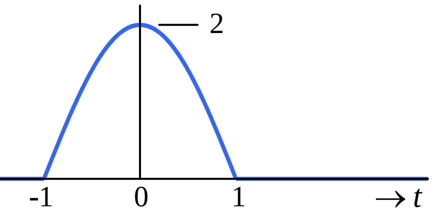
EE2023 TUTORIAL 3 (SOLUTIONS)

Solution to Q.1

$$x(t)=\begin{cases}2\cos(0.5\pi t); & |t|\le 1\\ 0; & |t|>1\end{cases}$$



(a)

Method 1: By applying direct Fourier transform:

$$\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} x(t)\exp(-j2\pi ft)\,dt = \int_{-1}^{1} 2\cos(0.5\pi t)\exp(-j2\pi ft)\,dt \\
&= 2\int_{-1}^{1}\underbrace{\cos(0.5\pi t)\cos(2\pi ft)}_{even\ \ function\ of\ t}dt - \cancel{j2\int_{-1}^{1}\underbrace{\cos(0.5\pi t)\sin(2\pi ft)}_{odd\ \ function\ of\ t}dt} \\
&= 4\int_{0}^{1}\cos(0.5\pi t)\cos(2\pi ft)\,dt \\
&= 2\int_{0}^{1}\cos\big((2\pi f-0.5\pi)t\big)+\cos\big((2\pi f+0.5\pi)t\big)\,dt \\
&= 2\left[\frac{\sin\big((2\pi f-0.5\pi)t\big)}{2\pi f-0.5\pi}+\frac{\sin\big((2\pi f+0.5\pi)t\big)}{2\pi f+0.5\pi}\right]_0^1 \\
&= 2\left(\frac{\sin(2\pi f-0.5\pi)}{2\pi f-0.5\pi}+\frac{\sin(2\pi f+0.5\pi)}{2\pi f+0.5\pi}\right) \\
&= \frac{2}{\pi}\left(\frac{-\cos(2\pi f)}{2f-0.5}+\frac{\cos(2\pi f)}{2f+0.5}\right) = \frac{2\cos(2\pi f)}{\pi\left(0.25-4f^2\right)}
\end{aligned}$$

Method 2: By applying Fourier transform properties:

$$x(t)=2\cos(0.5\pi t)\cdot\text{rect}(0.5t)$$

$$\Im\{2\cos(0.5\pi t)\}=\delta(f-0.25)+\delta(f+0.25)$$

$$\Im\{\text{rect}(0.5t)\}=2\text{sinc}(2f)$$

Applying the 'Multiplication in time-domain' property of the Fourier transform

$$\left[x(t)=\underbrace{2\cos(0.5\pi t)\cdot\text{rect}(0.5t)}_{\substack{\text{Multiplication in}\\ \text{time-domain}}}\right] \rightleftarrows \left[X(f)=\underbrace{\Im\{2\cos(0.5\pi t)\}\ast\Im\{\text{rect}(0.5t)\}}_{\substack{\text{Convolution in}\\ \text{frequency-domain}}}\right]$$

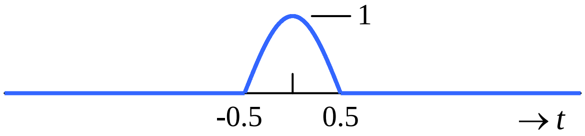
we get

$$\begin{aligned}
X(f) &= \left[\delta(f-0.25)+\delta(f+0.25)\right]\ast 2\text{sinc}(2f) \\
&= 2\text{sinc}(2f-0.5)+2\text{sinc}(2f+0.5) \\
&= 2\left(\frac{\sin(2\pi f-0.5\pi)}{\pi(2f-0.5)}+\frac{\sin(2\pi f+0.5\pi)}{\pi(2f+0.5)}\right) \quad \cdots\cdots \text{ Same result obtained by } \textbf{Method 1}
\end{aligned}$$

(b)

From Part (a): $X(f)=\dfrac{2\cos(2\pi f)}{\pi\left(0.25-4f^2\right)}$

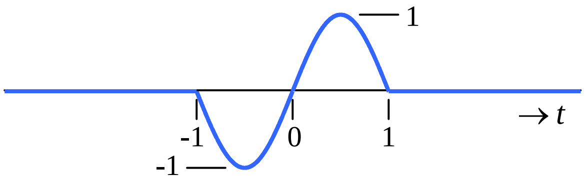
$\tilde{y}(t)=0.5x(2t)$



Applying the ***scaling property***:

$$\tilde{Y}(f)=0.5\left[\frac{1}{2}X\left(\frac{f}{2}\right)\right]=\frac{1}{4}X\left(\frac{f}{2}\right) \quad \cdots\cdots\cdots\cdots (*)$$

$y(t)=\tilde{y}(t-0.5)-\tilde{y}(t+0.5)$



Applying the ***time-shifting property***:

$$Y(f)=\tilde{Y}(f)\exp\left(-j2\pi f\left(\frac{1}{2}\right)\right)-\tilde{Y}(f)\exp\left(j2\pi f\left(\frac{1}{2}\right)\right) \quad \cdots\cdots\cdots (**)$$

Substituting $(*)$ into $(**)$:

$$\begin{cases} Y(f)=\dfrac{1}{4}X\left(\dfrac{f}{2}\right)\exp(-j\pi f)-\dfrac{1}{4}X\left(\dfrac{f}{2}\right)\exp(j\pi f) \\ \quad =-j\dfrac{1}{2}X\left(\dfrac{f}{2}\right)\sin(\pi f) \\ \quad =\dfrac{1}{j2}\left[\dfrac{2\cos(\pi f)}{\pi\left(0.25-f^2\right)}\right]\sin(\pi f) \\ \quad =\dfrac{1}{j2}\left[\dfrac{\sin(2\pi f)}{\pi\left(0.25-f^2\right)}\right] \end{cases}$$

Solution to Q.2

(a)

Fig.Q.2(a)(I) is a plot of $u(t-\gamma)$ against t:

$$\left[u(t)=\begin{cases}1; & t\ge 0\\ 0; & t<0\end{cases}\right] \rightarrow \left[u(t-\gamma)=\begin{cases}1; & t-\gamma\ge 0\\ 0; & t-\gamma<0\end{cases}\right] \rightarrow \underbrace{\left[u(t-\gamma)=\begin{cases}1; & t\ge \gamma\\ 0; & t<\gamma\end{cases}\right]}_{\substack{\text{Expressing } u(t-\gamma) \text{ as a}\\ \text{function of } t \text{ while}\\ \text{treating } \gamma \text{ as a parameter}}}$$

Fig.Q.2(a)(II) is a plot of $u(t-\gamma)$ against γ:

$$\left[u(t)=\begin{cases}1; & t\ge 0\\ 0; & t<0\end{cases}\right] \rightarrow \left[u(t-\gamma)=\begin{cases}1; & t-\gamma\ge 0\\ 0; & t-\gamma<0\end{cases}\right] \rightarrow \underbrace{\left[u(t-\gamma)=\begin{cases}1; & \gamma\le t\\ 0; & \gamma>t\end{cases}\right]}_{\substack{\text{Expressing } u(t-\gamma) \text{ as a}\\ \text{function of } \gamma \text{ while}\\ \text{treating } t \text{ as a parameter}}}$$

On the γ-axis , since $x(\gamma)=x(\gamma)u(t-\gamma)$ in the integration interval $(-\infty,t]$, we have

$$\int_{-\infty}^{t} x(\gamma)\,d\gamma = \underbrace{\int_{-\infty}^{t} x(\gamma)u(t-\gamma)\,d\gamma = \int_{-\infty}^{\infty} x(\gamma)u(t-\gamma)\,d\gamma}_{\because\ u(t-\gamma)=0 \ when\ \gamma>t} = x(t)\ast u(t)$$

(b)

$$\cos(t)u(t)\ast u(t) = \int_{-\infty}^{\infty}\cos(\gamma)u(\gamma)u(t-\gamma)\,d\gamma = \begin{cases}\int_0^t \cos(\gamma)\,d\gamma; & t\ge 0\\ 0; & t<0\end{cases}$$

$$=\begin{cases}\sin(t); & t\ge 0\\ 0; & t<0\end{cases}$$

$$=\sin(t)u(t)$$

(c)

Using the forward Fourier transform equation, it is straightforward to derive the Fourier transform pair:

$$\text{rect}\left(\frac{t}{\alpha}\right) \rightleftarrows \alpha\cdot\text{sinc}(\alpha f) \quad \cdots\cdots\cdots \quad (\ast)$$

Applying the ‘Duality’ property of the Fourier transform to $(\ast)$:

$$\alpha\cdot\text{sinc}(\alpha t) \rightleftarrows \text{rect}\left(\frac{f}{\alpha}\right) \quad \cdots\cdots \quad (\ast\ast)$$

Taking the limit $\alpha\to\infty$ on both sides of $(\ast\ast)$:

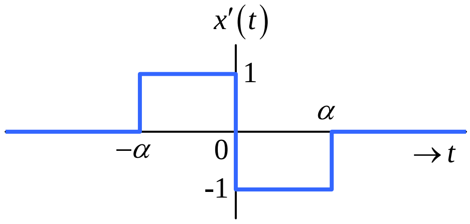
$$\lim_{\alpha\to\infty}\alpha\cdot\text{sinc}(\alpha t) \rightleftarrows \lim_{\alpha\to\infty}\text{rect}\left(\frac{f}{\alpha}\right)=1$$

Hence, $\boldsymbol{\lim_{\alpha\to\infty}\alpha\cdot\textbf{sinc}(\alpha t)=\Im^{-1}\{1\}=\delta(t)}$

Solution to Q.3

Spectrum of $x'(t) = \frac{dx(t)}{dt}$:

$$x'(t) = \frac{dx(t)}{dt} = \text{rect}\left(\frac{t+0.5\alpha}{\alpha}\right) - \text{rect}\left(\frac{t-0.5\alpha}{\alpha}\right)$$



Applying the 'Linearity' property of the Fourier transform:

$$\Im\{x'(t)\} = \Im\left\{\text{rect}\left(\frac{t+0.5\alpha}{\alpha}\right)\right\} - \Im\left\{\text{rect}\left(\frac{t-0.5\alpha}{\alpha}\right)\right\}$$

$$\Im\left\{\text{rect}\left(\frac{t}{\alpha}\right)\right\} = \alpha \cdot \text{sinc}(\alpha f)$$

Applying the 'Time-shifting' property of the Fourier transform:

$$\begin{aligned}\Im\{x'(t)\} &= \alpha \cdot \text{sinc}(\alpha f)\left[\exp(j\pi\alpha f) - \exp(-j\pi\alpha f)\right] \\ &= \alpha \cdot \text{sinc}(\alpha f)\left(j2\sin(\pi\alpha f)\right) \\ &= j2\pi f\alpha^2 \cdot \frac{\sin(\pi\alpha f)}{\pi\alpha f} \cdot \frac{\sin(\pi\alpha f)}{\pi\alpha f} \\ &= j2\pi f\alpha^2 \text{sinc}^2(\alpha f)\end{aligned}$$

Spectrum of $x(t)$:

$$\Im\{x(t)\} = \Im\left\{\int_{-\infty}^{t} x'(\tau)\,d\tau\right\} \quad \cdots \quad \text{Noting: } \int_{-\infty}^{\infty} x'dt = 0$$

Applying the 'Integration' property of the Fourier transform:

$$\begin{aligned}\Im\{x(t)\} &= \frac{1}{j2\pi f}\Im\{x'(t)\} \\ &= \frac{1}{j2\pi f} \cdot j2\pi f\alpha^2 \text{sinc}^2(\alpha f) \\ &= \alpha^2\text{sinc}^2(\alpha f)\end{aligned}$$

Expressing $x(t)$ as a function of $\text{rect}(\cdot)$:

$$\begin{aligned}\Im\{x(t)\} &= \alpha^2\text{sinc}^2(\alpha f) = \alpha\text{sinc}(\alpha f)\cdot\alpha\text{sinc}(\alpha f) \\ &= \Im\left\{\text{rect}\left(\frac{t}{\alpha}\right)\right\}\cdot\Im\left\{\text{rect}\left(\frac{t}{\alpha}\right)\right\} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (*)\end{aligned}$$

Applying the 'Convolution' property of the Fourier transform:

$$\Im\left\{\text{rect}\left(\frac{t}{\alpha}\right) \ast \text{rect}\left(\frac{t}{\alpha}\right)\right\} = \Im\left\{\text{rect}\left(\frac{t}{\alpha}\right)\right\}\cdot\Im\left\{\text{rect}\left(\frac{t}{\alpha}\right)\right\} \quad \cdots\cdots \quad (**)$$

Comparing $(*)$ and $(**)$, we have $x(t) = \text{rect}\left(\frac{t}{\alpha}\right) \ast \text{rect}\left(\frac{t}{\alpha}\right)$

Solution to Q.4

Given: $X(f) = \exp(-\alpha|f|); \quad \alpha > 0$

(a) ***Energy Spectral Density of $x(t)$:***

$$E_x(f) = |X(f)|^2 = \exp(-2\alpha|f|)$$

Energy of $x(t)$ contained within a bandwidth of B:

$$E_B = \int_{-B}^{B} E_x(f)\,df = 2\int_0^B \exp(-2\alpha f)\,df = 2\left[\frac{\exp(-2\alpha f)}{-2\alpha}\right]_0^B = \frac{1}{\alpha}\left[1-\exp(-2\alpha B)\right]$$

Total energy of $x(t)$:

$$E = \underbrace{\int_{-\infty}^{\infty}|x(t)|^2\,dt = \int_{-\infty}^{\infty}|X(f)|^2\,df}_{\text{Rayleigh Energy Theorem}} = \int_{-\infty}^{\infty}E_x(f)\,df = E_B\big|_{B=\infty} = \frac{1}{\alpha}$$

99% energy containment bandwidth, B_{99}, of $x(t)$:

$$\left[\underbrace{\frac{1}{\alpha}\left[1-\exp(-2\alpha B_{99})\right]}_{E_{B_{99}}} = 0.99E = \frac{0.99}{\alpha}\right] \rightarrow \exp(2\alpha B_{99}) = 100$$

$$\rightarrow B_{99} = \frac{1}{\alpha}\ln(10) \text{ Hz}$$

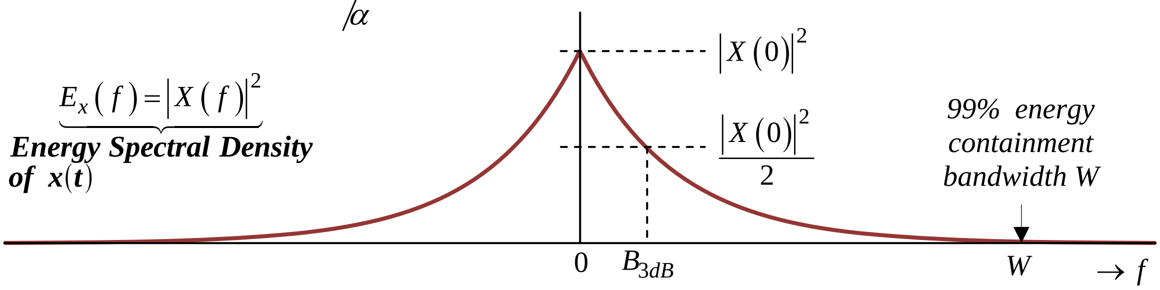
(b) ***3dB bandwidth, B_{3dB}, of $x(t)$:***

By definition, $|X(B_{3dB})| = \dfrac{|X(0)|}{\sqrt{2}}$.

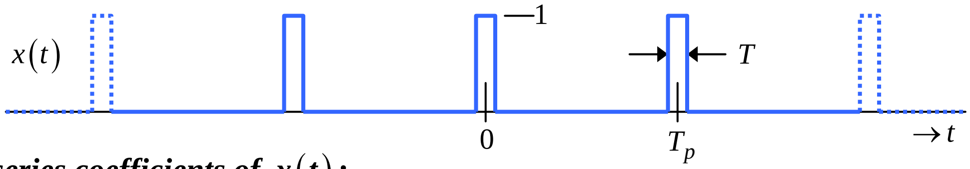
Solving:
$$\left\{\begin{array}{l} |X(f)| = \exp(-\alpha|f|) \\ \left.\begin{array}{l}|X(B_{3dB})| = \exp(-\alpha B_{3dB}) \\ |X(0)| = 1\end{array}\right\} \rightarrow \exp(-\alpha B_{3dB}) = \dfrac{1}{\sqrt{2}} \\ \qquad\qquad\qquad\qquad \rightarrow B_{3dB} = \dfrac{1}{2\alpha}\ln(2) \text{ Hz}\end{array}\right.$$

Percent energy contained within the 3dB bandwidth:

$$\frac{E_{B_{3dB}}}{E}\times 100 = \frac{\cancel{\frac{1}{\alpha}}\left[1-\exp\left(-\cancel{2\alpha}\dfrac{\ln(2)}{\cancel{2\alpha}}\right)\right]}{\cancel{\frac{1}{\alpha}}}\times 100 = 50\%.$$



Solution to Q.5



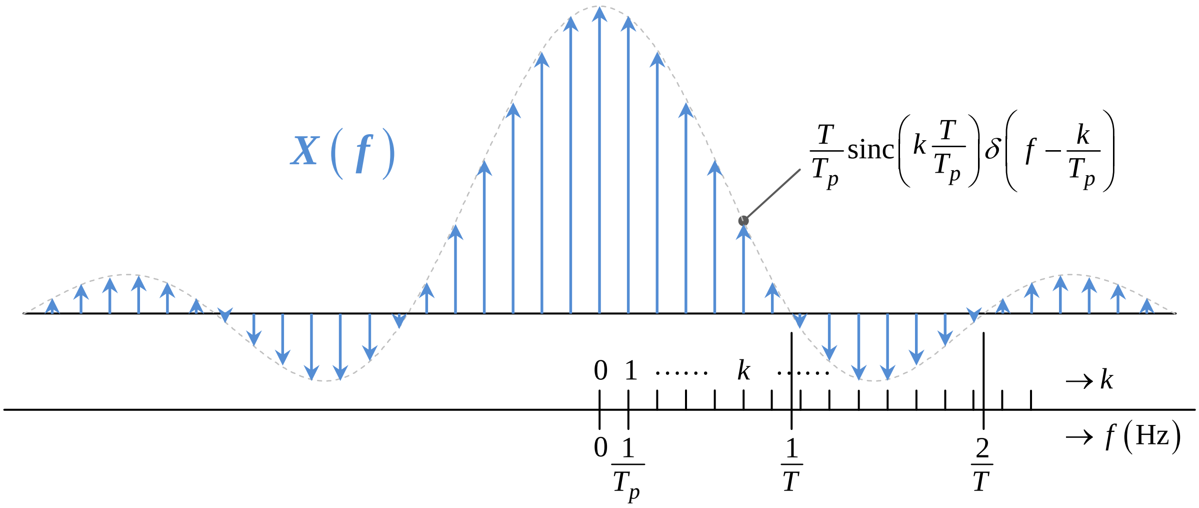
(a) ***Fourier series coefficients of $x(t)$:***

$$X_k = \frac{1}{T_p}\int_{-0.5T}^{T_p-0.5T} x(t)\exp\left(-j2\pi kt/T_p\right)dt = \frac{1}{T_p}\int_{-0.5T}^{0.5T} \exp\left(-j2\pi kt/T_p\right)dt$$

$$= \frac{1}{T_p}\left[\frac{\exp\left(-j2\pi kt/T_p\right)}{-j2\pi k/T_p}\right]_{-0.5T}^{0.5T} = \frac{T}{T_p}\left[\frac{\sin\left(\pi kT/T_p\right)}{\pi kT/T_p}\right] = \frac{T}{T_p}\mathrm{sinc}\left(k\frac{T}{T_p}\right)$$

Continuous-frequency spectrum (or Fourier transform) of $x(t)$:

$$X(f) = \sum_{k=-\infty}^{\infty} X_k \delta\left(f - \frac{k}{T_p}\right) = \sum_{k=-\infty}^{\infty} \frac{T}{T_p}\mathrm{sinc}\left(k\frac{T}{T_p}\right)\delta\left(f - \frac{k}{T_p}\right)$$



(b) ***Power Spectral Density of $x(t)$:***

$$P_x(f) = \sum_{k=-\infty}^{\infty} \left|X_k\right|^2 \delta\left(f - \frac{k}{T_p}\right) = \sum_{k=-\infty}^{\infty} \frac{T^2}{T_p^2}\mathrm{sinc}^2\left(k\frac{T}{T_p}\right)\delta\left(f - \frac{k}{T_p}\right)$$

Average power of $x(t)$:

$$P = \underbrace{\int_{-\infty}^{\infty} P_x(f)\,df = \frac{1}{T_p}\int_{-0.5T}^{T_p-0.5T}\left|x(t)\right|^2 dt}_{\textit{Parseval Power Theorem}} = \frac{1}{T_p}\int_{-0.5T}^{0.5T} dt = \frac{T}{T_p}$$

99% power containment bandwidth, B_{99}, of $x(t)$:

$$B_{99} = \frac{K}{T_p}(\mathrm{Hz}) \quad \cdots \left(\begin{array}{l} \text{where } K \text{ satisfies } \displaystyle\sum_{k=-K}^{K}\left|X_k\right|^2 \geq 0.99P > \sum_{k=-(K-1)}^{(K-1)}\left|X_k\right|^2 \\ \text{in which } \left|X_k\right|^2 = \dfrac{T^2}{T_p^2}\mathrm{sinc}^2\left(k\dfrac{T}{T_p}\right) \text{ and } P = \dfrac{T}{T_p}. \end{array}\right).$$

(c) *Average power of $y(t)$:*

$$
\begin{aligned}
P &= \frac{1}{T_p}\int_{-0.5T}^{T_p-0.5T} \left|y(t)\right|^2 dt \\
&= \frac{1}{T_p}\int_{-0.5T}^{T_p-0.5T} \left|x(t)\right|^2 \mu^2 \cos^2\left(2\pi f_c t\right) dt \\
&= 0.5\mu^2 \frac{1}{T_p}\int_{-0.5T}^{0.5T} \left[1+\cos\left(4\pi f_c t\right)\right] dt \\
&= 0.5\mu^2 \frac{1}{T_p}\int_{-0.5T}^{0.5T} dt + 0.5\mu^2 \frac{1}{T_p}\int_{-0.5T}^{0.5T} \cos\left(4\pi f_c t\right) dt \\
&\cdots\cdots \text{ since } T >> \frac{1}{f_c} \\
&\simeq 0.5\mu^2 \frac{1}{T_p}\int_{-0.5T}^{0.5T} dt = 0.5\mu^2 \frac{T}{T_p}
\end{aligned}
$$

(Assuming that μ cannot be changed, the laser pointer output power can only be controlled by changing the duty cycle T/T_p of the control signal) *(Allows estimation of the battery life)*