NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 EXAMINATION 2006/2007

MA2214 Combinatorial Analysis

April/May 2007 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains a total of **FIVE (5)** questions and comprises **THREE (3)** printed pages.

2. Answer **ALL** questions. The marks for each question are indicated at the beginning of the question.

3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Question 1 [20 marks]

(a) Among the 26 letters of the English alphabet, the letters a, e, i, o, u are called vowels and the rest are called consonants. Find the number of arrangements of all these 26 letters in such a way that any two vowels must be separated by at least two consonants if the arrangements are

(i) linear.

(ii) circular.

(b) Use a combinatorial method to find a formula for $\sum_{r=1}^{n} r^5$ in terms of some binomial coefficients involving n.

Question 2 [20 marks]

(a) Show by a combinatorial argument that for each positive integer n, the expression

$$\frac{(5n+7)!}{(3n+5)!(2n+3)!}$$

is an integer.

(b) Find the number of 4-digit integers $a_1a_2a_3a_4$ where a_1, a_2, a_3 and a_4 are the digits such that the following hold.

(i) $1 \leq a_1 \leq 6$, $3 \leq a_2 \leq 9$, $2 \leq a_3 \leq 8$ and a_4 is even.

(ii) Any two adjacent digits are distinct and in addition, a_1 and a_4 are also distinct.

Question 3 [20 marks]

(a) Let the universal set S be the set of 7-digit integers comprising some or all of the six digits, namely, 0, 1, 2, 3, 4 and 5. Let a be the number of those elements in S which contain neither a block of 12 nor a block of 21. Let b be the number of those elements in S which contain exactly **ONE** block of 12 and no block of 21. Use the principle of inclusion and exclusion to evaluate a and b.

(b) Find, in terms of n, the number of ways of distributing $3n$ distinct objects into n boxes if

(i) the n boxes are distinct and each box contains 3 objects.

(ii) the n boxes are identical and each box contains 3 objects.

(iii) the n boxes are distinct and no box is empty.

(iv) the n boxes are identical and no box is empty.

Question 4 [20 marks]

(a) For each positive integer n, let a_n be the number of n-digit integers comprising some or all of the six digits, namely, 0, 1, 2, 3, 4 and 5, such that these integers contain neither a block of 11 nor a block of 23.

(i) Find a recurrence relation for a_n with the necessary initial conditions.

(ii) Evaluate a_6.

(b) For each positive integer n, let D_n be the number of derangements of the n integers from 1 through n. Find a recurrence relation for D_n with exactly three necessary initial conditions, namely, the values of D_1, D_2 and D_3.

Question 5 [20 marks]

(a) For each integer $n \geq 5$, let a_n denote the number of ways of distributing n identical objects into 9 distinct boxes labelled from 1 through 9 such that the five boxes labelled from 1 through 5 altogether contain an even number of objects and the three boxes labelled 6, 7 and 8 each contain an odd number of objects, while box 9 must contain at least 2 objects.

(i) Find a suitable generating function for a_n.

(ii) Express a_n in terms of n.

(b) For each integer $n \geq 10$, let a_n denote the number of n-digit integers comprising all the five digits 1, 2, 3, 4 and 5 such that each of these five digits must occur at least twice in these integers.

(i) Find a suitable generating function for a_n.

(ii) Express a_n in terms of n.

END OF PAPER