National University of Singapore
Department of Electrical & Computer Engineering

EE2023 Signals & Systems
Tutorial 5 Solutions

Section I

1. (a) From the double angle formula $\cos 2\omega t = 2\cos^2 \omega t - 1$, $\cos^2 \omega t = \frac{1}{2}[\cos 2\omega t + 1]$

$$\begin{aligned} \mathcal{L}\{\cos^2 \omega t\} &= \frac{1}{2}\mathcal{L}\{\cos 2\omega t\} + \frac{1}{2}\mathcal{L}\{1\} \\ &= \frac{1}{2}\left[\frac{s}{s^2+4w^2} + \frac{1}{s}\right] \end{aligned}$$

(b) $$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s+2)(s+4)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{15(s-1)} - \frac{1}{6(s+2)} + \frac{1}{10(s+4)}\right\} \\ &= \frac{1}{15}e^{t} - \frac{1}{6}e^{-2t} + \frac{1}{10}e^{-4t} \end{aligned}$$

(c) $$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s_1^2}\right\} \text{ where } s_1 = s+1 \\ &= te^{-t} \end{aligned}$$

(d) $$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{s+9}{s^2+6s+13}\right\} &= \mathcal{L}^{-1}\left\{\frac{s+9}{(s+3)^2+4}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{s+3+6}{(s+3)^2+4}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+4} + 3\frac{2}{(s+3)^2+4}\right\} \\ &= e^{-3t}[\cos 2t + 3\sin 2t] \end{aligned}$$

(e) $$\begin{aligned} \mathcal{L}\left\{\frac{3}{5} - \frac{\sqrt{45}}{5}e^{-2t}\sin(t+\tan^{-1}0.5)\right\} &= \mathcal{L}\left\{\frac{3}{5} - \frac{6}{5}e^{-2t}\sin t - \frac{3}{5}e^{-2t}\cos t\right\} \\ &= \frac{3}{5s} - \frac{6}{5}\frac{1}{(s+2)^2+1} - \frac{3}{5}\frac{s+2}{(s+2)^2+1} \\ &= \frac{3}{s(s^2+4s+5)} \end{aligned}$$

(f) $$\mathcal{L}\left\{(t-1)^2U(t-1)\right\} = \frac{2}{s^3}e^{-s}$$

(g) $$\begin{aligned} \mathcal{L}\left\{t^2U(t-1)\right\} &= \mathcal{L}\left\{(t-1)^2U(t-1) + 2tU(t-1) - U(t-1)\right\} \\ &= \mathcal{L}\left\{(t-1)^2U(t-1) + 2(t-1)U(t-1) + U(t-1)\right\} \\ &= \frac{2}{s^3}e^{-s} + \frac{2}{s^2}e^{-s} + \frac{1}{s}e^{-s} \end{aligned}$$

(h) $$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{se^{-2s}}{s^2+\pi^2}\right\} &= \cos[\pi(t-2)]U(t-2) \\ &= \cos(\pi t)U(t-2) \end{aligned}$$

(i) $$\begin{aligned}\mathcal{L}\left\{te^{-t}\sin t\right\} &= -\frac{d}{ds}\left[\frac{1}{(s+1)^2+1}\right]\\ &= -\frac{d}{ds}\left[\frac{1}{(s^2+2s+s}\right]\\ &= \frac{2(s+1)}{(s^2+2s+2)^2}\end{aligned}$$

(j) As $\mathcal{L}\{\sin 3t\} = \dfrac{3}{s^2+9}$, $\dfrac{d}{ds}\dfrac{3}{s^2+9} = -\dfrac{6s}{(s^2+9)^2}$

$\therefore\ \mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+9)^2}\right\} = \dfrac{1}{6}t\sin 3t$

Section II

1. In the given circuit, the input signal is the voltage source and the output signal is the current flowing in the circuit, $i(t)$. To solve the problem,

   - Derive the differential equation relating the input and output signals using Kirchoff Voltage Law :

$$100i(t) + 2\frac{di(t)}{dt} = 100, t < 0 \tag{1a}$$
$$25i(t) + 2\frac{di(t)}{dt} = 100, t \geq 0 \tag{1b}$$

   - The problem states that the circuit is operating in steady-state with the switch open prior to t = 0. This statement indicates that only the steady-state solution is needed for $t < 0$. Since input signal is a constant, the steady-state output should also be a constant. Hence, $\frac{di(t)}{dt}$ in Equation 1(a) should be zero. Equation 1(a) reduces to $100i(t) = 100$ or $i(t) = 1$ when $t < 0$
     Another way to find the steady-state current is to use the property that an inductor reduces to a short circuit at steady-state if the input signal is a constant. The steady-state current can then be found by solving a resistive circuit.
   - Lastly, solve Equation 1(b) using Laplace Transform and the initial condition $i(0) = 1$.

$$\begin{aligned}25i(t) + 2\frac{di(t)}{dt} &= 100 \cdot U(t)\\ 25I(s) + 2[sI(s) - i(0)] &= \frac{100}{s}\\ I(s) &= \frac{2i(0)}{2s+25} + \frac{100}{s\,(2s+25)}\\ &= 4 - 3e^{-12.5t}\end{aligned}$$

   Note that only one initial condition is needed as the system model is a first order differential equation.

2. (a) Using KVL, the differential equation relating $i(t)$ to $E(t)$ is

$$\begin{aligned} L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau &= E(t) \\ L\frac{d^2i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{C}i(t) &= \frac{dE(t)}{dt} \end{aligned}$$

(b) Transforming the differential equation into the s-domain using Laplace Transform,

$$s^2I(s) - si(0) - i'(0) + 6[sI(s) - i(0)] + 25I(s) = \frac{120s}{s^2+25}$$

Since initial conditions $i(0)$ and $i'(0)$ are zero,

$$\begin{aligned} s^2I(s) + 6sI(s) + 25I(s) &= \frac{120s}{s^2+25} \\ I(s) &= \frac{120s}{(s^2+6s+25)(s^2+25)} \\ &= -\frac{20}{s^2+6s+25} + \frac{20}{s^2+25} \\ i(t) &= -5e^{-3t}\sin 4t + 4\sin 5t \end{aligned}$$

3. (a) Drugs taken in tablet form can be modelled by impulse function whose strength is equal to the quantity of drug ingested. To obtain the necessary mathematical function, first consider the definition of *impulse function* and *strength* of an impulse function. By definition, $\delta(t-a)$ is non-zero when $t = a$. Since system is linear time invariant, the instant when first tablet is ingested can be assumed to be $t = 0$ and the second tablet is taken at $t = 1$ (1 day later or 24 hours later). Hence, $f(t) = 100\delta(t-0) + 50\delta(t-1) = 100\delta(t) + 50\delta(t-1)$.

(b) System is modelled by a second order differential equation so two initial conditions are needed. Given that there is no stress relief drug in Ah Kow's bloodstream, $y(0^-) = \dot{y}(0^-) = 0$

(c) To solve the problem, start by performing Laplace Transform on both sides of the differential equation. Laplace transform is linear so take the Laplace transform of the individual terms.

- Laplace Transform of the derivative terms can be found using the Transform of derivative rule.
- From the Laplace Transform table, $\mathcal{L}\{\delta(t)\} = 1$.
- $\delta(t-1)$ is the impulse function, $\delta(t)$, shifted by one unit along the time axis. This is a clue that the shift in time-domain function rule $\mathcal{L}\{f(t-t_0)U(t-t_0)\} = e^{-st_0}F(s)$ where $\mathcal{L}\{f(t)\} = F(s)$ can be used to solve the problem. As the term in the Laplace Transform rule is $f(t-t_0)U(t-t_0)$, the first step is to express $\delta(t-1)$ in the same form. Since $\delta(t-1) = \delta(t-1)U(t-1)$, $\mathcal{L}\{\delta(t-1)\} = \mathcal{L}\{\delta(t-1)U(t-1)\} = e^{-s}\cdot 1$.

- Manipulate the resulting algebric equation to obtain

$$Y(s) = \frac{50\left(2 + e^{-s}\right)}{s^2 + 3s + 2}$$

- Perform inverse Laplace Transform to obtain

$$y(t) = 100\left[e^{-t} - e^{-2t}\right]U(t) + 50\left[e^{-(t-1)} - e^{-2(t-1)}\right]U(t-1)$$

Section III

1. - Input signal is the current source, $i(t)$, while the desired output signal is the voltage across the resistor, $v(t)$. Hence, objective is to derive a differential equation involving $v(t)$ and $i(t)$.
   - Other concepts
     - Kirchoff Currrent Law : Sum of current flowing through resistor and capacitor is equal to the current provided by the source.
     - Current flowing through the resistor is $\frac{v(t)}{R}$
     - Current flowing through the capacitor is $C\frac{dv(t)}{dt}$

   Hence,

$$C\frac{dv(t)}{dt} + \frac{v(t)}{R} = i(t)$$

   Solve the differential equation to derive $v(t)$