1. $\begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}$ $Tr = -1, \det = -5 \Rightarrow$ SADDLE

$\begin{bmatrix} 2 & -2 \\ 4 & 0 \end{bmatrix}$ $Tr = 2$, $\det = 8$, $T_r^2 - 4\det = -28 < 0 \Rightarrow$ SPIRAL SOURCE

$\begin{bmatrix} -2 & -4 \\ 10 & 0 \end{bmatrix}$ $Tr = -2$, $\det = 40$, $T_r^2 - 4\det = -36 < 0 \Rightarrow$ SPIRAL SINK

$\begin{bmatrix} -5 & 4 \\ -2 & 1 \end{bmatrix}$ $Tr = -4$, $\det = 3$, $T_r^2 - 4\det = 4 > 0 \Rightarrow$ NODAL SINK

$\begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix}$ $Tr = 4$, $\det = 3$, $T_r^2 - 4\det = 4 > 0 \Rightarrow$ NODAL SOURCE

$\begin{bmatrix} 0 & 1 \\ -10 & 0 \end{bmatrix}$ $Tr = 0$, $\det = 10$, $T_r^2 - 4\det = -40 < 0 \Rightarrow$ CENTRE

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2. Let $E(t)$ be the number of elves, $D(t)$ the number of Dwarves. Let B_E, D_E be the birth and death rates per capita for Elves, similarly B_D, D_D for Dwarves. We are told that $B_E > B_D$ and $D_E < D_D$. So $B_E - D_E > B_D - D_D > 0$ (more births than deaths).

Now $\frac{dE}{dt}$ is controlled by $(B_E - D_E)E$ in the usual Malthus model, but here we are told that $\frac{dE}{dt}$ is reduced by the presence of Dwarves, so we propose

$$\frac{dE}{dt} = (B_E - D_E)E - P_E D$$

where P_E is a constant that measures the prejudice of the elves. Similarly

$$\frac{dD}{dt} = (B_D - D_D)D - P_D E$$

Where P_D represents the prejudice of the Dwarves. So

$$\begin{pmatrix} \frac{dE}{dt} \\ \frac{dD}{dt} \end{pmatrix} = \begin{bmatrix} (B_E - D_E) & -P_E \\ -P_D & (B_D - D_D) \end{bmatrix} \begin{pmatrix} E \\ D \end{pmatrix}$$

So in a concrete case, the constants in the matrix should satisfy $B_E - D_E > B_D - D_D > 0$, and we are told that $P_E > P_D$. Indeed $\begin{bmatrix} 5 & -4 \\ -1 & 2 \end{bmatrix}$ satisfies all of these. $Tr = 7$, det=6, $T_r^2 - 4\det = 49 - 24 = 25 > 0$ so we have a nodal source. Eigenvalues

$$(5 - \lambda)(2 - \lambda) - 4 = 0 \Rightarrow \lambda = 1 \quad \text{or} \quad 6.$$

Eigenvectors are $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1/4 \end{pmatrix}$.
So the phase plane diagram
(we only care about the
first quadrant, since
$D \geq 0, E \geq 0$) is

bisected by the line $D = E$. All points ABOVE that line will move along trajectories that eventually hit the D axis (that is, $E = 0$). So if at any time $D > E$, then the Elf population may increase for a while, but eventually it will reach a maximum and then collapse to zero. Rivendell is completely taken over by Dwarves, EVEN THOUGH $B_E > B_D$ AND $D_E < D_D$. The prejudice of the Elves cancels out their other advantages and causes them to lose the competition.

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3. (1) Use the rule:
Rate of change of the amount of UF_6= (concentration in) (flow rate in) - (concentration out) (flow rate out), where "concentration" is the mass of uranium hexafluoride per unit volume of water $-$ you can understand this equation by looking at the units:

$$\frac{lbs}{sec} = \frac{lbs}{gallons} \times \frac{gallons}{sec}$$

Let x_A be the mass of UF_6 in the first tank, so the concentration in that tank is $x_A/100$. Similarly let x_B be the mass of UF_6 in the second tank. Then from the given data we have

$$\dot{x}_A = \frac{2}{100}x_B - \frac{6}{100}x_A$$

and

$$\dot{x}_B = \frac{6}{100}x_A - \frac{6}{100}x_B.$$

Notice that the 4 gallons/min of pure water flowing into the first tank does not appear here $-$ it contains no UF_6; it is just there so that the amounts of water in the tanks remain constant. The initial conditions are $x_A(0) = 25$, $x_B(0) = 0$.

In matrix form, we have

$$\begin{bmatrix} \dot{x}_A \\ \dot{x}_B \end{bmatrix} = \frac{1}{100} \begin{bmatrix} -6 & 2 \\ 6 & -6 \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix}.$$

The eigenvalues of the coefficient matrix can be found as $\lambda_1 = \dfrac{-6+2\sqrt{3}}{100}$ and $\lambda_2 = \dfrac{-6-2\sqrt{3}}{100}$. The corresponding eigenvectors are $\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}$.

Hence

$$\begin{bmatrix} x_A \\ x_B \end{bmatrix} = c_1 e^{\lambda_1 t} \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} + c_2 e^{\lambda_2 t} \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}.$$

Substituting the initial values, we have $c_1 = 12.5$, $c_2 = 12.5$, so we get $x_A(t) = 12.5(e^{\lambda_1 t} + e^{\lambda_2 t})$, $x_B(t) = 12.5(\sqrt{3}e^{\lambda_1 t} - \sqrt{3}e^{\lambda_2 t})$.

b) In the java applet, x corresponds to x_A and y to x_B. You have to click on a point on the x axis because we are told that x_B [that is, y] is zero initially. You should see the red curve going steadily down, but the green curve goes up at first and then down.

c) Yes. The two curves intersect when $t_0 = \frac{25}{\sqrt{3}} ln(2+\sqrt{3}) \approx 19$, (which is in fact the maximum point of x_B). Before t_0, the uranium hexafluoride in tank A is more than that in tank B; after that, it's always less.

d) Here Trace $= -12/100$ and det $= 24/10000$, so Trace$^2 - 4$ det $=$ (144-96)/10000 which is positive; hence we have a nodal sink. That is also clear from the java applet.

4. Differentiating

$$u(x, y) = F(y - 3x) \qquad (*)$$

w.r.t. x and y separately using chain rule, we get

$$u_x = -3F'(y - 3x) \quad \text{and} \quad u_y = F'(y - 3x).$$

Therefore,

$$u_x + 3u_y = \left[-3F'(y - 3x)\right] + 3\left[F'(y - 3x)\right] = 0.$$

i.e. the p.d.e. is satisfied and $(*)$ is a general solution.

To find the particular solution to the p.d.e., we need to find the specific single variable function $F(t)$ that satisfies the given initial condition.

(a) Substituting $x = 0$ in $(*)$, we get $u(0, y) = F(y)$. Comparing with the given condition, we have

$$F(y) = 4\sin(y) \qquad (\dagger).$$

Substituting $(\dagger)$ into $(*)$, we get the particular solution $u(x, y) \; = 4\sin(y - 3x)$.

(b) Substituting $y = 0$ in $(*)$, we get $u(x, 0) = F(-3x)$. Comparing with the given condition, we have

$$F(-3x) = e^{x+1} \qquad (\triangle).$$

We set $t = -3x$ to get $x = -t/3$. So equation $(\triangle)$ can be rewritten as

$$F(t) = e^{\left(-\frac{t}{3}+1\right)} \qquad (\diamondsuit).$$

Substituting $(\diamondsuit)$ into $(*)$, we get the particular solution $u(x, y) \; = e^{\left(-\frac{(y-3x)}{3}+1\right)}$.

5. (a) Setting $u_x = p$, we have $p_y = p$, which we write as $\dfrac{1}{p}p_y = 1$. We treat this as a *separable* o.d.e. in p and y to obtain $\ln|p| \; = \; y + c_1(x)$, i.e. $p \; = \; \pm e^{c_1(x)}e^y \; = \; c_2(x)e^y$. Integrating w.r.t. x yields

$$u(x, y) \; = \; c(x)e^y + h(y),$$

where $c(x)$ and $h(y)$ are arbitrary functions of x and y respectively.

(b) We write $u_x = 2xyu$ as $\dfrac{1}{u}u_x = 2xy$. Treating this as a *separable* o.d.e. in u and x, we obtain $\ln|u| \; = \; x^2y + c_1(y)$, i.e.

$$u(x, y) \; = \; \pm e^{c_1(y)}e^{x^2y} \; = \; c(y)e^{x^2y},$$

where $c(y)$ is an arbitrary function of y.

6. (a) Setting $u(x,y) = X(x)Y(y)$, the p.d.e. $yu_x - xu_y = 0$ becomes $yX'Y - xXY' = 0$. Dividing by XY gives

$$\begin{aligned} y\frac{X'}{X} - x\frac{Y'}{Y} &= 0 \\ \text{i.e.} \qquad \frac{1}{x}\cdot\frac{X'}{X} &= \frac{1}{y}\cdot\frac{Y'}{Y} = k_1 \quad \text{(constant)} \end{aligned}$$

This gives two *separable* o.d.e., the first of which is $\frac{1}{x}\cdot\frac{X'}{X} = k_1$. Integrating this yields

$$\begin{aligned} \ln|X| &= \frac{1}{2}k_1x^2 + c_1 \\ \text{i.e.} \qquad X &= \pm e^{c_1}e^{cx^2}, \text{where } c = \frac{1}{2}k_1 \end{aligned}$$

Similarly, integrating the second *separable* o.d.e. $\frac{1}{y}\cdot\frac{Y'}{Y} = k_1$ gives $Y = \pm e^{c_2}e^{cy^2}$.

Thus,

$$\begin{aligned} u(x,y) &= XY = \pm e^{c_1}e^{cx^2}e^{c_2}e^{cy^2} \\ &= ke^{c(x^2+y^2)}. \end{aligned}$$

(b) Setting $u(x,y) = X(x)Y(y)$, the p.d.e. $u_x = yu_y$ becomes $X'Y = yXY'$. Dividing by XY gives

$$\frac{X'}{X} = y\frac{Y'}{Y} = c \quad \text{(constant)}$$

This gives two *separable* o.d.e., the first of which is $\frac{X'}{X} = c$. Integrating this yields

$$\begin{aligned} \ln|X| &= cx + k_1 \\ \text{i.e.} \qquad X &= \pm e^{k_1}e^{cx}. \end{aligned}$$

Similarly, integrating the second *separable* o.d.e. $\frac{Y'}{Y} = \frac{c}{y}$ gives

$$\begin{aligned} \ln|Y| &= c\ln|y| + k_2 \\ \text{i.e.} \qquad Y &= \pm e^{k_2}y^c. \end{aligned}$$

Thus,

$$\begin{aligned} u(x,y) &= XY = \pm e^{k_1}e^{cx}e^{k_2}y^c \\ &= ky^ce^{cx}. \end{aligned}$$

(c) Setting $u(x,y) = X(x)Y(y)$, the p.d.e. $u_{xy} = u$ becomes $X'Y' = XY$. Dividing by XY gives

$$\frac{X'}{X}\cdot\frac{Y'}{Y} = 1.$$

This implies that both $\dfrac{X'}{X}$ and $\dfrac{Y'}{Y}$ are nonzero constants and we set $\dfrac{X'}{X} = c$ and $\dfrac{Y'}{Y} = \dfrac{1}{c}$. This gives two *separable* o.d.e., the first of which is $\dfrac{X'}{X} = c$. Integrating this yields

$$\begin{aligned} \ln|X| &= cx + k_1 \\ \text{i.e.} \qquad X &= \pm e^{k_1} e^{cx}. \end{aligned}$$

Similarly, integrating the second *separable* o.d.e. $\dfrac{Y'}{Y} = \dfrac{1}{c}$ gives

$$\begin{aligned} \ln|Y| &= \frac{y}{c} + k_2 \\ \text{i.e.} \qquad Y &= \pm e^{k_2} e^{y/c}. \end{aligned}$$

Thus,

$$\begin{aligned} u(x,y) &= XY = \pm e^{k_1} e^{cx} e^{k_2} e^{y/c} \\ &= k e^{cx+y/c}. \end{aligned}$$

(d) Setting $u(x,y) = X(x)Y(y)$, the p.d.e. $xu_{xy} + 2yu = 0$ becomes $xX'Y' + 2yXY = 0$. Dividing by $-2yXY$ gives

$$\left(x\frac{X'}{X}\right)\left(-\frac{1}{2y}\cdot\frac{Y'}{Y}\right) = 1.$$

This implies that both $x\dfrac{X'}{X}$ and $\dfrac{-1}{2y}\cdot\dfrac{Y'}{Y}$ are nonzero constants. We set $x\dfrac{X'}{X} = c$ and $\dfrac{-1}{2y}\cdot\dfrac{Y'}{Y} = \dfrac{1}{c}$, which respectively give two *separable* o.d.e. $\dfrac{X'}{X} = \dfrac{c}{x}$ and $\dfrac{Y'}{Y} = -\dfrac{2y}{c}$. Integrating the first o.d.e. $\dfrac{X'}{X} = \dfrac{c}{x}$ yields

$$\begin{aligned} \ln|X| &= c\ln x + k_1 \\ \text{i.e.} \qquad X &= \pm e^{k_1} x^c. \end{aligned}$$

Similarly, integrating the second *separable* o.d.e. $\dfrac{Y'}{Y} = -\dfrac{2y}{c}$ gives

$$\begin{aligned} \ln|Y| &= -\frac{y^2}{c} + k_2 \\ \text{i.e.} \qquad Y &= \pm e^{k_2} e^{-y^2/c}. \end{aligned}$$

Thus,

$$\begin{aligned} u(x,y) &= XY = \pm e^{k_1} x^c e^{k_2} e^{-y^2/c} \\ &= k x^c e^{-y^2/c}. \end{aligned}$$