NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2002-2003

MA1506 MATHEMATICS II

April / May 2003 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided below.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **TEN (10)** questions and comprises **THIRTY EIGHT (38)** printed pages.
3. Answer **ALL** questions. Write your answers in the boxes and working in the spaces provided inside the booklet following that questions.
4. The marks for each question are indicated at the beginning of the question.
5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Matriculation Number:

For official use only. Do not write below this line.

Question	1	2	3	4	5	6	7	8	9	10
Marks										

Answer **all** *the questions.*

Question 1 [15 marks]

Multiple Choice Questions *(Write your answers here.)*

(i)	(ii)	(iii)	(iv)	(v)

(i) Which of the following does not represent the same line in the xyz-space as the others?

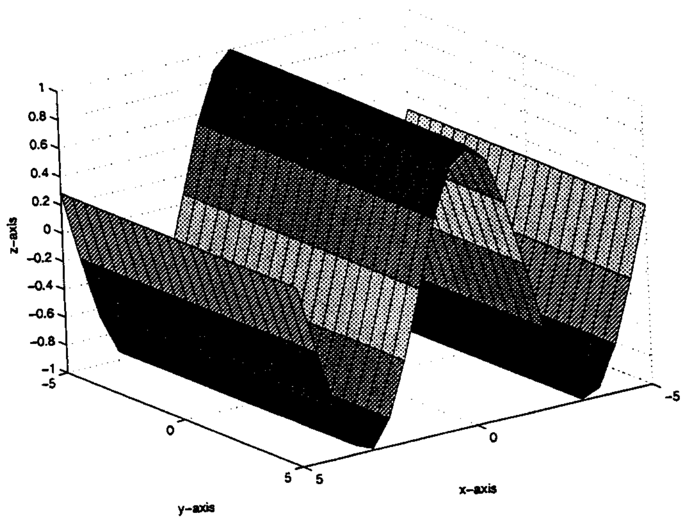
(A) $\dfrac{x-1}{1} = \dfrac{y-2}{2} = \dfrac{z-3}{3}$

(B) $x = 2-t, \quad y = 3-t, \quad z = 4-t \quad$ for $t \in \mathbb{R}$.

(C) $2y - z \;=\; x \;=\; 2z - y - 3$

(D) $\overrightarrow{r}(t) = \langle t+1, \;\; t+2, \;\; t+3 \rangle \quad$ for $t \in \mathbb{R}$.

(ii) Which of the following equations best describe the surface shown in the diagram?



(A) $z = \cos(x)$

(B) $z = y\cos(x)$

(C) $z = \cos(x) + y$

(D) $z = \cos(x) + \cos(y)$

(iii) Let A be a 5×3 matrix. Which of the following conditions is impossible?

(A) The nullspace of A has dimension 4.

(B) The row space of A has dimension 2.

(C) The reduced row echelon form of A has 3 pivotal columns.

(D) The reduced row echelon form of A has exactly 1 zero row.

(iv) Let A be an 3×3 matrix. Which of the following conditions does not guarantee that A is diagonalisable?

(A) A has 3 distinct eigenvalues.

(B) A has 3 distinct eigenvectors.

(C) A has an eigenvalue of geometric multiplicity 3.

(D) Every eigenvalue of A has algebraic multiplicity 1.

(v) Let $f(x, y) = \sqrt{x^2 + y^2}$. Which of the following is false?

(A) The partial derivatives of f at $(0, 0)$ with respect to x and y does not exist.

(B) $\lim_{(x,y)\to(0,0)} f(x, y)$ exists.

(C) f is continuous at $(0, 0)$.

(D) The directional derivative of f at $(0, 0)$ exists for some direction $\overrightarrow{u}$.

Question 2 **[10 marks]**

Let

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix}.$$

(i) Find *all* vectors $\mathbf{v}$ in $\mathbb{R}^3$ such that

$\mathbf{v} \bullet \mathbf{w}_1 = 0, \quad \mathbf{v} \bullet \mathbf{w}_2 = 0$ and $\mathbf{v} \bullet \mathbf{w}_3 = 0$ *simultaneously.*

Answer 2(i)	

(ii) Let V be the set of all $\mathbf{v}$ in (i). Explain briefly why V is a vector space. (You may use any definition or theorem in the lecture notes.)

Answer 2(ii) *Use work spaces* *if necessary.*	**Explanation:**

(iii) Find a basis for V.

Answer 2(iii)	

(iv) What is the rank of the 3×3 matrix A with $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$ as the three columns of A^T?

Answer 2(iv)	

Show your working on the next three pages.

(Working spaces for Question 2 - Indicate your parts clearly)

(More working spaces for Question 2)

(More working spaces for Question 2)

Question 3 [10 marks]

The matrix $A = \begin{bmatrix} 0 & 25 & -29 \\ 19 & -66 & 89 \\ 17 & -65 & 86 \end{bmatrix}$ has eigenvalues λ_1, λ_2, λ_3 with corresponding eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}.$$

(i) Find λ_1, λ_2, λ_3. (Hint: It is not necessary to find the characteristic polynomial of A.)

Answer 3(i)	λ_1 :	λ_2 :	λ_3 :

(ii) Write down a 3×3 matrix P such that $P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$.

You do not need to justify your answer for part (ii).

Answer 3(ii)	P :

Show your working for Question 3(i) below and the next page.

(More working spaces for Question 3(i))

3(iii) Solve the system of linear differential equations:

$$\begin{cases} y_1' = & & 25y_2 - 29y_3 \\ y_2' = & 19y_1 & - 66y_2 + 89y_3 \\ y_3' = & 17y_1 & - 65y_2 + 86y_3 \end{cases}$$

Answer 3(iii)	

Show your working for Question 3(iii) below and on the next page.

(More working spaces for Question 3(iii))

Question 4 [**9 marks**]

4(a) Find the local (or relative) maxima, minima and saddle points of the function

$$f(x,y) = \frac{1}{2}x^2 + 3y^3 + 9y^2 - 3xy + 9y - 9x.$$

Answer 4(a)	**local max:** **local min:** **saddle point:**

(Show your working below and on the next page.)

(More working spaces for Question 4(a))

4(b) Let

$$g(x,y) = \frac{2x^2y}{x^4+17y^2}.$$

(i) Find the limit of $g(x,y)$ as (x,y) approaches the origin $(0,0)$ *along the parabola* $y = kx^2$, where k is a real number.

Answer 4(b)(i)	Limit along $y = kx^2$:

(ii) Determine whether

$$\lim_{(x,y)\to(0,0)} g(x,y)$$

exists. Justify your answer in the work space.

Answer 4(b)(ii)	Existence of limit: Yes / No

(Show your working below and on the next page.)

(More working spaces for Question 4(b))

Question 5 [10 marks]

The plane Π is tangent to the ellipsoid

$$4x^2 + y^2 + z^2 = 1$$

at the point $P(x_0, y_0, z_0)$ in the first octant (i.e. $x_0 > 0$, $y_0 > 0$, $z_0 > 0$). The tetrahedral region R in the first octant is bounded by Π and the three planes $x = 0$, $y = 0$ and $z = 0$.

(i) Show that the Cartesian equation of Π can be expressed as

$$4x_0x + y_0y + z_0z = 1.$$

(Show your working in the work space.)

(ii) Find the values of x_0, y_0, z_0 such that R has the smallest volume.

(Hint: Volume of a right tetrahedron is $\frac{1}{3} \times$ base area $\times$ height.)

Answer 5(ii)	x_0 :	y_0 :	z_0 :

Show your working below and on the next three pages.

(More working spaces for Question 5)

(More working spaces for Question 5)

(More working spaces for Question 5)

Question 6 [10 marks]

6(a) Evaluate $\iint_R (3 - x^2 - 2y^2)\, dA$, where R is the region in the xy-plane given by $x^2 + y^2 \leq 1$.

Answer 6(a)	

(Show your working below and on the next page.)

(More working spaces for Question 6(a))

6(b) Evaluate $\int_0^6 \left[\int_{x/3}^2 x \sqrt[3]{y^3+1}\, dy \right] dx.$

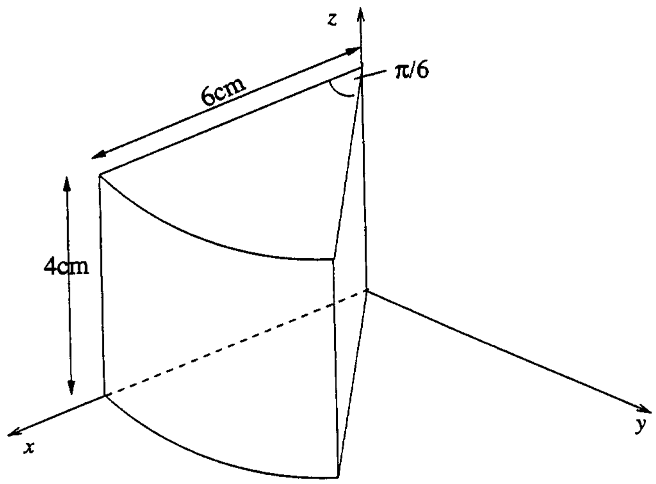
Answer 6(b)	

(Show your working below and on the next page.)

(More working spaces for Question 6(b))

Question 7 **[7 marks]**

A wedge of cheese is cut from a cylinder 4 cm high and 6 cm in radius; this wedge subtends an angle of $\pi/6$ at the center (refer to the figure below).



(i) Give the inequalities that describe the wedge in cylindrical coordinates.

Answer 7(i)	

(ii) Find the mass of the wedge of cheese given that the density is 1.2 grams per cm^3.

Answer 7(ii)	

(Show your working below and on the next page.)

(Working spaces for Question 7)

Question 8 **[10 marks]**

8(a) If $\vec{r}$ is the position vector of any point in $\mathbf{R}^3$ and $\vec{a} = \langle 1, 1, 0 \rangle$, compute

$$\mathrm{div}(\vec{a} \times \vec{r}).$$

Answer 8(a)	

(Show your working below.)

8(b) Let

$$\vec{F} = \langle e^y - ze^x, xe^y, -e^x \rangle.$$

Find f so that

$$\nabla f = \vec{F}.$$

Evaluate

$$\int_C \vec{F} \bullet d\vec{r}$$

on C, the line segment on the z-axis from (0,0,0) to (0,0,1).

Answer 8(b)	

(Show your working below and on the next two pages.)

(More working spaces for Question 8(b))

(More working spaces for Question 8(b))

Question 9 **[9 marks]**

Evaluate

$$\iint_S \left[(x^3 - yz)\, dy\, dz - 2x^2y\, dz\, dx + z\, dx\, dy\right]$$

on the surface S of a cube bounded by the planes $x = 1$, $y = 1$, $z = 1$ and the coordinate planes (that is the planes $x = 0$, $y = 0$ and $z = 0$).

(i) by direct integration, and

(ii) by using the divergence theorem of Gauss.

Answer 9	

(Show your working below and on the next three pages - Indicate your parts clearly.)

(More working spaces for Question 9)

(More working spaces for Question 9)

(More working spaces for Question 9)

Question 10 **[10 marks]**

10(a) Use the method of separation of variable to obtain solutions $u(x, y)$ of the equation

$$u_x + u_y = 2(x + y)u.$$

Answer 10(a)	

(Show your working below and on the next page.)

(Working spaces for Question 10(a))

10(b) Consider the wave equation

$$u_{tt} = 900\ u_{xx} \tag{1}$$

with boundary conditions

$$u(0,t) = 0, \quad u(2,t) = 0 \quad \text{for all} \ \ t, \tag{2}$$

and the initial condition

$$u(x,0) = 0. \tag{3}$$

(i) Verify that

$$u(x,t) = b_n\ \sin(15n\pi t)\ \sin\left(\frac{1}{2}n\pi x\right)$$

satisfies the partial differential equation (1) and the conditions (2) and (3) for all n.

(Show your working in the work space.)

(ii) Obtain the solution of (1) satisfying (2), (3) and the initial condition

$$u_t(x,0) = 300\ \sin(4\pi x)\,.$$

(iii) Given the conditions in (ii), determine the maximum value of u when $x = \frac{1}{8}$.

Answer 10(b)	(ii) (iii)

(Show your working below and on the next two pages -Indicate your parts claerly.)

(More working spaces for Question 10(b))

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[END OF PAPER]