CHAPTER 3. BASIC MATHEMATICAL MODELLING

3.1. What is Mathematical Modelling?

Mathematical Modelling is the art of using mathematics to analyse SIMPLE situations which are supposed to approximate VERY COMPLICATED realistic situations.

Some students find it a bit hard to understand the idea of modelling, so let's begin with some examples where the maths is trivial! Then we can focus on the modelling bit. Eventually we want to use our knowledge of differential equations to set up models, but there won't be any differential equations in the first few examples.

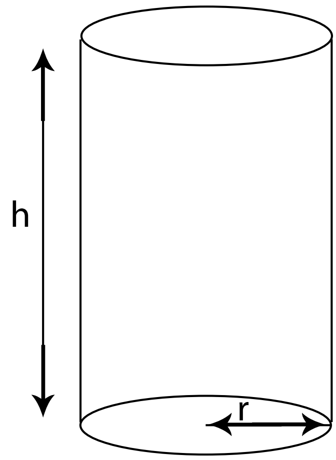
How should CANS be made?

Suppose you are an engineer helping to turn a factory making “tin” cans. Your objective is to MINIMIZE the cost of this process.

Now IN REALITY the function $C()$, which gives you the cost of making one can, depends on a large number of things: the price of “tin”, the shape of the can, cost of sticking its parts together, etc... Very complicated. So let’s start with the SIMPLEST POSSIBLE MODEL:

Model 1

In THIS [very simple] model, we only care about the amount of “tin” actually in the CAN.



If the height of the can is h and its radius is r, you can easily see that the area is:

$$A = 2\pi r^2 + 2\pi r h$$

This is minimized by setting $r = 0$... the cheapest way to make cans IS NOT TO MAKE THEM AT ALL! TRUE!

oh, OK, you want to put something inside those cans!

$$V = \pi r^2 h = CONSTANT$$

$$\Rightarrow h = \frac{V}{\pi r^2}$$

$$A(r) = 2\pi r^2 + \frac{2V}{r}$$

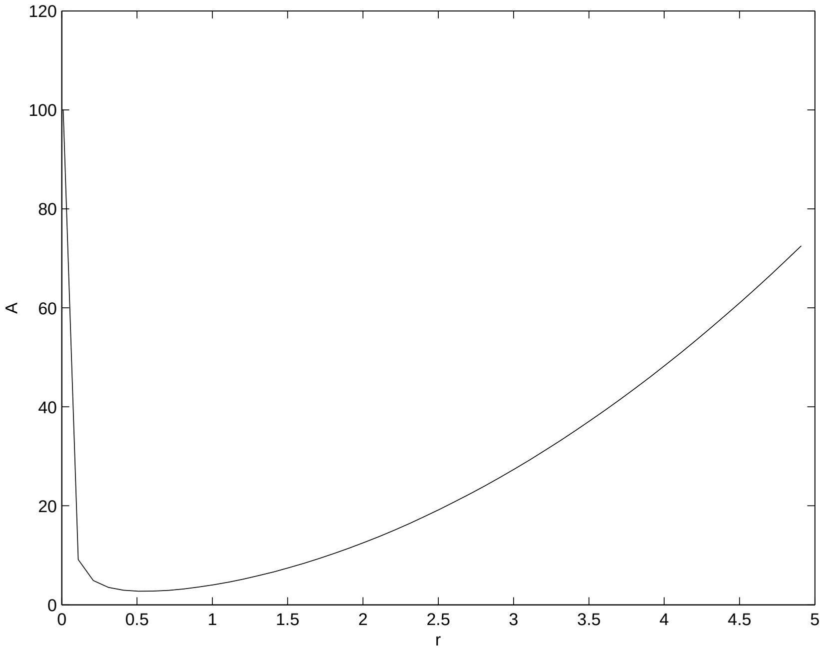
$$A' = 0 \Rightarrow 4\pi r - \frac{2V}{r^2} = 0$$

$$\pi r^2 h = V = 2\pi r^3$$

$$\Rightarrow h = 2r$$

We see that, when we impose the condition that the volume should be constant, the area becomes a function of r, as shown in the graph.

In the graph, we chose h = 1, and you can see that the minimum is indeed, as calculus shows, at r = 1/2: the radius should be half the height,



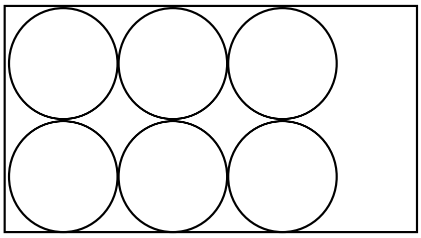
that is, the diameter should EQUAL the height.

SO CANS SHOULD EITHER NOT BE MADE **OR** THEY SHOULD ALWAYS BE EXACTLY AS HIGH AS THEY ARE WIDE

But can-manufacturers don't usually do this, except for LARGE cans, like cans of paint! So our model is predicting something wrong $\rightarrow$

WE NEED A MORE COMPLEX MODEL

Model 2 As with model 1, BUT also we care

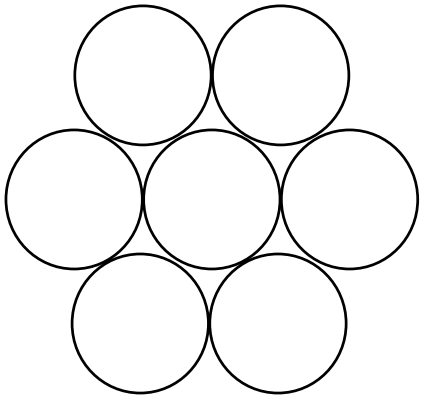


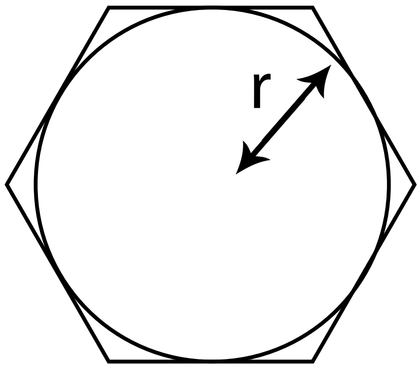
about WASTAGE. The top and bottom of the can are punched out of flat metal, perhaps in the way shown. YOU HAVE TO PAY for the whole sheet!

Well actually you should not punch the holes in this way, but rather in this way instead:

But still you have to pay for a whole HEXAGON, as shown:

An exercise in trigonometry should convince





you that the area of the hexagon shown is $2\sqrt{3}r^2$. So the amount of "tin" we have to pay for is not $2\pi r^2$ but rather $2 \times 2\sqrt{3}r^2$ (multiply by 2 because we need to make the top and bottom).

$$A = 4\sqrt{3}r^2 + \frac{2V}{r}$$

and now

$$\pi r^2 h = V = 4\sqrt{3}r^3$$

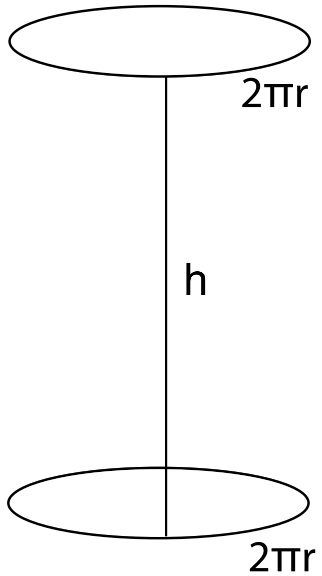
$$\Rightarrow h = \frac{4\sqrt{3}}{\pi} r \text{ instead of } 2r$$

BUT $\frac{4\sqrt{3}}{\pi} \approx 2.21 > 2$. So our NEW MODEL SAYS THAT CANS SHOULD BE ABOUT 10 % HIGHER THAN WIDE... BETTER!

Well this can't be the whole story, since it does not explain the difference of shapes of LARGE cans and SMALL ones! We need a STILL MORE COMPLEX MODEL:

Model 3

As in model 2, but now we care about the PROCESS OF MANUFACTURE, *ie* the cost of actually sticking the can together! The top and bottom have to be welded onto the wall, which itself has a seam. Let's **ASSUME** that the cost of this welding is proportional to its length with constant K (units: \$ / cm)



The total cost of the can is now:

$$C(r) = J\left[4\sqrt{3}r^2 + \frac{2V}{r}\right] + K\left[4\pi r + \frac{V}{\pi r^2}\right]$$

where J = cost of "tin"/cm^2. Now:

$$C'(r) = J\left[8\sqrt{3}r - \frac{2V}{r^2}\right] + K\left[4\pi - \frac{2V}{\pi r^3}\right] = 0$$

$\Rightarrow$ a bit of algebra $\rightarrow$

$$\frac{h}{r} = \frac{4\sqrt{3} + \frac{2\pi K}{rJ}}{\pi + \frac{K}{rJ}}$$

[Notice by the way that the units of K/J are centimetres:

$$\frac{K}{J} = \frac{\$/cm}{\$/cm^2} = cm$$

.

We say that K/J SETS THE SCALE of this problem: when we say that a can is "large", we mean that its diameter or height is large COM-

PARED TO K/J. In engineering problems, one of the most important things to do is to IDENTIFY THE SCALES of the problem, so you know what the words "large" and "small" actually MEAN!]

We see that the ideal shape now depends on (a) $\frac{K}{J}$ (relative cost of welding compared to that of tin) and (b) r (size of the can).

This is a monotonically decreasing function of r. If you examine this function, you see that:

If r is large, then we get back $\frac{4\sqrt{3}}{\pi}$, our old answer from Model 2!

If r is small, then we get h/r $= 2\pi$,

So LARGE CANS SHOULD BE ALMOST

"SQUARE", $h/2r \approx \frac{2\sqrt{3}}{\pi} \approx 1.1$. But SMALL CANS SHOULD BE MADE ABOUT π TIMES AS HIGH AS WIDE, ie they should be tall and thin!

Still not satisfied?

Model 4

And so on...

SUMMARY:

We have been constructing MODELS, that is, very simple versions of a real problem. The real problem is very very complicated, the model is just an APPROXIMATION, but it is easier to understand. BASIC PRINCIPLE: BE-

GIN WITH SIMPLE MODELS, UNDERSTAND THEIR WEAKNESSES, and only then MAKE THEM MORE COMPLICATED!

3.2. MALTHUS MODEL OF POPULATION

The total population of a country is clearly a function of time. Given the population now, can we predict what it will be in the future?

Suppose that B is a function giving the PER CAPITA BIRTH-RATE in a given society, *ie* B is the number of babies born per second, divided by the total population of the country at that moment. Note that B could be small in a big country and large in a small country - it depends on whether there is a strong social pressure on people to get married and have kids. Now B could depend on time (people might might gradually come to realise that large families are no fun, etc...) and it could depend on N (people might realise that it stupid to have many children on a small, crowded island, for instance...) But SUPPOSE YOU DON'T BELIEVE THESE THINGS: suppose you think that people will always have as many kids as they can, no matter what. Then B is constant. Now just as

$$\textsc{Distance} = \textsc{Speed} \times \textsc{Time}$$

when SPEED IS CONSTANT, so also we have

$$\#\text{babies born in time } \delta t = BN\delta t$$

Similarly let D be the death rate per capita; again, it could be a function of t (better medicine, fewer smokers) or N (overcrowding leads to famine/disease) but if we assume that it is constant, then

$$\#\text{deaths in time } \delta t = DN\delta t$$

So the change in N, δN, during δt is

$$\delta N = \#birth - \#deaths$$

PROVIDED there is no emigration or immigration. Thus,

$$\delta N = (B - D)N\delta t$$

and so $\frac{\delta N}{\delta t} = (B - D)N$ or in the limit as $\delta t \rightarrow$

0,

$$\frac{dN}{dt} = (B - D)N = kN \tag{1}$$

if $k = B - D$.

This model of society was put forward by THOMAS MALTHUS in 1798. Clearly Malthus was assuming a socially STATIC society in which human reproductive behaviour never changes with time or overcrowding, poverty etc... What does Malthus' model predict? Suppose that the population NOW is $\hat{N}$, and let $t = 0$ NOW. From $\frac{dN}{dt} = kN$ we have $\int \frac{dN}{N} = \int k dt = k \int dt = kt + c$

so $ln(N) = kt + c$ and thus $N(t) = Ae^{kt}$.

Since $\hat{N} = N(0) = A$, we get:

$$N(t) = \hat{N}e^{kt} \tag{2}$$

with graphs as shown on figure 1.

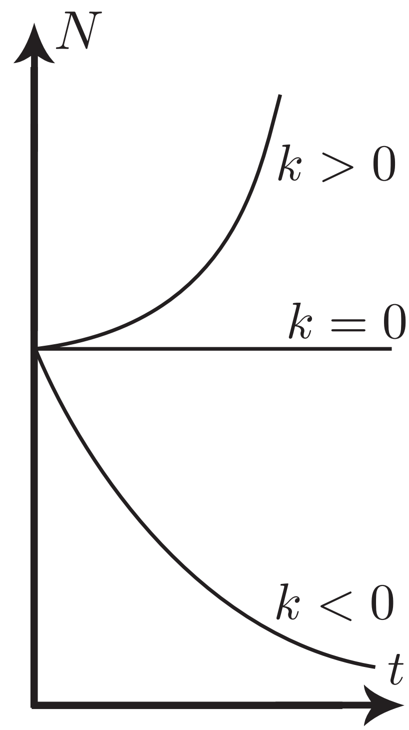


Figure 1: Graphs of $N(t)$, for different values of k

The population collapses if $k < 0$ (more deaths than births per capita), remains stable if (and only if) $k = 0$, and it EXPLODES if $k > 0$ (more births than deaths). Malthus observed that the population of Europe was increasing, so he predicted a catastrophic POPULATION EXPLOSION; since the food supply could not be expanded so fast, this would be disastrous.

In fact, this didn't happen (in Europe). So Malthus' model is wrong: many millions went to the US, many millions died in wars.

Second, the "static society" assumption has turned out to be wrong in many societies, with B and D both declining as time passed after WW2.

SUMMARY: The Malthus model of population is based on the idea that per capita birth and death rated are independent of time and N. It leads to EXPONENTIAL growth or decay of N.

3.2. IMPROVING ON MALTHUS

Malthus' model is interesting because it shows that static behaviour patterns can lead to disaster. But precisely because e^{kt} grows so quickly, Malthus' assumptions must eventually go wrong - obviously there is a limit to the possible population. Eventually, if we don't control B, then D will have to increase. So we have to assume that D is a function of N.

Clearly, D must be an increasing function of N... but WHICH function? Well, surely the SIMPLEST POSSIBLE CHOICE (Remember: always go for the SIMPLE model before trying a complicated one!) is

$$\boxed{D = sN, \quad \begin{matrix} \text{(LOGISTIC)} \\ \text{ASSUMPTION} \\ s = \text{constant} \end{matrix}} \tag{3}$$

This represents the idea that, in a world with FINITE RESOURCES, large N will eventually cause starvation and disease and so increase D.

Remark: In modelling, it is often useful to take note of $\boxed{\text{units}}$. Units of D are (#dead people) / second / (total # people) = $(\text{sec})^{-1}$.

Units of N are # (ie no units). So if $D = sN$, units of s must be $(\text{sec})^{-1}$.

As before, let $\hat{N}$ be the value of N at $t = 0$. We have to solve

$$\frac{dN}{dt} = BN - DN = BN - sN^2$$

with the condition $N(0) = \hat{N}$

We can and will solve this, but let's try to GUESS what the solution will look like (a useful skill - in many other cases you won't be able to solve exactly!). Suppose that $\hat{N}$ is very small. Then (by continuity) $N(t)$ will be very small for t near to zero.

Of course if N is small, N^2 i much smaller and can be neglected. So at early times, our

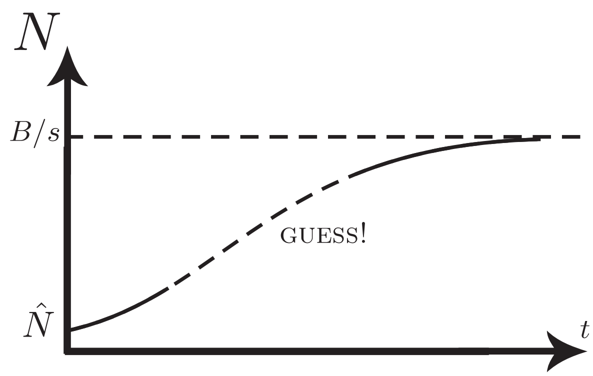
ODE is ALMOST linear and so

$$\frac{dN}{dt} \approx BN \rightarrow N(t) \approx \hat{N}e^{Bt}$$

So AT FIRST the population explodes, as Malthus predicted. On the other hand, if N continues to grow, since N^2 grows faster than N, we will reach a point where $sN^2 \approx BN$ ie $N \approx B/S$. At that point , since $\frac{dN}{dt} = BN - SN^2$, the population will stop growing. So B/S should measure the MAXIMUM population possible. So we GUESS that the solution should look like this:

ie it starts out exponentially and ends up approaching B/S asymptotically. The dotted part is a reasonable GUESS!

OK, now that we know what to expect, let's



actually solve it!

$$\frac{dN}{dt} = BN - sN^2 \rightarrow t = \int \frac{dN}{N(B-sN)} + c$$

Write $\frac{1}{N(B-sN)} = \frac{\alpha}{N} + \frac{\beta}{B-sN}$

$$\begin{aligned} 1 &= \alpha(B - sN) + \beta N \\ &= \alpha B + (\beta - \alpha s)N \rightarrow 1 = \alpha B \ , \ \beta = \alpha s \\ \alpha &= 1/B \ , \ \beta = s/B, \text{ so} \end{aligned}$$

$$\begin{aligned}\int \frac{dN}{N(B-sN)} &= \frac{1}{B}\int \frac{dN}{N} + \frac{s}{B}\int \frac{dN}{B-sN} \\ &= \frac{1}{B}\ln N - \frac{1}{B}\ln|B-sN|\end{aligned}$$

Now here we begin to feel uneasy - what if $N = B/s$ at some time? (ln(0) is not defined). In fact we should have worried about this when we first wrote $\frac{1}{B-sN}$ - how do we know that we are not dividing by zero?? Let's not worry about that just now: let's ASSUME (temporarily) THAT $B - sN$ IS NEVER ZERO. That is, we assume either that N is always either LESS THAN B/s or MORE THAN B/s. OK, let's take LESS THAN first. So $|B - sN| = B - sN$, and we get

$$\begin{aligned} t &= \frac{1}{B}\ln N - \frac{1}{B}\ln(B - sN) + c \\ &= \frac{1}{B}\ln\frac{N}{B - sN} + c \end{aligned}$$

So

$\frac{N}{B-sN} = Ke^{Bt}$. Since $\hat{N} = N(0)$, $\frac{\hat{N}}{B-s\hat{N}} = K$

so

$$\frac{N}{B - sN} = \frac{\hat{N}}{B - s\hat{N}}e^{Bt}$$

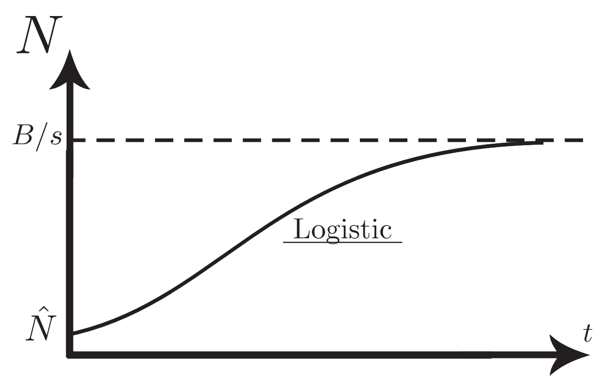
Solve for N,

$$N(t) = \frac{B}{s + \left(\frac{B}{\hat{N}} - s\right)e^{-Bt}} \tag{4}$$

REMARK: It is a very good habit to check that your solution agrees with your assumptions - to guard against mistakes!

Check: $N(0) = \frac{B}{s+\left(\frac{B}{\hat{N}}-s\right)} = \hat{N}$ correct! Check: If $B - sN > 0$ is true at $t = 0$ then $B - s\hat{N} > 0$ so $\left(\frac{B}{\hat{N}} - s\right) > 0$ so $\frac{B}{s+\left(\frac{B}{\hat{N}}-s\right)e^{-Bt}} < \frac{B}{s}$ for all t *ie* $N(t) < B/s$ which is consistent.

The graph of (4) is easy to sketch:



This is the famous LOGISTIC CURVE; $N(t)$ given by (4) is called the LOGISTIC FUNCTION; and $\frac{dN}{dt} = BN - sN^2$ is the LOGISTIC EQUATION.

It's easy to see what is happening here. Initially the population is small, plenty of food and space, so we get a Malthusian population explosion. But eventually the death rate rises until it is almost equal to the birth rate (*ie* $sN \approx B$ or $N \approx B/s$) and then the population approaches a fixed limit.

This situation is what people usually mean when they use the word "LOGISTIC". But we are not done yet: on page 12 we ASSUMED that $N(t) < B/s$. What if $N(t) > B/s$?

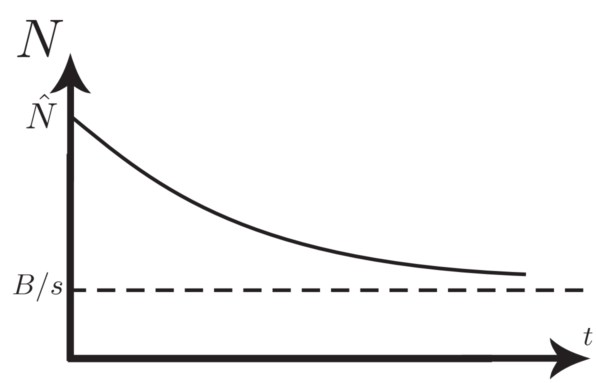
Then $|B - sN| = -(B - sN)$ so:

$$\begin{aligned} t &= \frac{1}{B}\ln N - \frac{1}{B}\ln(sN - B) + c \\ &= \frac{1}{B}\ln\frac{N}{sN - B} + c \Rightarrow \end{aligned}$$

$$N(t) = \frac{B}{s - \left(s - \frac{B}{\hat{N}}\right)e^{-Bt}} \tag{5}$$

Check $N(0) = \hat{N}$ and $N(t) > B/s$

And now the graph is



Again, the meaning is clear: the initial population was so big that the death rate exceeded the birth rate, so of course the population declines until it gets near to the long-term SUSTAINABLE value.

The number B/s is called the CARRYING CAPACITY or the SUSTAINABLE POPULATION - in all cases, it is the value approached by $N(t)$ as $t \to \infty$. If we set

$$N_\infty = B/s \tag{6}$$

then our solutions are:

$$N(t) = \frac{N_\infty}{1 + \left(\frac{N_\infty}{\hat{N}} - 1\right) e^{-Bt}} (\hat{N} < N_\infty) \tag{7}$$

$$N(t) = \frac{N_\infty}{1 - \left(1 - \frac{N_\infty}{\hat{N}}\right) e^{-Bt}} (\hat{N} > N_\infty) \tag{8}$$

(obtained by dividing numerator and denominator by s in 4 and 5)

BUT we aren't finished yet! We had to ASSUME (page 12) that N is never equal to B/s, *ie* to N_∞. So we have to think about this.

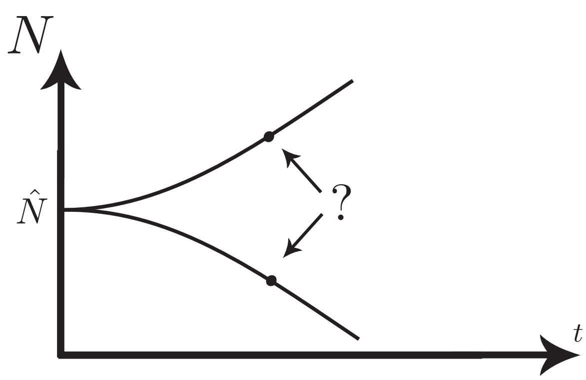
First, let's ask what happens if $\hat{N} = N_\infty$ *ie* $N(t)$ is initially N_∞.

Intuitively, since N_∞ is the sustainable population, you would expect that $N(t) = \text{constant} = \hat{N} = N_\infty$ should be possible ! Indeed, substiture $N = N_\infty$ into $\frac{dN}{dt} = BN - sN^2$ and the left side is zero while the right side is $BN_\infty - sN_\infty^2 = N_\infty[B - sN_\infty] = N_\infty[B - B] = 0$. So we have

$$N(t) = N_\infty \;\; (\hat{N} = N_\infty) \tag{9}$$

Clearly (7) (8) (9) cover all possible values of $\hat{N}$.

Now INTUITIVELY we believe that, in our model, a knowledge of $\hat{N}$ should allow us to find $N(t)$ for all later times given B and s. For B and s are the only parameters. A situation like the one in the diagram where there are TWO



SOLUTIONS FOR A GIVEN $\hat{N}$ would be completely CRAZY - what is the population at time

t? So in a real problem, THERE MUST BE A UNIQUE SOLUTION $N(t)$ FOR FIXED VALUES OF THE PARAMETERS (B, s) AND THE INITIAL DATA $(\hat{N})$.

QUESTION: Is it true that the problem

$$\frac{dy}{dt} = F(t, y) \ , \ y(a) = \alpha$$

ALWAYS has a unique solution?

ANSWER: NO! BUT it NEARLY ALWAYS DOES.

So we have to be careful about this when modelling with ODEs! If the ODE with fixed parameters and initial data DOESN'T have a unique solution, then we have to use our knowledge of the problem to PICK THE RIGHT SO-

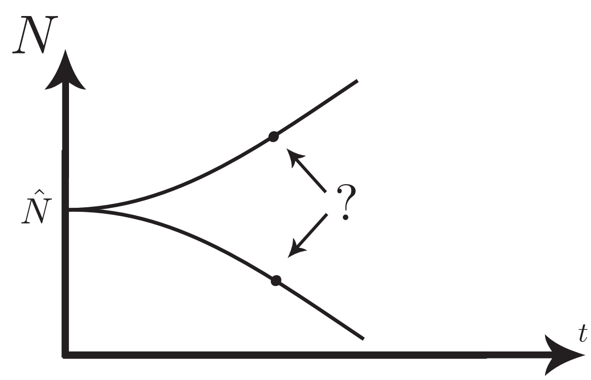
LUTION - or maybe we are using the wrong ODE! We will come back to this in a moment.

SUMMARY: A simple way to improve on Malthus is to replace his assumption $D =$ constant by the LOGISTIC ASSUMPTION $D = sN$, $s =$ constant. If $\hat{N} < B/s = N_\infty$, then the graph of $N(t)$ is the "S-shape" on page 14.

3.4. THE NO-CROSSING PRINCIPLE

As we know, a first-order ODE never has just ONE solution — it always has many, because of the arbitrary constant. That's why, when we graph solutions of ODEs, there are always many possible graphs.

We argued earlier that, in a real engineering problem, a situation like the one in the diagram



does not make sense. MATHEMATICALLY, it can happen, but in the real world it almost never does.

Now one way to describe the situation shown in the diagram is to say that the two curves are touching or CROSSING each other. That should not happen in a real problem. We call this the

$$\boxed{\begin{array}{c}\text{NO}\\\text{CROSSING}\\\text{PRINCIPLE}\end{array}}$$

It just says that DIFFERENT SOLUTION CURVES OF A FIRST-ORDER ODE SHOULD NEVER CROSS EACH OTHER. This is a useful thing to remember, because it tells you quite a lot about the shapes of the graphs. For example, in the logistic model, suppose that N(t) is initially smaller than B/s. Now N = B/s is actually a solution of the logistic equation. So

NO OTHER SOLUTION CURVE CAN EVER CROSS THE STRAIGHT LINE N = B/s [because that would contradict our no-crossing rule]. So if N is INITIALLY below that line, then it has to STAY below it forever! In general you should always remember the no-crossing rule whenever you have to graph the solution curves of an ODE.

3.5. HARVESTING

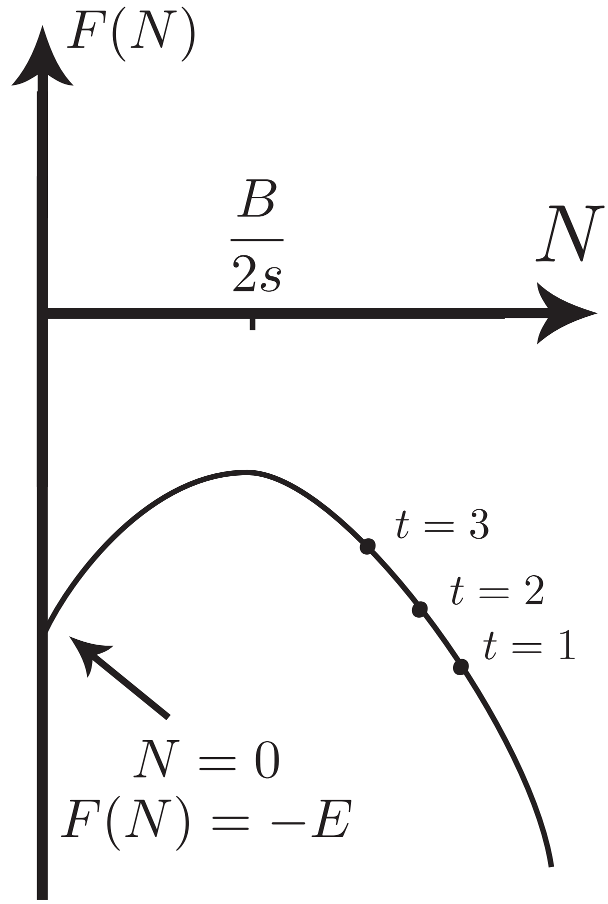
A major application of modelling is in dealing with populations of animals e.g. fish. We want to know how many we can eat without wiping them out. Let's build on our logistic model, *ie* $\boxed{\text{assume}}$ that the fish population WOULD follow that model if we didn't catch any. Next, $\boxed{\text{assume}}$ that we catch E (constant) fish per year. Then we have:

$$\frac{dN}{dt} = (B - sN)N - E \quad \boxed{\begin{array}{c}\text{BASIC}\\ \text{HARVESTING}\\ \text{MODEL}\end{array}}$$

We'll now try to guess what the solutions should look like. We are particularly interested in the long term - will the harvesting eventually exterminate the fish? Consider the function:

$$F(N) = (B - sN)N - E$$

. The graph of $F(N)$ can take one of 3 forms.



In the first case, the quadratic

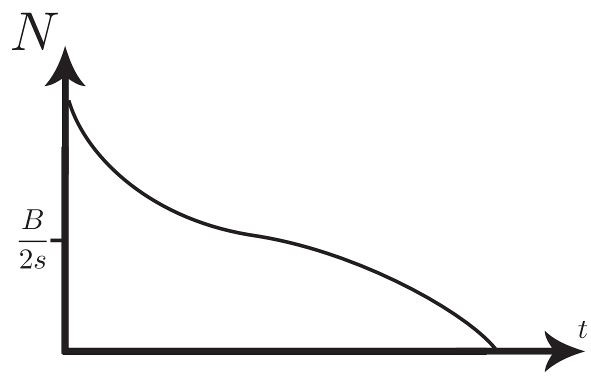
$$-sN^2 + BN - E$$

has no solutions, *ie*

$$B^2 - 4sE < 0 \text{ or } E > \frac{B^2}{4s}$$

.

Since $\frac{dN}{dt} = F(N)$, we see that in this case the population always declines. Note that there is no t-axis in this picture, but you should imagine time passing by thinking of a moving spot on the graph. Notice that as we move through $t = 1, 2, 3$ we have to move to the left since $\frac{dN}{dt} < 0$ always. But $|\frac{dN}{dt}|$ is DECREASING, so the rate of decline slows down as time goes on, until we pass $N = \frac{B}{2s}$. After that, $|\frac{dN}{dt}|$ increases and the value of N decreases rapidly to zero. Congratulations - you have wiped out your fish! [In drawing this picture, I assumed that $\hat{N}$, the initial value of N, was greater than $\frac{B}{2s}$, the max-



imum point on the graph of $F(N)$. Note that $\frac{dN}{dt} = F(N)$ implies

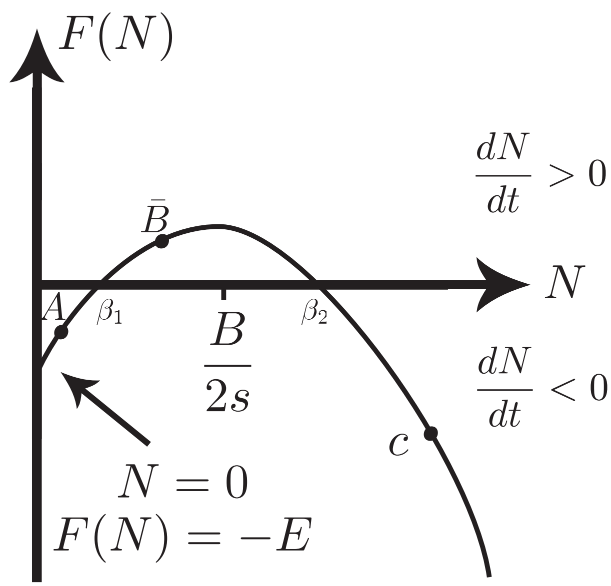
$$\frac{d^2N}{dt^2} = F'(N)\frac{dN}{dt} = F'(N)F(N)$$

so watch for "points of inflection" on the graph of $N(t)$ at values of N where $F'(N)$ or $F(N)$ vanish.]

Clearly it is NOT a good idea to harvest at a rate $E > \frac{B^2}{4s}$ (Check units). So let's assume that our fishermen ease off and harvest at a rate $E <$

$\frac{B^2}{4s}$. (The special case E EXACTLY EQUALS $\frac{B^2}{4s}$ is clearly impossible in reality, but we will come back to it later anyway!)

Now the graph of $F(N)$ is as shown in the next diagram.



Again, remembering that $\frac{dN}{dt} = F(N)$, we see that

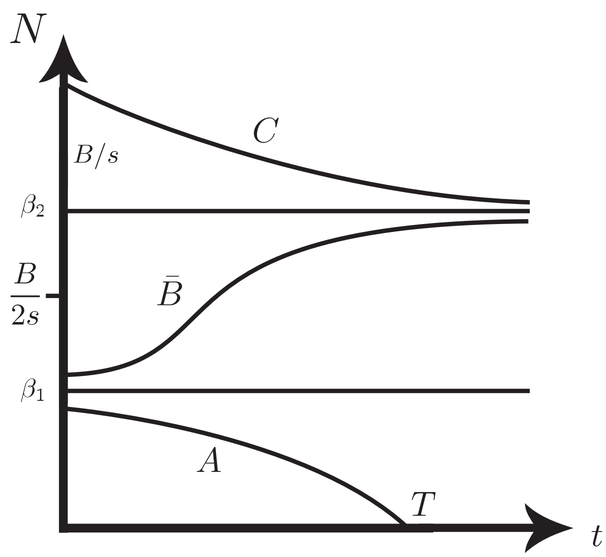
$$
\begin{aligned}
\frac{dN}{dt} &< 0 \text{ if } 0 < N < \beta_1 \\
\frac{dN}{dt} &> 0 \text{ if } \beta_1 < N < \beta_1 \\
\frac{dN}{dt} &< 0 \text{ if } N > \beta_2
\end{aligned}
$$

and of course $\frac{dN}{dt} = 0$ at $N = \beta_1$ and β_2, where

$$
\begin{matrix}\beta_1 \\ \beta_2\end{matrix} = \frac{-B \pm \sqrt{B^2 - 4Es}}{-2s} = \frac{B \mp \sqrt{B^2 - 4Es}}{2s}
$$

Now suppose $\hat{N} = N(0)$ is large, so we start at point C on the diagram. Then $\frac{dN}{dt} < 0$ so the fish stocks decline toward β_2. Suppose on the other hand that $\hat{N}$ is small, but still more than β_1. Then we might start at $\bar{B}$ and the number of fish will INCREASE until we reach β_2. If $\hat{N}$

is very small, however, then we are at a point like A, and the fish population will collapse to zero. So we get a picture like this:

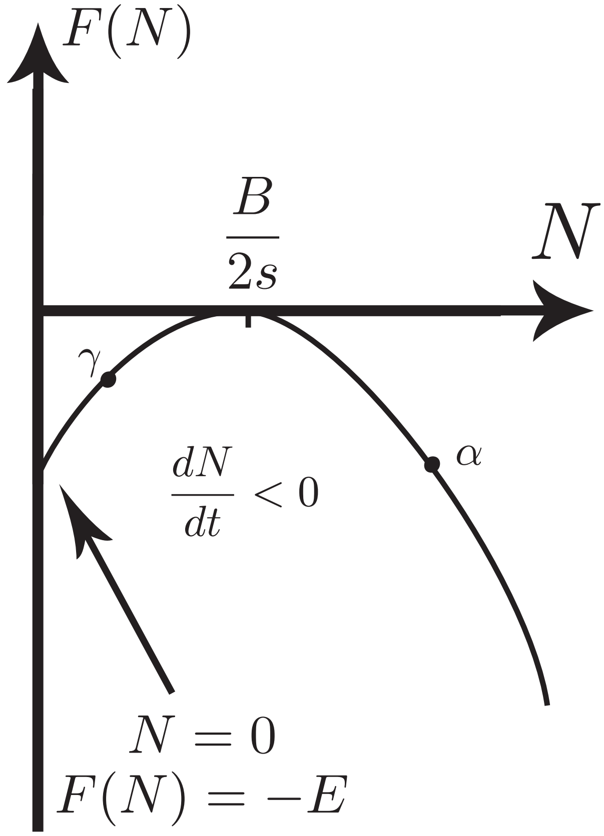


Of course, if $\hat{N} = \beta_1$ or β_2, then since $F(\beta_1) = 0$ and $F(\beta_2) = 0$, $\frac{dN}{dt} = F(N)$ has solutions $N(t) = \beta_1$ and $N(t) = \beta_2$, the constant solutions. We call β_1 and β_2 the EQUILIBRIUM POPULATIONS: given a fixed harvesting rate, if the initial population is either β_1 or β_2, then IN THEORY the population remains steady - which is good! BUT there is a vast DIFFERENCE between β_1 and β_2!! Look at the diagram and suppose you have exactly β_2 fish. Now suppose a SMALL number of new fish arrive from somewhere else. Then the diagram shows that the population will decline back to β_2. If some fish go away, the population will INCREASE back to β_2. Of course, such things happen all the time, so it's VERY GOOD to have this kind of behaviour! We say that β_2 is a STABLE EQUILIBRIUM POPULATION. BUT NOW LOOK AT β_1! If a few more fish arrive, also fine - in fact the population INCREASES (to β_2) BUT suppose a few fish decide to move on. THEN YOUR FISH STOCKS BECOME EXTINCT!! Note that this can happen though E is relatively small (we are assuming $E < \frac{B^2}{4s}$. We say that β_1 is an UNSTABLE EQUILIBRIUM POPULATION. The time required to reach $N = 0$ is called the EXTINCTION TIME. It can be computed: since $\frac{dN}{dt} = N(B - sN) - E$, we have:

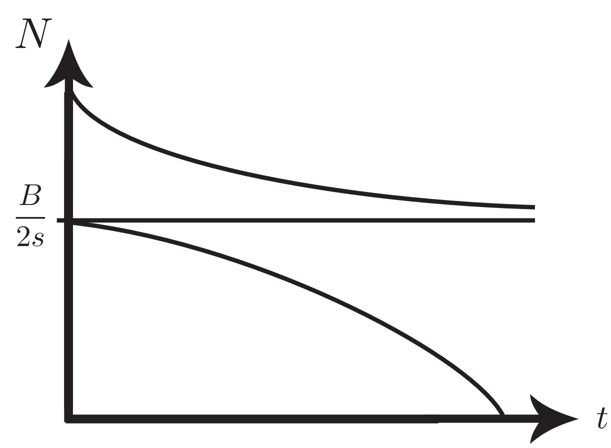
$$\int_0^T dt = T = \int_{\hat{N}}^0 \frac{dN}{N(B - sN) - E}$$

Of course it is very important to know T - it is the amount of time you have to save the situation!

Before we go on, let's consider the third possible graph for $F(N)$:



Clearly $\hat{N} = \frac{B}{2s}$ is a solution; it is the only equilibrium population. If $\hat{N} > \frac{B}{2s}$ (for example, at the point α) then the population declines to $\frac{B}{2s}$, asymptotically. But if we start at the point γ, then (since $\frac{dN}{dt} < 0$ everywhere below the axis - always remember that $\frac{dN}{dt} = F(N)$) the population will collapse to zero. So we have an UNSTABLE equilibrium at $\hat{N} = \frac{B}{2s}$. [We only call it stable if it is stable to perturbations in BOTH directions!]. The graph of N is as shown.



Clearly, the first and third cases are bad! The first case was $E > \frac{B^2}{4s}$. So we want the second case, with $E < \frac{B^2}{4s}$, and we want STABLE equilibrium, that is, a population which fluctuates around $\beta_2 = \frac{B+\sqrt{B^2-4Es}}{2s}$. Let's imagine you are a modeller for the Peruvian State Fishing Regulatory Organisation. What do you predict?

OVERFISHING IS BAD! Pretty obvious! But what is NOT so obvious is the shape of the graph in the first case, $E > \frac{B^2}{4s}$. Note that it is downhill all the way, BUT at first the rate of decline is DECREASING, *ie* the graph is CONCAVE UP. Now it is not surprising that fish stocks decline when you start to harvest them - notice that, EVEN IN THE "GOOD" CASE the curve marked C is concave up and decreasing.

(Notice that $\beta_2 = \frac{B+\sqrt{B^2-4Es}}{2s} < \frac{B+\sqrt{B^2-0}}{2s} = \frac{B}{s}$ = carrying capacity in the absence of harvesting (recall $N_\infty = \frac{B}{s}$). So even the stable equilibrium population under harvesting is less than the carrying capacity!)

So the mere fact that the fish numbers are declining is not a danger signal in itself. In fact, UNTIL YOU GET TO THE POINT OF INFLECTION, the graph might fool you into thinking that the population will settle down eventually to some non-zero asymptotic value!

MORAL OF THE STORY: CONCAVE UP $\boxed{\text{NOW}}$ DOESN'T MEAN YOU NEED NOT PANIC! Any decline must be monitored carefully. The point is that since $D = sN$, a reduction in N drives the death rate DOWN, so paradoxically AT FIRST overfishing may SEEM to BENEFIT the fish!! But not for long!

3.6. MODEL OF A PLUG FLOW REACTOR

In chemical engineering, a PLUG FLOW REACTOR is like a long tube into which you push some mixture of chemicals which move through the tube while they react with each other. These devices can be very complicated objects, and we are not going to pretend to describe how they really work. But we can set up a very simple MATHEMATICAL MODEL of such a gadget, with the understanding that a REALISTIC model would be very much more complicated!

For example, you push in hydrogen and oxygen at one end, and get water coming out the other end. Since Oxygen is cheap and Hydrogen is expensive, we assume that we pump in a lot of Oxygen compared to the Hydrogen, so there is plenty of Oxygen all the way along the PFR. The question we want to answer is: what happens to the concentration of Hydrogen as a function of position in the PFR? Of course it will decrease, but how rapidly? Assume

[a] That everything flows through the PFR at constant speed u, and the cross-sectional area is A, a constant.

[b] That there is no mixing upstream or downstream $\rightarrow$ everything that happens in a small region of the PFR is controlled by chemical reactions IN that region and by flows in and out from the neighbouring regions.

[c] All temperatures are constant in time.

Now at a point x along the PFR, how many molecules of H_2 are passing by per second? Let $C_{H_2}(x)$ be the CONCENTRATION of hydrogen (molecules per cubic metre) at x. Then in a time δt, let δN_{H_2} be the number that pass by. Then δN_{H_2} is the number of H_2 molecules in the cylinder shown in the diagram.

Since $\begin{pmatrix} Number\ of \\ molecules \end{pmatrix} = \left(\dfrac{molecules}{volume}\right) \times (volume)$ we have

$$\delta N = C_{H_2}(x) \times Au\delta t$$

because $A =$ area of base and $u\delta t$ is the height. So

$$\frac{dN}{dt} = \lim_{\delta t \to 0} \frac{\delta N}{\delta t} = C_{H_2}(x)Au$$

Now of course we are LOSING H_2 molecules as the gases move down the PFR, because of the reaction. Consider a small piece of

the tube, of length δx.

Then this PLUG satisfies

(1) H_2 molecules are flowing IN

at x, at a rate $C_{H_2}(x)Au$.

(2) H_2 molecules are flowing OUT at $x + \delta x$ at a rate $C_{H_2}(x + \delta x)Au$.

(3) H_2 molecules are being DESTROYED inside the small piece of tube, at a rate

$$-2rA\delta x.$$

Here r is RATE PER UNIT VOLUME at which the reaction

$$2H_2 + O_2 \rightarrow 2H_2O$$

happens in each unit of volume inside the PFR [so units of r are $\dfrac{1}{\sec \times m^3}$]. Note the MINUS because we are losing H_2, and the 2 because each reaction costs 2 molecules of H_2.

Since the total matter content cannot increase or decrease,

$$C_{H_2}(x)Au - C_{H_2}(x + \delta x)Au - 2rA\delta x = 0$$

or, since $\delta C_{H_2} \equiv C_{H_2}(x + \delta x) - C_{H_2}(x)$,

$$
\begin{aligned}
&- \delta C_{H_2} Au - 2rA\delta x = 0 \\
\Rightarrow \quad & u\frac{dC_{H_2}}{dx} = u \lim_{\delta x \to 0} \frac{\delta C_{H_2}}{\delta x} = -2r.
\end{aligned}
$$

Now r depends on many things, for example the concentration of H_2, the temperature, etc etc etc. Let's construct a simple MODEL of this situation. It's pretty clear that the main thing that controls the rate of a reaction is the concentration, and it's also clear that the higher the concentration, the faster the reaction will go — so the rate should be an increasing function of the concentration, and of course it should be zero when the concentration is zero. So the SIMPLEST POSSIBLE model we can think of is given by the equation

$$r = kC_{H_2}(x),$$

where k [units 1/sec — check this!] is a positive constant [because we always assume here that the temperature is constant — in general k will depend on the temperature of course]. That is, we assume that the rate is [approximately!] proportional to the concentration. Then

$$
\begin{aligned}
& u\frac{dC_{H_2}}{dx} = -2kC_{H_2} \\
& \rightarrow \frac{dC_{H_2}}{C_{H_2}} = -\frac{2k}{u} \\
& \rightarrow \ell n|C_{H_2}| = \ell n\ C_{H_2} = -\frac{2kx}{u} + \text{ constant} \\
& \rightarrow C_{H_2} = C_{H_2}(0)e^{\frac{-2kx}{u}},
\end{aligned}
$$

where x = 0 at the top [or "beginning"] of the PFR.

Check the units — remember that it does not make sense to take the exponential of something that has units, so kx/u should have no units [we say it is DIMENSIONLESS]. Check that our answer MAKES SENSE. For example, our result says that the concentration decreases more rapidly when u is small and k is large — is that sensible, based on your knowledge of chemistry [or common sense!]?

If T is the time from the reagents entering the PFR to their exit, $u = \frac{X}{T}$ where X is the full length of the

PFR, and we have

$$C_{H_2}(\text{exit}) = C_{H_2}(\text{entrance})e^{-2kT}.$$

What this relation is really telling us is that Plug Flow Reactors are a good idea, because they are very efficient. As you know, the exponential function decreases very rapidly, so the Hydrogen is being turned into water very efficiently — almost none of it is left by the time you come to the end of the PFR. Notice too that the important parameter here is the TIME the mixture spends inside the PFR. Finally, remember where that 2 came from — it came from the chemical formula for the reaction. So you have to know your Chemistry to use one of these things.

So PFRs are good, unless they blow up of course..... remember that we deliberately left out temperature. We need a more complicated model....

3.7. CANTILEVERED BEAMS: A MODEL OF A BALCONY.

Let's build a simple mathematical MODEL of a balcony. Balconies are tricky to make: let's see why! A BEAM is a long, thin object used in buildings etc, which bends when subjected to loads.

Take a small element as shown.

Then the element is subjected to both FORCES and TORQUES. All of these must balance out if you don't want the beam to move.

Both F, the SHEARING FORCE, and M, the TORQUE or BENDING MOMENT, are functions of x. We assume that there is a force, called the LOAD, acting on the beam. This load is also a function of x.

We measure the load as FORCE PER UNIT LENGTH, $w(x)$, so the force itself is $w(x)dx$.

Balancing forces [and using Taylor's theorem] on this small element, we get:

$$F(x) + w(x)dx = F(x + dx) = F(x) + \frac{dF}{dx}dx$$

$$\Rightarrow \quad \boxed{\tfrac{dF}{dx} = w(x)}$$

Next we must balance the torques (remember TORQUE = FORCE $\times$ PERPENDICULAR DISTANCE).

We get, if we take moments around the point O,

$$\begin{aligned} &M(x) + F(x + dx)dx(\text{clockwise}) \\ &= M(x + dx) + (w(x)dx)\frac{dx}{2}(\text{anticlockwise}). \end{aligned}$$

So

$$\begin{aligned} M(x) + F(x)dx + \frac{dF}{dx}(dx)^2 \\ = M(x) + \frac{dM}{dx}dx + \frac{1}{2}w(x)(dx)^2. \end{aligned}$$

[The term dx/2 comes about because, when calculating the torque due to the load, you can think of the load as being concentrated at the centre of the element. Actually, the element is so short that it doesn't really matter where you concentrate the load, so the half is just for convenience.]

Neglecting the very small terms involving $(dx)^2$, we get

$$\frac{dM}{dx} = F.$$

So

$$\frac{d^2M}{dx^2} = \frac{dF}{dx} = w(x),$$

$$\boxed{\frac{d^2M}{dx^2} = w(x)}$$

Now the amount of DEFLECTION of a beam depends on the LOAD and on HOW STIFF the beam is. The stiffness is measured as follows: given a bending moment M, what is the CURVATURE of the beam? If it curves a lot, that means that it isn't very stiff, while a stiff beam will curve very little. From your study of calculus you know that the way to measure the curvature of a graph is to use the SECOND DERIVATIVE of the defining function: large second derivative means that the graph is bending a lot.

The stiffness of the beam is measured by the ratio $M/\frac{d^2y}{dx^2}$, as you can see in the diagram.

For many beams, the stiffness is a constant; this constant depends on what the beam is made of, measured by a constant E called YOUNG'S MODULUS, and also it depends on the SHAPE of the crosssection of the beam, measured by a constant I is called the SECOND MOMENT OF AREA of the beam. So the stiffness is also measured by the product EI and we have

$$EI = M / \frac{d^2 y}{dx^2}$$

or

$$\boxed{\frac{d^2 y}{dx^2} = \frac{M}{EI}}$$

But now $\frac{d^4 y}{dx^4} = \frac{1}{EI} \frac{d^2 M}{dx^2} = \frac{w(x)}{EI}$,
So the deflection of the beam is governed by the FOURTH-ORDER ODE

$$\boxed{\frac{d^4 y}{dx^4} = \frac{w(x)}{EI}}$$

EXAMPLE. A CANTILEVER is a beam stuck into a wall, as shown. Suppose
it bends under its own

weight. We assume that it has a uniform mass per unit length, so the load function is $w(x) = \text{constant} = -\alpha$. (Remember that w is positive in the UPWARD direction.)

QUESTION: What is Δ, the maximum deflection?

Answer:

$$\frac{d^4y}{dx^4} = \frac{w}{EI} = -\frac{\alpha}{EI} \quad \rightarrow$$
$$\frac{d^3y}{dx^3} = -\frac{\alpha x}{EI} + A$$

by integration. But remember $\frac{d^3y}{dx^3} = \frac{1}{EI}\frac{dM}{dx} = \frac{F(x)}{EI}$, and clearly there is no shearing force at the right-hand END of the beam, so $F(L) = 0$. Thus

$$F(L) = 0 \; = \; EI\frac{d^3y}{dx^3}(L) = EI\left(-\frac{\alpha L}{EI} + A\right)$$

Hence we have

$$\boxed{A = \tfrac{\alpha L}{EI}} \quad .$$

So

$$\frac{d^3y}{dx^3} = -\frac{\alpha x}{EI} + \frac{\alpha L}{EI} \quad \rightarrow$$
$$\frac{d^2y}{dx^2} = -\frac{\alpha x^2}{2EI} + \frac{\alpha Lx}{EI} + B,$$

where we have integrated again. But $\frac{d^2y}{dx^2} = \frac{M}{EI}$ and $M(L) = 0$ since there is no bending moment at the END. So

$$M(L) = 0 = EI\frac{d^2y}{dx^2}(L) \;=\; EI\left[-\frac{\alpha L^2}{2EI} + \frac{\alpha L^2}{EI} + B\right]$$

$$\Rightarrow \boxed{B = \frac{\alpha L^2}{2EI} - \frac{\alpha L^2}{EI} = -\frac{\alpha L^2}{2EI}}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{\alpha x^2}{2EI} + \frac{\alpha Lx}{EI} - \frac{\alpha L^2}{2EI}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\alpha x^3}{6EI} + \frac{\alpha Lx^2}{2EI} - \frac{\alpha L^2 x}{2EI} + C.$$

But from the diagram, $\frac{dy}{dx}(0) = 0$ so

$$0 = -0 + 0 - 0 + C$$

$$\rightarrow \boxed{C = 0}$$

$$\rightarrow y = -\frac{\alpha x^4}{24EI} + \frac{\alpha Lx^3}{6EI} - \frac{\alpha L^2x^2}{4EI} + D,$$

where we have integrated yet again! But again the diagram tells us $y(0) = 0$, so

$$\boxed{D = 0}$$

Thus

$$y = \frac{\alpha L^4}{2EI}\left[-\frac{1}{12}\left(\frac{x}{L}\right)^4 + \frac{1}{3}\left(\frac{x}{L}\right)^3 - \frac{1}{2}\left(\frac{x}{L}\right)^2\right]$$

So now finally we can work out the maximum deflection of the beam: you can easily show that this maximum occurs at the end [x = L] in agreement with common sense, so all we have to do is to substitute x = L into y and get

$$\Delta = y(L) = \frac{\alpha L^4}{2EI}\left[-\frac{1}{12} + \frac{1}{3} - \frac{1}{2}\right]$$

and so we obtain the famous CANTILEVER DEFLECTION FORMULA

$$\Delta = -\frac{\alpha L^4}{8EI}.$$

The negative answer of course reflects the fact that the cantilever always bends DOWNWARDS. Check that the answer MAKES SENSE: the deflection is larger for larger loads α, of course that makes sense. The deflection is smaller if EI is large, that is, if the beam is very stiff: that too makes sense. Finally, the deflection is greater if the beam is very long [large L], which also makes sense. [Are you in the habit of checking that your answers MAKE SENSE??]

There is still a surprise here though: notice that if you make your cantilever twice as long, the downwards deflection does not double $---$ it increases by a factor of SIXTEEN! So watch out if you want to build balconies etc. [The power of 4 here is a relic of the fact that we solved a FOURTH-order ODE.]