1. From the results of Tutorial 9,

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^4 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 16 \end{pmatrix} \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 8 & 8 \\ 8 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1/2 & 1 \\ 1/2 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} & \\ & \end{pmatrix}^4 = \begin{pmatrix} 41 & 80 \\ 20 & 41 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} 2-i & 0 \\ 0 & 2+i \end{pmatrix} \begin{pmatrix} 1/2 & i/2 \\ 1/2 & -i/2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}^4 = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} (2-i)^4 & 0 \\ 0 & (2+i)^4 \end{pmatrix} \begin{pmatrix} 1/2 & i/2 \\ 1/2 & -i/2 \end{pmatrix}$$

$$= \begin{pmatrix} -7-24i & -7+24i \\ -24+7i & -24-7i \end{pmatrix} \begin{pmatrix} 1/2 & i/2 \\ 1/2 & -i/2 \end{pmatrix}$$

$$= \begin{pmatrix} -7 & 24 \\ -24 & -7 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}^4 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

2. By definition $x_{2+k} = x_{1+k} + x_k$

so $$\begin{pmatrix} x_{k+1} \\ x_{k+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_k \\ x_{k+1} \end{pmatrix}$$

We need eigenvalues and eigenvectors of $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - \lambda - 1 = 0$$

$$\Rightarrow \lambda = \tfrac{1}{2}\left[+1 \pm \sqrt{5}\right] = \lambda\pm$$

Eigenvectors are $\begin{pmatrix} 1 \\ \lambda_+ \end{pmatrix} \begin{pmatrix} 1 \\ \lambda_- \end{pmatrix}$

So

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{pmatrix} \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} \begin{pmatrix} -\lambda_- & 1 \\ \lambda_+ & -1 \end{pmatrix}$$

Hence

$$\begin{aligned} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^k &= \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{pmatrix} \begin{pmatrix} \lambda_+^k & 0 \\ 0 & \lambda_-^k \end{pmatrix} \begin{pmatrix} -\lambda_- & 1 \\ \lambda_+ & -1 \end{pmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{pmatrix} \begin{pmatrix} -\lambda_+^k \lambda_- & \lambda_+^k \\ \lambda_+ \lambda_-^k & -\lambda_-^k \end{pmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{pmatrix} \lambda_+ \lambda_-^k - \lambda_+^k \lambda_- & \lambda_+^k - \lambda_-^k \\ \lambda_+ \lambda_-^{k+1} - \lambda_+^{k+1} \lambda_- & \lambda_+^{k+1} - \lambda_-^{k+1} \end{pmatrix} \end{aligned}$$

Now $V_k = AV_{k-1} = A^2 V_{k-2} = A^3 V_{k-3} = \text{etc} \;\; \text{etc etc} \;= A^k V_0 = A^k \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

So

$$\begin{aligned} V_k = \begin{pmatrix} x_k \\ x_{k+1} \end{pmatrix} &= \begin{pmatrix} & \\ & \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{pmatrix} \lambda_+^k - \lambda_-^k \\ \\ \lambda_+^{k+1} - \lambda_-^{k+1} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \Rightarrow x_k &= \frac{1}{\sqrt{5}} \left(\lambda_+^k - \lambda_-^k \right) \\ x_{k+1} &= \frac{1}{\sqrt{5}} \left(\lambda_+^{k+1} - \lambda_-^{k+1} \right) \\ \Rightarrow \frac{x_{k+1}}{x_k} &= \left(\lambda_+^{k+1} - \lambda_-^{k+1} \right) / \left(\lambda_+^k - \lambda_-^k \right) \\ &= \frac{\lambda_+ - \lambda_- \left(\frac{\lambda_-}{\lambda_+} \right)^k}{1 - \left(\frac{\lambda_-}{\lambda_+} \right)^k} \end{aligned}$$

But $\left| \frac{\lambda_-}{\lambda_+} \right| = \left| \frac{1-\sqrt{5}}{1+\sqrt{5}} \right| < 1$ so $\lim_{k \to \infty} \left(\frac{\lambda_-}{\lambda_+} \right)^k = 0$, hence $\frac{x_{k+1}}{x_k} \to \lambda_+$.

3.

$$\begin{pmatrix} \Sigma_{k+1} \\ \sigma_{k+1} \end{pmatrix} = \begin{pmatrix} 1/2 & 1/100 \\ -50/4 & 5/4 \end{pmatrix} \begin{pmatrix} \Sigma_k \\ \sigma_k \end{pmatrix}$$

Usual procedure $\Rightarrow$

$$\begin{pmatrix} 1/2 & 1/100 \\ -50/4 & 5/4 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 25 & 50 \end{pmatrix} \begin{pmatrix} 3/4 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1/25 \\ -1 & 1/25 \end{pmatrix}$$

Now clearly

$$\begin{pmatrix} \Sigma_{k+1} \\ \sigma_{k+1} \end{pmatrix} = \begin{pmatrix} 1/2 & 1/100 \\ -50/4 & 5/4 \end{pmatrix}^{k+1} \begin{pmatrix} \Sigma_0 \\ \sigma_0 \end{pmatrix}$$

"Long run" means $k \to \infty$ so since $(3/4)^k \to 0$ as $k \to \infty$ what we need is

$$\begin{pmatrix} 1 & 1 \\ 25 & 50 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1/25 \\ -1 & 1/25 \end{pmatrix} \begin{pmatrix} 50 \\ 1600 \end{pmatrix} = \begin{pmatrix} -1 & 1/25 \\ -50 & 2 \end{pmatrix} \begin{pmatrix} 50 \\ 1600 \end{pmatrix} = \begin{pmatrix} 14 \\ 700 \end{pmatrix}.$$

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4. Let $R(t)$ and $D(t)$ be functions representing the feeling of Romeo and Desdemona. We have $\frac{dR}{dt} = aD$, $\frac{dD}{dt} = bR$ where a and b are BOTH positive. Also we know that $D(0) = 0$, $R(0) = \alpha > 0$ from the wording of the question. Since $D(0) = 0$, this $\Rightarrow \frac{dR}{dt}(0) = 0$. Clearly $\ddot{R} = a\dot{D} = abR \Rightarrow R = Ae^{\lambda t} + Be^{-\lambda t}$, $\lambda = \sqrt{ab}$. Initial conditions $\Rightarrow R = \alpha \cosh(\lambda t)$. Also $D = \frac{1}{a}\dot{R} = \frac{\alpha\lambda}{a} \sinh(\lambda t)$. Now we have an expression for D/R and we can see what happens as t tends to infinity:

$$\frac{D}{R} = \frac{\lambda}{a} \tanh(\lambda t) \to \frac{\lambda}{a};$$

that is,

$$\frac{D}{R} \to \sqrt{b/a}$$

Question is whether D can be $> R$ i.e. whether $\frac{D}{R} > 1$. Clearly the answer is yes if $b > a$.

Note that

$\frac{R^2}{a} - \frac{D^2}{b} = \frac{\alpha^2}{a} \cosh^2 \lambda t - \frac{\alpha^2\lambda^2}{a^2 b} \sinh^2 \lambda t$

$= \frac{\alpha^2}{a}(\cosh^2 \lambda t - \sinh^2 \lambda t) = \frac{\alpha^2}{a}$

$=$ const $\to$ Hyperbola.

If b is large then D will eventually exceed R.
But if b is small, that will never happen. Clearly no stable relationship is possible under any circumstances: there is a point of equilibrium at D = R = 0, but it is obviously unstable. With the given initial conditions, the whole relationship gets totally out of control. But in a good way.

5. Let $J(t)$ represent Juliet and $Z(t)$ Othello. Notice that if we reverse the direction of time, $t \to -t$, then $dt \to -dt$. So the equations here,

$$\frac{dJ}{dt} = -aZ, \quad \frac{dZ}{dt} = -bJ$$

are actually the SAME as in Question 5, but with the direction of time reversed. So to get the phase plane diagrams here we just have to reverse the arrows.

We are told that $J(0) > 0$ and $Z(0) > 0$ so we start in the first quadrant. From the diagram [which I am afraid you have to draw for yourself — it's a typical saddle diagram with the contracting axis in the first and third quadrants — we see that there are two possible outcomes. In one case $J \to -\infty$ while $Z \to +\infty$ while in the other $J \to +\infty$ while $Z \to -\infty$. In both cases one of them is consumed by hatred but the other one refuses to save him/herself by running away. In both cases the result is tragic. [In principle, the initial conditions could be such that J and Z move along the unique trajectory passing through the origin, in which case both J and Z tend to zero as time tends to infinity. This rather depressing possibility would however require the initial conditions to be chosen with infinite accuracy, so it would never happen in reality.]

Question 6. We have

$$\vec{V(s)} = [sI - B]^{-1}\vec{v(0)},$$

or

$$\vec{V(s)} = \begin{pmatrix} s-4 & 5 \\ -2 & s+2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \frac{1}{s^2 - 2s + 2} \begin{pmatrix} s+2 & -5 \\ 2 & s-4 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

and so

$$X(s) = \frac{-10}{(s-1)^2+1}, \quad Y(s) = \frac{2(s-1)}{(s-1)^2+1} - \frac{6}{(s-1)^2+1}.$$

Computing the inverse Laplace transforms in the usual manner gives the stated answers.