Problem 4.5 Find the total charge on a circular disk defined by $r \leq a$ and $z = 0$ if:

(a) $\rho_s = \rho_{s0} \cos\phi$ (C/m^2)

(b) $\rho_s = \rho_{s0} \sin^2\phi$ (C/m^2)

(c) $\rho_s = \rho_{s0} e^{-r}$ (C/m^2)

(d) $\rho_s = \rho_{s0} e^{-r} \sin^2\phi$ (C/m^2)

where ρ_{s0} is a constant.

Solution:

(a)

$$Q = \int \rho_s \, ds = \int_{r=0}^{a} \int_{\phi=0}^{2\pi} \rho_{s0} \cos\phi \;\; r\, dr\, d\phi = \rho_{s0} \left. \frac{r^2}{2} \right|_0^a \left. \sin\phi \right|_0^{2\pi} = 0.$$

(b)

$$\begin{aligned} Q = \int_{r=0}^{a} \int_{\phi=0}^{2\pi} \rho_{s0} \sin^2\phi \;\; r\, dr\, d\phi &= \rho_{s0} \left. \frac{r^2}{2} \right|_0^a \int_0^{2\pi} \left(\frac{1-\cos 2\phi}{2} \right) d\phi \\ &= \frac{\rho_{s0} a^2}{4} \left. \left(\phi - \frac{\sin 2\phi}{2} \right) \right|_0^{2\pi} = \frac{\pi a^2}{2} \rho_{s0}. \end{aligned}$$

(c)

$$\begin{aligned} Q = \int_{r=0}^{a} \int_{\phi=0}^{2\pi} \rho_{s0} e^{-r} r\, dr\, d\phi &= 2\pi\rho_{s0} \int_0^a r e^{-r}\, dr \\ &= 2\pi\rho_{s0} \left[-re^{-r} - e^{-r} \right]_0^a \\ &= 2\pi\rho_{s0} [1 - e^{-a}(1+a)]. \end{aligned}$$

(d)

$$\begin{aligned} Q &= \int_{r=0}^{a} \int_{\phi=0}^{2\pi} \rho_{s0} e^{-r} \sin^2\phi \;\; r\, dr\, d\phi \\ &= \rho_{s0} \int_{r=0}^{a} r e^{-r}\, dr \int_{\phi=0}^{2\pi} \sin^2\phi \, d\phi \\ &= \rho_{s0} [1 - e^{-a}(1+a)] \cdot \pi = \pi\rho_{s0} [1 - e^{-a}(1+a)]. \end{aligned}$$

Problem 4.16 A line of charge with uniform density ρ_l extends between $z = -L/2$ and $z = L/2$ along the z-axis. Apply Coulomb's law to obtain an expression for the electric field at any point $P(r, \phi, 0)$ on the x–y plane. Show that your result reduces to the expression given by (4.33) as the length L is extended to infinity.

Solution:

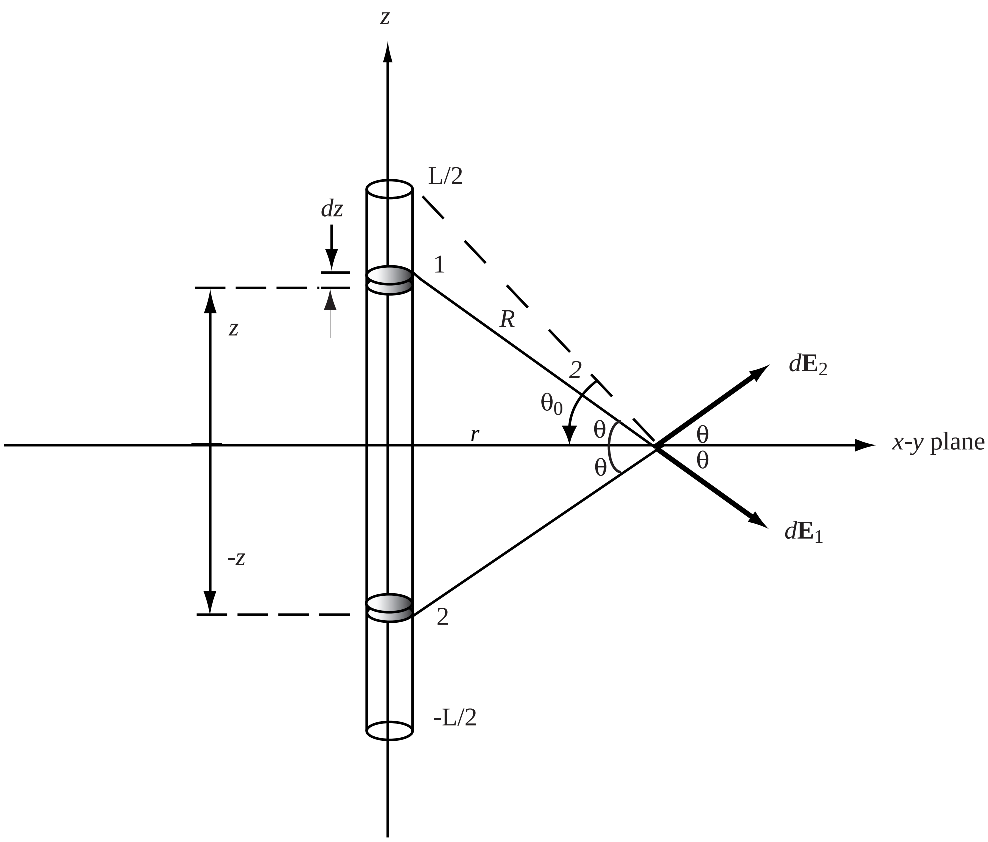

Figure P4.16: Line charge of length L.

Consider an element of charge of height dz at height z. Call it element 1. The electric field at P due to this element is $d\mathbf{E}_1$. Similarly, an element at $-z$ produces $d\mathbf{E}_2$. These two electric fields have equal z-components, but in opposite directions, and hence they will cancel. Their components along $\hat{\mathbf{r}}$ will add. Thus, the net field due to both elements is

$$d\mathbf{E} = d\mathbf{E}_1 + d\mathbf{E}_2 = \hat{\mathbf{r}}\frac{2\rho_l \cos\theta \, dz}{4\pi\varepsilon_0 R^2} = \frac{\hat{\mathbf{r}}\rho_l \cos\theta \, dz}{2\pi\varepsilon_0 R^2} \, .$$

where the $\cos\theta$ factor provides the components of $d\mathbf{E}_1$ and $d\mathbf{E}_2$ along $\hat{\mathbf{r}}$.

Our integration variable is z, but it will be easier to integrate over the variable θ from $\theta = 0$ to

$$\theta_0 = \sin^{-1} \frac{L/2}{\sqrt{r^2 + (L/2)^2}} \, .$$

Hence, with $R = r/\cos\theta$, and $z = r\tan\theta$ and $dz = r\sec^2\theta \; d\theta$, we have

$$\begin{aligned}
\mathbf{E} = \int_{z=0}^{L/2} d\mathbf{E} = \int_{\theta=0}^{\theta_0} d\mathbf{E} &= \int_0^{\theta_0} \hat{\mathbf{r}} \, \frac{\rho_l}{2\pi\varepsilon_0} \, \frac{\cos^3\theta}{r^2} \, r\sec^2\theta \; d\theta \\
&= \hat{\mathbf{r}} \, \frac{\rho_l}{2\pi\varepsilon_0 r} \int_0^{\theta_0} \cos\theta \; d\theta \\
&= \hat{\mathbf{r}} \, \frac{\rho_l}{2\pi\varepsilon_0 r} \sin\theta_0 = \hat{\mathbf{r}} \, \frac{\rho_l}{2\pi\varepsilon_0 r} \, \frac{L/2}{\sqrt{r^2 + (L/2)^2}} \, .
\end{aligned}$$

For $L \gg r$,

$$\frac{L/2}{\sqrt{r^2 + (L/2)^2}} \approx 1,$$

and

$$\mathbf{E} = \hat{\mathbf{r}} \, \frac{\rho_l}{2\pi\varepsilon_0 r} \qquad \text{(infinite line of charge).}$$

Problem 4.18 Multiple charges at different locations are said to be in equilibrium if the force acting on any one of them is identical in magnitude and direction to the force acting on any of the others. Suppose we have two negative charges, one located at the origin and carrying charge $-9e$, and the other located on the positive x-axis at a distance d from the first one and carrying charge $-36e$. Determine the location, polarity and magnitude of a third charge whose placement would bring the entire system into equilibrium.

Solution: If

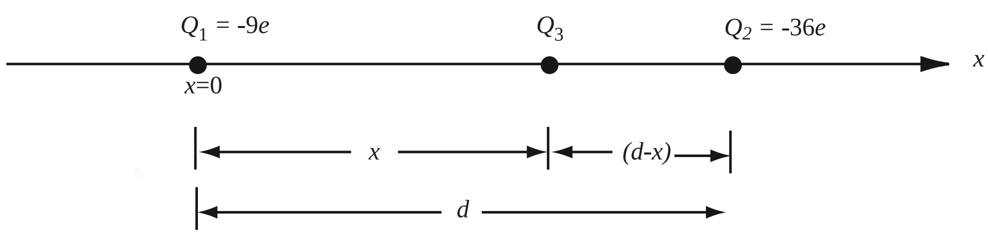

Figure P4.18: Three collinear charges.

$$\mathbf{F}_1 = \text{force on } Q_1,$$
$$\mathbf{F}_2 = \text{force on } Q_2,$$
$$\mathbf{F}_3 = \text{force on } Q_3,$$

then equilibrium means that

$$\mathbf{F}_1 = \mathbf{F}_2 = \mathbf{F}_3.$$

The two original charges are both negative, which mean they would repel each other. The third charge has to be positive and has to lie somewhere between them in order to counteract their repulsion force. The forces acting on charges Q_1, Q_2, and Q_3 are respectively

$$\mathbf{F}_1 = \frac{\hat{\mathbf{R}}_{21} Q_1 Q_2}{4\pi\varepsilon_0 R_{21}^2} + \frac{\hat{\mathbf{R}}_{31} Q_1 Q_3}{4\pi\varepsilon_0 R_{31}^2} = -\hat{\mathbf{x}} \frac{324e^2}{4\pi\varepsilon_0 d^2} + \hat{\mathbf{x}} \frac{9eQ_3}{4\pi\varepsilon_0 x^2},$$

$$\mathbf{F}_2 = \frac{\hat{\mathbf{R}}_{12} Q_1 Q_2}{4\pi\varepsilon_0 R_{12}^2} + \frac{\hat{\mathbf{R}}_{32} Q_3 Q_2}{4\pi\varepsilon_0 R_{32}^2} = \hat{\mathbf{x}} \frac{324e^2}{4\pi\varepsilon_0 d^2} - \hat{\mathbf{x}} \frac{36eQ_3}{4\pi\varepsilon_0 (d-x)^2},$$

$$\mathbf{F}_3 = \frac{\hat{\mathbf{R}}_{13} Q_1 Q_3}{4\pi\varepsilon_0 R_{13}^2} + \frac{\hat{\mathbf{R}}_{23} Q_2 Q_3}{4\pi\varepsilon_0 R_{23}^2} = -\hat{\mathbf{x}} \frac{9eQ_3}{4\pi\varepsilon_0 x^2} + \hat{\mathbf{x}} \frac{36eQ_3}{4\pi\varepsilon_0 (d-x)^2}.$$

Hence, equilibrium requires that

$$-\frac{324e}{d^2} + \frac{9Q_3}{x^2} = \frac{324e}{d^2} - \frac{36Q_3}{(d-x)^2} = -\frac{9Q_3}{x^2} + \frac{36Q_3}{(d-x)^2}.$$

Solution of the above equations yields

$$Q_3 = 4e, \qquad x = \frac{d}{3}\,.$$