4. Sampling Theorem

Sampling is usually done to convert an analog signal to a digital signal, so we can use digital techniques to do some processing on it. This is what is done for recording on CDs and it is done to your voice for transmission over radio on your digital cell Phone.

In practice, when we sample a signal, we capture the value of the signal at regular time intervals. In this way, we obtain a sequence of numbers which is a discrete-time representation of the analog signals. A proper study of this process would require the knowledge of discrete-time Fourier transform (DTFT).

In this chapter, we avoid DTFT and expound on the concept and theory of sampling based on **continuous-time sampling** and **ideal reconstruction filters**.

Nyquist Sampling Theorem :

1. A band-limited signal of finite energy, which has no frequency components higher than W Hz, is completely described by specifying the values of the signal at instants of time separated by $\dfrac{1}{2W}$ secs.

2. A band-limited signal of finite energy, which has no frequency components higher than W Hz, may be completely recovered from a knowledge of its samples taken at the rate of $2W$ samples/sec.

4.1 Ideal Reconstruction Filters

The word “filter” is used to denote an LTI system that exhibits some sort of frequency-selective behavior.

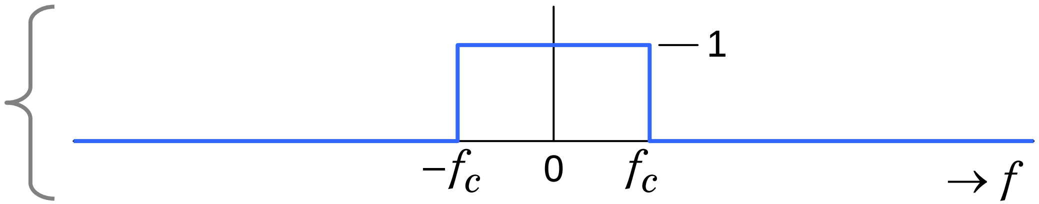
The band of frequencies passed by the filter is referred to as the **pass band**, and the band of frequencies rejected by the filter is called the **stop band**.

- **Ideal Low-Pass Filter (LPF)**

  An ideal low-pass filter completely eliminates all frequencies above a certain cutoff frequency while passing those below unchanged.

$$|H(f)| = \begin{cases} 1; & |f| < f_c \\ 0; & |f| > f_c \end{cases}$$

$$\angle H(f) \propto f \quad \cdots \; Linear\ Phase$$

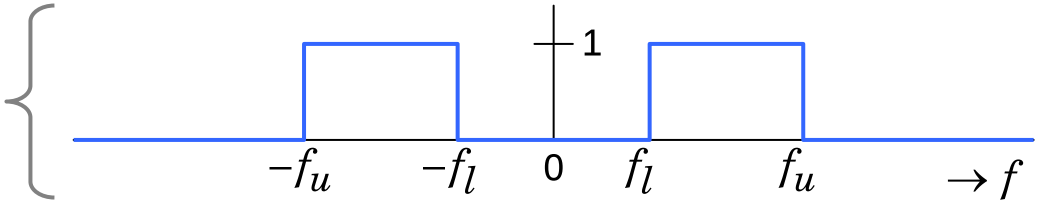


- **Ideal Band-Pass Filter (BPF)**

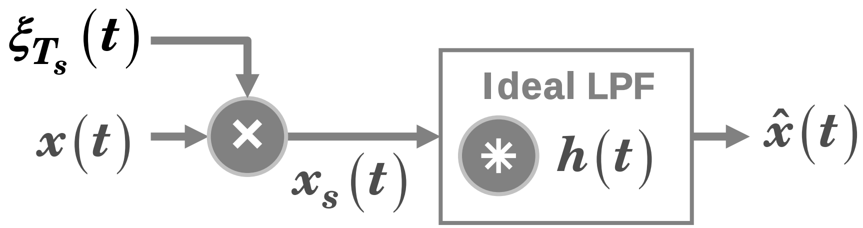
  An ideal band-pass filter completely eliminates all frequencies outside a certain contiguous frequency band while passing those within unchanged.

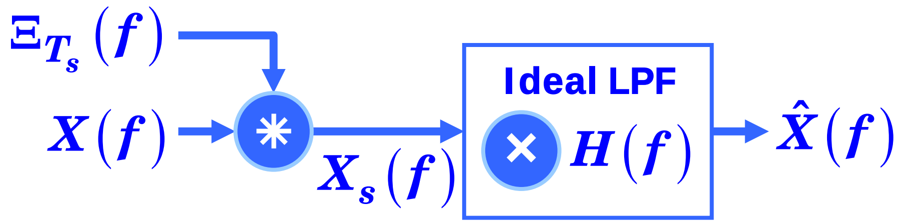
$$|H(f)| = \begin{cases} 1; & f_l < |f| < f_u \\ 0; & \text{otherwise} \end{cases}$$

$$\angle H(f) \propto f \quad \cdots \; Linear\ Phase$$

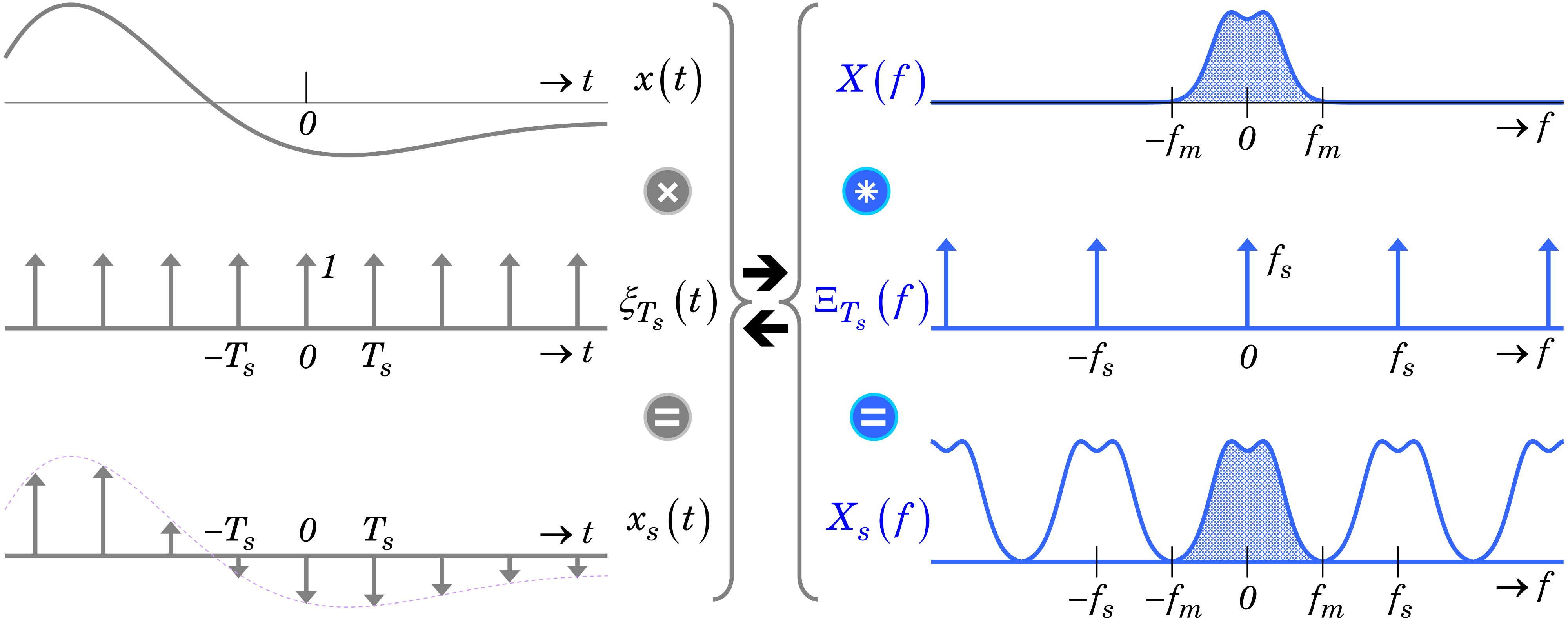


4.2 Continuous-time Sampling

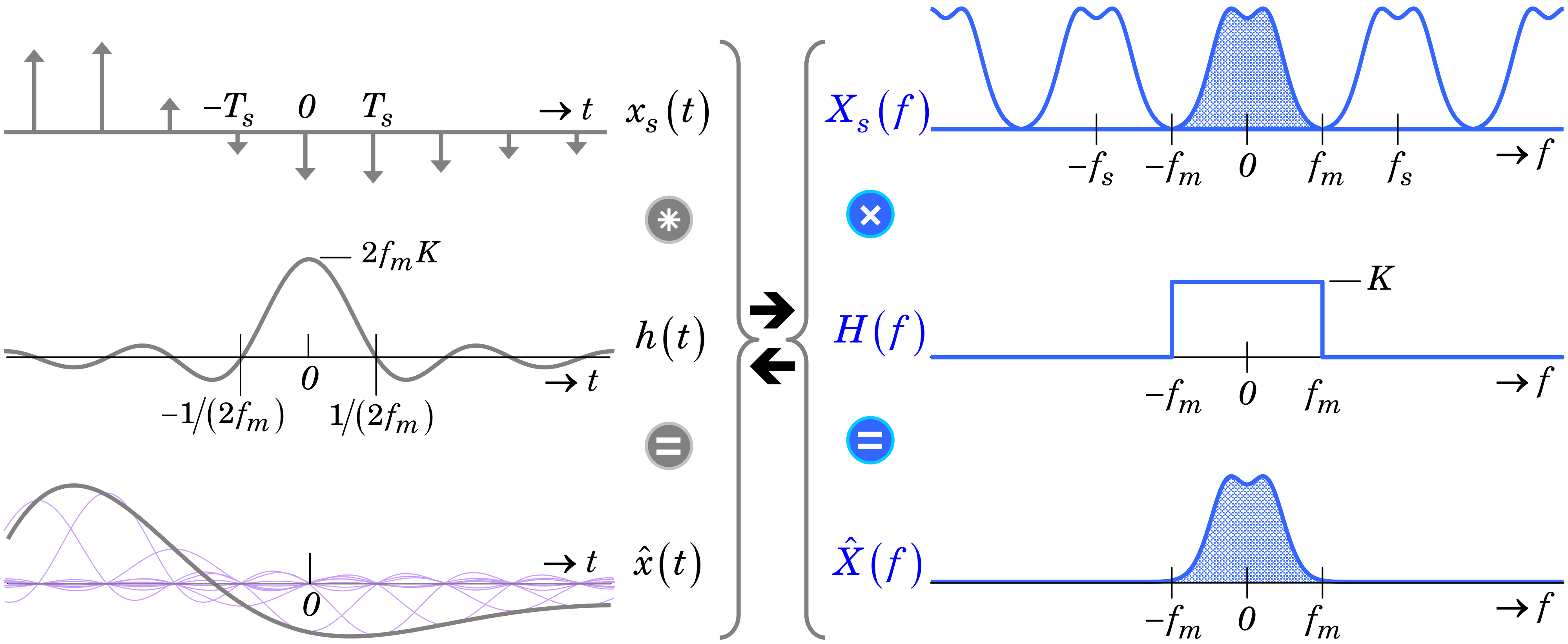




- ***Sampling*** (*Sampling frequency* $f_s = \dfrac{1}{T_s} = 2f_m$)



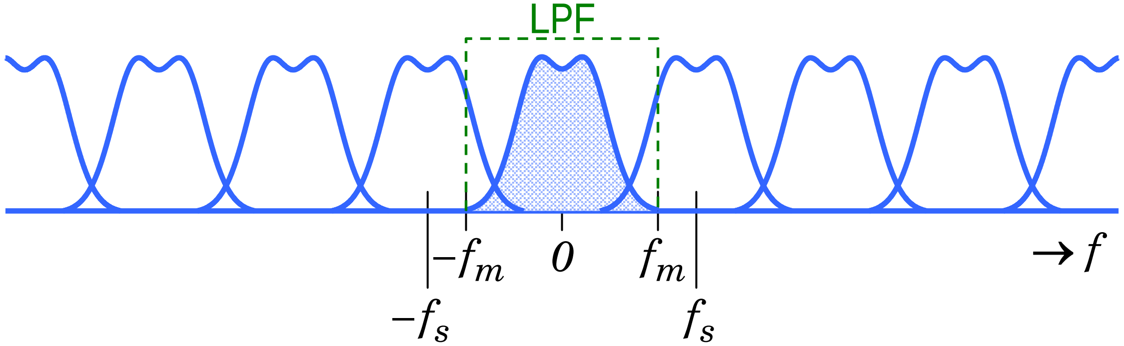
- ***Reconstruction***



What if sampling frequency $f_s = \dfrac{1}{T_s} \neq 2f_m$ ***?***

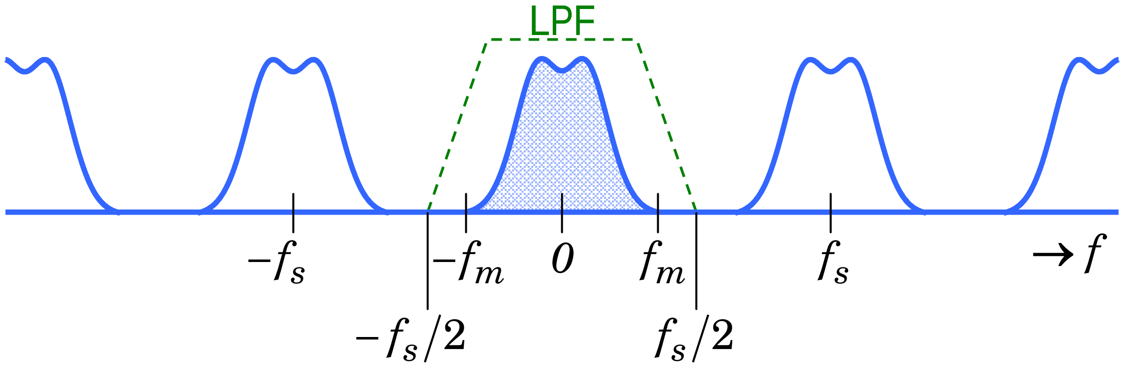
Sampling frequency $f_s = \dfrac{1}{T_s} < 2f_m$

- Spectral images overlap. This is called ***frequency aliasing.***
- Perfect reconstruction is not possible.



Sampling frequency $f_s = \dfrac{1}{T_s} > 2f_m$

- Gaps appear between spectral images due to ***oversampling.***
- Perfect reconstruction is possible.
- Relaxes LPF design.



Conditions for ***perfect reconstruction*** :

$$\left(\begin{array}{ll} x(t) \text{ must be bandlimited} & : X(f) = 0; \quad |f| > f_m \\ \text{Sampling rate of } x(t) & : f_s \ge 2f_m \end{array} \right) \Leftarrow$$

Nyquist Sampling Theorem

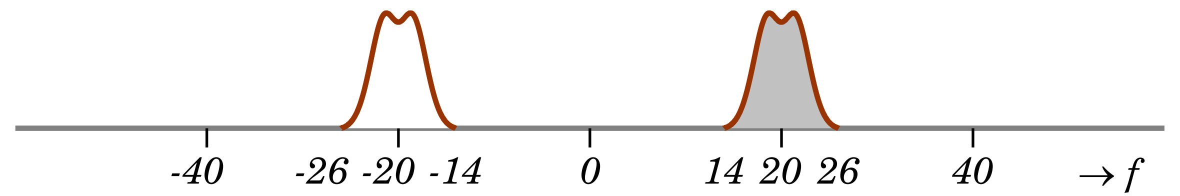
The sampling frequency should be at least twice the highest frequency contained in the signal.

$$\text{Reconstruction LPF} : \left(\begin{array}{ll} \left. \begin{array}{l} |H(f)| = \text{constant} \\ \angle H(f) = \text{linear} \end{array} \right\}; & |f| \le f_m \\ H(f) = 0; & |f| > f_s/2 \end{array} \right) \quad \text{assuming } f_s \ge 2f_m$$

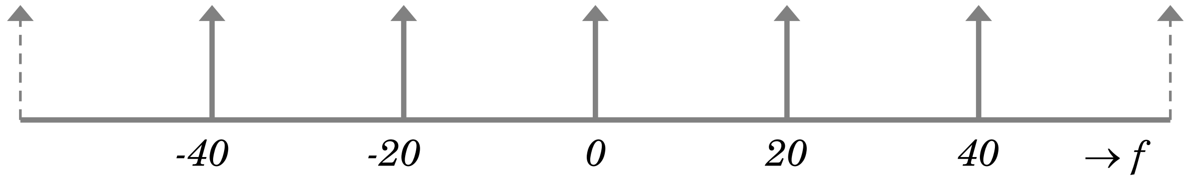
Example 4-1 (Sampling below Nyquist Rate)

$x(t):\left[\text{Bandpass}\left(\text{Centre - frequency}=20(\text{Hz}),\ \text{Bandwidth}=12(\text{Hz})\right)\right]$ and $\left[\Im\{x(t)\}=X(f)\right]$

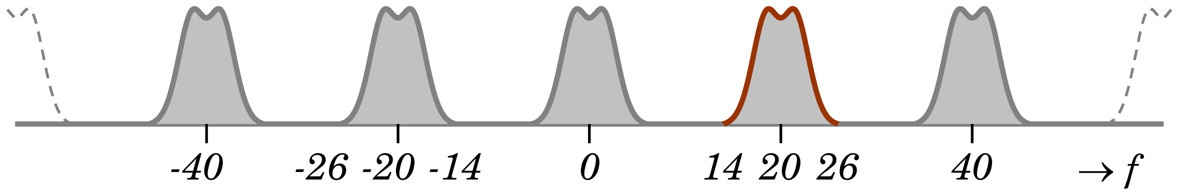
Nyquist rate: $26\times 2=52(\text{Hz})$. *Sample rate:* $20(\text{Hz})$.



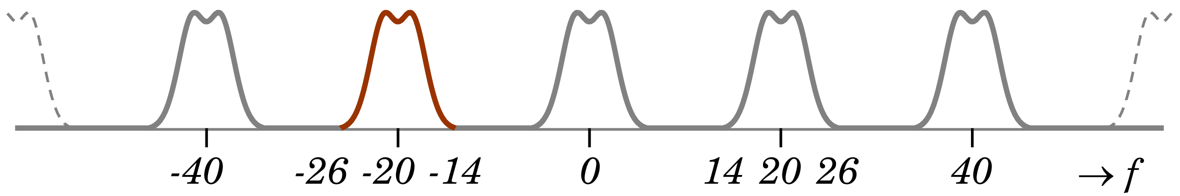
Spectrum of ORIGINAL SIGNAL $\boldsymbol{X(f)}$



Spectrum of SAMPLING FUNCTION $\boldsymbol{\Xi_T(f);\ \ T=\frac{1}{20}s}$

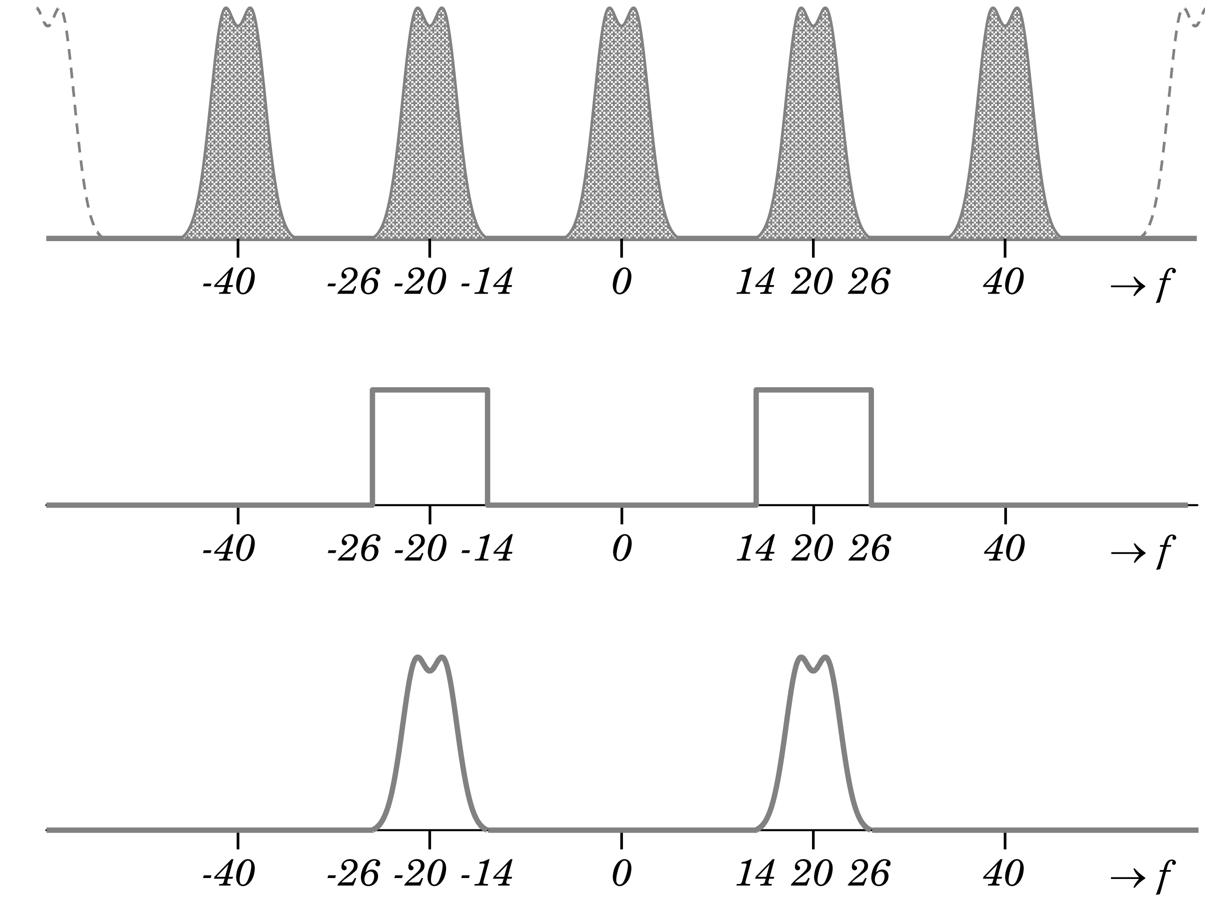


Images of +ve frequency band (A)



Images of -ve frequency band (B)

PgDn



Spectrum of SAMPLED SIGNAL
$(A)+(B)$

Ideal BANDPASS FILTER
Bandwidth : 12(Hz)
Center freq: 20(Hz)

Spectrum of
RECONSTRUCTED SIGNAL
$\boldsymbol{kX(f)}$

It is may be possible to sample a bandpass signal below Nyquist rate and yet achieve perfect reconstruction. This is NOT at all possible for LOWPASS signals.