

NATIONAL UNIVERSITY OF SINGAPORE
FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2003-2004

MA1505 MATHEMATICS I

November 2003 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided below.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **TEN (10)** questions and comprises **FORTY ONE (41)** printed pages.
3. Answer **ALL** questions. For each question, write your answer in the box and your working in the space provided inside the booklet following that question.
4. The marks for each question are indicated at the beginning of the question.
5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Matriculation Number:

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For official use only. Do not write below this line.

Question	1	2	3	4	5	6	7	8	9	10
Marks										

Question 1 (a) [5 marks]

For what value of m is the line $y = mx + c$ perpendicular to the tangent line of the graph of the function $y = \sqrt{x^2 + 16}$ at the point $(3, 5)$?

Answer 1(a)	
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(Show your working below and on the next page.)

(More working space for Question 1(a))

Question 1 (b) [5 marks]

Find $f'(\sqrt{3})$ if $f(x) = \frac{x(1-x^2)^2}{\sqrt{1+x^2}}$.

Answer 1(b)	
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(Show your working below and on the next page.)

(More working space for Question 1(b))

Question 2 (a) [5 marks]

Two points A and B start at time $t = 0$ at the origin and move along the positive x -axis with B moving 3 times as fast as A. Let C denote the fixed point $(0, 1)$ on the y -axis. Let θ denote the value of the angle $\angle ACB$ at any time t later. What is the maximum value of $\tan \theta$?

Answer 2(a)	
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(Show your working below and on the next page.)

(More working space for Question 2(a))

Question 2 (b) [5 marks]

Find the value of

$$\lim_{x \rightarrow 3^+} \frac{x^2 \int_3^x \sqrt{t^3 + 9} dt}{|3 - x|}.$$

Answer 2(b)	
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(Show your working below and on the next page.)

(More working space for Question 2(b))

Question 3 (a) [5 marks]

Let θ be the angle in the first quadrant such that $\sin \theta = \frac{\sqrt{15}}{4}$.
Find the value of

$$\int_0^\theta \frac{80 \sin^3 x}{\sqrt{\cos x}} dx.$$

Answer 3(a)	
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(Show your working below and on the next page.)

(More working space for Question 3(a))

Question 3 (b) [5 marks]

Let $x > 1$. Find

$$\int \left(\frac{1}{\ln x} - \frac{1}{(\ln x)^2} \right) dx.$$

Answer 3(b)	
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(Show your working below and on the next page.)

(More working space for Question 3(b))

Question 4 (a) [5 marks]

By using the Ratio Test, or otherwise, determine whether the series

$$\sum_{n=1}^{\infty} \frac{6^n (n!)^2}{(2n)!}$$

is convergent or divergent. Show clearly all your steps.

Answer 4(a)	
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(Show your working below and on the next page.)

(More working space for Question 4(a))

Question 4 (b) [5 marks]

By using the Root Test, or otherwise, determine whether the series

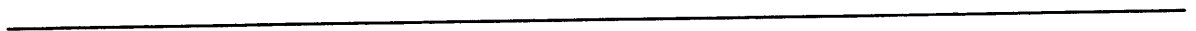
$$\sum_{n=1}^{\infty} \left(\frac{n}{3n-1} \right)^{2n}$$

is convergent or divergent. Show clearly all your steps.

Answer 4(b)	
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(Show your working below and on the next page.)

(More working space for Question 4(b))



Question 5 (a) [5 marks]

Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(-3)^{n+1} x^n}{\sqrt{n+1}}.$$

Answer 5(a)	
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(Show your working below and on the next page.)

(More working space for Question 5(a))

Question 5 (b) [5 marks]

Let

$$f(x) = \frac{1}{x^2 + x + 1}.$$

Let $f(x) = \sum_{n=0}^{\infty} c_n x^n$ be the Maclaurin series representation for $f(x)$. Find the value of $c_{36} - c_{37} + c_{38}$.

Answer 5(b)	
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(Show your working below and on the next page.)

(More working space for Question 5(b))

Question 6 (a) [5 marks]

Let

$$f(x) = \begin{cases} 0 & \text{if } -1 < x < 0 \\ 1 & \text{if } 0 < x < 1. \end{cases}$$

Let $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$ be the Fourier Series representation for $f(x)$. Find the value of

$$a_0 - \pi a_3 + \pi b_5.$$

Answer 6(a)	
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(Show your working below and on the next page.)

(More working space for Question 6(a))

Question 6 (b) [5 marks]

Let $f(x) = x(\pi - x)$ for $0 < x < \pi$. Let $\sum_{n=1}^{\infty} b_n \sin nx$ be the Fourier Sine Series which represents $f(x)$. Find the value of the coefficient b_3 .
Give your answer in terms of π .

Answer 6(b)	
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(Show your working below and on the next page.)

(More working space for Question 6(b))

Question 7 (a) [5 marks]

Find $f(x)$ which satisfies the differential equation

$$f'(x) + \frac{2}{x}f(x) = 8x, \quad x > 0$$

and the initial condition $f(1) = 3$.

Answer 7(a)	
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(Show your working below and on the next page.)

(More working space for Question 7(a))

Question 7 (b) [5 marks]

Solve the differential equation

$$y' = \frac{2y^4 + x^4}{xy^3}, \quad y(1) = 2.$$

Answer 7(b)	
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(Show your working below and on the next page.)

(More working space for Question 7(b))

Question 8 (a) [5 marks]

Find the general solution of the differential equation

$$y'' - 2y' + 10y = 0.$$

Answer 8(a)	
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(Show your working below and on the next page.)

(More working space for Question 8(a))

Question 8 (b) [5 marks]

Solve the differential equation

$$y'' - y = e^x, \quad y(0) = 2, \quad y'(0) = \frac{1}{2}.$$

Answer 8(b)	
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(Show your working below and on the next page.)

(More working space for Question 8(b))

Question 9 (a) [5 marks]

A tank initially holds 100 litres of a brine solution containing 20 kg of salt. Starting from time $t = 0$, fresh water is continuously poured into the tank at the rate of 5 litres/minute, while the well-stirred mixture leaves the tank continuously at the same rate. Find an expression for the amount (in kg) of salt in the tank at any time t .

Answer 9(a)	
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(Show your working below and on the next page.)

(More working space for Question 9(a))

Question 9 (b) [5 marks]

Find the inverse Laplace transform

$$L^{-1} \left\{ \frac{1}{s(s^2 + 4)} \right\}.$$

Answer 9(b)	
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(Show your working below and on the next page.)

(More working space for Question 9(b))

Question 10 (a) [5 marks]

Given that the Laplace transform of \sqrt{t} is $L(\sqrt{t}) = \frac{1}{2}\sqrt{\pi}s^{-3/2}$.

Find the Laplace transform of the function

$$f(t) = t^{3/2}.$$

Answer 10(a)	
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(Show your working below and on the next page.)

(More working space for Question 10(a))

Question 10 (b) [5 marks]

Find the functions $x(t)$ and $y(t)$ which satisfy

$$\begin{cases} \frac{dx}{dt} = -2x + y \\ \frac{dy}{dt} = 2x - 3y \end{cases}$$

and the initial conditions $x(0) = 1, y(0) = 0$.

Answer 10(b)	
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(Show your working below and on the next page.)

(More working space for Question 10(b))

END OF PAPER

