Outline

1 When We Add
2 Permutations
3 Binomial Coefficients
4 Permutations with Repetition
5 Compositions
6 Set Partitions
7 Integer Partitions
8 The Twelvefold Way
9 The Pigeonhole Principle
10 The Inclusion-Exclusion Principle
11 Generating Functions
12 Arithmetic Progressions

Weak Compositions

Definition 5.1

Let $a_1, a_2, \ldots, a_k$ *be nonnegative integers satisfying*

$$a_1 + a_2 + \cdots + a_k = n,$$

then the k*-tuple* $(a_1, a_2, \ldots, a_k)$ *is called a weak composition of* n *into* k *parts.*

First Bijection

# weak compositions of n into k parts	=	# ways to put n identical balls into k distinct boxes (some boxes may be empty)

Weak Compositions

Example: 7 balls into 3 boxes

○ ○ ○	○ ○ ○○		$\leftrightarrow$	000100001
○		○ ○ ○ ○ ○○	$\leftrightarrow$	011000000
○ ○ ○	○○	○○	$\leftrightarrow$	000100100

Second Bijection

ways to put n identical balls into k boxes (some boxes may be empty) $=$ # binary sequences of length $n+k-1$ with exactly n zeroes and $k-1$ ones.

Weak Compositions

Theorem 5.2

The # of weak compositions of n into k parts is

$$\binom{n+k-1}{n} = \binom{n+k-1}{k-1}.$$

Proof: Consider the problem of putting n identical balls into k boxes. Each distribution is equivalent to a weak composition. We map each distribution to a binary sequence using 0s for balls and 1s for the walls of each box. Ignoring the extreme left and right 1s, this is a bijection to binary sequences of length $n+k-1$ with exactly n zeroes and $k-1$ ones.$\square$

Weak Compositions

Example 5.3

Recall from example 4.6, the # of weak compositions of 3 *into* 3 *parts is*

$$\binom{3+3-1}{3} = \binom{5}{2} = 10.$$

Weak Compositions

Example 5.4

There are three types of sandwiches: chicken, ham and egg. If you wish to order 6 sandwiches, how many orders are possible.

- Assume the shop has enough sandwiches to fulfill any other.
- Weak composition of 6 into 3 parts.
- Answer $\binom{6+3-1}{3-1} = 28$.

Weak Compositions

Example 5.5

There are three types of sandwiches: chicken, ham and egg. If you wish to order 6 sandwiches, how many orders are possible given that the shop has only 3 egg sandwiches available.

Assume the shop has enough chicken and ham sandwiches. We fix # of egg sandwiches from 0 to 3 and count the choices for each case. If k egg sandwiches are chosen, we have $\binom{6-k+1}{1}$ ways to choose the remaining two types.

$$\# = \sum_{k=0}^{3} \binom{6-k+1}{1} = 4 + 5 + 6 + 7 = 22.$$

Weak Compositions

Example 5.6

Find the # of ways to create a set of 10 letters from the letters A, B, C and D, without restriction on the number of appearances of each letter.

- This is generally called a multiset
- There is no order within the set
- This is bijective to weak compositions of 10 into 4 parts.
- Answer $\binom{10+4-1}{4-1} = \binom{13}{3}$.

Compositions

Definition 5.7

Let $a_1, a_2, \dots, a_k$ *be positive integers satisfying*

$$a_1 + a_2 + \cdots + a_k = n,$$

then the k*-tuple* $(a_1, a_2, \dots, a_k)$ *is called a composition of* n *into* k *parts.*

Compositions

Theorem 5.8

The # of compositions of n into k parts is $\binom{n-1}{k-1}$.

Proof: There is a bijection from the set of weak compositions of $n-k$ into k parts to the set of compositions of n into k parts, by adding 1 to each part. Hence by the bijection principle, # of compositions is

$$\binom{n-k+k-1}{k-1} = \binom{n-1}{k-1}.$$

□

Example 5.9

Find the # of integer solutions to

$$x_1 + x_2 + x_3 + x_4 = 5,$$

subject to $x_1 \geq 0$, $x_2 \geq 1$, $x_3 \geq 1$ *and* $x_4 \geq 0$.

Replacing $x_2 \mapsto y_2 + 1$ and $x_3 \mapsto y_3 + 1$, the equation becomes

$$x_1 + y_2 + y_3 + x_4 = 3,$$

subject to $x_1 \geq 0$, $y_2 \geq 0$, $y_3 \geq 0$ and $x_4 \geq 0$.
$\#$ of solutions $= \#$ weak compositions $= \binom{3+4-1}{3} = 20$.

Example 5.10

Find the # of ways to arrange the letters of the word VISITING, if the Is cannot be adjacent.

Method 1: Arrange the consonants in a line. (5! ways). There are 6 spaces between the consonants, from which we need to choose 3 to insert Is. By the product principle, $\# = 5!\binom{6}{3}$.
Method 2: Arrange the Is in a line. (1 way).

$$\square I \square I \square I \square.$$

We need to distribute the 5 consonants into the 4 boxes such that the middle two boxes are non-empty. This was computed in the previous example to be $\binom{6}{3}$ ways. Finally, permute the consonants to arrive at $\# = 5!\binom{6}{3}$.

Example 5.11

Find the # of compositions of 50 into 4 odd parts.

By bijection principle, it is equivalent to solving

$$(2x_1+1)+(2x_2+1)+(2x_3+1)+(2x_4+1)=50$$
$$\implies x_1+x_2+x_3+x_4=23,$$

subject to $x_i \geq 0$.
Note how we transformed a composition to a weak composition.
$\#$ of solutions $=\#$ weak compositions $=\binom{23+4-1}{3}$.

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Set Partitions

Definition 6.1

Let n and $k \leq n$ be positive integers. Let $B = \{B_1, \ldots B_k\}$ be a set where $B_i \subseteq [n]$, B_i are nonempty and pairwise disjoint, and $\bigcup_{i=1}^{k} B_i = [n]$. Then we say that B is a partition of $[n]$ into k blocks.

Example 6.2

There are six partitions of $[4]$ into three blocks,

$\{\{1,2\},\{3\},\{4\}\}$, $\{\{1,3\},\{2\},\{4\}\}$, $\{\{1,4\},\{2\},\{3\}\}$
$\{\{2,3\},\{1\},\{4\}\}$, $\{\{2,4\},\{1\},\{3\}\}$, $\{\{3,4\},\{1\},\{2\}\}$

Set Partitions

Example 6.3

There are 25 partitions of [5] into three blocks.

- Order within the blocks and within B does not matter
- Block sizes of 3,1,1 or 2,2,1
- $\binom{5}{3} = 10$ ways to form a block of 3 elements
- $\binom{5}{2} \times \binom{3}{2} = 30$ ways to form two blocks of 2 elements but this introduces order to the two blocks.
- By division principle there should be $30/2 = 15$ ways to form blocks of sizes 2,2,1.

Set Partitions

Definition 6.4

Let n and $k \leq n$ be positive integers. Then the $\#$ of partitions of $[n]$ into k blocks is denoted by $\left\{ {n \atop k} \right\}$ and is called a Stirling number of the second kind.

- $\left\{ {0 \atop 0} \right\} = 1$
- $\left\{ {n \atop k} \right\} = 0$ if $k > n$
- $\left\{ {n \atop 0} \right\} = 0$ if $n > 0$
- $\left\{ {n \atop 1} \right\} = 1$, $\left\{ {n \atop n} \right\} = 1$

Remark: There is also a Stirling number of the first kind.

Stirling Numbers: Recurrence

Theorem 6.5

For all positive integers n and $k \leq n$,

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}.$$

Proof: Both sides counts the number of partitions of $[n]$ into k blocks. If n is in a block by itself, then there are $\left\{ {n-1 \atop k-1} \right\}$ ways to partition the remaining $[n-1]$ into $k-1$ blocks. If n does not occur in a single block, we can partition $[n-1]$ into k blocks, then there are k choices to insert n into one of the blocks. $\square$

Stirling Numbers in a triangle

$\left\{ {n \atop k} \right\}$											k
$n=0$						1					
$n=1$					0		1				
$n=2$				0		1		1			
$n=3$			0		1		3		1		
$n=4$		0		1		7		6		1	
$n=5$	0		1		15		25		10		1

Stirling Numbers: More properties

Example 6.6

For all positive integers n, $\left\{ {n \atop 2} \right\} = 2^{n-1} - 1.$

- Let X be the set of partitions of $[n]$ into two non-empty blocks.
- Consider the map from binary sequences of length n to X:
  if 0 appears in the k-th position,put k into first block,
  if 1 appears put k into second block
- We must exclude the sequence of all 0s and that of all 1s.
- This is a 2-to-1 mapping since the blocks are ordered, i.e. associated with 0 or 1.
- $|X| = \frac{2^n-2}{2} = 2^{n-1} - 1.$
- If $n = 1$, X is empty and binary sequences= 0,1.

Example 6.6: Alternative

For all positive integers n, $\left\{ {n \atop 2} \right\} = 2^{n-1} - 1$.

- We have $\left\{ {n \atop 2} \right\} = \left\{ {n-1 \atop 1} \right\} + 2\left\{ {n-1 \atop 2} \right\}$
- Or $f(n) = 1 + 2f(n-1)$ for $n \geq 2$, $f(1) = 0, f(2) = 1$.

$$\begin{aligned} f(n) &= 1 + 2\big(1 + 2f(n-2)\big) \\ &= 1 + 2 + 2^2 f(n-2) \\ &= 1 + 2 + 4 + 2^3 f(n-3) \\ &= \cdots \\ &= 1 + 2 + 4 + 8 + \cdots + 2^{n-2} + 2^{n-1} f(1) \\ &= \frac{2^{n-1} - 1}{2 - 1} \end{aligned}$$

Stirling Numbers: More properties

Example 6.7

For all positive integers n, $\left\{ {n \atop n-1} \right\} = \binom{n}{2}$.

- If $n = 1$, both sides give 0.
- To partition $[n]$ into $n-1$ subsets, one block must have two elements, the remaining are singletons.
- There are $\binom{n}{2}$ ways to choose the block with two elements, and exactly one way to choose the other elements as singletons.

Example 6.7: Alternative

For all positive integers n, $\left\{ {n \atop n-1} \right\} = \binom{n}{2}$.

- We have $\left\{ {n \atop n-1} \right\} = \left\{ {n-1 \atop n-2} \right\} + (n-1)\left\{ {n-1 \atop n-1} \right\}$
- Or $g(n) = g(n-1) + (n-1)$ for $n \geq 2$, $g(1) = 0, g(2) = 1$.

Hence

$$\begin{aligned} g(n) &= g(n-1) + (n-1) \\ &= g(n-2) + (n-2) + (n-1) \\ &= \cdots \\ &= g(2) + 2 + 3 + \cdots + (n-2) + (n-1) \\ &= \frac{n(n-1)}{2} = \binom{n}{2} \end{aligned}$$

Stirling Numbers: More properties

Example 6.8

For all positive integers n, $\left\{ {n \atop n-2} \right\} = \binom{n}{3} + 3\binom{n}{4}$.

- If $n = 1, 2$, both sides give 0.
- Check $n = 3, 4$ case.
- To partition $[n]$ into $n-2$ subsets, either one block has three elements or two blocks have two each with the remaining singletons.
- $\binom{n}{3}$ ways to choose the block with three elements
- $\binom{n}{2}\binom{n-2}{2}$ to choose two blocks with order

By addition principle, $\left\{ {n \atop n-2} \right\} = \binom{n}{3} + \frac{1}{2}\frac{n!}{2^2(n-4)!} = \binom{n}{3} + 3\binom{n}{4}$.

Stirling Numbers: Second Recurrence

Theorem 6.9

For all positive integers n and $k \leq n$,

$$\left\{ {n+1 \atop k} \right\} = \sum_{i=0}^{n} \binom{n}{i} \left\{ {n-i \atop k-1} \right\}.$$

Proof: Both sides counts the number of partitions of $[n+1]$ into k blocks. RHS counts partitions when $n+1$ is in a block of size $i+1$. There are $\binom{n}{i}$ ways to pick the other i elements in this block. The remaining $n+1-(i+1) = n-i$ elements can be partitioned into $k-1$ blocks in $\left\{ {n-i \atop k-1} \right\}$ ways. $\square$

Application to finding power sums

Theorem 6.10

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} (x)_{\underline{k}}.$$

Two Lemmas

$$x(x)_{\underline{k}} = (x)_{\underline{k+1}} + k(x)_{\underline{k}},$$
$$n(x)_{\underline{n-1}} = (x+1)_{\underline{n}} - (x)_{\underline{n}}.$$

Application to finding power sums

Proof of Theorem 6.10 by induction:

Base case: $n = 0$ is true. For general n, consider $x^{n+1} = x(x^n)$,

$$\begin{aligned}
x^{n+1} &= x \sum_{k=0}^{n} \left\{ {n \atop k} \right\} (x)_{\underline{k}} \\
&= \sum_{k=0}^{n} \left\{ {n \atop k} \right\} (x)_{\underline{k+1}} + \sum_{k=0}^{n} \left\{ {n \atop k} \right\} k (x)_{\underline{k}} \quad \text{(Lemma)} \\
&= \sum_{k=1}^{n+1} \left\{ {n \atop k-1} \right\} (x)_{\underline{k}} + \sum_{k=0}^{n} \left\{ {n \atop k} \right\} k (x)_{\underline{k}} \\
&= \sum_{k=1}^{n} \left\{ {n+1 \atop k} \right\} (x)_{\underline{k}} + (x)_{\underline{n+1}} + \left\{ {n \atop 0} \right\} (x)_{\underline{0}} \quad \text{(Thm 6.5)} \\
&= \sum_{k=0}^{n+1} \left\{ {n+1 \atop k} \right\} (x)_{\underline{k}}. \qquad \square
\end{aligned}$$

Application to finding power sums

Example 6.11

$$\sum_{x=0}^{n} x = \frac{1}{2}n(n+1).$$

$$\begin{aligned} x^1 = (x)_{\underline{1}} &= \frac{1}{2}\left((x+1)_{\underline{2}} - (x)_{\underline{2}}\right) \quad \text{(Lemma)} \\ \implies \sum_{x=0}^{n} x &= \frac{1}{2}\sum_{x=0}^{n}\left((x+1)_{\underline{2}} - (x)_{\underline{2}}\right) \\ &= \frac{1}{2}(n+1)_{\underline{2}}. \end{aligned}$$

Application to finding power sums

Example 6.12

$$\sum_{x=0}^{n} x^2 = \frac{1}{3}(n+1)_{\underline{3}} + \frac{1}{2}(n+1)_{\underline{2}}$$

$$x^2 = \left\{ {2 \atop 2} \right\}(x)_{\underline{2}} + \left\{ {2 \atop 1} \right\}(x)_{\underline{1}}$$

$$\implies \sum_{x=0}^{n} x^2 = \frac{1}{3}\sum_{x=0}^{n}\left((x+1)_{\underline{3}} - (x)_{\underline{3}}\right) + \frac{1}{2}\sum_{x=0}^{n}\left((x+1)_{\underline{2}} - (x)_{\underline{2}}\right).$$

Application to finding power sums

Theorem 6.13

$$\sum_{x=0}^{m} x^n = \sum_{k=0}^{n} \frac{1}{k+1} \left\{ {n \atop k} \right\} (m+1)_{\underline{k+1}}.$$

Proof.

$$\begin{aligned}\sum_{x=0}^{m} x^n &= \sum_{x=0}^{m} \sum_{k=0}^{n} \left\{ {n \atop k} \right\} (x)_{\underline{k}} \\ &= \sum_{k=0}^{n} \sum_{x=0}^{m} \left\{ {n \atop k} \right\} \frac{1}{k+1} \left((x+1)_{\underline{k+1}} - (x)_{\underline{k+1}} \right). \qquad \square\end{aligned}$$

Bell Numbers

Definition 6.14

The number of partitions of $[n]$ is denoted by $B(n)$ and is called a Bell number.

$$B(n) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}.$$

n	0	1	2	3	4	5	6	7	8
$B(n)$	1	1	2	5	15	52	203	877	4140

Bell Numbers

Theorem 6.15

$$B(n+1) = \sum_{k=0}^{n} B(k)\binom{n}{k}.$$

Proof:

• LHS counts # of partitions of $[n+1]$

• Element $n+1$ may be in the same block with $n-k$ other elements and there are $\binom{n}{n-k} = \binom{n}{k}$ ways

• The remaining k elements can be partitioned in $B(k)$ ways. $\square$

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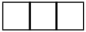
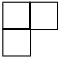
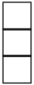
Integer Partitions

Definition 7.1

If a finite sequence $(a_1, a_2, \ldots, a_k)$ of positive integers satisfies $a_1 \geq a_2 \geq \cdots \geq a_k \geq 1$ and $a_1 + a_2 + \cdots + a_k = n$, then we say that the sequence is a partition of the integer n into k parts.

The partitions of 3 are:

3  2+1  1+1+1 

Note the difference between integer partitions, set partitions and compositions.

Integer Partitions

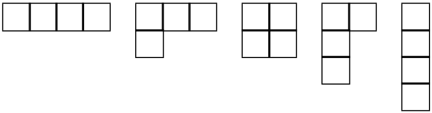
Definition 7.2

We use $p(n,k)$ to denote # of partitions of n into k parts and $p(n)$ to denote # of partitions of n in general

For example $p(4) = 5$, $p(4,2) = 2$ and $p(4,i) = 1$ for $i = 1,3,4$.

$$4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1.$$

With Ferrers diagrams



Values of $p(n)$ for $n = 1 \dots 20$

n	$p(n)$	n	$p(n)$	n	$p(n)$	n	$p(n)$	n	$p(n)$
1	1	2	2	3	3	4	5	5	7
6	11	7	15	8	22	9	30	10	42
11	56	12	77	13	101	14	135	15	176
16	231	17	297	18	385	19	490	20	627


MacMahon


Ramanujan

Formula for $p(n)$?

Hardy-Ramanujan-Rademacher

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n) k^{\frac{1}{2}} \left[\frac{d}{dx} \frac{\sinh\left(\frac{\pi}{k}\left(\frac{2}{3}\left(x - \frac{1}{24}\right)\right)^{\frac{1}{2}}\right)}{\left(x - \frac{1}{24}\right)^{\frac{1}{2}}} \right]_{x=n},$$

$$A_k(n) = \sum_{\substack{h \mod k \\ (h,k)=1}} \omega_{h,k} exp\left(-2\pi i \frac{nh}{k}\right), \qquad \omega_{h,k}^{24} = 1$$

First eight terms give

$$p(200) = 3,972,999,029,388.004.$$

Conjugate Partitions

Definition 7.3

The conjugate of a partition can be obtained by reflecting the Ferrers diagram through the main diagonal.

Example 7.4

$(4, 3, 1)$ *and* $(3, 2, 2, 1)$ *are conjugates*

Ferrers diagrams:

$$\begin{array}{llll} \square & \square & \square & \square \\ \square & \square & \square & \\ \square & & & \end{array} \qquad \begin{array}{lll} \square & \square & \square \\ \square & \square & \\ \square & \square & \\ \square & & \end{array}$$

Conjugate Partitions

Theorem 7.5

The # of partitions of n with at least k parts is equal to # of partitions of n in which the largest part is at least k.

Example 7.6

Partitions of 5*:*

$$5 = 4{+}1 = 3{+}2 = 3{+}1{+}1 = 2{+}2{+}1 = 2{+}1{+}1{+}1 = 1{+}1{+}1{+}1{+}1.$$

- partitions of 5 with at least 3 parts $= 4$
- partitions of 5 with largest part $\geq 3 = 4$

Conjugate Partitions

Theorem 7.7

The # of partitions of n where the first two parts are equal is equal to # of partitions of n in which each part is at least 2.

Proof: The first two parts of a partition are equal if the first two rows of its Ferrers diagram has the same # of boxes. On the other hand, each part is at least 2 if the first two columns of its Ferrers diagram has the same # of boxes. Taking conjugate establishes a bijection between the two classes of partitions.$\square$

Franklin's bijection

Theorem 7.8

Let $m > k \geq 1$. Define
1) S: partitions of n into m parts, with smallest part $= k$
2) T: partitions of n into $m-1$ parts, with k-th part larger than $(k+1)$-th part and the smallest part is at least k.
Then $|S| = |T|$.

Note: if $k = m-1$, then $(k+1)$-th part in T is 0. In this case, we need the additional condition that the k-th part is larger than k.

Example 7.9

Let $n = 10, m = 2$ and $k = 1$. Then $S = \{(9,1)\}$ and $T = \{(10)\}$.

Franklin's bijection

Example 7.10

Let $n = 16, m = 5$ *and* $k = 3$. *Then* $S = \{(4, 3, 3, 3, 3)\}$ *and* $T = \{(5, 4, 4, 3)\}$.

Example 7.11

Let $n = 10, m = 4$ *and* $k = 2$. *Then* $S = \{(4, 2, 2, 2), (3, 3, 2, 2)\}$ *and* $T = \{(5, 3, 2), (4, 4, 2)\}$.

Example 7.12

Let $n = 10, m = 3$ *and* $k = 2$. *Then* $S = \{(6, 2, 2), (5, 3, 2), (4, 4, 2)\}$ *and* $T = \{(7, 3), (6, 4), (5, 5)\}$.

Franklin's bijection

Proof of Theorem 7.8: Define a bijection $f : S \mapsto T$ by removing the last row of the Ferrers diagram and distributing the k boxes into each of the first k rows.

Check well defined: If $s \in S$,

i) $f(s)$ is a partition of n

ii) $f(s)$ has $m-1$ parts since s has m parts

iii) $f(s)$ has k-th part larger than $(k+1)$-th part

iv) $f(s)$ has smallest part $\geq k$

$\implies f(s) \in T$.

Check bijection: Do this by showing that an inverse of f exists. Let $t \in T$, we remove the last box from the first k rows and add this as the last row to get a partition in S. Since smallest part of t is at least k, we can add a row of k to get a partition with m parts and smallest part $= k$. The condition k-th part is larger than $(k+1)$-th part ensures the partition is non-decreasing. $\square$

Euler's Pentagonal Number Theorem

Definition 7.13

Let $p_{d,e}(n)$ and $p_{d,o}(n)$ denote respectively, the # of partitions of n into an even (resp. odd) # of parts, where each part is distinct.

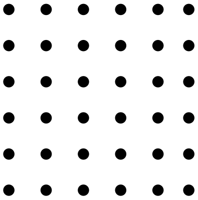
Example 7.14

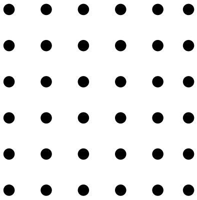
n	1	2	3	4	5	6	7
$p_{d,e}(n)$	0	0	1	1	2	2	3
$p_{d,o}(n)$	1	1	1	1	1	2	2

$p_{d,e}(5)$ counts $\{(4,1),(3,2)\}$ while $p_{d,o}(5)$ counts $\{(5)\}$
$p_{d,e}(6)$ counts $\{(5,1),(4,2)\}$ while $p_{d,o}(6)$ counts $\{(3,2,1),(6)\}$
$p_{d,e}(7)$ counts $\{(6,1),(5,2),(4,3)\}$
while $p_{d,o}(7)$ counts $\{(4,2,1),(7)\}$

Triangular, Square and Pentagonal Numbers

n	1	2	3	4	5	6	7
$\frac{n^2+n}{2}$	1	3	6	10	15	21	28
n^2	1	4	9	16	25	36	49
$\frac{3n^2-n}{2}$	1	5	12	22	35	51	70





Euler's Pentagonal Number Theorem

Theorem 7.15

Let n be a positive integer, then

$$p_{d,e}(n) - p_{d,o}(n) = \begin{cases} (-1)^j & \text{if } n = \frac{3j^2 \pm j}{2} \\ 0 & \text{otherwise} . \end{cases}$$

Example 7.16

- $12 = \frac{1}{2}(3(3)^2 - 3)$
- $p_{d,e}(12) = 7$ *counts* $\{(6,3,2,1),(5,4,2,1),(11,1),(10,2),(9,3),(8,4),(7,5)\}$
- $p_{d,o}(12) = 8$ *counts* $\{(9,2,1),$ $(8,3,1),(7,4,1),(7,3,2),(6,5,1),(6,4,2),(5,4,3),(12)\}$

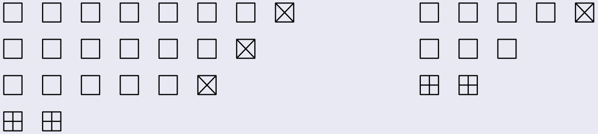
Euler's Pentagonal Number Theorem

Definition 7.17

Let D be the partitions of n into m distinct parts. If $\lambda \in D$,
$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, $\lambda_i > \lambda_j$ if $i < j$.
Define $last(\lambda) = \lambda_m$, the last part of λ
and $stair(\lambda) = s$ to be largest integer such that $\lambda_1, \lambda_2, \ldots, \lambda_s$ is consecutive.

Example 7.18

$\lambda = (8, 7, 6, 2)$ *has* $stair(\lambda) = 3$ *and* $last(\lambda) = 2$
$\mu = (5, 3, 2)$ *has* $stair(\mu) = 1$ *and* $last(\mu) = 2$



Euler's Pentagonal Number Theorem

Theorem 7.19

Let n be a positive integer, then

$$p_{d,e}(n) - p_{d,o}(n) = \begin{cases} (-1)^j & \text{if } n = \frac{3j^2 \pm j}{2} \\ 0 & \text{otherwise} \end{cases}.$$

Proof: (Idea)

- Let $D = D_e \cup D_o$ be the partitions of n into distinct parts, where $|D_e| = p_{d,e}(n)$ and $|D_o| = p_{d,o}(n)$.
- Define $F : D_e \to D_o$
- For n not pentagonal number, this is a bijection.
- For n pentagonal, there is exactly one partition that is not mapped.

Proof (cont'd) : Let $\lambda \in D_e$.

Case: $t = last(\lambda) \leq stair(\lambda)$:

- Define $F(\lambda)$ by removing the smallest part and adding a box to the first t rows.
- This is Franklin's bijection (Theorem 7.8)
- F changes the number of parts from even to odd but parts remain distinct.

Case: $last(\lambda) > stair(\lambda) = s$:

- Define $F(\lambda)$ by removing removing a box from the first s rows and using these s boxes to create a new smallest part
- This is the inverse of Franklin's bijection
- F changes the number of parts from even to odd but parts remain distinct.

Hence $F : D_e \to D_o$ is well-defined with one exception.

Case: $t = last(\lambda) \leq stair(\lambda)$:

- Define $F(\lambda)$ by removing the smallest part and adding a box to the first t rows.
- This is Franklin's bijection (Theorem 7.8)
- F changes the number of parts from even to odd but parts remain distinct.

Example: $\lambda = (8, 7, 6, 2)$, $F(\lambda) = (9, 8, 6)$

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Important to check: now $stair(F(\lambda)) = last(\lambda)$.
$\implies stair(F(\lambda)) < last(F(\lambda))$

Exception: $last(\lambda) = stair(\lambda) = m$:

- $F(\lambda) \notin D_o$
- $\lambda = (2m-1, 2m-2, \ldots, m)$
- $n = \frac{1}{2}(2m-1+m)(m)$, pentagonal number, m even

Example: $\lambda = (7,6,5,4)$, $F(\lambda) = (8,7,6,1)$

Case: $last(\lambda) > stair(\lambda) = s$:

- Define $F(\lambda)$ by removing removing a box from the first s rows and using these s boxes to create a new smallest part
- This is the inverse of Franklin's bijection
- F changes the number of parts from even to odd but parts remain distinct.

Example: $\lambda = (8, 7, 4, 3)$, $F(\lambda) = (7, 6, 4, 3, 2)$

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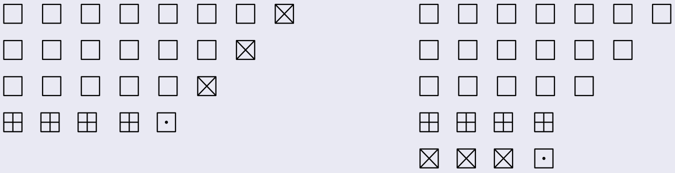
Important to check: now $last(F(\lambda)) = stair(\lambda)$.

$\implies last(F(\lambda)) \leq stair(F(\lambda))$

Exception: $last(\lambda) = stair(\lambda) + 1$:

- $F(\lambda) \notin D_o$
- $stair(\lambda) = m$ and $last(\lambda) = m + 1$
- $\lambda = (2m, 2m - 1, \ldots, m + 1)$
- $n = \frac{1}{2}(2m + m + 1)(m)$, pentagonal number, m even

Example: $\lambda = (8, 7, 6, 5)$, $F(\lambda) = (7, 6, 5, 4, 4)$



Summary $F : D_e \to D_o$

1. is well-defined except when $n = \frac{m(3m\pm1)}{2}$, where m is the number of parts (even). In this case, there is one λ that fails.
2. is onto.
   - If $last(\lambda) \leq stair(\lambda)$ then $last(F(\lambda)) > stair(F(\lambda))$
   - If $last(\lambda) > stair(\lambda)$ then $last(F(\lambda)) \leq stair(F(\lambda))$
   - Hence (with one exception) for every $\mu \in D_o$, we can find $\lambda \in D_e$ such that $F(\lambda) = \mu$. Thus F is onto.
   - Exception occurs when $n = \frac{m(3m\pm1)}{2}$, where m is the number of parts (odd). In this case, there is one λ that fails.
3. is 1-to-1.

Conclusion

$|D_e| = |D_o| + 1$ if $n = \frac{m(3m\pm1)}{2}$, m even.
$|D_e| + 1 = |D_o|$ if $n = \frac{m(3m\pm1)}{2}$, m odd.
$|D_e| = |D_o|$ otherwise. $\square$