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NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2005-2006

MA1506 MATHEMATICS II

April 2006 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided at the top of this page.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **Fourteen (14)** questions and comprises **Forty three (43)** printed pages.
3. Answer **ALL** questions. Write your answers and working in the spaces provided inside the booklet following each question.
4. The marks for each question are indicated at the beginning of the question.
5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

For official use only. Do not write below this line.

Question	1	2	3	4	5	6	7	Subtotal
Marks								
Question	8	9	10	11	12	13	14	Subtotal
Marks								
							Total Score	

Question 1 [7 marks]

Use Green's Theorem to evaluate

$$\oint_C (1 + 10xy + y^2)dx + (6xy + 5x^2)dy,$$

where C is the positively oriented triangle with vertices at $(0, 0)$, $(a, 0)$ and $(0, a)$ with $a > 0$.

Show your working below and on the next page.

(Working spaces for Question 1)

Question 2 [7 marks]

Let S be the surface $x^2 + y^2 = 9, \quad 0 \leq z \leq 3$ oriented with outward normal vector. Compute the surface integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S},$$

where $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Show your working below and on the next page.

(Working spaces for Question 2)

Question 3 [7 marks]

Let $\mathbf{F}(x, y, z) = e^x\mathbf{i} + \cos y\mathbf{j} + 2z\mathbf{k}$ and C the curve of intersection of the plane $2y + z = 5$ and the cylinder $x^2 + 4y^2 = 4$, oriented counterclockwise when viewed from above.

(i) Use Stoke's Theorem to evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

(ii) Suppose $\mathbf{F}$ represents a force field. Find the work done by $\mathbf{F}$ in moving a particle from $(0, 0, 0)$ to $(1, 0, 0)$.

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(Working spaces for Question 3)

Question 4 [7 marks]

Use the method of separation of variables to find $u(x, y)$ that satisfies the partial differential equation

$$u_{xy} + \frac{\sin y}{x+2} u = 0,$$

given that $u\left(2, \frac{\pi}{2}\right) = 10$ and $u\left(7, \frac{\pi}{2}\right) = 15.$

Show your working below and on the next page.

(Working spaces for Question 4)

Question 5 [7 marks]

Use Laplace transforms to solve for $w(x,t)$ in the boundary value problem

$$w_x + 2xw_t = 2x,$$

where $w(x,0) = 0$, $w(0,t) = t + e^t$ for $x \geq 0$, $t \geq 0$.
You may refer to the tables on page 43 of this booklet.

Show your working below and on the next 3 pages.

(Working spaces for Question 5)

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Question 6 [7 marks]

Let c be a positive constant. The motion of a string is described by the wave equation $u_{tt} = c^2 u_{xx}$, with boundary conditions $u(0,t) = 0, \quad u(\pi, t) = 0$ for all t, and the initial conditions $u(x, 0) = \sin(14x), \quad u_t(x, 0) = \sin(14x)$ for $0 \leq x \leq \pi$.

(i) Find $u(x,t)$. (*Leave your answer in terms of c.*)

(ii) Let $c = \frac{\sqrt{3}}{14}$. Find the first instant t when the string has no deflection (at any point).

Show your working below and on the next page.

(Working spaces for Question 6)

Question 7 [7 marks]

Let $\mathbf{A}$ be the 3×3 matrix $\begin{bmatrix} 1 & -2 & 1 \\ k & k+2 & k \\ 1 & -1 & 2 \end{bmatrix}$ and $\mathbf{v_1},\ \mathbf{v_2},\ \mathbf{v_3}$ denote the three columns of $\mathbf{A}$.

(i) Compute the determinant of $\mathbf{A}$ in terms of k.

(ii) Use (i) to determine the value(s) of k for which $\mathbf{v_1},\ \mathbf{v_2},\ \mathbf{v_3}$ are linearly *dependent*.

(iii) When $\mathbf{v_1},\ \mathbf{v_2},\ \mathbf{v_3}$ are linearly dependent, express $\mathbf{v_1}$ in terms of $\mathbf{v_2}$ and $\mathbf{v_3}$.

Show your working below and on the next page.

(Working spaces for Question 7)

Question 8 [7 marks]

Given that the eigenvalues of the matrix $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ are 5 and -1, solve the linear system

$$\begin{aligned} y_1' &= y_1 + 2y_2 + e^t \\ y_2' &= 4y_1 + 3y_2 \end{aligned}$$

where y' denotes $\dfrac{\mathrm{d}y}{\mathrm{d}t}$.

Show your working below and on the next page.

(Working spaces for Question 8)

Question 9 [7 marks]

A man is ordered by his doctor to take 5 units of vitamin A, 13 units of vitamin B, and 23 units of vitamin C each day. Three brands of vitamin pills are available, and the number of units of each vitamin per pill are shown in the table below.

	Vitamin (number of units per pill)		
Pill	A	B	C
Brand I	1	2	4
Brand II	1	1	3
Brand III	0	1	1

(i) Let x, y, z denote the number of pills of brand I, II, III respectively. The combinations (x, y, z) that provide the exact required daily amount of vitamins can be solved by a linear system. Write down such a linear system.

(ii) Write down all possible combinations of the number of pills that satisfy the linear system in (i) (no partial pills allowed).

(iii) If brand I costs 3 cents per pill, brand II costs 2 cents per pill, and brand III costs 5 cents per pill, find the least expensive combination in (ii).

Show your working below and on the next 3 pages.

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Question 10 [7 marks]

Let D be the smaller solid region bounded by the sphere $x^2 + y^2 + z^2 = 12$ and the horizontal plane $z = 3$. Set up triple integrals that give the volume of D in the form

(i) $\displaystyle\int_a^b \int_c^d \int_e^f f(\rho, \phi, \theta)\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\theta$ where ρ, ϕ, θ are the spherical coordinates; and

(ii) $\displaystyle\int_s^t \int_p^q \int_m^n g(r, \theta, z)\, \mathrm{d}z\, \mathrm{d}r\, \mathrm{d}\theta$ where r, θ, z are the cylindrical coordinates.

You do not need to find the volume of D.

Show your working below and on the next page.

(Working spaces for Question 10)

Question 11 [9 marks]

Let D be the solid region given by

$$x^2 + y^2 \leq z \leq \sqrt{100 - x^2 - y^2}, \quad -\sqrt{6 - x^2} \leq y \leq \sqrt{6 - x^2}, \quad -\sqrt{6} \leq x \leq \sqrt{6}.$$

Find the total surface area of D.

Show your working below and on the next 3 pages.

(Working spaces for Question 11)

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Question 12 [7 marks]

Suppose that $\mathbf{F}(x, y, z)$ is a vector field with the property that

$$\text{div } \mathbf{F} = 3 \quad \text{for} \quad 1 \leq x^2 + y^2 + z^2 \leq 30$$

and the (outward pointing) flux of $\mathbf{F}$ through the sphere of radius 3 centered at the origin is 8π. Find the (outward pointing) flux of $\mathbf{F}$ through the sphere of radius 5 centered at the origin.

Show your working below and on the next page.

(Working spaces for Question 12)

Question 13 **[7 marks]**

Let $f(x,y) = 49x^2 + 16y^2 - 784$.

(i) Find a plane region R in the xy-plane such that the double integral

$$\iint_R f(x,y)dA$$

has the smallest value.

Give your answer by describing R in terms of ranges of x and y.

(ii) Explain briefly how you get the answer in (i).

You are not required to evaluate the double integral over the region R.

Show your working below and on the next page.

(Working spaces for Question 13)

Question 14 [7 marks]

Let $\mathbf{A}$ be a 4×4 matrix satisfying

$$\mathbf{A}\begin{bmatrix}1\\0\\1\\0\end{bmatrix} = \begin{bmatrix}2\\0\\2\\0\end{bmatrix}, \quad \mathbf{A}\begin{bmatrix}0\\1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}, \quad \mathbf{A}\begin{bmatrix}1\\1\\0\\0\end{bmatrix} = \begin{bmatrix}-1\\-1\\0\\0\end{bmatrix}, \quad \mathbf{A}\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$$

Find all eigenvalues of $\mathbf{A}$ and write down a matrix $\mathbf{P}$ that diagonalizes $\mathbf{A}$.
You are not required to find the matrix $\mathbf{A}$.

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Laplace Transform

Standard Functions

$f(t)$	$F(s) = L(f)$	$f(t)$	$F(s) = L(f)$
1	$\frac{1}{s}$ $(s > 0)$	$\cos at$	$\frac{s}{s^2 + a^2}$ $(s > 0)$
e^{at}	$\frac{1}{s - a}$ $(s > 0)$	$\sin at$	$\frac{a}{s^2 + a^2}$ $(s > 0)$
t^n	$\frac{n!}{s^{n+1}}$ $(s > 0)$	$\cosh at$	$\frac{s}{s^2 - a^2}$ $(s > \|a\|)$
$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$	$\frac{e^{-as}}{s}$ $(s > 0)$	$\sinh at$	$\frac{a}{s^2 - a^2}$ $(s > \|a\|)$

Properties

Linear	$L(af + bg) = aL(f) + bL(g)$
Linear (inv)	$L^{-1}(af + bg) = aL^{-1}(f) + bL^{-1}(g)$
Derivative	$L(f') = sL(f) - f(0)$
2nd Derivative	$L(f'') = s^2 L(f) - sf(0) - f'(0)$
Integral	$L\left(\int_0^t f(x)dx\right) = \frac{1}{s}L(f)$
s-shift	$L(e^{ct}f(t)) = F(s-c)$ $L^{-1}(F(s-c)) = e^{ct}f(t)$
t-shift	$L(f(t-a)u(t-a)) = e^{-as}F(s)$ $L^{-1}(e^{-as}F(s)) = f(t-a)u(t-a)$

[END OF PAPER]