NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 EXAMINATION 2005-2006

MA1506 Mathematics II

November/December 2005 — Time allowed : $2\frac{1}{2}$ hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **ONE (1)** sections. It contains a total of **TEN (10)** questions and comprises **FIVE (5)** printed pages.
2. Answer **ALL** questions. The marks for each questions are indicated at the beginning of the question.
3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Question 1 [10 marks]

(a) Find a potential function for the gradient vector field

$$\mathbf{F} = e^x\ \mathbf{i} + \frac{z}{y}\ \mathbf{j} + \ln y\ \mathbf{k},$$

and evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is given by the vector function

$$\mathbf{r}(t) = t\ \mathbf{i} + (t^2+1)\ \mathbf{j} + (t^3+2)\ \mathbf{k},$$

for $0 \le t \le 1$.

(b) Show that area of the region bounded by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is πab, where a and b are positive real numbers.

(c) Let C be the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$. Evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F} = (x^2 - 3y)\ \mathbf{i} + (2y^3 + 4x)\ \mathbf{j}.$$

You may use Green's theorem to simplify the computation.

Question 2 [10 marks]

(a) Fix a point (x_0, y_0) in the plane. Let $d(x,y)$ be the function which gives the distance from (x,y) to the point (x_0,y_0). Write a formula for $d(x,y)$.

(b) Use the method of Lagrange multipliers to find the point(s) on the hyperbola $x^2 - \frac{1}{4}y^2 = 1$ closest to the point $(0,-5)$. The function you need to minimize is the *square* of the distance from (x,y) to $(0,-5)$.

(c) Draw a picture that includes the point $(0,-5)$, the graph of the hyperbola above, together with the point(s) you found in part (a).

Question 3 [10 marks]

Let $f(x,y) = x^2y - y^2 + 2\sqrt{y}$.

(a) Find the domain of $f(x,y)$.

(b) Find the maximum rate of change of $f(x,y)$ at the point $(2,1)$ and the direction in which it occurs.

(c) Find a unit vector $\mathbf{u}$ such that $D_{\mathbf{u}}f(2,1) = -3$.

(d) Find the maximum value of $f(x,y)$ along the parabola $y = 2x^2$.

Question 4 [10 marks]

Let $\mathbf{F} = y\ \mathbf{i} + xz\ \mathbf{j} + z^2\ \mathbf{k}$.

(a) Find curl $\mathbf{F}$.

(b) Use Stokes' theorem to evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the triangle with vertices $(-1, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 3)$, oriented *clockwise* when viewed from above.

(c) Is there a vector field $\mathbf{G}$ such that

$$\text{curl } \mathbf{G} = xy\ \mathbf{i} + zx^2\ \mathbf{j} + xyz\ \mathbf{k}\ ?$$

Justify your answer.

Question 5 [10 marks]

Let P be the plane given by $z = k$. Assume that $0 < k < 1$, so that P intersects the unit sphere centered at the origin in some curve C at height k. Let S denote the part of the sphere lying above the plane P, which has boundary C.

(a) Find a parametrization for the curve C, and describe the projection of S onto the xy-plane. Your answers will depend on k.

(b) Write down and evaluate an integral which calculates the surface area of S in terms of k.

(c) Find the value of k for which the surface area of S is equal to π.

(d) For the value of k you found in (c), describe in spherical coordinates (by giving the ranges for ρ, θ, ϕ) the solid region D bounded on top by S and below by P.

Question 6 [10 marks]

Let S be the surface given by $x^2 + y^2 + z^2 - 2x - 4y + 1 = 0$, oriented with the outward pointing normal vector.

(a) What kind of quadric surface is S? Justify your answer.

(b) Using the divergence theorem, compute the flux integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S},$$

where $\mathbf{F} = (2x + y)\ \mathbf{i} + (x^2 - 3z)\ \mathbf{j} + xy\ \mathbf{k}$.

Question 7 [10 marks]

A piano string of length 3π with fixed endpoints is initially undeflected. When the corresponding key is played, the string is struck in the middle by a hammer, and acquires the velocity profile

$$g(x) = \begin{cases} 0 & \text{if } 0 \le x \le \pi \\ 1 & \text{if } \pi < x < 2\pi \\ 0 & \text{if } 2\pi \le x \le 3\pi \end{cases}$$

Suppose that the solution $u(x,t)$ describing the string's motion satisfies the wave equation $u_{tt} = u_{xx}$. The solution $u(x,t)$ can be expressed as an infinite sum $\sum_{n=1}^{\infty} u_n(x,t)$, where

$$u_n(x,t) = \left(a_n \cos\frac{nt}{3} + b_n \sin\frac{nt}{3}\right)\sin\frac{nx}{3},$$

for appropriate constants a_n and b_n. Find the first three terms in the series for $u(x,t)$, that is, find a_n and b_n for $n = 1, 2, 3$.

Question 8 [10 marks]

Let $w(x,t)$ be a function of x and t which satisfies the differential equation

$$xw_x + w_t = x^2,$$

together with the initial conditions $w(x,0) = 3$. Find $w(x,t)$ using the method of Laplace transform.

Question 9 [10 marks]

$$A = \begin{bmatrix} -1 & 1 & 2 & 0 \\ 4 & 0 & -2 & 3 \\ 2 & 0 & a & -1 \\ 1 & 2 & 1 & 1 \end{bmatrix}.$$

(a) Find the value of a for which the inverse matrix A^{-1} fails to exist.

(b) Substitute this value of a into A. Since A is not invertible, the rows R_1, R_2, R_3, R_4 of A (regarded now as vectors in $\mathbb{R}^4$) cannot be linearly independent. Thus we can find scalars λ_1, λ_2, and λ_3 such that

$$R_4 = \lambda_1 R_1 + \lambda_2 R_2 + \lambda_3 R_3.$$

Find λ_1, λ_2, and λ_3.

Question 10 [10 marks]

(a) Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 10 & -4 \\ 18 & -12 \end{bmatrix}.$$

(b) Use this information to solve the linear system of differential equations

$$y_1' = 10y_1 - 4y_2, \qquad y_2' = 18y_1 - 12y_2,$$

given the initial conditions

$$y_1(0) = 1, \qquad y_2(0) = 8.$$

END OF PAPER