Chapter 7. Inference concerning the mean (C)

March 29, 2011

More Example

A department store manager determines that a new billing system will be cost-effective only if the mean monthly account is more than \$170.

A random sample of 400 monthly accounts is drawn, for which the sample mean is \$178. The accounts are approximately normally distributed with a standard deviation of \$65.

Can we conclude that the new system will be cost-effective?

We express this belief as a our research hypothesis, that is:

H$_1$: $\mu > 170$ (this is what we want to determine)

Thus, our null hypothesis becomes:

H$_0$: $\mu = 170$ (this specifies a single value, $\mu_0 = 170$, for the parameter of interest) [1]

We know:

$$n = 400, \quad \bar{x} = 178, \quad \sigma = 65, \quad \alpha = 0.05$$

We use the following test statistic

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$

Under H_0 (i.e. if H_0 is true), $Z \sim N(0, 1)$.

[1] Actually we mean $H_0 : \leq 170$, but usually we prefer the former

What will happen if H_1 is true instead?

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} = \frac{\bar{X} - \mu + \mu - \mu_0}{\sigma/\sqrt{n}} = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} + \frac{\mu - \mu_0}{\sigma/\sqrt{n}}$$

Under H_1 (i.e. if H_1 is true), $Z \sim N(c, 1)$ where

$$c = \frac{\mu - \mu_0}{\sigma/\sqrt{n}} > 0.$$

At a 5% significance level (i.e. $\alpha = 0.05$), we get [all α in one tail]

$$z_\alpha = z_{0.05} = 1.65$$

Thus

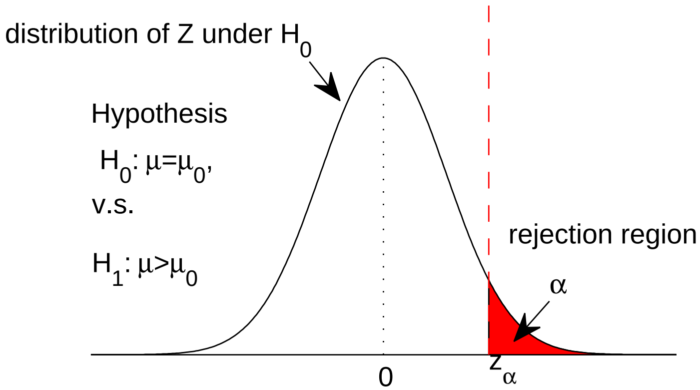
$$z \;=\; \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{178 - 170}{65/\sqrt{400}} = 2.46 (> z_\alpha)$$

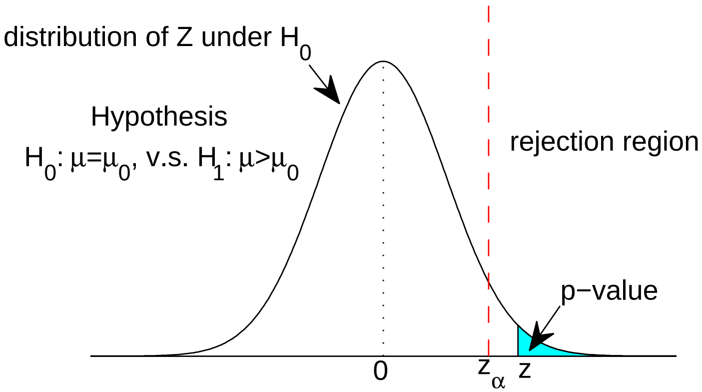
Therefore, reject the null hypothesis

OR

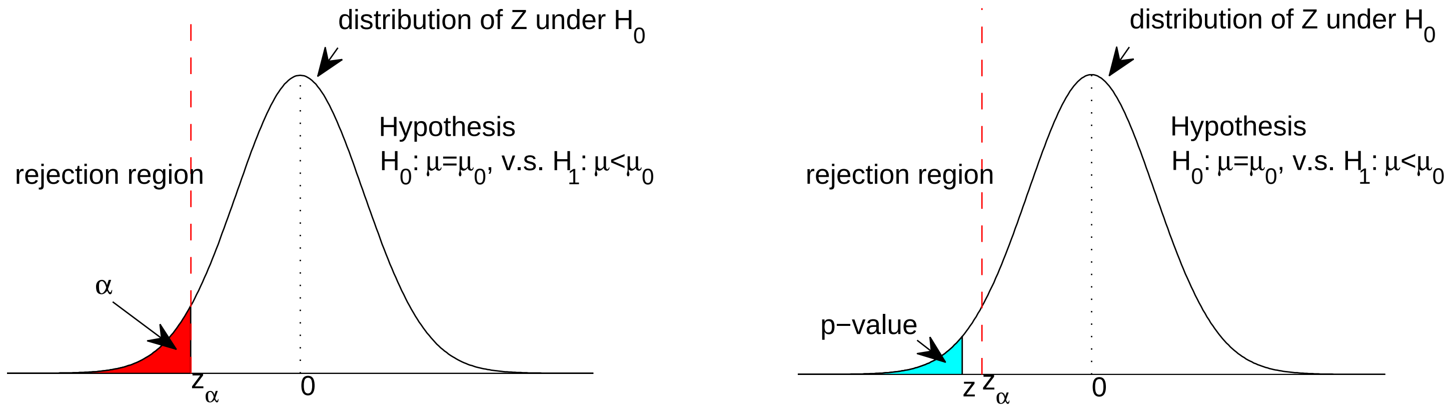
$$\text{p-value} = P(\bar{X} > 178) = P(Z > 2.46) = 0.0069 < \alpha$$

Therefore, reject the null hypothesis

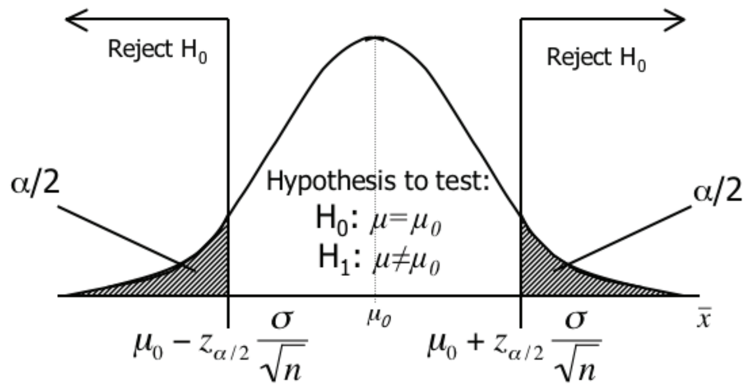




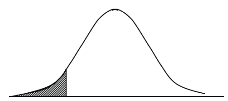
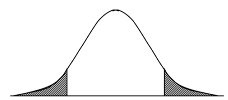
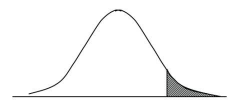
One tail test with rejection region on left, the rejection region will be in the left tail.



Two tail test with rejection region in both tails, The rejection region is split equally between the two tails. [the following plot is based on $\bar{X}$ instead of Z, the plot based on Z can be drawn similarly]



Summary of One- and Two-Tail Tests...

One-Tail Test (left tail)	Two-Tail Test	One-Tail Test (right tail)
$H_0 : \mu = \mu_0$ $H_1 : \mu < \mu_0$	$H_0 : \mu = \mu_0$ $H_1 : \mu \neq \mu_0$	$H_0 : \mu = \mu_0$ $H_1 : \mu > \mu_0$
		

and again

H1	Rejection Region	p-value
$\mu > \mu_0$	$z > z_\alpha$	1-F(z)
$\mu < \mu_0$	$z < -z_\alpha$	F(z)
$\mu \neq \mu_0$	$z < -z_{\alpha/2}$ or $z > z_{\alpha/2}$	2F(-|z|)

Case II: σ Known, n Large

- For Case II, similar arguments apply & results are summarized as follows

- H_0: $\mu = \mu_0$. The level of significance α. Test statistic and its distribution:

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \approx N(0,1)$$

H1	Rejection Region	p-value
$\mu > \mu_0$	$z > z_\alpha$	1-F(z)
$\mu < \mu_0$	$z < -z_\alpha$	F(z)
$\mu \neq \mu_0$	$z < -z_{\alpha/2}$ or $z > z_{\alpha/2}$	2F(-|z|)

Case III: σ Unknown, Data Normal, n Small

- H_0: $\mu = \mu_0$. The level of significance α. Test statistic and its distribution: $t = \frac{\bar{X}-\mu_0}{S/\sqrt{n}} \sim t_{n-1}$ or $(t(n-1))$, the t distribution with degrees of freedom n-1. Denote the computed value as .

H_1	Rejection Region	p-value
$\mu > \mu_0$	$t^* > t_\alpha$	$P(t \geq t^*)$
$\mu < \mu_0$	$t^* < -t_\alpha$	$P(t \leq t^*)$
$\mu \neq \mu_0$	$t^* > t_{\alpha/2}$ or $t^* < -t_{\alpha/2}$	$P(\lvert t \rvert \geq \lvert t^* \rvert)$

Example: Midterm Exam Scores

- We still use the midterm exam scores example. Assume that the lecturer didn't announce the variance. i.e. σ is unknown. Case III should be used.

- Step 1 and Step 2 are not changed.

- Step 3: Now that σ is known, data are normal, n=5.

$$t = \frac{\bar{X} - 16}{S/\sqrt{n}} \sim t(n-1) = t(4)$$

  rejection region: $t^* > t_\alpha$.

- Step 4: $t^* = (22 - 16)/(2.55/\sqrt{5}) = 5.26$.

- Step 5: 5.26 is contained within rejection region, we reject H$_0$ (or p-value $= 0.00313 < \alpha$, reject H$_0$)

Case IV: σ Unknown, n Large

- H_0: $\mu = \mu_0$. The level of significance α. Test statistic and its distribution:

$$Z = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} \approx N(0,1)$$

H1	Rejection Region	p-value
$\mu > \mu_0$	$z > z_\alpha$	1-F(z)
$\mu < \mu_0$	$z < -z_\alpha$	F(z)
$\mu \neq \mu_0$	$z < -z_{\alpha/2}$ or $z > z_{\alpha/2}$	2F(-|z|)

The Relation between Tests and Confidence Intervals

- We use Case III as an example. Recall that in Case III, the 100(1-α)% C.I. for μ is given by

$$\bar{x} - t_{\alpha/2}\frac{s}{\sqrt{n}} < \mu < \bar{x} + t_{\alpha/2}\frac{s}{\sqrt{n}}$$

- If we rearrange the above inequality, we obtain

$$-t_{\alpha/2} \leq \frac{\bar{x} - \mu}{s/\sqrt{n}} \leq t_{\alpha/2}$$

- On the other hand, the rejection region of two-sided test of hypothesis problem in case III is $t^* > t_{\alpha/2}$

- The relation between tests and C.I. can then be concluded

- When the $100(1-\alpha)\%$ C.I. contains μ_0, i.e.

$$-t_{\alpha/2} \leq \frac{\bar{x}-\mu_0}{s/\sqrt{n}} \leq t_{\alpha/2}$$

that is, the computed statistic value $-t_{\alpha/2} \leq t^* \leq t_{\alpha/2}$, which in turn means t^* is not located within the rejection region. H_0 will not be rejected at level α.

- Similarly, when C.I. does not contain μ_0 , then, $t^* > t_{\alpha/2}$ or $t^* < -t_{\alpha/2}$, indicating that t^* is contained by the rejection region. H_0 will be rejected.

- Therefore, C.I. can be used to perform a two-sided test.

Example: Use C.I. to Perform A Test

- We still use the midterm exam scores example. Assume that the lecturer didn't announce the variance. i.e. σ is unknown. Case III should be used. The student constructed 99% (i.e. $\alpha = 0.01$) C.I. for the average score of students for the midterm:

$$\bar{x} \pm t_{\alpha/2} \frac{s}{\sqrt{n}} = 22 \pm 4.604 \frac{2.55}{\sqrt{5}} = (16.75, 27.25)$$

- Note that the interval does not contain 16, therefore, the following test of hypothesis problem should be rejected: H_0: μ=16 versus H_1: $\mu \neq$16.

- However, the following test of hypothesis problem should not be rejected: H_0: μ=17 versus H_1: $\mu \neq$17.

Power[2]

- Definition: The **power of a statistical test**, given as

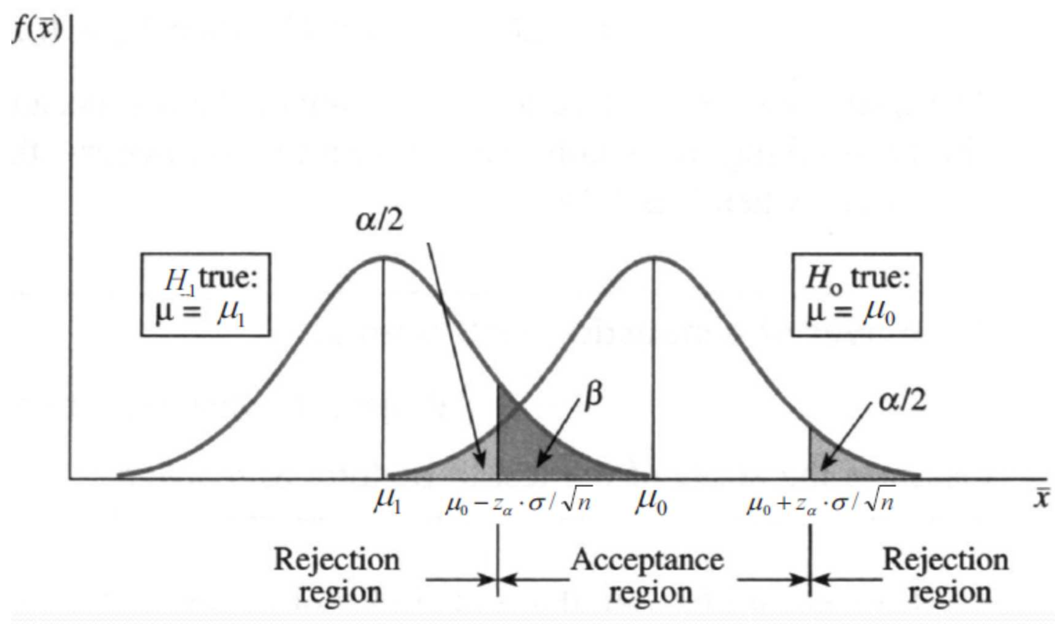
$$1 - \beta = P(\text{reject H}_0 \text{ when H}_1 \text{ is true})$$

  measures the ability of the test to perform as required.

- β is a function of true value of μ.

- The Figure in the next slide illustrates the meaning of $\beta(\mu = \mu_1)$, when H$_1$: $\mu \neq \mu_1$.

[2] advanced topic, you can ignore it



$f(\bar{x})$
$\alpha/2$
H_1 true:
$\mu = \mu_1$
H_0 true:
$\mu = \mu_0$
β
$\alpha/2$
μ_1
$\mu_0 - z_\alpha \cdot \sigma / \sqrt{n}$
μ_0
$\mu_0 + z_\alpha \cdot \sigma / \sqrt{n}$
$\bar{x}$
Rejection region
Acceptance region
Rejection region