

MA 1505 Mathematics I

Tutorial 2 Solutions

1. (a)  $\lim_{x \rightarrow \pi/2} \frac{1 - \sin x}{1 + \cos 2x} = \lim_{x \rightarrow \pi/2} \frac{-\cos x}{-2 \sin 2x} = \lim_{x \rightarrow \pi/2} \frac{\sin x}{-4 \cos 2x} = \frac{1}{4}.$
- (b)  $\lim_{x \rightarrow 0} \frac{\ln(\cos ax)}{\ln(\cos bx)} = \lim_{x \rightarrow 0} \frac{\frac{-a \sin ax}{\cos ax}}{\frac{-b \sin bx}{\cos bx}} = \lim_{x \rightarrow 0} \frac{a \sin ax \cos bx}{b \sin bx \cos ax} = \frac{a^2}{b^2}.$
- (c)  $\lim_{x \rightarrow \infty} x \tan \frac{1}{x} = \lim_{x \rightarrow \infty} \frac{\tan(x^{-1})}{x^{-1}} = \lim_{x \rightarrow \infty} \frac{-x^{-2} \sec^2(x^{-1})}{-x^{-2}} = \lim_{x \rightarrow \infty} \cos^{-2}(x^{-1}) = 1.$
- (d)  $\lim_{x \rightarrow 0+} x^a \ln x = \lim_{x \rightarrow 0+} \frac{\ln x}{x^{-a}} = \lim_{x \rightarrow 0+} \frac{\frac{1}{x}}{-ax^{-a-1}} = \lim_{x \rightarrow 0+} \frac{x^a}{-a} = 0.$
- (e)  $\lim_{x \rightarrow 1} \ln x^{\frac{1}{1-x}} = \lim_{x \rightarrow 1} \frac{\ln x}{1-x} = \lim_{x \rightarrow 1} \frac{\frac{1}{x}}{-1} = -1.$  So  $\lim_{x \rightarrow 1} x^{\frac{1}{1-x}} = e^{-1}.$
- (f) Using (1d) we have

$$\lim_{x \rightarrow 0+} \ln x^{\sin x} = \lim_{x \rightarrow 0+} \sin x \ln x = \lim_{x \rightarrow 0+} \frac{\sin x}{x} \cdot x \ln x = \lim_{x \rightarrow 0+} \frac{\sin x}{x} \lim_{x \rightarrow 0+} x \ln x = 0.$$

$$\text{So } \lim_{x \rightarrow 0+} x^{\sin x} = e^0 = 1.$$

$$\begin{aligned} \text{(g)} \quad \lim_{x \rightarrow 0} \ln \left[ \left( \frac{\sin x}{x} \right)^{\frac{1}{x^2}} \right] &= \lim_{x \rightarrow 0} \frac{\ln \left( \frac{\sin x}{x} \right)}{x^2} = \lim_{x \rightarrow 0} \frac{\left( \frac{x}{\sin x} \right) \cdot \frac{x \cos x - \sin x}{x^2}}{2x} \\ &= \frac{1}{2} \lim_{x \rightarrow 0} \frac{x}{\sin x} \lim_{x \rightarrow 0} \frac{x \cos x - \sin x}{x^3} = \frac{1}{2} \lim_{x \rightarrow 0} \frac{\cos x - x \sin x - \cos x}{3x^2} \\ &= -\frac{1}{6} \lim_{x \rightarrow 0} \frac{\sin x}{x} = -\frac{1}{6}. \end{aligned}$$

$$\text{So } \lim_{x \rightarrow 0} \left( \frac{\sin x}{x} \right)^{\frac{1}{x^2}} = e^{-1/6}.$$

2. (a)

$$\begin{aligned} \int_1^{\sqrt{2}} \frac{s^2 + \sqrt{s}}{s^2} ds &= \int_1^{\sqrt{2}} (1 + s^{-3/2}) ds = (\sqrt{2} - 1) - 2s^{-1/2} \Big|_1^{\sqrt{2}} \\ &= (\sqrt{2} - 1) - \frac{2}{\sqrt{2}} + 2 = 1 + \sqrt{2} - 2^{3/4}. \end{aligned}$$

- (b)

$$\int_{-4}^4 |x| dx = \int_0^4 x dx + \int_{-4}^0 (-x) dx = \frac{1}{2}4^2 + \frac{1}{2}4^2 = 16.$$

(c)

$$\begin{aligned}\int_0^\pi \frac{1}{2}(\cos x + |\cos x|) dx &= \int_0^{\pi/2} \frac{1}{2}(\cos x + |\cos x|) dx + \int_{\pi/2}^\pi \frac{1}{2}(\cos x + |\cos x|) dx \\ &= \int_0^{\pi/2} \cos x dx + 0 = \sin x \Big|_0^{\pi/2} = 1.\end{aligned}$$

(d)

$$\begin{aligned}\int_0^\pi \sin^2\left(1 + \frac{\theta}{2}\right) d\theta &= \int_0^\pi \frac{1}{2}[1 - \cos(2 + \theta)] d\theta = \frac{1}{2}\pi - \frac{1}{2}\sin(2 + \theta) \Big|_0^\pi \\ &= \frac{1}{2}\pi - \frac{1}{2}[\sin(2 + \pi) - \sin 2] = \frac{1}{2}\pi + \sin 2.\end{aligned}$$

3. The Fundamental Theorem of Calculus (I) says that

$$\frac{d}{du} \int_a^u f(t) dt = f(u)$$

for a continuous function  $f$ . Here  $a$  is a fixed number. It is a sort of *chain rule* to find

$$\frac{d}{dx} \int_a^{g(x)} f(t) dt.$$

To see this, let

$$F(u) = \int_a^u f(t) dt \quad \text{and} \quad u = g(x).$$

It follows that

$$\frac{dF}{du} = \frac{d}{du} \int_a^u f(t) dt = f(u).$$

Furthermore,

$$F \circ g(x) = F(g(x)) = \int_a^{g(x)} f(t) dt.$$

By the chain rule, we have

$$\frac{dF(g(x))}{dx} = \frac{dF}{du} \frac{dg(x)}{dx} = f(u) g'(x) = f(g(x)) g'(x).$$

$$(a) \quad y = \int_0^{\sqrt{x}} \cos t dt; \quad \cos \sqrt{x} \cdot \frac{d}{dx} \sqrt{x} = \frac{\cos \sqrt{x}}{2\sqrt{x}}.$$

$$(b) \quad y = \int_0^{x^2} \cos \sqrt{t} dt; \quad \cos \sqrt{x^2} \cdot 2x = 2x \cos |x| = 2x \cos x.$$

$$(c) \quad y = \int_0^{\sin x} \frac{dt}{\sqrt{1-t^2}}, \quad |x| < \frac{\pi}{2}. \quad \frac{1}{\sqrt{1-\sin^2 x}} \cdot \frac{d}{dx} \sin x = \frac{1}{\cos x} \cos x = 1.$$

4. (a)  $\int x^{1/2} \sin(x^{3/2} + 1) dx = \int \sin(x^{3/2} + 1) \cdot \frac{2}{3} d(x^{3/2} + 1) = -\frac{2}{3} \cos(x^{3/2} + 1) + C.$
- (b)  $\int \csc^2 2t \cot 2t dt = \int \cot 2t \cdot \left(-\frac{1}{2}\right) d(\cot 2t) = -\frac{1}{4} \cot^2 2t + C.$
- (c)  $\int \frac{1}{\theta^2} \sin \frac{1}{\theta} \cos \frac{1}{\theta} d\theta = \int \sin \frac{1}{\theta} \cos \frac{1}{\theta} \cdot (-1) d\left(\frac{1}{\theta}\right) = -\int \sin \frac{1}{\theta} d\left(\sin \frac{1}{\theta}\right) = -\frac{1}{2} \sin^2 \frac{1}{\theta} + C.$
- (d)  $\int \frac{18 \tan^2 x \sec^2 x}{(2 + \tan^3 x)} dx = \int \frac{18 \tan^2 x d(\tan x)}{(2 + \tan^3 x)} = \int \frac{6 d(\tan^3 x + 2)}{(2 + \tan^3 x)} = 6 \ln |\tan^3 x + 2| + C.$
- (e)  $\int \frac{\sin \sqrt{\theta}}{\sqrt{\theta} \cos^3 \sqrt{\theta}} d\theta = -2 \int (\cos \sqrt{\theta})^{-3} d(\cos \sqrt{\theta}) = (\cos \sqrt{\theta})^{-2} + C = \sec^2 \sqrt{\theta} + C.$

5. (a)

$$\begin{aligned} \int x \sin\left(\frac{x}{2}\right) dx &= -2 \int x d\left[\cos\left(\frac{x}{2}\right)\right] = -2 \left[ x \cos\left(\frac{x}{2}\right) - \int \cos\left(\frac{x}{2}\right) dx \right] + C \\ &= -2 \left[ x \cos\left(\frac{x}{2}\right) - 2 \int \cos\left(\frac{x}{2}\right) d\left(\frac{x}{2}\right) \right] + C \\ &= -2 \left[ x \cos\left(\frac{x}{2}\right) - 2 \sin\left(\frac{x}{2}\right) \right] + C. \end{aligned}$$

(b)

$$\begin{aligned} \int t^2 e^{4t} dt &= \frac{1}{4} \int t^2 d(e^{4t}) = \frac{1}{4} \left[ t^2 e^{4t} - 2 \int t e^{4t} dt \right] + C = \frac{1}{4} \left[ t^2 e^{4t} - \frac{1}{2} \int t d(e^{4t}) \right] + C \\ &= \frac{1}{4} \left[ t^2 e^{4t} - \frac{1}{2} \left( t e^{4t} - \int e^{4t} dt \right) \right] + C \\ &= \frac{1}{4} \left[ t^2 e^{4t} - \frac{1}{2} \left( t e^{4t} - \frac{e^{4t}}{4} \right) \right] + C \quad (\text{continue to simplify}). \end{aligned}$$

(c)

$$\begin{aligned} \int e^{-y} \cos y dy &= \int e^{-y} d(\sin y) = e^{-y} \sin y + \int e^{-y} \sin y dy + C \\ &= e^{-y} \sin y - \int e^{-y} d(\cos y) + C = e^{-y} \sin y - e^{-y} \cos y - \int e^{-y} \cos y dy \\ &\Rightarrow \int e^{-y} \cos y dy = \frac{e^{-y}}{2} (\sin y - \cos y) + C. \end{aligned}$$

(There is no harm to rename  $C/2$  as  $C$ .)

(d)

$$\begin{aligned} \int \theta^2 \sin(2\theta) d\theta &= -\frac{1}{2} \int \theta^2 d[\cos(2\theta)] = -\frac{1}{2} \left[ \theta^2 \cos(2\theta) - 2 \int \theta \cos(2\theta) d\theta \right] + C \\ &= -\frac{1}{2} \left[ \theta^2 \cos(2\theta) - \int \theta d[\sin(2\theta)] \right] + C \\ &= -\frac{1}{2} \left[ \theta^2 \cos(2\theta) - \theta \sin(2\theta) + \int \sin(2\theta) d\theta \right] + C \\ &= -\frac{1}{2} \left[ \theta^2 \cos(2\theta) - \theta \sin(2\theta) - \frac{1}{2} \cos(2\theta) \right] + C. \end{aligned}$$

(e)

$$\begin{aligned}\int z(\ln z)^2 dz &= \frac{1}{2} \int (\ln z)^2 d(z^2) = \frac{1}{2} \left[ z^2(\ln z)^2 - 2 \int z(\ln z) dz \right] + C \\&= \frac{1}{2} \left[ z^2(\ln z)^2 - \int (\ln z) d(z^2) \right] + C \\&= \frac{1}{2} \left[ z^2(\ln z)^2 - z^2(\ln z) + \int z dz \right] + C \\&= \frac{1}{2} \left[ z^2(\ln z)^2 - z^2(\ln z) + \frac{z^2}{2} \right] + C.\end{aligned}$$

(f)

$$\begin{aligned}\int \sin e^{-x} dz &= \int e^x d(\cos e^{-x}) \\&= e^x \cos e^{-x} - \int e^x \cos e^{-x} dx + C.\end{aligned}$$