MA1506 TUTORIAL 7

1. Find the Laplace transforms of the following functions [where u denotes the unit step function and the answers are given in brackets]:

   (a) $t^2 e^{-3t}$. $\quad [\frac{2}{(s+3)^3}]$

   (b) $tu(t-2)$. $\quad [e^{-2s}\{\frac{1}{s^2} + \frac{2}{s}\}]$

2. Find the inverse Laplace transforms of the following functions:

   (a) $\frac{s}{s^2 + 10s + 26}$. $\quad [e^{-5t}(\cos t - 5\sin t)]$

   (b) $e^{-2s}\,\frac{1+2s}{s^3}$. $\quad [(\frac{1}{2}t^2 - 2)u(t-2)]$

3. Solve the following initial value problems using Laplace transforms:

   (a) $y' = tu(t-2), \;\; y(0) = 4$. $\quad [(\frac{1}{2}t^2 - 2)u(t-2) + 4]$

   (b) $y'' - 2y' = 4, \;\; y(0) = 1, \;\; y'(0) = 0$. $\quad [e^{2t} - 2t]$

4. (i) Show, from the definition of the Laplace transform, that for any function f(t),

$$L[tf(t)] = -F'(s),$$

   where F(s) is the Laplace transform of f(t). Hence find the Laplace transform of t sin(t).

   (ii) Use Laplace transforms to solve the resonance equation

$$\ddot{y} + y = cos(t),$$

   where $y(0) = \dot{y}(0) = 0$. You should recognise the solution!

5. The oil tanker in Tutorial 4 is at rest in an almost calm sea. Suddenly, at time t = T > 0, it is hit by a single rogue wave [http://en.wikipedia.org/wiki/Rogue_wave] which imparts to it a vertical [upward] momentum P, doing so almost instantaneously. Neglecting friction, solve for x(t), the downward displacement of the ship, and graph it. How far down

does the ship go [if it doesn't sink!]? [Hint: according to Newton's second law, momentum is the time integral of force. So to get the force as a function of time in this problem, you have to find a function which is zero except at t = T, and which has an integral equal to P. Note that the delta function has units of 1/time.] See the "Laplace solver" at http://www.aw-bc.com/ide/ and try putting a delta function on the right side. [Answer: the ship goes down either to P/ωM or to Davy Jones http://en.wikipedia.org/wiki/Davy_Jones_Locker]

6. In Question 5, suppose that you don't want to assume that the wave hits instantaneously: you want to model the situation by assuming that the momentum P is imparted to the ship over a short but non-zero period of time τ, starting at t = T. Explain how you would do this, using the step function. [Remember how to use the step function to model situations where something is turned on, then turned off.] Compute the Laplace transform of the displacement function. Show, using L'Hopital's rule, that, in the limit $\tau \to 0$, this Laplace transform tends to the one you found in Question 5. That is reasonable, right?