_c}{dt} + v_c = 0$$
$$RC\frac{dv_c}{dt} + v_c = V_s$$

Assuming the capacitor was discharged before the switch was closed, and the fact that capacitor voltage cannot change instantaneously:

$$v_c(t+) = v_c(t-) = 0$$

Trying the expression of $K_1 + K_2 e^{st}$, where K_1, K_2, s are constants to be determined, we get:

$$(1 + RCs)K_2 e^{st} + K_1 = V_s$$

As the coefficient of e^{st} must be zero i.e. $s = \frac{-1}{RC}$ and $K_1 = V_s$.

Thus the solution becomes, $v_c(t) = V_s + K_2 e^{-t/RC}$

This solution is valid at time t=0 i.e. $v_c(t=0) = 0 = V_s + K_2 e^0 \Rightarrow K_2 = -V_s$.

With K_1, K_2 known the final solution the capacitor voltage is $v_c(t) = V_s - V_s e^{-t/RC}$.

The solution contains two terms. The first term represents the steady-state response and the second term represents the transient-response.

The capacitor voltage starts with zero and slowly grows to the full DC value.

DC Steady State

The transient terms in the expressions for voltage and current in RLC circuits decay to zero with time, except with LC circuits having R=0.

For DC sources, the steady state voltage and currents will also be zero.

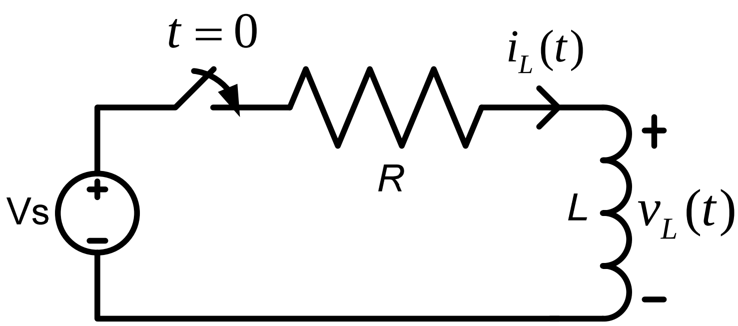
From $i_c(t) = C\frac{dv_c(t)}{dt}$, if steady-state capacitor voltage is constant then the steady-state capacitor current will be zero as well. In other words, capacitor behaves as an open circuit.

For inductance, $v_L(t) = L\frac{di_L(t)}{dt}$. As steady-state inductor current will be constant, the voltage drop across the inductor will be zero. In other words, inductor behaves like a short circuit.

Thus, when obtaining steady-state solutions of RLC circuits with DC sources, at first the capacitors are replaced by open circuits and the inductors are replaced by short circuits. This leads the circuit being reduced to a purely resistive circuit and hence can be analyzed using earlier mentioned methods.

RL circuits

The method of finding the current and voltage is similar to the method used for RC circuits.



Steps:

1. Apply Kirchoff's current and voltage laws to write the circuit equation.
   $$V_s = Ri_L + L\frac{di_L}{dt}$$
2. Rearrange the equation to obtain a first order differential equation with the inductor current as the variable of interest.
   $$L\frac{di_L}{dt} + Ri_L = V_s$$
3. Assume a solution of the form $i_L(t) = K_1 + K_2 e^{st}$.
   $$LK_2 se^{st} + R(K_1 + K_2 e^{st}) = V_s$$
   $$RK_1 + K_2(Ls + R)e^{st} = V_s$$
4. Substitute the solution into the differential equation to determine the value of K_1 and s.
   $$RK_1 = V_s \Rightarrow K_1 = \frac{V_s}{R}$$
   $$Ls + R = 0 \Rightarrow s = -\frac{R}{L}$$
5. Use the initial conditions to determine the value of K_2.
   As the inductor current cannot change instantaneously, $i_L(t=0) = 0$
   $$0 = \frac{V_s}{R} + K_2 \Rightarrow K_2 = -\frac{V_s}{R}$$

6. Write the final solution.

$$i_L(t) = \frac{V_S}{R} - \frac{V_S}{R} e^{-\frac{R}{L}t}$$

Here $\tau = \dfrac{L}{R}$ is the time constant of the RL circuit similar to the RC circuit.

The value of the inductor current after time equal to the time constant would be 0.632 times its steady-state value.

After 5 times the time constant, the inductor current will be almost equal to the steady-state current.

RC and RL circuits with general sources

Solution of the Differential Equation

The general differential equation of a circuit containing resistance and inductance or capacitance is:

$$\tau \frac{dx(t)}{dt} + x(t) = f(t)\,.$$

The general solution to this will consist of two parts:

1. The first pat is called a **particular solution** or **forced response** because it depends on the forcing function.
2. The second part is called the complementary solution and is the solution of the homogenous equation viz.

$$\tau \frac{dx_c(t)}{dt} + x_c(t) = 0\,.$$

The particular solution satisfies the differential equation, but it may not be consistent with the initial conditions. By adding the complementary solution, the general solution satisfies both the differential equation and the initial conditions.

The particular solution is obtained from the forcing function. It is normally of the same functional form as the forcing function and its derivatives. A table containing various forcing functions and their corresponding particular solutions are readily available.

The complementary solution does not depend on the forcing functions and is also called the natural response of the circuit as it depends on the passive circuit elements.

Rearranging the homogenous equation we get,

$$\frac{dx_c(t)/dt}{x_c(t)} = \frac{-1}{\tau}$$

Integrating both sides, we get:

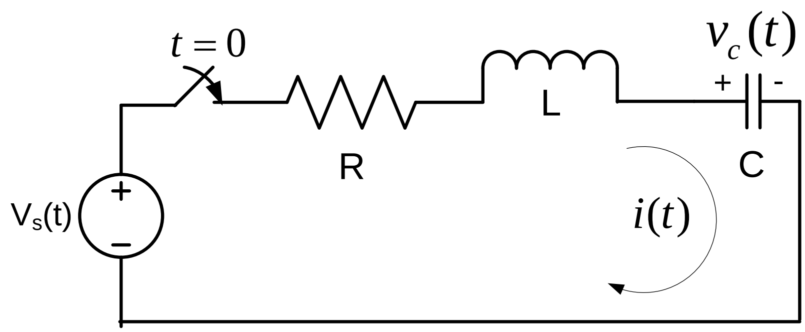
$\ln\left[x_c(t)\right] = \frac{-t}{\tau} + c$ where *c* is the constant of integration. Thus, $x_c(t) = e^c e^{-t/\tau} = Ke^{-t/\tau}$.

The given initial conditions can then be used to find the value of K.

Final solution then will be $x(t) = x_p(t) + x_c(t)$.

Second order circuits

A circuit containing a dc source and R,L, C in series as shown in the figure:



Writing KVL equation, we have

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(t)dt + v_c(0) = v_s(t)$$

By differentiating again with respect to time, we remove the integrals and convert it to a purely differential equation. We get

$$\frac{d^2 i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC}i(t) = \frac{1}{L}\frac{dv_s(t)}{dt}$$

Solution of the second-order equation

General second order differential equation is :

$$\frac{d^2 x(t)}{dt^2} + 2\alpha\frac{dx(t)}{dt} + \omega_0^2 x(t) = f(t)$$

with $\alpha = \frac{R}{2L}, \omega_0^2 = \frac{1}{LC}$.

The general solution to above equation is

$$x(t) = x_p(t) + x_c(t)$$

The particular equation will be dependent on the forcing function as in case of first order differential equation.

However, the complementary solution will be a solution to the second order homogenous differential equation:

$$\frac{d^2x_c(t)}{dt^2} + 2\alpha\frac{dx_c(t)}{dt} + \omega_0^2 x_c(t) = 0 .$$

We start by substituting $x_c(t) = Ke^{st}$, and get $(s^2 + 2\alpha s + \omega_0^2)Ke^{st} = 0$.

As we are interested in $Ke^{st} \neq 0$, $s^2 + 2\alpha s + \omega_0^2 = 0$, which is called the characteristics equation.

The two roots of the characteristics equation are

$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$

$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$

The solution will then be $x_c(t) = K_1e^{s_1t} + K_2e^{s_2t}$.

The constants K_1, K_2 can be solved from the initial conditions of $x_c(0+), \dot{x}_c(0+)$.

The damping ratio is defined as $\zeta = \frac{\alpha}{\omega_0}$, its value decided the shape of the response.

Overdamped: $\zeta > 1$. Critically damped: $\zeta = 1$and under damped: $\zeta < 1$. The under damped case will have oscillatory response with a frequency called natural frequency $\omega_n = \sqrt{\omega_0^2 - \alpha^2}$.