Chapter 5
Fourier Series

Overview

- <u>Introduction</u>

- <u>Periodic Functions</u>
  - Fourier Series
  - Euler Formulae
  - Representation by a Fourier Series
  - Periodic Functions of Period $p = 2L$
  - Fourier Sine & Cosine Series
  - Sum and Scalar Multiplications

- <u>Half-range Expansions</u>
  - Extension of $f(x)$

Introduction

Taylor Polynomials

The n-th order Taylor polynomial of f at a is

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$= f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

It provides the best polynomial approximation of degree n.

Taylor polynomials ------ Good approximation for points near $x = a$.

Taylor polynomials ------ No good for points far away from $x = a$.

Taylor polynomials ------ Involve $f'(a), f''(a), \ldots, f^{(n)}(a)$,

function must be differentiable.

Result : If $f(x)$ is differentiable, then $f(x)$ is continuous

Taylor polynomials ------ function must be continuous

Fourier Series ---- Objective

Let f be a periodic function of period 2π.

To express f in terms of

$$\begin{cases} 1, \ \sin x, \ \sin 2x, \ \cdots, \ \sin nx, \ \cdots, \\ \quad \cos x, \ \cos 2x, \ \cdots, \ \cos nx, \ \cdots, \end{cases} \tag{2}$$

that is,

$$a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$$

$$= a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{3}$$

where $a_0, \ a_1, a_2, \ \cdots, b_1, \ b_2, \ \cdots$ are real constants.

Fourier Series ---- Objective

Let f be a periodic function of period 2π.

To express f as

$$f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$$

$$= a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad (3)$$

where $a_0,\ a_1, a_2,\ \cdots, b_1,\ b_2,\ \cdots$ are real constants.

For a given function $f(x),$ to find the ***Fourier series,***

you just need to find the values of:

$$a_0, a_{1,} a_2, \cdots, b_1,\ b_2, \cdots$$

and then substitute them into the Fourier series formula.

Euler Formulae

To find the values of:

$$a_0, a_{1,} a_2, \cdots, \; b_1, \; b_2, \cdots,$$

we use Euler Formulae (7).

$$\left.\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx,\; n = 1, 2, \cdots \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx,\; n = 1, 2, \cdots
\end{aligned}\right\} \quad (7)$$

In ***Fourier series,***

we use integration instead of differentiation to find the values of:

$$a_0, a_{1,} a_2, \cdots, \; b_1, \; b_2, \cdots.$$

Fourier series - - - - - -

gives good approximation on wider intervals

often works for discontinuous functions

(Taylor polynomials fail to apply)

Uses sine & cosine functions

instead of $1,\ x,\ x^2,\ \cdots,\ x^n, \cdots$

very suitable for studying periodic functions

(e.g., alternating currents, radio signals)

Fourier Series

- Is a good tool for solving problems such as ***heat transfer problems*** & many others in Engineering & Science

- ***Applications*** – electrical engineering, vibration analysis, acoustic, optics, signal and image processing & data compression

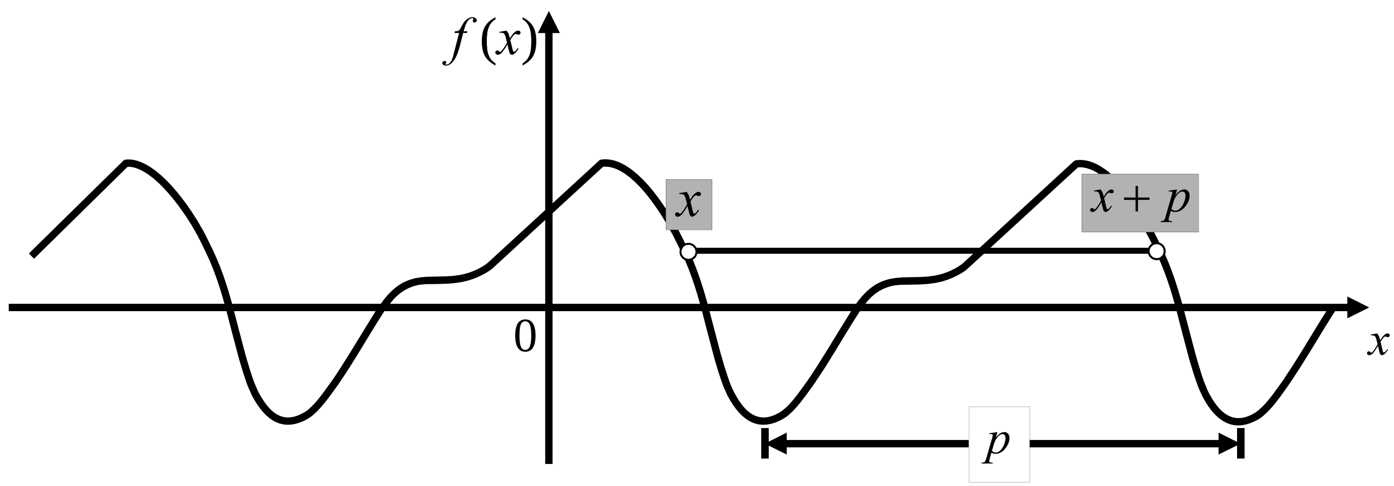
Periodic Functions

Periodic Functions

$f : R \to R$ is ***periodic***:

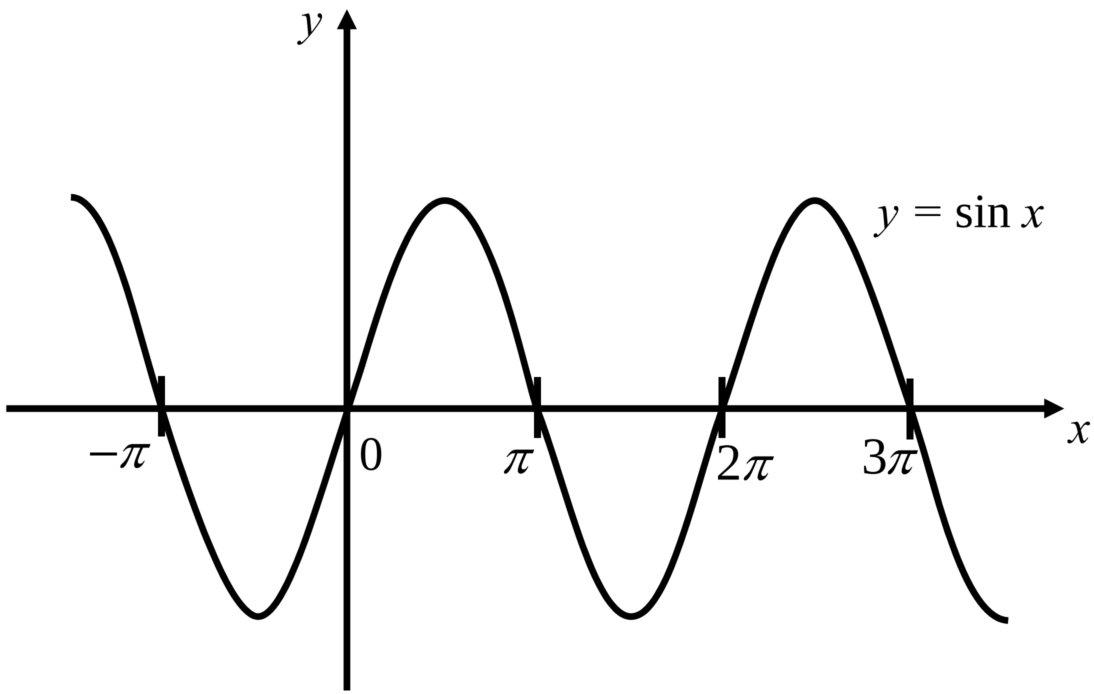
$$f(x+p) = f(x) \quad \text{for all } x \in R \tag{1}$$

where p is the period of f



Periodic Functions - Examples

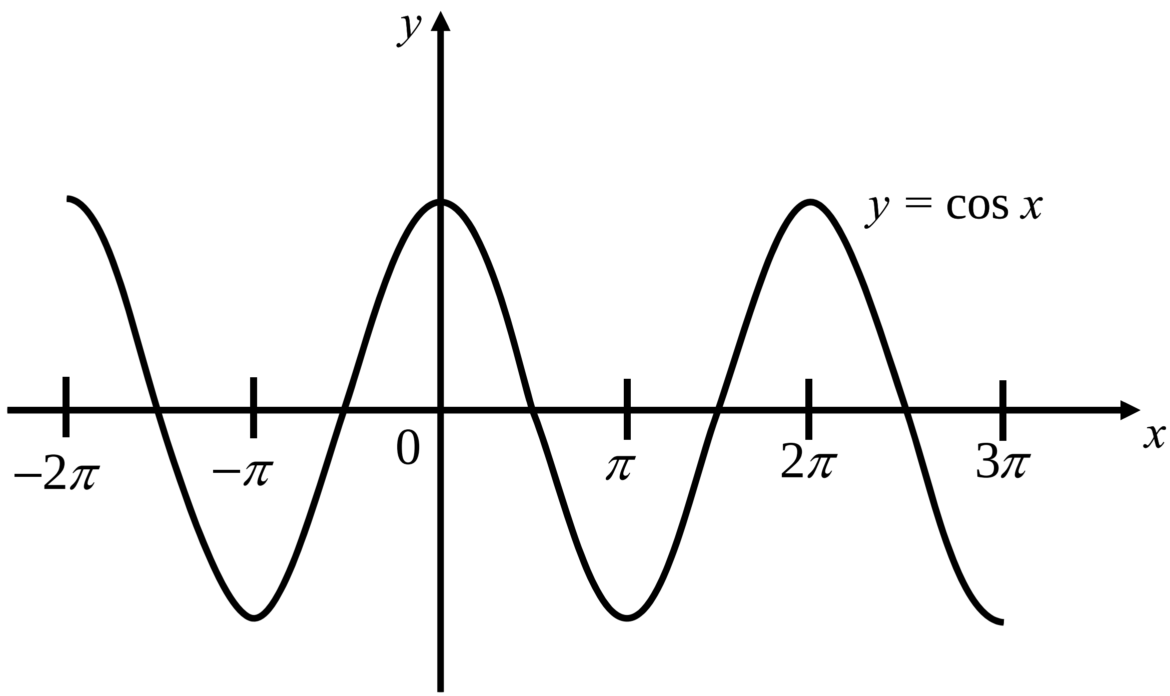
$$\sin(x + 2\pi) = \sin x$$



$$\text{Period} = 2\pi$$

Periodic Functions - Examples

$$\cos(x+2\pi)=\cos x$$



$$\text{Period}=2\pi$$

Periodic Functions - Examples

$$f(x) = k$$

where k is a constant, has any non-zero period.

Periodic Functions - Examples

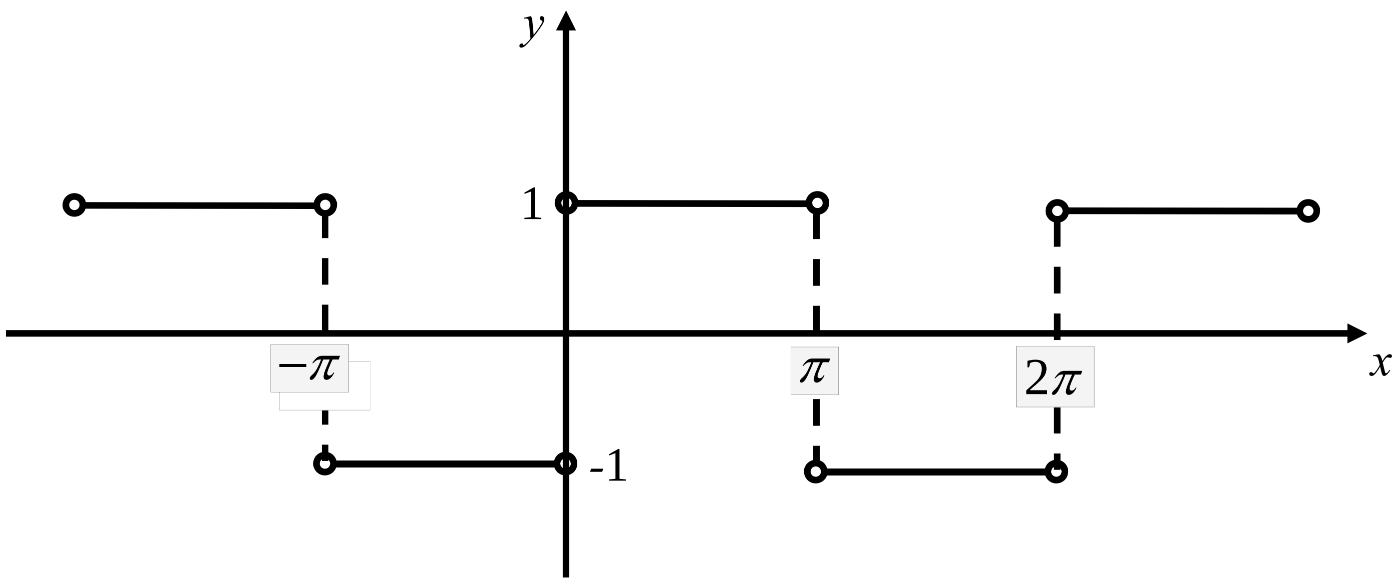
$$1,\ \sin x,\ \sin 2x,\ \cdots,\ \sin nx,\ \cdots,$$

$$\cos x,\ \cos 2x,\ \cdots,\ \cos nx,\ \cdots,$$

$$\text{all have } \boldsymbol{\mathit{period}}\ 2\pi.$$

Periodic Functions

$$f(x)=\begin{cases} 1 \;, & 0<x<\pi \\ -1 \;, & -\pi<x<0 \end{cases}$$

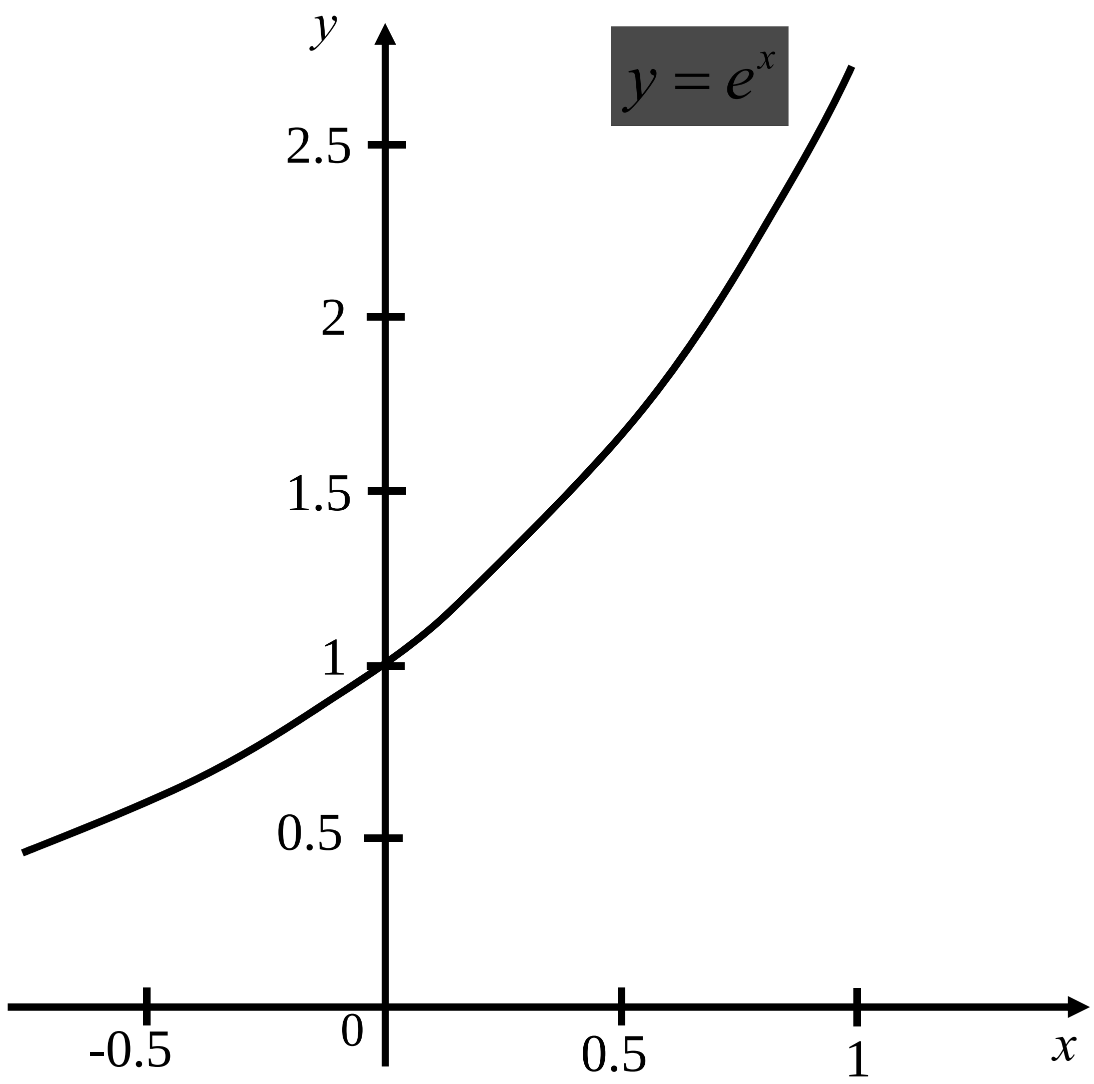


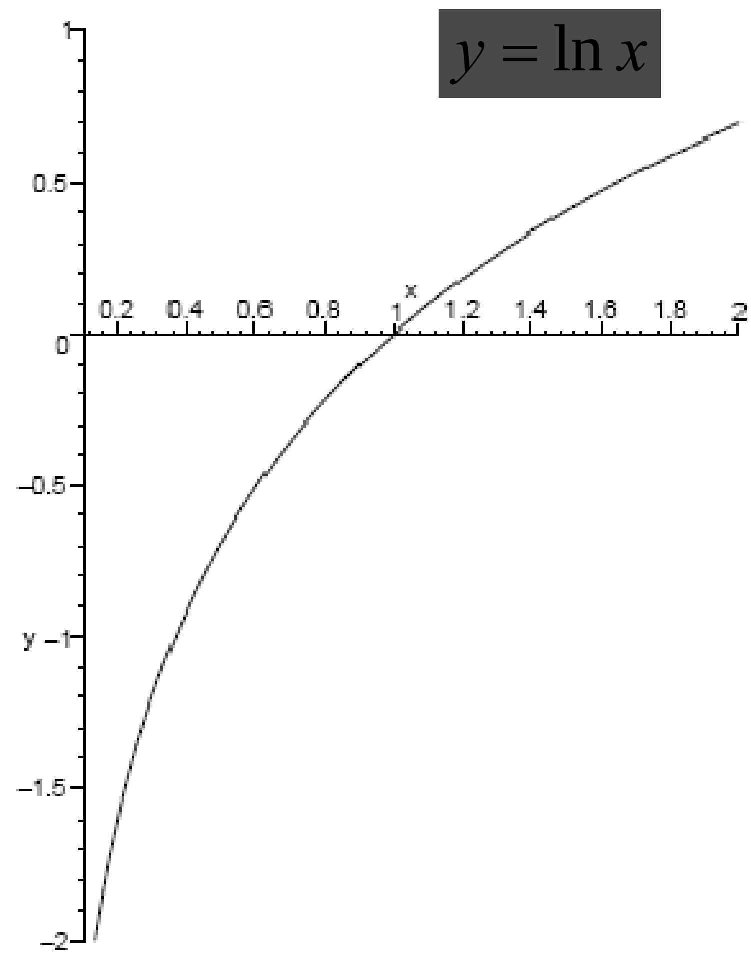
$$\text{Period} = 2\pi$$

Periodic Functions - Examples

On the other hand,

$x^n\,(n \geq 1)$, $\ln x$, e^x, etc, are ***not periodic***.





Properties of Periodic Functions

If f is of ***period*** p, then

$$f(x+np) = f(x), \quad \text{for all } x \in R,$$

that is, f is also of ***period*** $2p,\ 3p,\ \cdots$

If f and g are of period p, then for any constants a and b, the function

$$af + bg$$

is also ***periodic*** of ***period*** p.

Fourier Series --- Objective

Let f be a periodic function of period 2π.

To express f in terms of

$$\begin{cases} 1, \sin x, \sin 2x, \cdots, \sin nx, \cdots, \\ \quad \cos x, \cos 2x, \cdots, \cos nx, \cdots, \end{cases} \tag{2}$$

that is,

$$a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$$

$$= a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{3}$$

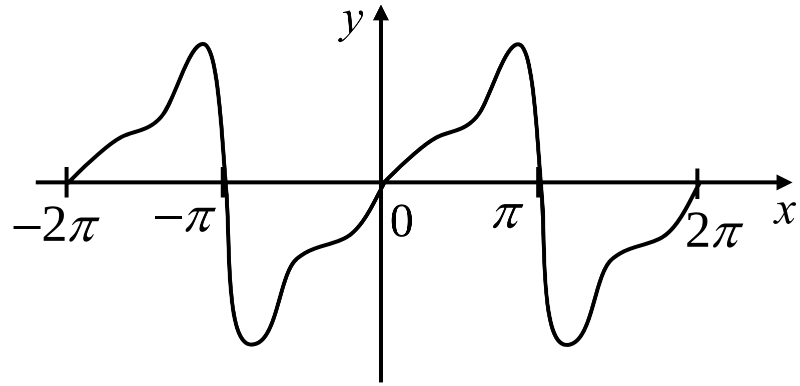
where $a_0, a_1, a_2, \cdots, b_1, b_2, \cdots$ are real constants.

The set of functions (2) is often called a ***trigonometric*** system.

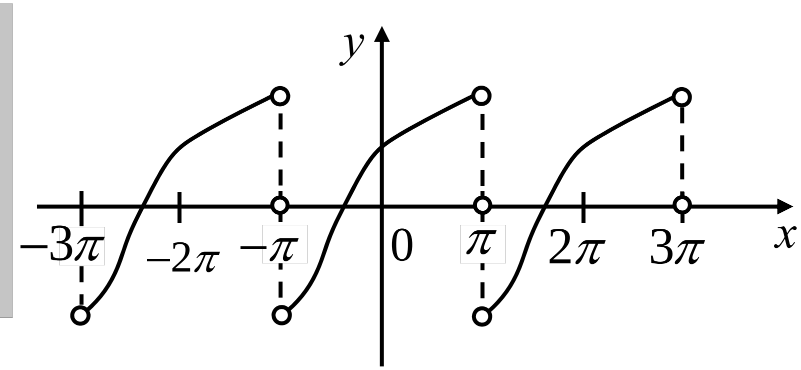
Series (3) is called a ***trigonometric series***, and a_n and b_n are called the ***coefficients*** of the series.

Fourier Series

Let f be a periodic function of ***period*** 2π (from $-\pi$ to π as shown).



Note : f need not be a continuous function.



Suppose

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad (4)$$

is the ***Fourier series*** of f.

To find the values of:

$$a_0, a_1, a_2, \cdots, b_1, \ b_2, \cdots,$$

we use Euler Formulae (7).

$$\left.\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx,\ n = 1, 2, \cdots \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx,\ n = 1, 2, \cdots \end{aligned}\right\} \quad (7)$$

Fourier Series

Let f be a periodic function of ***period*** 2π (from $-\pi$ to π as shown).

Suppose

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad (4)$$

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Question : How to derive the Euler Formulae (7) ???

Let f be a periodic function of ***period*** 2π (from $-\pi$ to π).

Suppose

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad (4)$$

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To find the values of:

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Question : How to derive the Euler Formulae (7) ???

1. To find a_0, we integrate both sides of (4) term by term from $-\pi$ to π

2. To find a_n, we multiply both sides of (4) by $\cos mx$ and integrate term by term from $-\pi$ to π :

3. To find b_n, we multiply both sides of (4) by $\sin mx$ and integrate term by term from $-\pi$ to π :

Question : How to derive the Euler Formulae (7) ???

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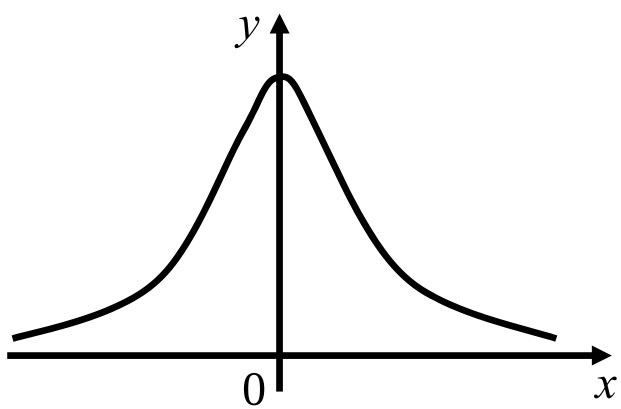
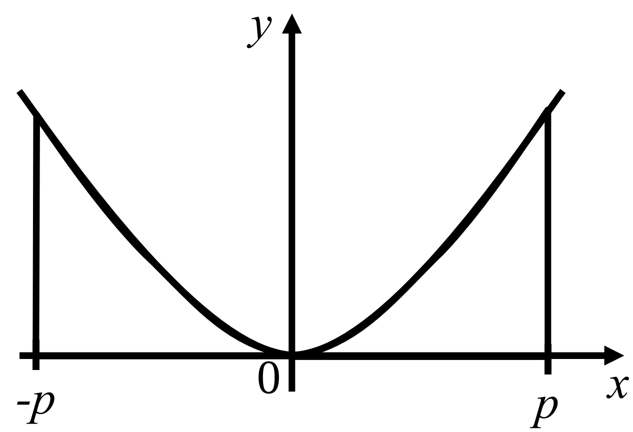
3. To find b_n, we multiply both sides of (4) by $\sin mx$ and integrate term by term from $-\pi$ to π :

Before we find a_n and b_n,
we shall firsts collect some useful results
needed in finding a_n and b_n.

Even function

$$\text{Even function}: f(-x) = f(x)$$



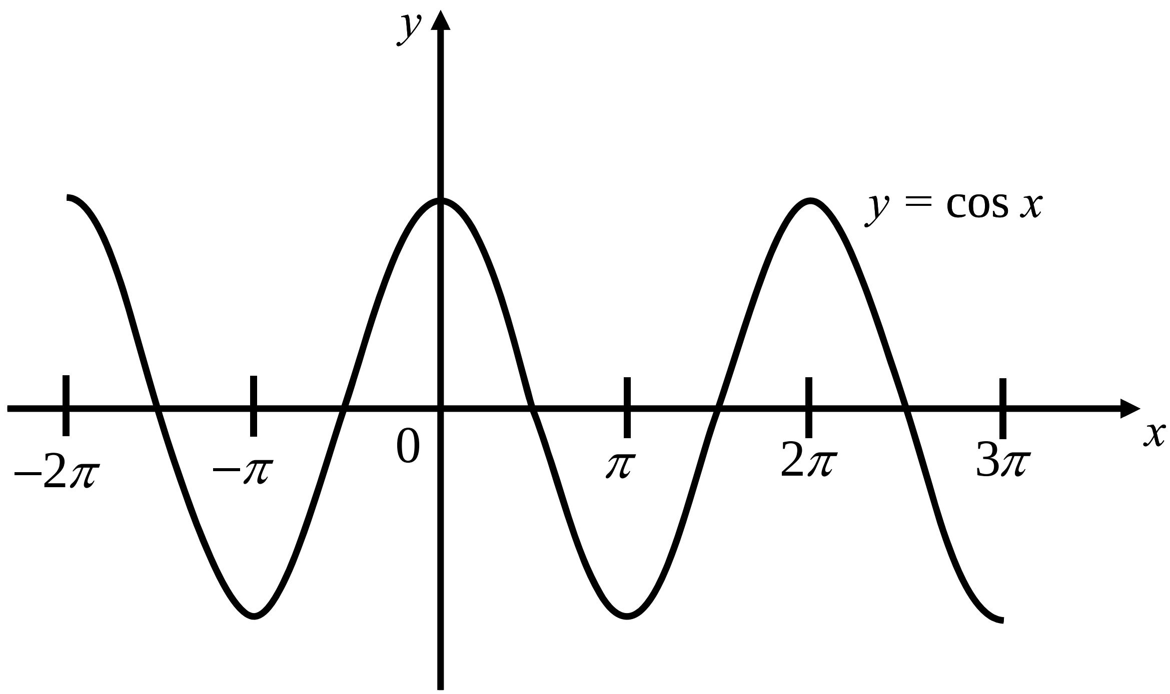


$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

$$\int_{-p}^{p} (\text{even function})\,dx = 2\int_{0}^{p} (\text{even function})\,dx$$

Even function

$$\text{Even function}: f(-x) = f(x)$$

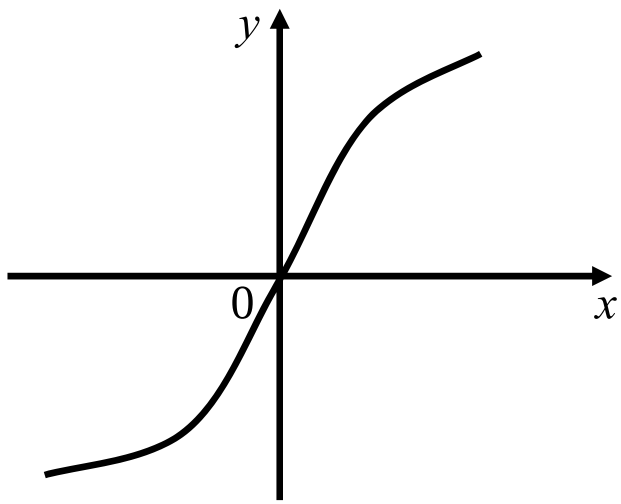


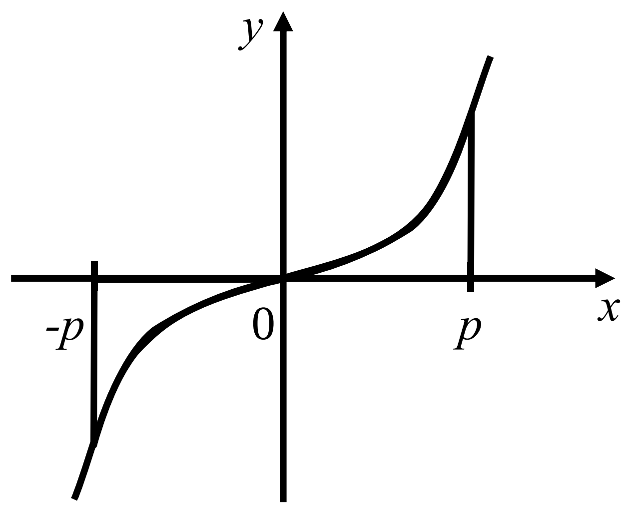
$$\text{Even function}: \cos(-\theta) = \cos\theta$$

$$\text{In general}: \quad \cos(-n\theta) = \cos n\theta$$

Odd function

$$\text{Odd function}: f(-x) = -f(x)$$



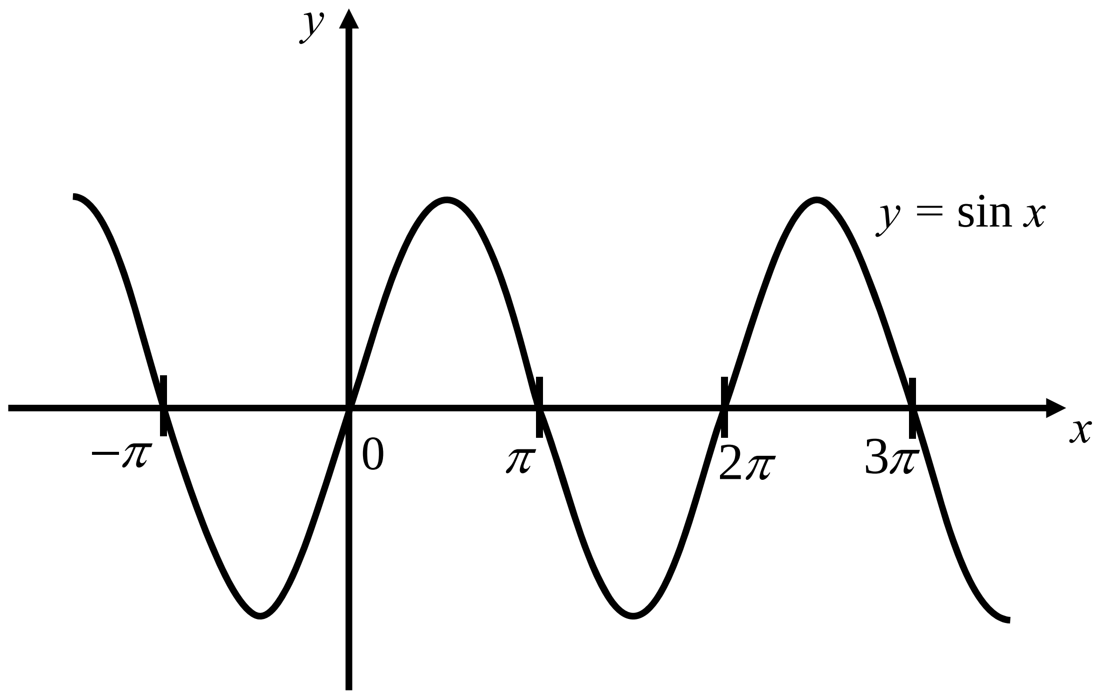


$$\int_{-a}^{a} f(x)\,dx = 0$$

$$\int_{-p}^{p} (\text{odd function})\,dx = 0$$

Odd function

$$\text{Odd function}: f(-x) = -f(x)$$



$$\text{Odd function}: \sin(-\theta) = -\sin\theta$$

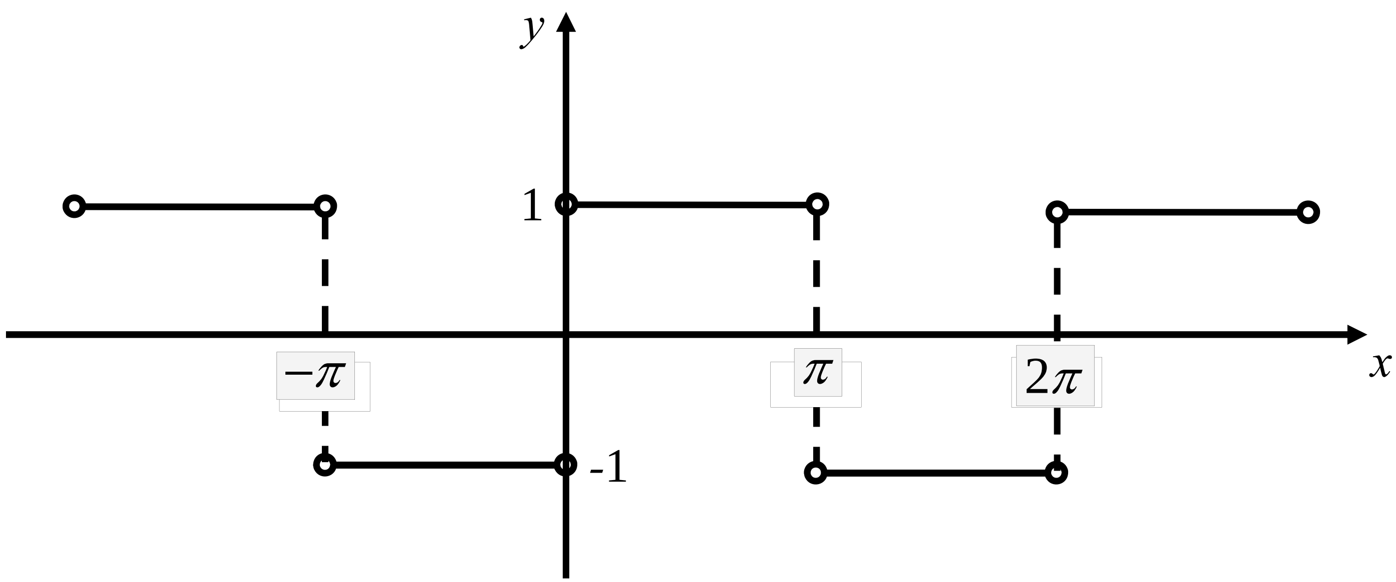
$$\text{In general}: \quad \sin(-n\theta) = -\sin n\theta$$

Pause and Think !!!

Odd function or even function ???

$$f(x) = \begin{cases} 1 \ , & 0 < x < \pi \\ -1 \ , & -\pi < x < 0 \end{cases}$$

$\text{Period} = 2\pi$



Odd function and Even function

1. $\text{(Odd function)} \times \text{(Odd function)} = \text{(Even function)}$

2. $\text{(Odd function)} \times \text{(Even function)} = \text{(Odd function)}$

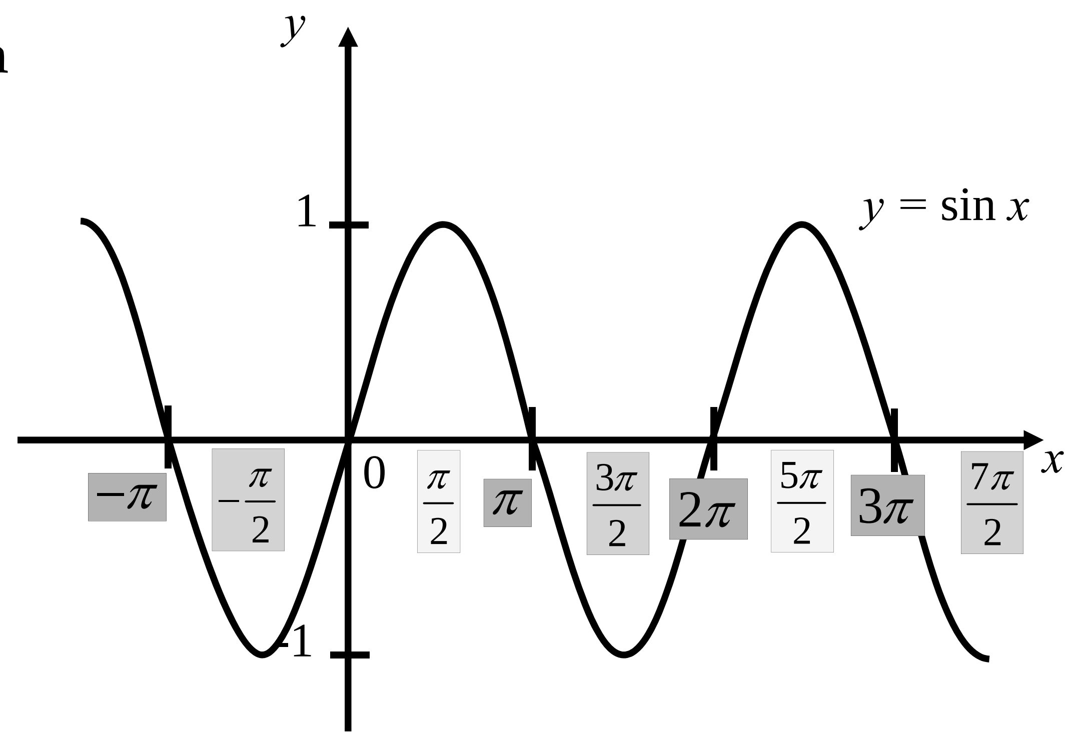
$$\int_{-p}^{p} \text{(Odd)(Even)}\, dx = \int_{-p}^{p} \text{Odd}\, dx = 0$$

3. $\text{(Even function)} \times \text{(Even function)} = \text{(Even function)}$

Some results on Sine function

Odd function: $\sin(-nx) = -\sin nx$

$$\int_{-a}^{a} \sin nx \, dx = 0$$



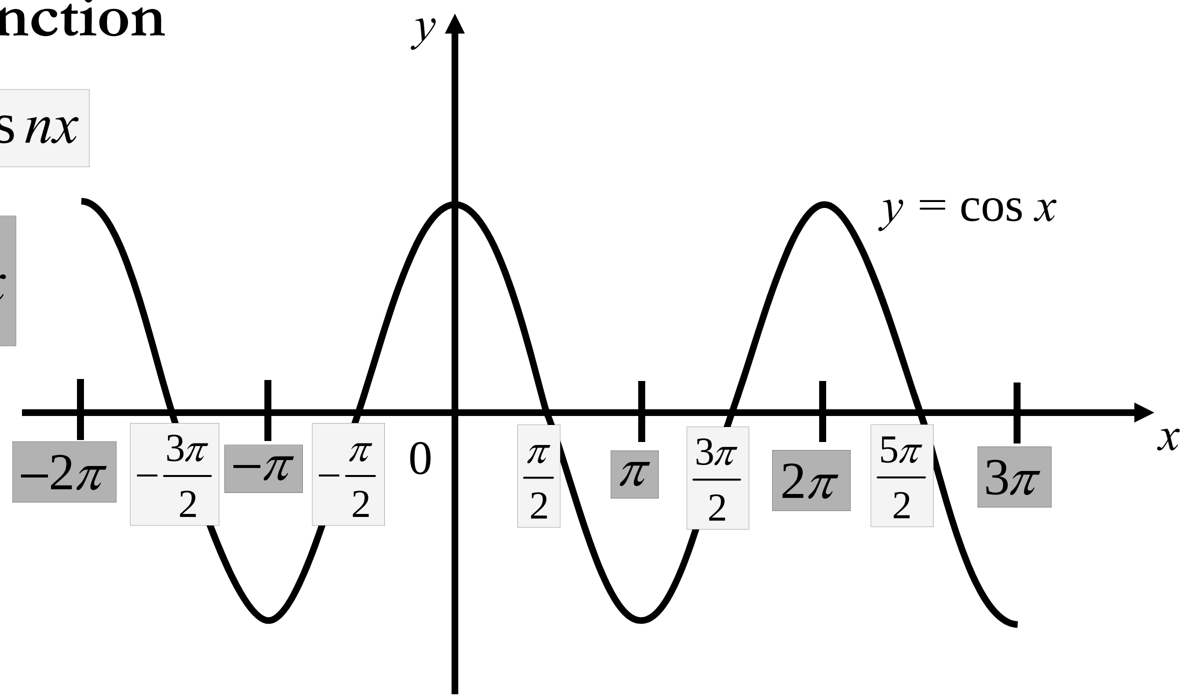
$$\sin n\pi = 0, \quad n = \cdots, -2, -1, 0, 1, 2, \cdots$$

$$\sin\frac{n\pi}{2} = \begin{cases} 1 \text{ if } n = 1, 5, 9, 13, \cdots \\ 0 \text{ if } n = 2, 4, 6, 8, \cdots \\ -1 \text{ if } n = 3, 7, 11, 15, \cdots \end{cases}$$

Some results on Cosine function

Even function: $\cos(-nx) = \cos nx$

$$\int_{-a}^{a} \cos nx \; dx = 2\int_{0}^{a} \cos nx \; dx$$



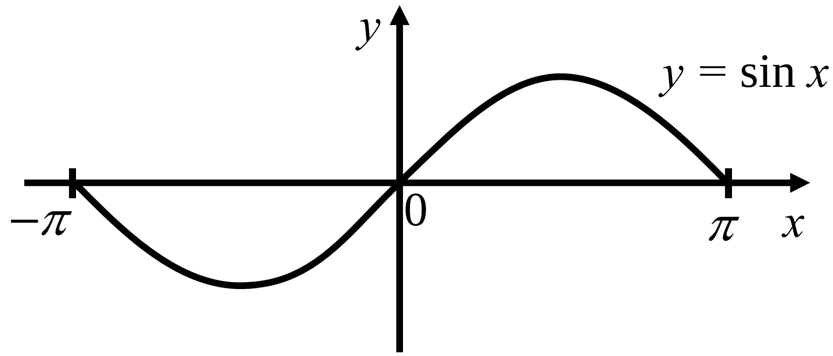
$$\cos \pi = -1, \;\; \cos 2\pi = 1, \;\; \cos 3\pi = -1, \;\; \cos 4\pi = 1, \;\; \cdots$$

Note that : $\cos n\pi = (-1)^n$

$$\cos \frac{n\pi}{2} = \begin{cases} 0 \;\text{ if }\; n = 1, 3, 5, 7, \cdots \\ 1 \;\text{ if }\; n = 0, 4, 8, 12, \cdots \\ -1 \;\text{ if }\; n = 2, 6, 10, 14, \cdots \end{cases}$$

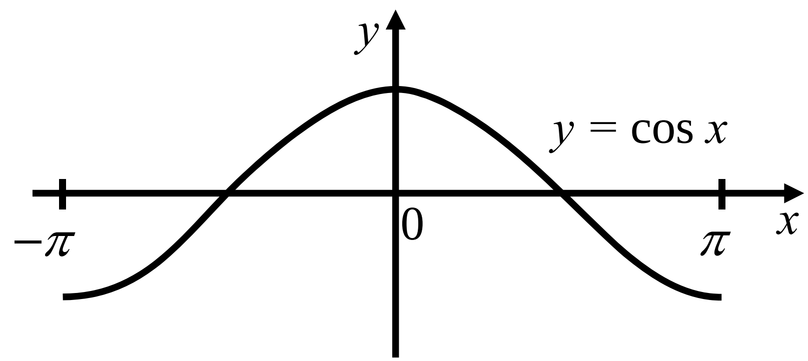
Another useful result

Odd function: $\sin(-nx) = -\sin nx$



$$\int_{-\pi}^{\pi} \sin nx \; dx = 0$$

Even function: $\cos(-nx) = \cos nx$



$$\int_{-\pi}^{\pi} \cos nx \; dx = 2\int_{0}^{\pi} \cos nx \; dx$$

Although $\cos nx$ is an even function, we also have

$$\int_{-\pi}^{\pi} \cos nx \; dx = 0$$

Useful Trigonometric Identities

$$2\cos A\cos B = \cos(A+B)+\cos(A-B)$$

$$2\sin A\sin B = \cos(A-B)-\cos(A+B)$$

Let f be a periodic function of ***period*** 2π (from $-\pi$ to π).

Suppose

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \qquad (4)$$

is the ***Fourier series*** of f.

To find the values of:

$$a_0, a_1, a_2, \cdots, b_1, b_2, \cdots,$$

we use Euler Formulae (7).

$$\left.\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx,\ n = 1, 2, \cdots \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx,\ n = 1, 2, \cdots \end{aligned}\right\} \quad (7)$$

Question : How to derive the Euler Formulae (7) ???

1. To find a_0, we integrate both sides of (4) term by term from $-\pi$ to π

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3. To find b_n, we multiply both sides of (4) by $\sin mx$ and integrate term by term from $-\pi$ to π :

Suppose

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad (4)$$

is the ***Fourier series*** of f.

$$\left.\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx,\ n = 1, 2, \cdots \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx,\ n = 1, 2, \cdots \end{aligned}\right\} \quad (7)$$

To find a_0, we integrate both sides of (4) term by term from $-\pi$ to π

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} [a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)]\,dx$$

$$\int_{-\pi}^{\pi} f(x)\,dx = a_0 \int_{-\pi}^{\pi} 1\,dx + \sum_{n=1}^{\infty} (a_n \int_{-\pi}^{\pi} \cos nx\,dx + b_n \int_{-\pi}^{\pi} \sin nx\,dx)$$

$$= 2\pi\, a_0$$

$$\int_{-\pi}^{\pi} \cos nx\,dx = 0$$

$$\int_{-\pi}^{\pi} \sin nx\,dx = 0$$

$$2\pi a_0 = \int_{-\pi}^{\pi} f(x)\,dx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$\int_{-\pi}^{\pi} 1\,dx = [x]_{-\pi}^{\pi} = 2\pi$$

Suppose

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad (4)$$

is the ***Fourier series*** of f.

$$\left.\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx,\ n=1,2,\cdots \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx,\ n=1,2,\cdots \end{aligned}\right\} \quad (7)$$

To find a_m, we multiply both sides of (4) by $\cos mx$ and integrate term by term from $-\pi$ to π:

$$\begin{aligned} \int_{-\pi}^{\pi} f(x)\cos mx\ dx = a_0 \int_{-\pi}^{\pi} \cos mx\ dx \ + \\ \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \cos mx\ dx + \\ \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \cos mx\ dx \end{aligned} \qquad (5)$$

$$\int_{-\pi}^{\pi} \cos mx\ dx = 0$$

$\sin nx \cos mx$

is an odd function

$$\int_{-\pi}^{\pi} \sin nx \cos mx\ dx = 0$$

Thus, $\displaystyle \int_{-\pi}^{\pi} f(x)\cos mx\ dx = \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \cos mx\ dx$

$$\int_{-\pi}^{\pi} f(x)\cos mx\ dx = \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \cos mx\ dx$$

Put $n = 1$ $$= a_1 \int_{-\pi}^{\pi} \cos 1x \cos mx\ dx$$

Put $n = 2$ $$+\ a_2 \int_{-\pi}^{\pi} \cos 2x \cos mx\ dx$$

Put $n = 3$ $$+\ a_3 \int_{-\pi}^{\pi} \cos 3x \cos mx\ dx$$

$$\vdots$$

$$\vdots$$

Put $n = m$ $$+\ a_m \int_{-\pi}^{\pi} \cos mx \cos mx\ dx$$

$$\vdots$$

$$\vdots$$

Need to find $\int_{-\pi}^{\pi} \cos nx \cos mx\ dx$

Need to find $\int_{-\pi}^{\pi} \cos nx \cos mx \; dx$

$$2\cos A\cos B = \cos(A+B) + \cos(A-B)$$

$$\cos nx \cos mx = \frac{1}{2}\left[\cos(m+n)x + \cos(m-n)x\right]$$

$$\int_{-\pi}^{\pi} \cos nx \cos mx \; dx = \frac{1}{2}\int_{-\pi}^{\pi} [\cos(m+n)x + \cos(m-n)x] \; dx$$

$$\int \cos nx \; dx = \frac{\sin nx}{n} + C$$

$$= \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left[\frac{\sin(m+n)\pi}{m+n} + \frac{\sin(m-n)\pi}{m-n} - \frac{\sin(m+n)(-\pi)}{m+n} - \frac{\sin(m-n)(-\pi)}{m-n}\right]$$

$$= 0$$

$$\sin n\pi = 0, \;\; n = \cdots, -2, -1, 0, 1, 2, \cdots$$

Pause and Think !!!

Is the answer complete ??

$$\int_{-\pi}^{\pi} \cos nx \cos mx \; dx = \frac{1}{2}\int_{-\pi}^{\pi} [\cos(m+n)x + \cos(m-n)x] \; dx$$

$$\int \cos nx \; dx = \frac{\sin nx}{n} + C$$

$$= \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left[\frac{\sin(m+n)\pi}{m+n} + \frac{\sin(m-n)\pi}{m-n} - \frac{\sin(m+n)(-\pi)}{m+n} - \frac{\sin(m-n)(-\pi)}{m-n}\right]$$

$$= 0$$

Pause and Think !!!

Is the answer complete ??

Case $m = n$.

$$\int \cos nx \; dx = \frac{\sin nx}{n} + C$$

$$\int_{-\pi}^{\pi} \cos nx \cos mx \; dx = \frac{1}{2}\int_{-\pi}^{\pi} [\cos(m+n)x + \cos(m-n)x] \; dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} [\cos(m+m)x + \cos(m-m)x] \; dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} [\cos 2mx + \cos 0] \; dx$$

$$\cos 0 = 1$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} 1 + \cos 2mx \; dx$$

$$= \frac{1}{2}\left[x + \frac{\sin 2mx}{2m}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left[\pi + \frac{\sin 2m\pi}{2m} - (-\pi) - \left(\frac{\sin 2m(-\pi)}{2m}\right)\right]$$

$$= \pi$$

$$\sin n\pi = 0, \quad n = \cdots, -2, -1, 0, 1, 2, \cdots$$

$$\int_{-\pi}^{\pi} \cos nx \cos mx \; dx = \frac{1}{2}\int_{-\pi}^{\pi} [\cos(m+n)x + \cos(m-n)x] \; dx$$

$$= \begin{cases} 0 \,, & \text{if } m \neq n \\ \pi \,, & \text{if } m = n \end{cases}$$

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} [\cos(m+n)x + \cos(m-n)x] \, dx = \begin{cases} 0\,, & \text{if } m \neq n \\ \pi\,, & \text{if } m = n \end{cases}$$

$$\int_{-\pi}^{\pi} f(x) \cos mx \, dx = \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \cos mx \, dx$$

Put $n=1$ $\quad = a_1 \int_{-\pi}^{\pi} \cos 1x \cos mx \, dx \quad = 0$

Put $n=2$ $\quad + a_2 \int_{-\pi}^{\pi} \cos 2x \cos mx \, dx \quad = 0$

Put $n=3$ $\quad + a_3 \int_{-\pi}^{\pi} \cos 3x \cos mx \, dx \quad = 0$

$\vdots$

Put $n=m$ $\quad + a_m \int_{-\pi}^{\pi} \cos mx \cos mx \, dx \quad = a_m \pi$

$\vdots$

$$\Bigg\} = a_m \pi$$

Thus, $\int_{-\pi}^{\pi} f(x) \cos mx \, dx = a_m \pi$

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx \, dx, \quad m \geq 1$$

Suppose

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad (4)$$

is the ***Fourier series*** of f.

$$\left.\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx,\ n = 1, 2, \cdots \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx,\ n = 1, 2, \cdots \end{aligned}\right\} \quad (7)$$

To find b_m, we multiply both sides of (4) by $\sin mx$ and integrate term by term from $-\pi$ to π:

$$\int_{-\pi}^{\pi} f(x) \sin mx\ dx = a_0 \int_{-\pi}^{\pi} \sin mx\ dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \sin mx\ dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \sin mx\ dx \qquad (6)$$

$$\int_{-\pi}^{\pi} \sin mx\ dx = 0$$

$\cos nx \sin mx$

is an odd function

$$\int_{-\pi}^{\pi} \cos nx \sin mx\ dx = 0$$

Thus, $\displaystyle\int_{-\pi}^{\pi} f(x) \sin mx\ dx = \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \sin mx\ dx$

$$\int_{-\pi}^{\pi} f(x)\sin mx\ dx = \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \sin mx\ dx$$

Put $n = 1$

$$= b_1 \int_{-\pi}^{\pi} \sin 1x \sin mx\ dx$$

Put $n = 2$

$$+ \mathrm{b}_2 \int_{-\pi}^{\pi} \sin 2x \sin mx\ dx$$

Put $n = 3$

$$+ \mathrm{b}_3 \int_{-\pi}^{\pi} \sin 3x \sin mx\ dx$$

$$\vdots$$

$$\vdots$$

Put $n = m$

$$+ \mathrm{b}_m \int_{-\pi}^{\pi} \sin mx \sin mx\ dx$$

$$\vdots$$

$$\vdots$$

Need to find $\int_{-\pi}^{\pi} \sin nx \sin mx\ dx$

Need to find $\int_{-\pi}^{\pi} \sin nx \sin mx \; dx$

$$2\sin A \sin B = \cos(A-B) - \cos(A+B)$$

$$\sin nx \sin mx = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right]$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx \; dx = \frac{1}{2}\int_{-\pi}^{\pi} [\cos(m-n)x - \cos(m+n)x] \; dx$$

$$\int \cos nx \; dx = \frac{\sin nx}{n} + C$$

$$= \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_{-\pi}^{\pi}$$

$$m \neq n.$$

$$= \frac{1}{2}\left[\frac{\sin(m-n)\pi}{m-n} - \frac{\sin(m+n)\pi}{m+n} - \frac{\sin(m-n)(-\pi)}{m-n} + \frac{\sin(m+n)(-\pi)}{m+n}\right]$$

$$= 0$$

$$\sin n\pi = 0, \;\; n = \cdots, -2, -1, 0, 1, 2, \cdots$$

Case $m = n$.

$$\int \cos nx \; dx = \frac{\sin nx}{n} + C$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx \; dx = \frac{1}{2}\int_{-\pi}^{\pi} [\cos(m-n)x - \cos(m+n)x] \; dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} [\cos(m-m)x - \cos(m+m)x] \; dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} [\cos 0 - \cos 2mx] \; dx$$

$$\cos 0 = 1$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} 1 - \cos 2mx \; dx$$

$$= \frac{1}{2}\left[x - \frac{\sin 2mx}{2m} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left[\pi - \frac{\sin 2m\pi}{2m} - (-\pi) + (\frac{\sin 2m(-\pi)}{2m}) \right]$$

$$= \pi$$

$$\sin n\pi = 0, \quad n = \cdots, -2, -1, 0, 1, 2, \cdots$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx \; dx = \frac{1}{2}\int_{-\pi}^{\pi} [\cos(m-n)x - \cos(m+n)x] \; dx$$

$$= \begin{cases} 0\,, & \text{if } m \neq n \\ \pi\,, & \text{if } m = n \end{cases}$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} [\cos(m-n)x - \cos(m+n)x] \, dx = \begin{cases} 0 \,, & \text{if } m \neq n \\ \pi \,, & \text{if } m = n \end{cases}$$

$$\int_{-\pi}^{\pi} f(x) \sin mx \, dx = \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \sin mx \, dx$$

$$\left.\begin{array}{lll}
\text{Put } n=1 & = b_1 \int_{-\pi}^{\pi} \sin 1x \sin mx \, dx & = 0 \\
\text{Put } n=2 & + \, \mathrm{b}_2 \int_{-\pi}^{\pi} \sin 2x \sin mx \, dx & = 0 \\
\text{Put } n=3 & + \, \mathrm{b}_3 \int_{-\pi}^{\pi} \sin 3x \sin mx \, dx & = 0 \\
 & \vdots & \\
\text{Put } n=m & + \, \mathrm{b}_m \int_{-\pi}^{\pi} \sin mx \sin mx \, dx & = b_m \pi \\
 & \vdots &
\end{array}\right\} = b_m \pi$$

$$\text{Thus,} \quad \int_{-\pi}^{\pi} f(x) \sin mx \, dx = b_m \pi$$

$$\mathrm{b}_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx \, dx, \quad m \geq 1$$

Euler Formulae

Let f be a periodic function of period 2π (from $-\pi$ to π) with Fourier series

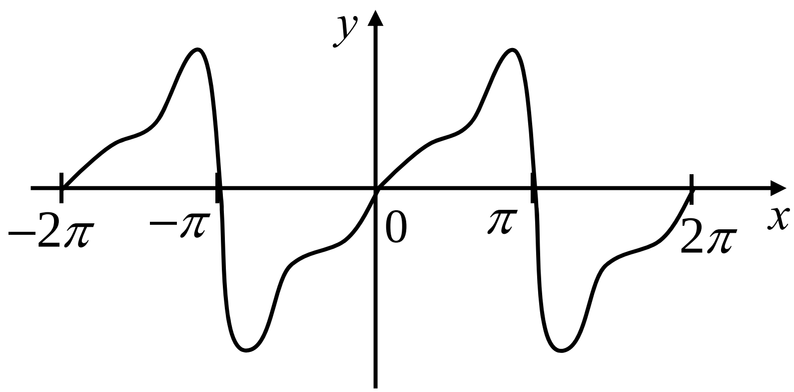
$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

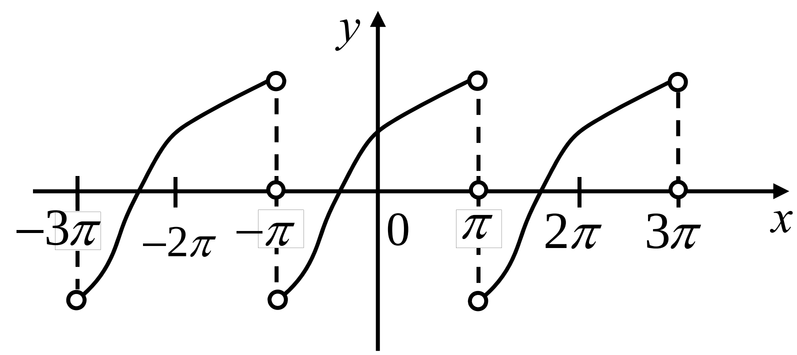
To find the values of:

$$a_0, a_1, a_2, \cdots, b_1, b_2, \cdots,$$

we use Euler Formulae (7).

$$\left.\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx,\ n = 1, 2, \cdots \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx,\ n = 1, 2, \cdots
\end{aligned}\right\} \quad (7)$$





$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots, \quad -1 < x < 1.$$

Put $x = 2$

$$\text{Left hand side } = \frac{1}{1-x} = \frac{1}{1-2} = -1$$

Right hand side

$$= 1 + x + x^2 + \cdots + x^n + \cdots$$

$$= 1 + 2 + 4 + 8 + \ldots$$

$$> 0$$

Left hand side and Right hand side are not consistent!!

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots, \quad -1 < x < 1.$$

Put $x = -3$

Left hand side $= \dfrac{1}{1-x}$

$$= \frac{1}{1-(-3)}$$

$$= \frac{1}{4}$$

Right hand side

$= 1 + x + x^2 + \cdots + x^n + \cdots$

$= 1 - 3 + 9 - 27 + \ldots$

(integer)

Left hand side and Right hand side are not consistent!!

Problem

Given a ***power series*** about $x = a$,

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

we want to know for what values of x the power series is convergent.

The number a is called the centre of the power series.

We are interested in finding out

(1) interval of convergence ($x = a$ is the centre of the interval)

(2) radius of convergence R

Representation by a Fourier Series

Let f be a periodic function of period 2π (from $-\pi$ to π).

$$\left.\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx,\ n=1,2,\cdots \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx,\ n=1,2,\cdots \end{aligned}\right\} \quad (7)$$

Suppose we find the values of $a_0, a_{1,} a_2, \cdots, b_1,\ b_2, \cdots,$ using Euler Formulae (7) and obtain the Fourier series

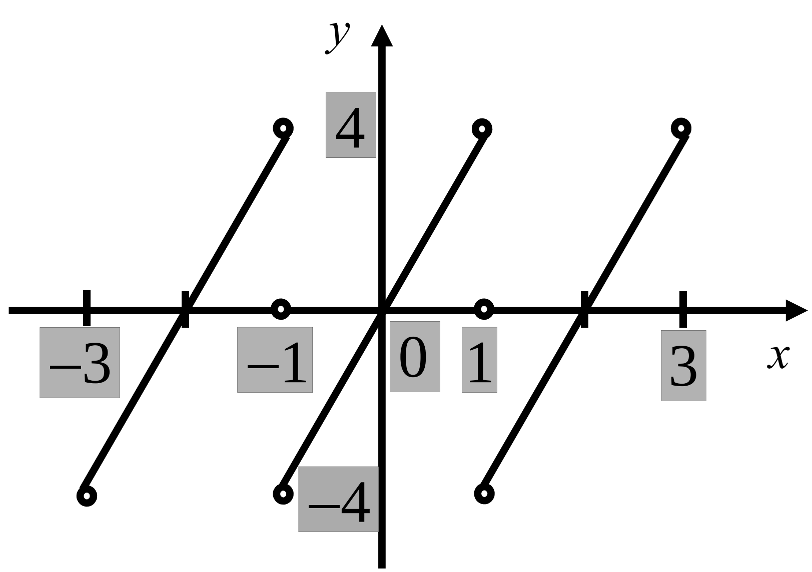
$$a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

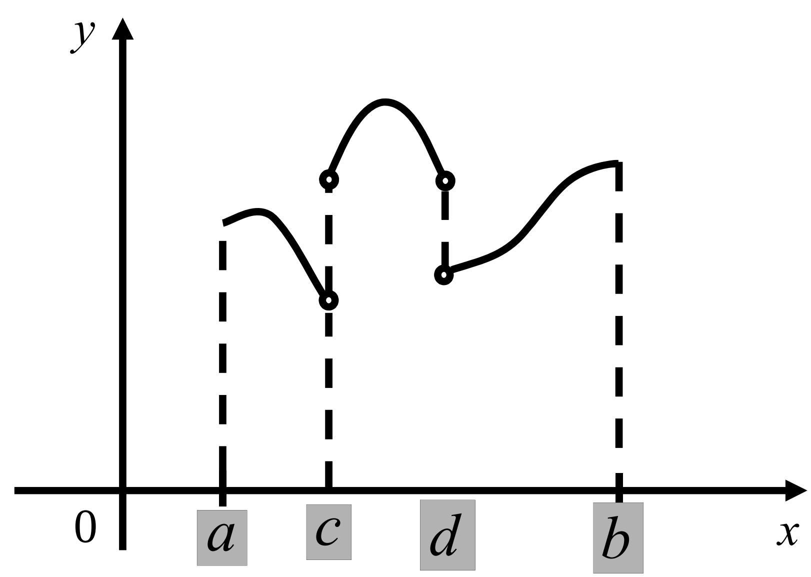
We are interested in finding out

When will the Fourier series converge back to $f(x)$??

Representation by a Fourier Series

A ***piecewise continuous*** function on $[a,b]$ is a function which is continuous except at a ***finite number of points*** where it has ***jumps*** (one-sided limits exist from each side).



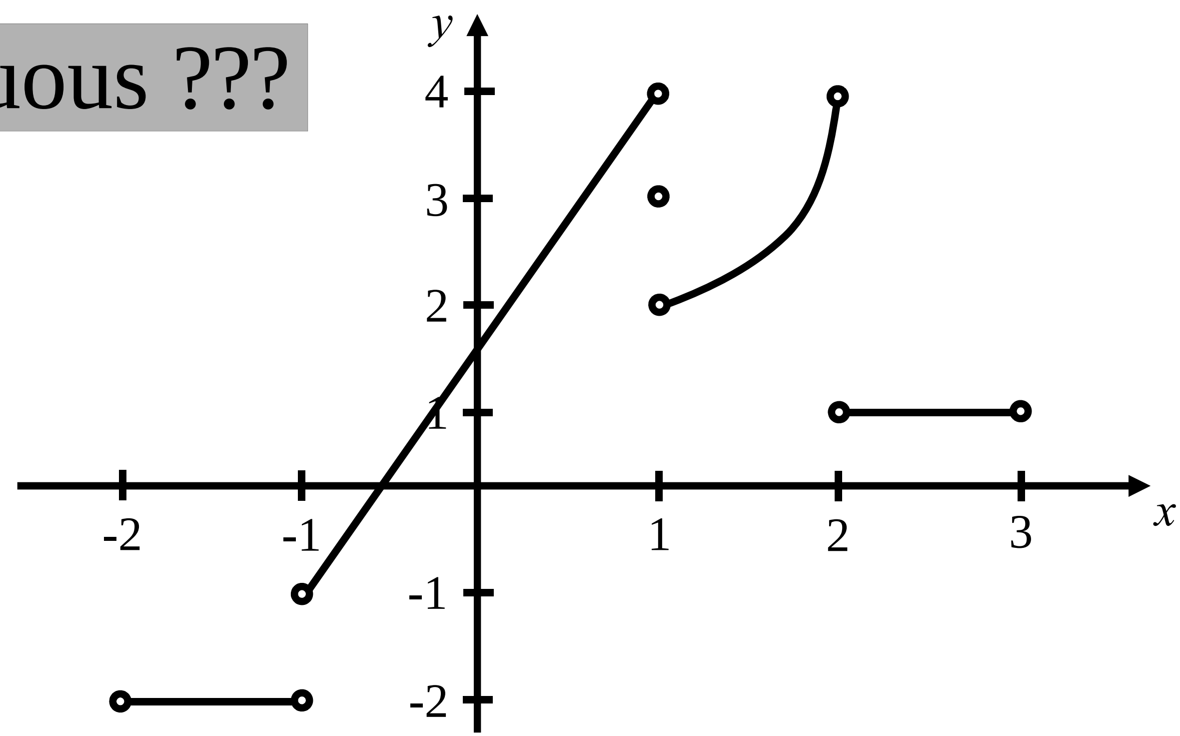


Representation by a Fourier Series

A ***piecewise continuous*** function on $[a, b]$ is a function which is continuous except at a ***finite number of points*** where it has ***jumps*** (one-sided limits exist from each side).

Pause and Think !!!

Piecewise continuous ???



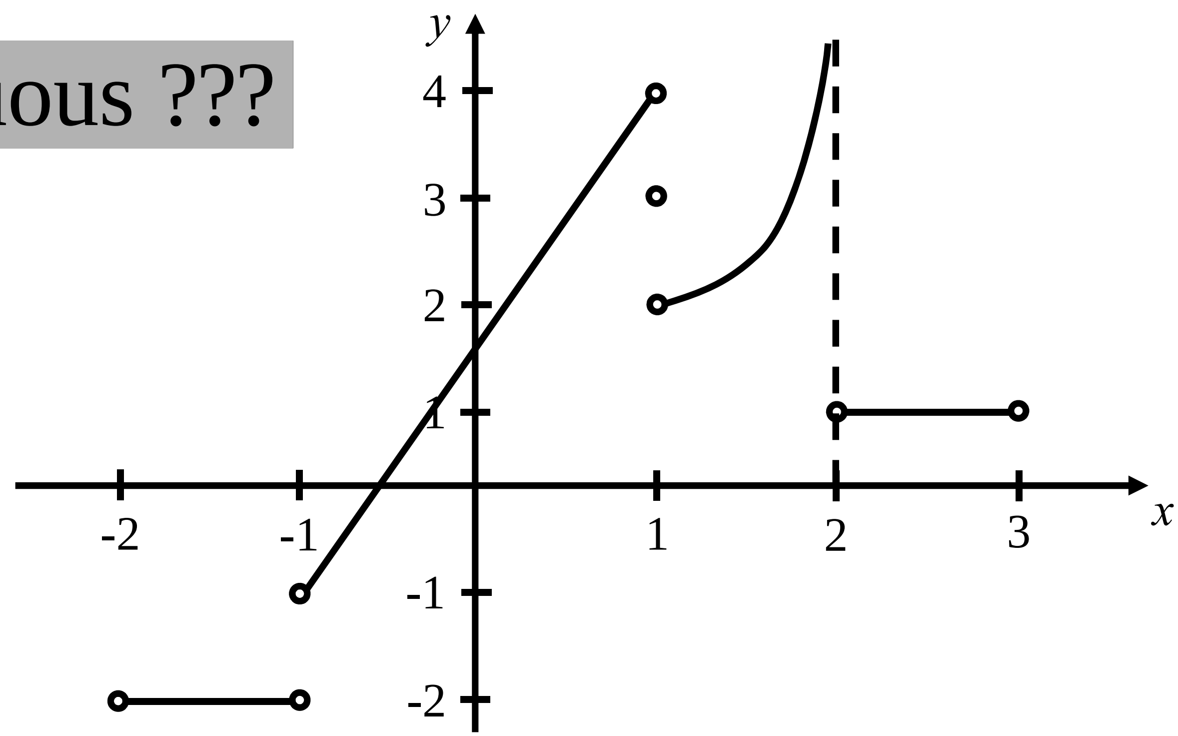
Representation by a Fourier Series

A ***piecewise continuous*** function on $[a,b]$ is a function which is continuous except at a ***finite number of points*** where it has ***jumps*** (one-sided limits exist from each side).

Pause and Think !!!

Piecewise continuous ???

Why ???



Representation by a Fourier Series

Let f be a function such that f and f' are ***piecewise continuous*** on $[-\pi, \pi]$. Then

(1) at any point x where f is ***continuous***,
$f(x)$ equals to its Fourier series.

(2) at c where f is ***discontinuous***,
the Fourier series converges to

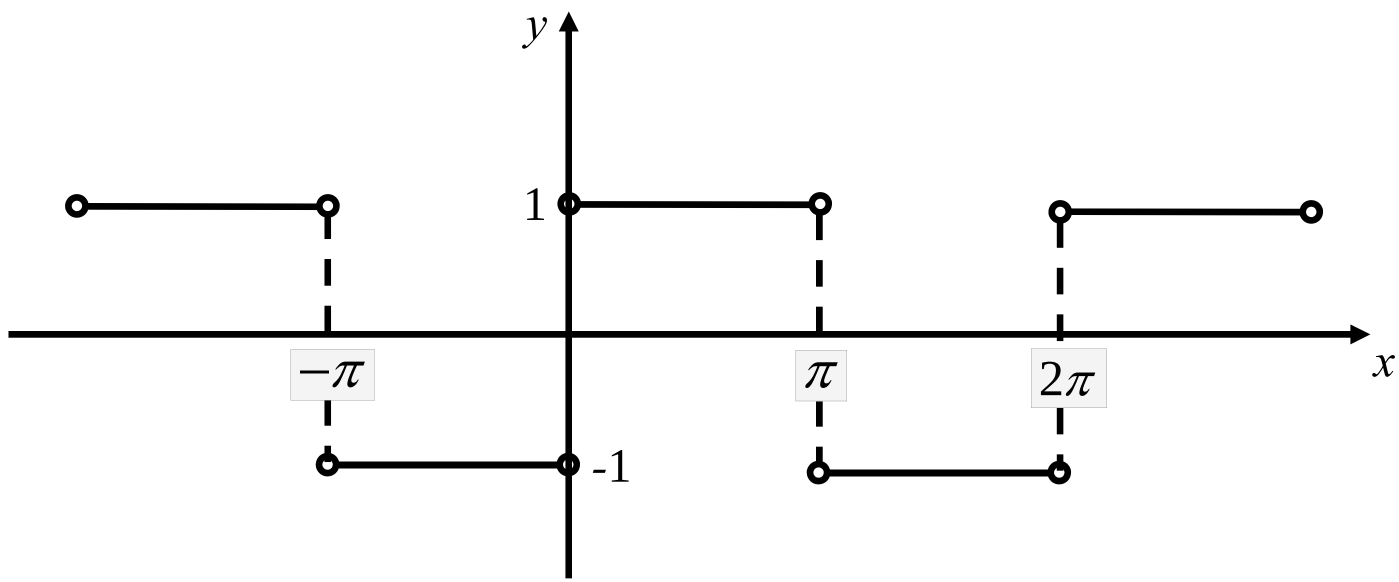
$$\frac{1}{2}[f(c^+) + f(c^-)]$$

where $f(c^+)$ is the Right-Hand limit of f at c
and $f(c^-)$ is the Left-Hand limit of f at c.

Example

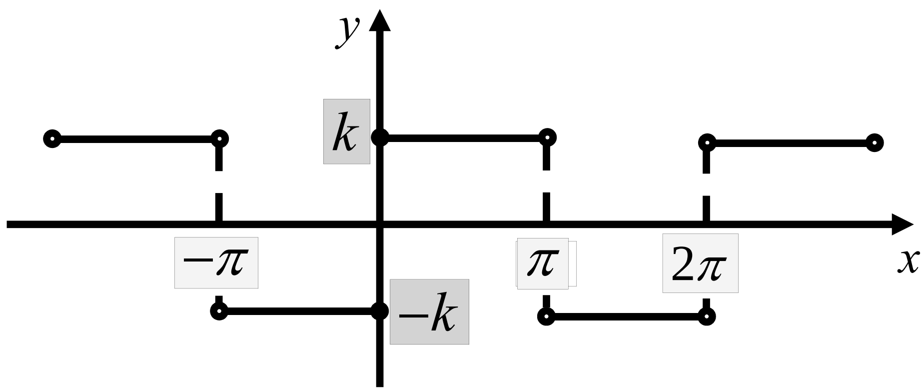
Find the Fourier series of the ***square wave*** which is a function f defined by

$$f(x)=\begin{cases} -k & , -\pi<x<0 \\ k & , 0<x<\pi \end{cases} \quad \text{and} \quad f(x)=f(x+2\pi).$$



Example

$$f(x)=\begin{cases}-k & , -\pi<x<0\\ k & , 0<x<\pi\end{cases}$$



Note that f is an odd function.

$$\int_{-\pi}^{\pi} f(x)\,dx=0$$

Thus, $a_0=\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx=0$

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos nx\,dx &= \int_{-\pi}^{\pi}(\text{odd})(\text{even})\,dx\\ &= \int_{-\pi}^{\pi}(\text{odd})\,dx\\ &=0\end{aligned}$$

Hence, $a_n=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos nx\,dx=0\ , n=1,2,\ldots$

By (7),

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx\quad (n\ge 1)$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx\quad (n\ge 1)$$

$$f(x)=\begin{cases}-k & , \; -\pi<x<0 \\ k & , \; 0<x<\pi\end{cases}$$

$$\int_{-\pi}^{\pi} f(x)\sin nx\,dx=\int_{-\pi}^{\pi}(\text{odd})(\text{odd})\,dx$$
$$=\int_{-\pi}^{\pi}(\text{even})\,dx$$
$$=2\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$=\frac{2}{\pi}\int_{0}^{\pi} k\sin nx\,dx$$

$$=\frac{2k}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi}$$
$$=\frac{2k}{n\pi}(1-\cos n\pi)$$

$$\cos n\pi=(-1)^n$$

$$=\frac{2k}{n\pi}[1-(-1)^n]$$

$$1-(-1)^n=2 \quad \text{if } n \text{ is odd}$$

$$1-(-1)^n=0 \quad \text{if } n \text{ is even}$$

$$\text{Thus,}\quad b_1=\frac{4k}{\pi},\; b_2=0,\; b_3=\frac{4k}{3\pi},\; b_4=0,\; b_5=\frac{4k}{5\pi},\; b_6=0,\; \cdots.$$

$$\text{Fourier series } = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = 0 \ , n = 1, 2, \ldots$$

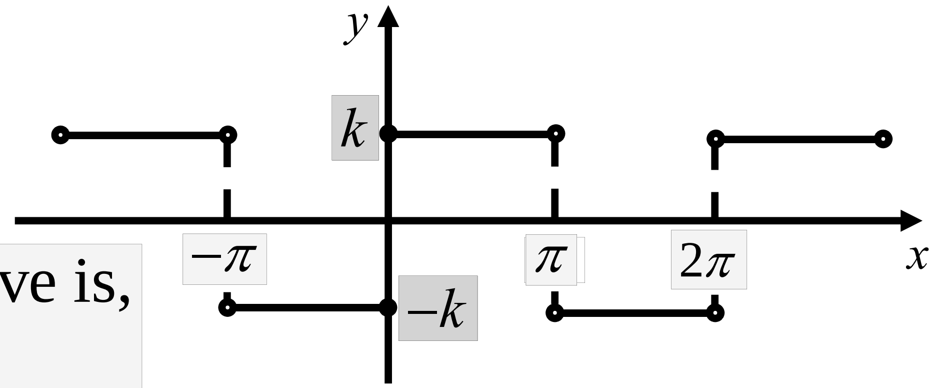
$$b_1 = \frac{4k}{\pi}, \ b_2 = 0, \ b_3 = \frac{4k}{3\pi}, \ b_4 = 0, \ b_5 = \frac{4k}{5\pi}, \ b_6 = 0, \ \cdots.$$

The Fourier series for the square wave is, therefore,

$$\frac{4k}{\pi} \sin x + \frac{4k}{3\pi} \sin 3x + \frac{4k}{5\pi} \sin 5x + \cdots$$

$$= \frac{4k}{\pi} \left(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots \right).$$

$$f(x)=\begin{cases}-k & , -\pi<x<0\\ k & , 0<x<\pi\end{cases}$$



The Fourier series for the square wave is,

$$=\frac{4k}{\pi}\left(\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+\cdots\right).$$

Points of discontinuity of f are

$x=\ldots,-3\pi,-2\pi,-\pi,0,\pi,2\pi,3\pi,\ldots$

At $x=0$, the sum of the series is equal to 0.

$\sin 0=0$

Right-Hand limit $f(0^+)=k$

Left-Hand limit $f(0^-)=-k$

$$\frac{1}{2}[f(0^+)+f(0^-)]=\frac{1}{2}[k+(-k)]$$
$$=0$$

(2) at c where f is ***discontinuous***,

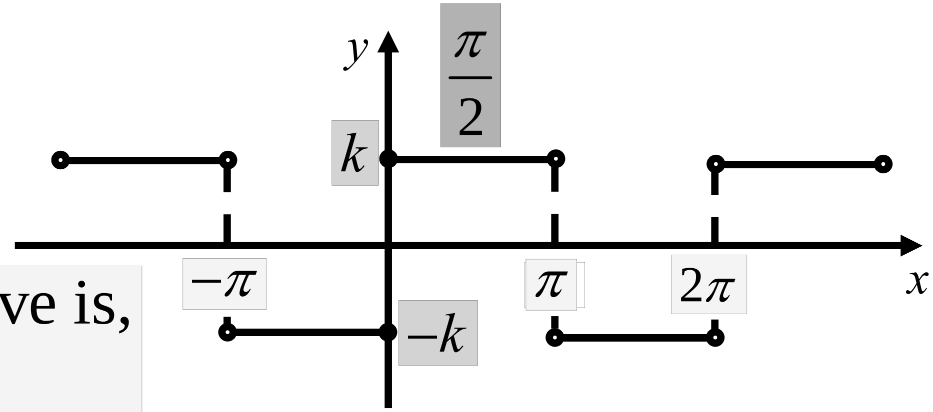
the Fourier series converges to

$$\frac{1}{2}[f(c^+)+f(c^-)]$$

where $f(c^+)$ is the Right-Hand of f at c

and $f(c^-)$ is the Left-Hand limits of f at c.

$$f(x)=\begin{cases}-k & , -\pi<x<0\\ k & , 0<x<\pi\end{cases}$$



The Fourier series for the square wave is,

$$=\frac{4k}{\pi}(\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+\cdots).$$

At $x=\frac{\pi}{2}$, f is continuous.

(1) at any point x where f is ***continuous***, $f(x)$ equals to its Fourier series.

Thus, $f(\frac{\pi}{2})=\frac{4k}{\pi}(\sin\frac{\pi}{2}+\frac{1}{3}\sin\frac{3\pi}{2}+\frac{1}{5}\sin\frac{5\pi}{2}+\cdots).$

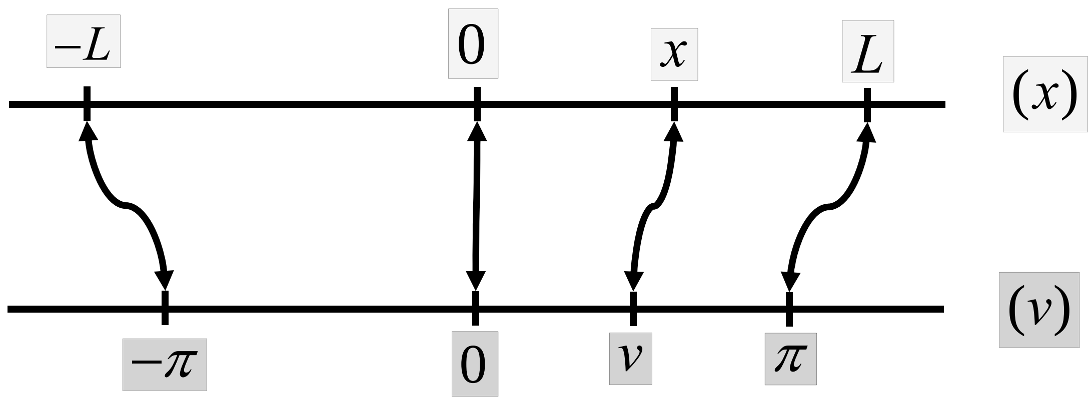
$$k=\frac{4k}{\pi}(1-\frac{1}{3}+\frac{1}{5}-\cdots)$$

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\cdots$$

$$\pi=4(1-\frac{1}{3}+\frac{1}{5}-\cdots)$$

Periodic Functions of Period $p = 2L$

Let f be a periodic function of period $p = 2L$ (from $-L$ to L).



Let $\dfrac{x}{L} = \dfrac{v}{\pi}$ and $g(v) = f(x)$

$$v = \frac{\pi x}{L}$$

When $x = -L,\ v = \dfrac{-\pi L}{L} = -\pi.$

When $x = L,\ v = \dfrac{\pi L}{L} = \pi.$

$g(v)$ is a function with period 2π

Periodic Functions of Period $p = 2L$

Let f be a periodic function of period $p = 2L$ (from $-L$ to L).

Let $\frac{x}{L} = \frac{v}{\pi}$ and $g(v) = f(x)$

$$v = \frac{\pi x}{L}$$

$$x = \frac{vL}{\pi}$$

To show $g(v)$ is a function with period 2π

$$g(v) = f(x) = f(\frac{vL}{\pi})$$

$$\begin{aligned} g(v+2\pi) &= f(\frac{(v+2\pi)L}{\pi}) \\ &= f(\frac{vL}{\pi} + 2L) \\ &= f(\frac{vL}{\pi}) \\ &= g(v) \end{aligned}$$

$$f(x+2L) = f(x)$$

Periodic Functions of Period $p = 2L$

Let f be a periodic function of period $p = 2L$ (from $-L$ to L).

$$v = \frac{\pi x}{L}$$

Then $g(v)$ is a periodic function of ***period*** 2π.

Thus,

$$g(v) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nv + b_n \sin nv)$$

with $\quad a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(v)\, dv \qquad$ Recall: $v = \frac{\pi x}{L}$

$$= \frac{1}{2\pi} \int_{-L}^{L} g(v) \frac{\pi}{L}\, dx$$

$$= \frac{1}{2L} \int_{-L}^{L} f(x)\, dx$$

Periodic Functions of Period $p = 2L$

Let f be a periodic function of period $p = 2L$ (from $-L$ to L).

$$v = \frac{\pi x}{L}$$

Then $g(v)$ is a periodic function of ***period*** 2π.

For $n = 1,\ 2,\ 3,\ \cdots$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(v)\cos nv\ dv = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\ dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(v)\sin nv\ dv = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\ dx$$

Periodic Functions of Period $p = 2L$

Let f be a periodic function of period $p = 2L$ (from $-L$ to L).

$$v = \frac{\pi x}{L}$$

Then $g(v)$ is a periodic function of ***period*** 2π.

$$g(v) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nv + b_n \sin nv)$$

Since $g(v) = f(x)$, we get

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right)$$

with a_0, a_n and b_n as given earlier.

Periodic Functions of Period $p = 2L$

Let f be a periodic function of period $p = 2L$ (from $-L$ to L).

Then

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$$

with

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\ dx$$

and for $n = 1,\ 2,\ 3,\ \cdots$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\ dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\ dx$$

Suppose

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

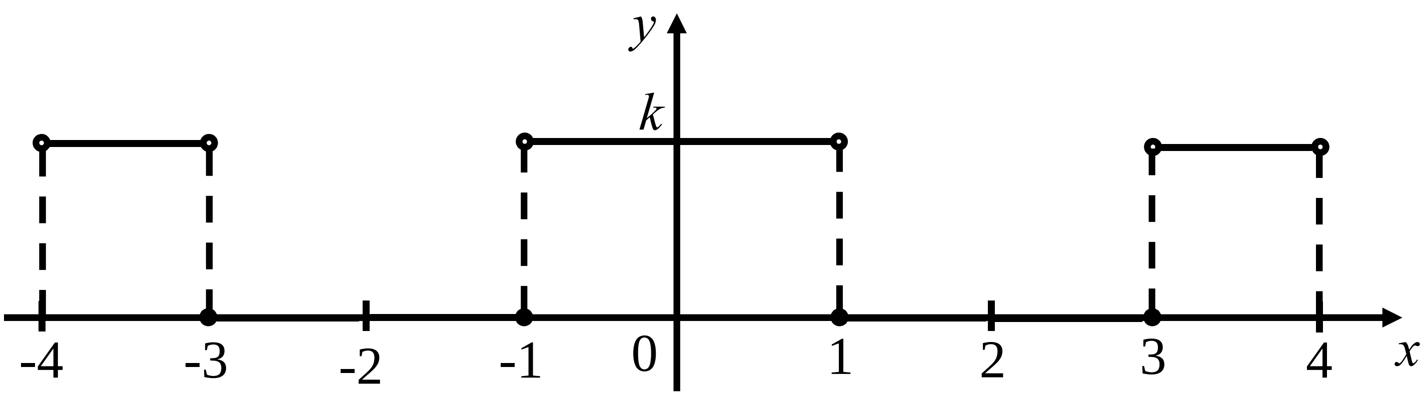
is the ***Fourier series*** of f.

$$\left.\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx,\ n = 1,\ 2,\ \cdots \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx,\ n = 1,\ 2,\ \cdots \end{aligned}\right\}\ (7)$$

Example

Let f be a periodic square wave of period $p = 2L = 4$ defined as follows:

$$f(x) = \begin{cases} 0, & \text{if } -2 < x < -1 \\ k, & \text{if } -1 < x < 1 \\ 0, & \text{if } \quad 1 < x < 2 \end{cases}$$



Pause and Think !!!

Odd function or even function ???

$$f(x)=\begin{cases}0, & \text{if } -2<x<-1\\ k, & \text{if } -1<x<1\\ 0, & \text{if } \quad 1<x<2\end{cases}$$

Period $p=2L=4$ $\quad L=2$

$f(x)$ is an even function

For $n=1,\ 2,\ 3,\ \cdots$

$$\begin{aligned}b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\ dx\\ &= \frac{1}{L}\int_{-L}^{L}(\text{even})(\text{odd})\ dx\\ &= \frac{1}{L}\int_{-L}^{L}(\text{odd})\ dx\\ &= 0\end{aligned}$$

$$a_0=\frac{1}{2L}\int_{-L}^{L} f(x)\ dx$$

and for $n=1,\ 2,\ 3,\ \cdots$

$$a_n=\frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\ dx$$

$$b_n=\frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\ dx$$

Thus, just need to find $a_0, a_1, a_2, \ldots$

$$f(x)=\begin{cases}0, & \text{if } -2<x<-1\\ k, & \text{if } -1<x<1\\ 0, & \text{if } \quad 1<x<2\end{cases}$$

Period $p=2L=4$

$L=2$

$$a_0=\frac{1}{2L}\int_{-L}^{L} f(x)\ dx$$

and for $n=1,\ 2,\ 3,\ \cdots$

$$a_n=\frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\ dx$$

$$b_n=\frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\ dx$$

$$\begin{aligned}a_0&=\frac{1}{2(2)}\int_{-2}^{2} f(x)\ dx\\ &=\frac{1}{4}\left(\int_{-2}^{-1} 0\ dx+\int_{-1}^{1} k\ dx+\int_{1}^{2} 0\ dx\ \right)\\ &=\frac{k}{2}\end{aligned}$$

$$f(x)=\begin{cases}0, & \text{if } -2<x<-1\\ k, & \text{if } -1<x<1\\ 0, & \text{if } \quad 1<x<2\end{cases}$$

Period $p = 2L = 4$

$L = 2$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx$$
$$= \frac{1}{2}\int_{-1}^{1} k\cos\frac{n\pi x}{2}\,dx$$
$$= \frac{k}{n\pi}\left[\sin\frac{n\pi x}{2}\right]_{-1}^{1}$$

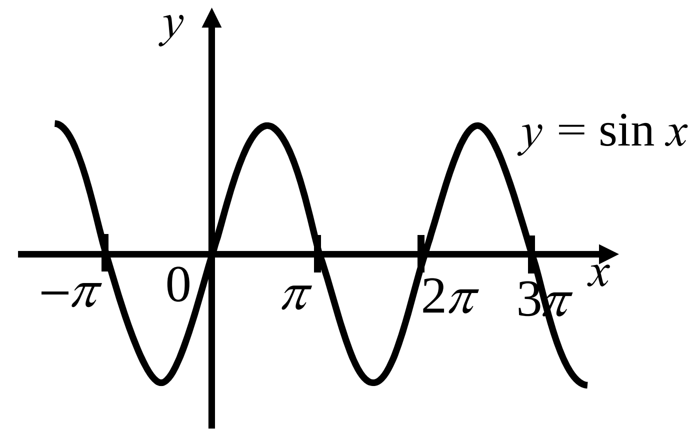
$$= \frac{2k}{n\pi}\sin\frac{n\pi}{2}$$

$$= \begin{cases} 0\,, & \text{if } n \text{ is even}\\ \dfrac{2k}{n\pi}, & \text{if } n = 1,\ 5,\ 9,\ \cdots\\ -\dfrac{2k}{n\pi}, & \text{if } n = 3,\ 7,\ 11,\ \cdots\end{cases}$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\ dx$$

and for $n = 1,\ 2,\ 3,\ \cdots$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\ dx$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\ dx$$



$$f(x) = \begin{cases} 0, & \text{if } -2 < x < -1 \\ k, & \text{if } -1 < x < 1 \\ 0, & \text{if } \quad 1 < x < 2 \end{cases}$$

Period $p = 2L = 4$ $\quad L = 2$

$$a_0 = \frac{k}{2}$$

For $n = 1, 2, 3, \ldots$

$$a_n = \begin{cases} 0\ , & \text{if } n \text{ is even} \\ \dfrac{2k}{n\pi}, & \text{if } n = 1,\ 5,\ 9,\ \cdots \\ -\dfrac{2k}{n\pi}, & \text{if } n = 3,\ 7,\ 11,\ \cdots \end{cases}$$

For $n = 1,\ 2,\ 3,\ \cdots$

$$b_n = 0$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right)$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x$$

$a_0 = \dfrac{k}{2}$ $\quad a_1 = \dfrac{2k}{\pi}$ $\quad a_2 = 0$

$a_3 = -\dfrac{2k}{3\pi}$ $\quad a_4 = 0$

$a_5 = \dfrac{2k}{5\pi}$ $\quad a_6 = 0$

$a_7 = -\dfrac{2k}{7\pi}$ $\quad a_8 = 0$

$$f(x)=\begin{cases}0, & \text{if } -2<x<-1\\ k, & \text{if } -1<x<1\\ 0, & \text{if } \quad 1<x<2\end{cases}$$

Period $p=2L=4$ $\quad L=2$

$a_0=\dfrac{k}{2}$

$a_1=\dfrac{2k}{\pi}$ $\quad a_2=0$ $\quad a_3=-\dfrac{2k}{3\pi}$ $\quad a_4=0$

$a_5=\dfrac{2k}{5\pi}$ $\quad a_6=0$ $\quad a_7=-\dfrac{2k}{7\pi}$ $\quad a_8=0$

Fourier series of $f(x)$ is

$$a_0+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi}{L}x$$

$$=a_0+\ a_1\cos\frac{\pi}{L}x+a_2\cos\frac{2\pi}{L}x+a_3\cos\frac{3\pi}{L}x+a_4\cos\frac{4\pi}{L}x+a_5\cos\frac{5\pi}{L}x+\ldots$$

$$=\frac{k}{2}+\ \frac{2k}{\pi}\cos\frac{\pi}{2}x+0\cos\frac{2\pi}{2}x-\frac{2k}{3\pi}\cos\frac{3\pi}{2}x+0\cos\frac{4\pi}{2}x+\frac{2k}{5\pi}\cos\frac{5\pi}{2}x+\ldots$$

$$=\frac{k}{2}+\frac{2k}{\pi}\left(\cos\frac{\pi}{2}x-\frac{1}{3}\cos\frac{3\pi}{2}x+\frac{1}{5}\cos\frac{5\pi}{2}x-\cdots\right).$$

Fourier Cosine & Fourier Sine Series

Let f be a periodic function of period $p = 2L$ (from $-L$ to L).

If f is ***even***, then $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L},$

(Fourier cosine series)

with
$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx,$$
$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx,\ n = 1, 2, \cdots$$

If f is ***odd***, then $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L},$

(Fourier Sine series)

with
$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx,\ n = 1, 2, \cdots$$

Sum and Scalar multiplication

The Fourier coefficients of $f_1 + f_2$ are the sums of corresponding Fourier cofficients of f_1 and f_2.

The Fourier coefficients of cf, where c is a constant, are c times the corresponding Fourier cofficients of f.

If the Fourier coefficients of f are :

$$a_0, a_1, a_2, ..., b_1, b_2, ...,$$

then the Fourier coefficients of cf are :

$$ca_0, ca_1, ca_2, ..., cb_1, cb_2, ...,$$

Pause and Think !!!

Question:

What is the Fourier Series of a constant function ??

Suppose $f(x)=1$, what is the Fourier Series of f ??

Suppose $f(x)=1$, what is the Fourier Series of f ??

Find the Fourier Series of $f(x)=1, \ -L \le x \le L.$

$f(x)=1$ is an even function, therefore $b_n=0.$

$$a_0=\frac{1}{L}\int_0^L 1\,dx=1$$

$$\begin{aligned}
a_n &= \frac{2}{L}\int_0^L 1\cdot\cos\frac{n\pi x}{L}\,dx \\
&= \frac{2}{L}\left(\frac{L}{n\pi}\right)\left[\sin\frac{n\pi x}{L}\right]_0^L \\
&= \frac{2}{n\pi}\left(\sin\frac{n\pi L}{L}-\sin 0\right) \\
&= 0, \quad \text{for } n=1,\ 2,\ \cdots
\end{aligned}$$

$\sin n\pi = 0$

Thus, Fourier series of $f(x)=1$ is

$$a_0+\sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{L}=1$$

Answer:

The Fourier Series of $f(x)=1$ is just 1.

Pause and Think !!!

Question:

What is the Fourier Series of a constant function ??

Suppose $f(x) = 1,$ what is the Fourier Series of f ??

Suppose $f(x) = c,$ what is the Fourier Series of f ??

Pause and Think !!!

Question:

Suppose $f(x) = c,$ what is the Fourier Series of f ??

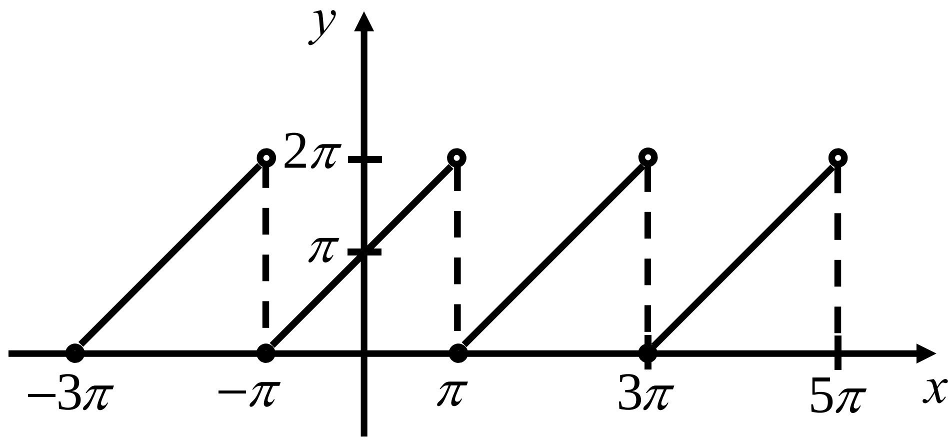
The Fourier coefficients of cf, where c is a constant, are c times the corresponding Fourier cofficients of f.

The Fourier Series of $f(x) = 1$ is just 1.

Example

Find the Fourier series of the saw tooth function f defined by

$f(x) = x + \pi, \quad -\pi < x < \pi \quad \& \quad f(x) = f(x + 2\pi).$



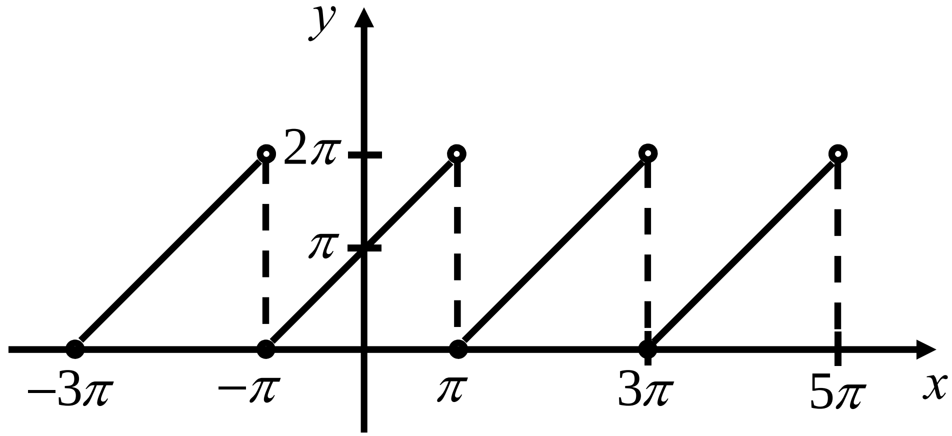
Pause and Think !!!

Odd function or even function ???

Example

Find the Fourier series of the saw tooth function f defined by

$f(x) = x + \pi, \quad -\pi < x < \pi \quad \& \quad f(x) = f(x + 2\pi).$



Note that $f = f_1 + f_2$, where $f_1 = x$ and $f_2 = \pi$.

$f_1(x) = x$ is an odd function

Just need to find $b_1, b_2, \ldots$

$f_2(x) = \pi$ is a constant function

Fourier Series of $f_2(x) = \pi$ is just π.

$f_1(x) = x$, is an odd function on $(-\pi, \pi)$.

Thus $a_n = 0$ for all n, and

$$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx$$

$$= \frac{2}{\pi}\left\{\left[x\left(\frac{-\cos nx}{n}\right)\right]_0^{\pi} - \int_0^{\pi} \frac{-\cos nx}{n} \, dx\right\}$$

$$= \frac{2}{\pi}\left\{\frac{-(-1)^n \pi}{n} - \left[\frac{-\sin nx}{n^2}\right]_0^{\pi}\right\} = \frac{2(-1)^{n+1}}{n}$$

$\cos n\pi = (-1)^n$

$\sin n\pi = 0$

Fourier series of $f_1(x)$ is $\displaystyle\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin \frac{n\pi x}{\pi}$

Fourier Series of $f_2(x) = \pi$ is just π.

The Fourier coefficients of $f_1 + f_2$ are the sums of corresponding Fourier cofficients of f_1 and f_2.

Find the Fourier series of the saw tooth function f defined by

$$f(x) = x + \pi, \quad -\pi < x < \pi \quad \& \quad f(x) = f(x + 2\pi).$$

Note that $f = f_1 + f_2$, where $f_1 = x$ and $f_2 = \pi$.

Fourier series of $f_1(x)$ is $\displaystyle\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin \frac{n\pi x}{\pi}$

Fourier Series of $f_2(x) = \pi$ is just π.

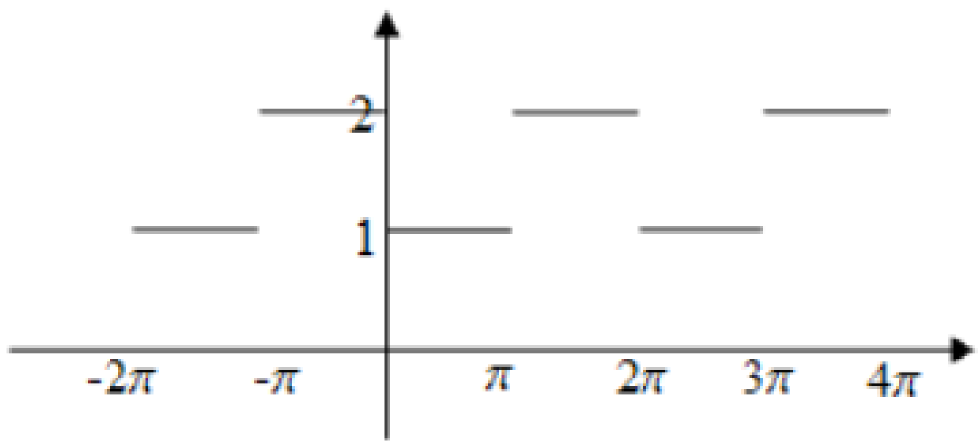
Thus, the Fourier series of f is

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin \frac{n\pi x}{\pi} + \pi = \pi + 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$

Pause and Think !!!

Tutorial 4 Question 2

2. Find the Fourier series that represent the following graph:

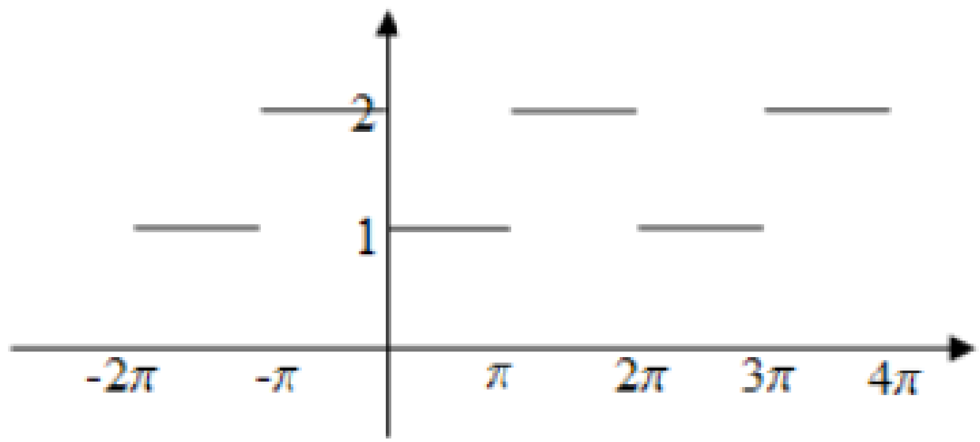


Odd function or even function ???

Pause and Think !!!

Tutorial 4 Question 2

2. Find the Fourier series that represent the following graph:



Is it possible to write $f(x) = f_1(x) + f_2(x)$,
where $f_1(x)$ is an odd function and $f_2(x)$ is a constant function ??

Is it possible to write $f(x) = f_1(x) + f_2(x)$,
where $f_1(x)$ is an even function and $f_2(x)$ is a constant function ??

Pause and Think !!!

Nov 2005 Exam Question 4(b)

Let $f(x) = 2x + 1$ for all $x \in (-\pi, \pi)$ and $f(x) = f(x + 2\pi)$. Let

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be the Fourier Series which represents $f(x)$. Find the value of $a_0 + a_5 + b_5$.

What should you find ??

Let $f(x) = 2x + 1$ for all $x \in (-\pi, \pi)$ and $f(x) = f(x + 2\pi)$. Let

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be the Fourier Series which represents $f(x)$. Find the value of $a_0 + a_5 + b_5$.

Note : $f(x) = 2 f_1(x) + f_2(x)$, where $f_1(x) = x$ and $f_2(x) = 1$.

$f_1(x) = x$, is an odd function on $(-\pi, \pi)$.

Thus $a_n = 0$ for all n, and

$$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx$$

$$= \frac{2}{\pi}\left\{\left[x\left(\frac{-\cos nx}{n}\right)\right]_0^{\pi} - \int_0^{\pi} \frac{-\cos nx}{n}\, dx\right\}$$

$$= \frac{2}{\pi}\left\{\frac{-(-1)^n \pi}{n} - \left[\frac{-\sin nx}{n^2}\right]_0^{\pi}\right\} = \frac{2(-1)^{n+1}}{n}$$

Fourier series of $f(x)$ is

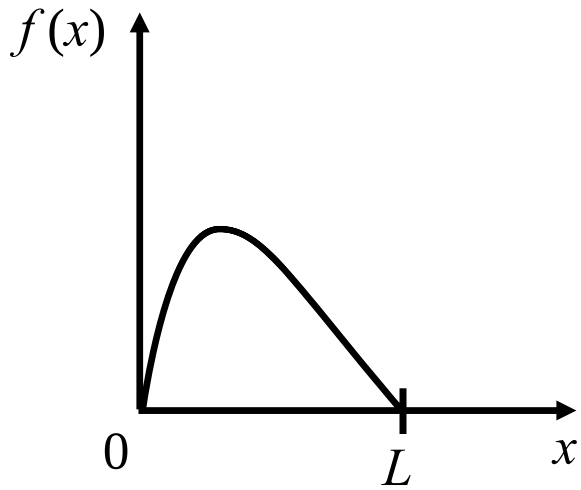
$$1 + 2\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin \frac{n\pi x}{\pi}$$

Fourier series of $f_1(x)$ is $\displaystyle\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin \frac{n\pi x}{\pi}$

Half-range Expansions

Half-range Expansions

Assume that f is defined on $[0, L]$ as shown below & we wish to expand it in a Fourier series.



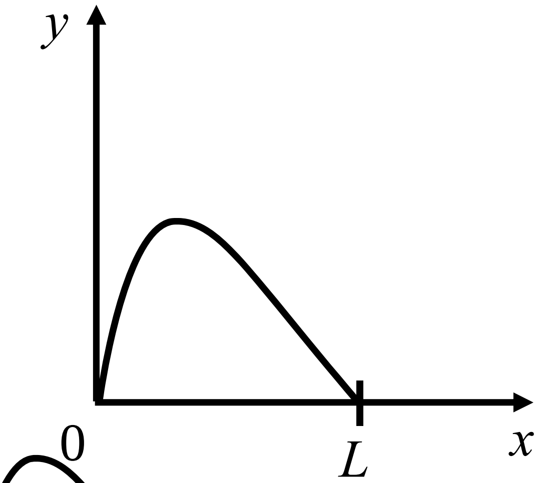
We extend the definition of f to $[-L, L]$ so that

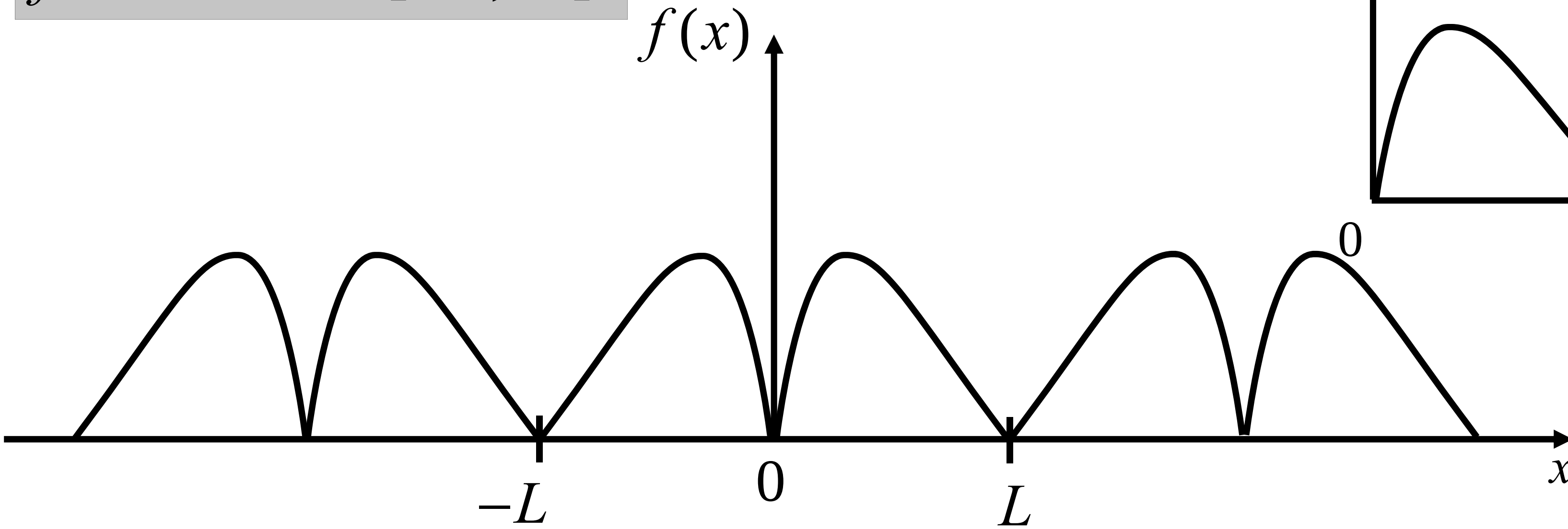
(1) f is ***even*** on $[-L, L]$ or

(2) f is ***odd*** on $[-L, L]$.

Half-range Expansions

f is ***even*** on $[-L, L]$





We can then represent it by a Fourier cosine series.

If f is ***even***, then $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$,

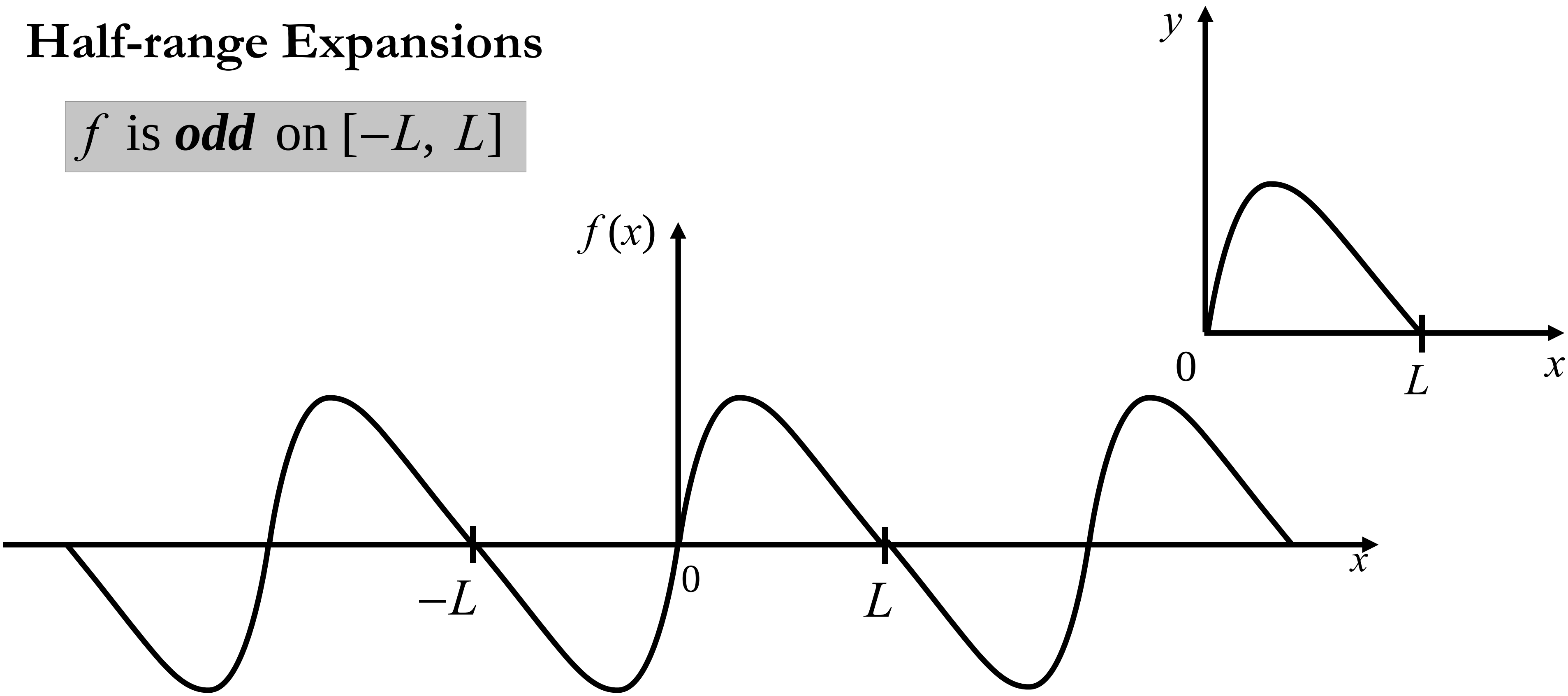
with $\quad a_0 = \frac{1}{L}\int_0^L f(x)\,dx,$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx,\ n = 1, 2, \cdots$$

(Fourier cosine series)

Half-range Expansions

f is ***odd*** on $[-L, L]$



We can then represent it by a Fourier sine series.

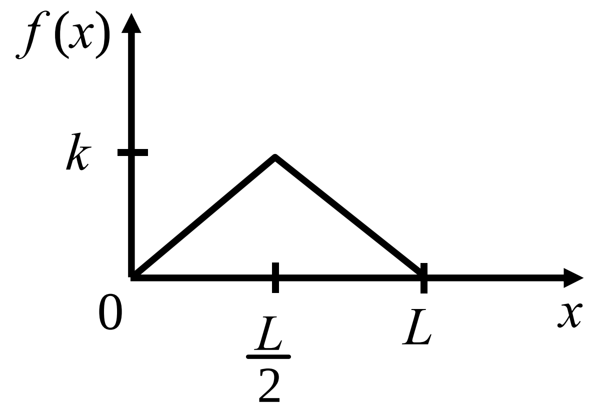
If f is ***odd***, then $f(x) = \sum_{n=1}^{\infty} b_n \sin \dfrac{n\pi x}{L},$ (Fourier Sine series)

with $$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\, dx,\ n = 1, 2, \cdots$$

Half-range Expansions - Example

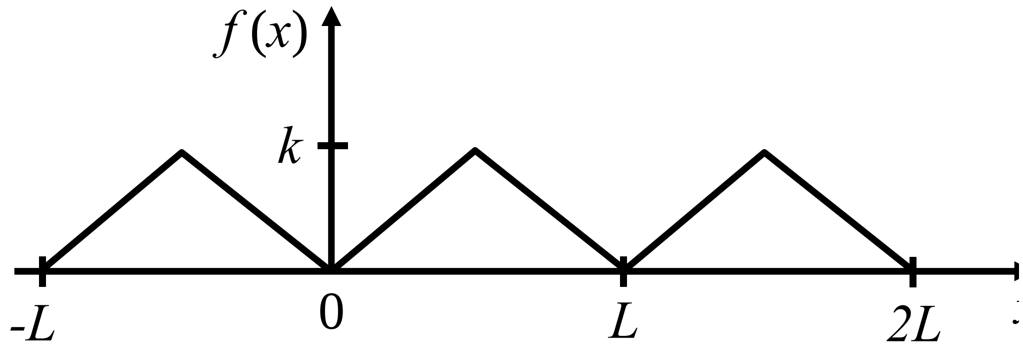
Find the two half range expansions for the 'Triangle' function f defined by

$$f(x) = \begin{cases} \dfrac{2}{L}kx & , \quad 0 < x < \dfrac{L}{2} \\ \dfrac{2k}{L}(L-x) & , \quad \dfrac{L}{2} < x < L \end{cases}$$



Half-range Expansions - Example

$$f(x) = \begin{cases} \dfrac{2}{L} kx & , \quad 0 < x < \dfrac{L}{2} \\ \dfrac{2k}{L}(L-x) & , \quad \dfrac{L}{2} < x < L \end{cases}$$



Cosine Half-Range Extension

For the cosine half range expansion, we have

$$a_0 = \frac{1}{L}\left[\int_0^{L/2} \frac{2k}{L} x \, dx + \int_{L/2}^{L} \frac{2k}{L}(L-x)\, dx\right]$$
$$= \frac{k}{2}$$

$$a_n = \frac{2}{L}\left[\int_0^{L/2} \frac{2k}{L} x \cos\frac{n\pi x}{L}\, dx + \int_{L/2}^{L} \frac{2k}{L}(L-x)\cos\frac{n\pi x}{L}\, dx\right]$$
$$= \frac{4k}{L^2}\left[\int_0^{L/2} x\cos\frac{n\pi x}{L}\, dx + \int_{L/2}^{L}(L-x)\cos\frac{n\pi x}{L}\, dx\right]$$

Half-range Expansions - Example

$$a_n = \frac{4k}{L^2}\left[\int_0^{L/2} x\cos\frac{n\pi x}{L}\,dx + \int_{L/2}^{L}(L-x)\cos\frac{n\pi x}{L}\,dx\right]$$

Integrating by parts, the first integral becomes

$$\int_0^{L/2} x\cos\frac{n\pi x}{L}\,dx = \left[\frac{Lx}{n\pi}\sin\frac{n\pi x}{L}\right]_0^{L/2} - \frac{L}{n\pi}\int_0^{L/2}\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{L^2}{2n\pi}\sin\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)$$

Half-range Expansions - Example

$$a_n = \frac{4k}{L^2}\left[\int_0^{L/2} x\cos\frac{n\pi x}{L}\,dx + \int_{L/2}^{L}(L-x)\cos\frac{n\pi x}{L}\,dx\right]$$

Similarly, the second integral becomes

$$\int_{L/2}^{L}(L-x)\cos\frac{n\pi x}{L}\,dx$$

$$= \left[\frac{L}{n\pi}(L-x)\sin\frac{n\pi x}{L}\right]_0^{L/2} + \frac{L}{n\pi}\int_{L/2}^{L}\sin\frac{n\pi x}{L}\,dx$$

$$= -\frac{L^2}{2n\pi}\sin\frac{n\pi}{2} - \frac{L^2}{n^2\pi^2}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)$$

Half-range Expansions - Example

$$\int_0^{L/2} x\cos\frac{n\pi x}{L}\,dx = \frac{L^2}{2n\pi}\sin\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)$$

$$\int_{L/2}^{L} (L-x)\cos\frac{n\pi x}{L}\,dx = -\frac{L^2}{2n\pi}\sin\frac{n\pi}{2} - \frac{L^2}{n^2\pi^2}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)$$

$$a_n = \frac{4k}{L^2}\left[\int_0^{L/2} x\cos\frac{n\pi x}{L}\,dx + \int_{L/2}^{L} (L-x)\cos\frac{n\pi x}{L}\,dx\right]$$

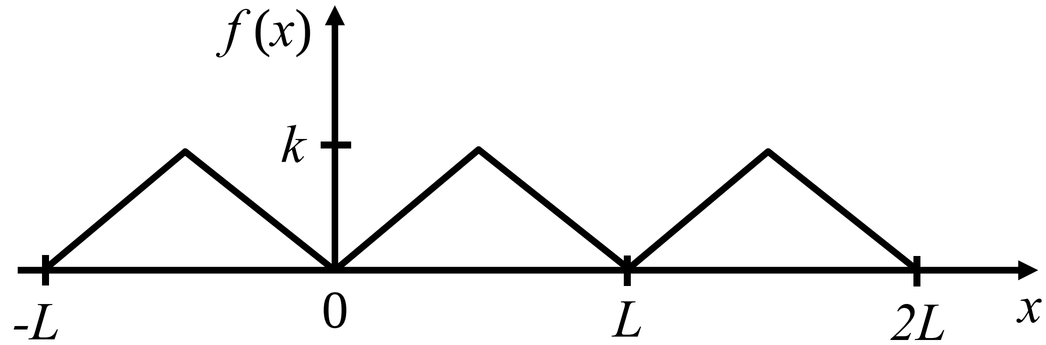
Thus a_n simplifies to $\quad a_n = \dfrac{4k}{n^2\pi^2}\left(2\cos\dfrac{n\pi}{2} - \cos n\pi - 1\right).$

Indeed, $\quad a_2 = \dfrac{-16k}{2^2\pi^2},\ a_6 = \dfrac{-16k}{6^2\pi^2},\ a_{10} = \dfrac{-16k}{10^2\pi^2},\ \cdots$

and $\quad a_n = 0$ if $n \geq 1$ and $n \neq 2, 6, 10, \cdots$

Half-range Expansions - Example

$$f(x)=\begin{cases}\dfrac{2}{L}kx & , \quad 0<x<\dfrac{L}{2}\\[2ex] \dfrac{2k}{L}(L-x) & , \quad \dfrac{L}{2}<x<L\end{cases}$$



Cosine Half-Range Extension

$$a_0=\frac{k}{2}$$

$$a_2=\frac{-16k}{2^2\pi^2},\ a_6=\frac{-16k}{6^2\pi^2},\ a_{10}=\frac{-16k}{10^2\pi^2},\ \cdots$$

and $a_n=0$ if $n\geq 1$ and $n\neq 2, 6, 10, \cdots$

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{L}$$

Consider $4m-2$

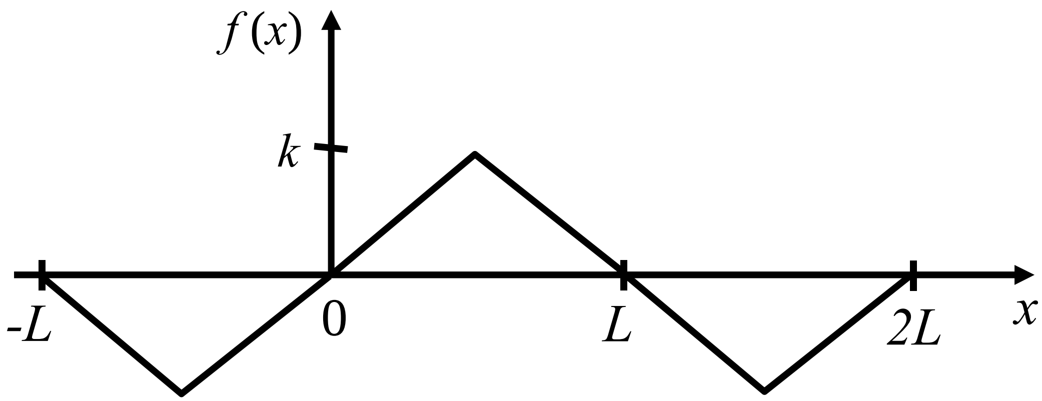
The cosine half range expansion is

$$f(x)=\frac{k}{2}-\frac{16k}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(4m-2)^2}\cos\frac{(4m-2)\pi x}{L}$$

$$=\frac{k}{2}-\frac{4k}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}\cos\frac{2(2m-1)\pi x}{L}$$

Half-range Expansions - Example

$$f(x)=\begin{cases}\dfrac{2}{L}kx & , \quad 0<x<\dfrac{L}{2}\\[2ex] \dfrac{2k}{L}(L-x) & , \quad \dfrac{L}{2}<x<L\end{cases}$$



Sine Half-Range Extension

For the sine half range expansion, we have

$$b_n=\frac{2}{L}\left[\int_0^{L/2}\frac{2k}{L}x\sin\frac{n\pi x}{L}\,dx+\int_{L/2}^{L}\frac{2k}{L}(L-x)\sin\frac{n\pi x}{L}\,dx\right]$$

$$=\frac{4k}{L^2}\left[\int_0^{L/2}x\sin\frac{n\pi x}{L}\,dx+\int_{L/2}^{L}(L-x)\sin\frac{n\pi x}{L}\,dx\right]$$

Half-range Expansions - Example

$$b_n = \frac{4k}{L^2}\left[\int_0^{L/2} x\sin\frac{n\pi x}{L}\,dx + \int_{L/2}^{L}(L-x)\sin\frac{n\pi x}{L}\,dx\right]$$

Integrating by parts, the first integral becomes

$$\int_0^{L/2} x\sin\frac{n\pi x}{L}\,dx = \left[-\frac{Lx}{n\pi}\cos\frac{n\pi x}{L}\right]_0^{L/2} - \frac{L}{n\pi}\int_0^{L/2} -\cos\frac{n\pi x}{L}\,dx$$

$$= -\frac{L^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\left(\sin\frac{n\pi}{2}\right)$$

Half-range Expansions - Example

$$b_n = \frac{4k}{L^2}\left[\int_0^{L/2} x \sin\frac{n\pi x}{L}\, dx + \int_{L/2}^{L} (L-x)\sin\frac{n\pi x}{L}\, dx\right]$$

Similarly, the second integral becomes

$$\int_{L/2}^{L} (L-x)\sin\frac{n\pi x}{L}\, dx$$

$$= \left[-\frac{L}{n\pi}(L-x)\cos\frac{n\pi x}{L}\right]_0^{L/2} + \frac{L}{n\pi}\int_{L/2}^{L} -\cos\frac{n\pi x}{L}\, dx$$

$$= \frac{L^2}{2n\pi}\cos\frac{n\pi}{2} - \frac{L^2}{n^2\pi^2}\left(\sin n\pi - \sin\frac{n\pi}{2}\right)$$

$$= \frac{L^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\int_0^{L/2} x \sin\frac{n\pi x}{L}\, dx = -\frac{L^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\left(\sin\frac{n\pi}{2}\right)$$

$$\int_{L/2}^{L} (L-x)\sin\frac{n\pi x}{L}\, dx = \frac{L^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$b_n = \frac{4k}{L^2}\left[\int_0^{L/2} x\sin\frac{n\pi x}{L}\, dx + \int_{L/2}^{L}(L-x)\sin\frac{n\pi x}{L}\, dx\right]$$

Thus b_n simplifies to $\quad b_n = \frac{4k}{L^2}\cdot\frac{2L^2}{n^2\pi^2}\sin\frac{n\pi}{2}$

The sine half range expansion is

$$f(x) = \frac{8k}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{L}$$

$$= \frac{8k}{\pi^2}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{(2m-1)^2}\sin\frac{(2m-1)\pi x}{L}$$

$$\sin\frac{n\pi}{2} = \begin{cases} 1 & \text{if } n = 1, 5, 9, 13, \cdots \\ 0 & \text{if } n = 2, 4, 6, 8, \cdots \\ -1 & \text{if } n = 3, 7, 11, 15, \cdots \end{cases}$$

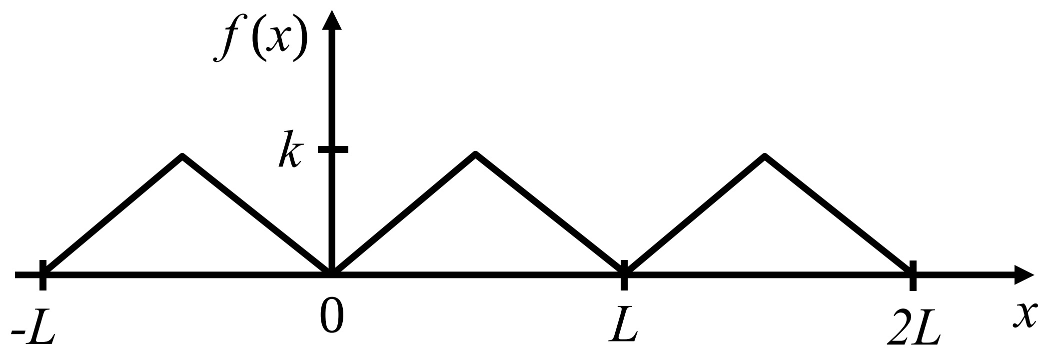
$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

Pause and Think !!!

How to use the Fourier Cosine series to find

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \quad ???$$

$$f(x) = \begin{cases} \dfrac{2}{L}kx & , \quad 0 < x < \dfrac{L}{2} \\ \dfrac{2k}{L}(L-x) \, , & \dfrac{L}{2} < x < L \end{cases}$$



Cosine Half-Range Extension

The cosine half range expansion is

$$f(x) = \frac{k}{2} - \frac{4k}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^2} \boxed{\cos \frac{2(2m-1)\pi x}{L}}$$

Put $x =$???

Representation by a Fourier Series

Let f be a function such that f and f' are ***piecewise continuous*** on $[-\pi, \pi]$. Then

(1) at any point x where f is ***continuous***,

$f(x)$ equals to its Fourier series.

(2) at c where f is ***discontinuous***,

the Fourier series converges to

$$\frac{1}{2}[f(c^+) + f(c^-)]$$

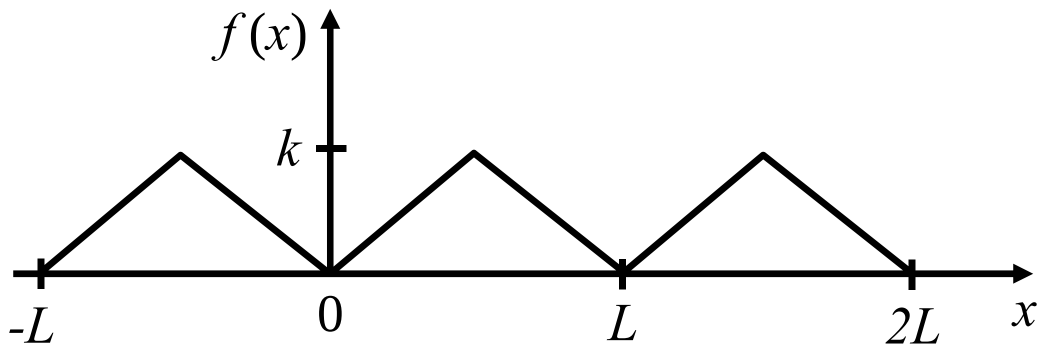
where $f(c^+)$ is the Right-Hand limit of f at c

and $f(c^-)$ is the Left-Hand limit of f at c.

How to use the Fourier Cosine series to find

$$\sum_{n=1}^{\infty}\frac{1}{n^2} \quad ???$$

$$f(x)=\begin{cases}\dfrac{2}{L}kx & , \ 0<x<\dfrac{L}{2}\\[2ex] \dfrac{2k}{L}(L-x) & , \ \dfrac{L}{2}<x<L\end{cases}$$



Cosine Half-Range Extension

The cosine half range expansion is

$$f(x)=\frac{k}{2}-\frac{4k}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}\boxed{\cos\frac{2(2m-1)\pi x}{L}}$$

Put $x=$???

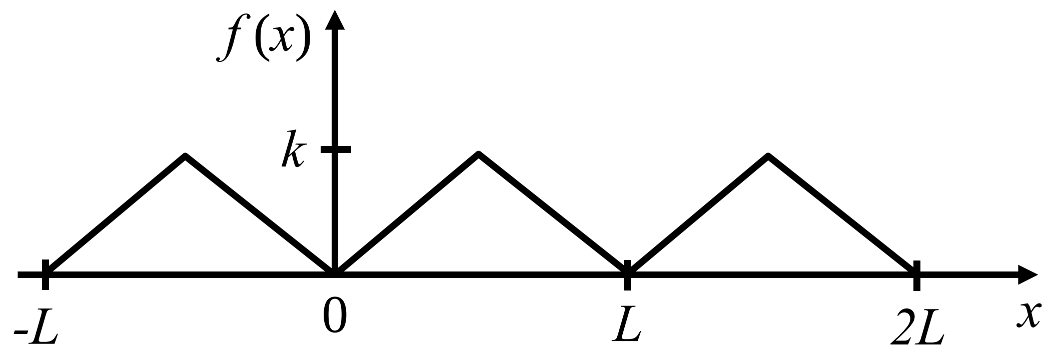
Choose x such that $\cos\dfrac{2(2m-1)\pi x}{L}=1$

$\cos 0 = 1$

Put $x = 0$

How to find $\sum_{n=1}^{\infty} \frac{1}{n^2}$???

$$f(x)=\begin{cases}\frac{2}{L}kx & , \quad 0<x<\frac{L}{2}\\ \frac{2k}{L}(L-x) & , \quad \frac{L}{2}<x<L\end{cases}$$



The cosine half range expansion is

$$f(x)=\frac{k}{2}-\frac{4k}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}\boxed{\cos\frac{2(2m-1)\pi x}{L}}$$

Put $x=0$

$\cos 0 = 1$

$$\frac{1}{2}[f(0^+)+f(0^-)]=\frac{k}{2}-\frac{4k}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}$$

$$\frac{1}{2}[f(0^+)+f(0^-)]=0$$

$$0=\frac{k}{2}-\frac{4k}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}$$

$$\boxed{\frac{k}{2}=\frac{4k}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}}$$

$$\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}=\frac{\pi^2}{8}$$

How to find $\sum_{n=1}^{\infty}\frac{1}{n^2}$???

$$\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}=\frac{\pi^2}{8}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}+\sum_{m=1}^{\infty}\frac{1}{(2m)^2}$$

$$=\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}+\sum_{m=1}^{\infty}\frac{1}{4m^2}$$

$$=\frac{\pi^2}{8}+\frac{1}{4}\sum_{m=1}^{\infty}\frac{1}{m^2}$$

$$\sum_{m=1}^{\infty}\frac{1}{m^2}=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2}-\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{8}$$

$$\frac{3}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{8}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

Jean Baptiste Joseph Fourier

Joseph Fourier
(1768-1830)

End