Question 2 (a) [5 marks]

Let

$$f(x) = \frac{x^2 + 1}{x + 1}$$

and let

$$\sum_{n=0}^{\infty} c_n (x+3)^n$$

be the Taylor series for f at $x = -3$. Find the **exact value** of $c_0 + c_1 + c_{101}$.

$$f(x) = \frac{x^2+1}{x+1}$$

Taylor series for f at $x = -3$. $\quad \sum_{n=0}^{\infty} c_n (x+3)^n$

Find $\quad c_0 + c_1 + c_{101}$

$$f(x) = \frac{x^2+1}{x+1}$$
$$= x - 1 + \frac{2}{x+1}$$

$$\begin{array}{r} x-1 \\ x+1 \overline{) x^2 \quad +1} \\ x^2+x \\ \hline -x+1 \\ -x-1 \\ \hline +2 \end{array}$$

$$f(x) = \frac{x^2+1}{x+1}$$
$$= x - 1 + \frac{2}{x+1}$$
$$= (x+3) - 4 + \frac{2}{(x+3)-2}$$

The ***Taylor series*** of f at $x = a$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

$$f(x) = \frac{x^2+1}{x+1}$$

Taylor series for f at $x = -3$. $\sum_{n=0}^{\infty} c_n (x+3)^n$

Find $c_0 + c_1 + c_{101}$

$$f(x) = \frac{x^2+1}{x+1}$$

$$= x - 1 + \frac{2}{x+1}$$

$$= (x+3) - 4 + \frac{2}{(x+3)-2}$$

$$= -4 + (x+3) - \frac{1}{1-\left(\frac{x+3}{2}\right)}$$

$$\frac{1}{1-r} = 1 + r + r^2 + r^3 + \cdots = \sum_{n=0}^{\infty} r^n, \quad |r| < 1$$

$$r = \frac{x+3}{2}$$

$$f(x) = \frac{x^2+1}{x+1}$$

Taylor series for f at $x=-3$. $\sum_{n=0}^{\infty} c_n (x+3)^n$

Find $c_0 + c_1 + c_{101}$

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$$= -4 + (x+3) - \sum_{n=0}^{\infty} \frac{1}{2^n}(x+3)^n$$

$$= -5 + \frac{1}{2}(x+3) - \sum_{n=2}^{\infty} \frac{1}{2^n}(x+3)^n$$

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Taylor series for f at $x = -3$. $\sum_{n=0}^{\infty} c_n (x+3)^n$

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$$\begin{array}{r} x-1 \\ x+1 \overline{\big) x^2 \quad +1} \\ \underline{x^2+x} \\ -x+1 \\ \underline{-x-1} \\ +2 \end{array}$$

$$\frac{1}{2^n}(x+3)^n$$

Put $n = 0$, $\frac{1}{2^0}(x+3)^0 = 1$

Put $n = 1$, $\frac{1}{2^1}(x+3)^1 = \frac{1}{2}(x+3)$

$$f(x) = \frac{x^2+1}{x+1}$$

Taylor series for f at $x = -3$. $\displaystyle\sum_{n=0}^{\infty} c_n (x+3)^n$

Find $c_0 + c_1 + c_{101}$

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$$= -4 + (x+3) - \sum_{n=0}^{\infty} \frac{1}{2^n}(x+3)^n$$

$$= -5 + \frac{1}{2}(x+3) - \sum_{n=2}^{\infty} \frac{1}{2^n}(x+3)^n$$

$$\begin{array}{r|l} & x-1 \\ \hline x+1 & x^2 \quad +1 \\ & x^2+x \\ \hline & -x+1 \\ & -x-1 \\ \hline & +2 \end{array}$$

$c_0 = -5$ $\quad c_1 = \dfrac{1}{2}$ $\quad c_{101} = -\dfrac{1}{2^{101}}$

$$\boxed{\therefore\ c_0 + c_1 + c_{101} = -5 + \frac{1}{2} - \frac{1}{2^{101}} = -\frac{9}{2} - \frac{1}{2^{101}}}$$

Question 2 (b) [5 marks]

A car is moving with speed 20 m/s and acceleration α m/s^2 at a given instant. The car is observed to have moved a distance of 29 m in the next second. Using a second degree Taylor polynomial, estimate the value of α.

We may assume that the car is at the origin with $t=0$ when $v = 20$ m/s and acceleration $= \alpha$ m/s^2.

Let $x =$ distance from origin at time t.

$$\therefore \quad \frac{dx}{dt}(0) = 20, \quad \frac{d^2x}{dt^2}(0) = \alpha$$

When $t = 0$, $x = 0$ since the car is at the origin.

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

$$= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

$$\therefore \quad x \simeq 0 + 20t + \frac{\alpha}{2!}t^2 = 20t + \frac{\alpha}{2}t^2$$

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$$= f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \cdots + \frac{f^{(n)}(0)}{n!} x^n + \cdots$$

$\therefore \quad x \simeq 0 + 20t + \frac{\alpha}{2!} t^2 = 20t + \frac{\alpha}{2} t^2$

$x = 29$ when $t = 1 \Rightarrow 29 = 20 + \frac{\alpha}{2}$

$\Rightarrow \alpha = 18$

Question 3 (a) [5 marks]

Let

$$f(x) = x^2\sqrt{\pi^2 - x^2}, \quad -\pi \le x \le \pi,$$

and $f(x + 2\pi) = f(x)$ for all x. Let

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be the Fourier Series which represents $f(x)$. Find the **exact value** of $b_2 + b_3 + \sum_{n=1}^{\infty} a_n$.

$\because f$ is even

$\therefore b_n = 0 \quad \forall n = 1, 2, 3, \cdots$

Put $x = 0 \Rightarrow \quad a_0 + \sum_{n=1}^{\infty} a_n = f(0) = 0$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi} x^2\sqrt{\pi^2-x^2}\,dx \qquad (\text{let } x=\pi\sin\theta)$$

$$= \frac{1}{\pi}\int_0^{\pi/2} (\pi^2\sin^2\theta)(\pi\cos\theta)(\pi\cos\theta\, d\theta)$$

$$= \frac{\pi^3}{4}\int_0^{\pi/2} \sin^2 2\theta\, d\theta$$

$$= \frac{\pi^3}{8}\int_0^{\pi/2} (1-\cos 4\theta)\, d\theta = \frac{\pi^4}{16}$$

$$b_2 + b_3 + \sum_{n=1}^{\infty} a_n$$

$$a_0 + \sum_{n=1}^{\infty} a_n = f(0) = 0$$

$\because f$ is even

$\therefore b_n = 0 \quad \forall n=1,2,3,\cdots$

$$\therefore b_2+b_3+\sum_{n=1}^{\infty} a_n = -a_0 = \underline{\underline{\frac{-\pi^4}{16}}}$$

Question 4 (b) [5 marks]

Let $\mathbf{A}$ and $\mathbf{B}$ be two non-zero constant vectors and $||\mathbf{B}|| = 2$. If

$$\lim_{x\to\infty} (||x\mathbf{A} + \mathbf{B}|| - ||x\mathbf{A}||) = -\frac{1}{5},$$

find the **exact value** of $\cos\theta$, where θ is the angle between $\mathbf{A}$ and $\mathbf{B}$.

$$\|\mathbf{v}\|^2 = \mathbf{v}.\mathbf{v}$$

$$\|\mathbf{A}\|^2 = \mathbf{A}.\mathbf{A}$$

$$\|\mathbf{B}\|^2 = \mathbf{B}.\mathbf{B}$$

$$\|x\mathbf{A}\|^2 = x\mathbf{A}.x\mathbf{A}$$

$$\|x\mathbf{A} + \mathbf{B}\|^2 = (x\mathbf{A} + \mathbf{B}).(x\mathbf{A} + \mathbf{B})$$

$$u - v = (u - v) \times \frac{(u + v)}{(u + v)}$$

$$= \frac{u^2 - v^2}{u + v}$$

$$\lim_{x\to\infty}\left(\|x\mathbf{A}+\mathbf{B}\|-\|x\mathbf{A}\|\right)=-\frac{1}{5}$$

$$\lim_{x\to\infty}\left(\|x\mathbf{A}+\mathbf{B}\|-\|x\mathbf{A}\|\right)=\lim_{x\to\infty}\frac{\|x\mathbf{A}+\mathbf{B}\|^2-\|x\mathbf{A}\|^2}{\|x\mathbf{A}+\mathbf{B}\|+\|x\mathbf{A}\|}$$

$$u-v=(u-v)\times\frac{(u+v)}{(u+v)}$$

$$=\frac{u^2-v^2}{u+v}$$

$$\|x\mathbf{A}+\mathbf{B}\|^2=(x\mathbf{A}+\mathbf{B}).(x\mathbf{A}+\mathbf{B})$$

$$=x\mathbf{A}.x\mathbf{A}+2x\mathbf{A}.\mathbf{B}+\mathbf{B}.\mathbf{B}$$

$$=\|x\mathbf{A}\|^2+2x\mathbf{A}.\mathbf{B}+\|\mathbf{B}\|^2$$

$$\lim_{x\to\infty}(||x\mathbf{A}+\mathbf{B}||-||x\mathbf{A}||)=-\frac{1}{5}$$

$$L.H.S. = \lim_{x\to\infty}\frac{||xA+B||^2-||xA||^2}{||xA+B||+||xA||}$$

$$= \lim_{x\to\infty}\frac{(xA+B)\cdot(xA+B)-||xA||^2}{||xA+B||+||xA||}$$

$$= \lim_{x\to\infty}\frac{2xA\cdot B+||B||^2}{||xA+B||+||xA||}$$

$$= \lim_{x\to\infty}\frac{2A\cdot B+||B||^2/x}{||A+\frac{B}{x}||+||A||}=\frac{A\cdot B}{||A||}=||B||\cos\theta$$

$$= 2\cos\theta$$

$$\|x\mathbf{A}+\mathbf{B}\|^2=(x\mathbf{A}+\mathbf{B}).(x\mathbf{A}+\mathbf{B})$$
$$=x\mathbf{A}.x\mathbf{A}+2x\mathbf{A}.\mathbf{B}+\mathbf{B}.\mathbf{B}$$
$$=\|x\mathbf{A}\|^2+2x\mathbf{A}.\mathbf{B}+\|\mathbf{B}\|^2$$

divide by x

$$\mathbf{A}.\mathbf{B}=\|\mathbf{A}\|\|\mathbf{B}\|\cos\theta$$

$$\therefore\ 2\cos\theta=-\frac{1}{5}$$

$$\cos\theta=-\frac{1}{10}$$

Question 6 (a) [5 marks]

Find the **exact value** of the integral

$$\int_0^4 \int_{-2}^{-\sqrt{y}} e^{x^3} \, dx\, dy.$$

$-2 \le x \le -\sqrt{y}$

$0 \le y \le 4$

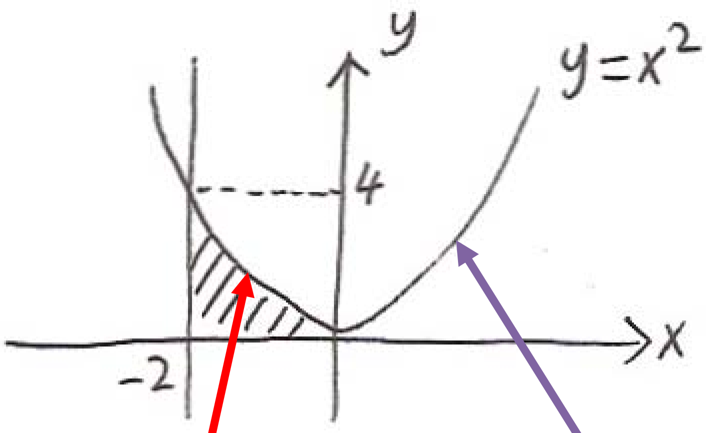
From $x = -\sqrt{y}$, we have $y = x^2$

$$\int_0^4 \int_{-2}^{-\sqrt{y}} e^{x^3} \, dx\, dy = \int_{-2}^{0} \int_{0}^{x^2} e^{x^3} \, dy\, dx$$

$$= \int_{-2}^{0} x^2 e^{x^3} \, dx$$

$$= \frac{1}{3} e^{x^3} \Big|_{-2}^{0}$$

$$= \underline{\underline{\frac{1}{3} - \frac{1}{3} e^{-8}}}$$



$x = -\sqrt{y}$

$x = \sqrt{y}$