# CHAPTER 2. OSCILLATIONS

## 2.1. THE HARMONIC OSCILLATOR

Consider the pendulum shown.

The small object, mass $m$,

at the end of the pendulum,

is moving on a circle of radius

$L$, so the component of its velocity

tangential to the circle is $L\dot{\theta}$

Hence its tangential acceleration is

$L\ddot{\theta}$ and so by $\overrightarrow{F} = M\overrightarrow{a}$ we have

$$mL\ddot{\theta} = -mg\sin\theta.$$

An obvious solution is $\theta = 0$. This is called an EQUI-

LIBRIUM solution, meaning that $\theta$ is a CONSTANT function. This means that if you set $\theta = 0$ initially, then $\theta$ will remain at 0 and the pendulum will not move — which of course we know is correct. There is ANOTHER equilibrium solution, $\theta = \pi$. Again, IN THEORY, if you set the pendulum EXACTLY at $\theta = \pi$, then it will remain in that position forever. IN REALITY, of course, it won't! Because the slightest puff of air will knock it over! So this equilibrium is very different from the one at $\theta = 0$. This is a very important distinction!

Equilibrium is said to be STABLE if a SMALL push away from equilibrium REMAINS small. If the small push tends to grow large, then the equilibrium is UNSTABLE. Obviously this is important for engineers! Especially you want vibrations of structures, engines, etc to remain small.

Let's look at $\theta = \pi$. By Taylor's theorem, near $\theta = \pi$, we have

$$f(\theta) = f(\pi) + f'(\pi)(\theta - \pi) + \frac{1}{2}f''(\pi)(\theta - \pi)^2 + ...$$

Now $\sin(\pi) = 0$, $\sin'(\pi) = \cos(\pi) = -1$, $\sin''(\pi) = -\sin(\pi) = 0$ etc so

$$\sin(\theta) = 0 - (\theta - \pi) - 0 + \frac{1}{6}(\theta - \pi)^3 \;\; \text{etc}$$

For small deviations away from $\pi$, $\theta - \pi$ is small,

$(\theta - \pi)^3$ is much smaller, etc, so we can approximate

$$\sin(\theta) \approx -(\theta - \pi)$$

so our equation is approximately

$$ML\ddot{\theta} = -mg\sin\theta = mg(\theta - \pi).$$

Let $\phi = \theta - \pi$, so $\ddot{\phi} = \ddot{\theta}$, and now

$$\ddot{\phi} = \frac{g}{L}\phi.$$

The general solution is

$$\phi = Ae^{(\sqrt{g/L})t} + Be^{-(\sqrt{g/L})t}$$

so $\theta = \phi + \pi = Ae^{(\sqrt{g/L})t} + Be^{-(\sqrt{g/L})t} + \pi$.

As you know, the exponential function grows very quickly; so even if $\theta$ is close to $\pi$ initially, it won't stay near to it very long! Very soon, $\theta$ will arrive either at $\theta = 0$ or $2\pi$, far away from $\theta = \pi$. The equilibrium is UNSTABLE! How long does it take for things to get out of control? That is determined by $\sqrt{g/L}$ or rather $\sqrt{L/g}$, which has units of TIME. Note that it takes longer to fall over if $L$ is large.

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EXAMPLE:

An eccentric professor likes to balance pendula near their unstable equilibrium point. In a given performance, the pendulum is initially slightly away from that point, and is initially at rest. The prof's skill is such that he can stop the pendulum from falling provided that the angular deviation from the vertical angle does not double. If the shortest pendulum for which he can perform this trick is 9.8 centimetres long, estimate the speed of his reflexes.

Solution: The problem is saying that the angle $\phi$ [the deviation from the vertical] is initially very small, and its initial rate of change is zero. So $\phi(0) = \epsilon$ [some very small number] and $\dot{\phi}(0) = 0$. Differentiating our solution for $\phi(t)$ above and substituting we get A + B = $\epsilon$ and A - B = 0, so in fact

A = B = $\epsilon/2$ and so

$$\phi = \epsilon\, cosh((\sqrt{g/L})t) = \epsilon\, cosh(10t)$$

since from the given data g/L = 100/sec$^2$. Now our objective is to calculate how long it takes for $\phi$ to double, so we need to find t such that $2\epsilon = \epsilon\, cosh(10t)$. Clearly t = $\cosh^{-1}(2)/10 \approx 0.132$ sec.

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**SUMMARY**: The equation $\ddot{\phi} = +\frac{g}{L}\phi$ is a symptom of INSTABILITY. The system is at equilibrium, but it will run away uncontrollably on a time scale fixed by $\sqrt{L/g}$.

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Now what about $\theta = 0$? Here of course we use Taylor's theorem around zero,

$$f(\theta) = f(0) + f'(0)\theta + \frac{1}{2}f''(0)\theta^2 + ...$$
$$\sin(\theta) = 0 + \theta - 0 - \frac{1}{6}\theta^3 + ...$$

so $\sin(\theta) \approx \theta$ and we have approximately

$$mL\ddot{\theta} = -mg\theta \quad \text{or}$$

$$\ddot{\theta} = -\frac{g}{L}\theta = -\omega^2\theta$$

with $\omega^2 = g/L$. That minus sign is crucial!

General solution is $C\cos(\omega\, t) + D\sin(\omega\, t)$ where C and D are arbitrary constants.

Now using trigonometric identities you can show that ANY expression of the form $C\cos(x) + D\sin(x)$ can be written as

$$C\cos(x) + D\sin(x) = \sqrt{C^2 + D^2}\cos(x - \gamma)$$

where $\tan(\gamma) = D/C$. [You can see this easily by taking the scalar product of the vectors $\begin{bmatrix} C \\ D \end{bmatrix}$ and $\begin{bmatrix} \cos(x) \\ \sin(x) \end{bmatrix}$.]

So now we can write our general solution as

$$\theta = A\cos(\omega t - \delta)$$

[**Check**: this does satisfy $\ddot{\theta} = -\omega^2\theta$ and it does contain TWO arbitrary constants, $A$ and $\delta$]. In this case, $\theta$ is never larger than $A$, never smaller than $-A$, so IF $\theta$ WAS SMALL INITIALLY, it REMAINS

SMALL! [We call A the AMPLITUDE.] So the equilibrium in this case is STABLE. This is called SIMPLE HARMONIC MOTION. Clearly $\theta$ repeats its values every time $\omega t$ increases by $2\pi$ [since cos is periodic with period $2\pi$]. Now

$$\omega t \rightarrow \omega t + 2\pi$$

means

$$t \rightarrow t + \frac{2\pi}{\omega}$$

So $\frac{2\pi}{\omega} = 2\pi\sqrt{L/g}$ is the time taken for $\theta$ to return to its initial value, the PERIOD. Again it takes a long time if $L$ is large. The number $\omega$ is called the ANGULAR FREQUENCY.

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**SUMMARY**: The equation $\ddot{\theta} = -\omega^2\theta$ is a symptom of STABILITY. The system oscillates, with a constant amplitude, around equilibrium on a time scale fixed by $\sqrt{L/g}$. The angular frequency is $\omega$.

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**REMARK**: Let $\ddot{x} = f(x)$, where f(x) is some function such that $f(0) = 0$. Then $x(t) = 0$ [identically] is an equilibrium solution.

Also

$$f(x) = f(0) + f'(0)x + ...$$

$$= f'(0)x + ...$$

Approximating this by keeping only the first term, and substituting into $\ddot{x} = f(x)$, we will get an ODE exactly like the ones we have been discussing. By our discussion, we will get UNSTABLE equilibrium if $f'(0) > 0$, and STABLE equilibrium if $f'(0) < 0$. [$f'(0) = 0$ is very rare.] In both cases, $f'(0)$ tells you something about the time it takes for things to happen. So basically Simple Harmonic Motion comes up WHENEVER you have motion near to a stable equilibrium point – it's NOT just about springs and the pendulum!

## 2.2. OSCILLATOR PHASE PLANE.

We all know that any curve in the plane can be given in PARAMETRIC form, $(x(t),\ y(t))$. Let $x(t)$ be the solution of a SHM problem, and define

$$y = \dot{x}.$$

Then as $t$ goes by, $x(t)$ and $y(t)$ trace out a curve in the $(x,\ y)$ plane. In fact, since

$$x = A\cos(\omega t - \delta) = A\cos(\delta - \omega t)$$

$$y = -A\omega\sin(\omega t - \delta) = A\omega\sin(\delta - \omega t)$$

we have

$$\frac{x^2}{A^2} + \frac{y^2}{A^2\omega^2} = \cos^2 + \sin^2 = 1,$$

an ELLIPSE.

Let $\psi = \delta - \omega t$, so

$x = A\cos(\psi)$

$y = A\omega \sin(\psi)$.

Note that y/x = $\omega\, tan(\psi)$. Clearly $\psi$ is somehow related to $\varphi$, the angle shown. In fact, from the diagram we see that y/x = $tan(\varphi)$, so

$tan(\varphi) = \omega\, tan(\psi)$.

So we see that $\varphi$ is not quite the same as $\psi$, but if you graph this relation you will see that [except at certain jumps] $\varphi$ is an increasing function of $\psi$. This means that WHEN $\psi$ DECREASES, SO DOES $\varphi$. Notice that since $\psi = \delta - \omega t$, $\psi$ DOES DECREASE as time goes on, and hence so does the angle $\varphi$; so the point $(x(t), y(t))$ moves around the ellipse in the CLOCKWISE direction. The SIZE of the ellipse is fixed by $A$. In fact if $\omega = 1$, $\dfrac{x^2}{A^2} + \dfrac{y^2}{A^2} = 1$ is the equation of a CIRCLE of radius $A$. Note that $A$ is determined by the initial conditions. So the full set of solutions for SHM is represented by a set of concentric ellipses in the $(x, \dot{x})$ plane. This plane is called the PHASE PLANE for this differential equation. From the phase plane diagram for $\ddot{x} = -\omega^2 x$ we see immediately where equilibrium must be (at $x = \dot{x} = 0$) because that is the only point that doesn't move. We also see right away that this equilibrium is stable – moving away from the origin just puts you on an ellipse which STAYS NEAR TO the origin.

If you take $\ddot{x} = +\omega^2 x$, say with $x(0) = \alpha$, $\dot{x}(0) = 0$, then you find

$$x(t) = \frac{1}{2}\alpha\left[e^{\omega t} + e^{-\omega t}\right] = \alpha\cosh(\omega t)$$

$$y(t) = \dot{x}(t) = \alpha\omega\sinh(\omega t).$$

Both of these functions grow without limit. The phase plane

diagram shows clearly that

the equilibrium here

[still at the origin]

is UNSTABLE.

## 2.3. DAMPED, FORCED OSCILLATORS.

When an object moves fairly slowly through air, the RESISTANCE DUE TO FRICTION is approximately proportional to its speed, and of course in the OPPOSITE DIRECTION. So in the case of the pendulum, where the speed of the object is $L\dot{\theta}$, the DAMPING FORCE is

$$-SL\dot{\theta}$$

where $S$ is some positive constant. We now have

$$mL\ddot{\theta} = -mg\sin\theta - SL\dot{\theta}$$

$$\approx -mg\theta - SL\dot{\theta}$$

if $\theta$ is close to zero. Thus

$$m\ddot{\theta} + S\dot{\theta} + \frac{mg}{L}\theta = 0,$$

and this is the equation of DAMPED HARMONIC MOTION. We can also attach a motor to the pendulum, that is, an external force $F(t)$ which may depend on time. Then

$$mL\ddot{\theta} = -mg\sin\theta - SL\dot{\theta} + F(t)$$

and if $\theta$ remains small, we get

$$m\ddot{\theta} + S\dot{\theta} + \frac{mg}{L}\theta = \frac{1}{L}F(t),$$

the equation of FORCED damped harmonic motion.

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## 2.4. MODELS OF ELECTRICAL CIRCUITS.

For an electrical circuit, voltage drops can occur in 3 ways:

[a] Across a RESISTOR:

$$V = RI$$

where $R$ is a constant called the RESISTANCE, $I(t)$ is the current, and $V(t)$ is the voltage drop. Resistors try to stop currents.

[b] Across an INDUCTOR:

$$V(t) = L\frac{dI}{dt}$$

where $L$ is a constant called the INDUCTANCE. Inductors try to stop CHANGES of currents.

[c] Across a CAPACITOR:

$$V(t) = \frac{1}{C}\int I(t)dt$$

where $C$ is a constant called the CAPACITANCE. Capacitors try to stop currents from building up an accumulation of charge.

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So in a circuit like the one shown, the source of voltage has to supply

$V(t) = RI + L\dot{I} + \frac{1}{c}\int^t I\ dt.$

If we define $Q = \int^t I\ dt$,

then $\dot{Q} = I$, $\ddot{Q} = \dot{I}$, so

$L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = V(t).$

But this is exactly the same equation as in forced, damped, harmonic motion! For example, if $R =$

$V(t) = 0$, then

$$\ddot{Q} = -\frac{1}{LC}Q = -\omega^2 Q$$

where $\omega = 1/\sqrt{LC}$, and we already know all about this equation. We see that inductance is like mass, resistance is like FRICTION, the voltage is like external force, etc.

## 2.5. DAMPED, UNFORCED OSCILLATORS.

We know that such things are governed by equations like

$$m\ddot{x} + b\dot{x} + kx = 0.$$

With the usual trick, $x = e^{\lambda t}$, we get

$$m\lambda^2 + b\lambda + k = 0.$$

Quadratic in $\lambda$, so 3 cases: both roots real, both complex, both equal.

[a] BOTH REAL : OVERDAMPING

Example, $\ddot{x} + 3\dot{x} + 2x = 0$

$$\lambda^2 + 3\lambda + 2 = 0 \to \lambda = -1, -2,$$

general solution $B_1 e^{-t} + B_2 e^{-2t}$. The motion very rapidly dies away to zero. Obviously we have too much friction.

[b] BOTH COMPLEX: UNDERDAMPING

Example, $\ddot{x} + 4\dot{x} + 13x = 0$

$$\lambda^2 + 4\lambda + 13 = 0 \to \lambda = -2 \pm 3i$$

general solution $B_1 e^{-2t}\cos(3t) + B_2 e^{-2t}\sin(3t)$, which can be written as

$$x = Ae^{-2t}\cos(3t - \delta).$$

The graph is obtained by “multiplying together“ the graphs of $e^{-2t}$ and $A\cos(3t - \delta)$:

This is like a simple harmonic oscillator such that the amplitude is a function of time.

In general if we have an equation for an unforced, damped harmonic oscillator, such that

$$m\ddot{x} + b\dot{x} + kx = 0,$$

$$m\lambda^2 + b\lambda + k = 0,$$

then if the solutions for $\lambda$ are COMPLEX, we say that the system is UNDERDAMPED. We get

$$x(t) = Ae^{\frac{-\,bt}{2m}} \cos(\beta t - \delta)$$

where $\beta = \frac{1}{2m}\sqrt{4mk - b^2}$. You can think of this as "SHM with frequency $\beta$ and amplitude $Ae^{\frac{-bt}{2m}}$."

Here $\beta$ is often called the QUASI-FREQUENCY and $\frac{2\pi}{\beta}$ is the QUASI-PERIOD. Notice that in this problem, UNLIKE in true SHM, there are actually TWO independent time scales: $\frac{2\pi}{\beta}$ has units of time but so does $\frac{2m}{b}$. This second time scale tells you how quickly the amplitude dies out. [They are INDEPENDENT because given $m$ and $b$ you can work out $\frac{2m}{b}$ but not $\frac{2\pi}{\beta}$ [since you have not been given $k$.]

Remember that the phase plane diagram for SHM was a set of ellipses. For underdamped harmonic motion, it must be like that, but now the point $(x, \dot{x}) \to (0, 0)$ [Check that $x, \dot{x}$ always $\to 0$ as $t \to \infty$.] So we get SPIRALS.

## 2.6. FORCED OSCILLATIONS

Suppose you have a mass $m$ which can move in a horizontal line. It is attached to the end of a spring which exerts a force

$$F_{\text{spring}} = -kx$$

where $x$ is the extension of the spring and $k$ is a constant (called the spring constant). This is Hooke's Law. Now we attach an external MOTOR to the mass $m$. This motor exerts a force $F_0\cos(\alpha t)$, where $F_0$ is the amplitude of the external force and $\alpha$ is the frequency. If $F_0 = 0$ we just have, from Newton,

$$m\ddot{x} = -kx,$$

so we get $\ddot{x} = -\omega^2 x, \ \ \omega = \sqrt{k/m}$. Here $\omega$ is the frequency that the system has if we leave it alone that is, it is the NATURAL frequency. It has NOTHING TO DO with $\alpha$ of course – we can choose $\alpha$ to suit ourselves.

If $F_0 \neq 0$, then we have

$$m\ddot{x} + kx = F_0 \cos\alpha t.$$

Let $z$ be a complex function satisfying

$$m\ddot{z} + kz = F_0 \; e^{i\alpha t}.$$

Clearly the real part, $Re\ z$, satisfies the above equation, so we can solve for $z$ and them take the real part. We try

$$z = C \; e^{i\alpha t}$$

and get

$$mC(i\alpha)^2 e^{i\alpha t} + C \; k \; e^{i\alpha t} = F_0 e^{i\alpha t}$$

$$\Rightarrow C = \frac{F_0}{k - m\alpha^2} = \frac{F_0/m}{\omega^2 - \alpha^2}$$

So $\qquad Re\ z = \frac{F_0/m}{\omega^2-\alpha^2} \cos(\alpha t)$

and the general solution is

$$x = A\cos(\omega t - \delta) + \frac{F_0/m}{\omega^2 - \alpha^2} \cos(\alpha t).$$

Note that

$$\dot{x} = -A\omega\sin(\omega t - \delta) - \frac{\alpha F_0/m}{\omega^2 - \alpha^2}\sin(\alpha t).$$

The arbitrary constants $A$ and $\delta$ are fixed by giving $x(0)$ and $\dot{x}(0)$ as usual. For example, suppose $x(0) = \dot{x}(0) = 0$, then

$$0 = A\cos(\delta) + \frac{F_0/m}{\omega^2 - \alpha^2}$$

$$0 = A\omega\sin(\delta).$$

Assuming $F_0 \neq 0$, we cannot have $A = 0, \Rightarrow \delta = 0$. So $A = -\frac{F_0/m}{\omega^2-\alpha^2}$,

$$x = \frac{F_0/m}{\omega^2 - \alpha^2}\left[\cos(\alpha t) - \cos(\omega t)\right].$$

Using the trigonometric identity

$$\cos A - \cos B = -2 \sin\left(\frac{A-B}{2}\right) \sin\left(\frac{A+B}{2}\right)$$

we find

$$x = \frac{2F_0/m}{\alpha^2 - \omega^2} \sin\left[\left(\frac{\alpha - \omega}{2}\right) t\right] \sin\left[\left(\frac{\alpha + \omega}{2}\right) t\right]$$

Now remember that $\alpha$ is under our control, $\omega$ is not. Suppose we adjust $\alpha$ to be very close to $\omega$, but not equal to it. Then we can think of this solution in the following way:

$$x = \left\{\frac{2F_0/m}{\alpha^2 - \omega^2} \sin\left[\left(\frac{\alpha - \omega}{2}\right) t\right]\right\} \times \sin\left[\left(\frac{\alpha + \omega}{2}\right) t\right]$$

The reason for splitting it like this is that $\frac{\alpha-\omega}{2}$ will be very SMALL, so LOW-FREQUENCY, while $\frac{\alpha+\omega}{2}$

is much larger, high frequency. So if we write

$$x = A(t)\sin\left[\left(\frac{\alpha+\omega}{2}\right)t\right]$$

then we have a sine function with an amplitude which is a (MUCH LOWER-FREQUENCY) sine function! We say that the system has a BEAT, and $\frac{\alpha-\omega}{2}$ is called the BEAT FREQUENCY.

Notice that the amplitude function

$$A(t) = \frac{2F_0/m}{\alpha^2-\omega^2}\sin\left[\left(\frac{\alpha-\omega}{2}\right)t\right]$$

has a maximum value $\left|\frac{2F_0/m}{\alpha^2-\omega^2}\right|$ which becomes very large when $\alpha$ is very close to $\omega$.

What happens if we let $\alpha \to \omega$? We have

$$A(t) = \frac{2F_0/m}{\alpha+\omega} \times \frac{\sin\left[\frac{\alpha-\omega}{2}t\right]}{\alpha-\omega}$$
$$\to \frac{F_0}{m\omega} \times \frac{t}{2} = \frac{F_0 t}{2m\omega}$$

by L'Hopital's rule. So in this limit

$$x = \frac{F_0 t}{2m\omega}\sin(\omega t)$$

and we see that the oscillations

go completely out of control.

This situation is called

RESONANCE. We see that IF A

SYSTEM IS FORCED IN A WAY

THAT AGREES WITH ITS OWN

NATURAL FREQUENCY, IT CAN OSCILLATE UNCONTROLLABLY.

This can be very dangerous! In reality resonance does not get completely out of control, because we cannot really ignore friction [or RESISTANCE in the case of an electrical circuit]. So we should really solve

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos \alpha t$$

$$m\ddot{z} + b\dot{z} + kz = F_0 e^{i\alpha t}$$

Set $z = ce^{i\alpha t}$ and take the real part at the end.

$$\rightarrow c = \frac{F_0}{k - m\alpha^2 + ib\alpha} = \frac{F_0[k - m\alpha^2 - ib\alpha]}{(k - m\alpha^2)^2 + b^2\alpha^2}$$

(Using the identity $\frac{1}{A+iB} = \frac{A-iB}{(A+iB)(A-iB)} = \frac{A-iB}{A^2+B^2}$.)

So we need the real part of this complex number times $e^{i\alpha t} = \cos(\alpha t) + i\sin(\alpha t)$. Now in general we have

$$(C + iD)(E + iF) = CE - DF + i[CF + DE]$$

Similarly here

$$\frac{F_0[k - m\alpha^2 - ib\alpha]}{(k - m\alpha^2)^2 + b^2\alpha^2} \times [\cos(\alpha t) + i\sin(\alpha t)]$$

has real part

$$x(t) = \frac{F_0(k - m\alpha^2)\cos(\alpha t) + F_0 b\alpha\sin(\alpha t)}{(k - m\alpha^2)^2 + b^2\alpha^2}.$$

To this we should add the general solution of $m\ddot{x} + bx + kx = 0$. But we know already what that looks like – whether overdamped or underdamped, the solution rapidly (exponentially) TENDS TO ZERO. We call that part of the solution the TRANSIENT. So after the transient dies off, we are left with this expression.

We saw earlier that ANY expression of the form $C\cos(x) + D\sin(x)$ can be written as

$$C\cos(x) + D\sin(x) = \sqrt{C^2 + D^2}\cos(x - \gamma)$$

where $\tan(\gamma) = D/C$.

So here we have

$$x(t) = \frac{\frac{1}{m}F_0\cos(\alpha t - \gamma)}{\sqrt{(\omega^2 - \alpha^2)^2 + \frac{b^2}{m^2}\alpha^2}}$$

So the system eventually settles down into a steady oscillation, BUT AT FREQUENCY $\alpha$, NOT $\omega$! Also, the AMPLITUDE of this oscillation is a FUNCTION OF $\alpha$,

$$A(\alpha) = \frac{F_0/m}{\sqrt{(\omega^2 - \alpha^2)^2 + \frac{b^2}{m^2}\alpha^2}}$$

The graph of this is called the AMPLITUDE RESPONSE CURVE. Depending on the values of the parameters, this curve might have a SHARP MAXIMUM, meaning that the system will suddenly respond strongly if $\alpha$ is chosen to be the value that gives that maximum. This is something to be avoided in some cases [things might break] but welcomed in others [you want your mobile phone to ignore all frequencies but one].

## 2.7. CONSERVATION.

Newton's 2nd law involves TIME derivatives, but sometimes it can be expressed in terms of SPATIAL derivatives, by means of the following trick:

$$\frac{d}{dx}\left(\frac{1}{2}\dot{x}^2\right) = \dot{x}\frac{d\dot{x}}{dx} = \frac{dx}{dt}\,\frac{d\dot{x}}{dx} = \ddot{x} \ \ \text{(chain rule)}.$$

For SHM we have

$$m\ddot{x} = -kx$$

so

$$m\frac{d}{dx}(\frac{1}{2}\dot{x}^2) = -kx.$$

But now we can integrate both sides:

$$\frac{1}{2}m\dot{x}^2 = -\frac{1}{2}kx^2 + E$$

where $E$ is a constant of integration. So we have

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2.$$

As you know, $\frac{1}{2}m\dot{x}^2$ is called the KINETIC ENERGY of the oscillator, and $\frac{1}{2}kx^2$ is called the POTENTIAL ENERGY. We call $E$ the TOTAL ENERGY. The fact that $E$ is CONSTANT is called the conservation of energy.

Of course, if friction is present then we do not expect $E$ to be constant. Instead we have, if the frictional force is proportional to the speed,

$$m\ddot{x} = -kx - b\dot{x}$$

where b is a positive constant. Keeping the same definition of E, you will find that this equation can be written as

$\frac{dE}{dx} = -b\dot{x}$

or $\frac{dE}{dt} = \frac{dx}{dt}\frac{dE}{dx} = \dot{x}\frac{dE}{dx} = -b(\dot{x})^2 \leq 0$,

so the energy decreases. Instead of ellipses, we will get a phase diagram that (in the underdamped case) consists of SPIRALS.

In general, for a ONE-DIMENSIONAL problem, we can define the potential energy to be $V(x) = -\int^x F(y)dy$, where $F(y)$ is the force function. Then [using the Fundamental Theorem of Calculus, which implies dV/dx = − F(x)], we have

$$
\begin{aligned}
& F = m\ddot{x} \\
& \rightarrow -\frac{dV}{dx} = \frac{d}{dx}\left(\frac{1}{2}m\dot{x}^2\right) \\
& \rightarrow \frac{d}{dx}\left(\frac{1}{2}m\dot{x}^2 + V(x)\right) = 0 \\
& \rightarrow \frac{1}{2}m\dot{x}^2 + V(x) = E = \text{constant}
\end{aligned}
$$

$\rightarrow$ Energy Conservation. This always works $IF$ $F$ is just a function of POSITION only. In two dimensions or more, the same trick works provided $\vec{F}$ is a function of position only AND $\vec{F}$ can be expressed as $\vec{F} = -\vec{\nabla} V$ for some function $V$. We say that $\vec{F}$ is CONSERVATIVE in this case.

Back to the one-dimensional case: notice that the energy equation gives the shape of the phase curves directly, for example, take

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = E,$$

and define $y = \dot{x}$, so

$$\left(\frac{1}{2}k\right)x^2 + \left(\frac{1}{2}m\right)y^2 = E,$$

and you should recognise this as an ellipse in the $xy$ plane. Similarly in the case of UNSTABLE MOTION,

$$\ddot{x} = +\omega^2 x$$

we have

$$\frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2 = E$$

which is the equation of a HYPERBOLA in the $(x,\ \dot{x})$ plane.