6. Laplace Transform (Re-visit)

Laplace transform (LT) provides an alternative representation for signals and systems. Many useful insights into the properties of continuous-time linear time-invariant (LTI) systems, as well as the study of many problems involving LTI systems, can be provided by application of the LT technique. Here, LT is reviewed in the context of continuous-time signal and system representations in the s-domain, where s is a complex variable.

6.1 Definitions of Laplace Transform

Bilateral (or two-sided) Laplace transform:

$$\tilde{F}(s) = \tilde{\mathcal{L}}\{f(t)\} = \int_{-\infty}^{\infty} f(t)\exp(-st)\,dt \tag{6.1}$$

Unilateral (or one-sided) Laplace transform:

$$F(s) = \mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} f(t)\exp(-st)\,dt \tag{6.2}$$

- $\boldsymbol{s}$ is complex variable generally expressed as $\boldsymbol{s = \sigma + j\omega}$.
- If $\boldsymbol{f(t) = 0;\ t < 0}$ (i.e. *right-sided*), then $\boldsymbol{\tilde{F}(s) = F(s)}$.

In this module, we are only concerned with the application of Laplace transform to **right-sided** functions. Hence, we focus our discussion on only the **unilateral** Laplace transform and simply refer to it as **Laplace transform** unless otherwise specified.

Example 6-1:

How to use (6.2) to find $\mathcal{L}\left\{\sin\left(\omega_o t\right)u\left(t\right)\right\}$.

$$\begin{aligned}
\mathcal{L}\left\{\sin\left(\omega_o t\right)u\left(t\right)\right\} &= \int_{0^-}^{\infty} \sin\left(\omega_o t\right)\exp\left(-st\right)dt \\
&= \frac{1}{2j}\int_{0^-}^{\infty}\left[\exp\left(j\omega_o t\right)-\exp\left(-j\omega_o t\right)\right]\exp\left(-st\right)dt \\
&= \frac{1}{2j}\int_{0^-}^{\infty}\left[\exp\left(\left(j\omega_o - s\right)t\right)-\exp\left(-\left(j\omega_o + s\right)t\right)\right]dt \\
&= \frac{1}{2j}\left[\frac{\exp\left(\left(j\omega_o - s\right)t\right)}{\left(j\omega_o - s\right)}-\frac{\exp\left(-\left(j\omega_o + s\right)t\right)}{-\left(j\omega_o + s\right)}\right]_{0^-}^{\infty} \\
&= \frac{1}{2j}\left[\frac{1}{j\left(\omega_o - s\right)}-\frac{1}{-j\left(\omega_o + s\right)}\right] \cdots \underbrace{\mathrm{Re}\left[s\right] > 0}_{\text{Region of convergence}} \\
&= \frac{\omega_o}{s^2 - \omega_o^2}
\end{aligned}$$

The range of values of s for which the Laplace transform converges is called the region of convergence (ROC).

Can you find $\mathcal{L}\left\{\cos\left(\omega_o t\right)u\left(t\right)\right\}$ *?*

Instead of evaluating the integral in (6.2) to obtain the transform of a given function, we can easily refer to standard **Laplace transform** tables and read out the desired transform as well as its inverse.

6.2 Properties of the Laplace Transform

In this section, we present some basic **properties of the Laplace transform**. The derivations of these properties can usually be obtain through straightforward algebraic manipulations of (6.1) and (6.2).

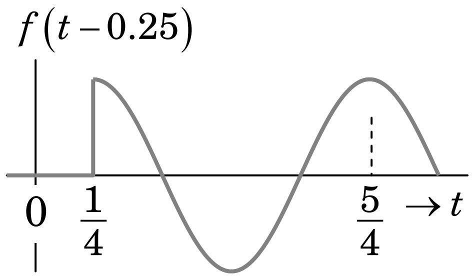
Let $\begin{cases} F(s) = \mathcal{L}\{f(t)\} & \text{denote the Laplace transform of } f(t) \\ f(t) \rightleftarrows F(s) & \text{denote a Laplace transform pair} \end{cases}$

A. ***Linearity*** $$\left\{ \alpha f(t) + \beta g(t) \rightleftarrows \alpha F(s) + \beta G(s) \right\} \tag{6.3}$$

B. ***Time Shifting*** $$\left\{ f(t-t_o) \rightleftarrows \exp(-st_o) F(s) \right\} \tag{6.4}$$

Example 6.2(B):

$$\left.\begin{array}{l} \underbrace{f(t) = \cos(2\pi t)u(t)}_{\text{1 Hz right-sided cosine wave}} \\ F(s) = \dfrac{s}{s^2 + 4\pi^2} \end{array}\right\} \mathcal{L}\underbrace{\left\{ f\left(t - \frac{1}{4}\right) \right\}}_{1/4\text{-cycle delay}} = \exp\left(-\frac{s}{4}\right)\frac{s}{s^2 + 4\pi^2}$$



C. ***Shifting in the s-Domian*** $\left\{ \ \exp(s_o t) f(t) \rightleftarrows F(s - s_o) \ \right\}$. (6.5)

Example 6.2(C):

$$\left.\begin{aligned} f(t) &= u(t) \\ F(s) &= \frac{1}{s} \end{aligned}\right\} \mathcal{L}\{\exp(-t) f(t)\} = F(s+1) = \frac{1}{s+1}$$

D. ***Time Scaling*** $\left\{ f(\alpha t) \rightleftarrows \frac{1}{|\alpha|} F\left(\frac{s}{\alpha}\right) \right\}$ (6.6)

Example 6.2(D):

$$\left.\begin{aligned} f(t) &= \cos(t) u(t) \\ F(s) &= \frac{s}{s^2+1} \end{aligned}\right\} \mathcal{L}\{f(8t)\} = \frac{1}{|8|} F\left(\frac{s}{8}\right) = \frac{1}{8} \frac{s/8}{\left(s^2/8^2+1\right)} = \frac{s}{\left(s^2+8^2\right)}$$

E. ***Integration in the Time Domain*** $\left\{ \int_{0^-}^{t} f(\tau) d\tau \rightleftarrows \frac{1}{s} F(s) \right\}$ (6.7)

Example 6.2(E):

$$\left.\begin{aligned} f(t) &= \sin(t) u(t) \\ F(s) &= \frac{1}{s^2+1} \end{aligned}\right\} \mathcal{L}\left\{\int_{0^-}^{t} f(\tau) d\tau\right\} = \frac{1}{s} \cdot F(s) = \frac{1}{s\left(s^2+1\right)}$$

F. *Differentiation in the Time Domain*

$$\frac{df(t)}{dt} \rightleftarrows sF(s) - f(0^-) \quad \left[\text{or } f^{(n)}(t) \rightleftarrows s^n F(s) - \sum_{k=0}^{n-1} s^{n-1-k} f^{(k)}(0^-)\right] \tag{6.8}$$

where $f^{(k)}(t)$ denotes the k^{th} derivative of $f(t)$ with respect to t.

Example 6.2(F):

$$\left.\begin{aligned} f(t) &= u(t) \\ F(s) &= \frac{1}{s} \end{aligned}\right\} \mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - \underbrace{f(0^-)}_{=0} = s\frac{1}{s} = 1$$

G. *Differentiation in the s-Domian*

$$-tf(t) \rightleftarrows \frac{dF(s)}{ds} \quad \left[\text{or } (-t)^n f(t) \rightleftarrows \frac{d^n F(s)}{ds^n}\right] \tag{6.9}$$

Example 6.2(G):

$$\left.\begin{aligned} f(t) &= \exp(-t)u(t) \\ F(s) &= \frac{1}{s+1} \end{aligned}\right\} \mathcal{L}\{-tf(t)\} = \frac{dF(s)}{ds} = -\frac{1}{(s+1)^2}$$

H. *Convolution in the Time Domain* (*or Multiplication in the s-Domain*)

$$\int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau \rightleftarrows F(s)G(s) \qquad (6.10)$$

Proof:

Begin with the bilateral Laplace transform:

$$\underbrace{\tilde{F}(s)\tilde{G}(s)}_{\text{Bilateral LT's}} = \int_{-\infty}^{\infty} f(\tau)\exp(-s\tau)d\tau \int_{-\infty}^{\infty} g(v)\exp(-sv)dv$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau)g(v)\exp\big(-s(\tau+v)\big)d\tau dv$$

$\cdots\cdots\cdots$ *letting* $t=\tau+v$ *and* $\therefore dt = dv$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau\right]\exp(-st)dt = \tilde{\mathcal{L}}\left\{\int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau\right\}$$

If $f(t)$ and $g(t)$ are right-sided functions, then

$$\underbrace{\tilde{F}(s)\tilde{G}(s)}_{\text{Bilateral LT's}} = \underbrace{F(s)G(s)}_{\text{Unilateral LT's}}.$$

* *see Property H of the Fourier transform in Chapter 2 for more details on convolution.*

I. *Initial Value Theorem*

$$\boldsymbol{f\left(0^{+}\right)=\lim_{s\to\infty} sF(s)} \tag{6.11}$$

Proof:

Begin with Property F (Differentiation in the Time Domain):

$$\begin{aligned}
sF(s)-f\left(0^{-}\right) &= \int_{0^-}^{\infty}\frac{df(t)}{dt}\exp(-st)\,dt \\
&= \int_{0^-}^{0^+}\frac{df(t)}{dt}\exp(-st)\,dt+\int_{0^+}^{\infty}\frac{df(t)}{dt}\exp(-st)\,dt \\
&= f(t)\Big|_{0^-}^{0^+}+\int_{0^+}^{\infty}\frac{df(t)}{dt}\exp(-st)\,dt \\
&= f\left(0^{+}\right)-f\left(0^{-}\right)+\int_{0^+}^{\infty}\frac{df(t)}{dt}\exp(-st)\,dt
\end{aligned}.$$

Thus

$$sF(s)=f\left(0^{+}\right)+\int_{0^+}^{\infty}\frac{df(t)}{dt}\exp(-st)\,dt$$

and

$$\lim_{s\to\infty} sF(s)=f\left(0^{+}\right)+\underbrace{\lim_{s\to\infty}\int_{0^+}^{\infty}\frac{df(t)}{dt}\exp(-st)\,dt}_{=0\ \ \left(\mathrm{Re}[s]>0\right)}=f\left(0^{+}\right).$$

J. *Final Value Theorem*

$$\lim_{t\to\infty} \boldsymbol{f}(\boldsymbol{t}) = \lim_{s\to 0} \boldsymbol{sF}(\boldsymbol{s}) \tag{6.12}$$

Proof:

Begin with Property F (Differentiation in the Time Domain):

$$\lim_{s\to 0}\left[sF(s) - f\left(0^-\right)\right] = \lim_{s\to 0}\int_{0^-}^{\infty}\frac{df(t)}{dt}\exp(-st)\,dt = \int_{0^-}^{\infty}\frac{df(t)}{dt}dt$$

$$= f(t)\Big|_{0^-}^{\infty} = \lim_{t\to\infty} f(t) - f\left(0^-\right)$$

Since

$$\lim_{s\to 0}\left[sF(s) - f\left(0^-\right)\right] = \lim_{s\to 0} sF(s) - f\left(0^-\right)$$

we conclude that

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Remarks:

- This is a convenient theorem to use when you want to determine the steady state value of a system's output response. You can obtain this steady state value from $F(s)$ instead of from $f(t)$.
- This theorem is applicable only when the $f(t)$ has a constant steady state value. For instance, it will fail when $f(t) = \sin(\omega_o t)u(t)$, $f(t) = tu(t)$, etc.

6.3 The Inverse Laplace Transform

Inverse Laplace transform is used to recover the time domain signal from its *s-domain* equivalent. The inverse Laplace transform is defined as:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)\exp(st)\,ds. \tag{6.13}$$

In this complex integral, the real c is to be selected such that if the Region of Convergence (ROC) is $\sigma_1 < \mathrm{Re}[s] < \sigma_2$, then $\sigma_1 < c < \sigma_2$. The evaluation of (6.13) requires understanding of complex variable theory.

For cases where $F(s)$ is a rational function of the form $\dfrac{N(s)}{D(s)}$ where $N(s)$ and $D(s)$ are polynomials in s, we do not need to find the inverse Laplace transform using this integral. Instead, we attempt to express $F(s)$ as a sum

$$F(s) = F_1(s) + F_2(s) + \cdots + F_n(s) \tag{6.14}$$

where $F_1(s), F_2(s), \cdots, F_n(s)$ are functions with known inverse transforms $f_1(t), f_2(t), \cdots, f_n(t)$ such as those given in standard Laplace transform tables. From the Linearity property (6.3) it follows that

$$f(t) = f_1(t) + f_2(t) + \cdots + f_n(t). \tag{6.15}$$

6.3.1 Partial-Fraction Expansion

Let $F(s)$ be a ratio of two polynomials in s, which can be expresses as

$$F(s)=\frac{N(s)}{D(s)}=\frac{K'(s+z_1)(s+z_2)\cdots(s+z_M)}{(s+p_1)(s+p_2)\cdots\cdots(s+p_N)} \tag{6.16}$$

where

$$\left(\begin{array}{l} \boldsymbol{K'} \textbf{ is a constant} \\ \\ \boldsymbol{-z_m;\ m=1,2,\cdots,M} \textit{ are called } \textbf{\textit{zeros}} \textit{ of } \boldsymbol{F(s)} \\ \left[\textit{These are values of } \boldsymbol{s} \textit{ that result in } \boldsymbol{F(s)=0}\right] \\ \\ \boldsymbol{-p_n;\ n=1,2,\cdots,N} \textit{ are called } \textbf{\textit{poles}} \textit{ of } \boldsymbol{F(s)} \\ \left[\textit{These are values of } \boldsymbol{s} \textit{ that result in } \boldsymbol{F(s)=\infty}\right] \end{array}\right).$$

In the following, we shall examine how may ***partial-fraction expansion***, together with (6.14) and (6.15), be used to determine the inversion of $F(s)$ without applying (6.13) explicitly.

A. $F(s)$ *is a Proper Rational Function* $(M < N)$

- **Simple Pole Case**

If the poles $-p_n$ of $F(s)$ (or roots of $D(s)$), are simple (or distinct), then $F(s)$ can be written as

$$F(s) = \frac{N(s)}{D(s)} = \frac{\alpha_1}{s+p_1} + \frac{\alpha_2}{s+p_2} + \cdots + \frac{\alpha_N}{s+p_N} \tag{6.17}$$

where

$$\alpha_n = (s+p_n)F(s)\Big|_{s=-p_n}. \tag{6.18}$$

Example 6.3:

$$F(s) = \frac{2s+4}{(s+1)(s+3)} = \frac{\alpha_1}{(s+1)} + \frac{\alpha_2}{(s+3)}$$

$$\alpha_1 = (s+1)F(s)\Big|_{s=-1} = 1$$

$$\alpha_2 = (s+3)F(s)\Big|_{s=-3} = 1$$

$$\therefore F(s) = \frac{1}{(s+1)} + \frac{1}{(s+3)} \;\Rightarrow\; \left(f(t) = \mathcal{L}^{-1}\{F(s)\} = \left(e^{-t} + e^{-3t}\right)u(t) \right)$$

- **Multiple Pole Case**

If $D(s)$ has multiple roots, i.e., it contains factors of the form $(s+p_n)^r$, we say that $-p_n$ is a ***multiple pole of $F(s)$ with multiplicity r***. The expansion of $F(s)$ will consist of terms of the form

$$\frac{\gamma_1}{s+p_n}+\frac{\gamma_2}{(s+p_n)^2}+\cdots+\frac{\gamma_r}{(s+p_n)^r} \tag{6.19}$$

where

$$\gamma_{r-k}=\frac{1}{k!}\frac{d^k}{ds^k}\left[(s+p_n)^r F(s)\right]\Big|_{s=-p_n} \quad ; \quad k=0,1,\cdots,r-1. \tag{6.20}$$

Example 6.4:

$$F(s)=\frac{s^2+2s+5}{(s+3)(s+5)^2}=\frac{\alpha_1}{(s+3)}+\frac{\gamma_1}{(s+5)}+\frac{\gamma_2}{(s+5)^2}$$

Using (5.18) : $\alpha_1=(s+3)F(s)\big|_{s=-3}=2$

Using $\underbrace{(5.20)}_{\text{with } k=0}$: $\gamma_2=(s+5)^2 F(s)\Big|_{s=-5}=-10$

Using $\underbrace{(5.20)}_{\text{with } k=1}$: $\gamma_1=\frac{d}{ds}\left[(s+5)^2 F(s)\right]\Big|_{s=-5}=\frac{d}{ds}\left[\frac{s^2+2s+5}{s+3}\right]\Big|_{s=-5}=\frac{s^2+6s+1}{(s+3)^2}\Big|_{s=-5}=-1$

$$\therefore F(s)=\frac{2}{(s+3)}-\frac{1}{(s+5)}-\frac{10}{(s+5)^2} \;\Rightarrow\; \left(f(t)=\mathcal{L}^{-1}\{F(s)\}=\left(2e^{-3}-e^{-5}-10te^{-5t}\right)u(t)\right)$$

B. *$F(s)$ is an Improper Rational Function $(M \geq N)$*

If $M \geq N$, we can apply long division to express $F(s)$ in the form

$$F(s)=\frac{N(s)}{D(s)}=Q(s)+\frac{R(s)}{D(s)} \tag{6.21}$$

such that the $\begin{cases}\text{Quotient} \quad : \ Q(s) \text{ is a polynomial in } s \text{ with degree } (M - N), \\ \text{Remainder} : \ R(s) \text{ is a polynomial in } s \text{ with degree strictly less than } N.\end{cases}$

The inverse Laplace transform of $R(s)/D(s)$, which is now a ***proper partial fraction***, can be computed by first expanding into partial fractions.

The inverse Laplace transform of $Q(s)$ can be computed by using

$$\mathcal{L}^{-1}\left\{s^k\right\}=\frac{d^k}{dt^k}\delta(t); \quad k=0,1,2,\cdots\cdots\cdots. \tag{6.22}$$

Example 6.5:

$$F(s)=\underbrace{\frac{2s^2+10s+10}{(s+1)(s+3)}=2+\frac{2s+4}{(s+1)(s+3)}}_{\text{by long division}}$$

$$\therefore F(s)=2+\frac{1}{s+1}+\frac{1}{s+3} \ \Rightarrow \ \left(f(t)=\mathcal{L}^{-1}\left\{F(s)\right\}=2\delta(t)+e^{-t}u(t)+e^{-3t}u(t)\right)$$

6.4 Relationship between the Fourier Transform and the Laplace Transform

Fourier Transform $\}$: $\Im\{f(t)\} = \int_{-\infty}^{\infty} f(t)\exp(-j\omega t)\,dt$ (6.23)

Bilateral Laplace Transform $\}$: $\tilde{F}(s) = \int_{-\infty}^{\infty} f(t)\exp(-st)\,dt$ (6.24)

Unilateral Laplace Transform $\}$: $F(s) = \int_{0^-}^{\infty} f(t)\exp(-st)\,dt$ (6.25)

- From (6.23) and (6.24), we see that the Fourier transform is a special case of the Laplace transform in which $\boldsymbol{s} = \boldsymbol{j\omega}$, that is

$$\Im\{f(t)\} = \tilde{F}(s)\Big|_{s=j\omega} \tag{6.26}$$

- Setting $\boldsymbol{s} = \boldsymbol{\sigma} + \boldsymbol{j\omega}$ in (6.24), we have

$$\begin{aligned}\tilde{F}(\sigma + j\omega) &= \int_{-\infty}^{\infty} f(t)\exp\big(-(\sigma + j\omega)t\big)\,dt = \int_{-\infty}^{\infty}\big[f(t)\exp(-\sigma t)\big]\exp(-j\omega t)\,dt \\ &= \Im\{f(t)\exp(-\sigma t)\}\end{aligned} \tag{6.27}$$

which shows that the **bilateral** Laplace transform of $f(t)$ can be viewed as the Fourier transform of $f(t)e^{-\sigma t}$.

- Considering the Laplace transform as a generalization of the Fourier transform where the frequency variable is generalized from $j\omega$ to $s=\sigma+j\omega$, the complex variable $\boldsymbol{s}$ is often referred to as the ***complex frequency***.

- From the above, it is clear that we **cannot** automatically assume that the Fourier transform of a function $f(t)$ is the Laplace transform with s replaced by $j\omega$.

- If $\boldsymbol{f(t)}$ **is absolutely integrable, i.e.** $\int_{-\infty}^{\infty}\left|\boldsymbol{f(t)}\right|\boldsymbol{dt}<\infty$**,** then the Fourier transform of $f(t)$ can be obtained from the Laplace transform with s replaced by $j\omega$.

  [*Note: In this case,* $f(t)$ *satisfies Dirichlet's* 4^{th} *-Condition – see Chapter 2, Pg 2-12*]

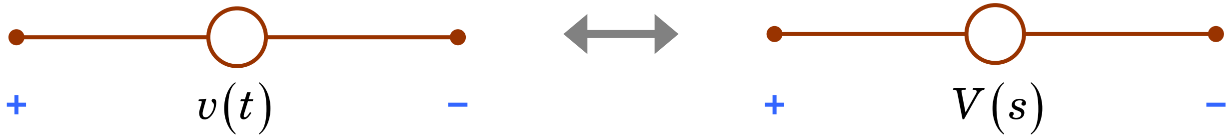
The above relationship between the Fourier Transform and the *Bilateral* Laplace Transform extends fully to *Unilateral* Laplace transform if $\boldsymbol{f(t)}$ is, in addition, a right-sided function.

Example 6.6:

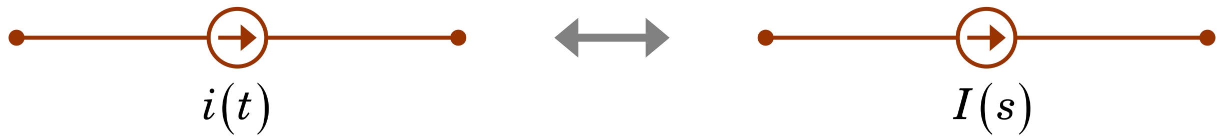
	$f(t)$	$F(s)$	$\Im\{f(t)\}$	Right-sided?	Absolutely Integrable?	$\Im\{f(t)\}=F(j\omega)$
Unit Impulse	$\delta(t)$	1	1	Yes	Yes	Yes
Unit Step	$u(t)$	$\dfrac{1}{s}$	$\pi\delta(\omega)+\dfrac{1}{j\omega}$	Yes	No	No
Exponential	$\exp(-\lvert\alpha\rvert t)u(t)$	$\dfrac{1}{s+\lvert\alpha\rvert}$	$\dfrac{1}{j\omega+\lvert\alpha\rvert}$	Yes	Yes	Yes

6.5 Transform Circuits

- ***Voltage Source:***

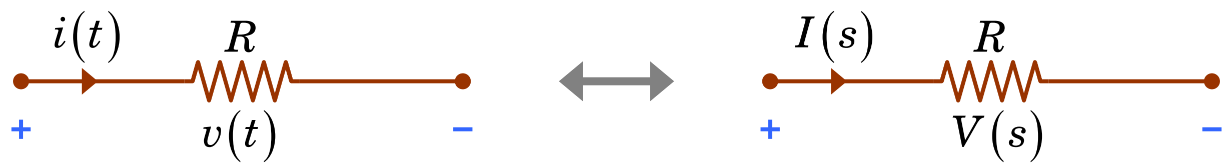


- ***Current Source:***



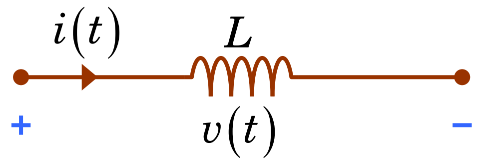
- ***Resistance R:***

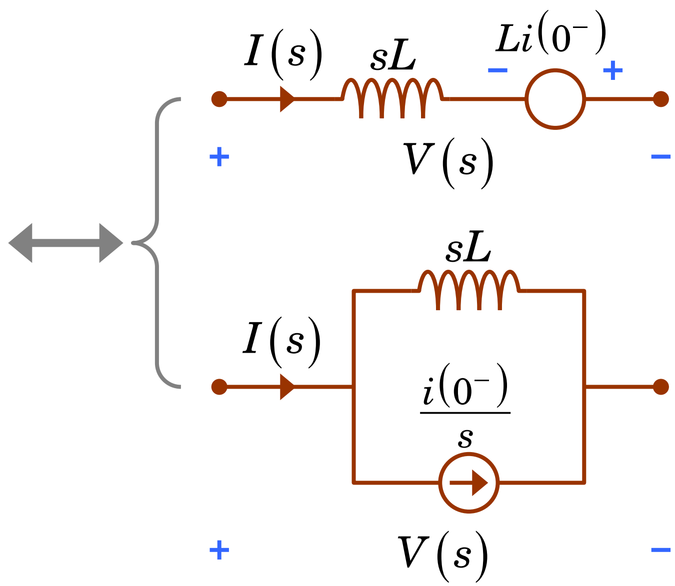
$$v(t) = Ri(t) \rightleftarrows V(s) = RI(s)$$



- ***Inductance L:***

$$v(t) = L\frac{di(t)}{dt} \rightleftarrows V(s) = sLI(s) - Li(0^-)$$

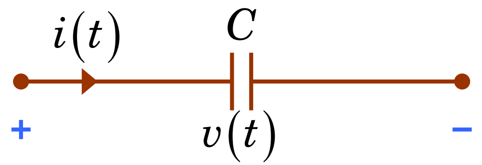


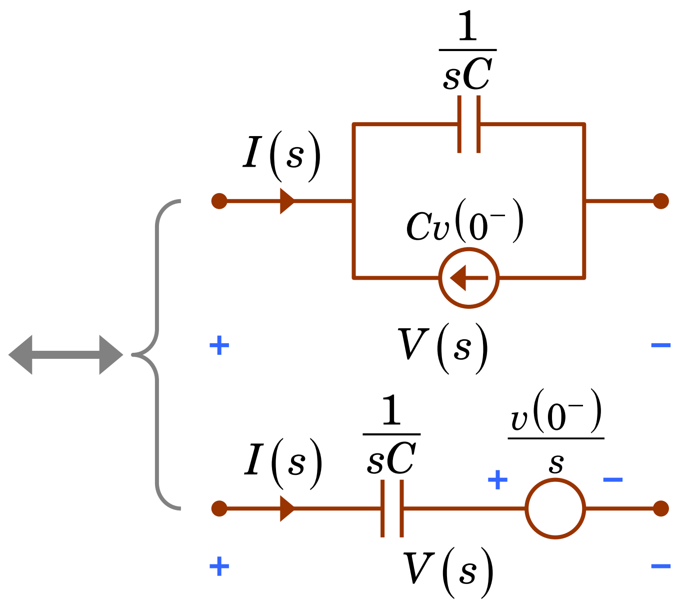


$$i(t) = \frac{1}{L}\int_{0^-}^{t} v(\tau)\,d\tau + i(0^-) \rightleftarrows I(s) = \frac{1}{sL}V(s) + \frac{1}{s}i(0^-)$$

- ***Capacitance C:***

$$i(t) = C\frac{dv(t)}{dt} \rightleftarrows I(s) = sCV(s) - Cv(0^-)$$

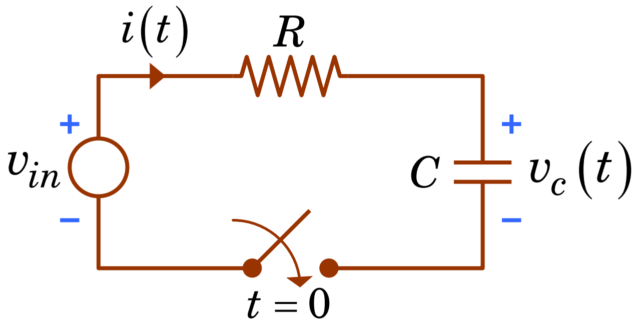




$$v(t) = \frac{1}{C}\int_{0^-}^{t} i(\tau)\,d\tau + v(0^-) \rightleftarrows V(s) = \frac{1}{sC}I(s) + \frac{1}{s}v(0^-)$$

Example 6.7: Series RC Circuit

For the RC circuit shown, find the voltage $v_c(t)$ across the capacitor C.



$$\underbrace{RC\frac{dv_c(t)}{dt}}_{i(t)}+v_c(t)=v_{in}$$

$$sRCV_c(s)-RCv_c(0^-)+V_c(s)=\frac{v_{in}}{s}$$

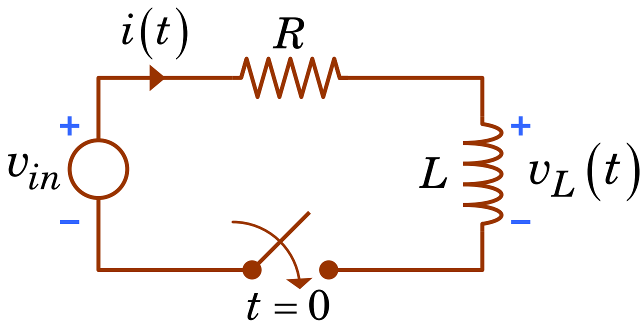
$$V_c(s)=\frac{RCv_c(0^-)}{sRC+1}+\frac{v_{in}}{s(sRC+1)}=\frac{RCv_c(0^-)}{sRC+1}+\frac{v_{in}}{s}-\frac{v_{in}RC}{sRC+1} \quad (\clubsuit)$$

$$\left.\begin{aligned} v_c(t)&=\mathcal{L}^{-1}\{V_c(s)\}=\mathcal{L}^{-1}\left\{\frac{v_c(0^-)}{s+\frac{1}{RC}}\right\}+\mathcal{L}^{-1}\left\{\frac{v_{in}}{s}-\frac{v_{in}}{s+\frac{1}{RC}}\right\}\\ &=\left[v_c(0^-)-v_{in}\right]\exp\left(-\frac{t}{RC}\right)+v_{in}\end{aligned}\right\} \quad (\heartsuit)$$

From $(\clubsuit)$ and $(\heartsuit)$, we observe that: $\underbrace{\lim_{t\to\infty}v_c(t)=\lim_{s\to 0}sV_c(s)}_{\text{Final Value Theorem}}=v_{in}$

Example 6.8: Series RL Circuit

For the RL circuit shown, find the current $i(t)$ flowing through the circuit.



$$\underbrace{L\frac{di(t)}{dt}}_{v_L(t)} + Ri(t) = v_{in}$$

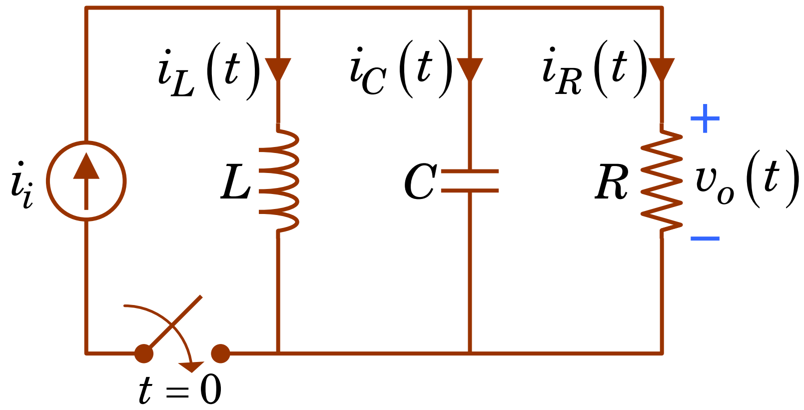
$$sLI(s) - Li(0^-) + RI(s) = \frac{v_{in}}{s}$$

$$I(s) = \frac{Li(0^-)}{sL+R} + \frac{v_{in}}{s(sL+R)} = \frac{Li(0^-)}{sL+R} + \frac{v_{in}/R}{s} - \frac{v_{in}L/R}{sL+R} \qquad (\clubsuit)$$

$$\left.\begin{aligned} i(t) &= \mathcal{L}^{-1}\{I(s)\} = \mathcal{L}^{-1}\left\{\frac{i(0^-)}{s+\frac{R}{L}}\right\} + \mathcal{L}^{-1}\left\{\frac{v_{in}}{R}\cdot\frac{1}{s} - \frac{v_{in}}{R}\cdot\frac{1}{s+\frac{R}{L}}\right\} \\ &= \left[i(0^-) - \frac{v_{in}}{R}\right]\exp\left(-\frac{Rt}{L}\right) + \frac{v_{in}}{R} \end{aligned}\right\} \qquad (\heartsuit)$$

From $(\clubsuit)$ and $(\heartsuit)$, we observe that: $\underbrace{\lim_{t\to\infty} i(t) = \lim_{s\to 0} sI(s)}_{\text{Final Value Theorem}} = \frac{v_{in}}{R}$

Example 6.9: Parallel RLC Circuit



For the RLC circuit shown, write down the differential equation for solving $v_o(t)$ and the expression for $V_o(s)$. Determine $v_o(t)$ for the case where the circuit is relaxed at $t=0^-$, and $i_i=1$, $L=C=1$, $R>0.5$.

Differential equation:

$$i_i = \underbrace{i_L(0^-) + \frac{1}{L}\int_{0^-}^{t} v_o(\tau)\,d\tau}_{i_L(t)} + \underbrace{C\frac{dv_o(t)}{dt}}_{i_C(t)} + \underbrace{\frac{1}{R}v_o(t)}_{i_R(t)}$$

Expression for $\mathbf{V_o(s)}$:

$$\frac{i_i}{s} = \frac{i_L(0^-)}{s} + \frac{1}{sL}V_o(s) + sCV_o(s) - Cv_o(0^-) + \frac{1}{R}V_o(s)$$

$$V_o(s) = \left[\frac{i_i}{s} - \frac{i_L(0^-)}{s} + Cv_o(0^-)\right]\frac{sLR}{s^2RLC + sL + R}$$

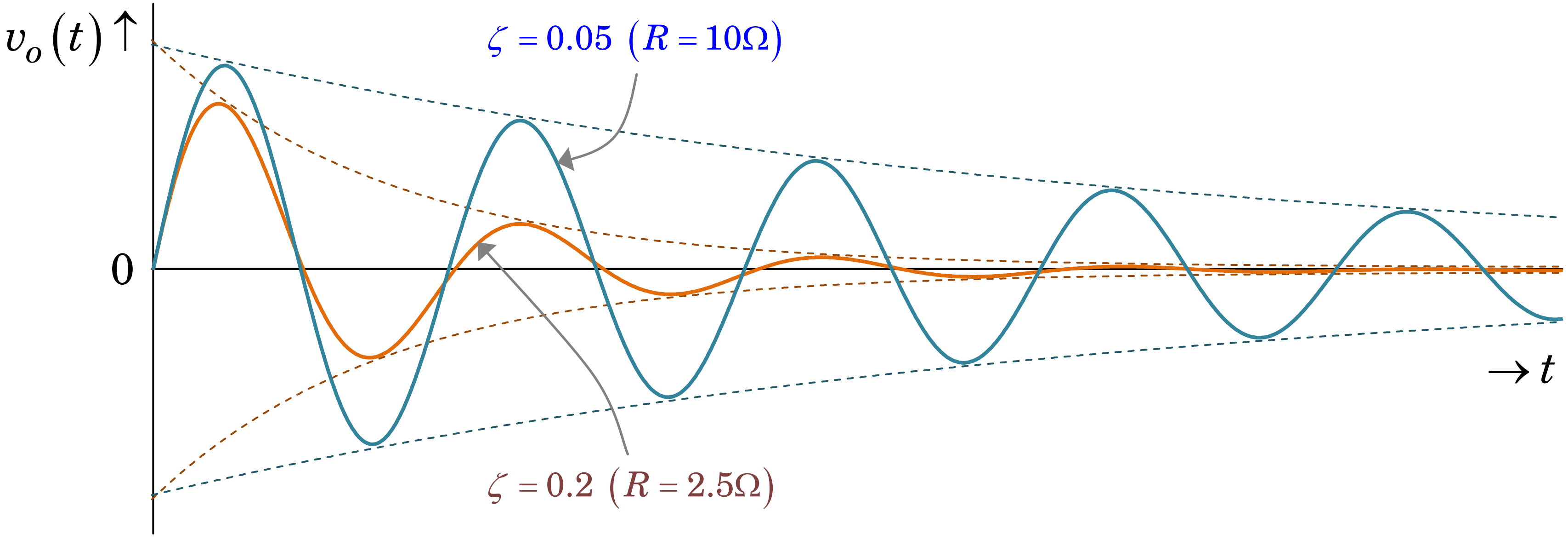
Finding $v_o(t)$: Given:
- Circuit is relax at $t = 0^-$ which implies $i_L(0^-) = v_o(0^-) = 0$.
- $L = C = 1,\ R > 0.5$ and $i_i = 1$.

Let $\zeta = \dfrac{1}{2R}$ $\cdots$ $(0 \le \zeta < 1$ because $R > 0.5)$.

$$V_o(s) = \frac{1}{s^2 + 2\zeta s + 1} = \frac{\left(1-\zeta^2\right)^{1/2}}{(s+\zeta)^2 + \left(1-\zeta^2\right)}\left(1-\zeta^2\right)^{-1/2}$$

Using Laplace transform tables:

$$v_o(t) = \left(1-\zeta^2\right)^{-1/2}\exp(-\zeta t)\sin\left(\left(1-\zeta^2\right)^{1/2} t\right)u(t)$$



RETURN

Unilateral Laplace Transform: $X(s)=\int_{0^-}^{\infty} x(t)\exp(-st)\,dt$

LAPLACE TRANSFORMS OF BASIC FUNCTIONS		
	$x(t)$	$X(s)$
Unit Impulse	$\delta(t)$	1
Unit Step	$u(t)$	$1/s$
Ramp	$t\,u(t)$	$1/s^2$
nth order Ramp	$t^n u(t)$	$\dfrac{n!}{s^{n+1}}$
Damped Ramp	$t\exp(-\alpha t)u(t)$	$1/(s+\alpha)^2$
Exponential	$\exp(-\alpha t)u(t)$	$1/(s+\alpha)$
Cosine	$\cos(\omega_o t)u(t)$	$s/(s^2+\omega_o^2)$
Sine	$\sin(\omega_o t)u(t)$	$\omega_o/(s^2+\omega_o^2)$

LAPLACE TRANSFORMS OF BASIC FUNCTIONS		
	$x(t)$	$X(s)$
Damped Cosine	$\exp(-\alpha t)\cos(\omega_o t)u(t)$	$\dfrac{s+\alpha}{(s+\alpha)^2+\omega_o^2}$
Damped Sine	$\exp(-\alpha t)\sin(\omega_o t)u(t)$	$\dfrac{\omega_o}{(s+\alpha)^2+\omega_o^2}$
Step response of 1st order system	$[1-\exp(-at)]u(t)$	$\dfrac{a}{s(s+a)}$
Step response of 2nd order underdamped system	$K\left\{1-\dfrac{\exp(-\sigma t)}{\sqrt{1-\zeta^2}}\sin[\omega_d t+\phi]\right\}$	$\dfrac{K\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$
	$\sigma=\omega_n\zeta,\quad \omega_d=\omega_n\sqrt{1-\zeta^2},\quad \phi=\tan^{-1}(\omega_d/\sigma)$	

RETURN

LAPLACE TRANSFORM PROPERTIES		
	Time-domain	***s*-domain**
Linearity	$\alpha x_1(t)+\beta x_2(t)$	$\alpha X_1(s)+\beta X_2(s)$
Time shifting	$x(t-t_o)$	$\exp(-st_o)X(s)$
Shifting in the s-domain	$\exp(s_o t)x(t)$	$X(s-s_o)$
Time scaling	$x(\alpha t)$	$\frac{1}{\lvert\alpha\rvert}X\left(\frac{s}{\alpha}\right)$
Integration in the time-domain	$\int_{0^-}^{t} x(\zeta)d\zeta$	$\frac{1}{s}X(s)$
Differentiation in the time-domain	$\frac{dx(t)}{dt}$	$sX(s)-x\left(0^-\right)$
	$\frac{d^n x(t)}{dt^n}$	$s^n X(s)-\sum_{k=0}^{n-1} s^{n-1-k}\left.\frac{d^k x(t)}{dt^k}\right\rvert_{t=0^-}$
Differentiation in the s-domain	$-tx(t)$	$\frac{dX(s)}{ds}$
	$(-t)^n x(t)$	$\frac{d^n X(s)}{ds^n}$
Convolution in the time-domain	$\int_{-\infty}^{\infty} x_1(\zeta)x_2(t-\zeta)d\zeta$	$X_1(s)X_2(s)$
Initial value theorem	$x\left(0^+\right)=\lim_{s\to\infty} sX(s)$	
Final value theorem	$\lim_{t\to\infty} x(t)=\lim_{s\to 0} sX(s)$	