NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2007-2008

MA1505 Mathematics 1

April/May 2008 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains a total of **EIGHT (8)** questions and comprises **SIX (6)** printed pages.
2. Answer **ALL** 8 questions. The marks for each question are indicated at the beginning of the question.
3. **Write your matriculation number neatly on the front page of the answer booklet provided.**
4. **Write your solutions in the answer booklet. Begin your solution to each question on a new page.**
5. Calculators may be used. However, you should lay out systematically the various steps in your calculations.
6. **This is a CLOSED BOOK examination. One A4-sized helpsheet is allowed.**

Question 1 [10 marks]

(a) Let $x = 2 + t^2 + \cos t$ and $y = 3 + 2t^4 + \sin t$.

Find the value of $\dfrac{dy}{dx}$ at the point when $t = \dfrac{\pi}{2}$.

(b) A (circular) cylindrical container with <u>no</u> top cover is to be constructed to hold a fixed volume V cm^3 of liquid. The cost of the material used for the base is 8 cents/cm^2, and the cost of the material used for the curved surface is 3 cents/cm^2. Find the radius r cm (in terms of V) of the least expensive container.

Question 2 [10 marks]

(a) The finite region R in the first quadrant is bounded by the curve $y = e^x$, and the lines $y = e^2$ and $x = 1$. Find the volume of the solid generated when R is revolved about the line $\underline{x = 1}$.

(*Give the exact volume in terms of* π *and* e.)

(b) Find the sum of the following infinite series inside the interval of convergence

$$1 - \frac{1}{3}(x-8) + \frac{1}{9}(x-8)^2 - + \cdots + \left(-\frac{x-8}{3}\right)^n + \cdots$$

Question 3 [10 marks]

(a) If

$$f(x) \;=\; \int_0^x t\,e^{t^3}\,dt,$$

use Taylor series to find $f^{(1505)}(0)$.

(*Leave your answer in terms of factorials.*)

(b) Let

$$f(x) = \begin{cases} x & \text{if } 0 < x < 1 \\ 2 - x & \text{if } 1 < x < 2 \end{cases}$$

The cosine half-range expansion of $f(x)$ is

$$f(x) \;=\; \frac{1}{2} \;-\; \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)\pi x$$

(*You need not derive this Fourier series.*)

Use the above cosine half-range expansion to find the sum of the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$. Hence find the sum of the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$.

(*Give the exact values in terms of* π.)

Question 4 [10 marks]

(a) Let Π_1 be the plane

$$x + 2y + 2z = 8.$$

If Π_2 is the plane parallel to Π_1 and which passes through the point $P(1, 6, 8)$, find the shortest distance between Π_1 and Π_2.

(b) The lines L_1 and L_2 are given parametrically by:

$L_1: \quad x = 3 + 2s, \quad y = 9 - 3s, \quad z = 10 + 4s$

$L_2: \quad x = 5 + 2t, \quad y = 4 - 4t, \quad z = 12 + 3t$

where s and t are real numbers.

The line L_3 is perpendicular to L_1 and L_2, and passes through the point of intersection Q of L_1 and L_2. Find the point of intersection R of L_3 with the xy-plane.

Question 5 [10 marks]

(a) Let C be the circle of radius 10 centred at the origin O in the xy-plane. If

$$f(x, y) = x^2y + xy^2 + 3x + 4y,$$

find the point(s) $P(x_0, y_0)$, if any, on C such that the directional derivative of f at O in the direction of the vector $\overrightarrow{OP}$ is <u>*zero*</u>.

(b) Find the local maximum, local minimum and saddle points (if any) of

$$f(x, y) = x^3 - 3x^2 - 4y^2 + 8.$$

Question 6 [10 marks]

(a) Find the value of the iterated integral

$$\int_0^4 \int_{\sqrt{x}}^2 x\sqrt{y^5+4}\, dy\, dx.$$

(b) Let b be a positive constant. The region R in the upper half of the xy-plane (where $y \geq 0$) is bounded by the two lines $y = x$ and $y = -x$, and the circle of radius b centred at the origin. Find the value of the integral

$$\iint_R \left(x^2+y^2\right) e^{x^2+y^2}\, dA,$$

leaving your answer in terms of b.

Question 7 [10 marks]

(a) Let C be the portion of the graph of

$$x^3 + y^3 \;=\; 8$$

in the first quadrant that joins the point $A(0,2)$ to the point $B(2,0)$. If

$$\mathbf{F}(x,y) \;=\; x^2\left(3y^2+5x^2\right)\mathbf{i} \;+\; 2y\left(x^3+1\right)\mathbf{j},$$

find the line integral $\displaystyle\int_C \mathbf{F} \bullet d\mathbf{r}$.

(b) Use the method of separation of variables to find $u(x,y)$ that satisfies the partial differential equation

$$u_x \;-\; u_y \;=\; 3x^2 u,$$

given that $u(0,0) = 3$ and $u(2,-3) = 3$.

Question 8 [10 marks]

(a) Let S be the cone described by

$$z = \sqrt{x^2 + y^2}, \quad \text{where } 0 \le z \le 4.$$

If

$$\mathbf{F}(x, y, z) = y\mathbf{i} - x\mathbf{j} + z^2\mathbf{k},$$

find the surface integral $\iint_S \mathbf{F} \bullet d\mathbf{S}$, where the orientation of S is given by the inner normal vector.

(b) Let S be the closed surface that consists of

(i) the upper hemisphere

$$x^2 + y^2 + z^2 = 1, \quad z \ge 0,$$

together with

(ii) the base of points $(x, y, 0)$, where $0 \le x^2 + y^2 \le 1$.

If

$$\mathbf{F}(x, y, z) = 4x\mathbf{i} + z^2\mathbf{j} + e^{xy}\mathbf{k},$$

use the Divergence Theorem to find the surface integral $\iint_S \mathbf{F} \bullet d\mathbf{S}$, where the orientation of S is given by the outer normal vector.

END OF PAPER