

**EE2023 TUTORIAL 4 (SOLUTIONS)****Solution to Q.1**

$$\tilde{x}(t) = \underbrace{\text{rect}\left(\frac{t-0.475}{0.45}\right)}_{x(t)} \cdot \underbrace{\sum_{n=-\infty}^{\infty} \delta(t-0.2n)}_{\Xi_{0.2}(t)}$$

$$= \delta(t-0.4) + \delta(t-0.6)$$

$$\mathfrak{T}\{\delta(t-\varsigma)\} = \exp(-j2\pi f\varsigma)$$

$$\tilde{X}(f) = \exp(-j2\pi f 0.4) + \exp(-j2\pi f 0.6)$$

$$= \begin{cases} \cos(0.8\pi f) - j\sin(0.8\pi f) + \\ \cos(1.2\pi f) - j\sin(1.2\pi f) \end{cases}$$

$$= \begin{cases} [\cos(0.8\pi f) + \cos(1.2\pi f)] - \\ j[\sin(0.8\pi f) + \sin(1.2\pi f)] \end{cases}$$

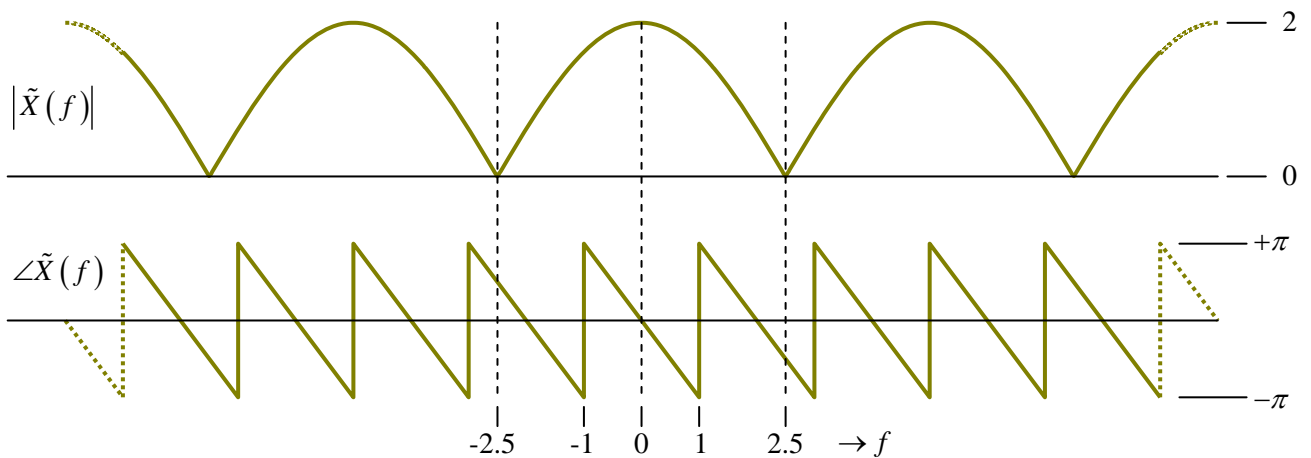
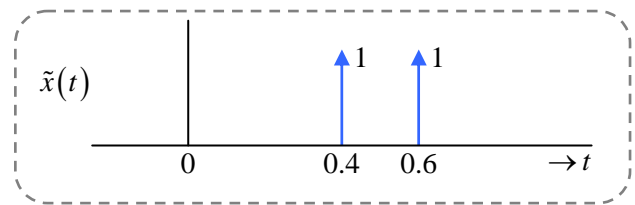
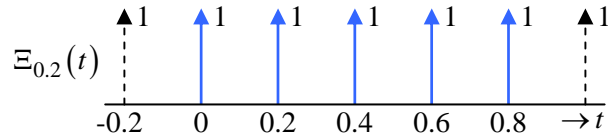
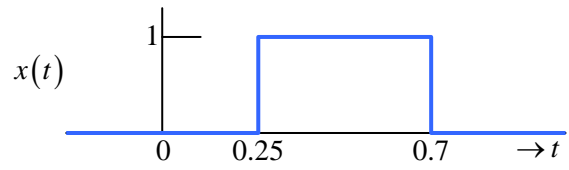
$$= 2\cos(\pi f)\cos(0.2\pi f) - j2\sin(\pi f)\cos(0.2\pi f)$$

$$|\tilde{X}(f)|^2 = 4\cos^2(\pi f)\cos^2(0.2\pi f) + 4\sin^2(\pi f)\cos^2(0.2\pi f)$$

$$= 4\cos^2(0.2\pi f)$$

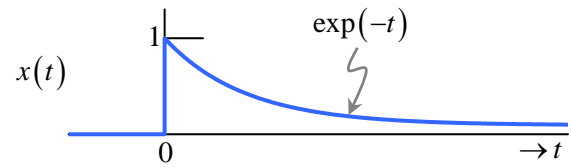
$$\bullet \quad |\tilde{X}(f)| = 2|\cos(0.2\pi f)|$$

$$\bullet \quad \angle\tilde{X}(f) = -\tan^{-1}\left(\frac{2\sin(\pi f)\cos(0.2\pi f)}{2\cos(\pi f)\cos(0.2\pi f)}\right) = -\tan^{-1}(\tan(\pi f)) = -\pi f$$

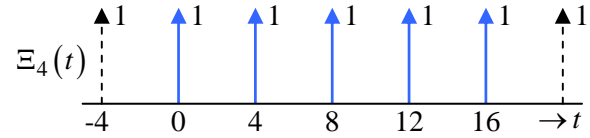


**Solution to Q.2**

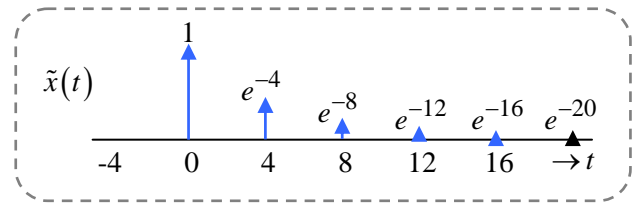
$$\begin{aligned}\tilde{x}(t) &= \underbrace{\exp(-t)u(t)}_{x(t)} \cdot \underbrace{\sum_{n=-\infty}^{\infty} \delta(t-4n)}_{\Xi_4(t)} \\ &= \sum_{n=0}^{\infty} \exp(-4n) \delta(t-4n)\end{aligned}$$



$$\mathfrak{I}\{\delta(t-\varsigma)\} = \exp(-j2\pi f\varsigma)$$



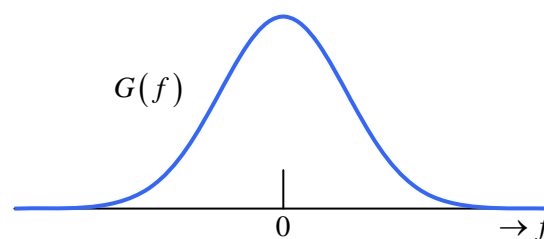
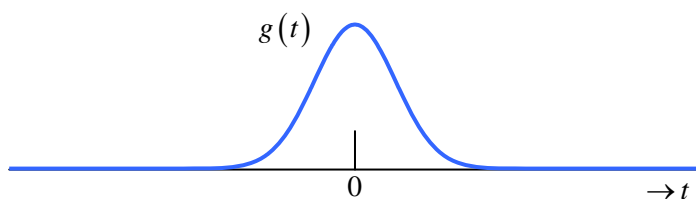
$$\begin{aligned}\tilde{X}(f) &= \sum_{n=0}^{\infty} \exp(-4n) \mathfrak{I}\{\delta(t-4n)\} \\ &= \sum_{n=0}^{\infty} \exp(-4n) \exp(-j2\pi f 4n) \\ &= \underbrace{\sum_{n=0}^{\infty} \exp(-4n(j2\pi f + 1))}_{\text{Geometric Series}} \\ &= \frac{1}{1 - \exp(-4(j2\pi f + 1))} = \frac{1}{1 - \exp(-4)\exp(-j8\pi f)}\end{aligned}$$



When a signal is sampled in the time domain, the spectrum of the sampled signal is periodic with a period equal to the sampling frequency, which in this case is 0.25 Hz.

Thus, to show that  $\tilde{X}(f)$  is periodic, all we need to show is  $\tilde{X}(f + 0.25) = \tilde{X}(f)$  as follows:

$$\begin{aligned}\tilde{X}(f + 0.25) &= \frac{1}{1 - \exp(-4)\exp(-j8\pi(f + 0.25))} \\ &= \frac{1}{1 - \exp(-4)\exp(-j8\pi f)\underbrace{\exp(-j2\pi)}_1} \\ &= \frac{1}{1 - \exp(-4)\exp(-j8\pi f)} = \tilde{X}(f)\end{aligned}$$

**Solution to Q.3**

Rewrite  $x(t) = \sum_{n=-\infty}^{\infty} g(t - nT_p)$  in convolution form:

$$x(t) = g(t) * \sum_{n=-\infty}^{\infty} \delta(t - nT_p) \quad \dots\dots\dots (*)$$

Applying the 'Convolution' property of the Fourier transform to (\*):

$$X(f) = G(f) \cdot \frac{1}{T_p} \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T_p}\right) \quad \dots\dots\dots (**)$$

**Conclusion:**

$X(f)$  can be obtained by sampling  $G(f)/T_p$  in the frequency-domain at regular spacings of  $1/T_p$  Hz.

**Relationship between  $X_k$  and  $G(f)$**

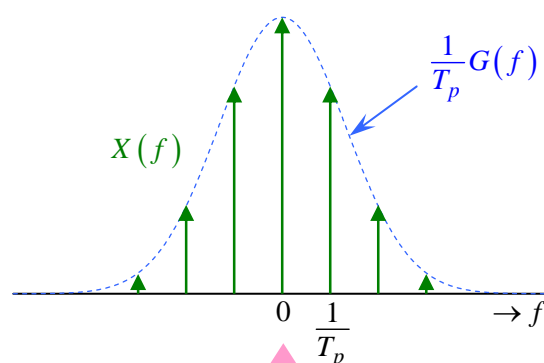
$$\text{Rewrite } (**) \text{ as: } X(f) = \sum_{k=-\infty}^{\infty} \frac{1}{T_p} G\left(\frac{k}{T_p}\right) \delta\left(f - \frac{k}{T_p}\right)$$

$$\text{In terms of } X_k: \quad X(f) = \sum_{k=-\infty}^{\infty} X_k \delta\left(f - \frac{k}{T_p}\right)$$

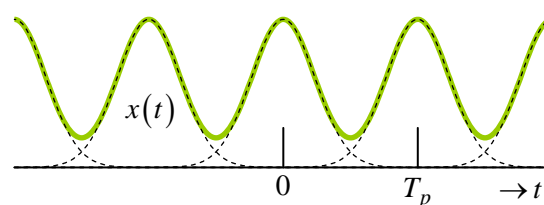
$$\text{Hence, } X_k = \frac{1}{T_p} G(f) \Big|_{f=k/T_p} = \frac{1}{T_p} G\left(\frac{k}{T_p}\right)$$

**Uniqueness of a generating function**

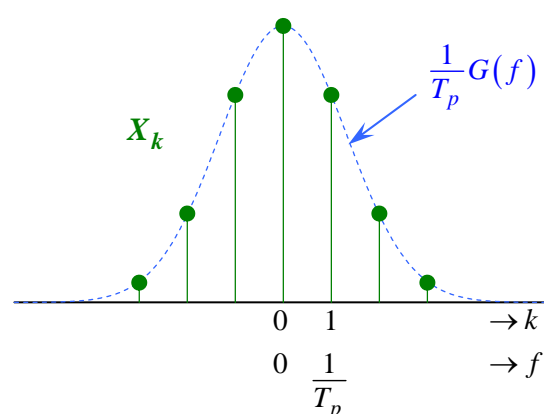
The generating function of a periodic signal is NOT unique. For instance, one period of a periodic signal can be used as its generating function.



Fourier Transform



Fourier Series Coefficients



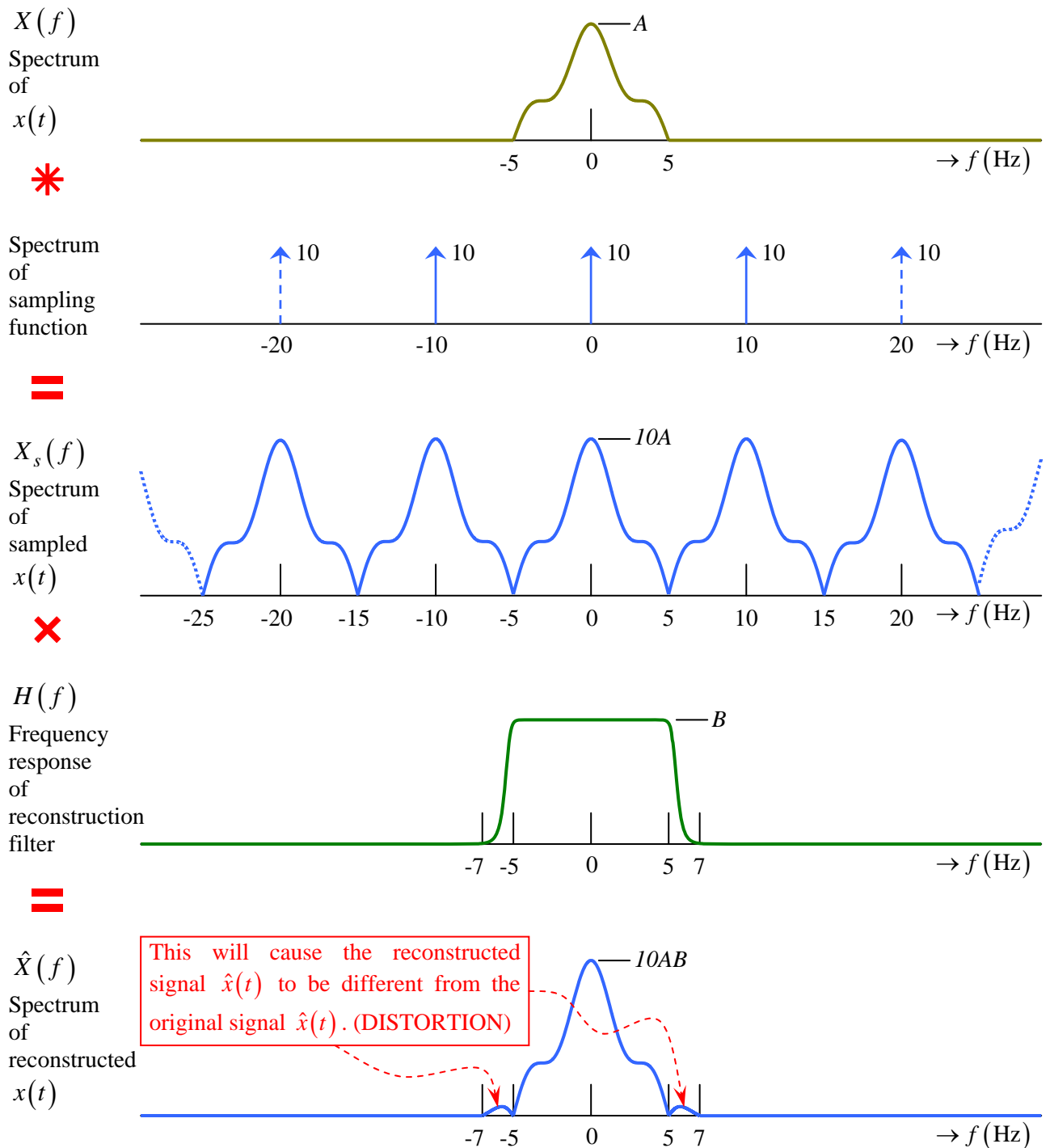
**Remarks:** From the sampling theorem, we observed that when a signal is sampled at regular time-intervals of  $T_s$  (sec), the spectrum corresponding to the sampled signal is periodic with a period of  $f_s = 1/T_s$  (Hz). In this exercise, we showed that when a spectrum is sampled at regular frequency-intervals of  $f_p$  (Hz), the signal corresponding to the sampled spectrum is periodic with a period of  $T_p = 1/f_p$  (sec). Hence, sampling a signal leads to a periodic spectrum and sampling a spectrum leads to periodic signal. This phenomenon is due to the convolution property of the Fourier Transform and the fact that  $\left[ \Xi_T(t) \Leftrightarrow \frac{1}{T} \Xi_{1/T}(f) \right]$ .

## Solution to Q.4

- Nyquist sampling frequency:  $2 \times 5 = 10$  Hz
- Recommended sampling frequency: 12 Hz

The excess 2 Hz is needed to prevent adjacent spectral images from contributing to the reconstruction process. See illustration below.

### Sampling frequency: 10 Hz (Nyquist Rate)

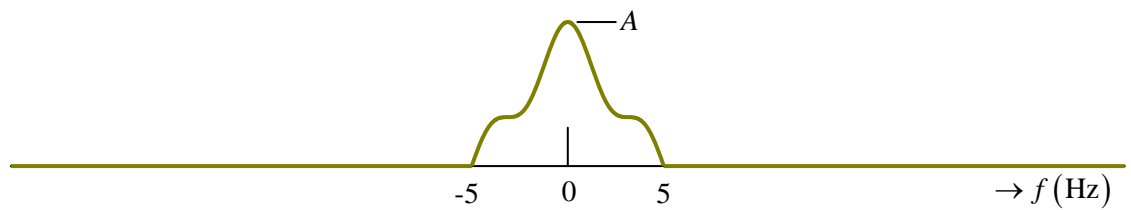


## Sampling frequency: 12 Hz

 $X(f)$ 

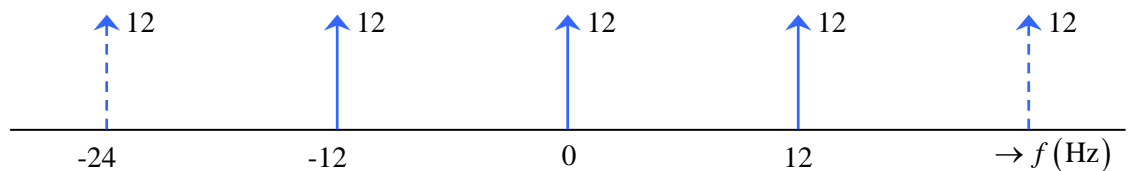
Spectrum  
of  
 $x(t)$

✱



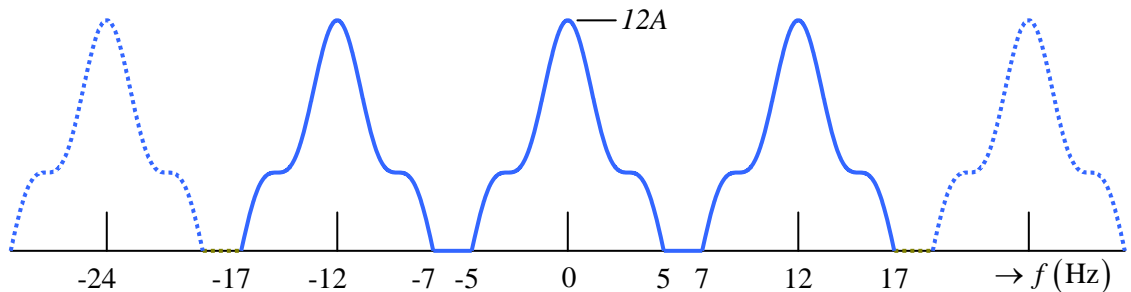
Spectrum  
of  
sampling  
function

=


 $X_s(f)$ 

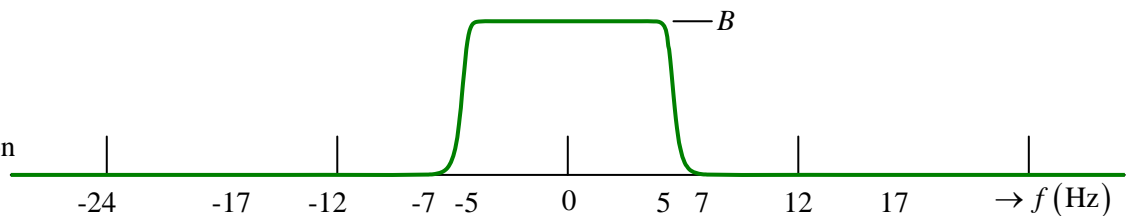
Spectrum  
of  
sampled  
 $x(t)$

✕

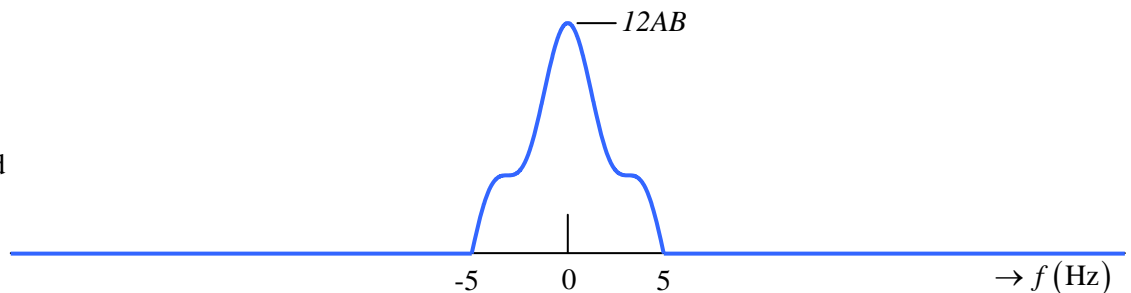

 $H(f)$ 

Frequency  
response  
of  
reconstruction  
filter

=


 $\hat{X}(f)$ 

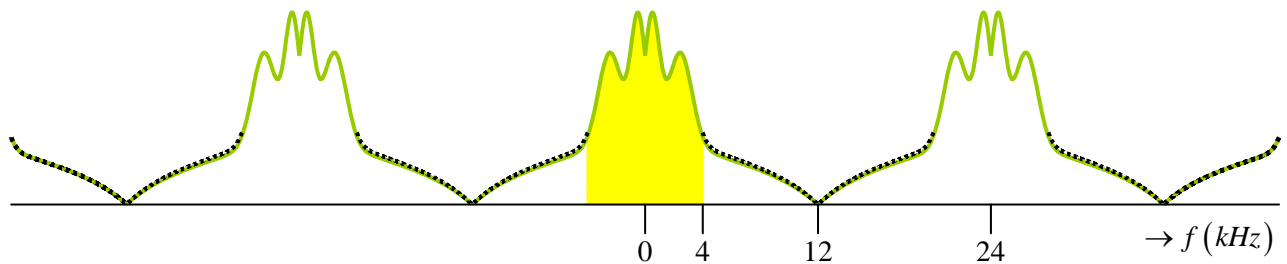
Spectrum  
of  
reconstructed  
 $x(t)$



*Remarks: Sampling frequencies higher than 12 Hz may be used to achieve the same result. However, high sampling frequency is usually matched by more costly data acquisition, storage and processing requirement.*

## Solution to Q.5

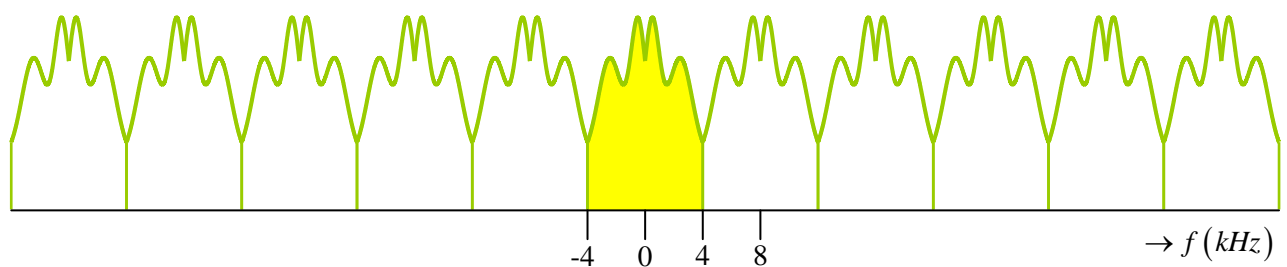
**Situation A:**  $\left( \begin{array}{l} \text{No anti-aliasing lowpass filter} \\ \text{and no frequency aliasing} \end{array} \right): \left\{ \begin{array}{l} \text{Sampling frequency} = 2 \times 12 = 24 \text{ kHz} \end{array} \right.$



*Advantage:* No anti-aliasing LPF needed.

*Disadvantage:* Sampling frequency is higher than necessary to preserve the *TQ-Band*.

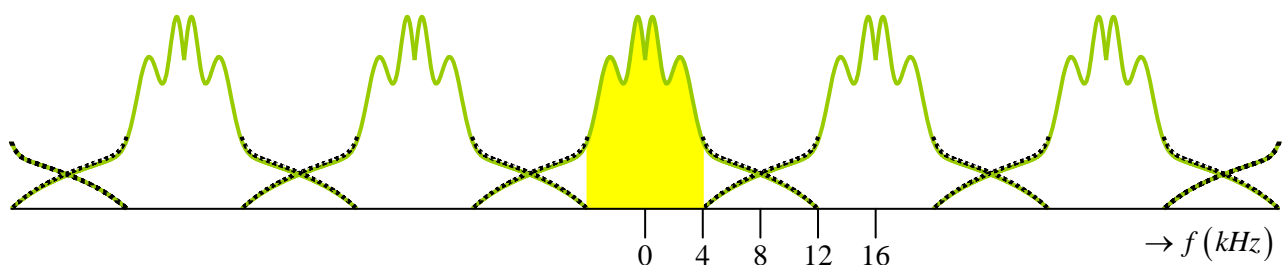
**Situation B:** *Lowest sampling frequency:*  $\left\{ \begin{array}{l} \text{Ideal anti-aliasing filter of bandwidth 4 kHz} \\ \text{Sampling frequency} = 2 \times 4 = 8 \text{ kHz} \end{array} \right.$



*Advantage:* Lowest possible sampling frequency to preserve the *TQ-Band*.

*Disadvantage:* Require anti-aliasing LPF with sharp cutoff.

**Situation C:** *No anti-aliasing lowpass filter:*  $\left\{ \text{Sampling frequency} = 2 \times \left( \frac{12+4}{2} \right) = 16 \text{ kHz} \right.$



*Advantage:* Lowest possible sampling frequency to preserve the *TQ-Band* without requiring anti-aliasing LPF with sharp cutoff.

*Disadvantage:* Sampling frequency is still higher than the minimum needed to preserve the *TQ-Band*.

*This problem illustrates the trade-off between oversampling and the requirement of expensive anti-aliasing LPF with sharp cutoff frequency.*