NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2006-2007

MA1505 MATHEMATICS I

November 2006 Time allowed: 2 hours

Matriculation Number:

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided above.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **EIGHT (8)** questions and comprises **THIRTY THREE (33)** printed pages.
3. Answer **ALL** questions. For each question, write your answer in the box and your working in the space provided inside the booklet following that question.
4. The marks for each question are indicated at the beginning of the question.
5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

For official use only. Do not write below this line.

Question	1	2	3	4	5	6	7	8
Marks								

Question 1 (a) [5 marks]

Given that $y = t + t^2 + t^5$ and $x = t^3 - t^2$, find the value of $\frac{dy}{dx}$ at the point corresponding to $t = 1$.

Answer 1(a)	8

(Show your working below and on the next page.)

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1+2t+5t^4}{3t^2-2t}$$

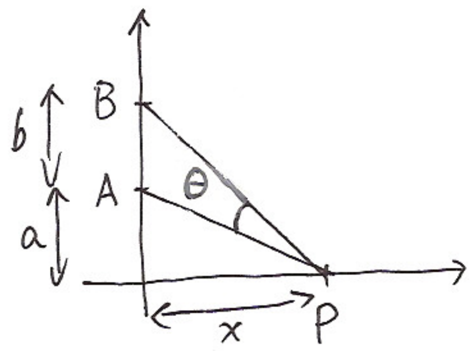
at $t=1$, $\quad \frac{dy}{dx} = \frac{1+2+5}{3-2} = \underline{\underline{8}}$

Question 1 (b) [5 marks]

Let A be the point $(0, a)$ and B be the point $(0, a+b)$, where a and b are two positive constants. Let P denote a variable point $(x, 0)$, where $x > 0$. Find the value of x (in terms of a and b) that gives the largest angle $\angle$APB.

Answer 1(b)	$\sqrt{a(a+b)}$

(Show your working below and on the next page.)



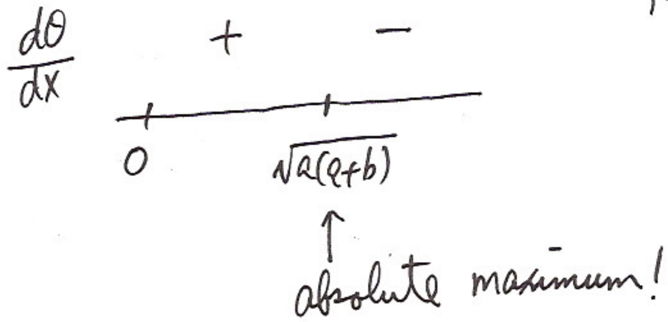
$$\theta = \tan^{-1}\frac{a+b}{x} - \tan^{-1}\frac{a}{x}$$

$$\frac{d\theta}{dx} = \frac{1}{1+\left(\frac{a+b}{x}\right)^2}\,\frac{-(a+b)}{x^2} + \frac{1}{1+\frac{a^2}{x^2}}\,\frac{a}{x^2}$$

$$= \frac{-(a+b)}{x^2+(a+b)^2} + \frac{a}{x^2+a^2}$$

$$= \frac{-(a+b)(x^2+a^2) + a\{x^2+(a+b)^2\}}{\{x^2+(a+b)^2\}(x^2+a^2)}$$

$$= \frac{b\{\sqrt{a(a+b)}-x\}\{\sqrt{a(a+b)}+x\}}{\{x^2+(a+b)^2\}(x^2+a^2)}$$

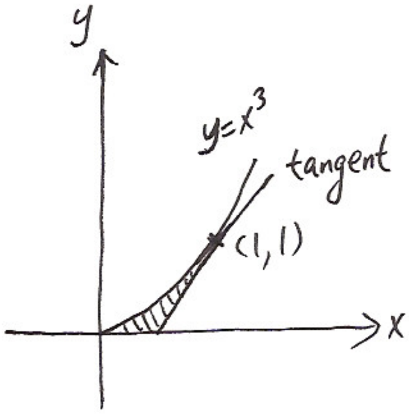


Question 2 (a) [5 marks]

The region R in the first quadrant of the xy-plane is bounded by the curve $y = x^3$, the x-axis and the tangent to $y = x^3$ at the point $(1, 1)$. Find the area of R.

Answer 2(a)	$\frac{1}{12}$

(Show your working below and on the next page.)



$$\frac{dy}{dx} = 3x^2$$

at $(1,1)$, $\frac{dy}{dx} = 3$

Equation of tangent at $(1,1)$ is

$$y - 1 = 3(x-1) = 3x - 3$$

i.e. $x = \frac{y+2}{3}$

$$\text{Area} = \int_0^1 \left(\frac{y+2}{3} - y^{1/3}\right) dy$$

$$= \left[\frac{y^2}{6} + \frac{2}{3}y - \frac{3}{4}y^{4/3}\right]_0^1$$

$$= \frac{1}{6} + \frac{2}{3} - \frac{3}{4} = \frac{2+8-9}{12} = \underline{\underline{\frac{1}{12}}}$$

Question 2 (b) [5 marks]

A thin rod of 2 unit length is placed on the x-axis from $x = 0$ to $x = 2$. Its density varies across the length given by the function

$$\delta(x) = \begin{cases} 6 + x & 0 \leq x < 1 \\ 9 - 2x & 1 \leq x \leq 2. \end{cases}$$

Find the x-coordinate of the center of gravity of the rod.

Answer 2(b)	$\frac{73}{75}$

(Show your working below and on the next page.)

$$\bar{x} = \frac{\int_0^2 x\,\delta(x)\,dx}{\int_0^2 \delta(x)\,dx} = \frac{\int_0^1 x(6+x)\,dx + \int_1^2 x(9-2x)\,dx}{\int_0^1 (6+x)\,dx + \int_1^2 (9-2x)\,dx}$$

$$= \frac{\left[3x^2 + \frac{1}{3}x^3\right]_0^1 + \left[\frac{9}{2}x^2 - \frac{2}{3}x^3\right]_1^2}{\left[6x + \frac{1}{2}x^2\right]_0^1 + \left[9x - x^2\right]_1^2}$$

$$= \frac{3 + \frac{1}{3} + 18 - \frac{16}{3} - \frac{9}{2} + \frac{2}{3}}{6 + \frac{1}{2} + 18 - 4 - 9 + 1} = \underline{\underline{\frac{73}{75}}}$$

Question 3 (a) [5 marks]

Find the radius of covergence of the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}(5x+2)^n.$$

Answer 3(a)	$\frac{1}{5}$

(Show your working below and on the next page.)

$$\left| \frac{\frac{(-1)^{n+1}}{n+2}(5x+2)^{n+1}}{\frac{(-1)^n}{n+1}(5x+2)^n} \right| = \frac{n+1}{n+2}|5x+2| \longrightarrow |5x+2|$$

$$|5x+2| < 1 \Rightarrow |x-(-\tfrac{2}{5})| < \underline{\underline{\tfrac{1}{5}}}$$

Question 3 (b) [5 marks]

Let $f(x) = \left|x - \frac{\pi}{2}\right|$ for all $x \in (0, \pi)$. Let

$$\sum_{n=1}^{\infty} b_n \sin nx$$

be the Fourier Sine Series which represents $f(x)$. Find the value of $b_1 + b_2$.

Answer 3(b)	$\frac{2}{\pi}(\pi - 2)$ (≈ 0.727)

(Show your working below and on the next page.)

$$b_n = \frac{2}{\pi}\int_0^{\pi} \left|x - \frac{\pi}{2}\right| \sin nx\, dx$$

$$= \frac{2}{\pi}\left\{ -\int_0^{\pi/2} \left(x - \frac{\pi}{2}\right) \sin nx\, dx + \int_{\pi/2}^{\pi} \left(x - \frac{\pi}{2}\right) \sin nx\, dx \right\}$$

$$= \frac{2}{\pi}\left\{ \frac{1}{n}\int_0^{\pi/2} \left(x - \frac{\pi}{2}\right) d(\cos nx) - \frac{1}{n}\int_{\pi/2}^{\pi} \left(x - \frac{\pi}{2}\right) d(\cos nx) \right\}$$

$$= \frac{2}{n\pi}\left\{ \left[\left(x - \frac{\pi}{2}\right)\cos nx\right]_0^{\pi/2} - \int_0^{\pi/2} \cos nx\, dx - \left[\left(x - \frac{\pi}{2}\right)\cos nx\right]_{\pi/2}^{\pi} + \int_{\pi/2}^{\pi} \cos nx\, dx \right\}$$

$$= \frac{2}{n\pi}\left\{ \frac{\pi}{2} - \frac{1}{n}\sin\frac{n\pi}{2} - \frac{\pi}{2}\cos n\pi - \frac{1}{n}\sin\frac{n\pi}{2} \right\}$$

$$= \frac{2}{n\pi}\left\{ \frac{\pi}{2}\left[1 - (-1)^n\right] - \frac{2}{n}\sin\frac{n\pi}{2} \right\}$$

$$\therefore b_1 + b_2 = \frac{2}{\pi}(\pi - 2) + 0 = \underline{\underline{\frac{2}{\pi}(\pi - 2)}}$$

Question 4 (a) [5 marks]

Find the distance from the point $(2, -1, 4)$ to the line

$$\mathbf{r}(t) = \mathbf{i} + 2\mathbf{j} + 7\mathbf{k} + t(-3\mathbf{i} + \mathbf{j} - 3\mathbf{k}).$$

Answer 4(a)	$\sqrt{\frac{352}{19}}$

(Show your working below and on the next page.)

Let $\vec{a} = (2,-1,4) - (1,2,7) = (1,-3,-3)$

$$\vec{u} = \frac{-3\vec{i} + \vec{j} - 3\vec{k}}{\sqrt{3^2+1^2+3^2}} = \frac{1}{\sqrt{19}}(-3\vec{i} + \vec{j} - 3\vec{k})$$

$$\vec{a} \times \vec{u} = \begin{vmatrix} i & j & k \\ 1 & -3 & -3 \\ -3 & 1 & -3 \end{vmatrix} \left(\frac{1}{\sqrt{19}}\right) = \frac{1}{\sqrt{19}}(12\vec{i} + 12\vec{j} + 8\vec{k})$$

$$\therefore \text{distance} = \|\vec{a} \times \vec{u}\| = \frac{1}{\sqrt{19}}(12^2 + 12^2 + 8^2)^{1/2}$$

$$= \underline{\underline{\sqrt{\frac{352}{19}}}}$$

Question 4 (b) [5 marks]

Let $f(x,y)$ be a differentiable function of two variables such that $f(2,1) = 1506$ and $\frac{\partial f}{\partial x}(2,1) = 4$. It was found that if the point Q moved from $(2,1)$ a distance 0.1 unit towards $(3,0)$, the value of f became 1505. Estimate the value of $\frac{\partial f}{\partial y}(2,1)$.

Answer 4(b)	18.14

(Show your working below and on the next page.)

Let $\frac{\partial f}{\partial y}(2,1) = a$.

$\vec{u}$ = unit vector from $(2,1)$ to $(3,0) = \frac{(3,0)-(2,1)}{\|(3,0)-(2,1)\|}$

$= \frac{1}{\sqrt{2}}(1, -1)$

$\therefore D_u f(2,1) = (4)\left(\frac{1}{\sqrt{2}}\right) + a\left(-\frac{1}{\sqrt{2}}\right) = \frac{4-a}{\sqrt{2}}$

$\therefore 1505 - 1506 \approx \frac{4-a}{\sqrt{2}}(0.1) = \frac{4-a}{10\sqrt{2}}$

$\therefore -10\sqrt{2} \simeq 4 - a$

$a \simeq 4 + 10\sqrt{2} \approx \underline{\underline{18.14}}$

Question 5 (a) [5 marks]

Find and classify all the critical points of

$$f(x,y) = 4xy - 2x^2 - y^4 - 81.$$

Answer 5(a)	$(-1,-1)$ and $(1,1)$ $\leftrightarrow$ loc. max. $(0,0)$ $\leftrightarrow$ saddle point

(Show your working below and on the next page.)

$f_x = 0 \Rightarrow 4y - 4x = 0 \Rightarrow x = y$ ---- ①

$f_y = 0 \Rightarrow 4x - 4y^3 = 0 \Rightarrow x = y^3$ ---- ②

① & ② $\Rightarrow y^3 - y = 0 \Rightarrow y = -1, 0, 1.$

$\therefore (-1,-1), (0,0), (1,1)$ are the critical points.

$f_{xx} = -4, \quad f_{xy} = 4, \quad f_{yy} = -12y^2$

critical point	f_{xx}	f_{yy}	f_{xy}	$f_{xx}f_{yy} - (f_{xy})^2$
$(-1,-1)$	-4	-12	4	$+$
$(0,0)$	-4	0	4	$-$
$(1,1)$	-4	-12	4	$+$

$\therefore (-1,-1)$ and $(1,1)$ are local maximums.

$(0,0)$ is a saddle point.

Question 5 (b) [5 marks]

Let k be a positive constant. Evaluate

$$\iint_D x^2 e^{xy}\,dx\,dy$$

where D is the plane region given by

$$D: \quad 0 \le x \le 2k \text{ and } 0 \le y \le \frac{1}{2k}.$$

Answer 5(b)	$2k^2$

(Show your working below and on the next page.)

$$\iint_D x^2 e^{xy}\,dx\,dy = \int_0^{2k}\left\{\int_0^{\frac{1}{2k}} x^2 e^{xy}\,dy\right\}dx$$

$$= \int_0^{2k}\left[x e^{xy}\right]_{y=0}^{y=\frac{1}{2k}} dx$$

$$= \int_0^{2k}\left[x e^{\frac{x}{2k}} - x\right] dx$$

$$= 2k\int_0^{2k} x\, d\left(e^{\frac{x}{2k}}\right) - \int_0^{2k} x\,dx$$

$$= 2k\left\{\left[x e^{\frac{x}{2k}}\right]_0^{2k} - \int_0^{2k} e^{\frac{x}{2k}}dx\right\} - \left[\frac{1}{2}x^2\right]_0^{2k}$$

$$= 2k\left\{2ke - 2k\left[e^{\frac{x}{2k}}\right]_0^{2k}\right\} - 2k^2$$

$$= 4k^2 - 2k^2$$

$$= \underline{\underline{2k^2}}$$

Question 6 (a) [5 marks]

Evaluate
$$\int_0^1 \left[\int_{\sqrt{x}}^1 \sin\left(\frac{y^3+1}{2}\right) dy \right] dx.$$

Answer 6(a)	$\frac{2}{3}(\cos\frac{1}{2} - \cos 1)$

(Show your working below and on the next page.)

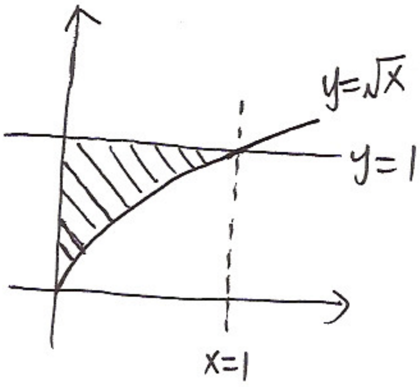
$$\int_0^1 \left(\int_{\sqrt{x}}^1 \sin\left(\frac{y^3+1}{2}\right) dy\right) dx$$

$$= \int_0^1 \left[\int_0^{y^2} \sin\left(\frac{y^3+1}{2}\right) dx\right] dy$$

$$= \int_0^1 y^2 \sin\left(\frac{y^3+1}{2}\right) dy$$

$$= \left[-\frac{2}{3}\cos\left(\frac{y^3+1}{2}\right)\right]_0^1$$

$$= \frac{2}{3}\left[\cos\frac{1}{2} - \cos 1\right]$$



Question 6 (b) [5 marks]

Evaluate

$$\iiint_D |x| \, dxdydz$$

where D is the spherical ball of radius 2 centered at the origin .

Answer 6(b)	8π

(Show your working below and on the next page.)

We use spherical coordinates

$x = r\sin\phi\cos\theta, \quad y = r\sin\phi\sin\theta, \quad z = r\cos\phi$

where $0 \le r \le 2, \quad 0 \le \phi \le \pi, \quad 0 \le \theta \le 2\pi$.

Then $dxdydz = r^2 \sin\phi \, dr \, d\phi \, d\theta$.

$$\iiint_D |x| \, dxdydz = \int_0^{2\pi}\int_0^{\pi}\int_0^{2} r\sin\phi \, |\cos\theta| \, r^2 \sin\phi \, dr\,d\phi\,d\theta$$

$$= \left(\int_0^2 r^3 dr\right)\left(\int_0^{\pi} \sin^2\phi \, d\phi\right)\left(\int_0^{2\pi} |\cos\theta| \, d\theta\right)$$

$$= \left[\frac{1}{4}r^4\right]_0^2 \left(\int_0^{\pi} \frac{1-\cos 2\phi}{2} d\phi\right)\left(4\int_0^{\pi/2} \cos\theta \, d\theta\right)$$

$$= (4)\left(\frac{\pi}{2}\right)(4)$$

$$= \underline{\underline{8\pi}}$$

Question 7 (a) [5 marks]

A force given by the vector field

$$\mathbf{F} = (y+z)\mathbf{i} + (x+2yz)\mathbf{j} + (x+y^2)\mathbf{k}$$

moves a particle from point $P(0,0,0)$ to point $Q(1,2,3)$. Find the work done by $\mathbf{F}$.

Answer 7(a)	17

(Show your working below and on the next page.)

$\because \frac{\partial}{\partial y}(y+z) = \frac{\partial}{\partial x}(x+2yz)$, $\frac{\partial}{\partial z}(y+z) = \frac{\partial}{\partial x}(x+y^2)$, $\frac{\partial}{\partial z}(x+2yz) = \frac{\partial}{\partial y}(x+y^2)$

$\therefore$ F is conservative.

<u>First Solution</u> Finding a potential function for F.

$f_x = y+z \Rightarrow f = xy + xz + g(y,z)$

$\therefore x+2yz = x + g_y \Rightarrow g_y = 2yz \Rightarrow g = y^2z + h(z)$

$\therefore f = xy + xz + y^2z + h(z)$

$f_z = x+y^2 \Rightarrow h'(z) = 0 \Rightarrow h \equiv$ Constant.

$\therefore f = xy + xz + y^2z$ is a potential function.

Workdone $= f(1,2,3) - f(0,0,0) = \underline{\underline{17}}$

<u>Second Solution</u> $\overrightarrow{PQ}$ is given by $\vec{r}(t) = (t, 2t, 3t)$, $0 \le t \le 1$

$\therefore$ Workdone $= \int_0^1 F(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$

$= \int_0^1 (36t^2 + 10t)\,dt$

$= \underline{\underline{17}}$

Question 7 (b) [5 marks]

Evaluate the line integral

$$\int_C \left(\ln\sqrt{1+x^2} - y^3\right) dx + \left(x^3 + \sqrt{1-\sin^3 y}\right) dy$$

where C is the boundary with positive orientation of the region between the circles $x^2+y^2=1$ and $x^2+y^2=4$.

Answer 7(b)	$\dfrac{45\pi}{2}$

(Show your working below and on the next page.)

Let D = region between $x^2+y^2=1$ and $x^2+y^2=4$.

Apply Green's Theorem, we have

$$\int_C (\ln\sqrt{1+x^2} - y^3)dx + (x^3 + \sqrt{1-\sin^3 y})\,dy$$

$$= \iint_R (3x^2+3y^2)\,dx\,dy$$

$$= 3\int_0^{2\pi}\int_1^2 r^3\,dr\,d\theta$$

$$= 6\pi\left[\frac{1}{4}r^4\right]_1^2$$

$$= \frac{90\pi}{4} = \frac{45\pi}{2}$$

Question 8 (a) [5 marks]

Evaluate $\int\int_S F \cdot dS$, where $F = y^2\mathbf{i} + x^2\mathbf{j} + z\mathbf{k}$ and S is the portion of the plane $x + y + z - 1 = 0$ in the first octant. The orientation of S is given by the upward normal vector.

Answer 8(a)	$\frac{1}{3}$

(Show your working below and on the next page.)

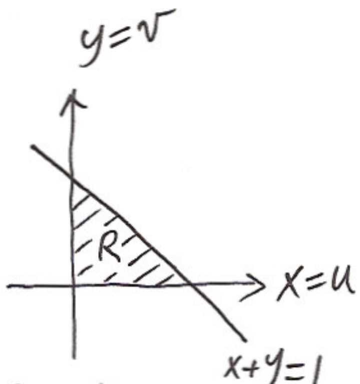
$x+y+z-1=0 \Rightarrow z = 1-x-y$

$\therefore$ a parametric representation of S is

$$\vec{r}(u,v) = u\vec{i} + v\vec{j} + (1-u-v)\vec{k}.$$

$\therefore \vec{r_u} = \vec{i} - \vec{k}$ and $\vec{r_v} = \vec{j} - \vec{k}$.

$$\vec{r_u} \times \vec{r_v} = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \vec{i} + \vec{j} + \vec{k}$$



Projection of S onto the xy-plane

$$\iint_S F \cdot dS = \iint_R F \cdot (\vec{r_u} \times \vec{r_v})\, du\, dv$$

$$= \iint_R \{v^2 + u^2 + (1-u-v)\}\, du\, dv$$

$$= \int_0^1 \int_0^{1-v} (v^2 + u^2 + 1 - u - v)\, du\, dv$$

$$= \int_0^1 \left[v^2 u + \frac{1}{3}u^3 + u - \frac{1}{2}u^2 - vu\right]_{u=0}^{u=1-v} dv$$

$$= \int_0^1 \{2v^2 - v^3 - v + \frac{1}{3}(1-v)^3 + (1-v) - \frac{1}{2}(1-v)^2\}\, dv$$

$$= \left[\frac{2}{3}v^3 - \frac{1}{4}v^4 - \frac{1}{2}v^2 - \frac{1}{12}(1-v)^4 - \frac{1}{2}(1-v)^2 + \frac{1}{6}(1-v)^3\right]_0^1$$

$$= \underline{\underline{\frac{1}{3}}}$$

Question 8 (b) [5 marks]

Using the method of separation of variables, solve the partial differential equation

$$xu_x - yu_y = 0,$$

where $x > 0$ and $y > 0$.

Answer 8(b)	$u = k(xy)^c$

(Show your working below and on the next page.)

Let $u = XY$

$\therefore xX'Y - yXY' = 0 \Rightarrow xX'Y = yXY'$

$\Rightarrow x\frac{X'}{X} = y\frac{Y'}{Y} = c$

$\therefore \frac{X'}{X} = \frac{c}{x}$ and $\frac{Y'}{Y} = \frac{c}{y}$

$\therefore \ln|X| = c\ln|x| + a$ and $\ln|Y| = c\ln|y| + b$

$\therefore X = k_1 x^c$ and $Y = k_2 y^c$

$\therefore u = XY = k(xy)^c$

where k and c are constants.