Electrical & Computer Engineering Department
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EE2011 Engineering Electromagnetics
Tutorial 5: Magnetic Fields

Q1(a) A charged particle is initially travelling with velocity $\vec{v} = v_1 \hat{u}_x + v_2 \hat{u}_z$ (where v_1 and v_2 are constants). Explain what you expect to observe after it enters a region with uniform magnetic field $\vec{B} = B_0 \hat{u}_z$ (where B_0 is a constant).

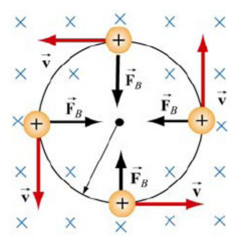
$v_2 \hat{u}_z$ component parallel to $B_0 \hat{u}_z$ $\rightarrow$ trajectory in z-direction unaffected

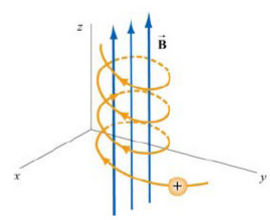
remaining component of $\vec{v}$ always normal to $\vec{B}$ $\rightarrow$ magnetic force $\vec{F} = q\,\vec{v} \times \vec{B}$

$\Rightarrow$ circular trajectory in x-y plane due to centripetal force $\frac{m v_1^2}{r} = q\, v_1 B$

expect to see helical trajectory with $r = \frac{m v_1}{q B}$

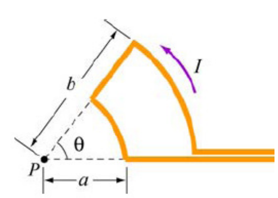
can also derive synchrotron frequency (*i.e.* angular velocity) $\omega = \frac{qB}{m}$





Q1(b) Figure 1(b) depicts a current-carrying wire formed by circular segments and radial lengths. Derive an expression for the magnetic flux density vector at P (which is the common center of the circular segments which have radii *a* and *b*).

apply Biot-Savart's Law $\vec{B} = \frac{\mu_o I}{4\pi} \int \frac{d\vec{s}\,' \times \hat{u}_{r''}}{(r'')^2}$



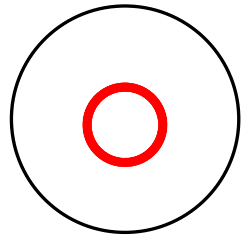
- no fields from I in radial lengths due to $d\vec{s}\,' \times \hat{u}_{r''} = \vec{0}$
- z-directed fields from I in circular arcs due to $d\vec{s}\,' \times \hat{u}_{r''}$

$$B_z = \pm \frac{\mu_o I}{4\pi} \int \frac{r'' d\theta'}{(r'')^2} = \pm \frac{\mu_o I}{4\pi r} \int d\theta'$$

opposite directions for field contributions from current flow in inner and outer arcs

$\therefore$ sum of both contributions $B_z = \frac{\mu_o I \theta}{4\pi} \left(\frac{1}{a} - \frac{1}{b} \right)$ into paper

Q1(c) A thin circular disk (with radius r_0) rotates with angular speed ω. Show that the magnetic field strength at the center of the disk (with uniform surface charge density σ) is given by $B = \frac{1}{2}\mu_0\sigma\omega r_0$.



consider $\mathrm{d}r$ strip containing charge $\mathrm{dq} = \sigma(2\pi r\,\mathrm{d}r)$

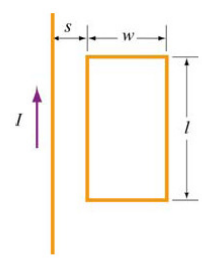
equivalent to current flow $\mathrm{dI} = \frac{\omega}{2\pi}\mathrm{dq} = \omega\sigma r\,\mathrm{d}r$

contributes $\mathrm{dB} = \mu_0 \frac{\mathrm{dI}}{2r}$ at center of circular loop

(see Appendix for derivation of B from current in circular wire loop)

integrate with respect to r $\rightarrow$ $B = \frac{1}{2}\mu_0\omega\sigma\int_0^{r_0}\mathrm{d}r = \frac{1}{2}\mu_0\omega\sigma r_0$

Q2. Figure 2 depicts a rectangular wire loop (of length l and width w) which is placed in the vicinity of a long straight wire. Determine the mutual impedance between these two (with separation s).



2 options**:** place current on *either* straight wire *or* wire loop

same answer for both but more troublesome to derive B of loop

$\therefore$ choose to place I on straight wire $\rightarrow$ can use Ampere's Law

$$B_\phi = \frac{\mu_0 I}{2\pi r} \qquad \textit{(valid only if wire is sufficiently long)}$$

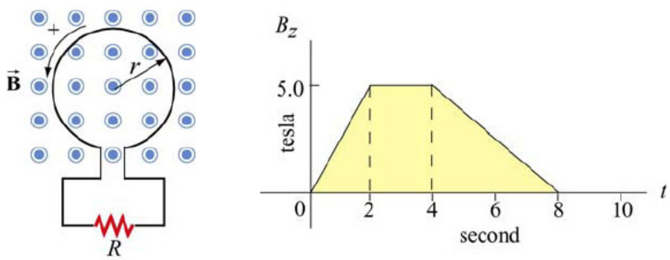
gives rise to flux linkage with loop

$$\mathrm{d}\Phi = B_\phi \mathrm{dA} = \frac{\mu_0 I}{2\pi r}(l\,\mathrm{dr}) = \frac{\mu_0 I l}{2\pi}\frac{\mathrm{dr}}{r}$$

divide total flux by current to obtain mutual impedance**:**

$$M = \frac{\Phi}{I} = \frac{\mu_0 l}{2\pi}\int_s^{s+w}\frac{\mathrm{dr}}{r} = \frac{\mu_0 l}{2\pi}\ln\left(1+\frac{w}{s}\right)$$

Q3. Figure 3(a) depicts a circular wire loop (with radius r = 50 cm) which is connected to a resistor (with resistance R = 100 Ω). The uniform magnetic field $\vec{B}$ in the vicinity varies with time t in accordance with the plot reproduced in Figure 3(b). Sketch the variation of the current flowing in R as a function of time t, given that $\vec{B}$ is in the +z direction (as denoted by the circles with enclosed dots) and the corresponding positive convention for the circular loop is given by the faint arrow.



current flow in R due to EMF given by Faraday's Law

$$I = \frac{V_{EMF}}{R} = -\frac{1}{R}\frac{d}{dt}\iint \vec{B} \bullet d\vec{A} = -\frac{\pi r^2}{R}\frac{dB_z}{dt}$$

Lenz's Law: negative sign → induced current in clockwise sense if $\frac{dB_z}{dt} > 0$

for $0 < t < 2$, EMF due to (linear) increase in B

$$\therefore\ I = -\frac{\pi\, 0.5^2}{100}\frac{5}{2} = -19.6 \text{ mA } (\textit{i.e.} \text{ clockwise})$$

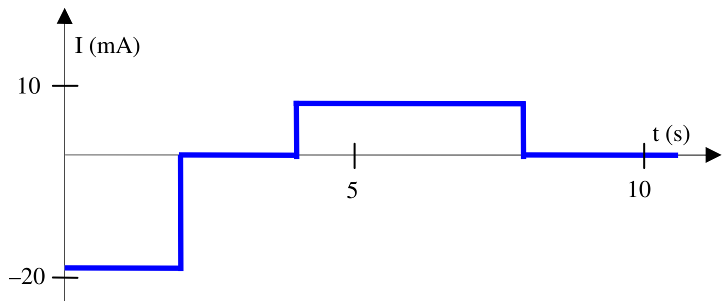
for $2 < t < 4$, no EMF due to constant B

$$\therefore\ I = -\frac{\pi\, 0.5^2}{100}\frac{0}{2} = 0$$

for $4 < t < 8$, EMF due to (linear) decrease in B

$$\therefore\ I = -\frac{\pi\, 0.5^2}{100}\left(-\frac{5}{4}\right) = +9.8 \text{ mA } (\textit{i.e.} \text{ anti-clockwise})$$

for $t > 8$, no EMF due to constant B with current thus reverting to zero



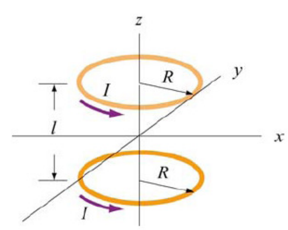
Q4. Engineers often employ Helmholz coils to provide a region with sufficiently uniform magnetic field. As shown in Figure 4, the set-up comprises two identical coils which are symmetrically equidistant from the origin O of the Cartesian coordinate system. Both coils have N turns of wire, radius R, current I and +z orientation.

Derive an expression for the magnetic field at any point on the z-axis and show that its first-order derivative is zero (*i.e.* $\frac{\partial \mathrm{B}}{\partial z} = 0$) at the origin O.

Derive the design condition for its second-order derivative to be zero (*i.e.* $\frac{\partial^2 \mathrm{B}}{\partial z^2} = 0$) as well at the origin O.

start from magnetic field expression for single coil
(see Appendix for derivation of B from current in circular wire loop)

use superposition to obtain total field at coordinate-system origin



$$\mathrm{B_z} = \frac{\mu_0 \mathrm{N I} R^2}{2\left\{\left(z-\frac{l}{2}\right)^2 + R^2\right\}^{\frac{3}{2}}} + \frac{\mu_0 \mathrm{N I} R^2}{2\left\{\left(z+\frac{l}{2}\right)^2 + R^2\right\}^{\frac{3}{2}}}$$

$$\frac{\partial \mathrm{B_z}}{\partial \mathrm{z}} = -\frac{3\mu_0 \mathrm{N I} R^2}{2}\left\{\frac{z-\frac{l}{2}}{\left\{\left(z-\frac{l}{2}\right)^2 + R^2\right\}^{\frac{5}{2}}} + \frac{z+\frac{l}{2}}{\left\{\left(z+\frac{l}{2}\right)^2 + R^2\right\}^{\frac{5}{2}}}\right\}$$

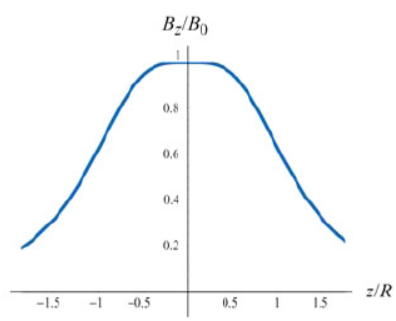
$$\frac{\partial \mathrm{B_z}}{\partial \mathrm{z}}(\mathrm{z}=0) = -\frac{3\mu_0 \mathrm{N I} R^2}{2}\left\{\frac{-\frac{l}{2}}{\left\{\left(\frac{l}{2}\right)^2 + R^2\right\}^{\frac{5}{2}}} + \frac{\frac{l}{2}}{\left\{\left(\frac{l}{2}\right)^2 + R^2\right\}^{\frac{5}{2}}}\right\} = 0$$

$$\frac{\partial^2 \mathrm{B_z}}{\partial \mathrm{z}^2} = -\frac{3\mu_0 \mathrm{N I} R^2}{2}\left\{\frac{1}{\left\{\left(z-\frac{l}{2}\right)^2 + R^2\right\}^{\frac{5}{2}}} + \frac{1}{\left\{\left(z+\frac{l}{2}\right)^2 + R^2\right\}^{\frac{5}{2}}} - \frac{5\left(z-\frac{l}{2}\right)^2}{\left\{\left(z-\frac{l}{2}\right)^2 + R^2\right\}^{\frac{7}{2}}} - \frac{5\left(z+\frac{l}{2}\right)^2}{\left\{\left(z+\frac{l}{2}\right)^2 + R^2\right\}^{\frac{7}{2}}}\right\}$$

$$\frac{\partial^2 \mathrm{B_z}}{\partial \mathrm{z}^2}(\mathrm{z}=0) = -\frac{3\mu_0 \mathrm{N I} R^2}{2}\left\{\frac{1}{\left\{\left(\frac{l}{2}\right)^2 + R^2\right\}^{\frac{5}{2}}} + \frac{1}{\left\{\left(\frac{l}{2}\right)^2 + R^2\right\}^{\frac{5}{2}}} - \frac{5\left(\frac{l}{2}\right)^2}{\left\{\left(\frac{l}{2}\right)^2 + R^2\right\}^{\frac{7}{2}}} - \frac{5\left(\frac{l}{2}\right)^2}{\left\{\left(\frac{l}{2}\right)^2 + R^2\right\}^{\frac{7}{2}}}\right\}$$

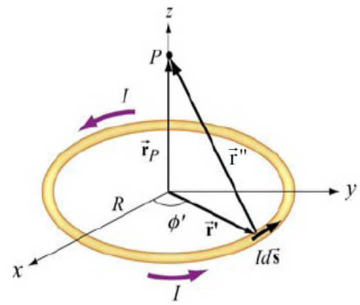
$$= -\frac{3\mu_0 \mathrm{N I} R^2\left(R^2 - l^2\right)}{\left\{\left(\frac{l}{2}\right)^2 + R^2\right\}^{\frac{7}{2}}} = 0 \quad \text{only if we choose } l = \mathrm{R}$$

almost uniform magnetic field for Helmholz coils in vicinity of mid-point



Appendix for use with Q1(c) and Q4

Derive an expression for the magnetic field along the axis of a circular wire loop (of radius *R*) carrying current I.



need to derive expression for $\vec{B}$ at P (0, 0, z) via Biot-Savart's Law $d\vec{B} = \frac{\mu_0 I}{4\pi}\frac{d\vec{s}' \times \hat{u}_{r''}}{(r'')^2}$

direction vector for elemental length $d\vec{s}'$ given by $\hat{u}_{\phi'} = -\sin\phi'\hat{u}_x + \cos\phi'\hat{u}_y$

position vector for elemental length $d\vec{s}'$ given by $\vec{r}' = R\left(\cos\phi'\hat{u}_x + \sin\phi'\hat{u}_y\right)$

$$\Rightarrow \vec{r}'' = \vec{r}_P - \vec{r}' = -R\left(\cos\phi\,\hat{u}_x + \sin\phi\,\hat{u}_y\right) + z\,\hat{u}_z$$

substituting into Biot-Savart's Law

$$d\vec{B} = \frac{\mu_0 I}{4\pi}\frac{d\vec{s}' \times \hat{u}_{r''}}{(r'')^2} = \frac{\mu_0 I}{4\pi}\frac{d\vec{s}' \times \vec{r}''}{(r'')^3} = \frac{\mu_0 I}{4\pi}\frac{d\vec{s}' \times (\vec{r}_P - \vec{r}')}{(r'')^3} = \frac{\mu_0 I R}{4\pi}\frac{z\cos\phi'\hat{u}_x + z\sin\phi'\hat{u}_y + R\hat{u}_z}{\left(R^2+z^2\right)^{\frac{3}{2}}}\,d\phi'$$

infer from cylindrical symmetry that $\vec{B} = B_z\hat{u}_z$

(or note from $d\vec{B}$ expression that $\oint \cos\phi' d\phi' = 0 \Rightarrow B_x = 0$ and $\oint \sin\phi' d\phi' = 0 \Rightarrow B_y = 0$)

$\therefore$ integrate only z-component in $d\vec{B}$ expression

$$B_z = \frac{\mu_0 I R^2}{4\pi\left(R^2+z^2\right)^{\frac{3}{2}}}\oint d\phi' = \frac{\mu_0 I R^2}{2\left(R^2+z^2\right)^{\frac{3}{2}}} \qquad \rightarrow \text{ to be used for Q4}$$

substitute z = 0 for field at center of loop

$$B_z\big|_{z=0} = \frac{\mu_0 I}{2R} \qquad \rightarrow \text{ to be used for Q1(c) after replacing } R \text{ by } r$$