

Geometric Series

For $a \neq 0$, the series

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

is called a geometric series, where a and r are fixed numbers,

a is called the first term and r is the (common) ratio.

Geometric Series

For this series, the n -th partial sum s_n is given by

$$s_n = a + \cancel{ar} + \cancel{ar^2} + \cdots + \cancel{ar^{n-1}}$$

$$rs_n = \cancel{ar} + \cancel{ar^2} + ar^3 + \cdots + \cancel{ar^{n-1}} + ar^n.$$

$$s_n - rs_n = a - ar^n$$

$$s_n = a \frac{1 - r^n}{1 - r}$$

$$r \neq 1$$

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$$

(i) $r = 1$

$$a + a + a + a + \cdots$$

Then $s_n = na \rightarrow \infty$ if $a > 0$ (or $-\infty$ if $a < 0$)

Thus, the series is *divergent*.

(ii) $r = -1$

$$a - a + a - a + \cdots$$

Then $\{s_n\}$ is $a, 0, a, 0, \cdots$

Thus, the series is *divergent*.

$$s_n = a \frac{1 - r^n}{1 - r}$$

(iii) If $|r| < 1$, then $r^n \rightarrow 0$.

Thus,
$$s_n \rightarrow \frac{a}{1 - r}.$$

Hence, the sum of the series is $\frac{a}{1 - r}$.

(iv) If $|r| > 1$, then $r^n \rightarrow \infty$ (or $-\infty$), and the series diverges.

Convergence of Geometric Series

The geometric series

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$$

with $a \neq 0$ converges to the sum

$$\frac{a}{1-r} \quad \text{if } |r| < 1$$

and

it diverges if $|r| \geq 1$.

Example

(i) $\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \dots$ is a geometric series

first term $a = \frac{1}{9}$ and common ratio $r = \frac{1}{3}$.

It converges to $\frac{a}{1-r} = \frac{\frac{1}{9}}{1-\frac{1}{3}}$
 $= \frac{1}{6}.$

Example

$$(ii) \quad 4 - 2 + 1 - \frac{1}{2} + \frac{1}{4} - \dots$$

first term $a = 4$ and common ratio $r = -\frac{1}{2}$.

$$\begin{aligned} 4 + 4\left(-\frac{1}{2}\right) + 4\left(-\frac{1}{2}\right)^2 + \dots &= \frac{a}{1-r} \\ &= \frac{4}{1 - \left(-\frac{1}{2}\right)} \\ &= \frac{8}{3} \end{aligned}$$

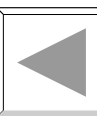
Some Rules on Series

If $\sum a_n = A$, and $\sum b_n = B$, then

(1) Sum rule: $\sum (a_n + b_n) = A + B.$

(2) Difference rule: $\sum (a_n - b_n) = A - B.$

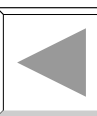
(3) Constant multiple rule: $\sum (ka_n) = kA.$



Question

$$\text{Infinite Series : } \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

How to check a given infinite series is convergent ???



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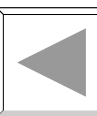
How to check a given infinite series is convergent ???

Consider the *partial sum* $s_n = a_1 + a_2 + \cdots + a_n$.

If $\lim_{n \rightarrow \infty} s_n = L$, then we have

$$a_1 + a_2 + \cdots + a_n + \cdots = L$$

$$\sum_{n=1}^{\infty} a_n = L$$



Ratio Test

Let $\sum a_n$ be a series, and let

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \mathbf{r}.$$

Then

(1) the series converges if $\mathbf{r} < 1$.

(2) the series diverges if $\mathbf{r} > 1$.

(3) no conclusion if $\mathbf{r} = 1$.

Ratio Test - Example

$$(i) \sum a_n \quad \text{where } a_1 = 1 \quad \text{and} \quad a_{n+1} = \frac{n}{2n+1} a_n$$

To find a_2 , put $n = 1$

$$a_{1+1} = \frac{1}{2(1)+1} a_1$$
$$a_2 = \frac{1}{3}$$

To find a_3 , put $n = 2$

$$a_{2+1} = \frac{2}{2(2)+1} a_2$$
$$a_3 = \frac{2}{5} \cdot \frac{1}{3}$$

The series is

$$\sum a_n = 1 + \frac{1}{3} + \frac{1 \cdot 2}{3 \cdot 5} + \frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7} + \dots$$

$$(i) \sum a_n \quad \text{where } a_1 = 1 \quad \text{and} \quad a_{n+1} = \frac{n}{2n+1} a_n$$

The series is

$$\sum a_n = 1 + \frac{1}{3} + \frac{1 \cdot 2}{3 \cdot 5} + \frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7} + \dots$$

$$\text{From } a_{n+1} = \frac{n}{2n+1} a_n, \quad \text{we have } \frac{a_{n+1}}{a_n} = \frac{n}{2n+1}$$

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \frac{n}{2n+1} \\ &= \frac{\frac{n}{n}}{\frac{2n}{n} + \frac{1}{n}} \\ &= \frac{1}{2 + \frac{1}{n}} \rightarrow \frac{1}{2} \quad \text{as } n \rightarrow \infty. \end{aligned}$$

By ratio test,
the given series is convergent.

$$\frac{1}{n} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

Note

The factorial of a non - negative integer n , denoted by $n!$, is given by

$$n! = 1 \times 2 \times 3 \times \cdots \times n$$

$$5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$$

Note that :

$$(n+1)! = 1 \times 2 \times 3 \times \cdots \times n \times (n+1) = n! \times (n+1)$$

$$\text{Thus, we have } \frac{(n+1)!}{n!} = \frac{n! \times (n+1)}{n!} = n+1.$$

Ratio Test - Example

(ii) Determine if $\sum \frac{(n!)^2}{(2n)!}$ is convergent.

$$a_n = \frac{(n!)^2}{(2n)!} = \frac{n!n!}{(2n)!}$$

Replace n by $n + 1$

$$a_{n+1} = \frac{(n+1)!(n+1)!}{(2n+2)!}$$

$$\frac{a_n}{a_{n+1}} = \frac{(n+1)!(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{n!n!}$$

$$= \frac{(n+1) \cdot n! (n+1) \cdot n!}{(2n+2)(2n+1) \cdot (2n)!} \cdot \frac{(2n)!}{n!n!}$$

$$= \frac{(n+1)(n+1)}{(2n+2)(2n+1)}$$

$$= \frac{n+1}{2(2n+1)} = \frac{1 + \frac{1}{n}}{2(2 + \frac{1}{n})} \rightarrow \frac{1}{2(2)} = \frac{1}{4}.$$

Note that :

$$(2n+2)! = (2n+2)(2n+1) \cdot (2n)!$$

By ratio test,
the given series
is convergent.

Ratio Test - Example

(iii) Determine if $\sum \frac{3^n}{2^n + 5}$ is convergent.

$$a_n = \frac{3^n}{2^n + 5}$$

Replace n by $n + 1$

$$a_{n+1} = \frac{3^{n+1}}{2^{n+1} + 5}$$

$$\frac{a_{n+1}}{a_n} = \frac{3^{n+1}}{2^{n+1} + 5} \cdot \frac{2^n + 5}{3^n}$$

$$= 3 \frac{2^n + 5}{2^{n+1} + 5}$$

$$= 3 \cdot \frac{1 + \frac{5}{2^n}}{2 + \frac{5}{2^n}} \rightarrow \frac{3}{2}$$

Divide by 2^n

$$\frac{5}{2^n} \rightarrow 0 \text{ as } n \rightarrow \infty$$

By ratio test,
the given series
is divergent.

Ratio Test - Example

(iv) Determine if the Harmonic series $\sum \frac{1}{n}$ is convergent.

$$a_n = \frac{1}{n}$$

Replace n by $n+1$

$$a_{n+1} = \frac{1}{n+1}$$

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \frac{n}{n+1} \\ &= \frac{1}{1 + \frac{1}{n}} \rightarrow 1. \end{aligned}$$

We cannot draw conclusion from ratio test.

Ratio Test - Example

(v) Determine if $\sum \frac{1}{n^2}$ is convergent.

$$a_n = \frac{1}{n^2}$$

Replace n by $n+1$

$$a_{n+1} = \frac{1}{(n+1)^2}$$

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \frac{n^2}{(n+1)^2} \\ &= \frac{1}{\left(1 + \frac{1}{n}\right)^2} \rightarrow 1. \end{aligned}$$

We cannot draw conclusion from ratio test.

Ratio Test - Example

(iv) Determine if $\sum \frac{1}{n}$ is convergent.

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{n}{n+1} \\ = \frac{1}{1 + \frac{1}{n}} \rightarrow 1.$$

(v) Determine if $\sum \frac{1}{n^2}$ is convergent.

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{n^2}{(n+1)^2} \\ = \frac{1}{\left(1 + \frac{1}{n}\right)^2} \rightarrow 1.$$

We cannot draw conclusion from ratio test.

It can be shown that

$\sum \frac{1}{n}$ is divergent.

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$\sum \frac{1}{n^2}$ is convergent.

Ratio Test - Example

To show the Harmonic series is divergent.

$$\sum \frac{1}{n} = 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots$$

$$> 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \dots$$

$$> 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) + \dots$$

Thus, the Harmonic series is divergent.

***p*-series**

The *p* - *series* is the series

$$\sum \frac{1}{n^p}$$

for any non-negative real number p .

(i) It diverges if $0 \leq p \leq 1$.

(ii) It converges if $p > 1$.

