Proofs for Fourier Transform Properties

Property A *(Linearity)*

$$\boldsymbol{\alpha x_1(t)+\beta x_2(t) \rightleftarrows \alpha X_1(f)+\beta X_2(f)}$$

Proof:

$$\begin{aligned}
\Im\{\alpha x_1(t)+\beta x_2(t)\} &= \int_{-\infty}^{\infty}\left[\alpha x_1(t)+\beta x_2(t)\right]\exp(-j2\pi ft)\,dt \\
&= \alpha\underbrace{\int_{-\infty}^{\infty}x_1(t)\exp(-j2\pi ft)\,dt}_{\Im\{x_1(t)\}}+\beta\underbrace{\int_{-\infty}^{\infty}x_2(t)\exp(-j2\pi ft)\,dt}_{\Im\{x_2(t)\}} \\
&= \alpha X_1(f)+\beta X_2(f)
\end{aligned}$$

Property B *(Time Scaling)*

$$\boldsymbol{x(\beta t) \rightleftarrows \frac{1}{|\beta|}X\left(\frac{f}{\beta}\right); \quad \beta\neq 0}$$

Proof:

$$\begin{aligned}
\Im\{x(\beta t)\} &= \int_{-\infty}^{\infty}x(\beta t)\exp(-j2\pi ft)\,dt \\
&\cdots\cdots \textit{ letting } \zeta=\beta t \textit{ and thus, } d\zeta=\beta dt \\
&= \frac{1}{\beta}\int_{-\beta.\infty}^{\beta.\infty}x(\zeta)\exp\left(-j2\pi\frac{f}{\beta}\zeta\right)d\zeta \\
&= \begin{cases} \dfrac{1}{\beta}\displaystyle\int_{-\infty}^{\infty}x(\zeta)\exp\left(-j2\pi\frac{f}{\beta}\zeta\right)d\zeta; & \beta>0 \\ -\dfrac{1}{\beta}\displaystyle\int_{-\infty}^{\infty}x(\zeta)\exp\left(-j2\pi\frac{f}{\beta}\zeta\right)d\zeta; & \beta<0 \end{cases} \\
&= \frac{1}{|\beta|}\int_{-\infty}^{\infty}x(\zeta)\exp\left(-j2\pi\frac{f}{\beta}\zeta\right)d\zeta; \quad \beta\neq 0 \\
&\cdots\cdots \textit{ letting } \zeta=t \textit{ and thus, } d\zeta=dt \\
&= \frac{1}{|\beta|}\underbrace{\int_{-\infty}^{\infty}x(t)\exp\left(-j2\pi\frac{f}{\beta}t\right)dt}_{\substack{\textit{Fourier transform of } x(t) \\ \textit{with } f \textit{ replaced by } f/\beta}} = \frac{1}{|\beta|}X\left(\frac{f}{\beta}\right)
\end{aligned}$$

Remarks:

$(1<|\beta|<\infty)$: *Time-compression and frequency-expansion by a factor of* β.

$(0<|\beta|<1)$: *Time-expansion and frequency-compression by a factor of* β.

$(\textit{negative } \beta)$: *Time-reversal and frequency-reversal.*

Property C *(Duality)*

$$\boldsymbol{X(t) \rightleftarrows x(-f)}$$

Proof:

$$x(t) = \int_{-\infty}^{\infty} X(f)\exp(j2\pi ft)df$$

Interchange the role of t and f:

$$x(f) = \int_{-\infty}^{\infty} X(t)\exp(j2\pi ft)dt$$

Negate f:

$$x(-f) = \int_{-\infty}^{\infty} X(t)\exp(-j2\pi ft)dt = \Im\{X(t)\}$$

Property D *(Time-shifting)*

$$\boldsymbol{x(t-t_0) \rightleftarrows X(f)\exp(-j2\pi ft_0)}$$

Proof:

$$\Im\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(t-t_0)\exp(-j2\pi ft)dt$$

$\cdots\cdots$ *letting* $\tilde{t} = t - t_0$ *and thus,* $d\tilde{t} = dt$

$$= \int_{-\infty}^{\infty} x(\tilde{t})\exp(-j2\pi f(\tilde{t}+t_0))d\tilde{t}$$

$$= \exp(-j2\pi ft_0)\int_{-\infty}^{\infty} x(\tilde{t})\exp(-j2\pi f\tilde{t})d\tilde{t}$$

$\cdots\cdots$ *letting* $\tilde{t} = t$ *and thus,* $d\tilde{t} = dt$

$$= \exp(-j2\pi ft_0)\int_{-\infty}^{\infty} x(t)\exp(-j2\pi ft)dt$$

$$= \exp(-j2\pi ft_0)X(f)$$

Property E *(Frequency-shifting or Modulation)*

$$\boldsymbol{x(t)\exp(j2\pi f_0 t) \rightleftarrows X(f-f_0)}$$

Proof:

$$\Im\left[x(t)\exp(j2\pi f_0 t)\right] = \int_{-\infty}^{\infty} x(t)\exp(j2\pi f_0 t)\exp(-j2\pi ft)dt$$

$$= \underbrace{\int_{-\infty}^{\infty} x(t)\exp(-j2\pi(f-f_0)t)dt}_{\substack{\textit{Fourier transform of } x(t) \\ \textit{with } f \textit{ replaced by } f-f_0}}$$

$$= X(f-f_0)$$

Property F *(Differentiation in time domain)*

$$\frac{d}{dt}x(t) \rightleftarrows j2\pi fX(f)$$

Proof:

$$\begin{aligned}\frac{d}{dt}x(t) &= \frac{d}{dt}\int_{-\infty}^{\infty} X(f)\exp(j2\pi ft)df \\ &= \int_{-\infty}^{\infty} X(f)\frac{d}{dt}\left[\exp(j2\pi ft)\right]df \\ &= \underbrace{\int_{-\infty}^{\infty}\left(j2\pi fX(f)\right)\exp(j2\pi ft)df}_{\substack{\text{Inverse Fourier transform} \\ \text{of } j2\pi fX(f)}} = \Im^{-1}\left\{j2\pi fX(f)\right\}\end{aligned}$$

Property G *(Integration in time domain)*

If $\int_{-\infty}^{\infty} x(t)dt = 0$, or equivalently $X(0) = 0$, then

$$\int_{-\infty}^{t} x(\tau)d\tau \rightleftarrows \frac{1}{j2\pi f}X(f)$$

Proof:

Assume: $\int_{-\infty}^{\infty} x(t)dt = 0$

$$\Im\left\{\int_{-\infty}^{t} x(\tau)d\tau\right\} = \int_{-\infty}^{\infty}\underbrace{\left(\int_{-\infty}^{t} x(\tau)d\tau\right)}_{u}\underbrace{\exp(-j2\pi ft)dt}_{dv}$$

$\cdots\cdots$ *performing integration by parts*

$$= \left[\underbrace{\left(\int_{-\infty}^{t} x(\tau)d\tau\right)}_{u}\underbrace{\frac{\exp(-j2\pi ft)}{-j2\pi f}}_{v}\right]_{t=-\infty}^{\infty} - \int_{-\infty}^{\infty}\underbrace{\frac{\exp(-j2\pi ft)}{-j2\pi f}}_{v}\underbrace{x(t)dt}_{du}$$

$$= \left[\begin{array}{l}\left(\underbrace{\left(\int_{-\infty}^{\infty} x(\tau)d\tau\right)}_{\substack{=0 \\ \text{from assumption}}} \times \underbrace{\lim_{t\to\infty}\frac{\exp(-j2\pi ft)}{-j2\pi f}}_{\substack{\text{Magnitude} \\ \text{is bounded}}}\right) - \\ \left(\underbrace{\left(\int_{-\infty}^{-\infty} x(\tau)d\tau\right)}_{\substack{=0 \\ \text{naturally}}} \times \underbrace{\lim_{t\to-\infty}\frac{\exp(-j2\pi ft)}{-j2\pi f}}_{\substack{\text{Magnitude} \\ \text{is bounded}}}\right) + \\ \frac{1}{j2\pi f}\underbrace{\int_{-\infty}^{\infty} x(t)\exp(-j2\pi ft)dt}_{X(f)}\end{array}\right] = \frac{1}{j2\pi f}X(f)$$

Property H *(Convolution in the time-domain)*

$$\underbrace{\int_{-\infty}^{\infty} \boldsymbol{x_1(\zeta)\, x_2(t-\zeta)\, d\zeta}}_{\text{Convolution in the time-domain}} \rightleftarrows \underbrace{\boldsymbol{X_1(f)\, X_2(f)}}_{\text{Multiplication in the frequency-domain}}$$

Proof:

$$\Im^{-1}\left\{\int_{-\infty}^{\infty} x_1(\zeta)\, x_2(t-\zeta)\, d\zeta\right\} = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x_1(\zeta)\, x_2(t-\zeta)\, d\zeta\right]\exp(-j2\pi ft)\, dt$$

$$\cdots\cdots \textit{ letting } \xi = t-\zeta \textit{ and thus, } d\xi = dt$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x_1(\zeta)\, x_2(\xi)\, d\zeta\right]\exp\big(-j2\pi f(\zeta+\xi)\big)\, d\xi$$

$$= \underbrace{\int_{-\infty}^{\infty} x_1(\zeta)\exp(-j2\pi f\zeta)\, d\zeta}_{\text{Fourier transform of } x_1(t)} \cdot \underbrace{\int_{-\infty}^{\infty} x_2(\xi)\exp(-j2\pi f\xi)\, d\xi}_{\text{Fourier transform of } x_2(t)}$$

$$= X_1(f)\, X_2(f)$$

Property I *(Multiplication in the time-domain)*

$$\underbrace{\boldsymbol{x_1(t)\, x_2(t)}}_{\text{Multiplication in the time-domain}} \rightleftarrows \underbrace{\int_{-\infty}^{\infty} \boldsymbol{X_1(\zeta)\, X_2(f-\zeta)\, d\zeta}}_{\text{Convolution in the frequency-domain}}$$

Proof:

$$\Im\{x_1(t)\, x_2(t)\} = \int_{-\infty}^{\infty} x_1(t)\, x_2(t)\exp(-j2\pi ft)\, dt$$

$$= \int_{-\infty}^{\infty} x_1(t) \underbrace{\int_{-\infty}^{\infty} X_2(\tilde{f})\exp(j2\pi \tilde{f}t)\, d\tilde{f}}_{x_2(t)} \exp(-j2\pi ft)\, dt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(t)\, X_2(\tilde{f})\exp\big(-j2\pi(f-\tilde{f})t\big)\, d\tilde{f}\, dt$$

$$\cdots\cdots \textit{ letting } \tilde{f} = f-\zeta \textit{ and thus, } d\tilde{f} = -d\zeta$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(t)\, X_2(f-\zeta)\exp(-j2\pi\zeta t)\, d\zeta\, dt$$

$$= \int_{-\infty}^{\infty} X_2(f-\zeta) \underbrace{\int_{-\infty}^{\infty} x_1(t)\exp(-j2\pi\zeta t)\, dt}_{\substack{\text{Fourier transform of } x_1(t) \\ \text{with } f \text{ replaced by } \zeta}} d\zeta$$

$$= \int_{-\infty}^{\infty} X_1(\zeta)\, X_2(f-\zeta)\, d\zeta$$