Engineering Electromagnetics

EE2011, Part CXD

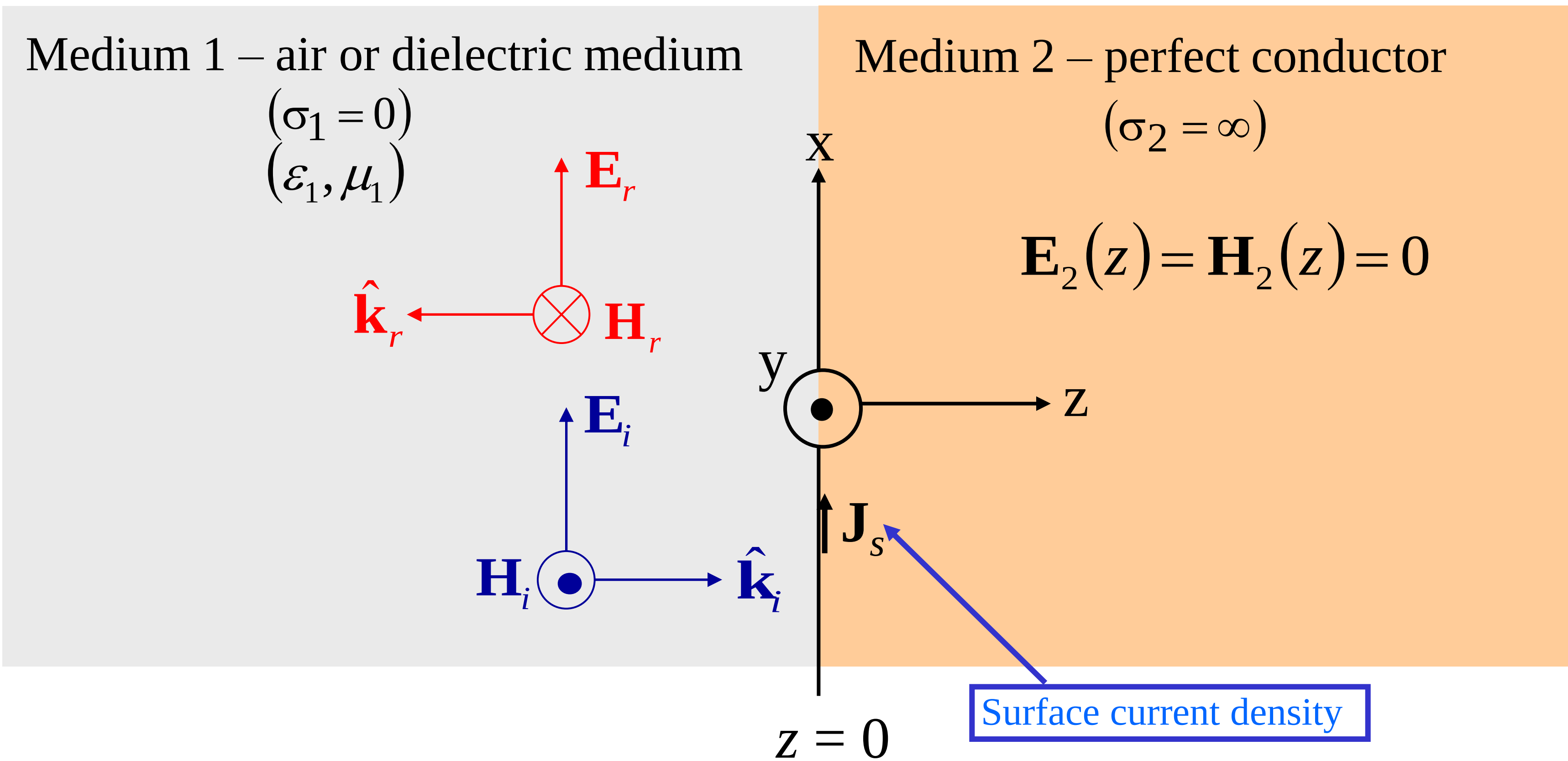
LECTURE 5

Chen Xudong

Dept. of Electrical and Computer Engineering
National University of Singapore

Plane Wave Reflection and Transmission

1 Normal Incidence at a Perfect Conductor



Given an incident fields:

Actual E-field may not be in $+\hat{\mathbf{x}}$ direction since $\hat{\mathbf{x}}E_{i0}$ determines the direction of E-field

$$\mathbf{E}_i(z) = \hat{\mathbf{x}}E_{i0}e^{-j\beta_1 z} \qquad \mathbf{H}_i(z) = \hat{\mathbf{y}}\frac{E_{i0}}{\eta_1}e^{-j\beta_1 z}$$

Reflected fields:

$$\mathbf{E}_r(z) = \hat{\mathbf{x}}E_{r0}e^{+j\beta_1 z} \qquad \mathbf{H}_r(z) = -\hat{\mathbf{y}}\frac{1}{\eta_1}E_{r0}e^{+j\beta_1 z}$$

Transmitted fields in medium 2:

$$\mathbf{E}_2(z) = \mathbf{H}_2(z) = 0$$

At $z = 0$, using the boundary condition:

Tangential component of **E**-field continuous

Since the **E**-field vanishes in perfect conductors, the tangential component of **E**-field in medium 1 must also vanish at the boundary.

At the boundary of a dielectric medium and a perfect conductor:
Only one equation is used to solve the reflected wave

Table 6-2: Boundary conditions for the electric and magnetic fields.

Field Components	General Form	Medium 1 Dielectric	Medium 2 Dielectric	Medium 1 Dielectric	Medium 2 Conductor
Tangential E	$\hat{\mathbf{n}}_2 \times (\mathbf{E}_1 - \mathbf{E}_2) = 0$	$E_{1t} = E_{2t}$		$E_{1t} = E_{2t} = 0$	
Normal D	$\hat{\mathbf{n}}_2 \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$	$D_{1n} - D_{2n} = \rho_s$		$D_{1n} = \rho_s$	$D_{2n} = 0$
Tangential H	$\hat{\mathbf{n}}_2 \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s$	$H_{1t} = H_{2t}$		$H_{1t} = J_s$	$H_{2t} = 0$
Normal B	$\hat{\mathbf{n}}_2 \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0$	$B_{1n} = B_{2n}$		$B_{1n} = B_{2n} = 0$	

Notes: (1) ρ_s is the surface charge density at the boundary; (2) $\mathbf{J}_s$ is the surface current density at the boundary; (3) normal components of all fields are along $\hat{\mathbf{n}}_2$, the outward unit vector of medium 2; (4) $E_{1t} = E_{2t}$ implies that the tangential components are equal in magnitude and parallel in direction; (5) direction of $\mathbf{J}_s$ is orthogonal to $(\mathbf{H}_1 - \mathbf{H}_2)$.

Total electric field in medium 1:

$$\mathbf{E}_1(z) = \mathbf{E}_i(z) + \mathbf{E}_r(z)$$

$$= \hat{\mathbf{x}} E_{i0} e^{-\mathrm{j}\beta_1 z} + \hat{\mathbf{x}} E_{r0} e^{+\mathrm{j}\beta_1 z}$$

At $z = 0$,

$$\mathbf{E}_1(0) = \hat{\mathbf{x}} E_{i0} e^{0} + \hat{\mathbf{x}} E_{r0} e^{0} = \hat{\mathbf{x}}\left(E_{i0} + E_{r0}\right)$$

At $z = 0$ boundary, tangential directions are x and y directions.

From the boundary condition, we have

$$E_{1,x}(0) = 0 \quad \Rightarrow \quad E_{i0} + E_{r0} = 0$$

$$\therefore \quad E_{r0} = -E_{i0}$$

Then reflection coefficient: $\Gamma = \dfrac{E_{r0}}{E_{i0}} = -1$

Reflected electric field :

$$\mathbf{E}_r(z) = -\hat{\mathbf{x}} E_{i0} e^{+j\beta_1 z}$$

Total electric field:

$$\begin{aligned}\mathbf{E}_1(z) &= \mathbf{E}_i(z) + \mathbf{E}_r(z) \\ &= \hat{\mathbf{x}} E_{i0}\left(e^{-j\beta_1 z} - e^{+j\beta_1 z}\right) \\ &= -\hat{\mathbf{x}} j 2 E_{i0} \sin(\beta_1 z)\end{aligned}$$

Reflected magnetic field:

$$\begin{aligned}\mathbf{H}_r(z) &= \frac{1}{\eta_1}(-\hat{\mathbf{z}}) \times \mathbf{E}_r(z) \\ &= \frac{1}{\eta_1} \hat{\mathbf{y}} E_{i0} e^{+j\beta_1 z}\end{aligned}$$

Total magnetic field in medium 1:

$$\begin{aligned}\mathbf{H}_1(z) &= \mathbf{H}_i(z) + \mathbf{H}_r(z) \\ &= \hat{\mathbf{y}} \frac{E_{i0}}{\eta_1} e^{-j\beta_1 z} + \frac{1}{\eta_1} \hat{\mathbf{y}} E_{i0} e^{+j\beta_1 z} = \hat{\mathbf{y}} \frac{E_{i0}}{\eta_1}\left(e^{-j\beta_1 z} + e^{+j\beta_1 z}\right) \\ &= \hat{\mathbf{y}} \frac{E_{i0}}{\eta_1} 2\cos(\beta_1 z)\end{aligned}$$

Instantaneous fields:

$$\boxed{\begin{aligned}\mathbf{E}_1(z,t) &= \mathrm{Re}\left\{\mathbf{E}_1(z)e^{j\omega t}\right\} = \hat{\mathbf{x}}2E_{i0}\sin(\beta_1 z)\sin(\omega t)\\ \mathbf{H}_1(z,t) &= \mathrm{Re}\left\{\mathbf{H}_1(z)e^{j\omega t}\right\} = \hat{\mathbf{y}}2\frac{E_{i0}}{\eta_1}\cos(\beta_1 z)\cos(\omega t)\end{aligned}}$$

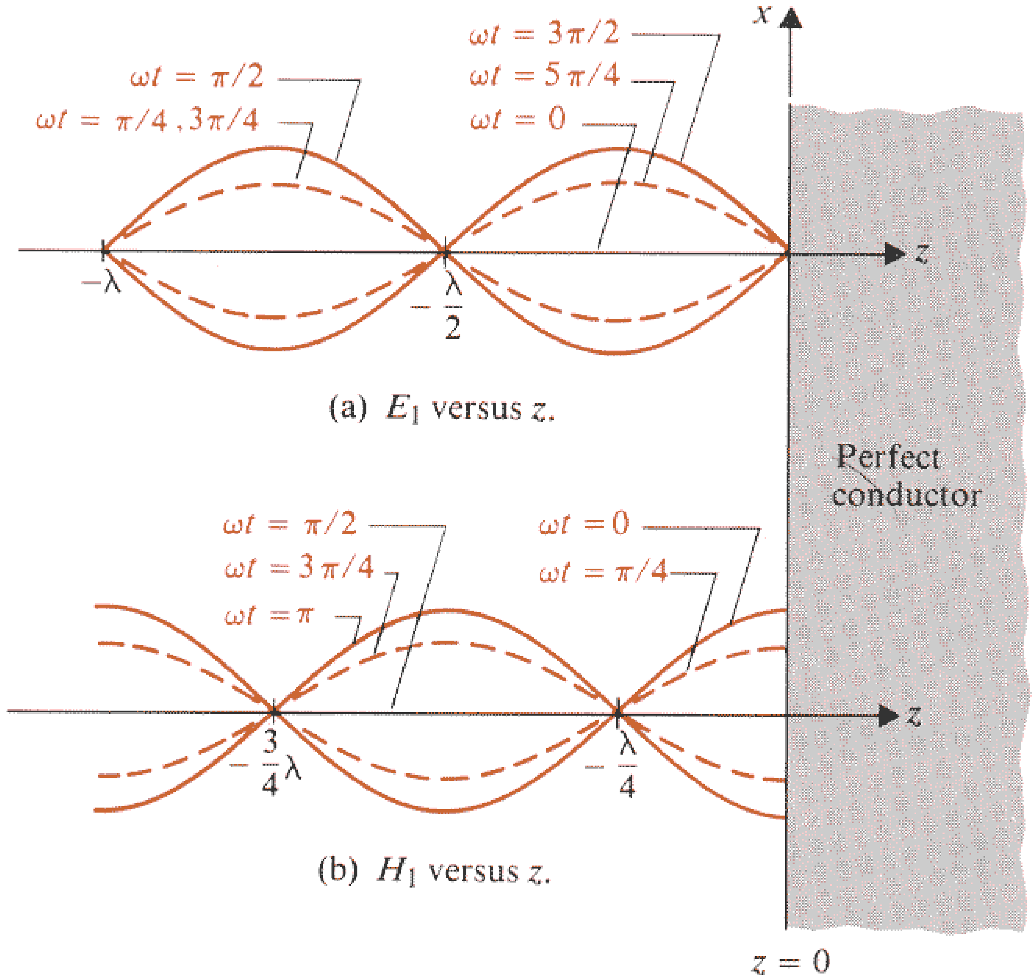
Note that both the total electric and total magnetic fields in medium 1 are standing waves.

By setting $\sin(\beta_1 z)=0$ in E_1

1. They are $\perp$ to each other and 90° out of phase.
2. The electric field vanishes at $z = -n\lambda/2$, $n = 0,1,2,$
3. The magnetic field vanishes at $z = -(\lambda/4 + n\lambda/2)$.

By setting $\cos(\beta_1 z)=0$ in H_1

Negative sign:
Medium 1 is on the left of origin: z <0



Total electric and magnetic fields in medium 1

Animation: http://www.walter-fendt.de/ph14e/stwaverefl.htm Press 'Start' button

Example 1

A uniform plane wave ($\mathbf{E}_i$, $\mathbf{H}_i$) at a frequency of 100 MHz travels in air in the $+x$ direction. The electric field is polarised in the y direction. The wave impinges normally on a perfectly conducting plane at $x = 0$. The magnitude of the incident electric field is 6×10^{-3} V/m and its initial phase is zero.

(a) Write phasor and instantaneous expressions for $\mathbf{E}_i$, $\mathbf{H}_i$.

(b) Write phasor and instantaneous expressions for $\mathbf{E}_r$, $\mathbf{H}_r$.

(c) Write phasor and instantaneous expressions for $\mathbf{E}_1$, $\mathbf{H}_1$ in air.

(d) Determine the position nearest to the conducting plane where $\mathbf{E}_1 = 0$.

Solutions

(a) Incident wave

$$\beta_1 = k_0 = \omega\sqrt{\mu_0\varepsilon_0} = \frac{\omega}{c} \approx \frac{2\pi\times10^8}{3\times10^8} = 2\pi/3 \quad \text{rad/m}$$

$$\eta_1 = \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi \quad \Omega$$

Phasor expressions:

$$\mathbf{E}_i(x) = \hat{\mathbf{y}}\, E_{i0}\, e^{-j\beta_1 x} = \hat{\mathbf{y}}\, 6\times10^{-3}\, e^{-j(2\pi/3)x} \quad \text{V/m}$$

$$\mathbf{H}_i(x) = \frac{1}{\eta_1}\hat{\mathbf{x}}\times\mathbf{E}_i(x) = \hat{\mathbf{z}}\,\frac{E_{i0}}{\eta_1}\, e^{-j\beta_1 x} = \hat{\mathbf{z}}\,\frac{1\times10^{-4}}{2\pi}\, e^{-j(2\pi/3)x} \quad \text{A/m}$$

Instantaneous expressions:

$$\mathbf{E}_i(x,t) = \mathrm{Re}\left[\mathbf{E}_i(x)\, e^{j\omega t}\right] = \hat{\mathbf{y}}\, 6\times10^{-3}\cos\left(2\pi\times10^8 t - \frac{2\pi x}{3}\right) \quad \text{V/m}$$

$$\mathbf{H}_i(x,t) = \mathrm{Re}\left[\mathbf{H}_i(x)\, e^{j\omega t}\right] = \hat{\mathbf{z}}\,\frac{1\times10^{-4}}{2\pi}\cos\left(2\pi\times10^8 t - \frac{2\pi x}{3}\right) \quad \text{A/m}$$

(b) Reflected wave:

Phasors:

$$\mathbf{E}_r(x) = \hat{\mathbf{y}}\,(-1)E_{i0}\; e^{+j\beta_1 x}$$
$$= -\hat{\mathbf{y}}\; 6\times10^{-3}\; e^{+j(2\pi/3)x} \quad \text{V/m}$$

$$\mathbf{H}_r(x) = \frac{1}{\eta_1}(-\hat{\mathbf{x}})\times\mathbf{E}_r(x)$$

$$= \hat{\mathbf{z}}\,\frac{E_{i0}}{\eta_1}\,e^{+j\beta_1 x} = \hat{\mathbf{z}}\,\frac{1\times10^{-4}}{2\pi}\,e^{+j(2\pi/3)x} \quad \text{A/m}$$

Instantaneous:

$$\mathbf{E}_r(x,t) = \text{Re}\left[\mathbf{E}_r(x)\, e^{j\omega t}\right]$$
$$= -\hat{\mathbf{y}}\; 6\times10^{-3}\,\cos\left(2\pi\times10^{8}t + \frac{2\pi x}{3}\right) \quad \text{V/m}$$

$$\mathbf{H}_r(x,t) = \text{Re}\left[\mathbf{H}_r(x)\, e^{j\omega t}\right]$$
$$= \hat{\mathbf{z}}\,\frac{1\times10^{-4}}{2\pi}\,\cos\left(2\pi\times10^{8}t + \frac{2\pi x}{3}\right) \quad \text{A/m}$$

(c) Total field:

Phasors:

$$\mathbf{E}_1(x) = \mathbf{E}_i(x) + \mathbf{E}_r(x) = \hat{\mathbf{y}}\ 6\times10^{-3}\ (e^{-j2\pi x/3} - e^{+j2\pi x/3})$$

$$= \hat{\mathbf{y}}\ (-j)\, 12\times10^{-3}\ \sin(2\pi x/3)$$

$$\mathbf{H}_1(x) = \mathbf{H}_i(x) + \mathbf{H}_r(x) = \hat{\mathbf{z}}\ \frac{1\times10^{-4}}{2\pi}\ (e^{-j2\pi x/3} + e^{+j2\pi x/3})$$

$$= \hat{\mathbf{z}}\ \frac{1\times10^{-4}}{\pi} \cos(2\pi x/3)$$

Instantaneous:

$$\mathbf{E}_1(x,t) = \mathrm{Re}\left[\mathbf{E}_1(x)\ e^{j\omega t}\right]$$

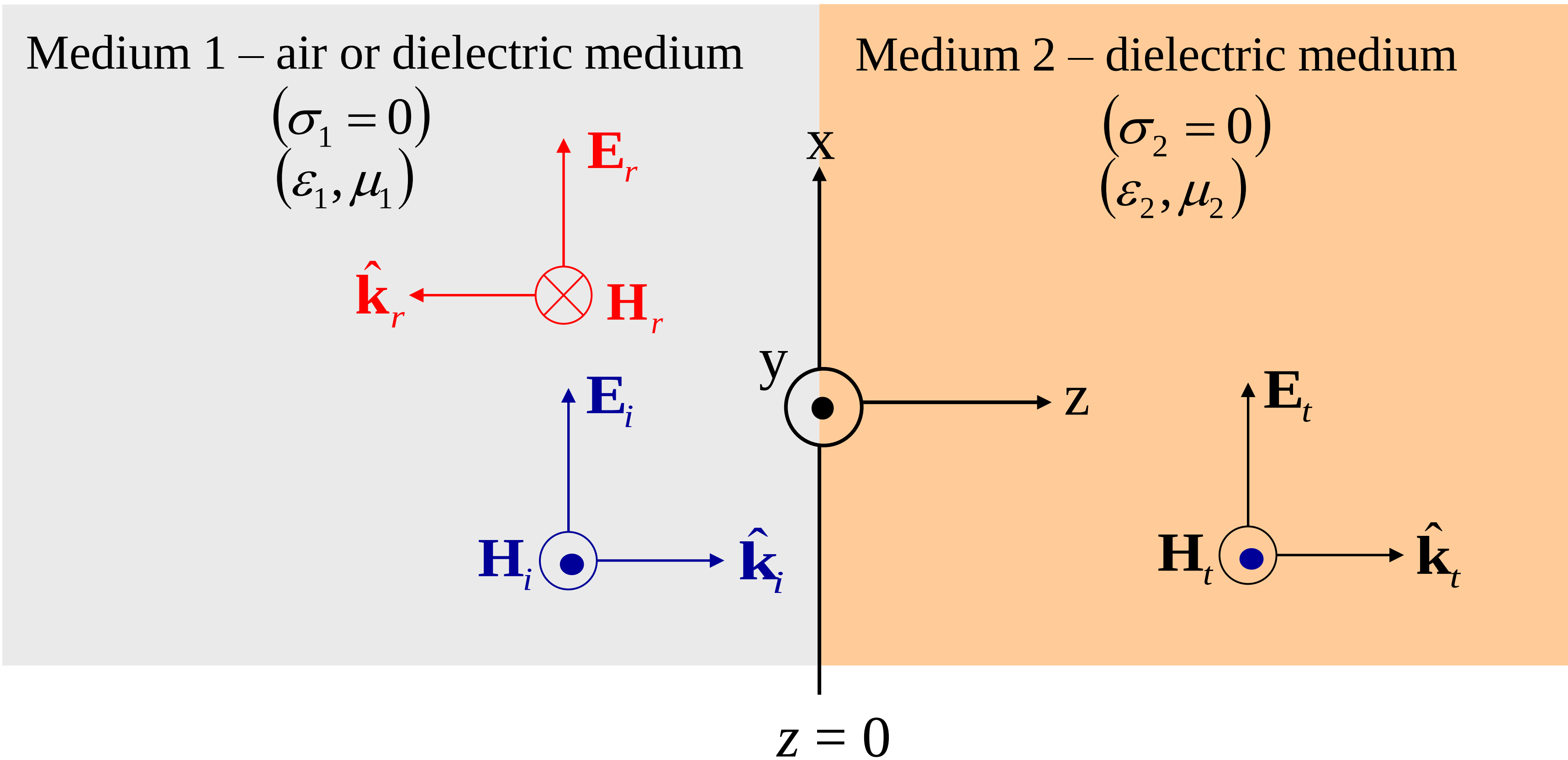
$$= \hat{\mathbf{y}}\ 12\times10^{-3}\ \sin(2\pi x/3)\sin(2\pi\times10^8 t) \quad \text{V/m}$$

$$\mathbf{H}_1(x,t) = \mathrm{Re}\left[\mathbf{H}_1(x)\ e^{j\omega t}\right]$$

$$= \hat{\mathbf{z}}\ \frac{1\times10^{-4}}{\pi} \cos(2\pi x/3)\cos(2\pi\times10^8 t) \quad \text{A/m}$$

(d) The electric field vanishes at $x = -n\lambda/2$, $n = 0,1,2,\ldots$. Excluding the boundary surface ($n = 0$), the nearest null will be at $n = 1$, i.e., $x = -\lambda/2 = -(2\pi/\beta_1)/2 = -1.5$ m.

2 Normal Incidence at a lossless Dielectric Boundary



Incident, reflected, and transmitted fields:

Actual E-field may not be in $+\hat{\mathbf{x}}$ direction since $\hat{\mathbf{x}}E_{i0}$ determines the direction of E-field

$$\mathbf{E}_i(z) = \hat{\mathbf{x}} E_{i0} e^{-j\beta_1 z} \qquad \mathbf{H}_i(z) = \hat{\mathbf{y}} \frac{E_{i0}}{\eta_1} e^{-j\beta_1 z}$$

$$\mathbf{E}_r(z) = \hat{\mathbf{x}} E_{r0} e^{j\beta_1 z} \qquad \mathbf{H}_r(z) = -\hat{\mathbf{y}} \frac{E_{r0}}{\eta_1} e^{j\beta_1 z}$$

$$\mathbf{E}_t(z) = \hat{\mathbf{x}} E_{t0} e^{-j\beta_2 z} \qquad \mathbf{H}_t(z) = \hat{\mathbf{y}} \frac{E_{t0}}{\eta_2} e^{-j\beta_2 z}$$

Medium parameters:

$$\beta_1 = \omega\sqrt{\varepsilon_1 \mu_1}, \qquad \beta_2 = \omega\sqrt{\varepsilon_2 \mu_2}$$

$$\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}, \qquad \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}}$$

At the boundary of two dielectric media:
2 equations are considered in solving the reflected and transmitted waves

Table 6-2: Boundary conditions for the electric and magnetic fields.

Field Components	General Form	Medium 1 Dielectric / Medium 2 Dielectric	Medium 1 Dielectric	Medium 2 Conductor
Tangential E	$\hat{\mathbf{n}}_2 \times (\mathbf{E}_1 - \mathbf{E}_2) = 0$	$E_{1t} = E_{2t}$	$E_{1t} = E_{2t} = 0$	
Normal D	$\hat{\mathbf{n}}_2 \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$	$D_{1n} - D_{2n} = \rho_s$	$D_{1n} = \rho_s$	$D_{2n} = 0$
Tangential H	$\hat{\mathbf{n}}_2 \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s$	$H_{1t} = H_{2t}$	$H_{1t} = J_s$	$H_{2t} = 0$
Normal B	$\hat{\mathbf{n}}_2 \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0$	$B_{1n} = B_{2n}$	$B_{1n} = B_{2n} = 0$	

Notes: (1) ρ_s is the surface charge density at the boundary; (2) $\mathbf{J}_s$ is the surface current density at the boundary; (3) normal components of all fields are along $\hat{\mathbf{n}}_2$, the outward unit vector of medium 2; (4) $E_{1t} = E_{2t}$ implies that the tangential components are equal in magnitude and parallel in direction; (5) direction of $\mathbf{J}_s$ is orthogonal to $(\mathbf{H}_1 - \mathbf{H}_2)$.

Boundary conditions:

Boundary z=0

$$E_{1\parallel}(0) = E_{2\parallel}(0) \qquad H_{1\parallel}(0) = H_{2\parallel}(0)$$

Explicitly:

$$E_{1\parallel}(0) = E_{i0}(0) + E_{r0}(0), \qquad E_{2\parallel}(0) = E_{t0}(0)$$

$$H_{1\parallel}(0) = \frac{E_{i0}(0)}{\eta_1} - \frac{E_{r0}(0)}{\eta_1}, \qquad H_{2\parallel}(0) = \frac{E_{t0}(0)}{\eta_2}$$

The boundary conditions lead to:

$$E_{i0} + E_{r0} = E_{t0} \qquad \frac{E_{i0}}{\eta_1} - \frac{E_{r0}}{\eta_1} = \frac{E_{t0}}{\eta_2}$$

Solving for E_{r0} and E_{t0},

$$E_{r0} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} E_{i0} \qquad E_{t0} = \frac{2\eta_2}{\eta_2 + \eta_1} E_{i0}$$

Define:

Similar to T.L: $\Gamma_L = \dfrac{Z_L - Z_0}{Z_L + Z_0}$

Reflection coefficient, $\Gamma = \dfrac{E_{r0}}{E_{i0}} = \dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$

Transmission coefficient, $\tau = \dfrac{E_{t0}}{E_{i0}} = \dfrac{2\eta_2}{\eta_2 + \eta_1}$

Note:

$$\boxed{1 + \Gamma = \tau}$$

$$\boxed{|\Gamma| \leq 1}$$

Using Γ and τ, the field expressions in the media can be expressed in terms of the incident field amplitude E_{i0}:

Incident	$\mathbf{E}_i(z) = \hat{\mathbf{x}} E_{i0} e^{-j\beta_1 z}$	$\mathbf{H}_i(z) = \hat{\mathbf{y}} \dfrac{E_{i0}}{\eta_1} e^{-j\beta_1 z}$
Reflected	$\mathbf{E}_r(z) = \hat{\mathbf{x}} \Gamma\, E_{i0} e^{j\beta_1 z}$	$\mathbf{H}_r(z) = -\hat{\mathbf{y}} \dfrac{\Gamma\, E_{i0}}{\eta_1} e^{j\beta_1 z}$
Transmitted	$\mathbf{E}_t(z) = \hat{\mathbf{x}} \tau E_{i0} e^{-j\beta_2 z}$	$\mathbf{H}_t(z) = \hat{\mathbf{y}} \dfrac{\tau E_{i0}}{\eta_2} e^{-j\beta_2 z}$

Power Density Relationship

Reflected power density, $\mathbf{S}_r$

$$\mathbf{S}_r = \frac{1}{2}\mathrm{Re}\left\{\mathbf{E}_r(z)\times\mathbf{H}_r^*(z)\right\} = \frac{1}{2}\mathrm{Re}\left\{(-\hat{\mathbf{z}})|\Gamma|^2\frac{|E_{i0}|^2}{\eta_1}\right\}$$

$$= (-\hat{\mathbf{z}})|\Gamma|^2\frac{1}{2}\frac{|E_{i0}|^2}{\eta_1}$$

$$\frac{1}{2}\frac{|E_{i0}|^2}{\eta_1} = \text{incident power density}$$

$$= (-\hat{\mathbf{z}})|\Gamma|^2 \times \text{incident power density}$$

$$\boxed{|\Gamma|^2 = \frac{\text{reflected power density}}{\text{incident power density}} = \text{fraction of power reflected}}$$

Transmitted power density, $\mathbf{S}_t$

$$\mathbf{S}_t = \frac{1}{2}\mathrm{Re}\left\{\mathbf{E}_t(z)\times\mathbf{H}_t^*(z)\right\} = \frac{1}{2}\mathrm{Re}\left\{\hat{\mathbf{z}}\left|\tau\right|^2\frac{\left|E_{i0}\right|^2}{\eta_2}\right\}$$

$$= \hat{\mathbf{z}}\left|\tau\right|^2\frac{\eta_1}{\eta_2}\frac{1}{2}\frac{\left|E_{i0}\right|^2}{\eta_1}$$

$$= \hat{\mathbf{z}}\left|\tau\right|^2\frac{\eta_1}{\eta_2}\times\text{incident power density}$$

$$\boxed{\left|\tau\right|^2\frac{\eta_1}{\eta_2} = \frac{\text{transmitted power density}}{\text{incident power density}} = \text{fraction of power transmitted}}$$

$$\boxed{\text{Fraction of reflected power + Fraction of transmitted power = 1}}$$

lossless

$$\boxed{\left|\Gamma\right|^2 + \left|\tau\right|^2\frac{\eta_1}{\eta_2} = 1}$$

Total average power density in medium 1, $\mathbf{S}_1$

Derivation is not required

$$\mathbf{S}_1 = \frac{1}{2}\mathrm{Re}\left\{\mathbf{E}_1(z)\times\mathbf{H}_1^*(z)\right\}$$

$$= \frac{1}{2}\mathrm{Re}\left\{\left[\mathbf{E}_i(z)+\mathbf{E}_r(z)\right]\times\left[\mathbf{H}_i^*(z)+\mathbf{H}_r^*(z)\right]\right\}$$

$$= \frac{1}{2}\mathrm{Re}\left\{\mathbf{E}_i(z)\times\mathbf{H}_i^*(z)\right\} + \frac{1}{2}\mathrm{Re}\left\{\mathbf{E}_r(z)\times\mathbf{H}_r^*(z)\right\}$$

$$= \hat{\mathbf{z}}\frac{\left|E_{i0}\right|^2}{2\eta_1} + (-\hat{\mathbf{z}})\frac{\left|E_{i0}\right|^2}{2\eta_1}\left|\Gamma\right|^2$$

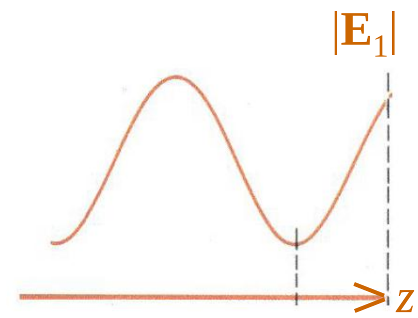
$$= \hat{\mathbf{z}}\left(1-\left|\Gamma\right|^2\right)\times \text{incident power density}$$

$$= \hat{\mathbf{z}}\left|\tau\right|^2\frac{\eta_1}{\eta_2}\times \text{incident power density}$$

$$= \hat{\mathbf{z}}\ \text{transmitted power density}$$

Total electric field in medium 1:

$$\mathbf{E}_1(z) = \mathbf{E}_i(z) + \mathbf{E}_r(z) = \hat{\mathbf{x}} E_{i0} e^{-j\beta_1 z} + \hat{\mathbf{x}} E_{r0} e^{j\beta_1 z}$$
$$= \hat{\mathbf{x}} E_{i0} \left(e^{-j\beta_1 z} + \Gamma e^{j\beta_1 z} \right)$$



The total electric field in medium 1 is a standing wave, and it has local maximum and minimum values but does not go to zero at any location (Note: this is different from the case of incidence upon a conductor).

Total magnetic field in medium 1:

$$\mathbf{H}_1(z) = \mathbf{H}_i(z) + \mathbf{H}_r(z)$$
$$= \hat{\mathbf{y}} \frac{E_{i0}}{\eta_1} \left(e^{-j\beta_1 z} - \Gamma e^{j\beta_1 z} \right)$$

Note:
In comparison, the fields in medium 2 are only transmitted waves and they are pure travelling waves.

We will determine the maxima/minima of EM fields, as we did for T-Lines

$$\mathbf{E}_1(z) = \hat{\mathbf{x}} E_{i0} e^{-j\beta_1 z}\left(1 + \Gamma e^{j2\beta_1 z}\right)$$

$\Gamma = |\Gamma| e^{j\theta}$

$$\mathbf{E}_1(z') = \hat{\mathbf{x}} E_{i0} e^{j\beta_1 z'}\left(1 + |\Gamma| e^{j\theta} e^{-j2\beta_1 z'}\right)$$

$$= \hat{\mathbf{x}} E_{i0} e^{j\beta_1 z'}\left[|\Gamma| e^{j(\theta - 2\beta_1 z')} - (-1)\right]$$

for convenience: $z' = -z$, where $z' \geq 0$

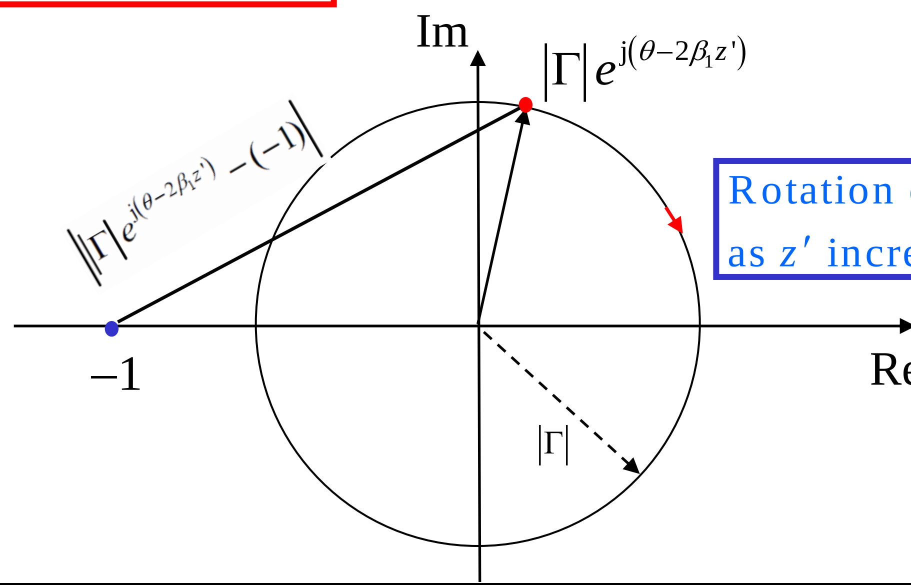
magnitude

$$\boxed{\left|\mathbf{E}_1(z')\right| = \left|E_{i0}\right| \left| |\Gamma| e^{j(\theta - 2\beta_1 z')} - (-1) \right|}$$

In T.L.: l is similar to z'

$\left| |\Gamma| e^{j(\theta - 2\beta_1 z')} - (-1) \right|$:

Distance between $|\Gamma| e^{j(\theta - 2\beta_1 z')}$ and -1 in complex plane



Rotation direction as z' increases

Obviously, from the figure, we know $\mathbf{E}_1$ achieves

(1) maximum at z'_M when $e^{j(\theta - 2\beta_1 z'_M)} = 1$ such that:

$$\left|\mathbf{E}_1(z'_M)\right| = |E_{i0}|\left(1+|\Gamma|\right)$$

i.e., when:

$$2\beta_1 z'_M - \theta = 2n\pi, \qquad n \text{ is an integer that makes } z'_M \geq 0$$

(2) minimum at z'_m when $e^{j(\theta - 2\beta_1 z'_m)} = -1$ such that:

$$\left|\mathbf{E}_1(z'_m)\right| = |E_{i0}|\left(1-|\Gamma|\right)$$

i.e., when:

$$2\beta_1 z'_m - \theta = 2n\pi + \pi, \qquad n \text{ is an integer that makes } z'_m \geq 0$$

Note:

We choose the convention that θ is specified in the range $[-\pi, \pi)$

If the media are lossless, η_1 and η_2 are both real.

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \begin{cases} > 0, & \text{when} \quad \eta_2 > \eta_1 \\ < 0, & \text{when} \quad \eta_2 < \eta_1 \end{cases}$$

$$\Gamma = |\Gamma| e^{j\theta} \begin{cases} \theta = 0, & \text{when} \quad \eta_2 > \eta_1 \\ \theta = -\pi, & \text{when} \quad \eta_2 < \eta_1 \end{cases}$$

Therefore,

$$\boxed{\begin{cases} 2\beta_1 z'_M = 2n\pi, & \text{when} \quad \eta_2 > \eta_1 \\ 2\beta_1 z'_M = (2n' - 1)\pi = (2n+1)\pi, & \text{when} \quad \eta_2 < \eta_1 \end{cases}}$$

Since $z'_M \geq 0$, we know n' starts from 1

$n = n' - 1$ starts from 0

$$\boxed{\begin{cases} 2\beta_1 z'_m = (2n+1)\pi, & \text{when} \quad \eta_2 > \eta_1 \\ 2\beta_1 z'_m = 2n\pi, & \text{when} \quad \eta_2 < \eta_1 \end{cases}}$$

Total magnetic field in medium 1:

$$\mathbf{H}_1(z) = \hat{\mathbf{y}}\frac{E_{i0}e^{-j\beta_1 z}}{\eta_1}\left(1-\Gamma e^{j2\beta_1 z}\right)$$

$$\mathbf{H}_1(z') = \hat{\mathbf{y}}\frac{E_{i0}e^{j\beta_1 z'}}{\eta_1}\left(1-|\Gamma|e^{j\theta}e^{-j2\beta_1 z'}\right)$$

$$= \hat{\mathbf{y}}\frac{E_{i0}e^{j\beta_1 z'}}{\eta_1}\left[1-|\Gamma|e^{j(\theta-2\beta_1 z')}\right]$$

$$\boxed{\left|\mathbf{H}_1(z')\right| = \frac{|E_{i0}|}{\eta_1}\left|1-|\Gamma|e^{j(\theta-2\beta_1 z')}\right|}$$

Distance between $|\Gamma|e^{j(\theta-2\beta_1 z')}$ and +1 in complex plane

Observations:

(1) $\mathbf{H}_1$'s maxima and minima are opposite to those of $\mathbf{E}_1$'s.

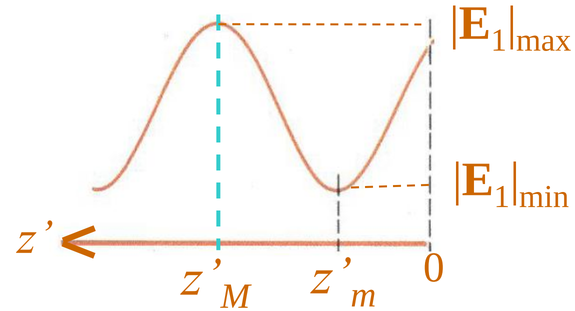
(2) Since $\beta_1\lambda = 2\pi$, when z' increases by λ, $\left|\mathbf{H}_1(z')\right|$ (as well as $\left|\mathbf{E}_1(z')\right|$) experiences 2 periods since $2\beta_1 z'$ is in the exponent.

Table for positions of the maxima and minima of the EM field in medium 1

	$\Gamma > 0 \quad (\eta_2 > \eta_1)$	$\Gamma < 0 \quad (\eta_2 < \eta_1)$
$\lvert\mathbf{E}_1\rvert_{max}$, $\lvert\mathbf{H}_1\rvert_{min}$	$\lvert E_{i0}\rvert(1+\lvert\Gamma\rvert)$, $\dfrac{\lvert E_{i0}\rvert}{\eta_1}(1-\lvert\Gamma\rvert)$	$\lvert E_{i0}\rvert(1+\lvert\Gamma\rvert)$, $\dfrac{\lvert E_{i0}\rvert}{\eta_1}(1-\lvert\Gamma\rvert)$
Condition	$2\beta_1 z'_M = 2n\pi$	$2\beta_1 z'_M = (2n+1)\pi$
Position	$z'_M = n\dfrac{\lambda_1}{2}$, $n = 0, 1, 2, ..$	$z'_M = \dfrac{\lambda_1}{4} + n\dfrac{\lambda_1}{2}$, $n = 0, 1, 2, ..$
$\lvert\mathbf{E}_1\rvert_{min}$, $\lvert\mathbf{H}_1\rvert_{max}$	$\lvert E_{i0}\rvert(1-\lvert\Gamma\rvert)$, $\dfrac{\lvert E_{i0}\rvert}{\eta_1}(1+\lvert\Gamma\rvert)$	$\lvert E_{i0}\rvert(1-\lvert\Gamma\rvert)$, $\dfrac{\lvert E_{i0}\rvert}{\eta_1}(1+\lvert\Gamma\rvert)$
Condition	$2\beta_1 z'_m = (2n+1)\pi$	$2\beta_1 z'_m = 2n\pi$
Position	$z'_m = \dfrac{\lambda_1}{4} + n\dfrac{\lambda_1}{2}$, $n = 0, 1, 2, ..$	$z'_m = n\dfrac{\lambda_1}{2}$, $n = 0, 1, 2, ..$

Attention

Note that : $$\lvert\mathbf{E}_1\rvert_{max} = \lvert E_{i0}\rvert\left(1+\lvert\Gamma\rvert\right)$$
$$\lvert\mathbf{E}_1\rvert_{min} = \lvert E_{i0}\rvert\left(1-\lvert\Gamma\rvert\right)$$



The ratio of $|E_1|_{max}$ to $|E_1|_{min}$ is called

the standing wave ratio *S*:

$$\boxed{S = \frac{|\mathbf{E}_1|_{max}}{|\mathbf{E}_1|_{min}} = \frac{1+|\Gamma|}{1-|\Gamma|}}$$

It is easy to find

$$\boxed{|\Gamma| = \frac{S-1}{S+1}}$$

Example 2

A beam of yellow light with a wavelength of 0.6 μm is normally incident from air ($z < 0$) on to a glass ($z > 0$) . If the glass surface is at the plane $z = 0$ and the relative permittivity of glass is 2.25, determine:

(a) the locations of the electric field maxima in medium 1 (air),
(b) the fraction of the incident power transmitted into the glass medium.

Solutions

(a) We determine the medium parameters,

$$\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \simeq 120\pi \ (\Omega)$$

$$\eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} = \sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{1}{\sqrt{\varepsilon_r}} \simeq \frac{120\pi}{\sqrt{2.25}} = 80\pi\,(\Omega)$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{80\pi - 120\pi}{80\pi + 120\pi} = -0.2$$

$$\tau = \frac{2\eta_2}{\eta_2 + \eta_1} = \frac{160\pi}{80\pi + 120\pi} = 0.8$$

Electric-field magnitude is a maximum at (with $\Gamma < 0$):

$$z'_M = \frac{\lambda_1}{4} + n\frac{\lambda_1}{2} \quad (n = 0,1,2,\ldots)$$

$$\text{with } \lambda_1 = 0.6\mu\text{m}$$

(b) The fraction of the incident power transmitted into the glass medium is

Note: $P_{avi} \neq P_{av1}$

$$\frac{P_{av_2}}{P_{avi}} = \tau^2 \frac{\left|E_0^i\right|^2}{2\eta_2} / \left[\frac{\left|E_0^i\right|^2}{2\eta_1}\right] = \tau^2 \frac{\eta_1}{\eta_2} = 0.8^2 \frac{120}{80} = 0.96$$

$$\text{Alternatively}, \frac{P_{av_2}}{P_{avi}} = 1 - \left|\Gamma\right|^2 = 1 - (0.2)^2 = 0.96 \text{ or } 96\%$$

Relationship between

1 Normal Incidence at a Perfect Conductor

and

2 Normal Incidence at a Dielectric Boundary

The former is a special case of the latter:

For a perfect conductor, $\varepsilon_c = \varepsilon - j\dfrac{\sigma}{\omega} \to -j\infty$, thus $\eta_2 = \sqrt{\dfrac{\mu}{\varepsilon_c}} \to 0$

Reflection coefficient, $\Gamma = \dfrac{E_{r0}}{E_{i0}} = \dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$

Transmission coefficient, $\tau = \dfrac{E_{t0}}{E_{i0}} = \dfrac{2\eta_2}{\eta_2 + \eta_1}$

It is easy to find: $\Gamma = -1$ $\tau = 0$

- **Textbooks:**
  - *Fundamentals of Applied Electromagnetics*,

    F. T. Ulaby, E. Michielssen, U. Ravaioli,

    Pearson Education, 2010, 6th edition

Suggested reading [textbook]:

- Section 6-8: Boundary Conditions for Electromagnetics
- Section 8-1.1: Boundary between Lossless Media
- Section 8-1.3: Power Flow in Lossless Media