**2006/2007 SEMESTER 2 MID-TERM TEST**

**MA1506 MATHEMATICS II**

**February 26, 2007**

**SESSION 2 : 7:30 - 8:30pm**

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**PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY:**

1. This test paper consists of **TEN (10)** multiple choice questions and comprises **Twelve (12)** printed pages.
2. Answer all 10 questions. 1 mark for each correct answer. No penalty for wrong answers. Full mark is 10.
3. All answers (Choices A, B, C, D, E) are to be submitted using the pink form (FORM CC1).
4. Use **only 2B pencils** for FORM CC1.
5. On FORM CC1 (section B), **write** your **matriculation number** and **shade** the corresponding numbered circles carefully. Your FORM CC1 will be graded by a computer and it will record a **ZERO** for your score if your matriculation number is not correct.
6. Write your full name in section A of FORM CC1.
7. Only circles for answers 1 to 10 are to be shaded.
8. For each answer, the circle corresponding to your choice should be properly and completely shaded. If you change your answer later, you must make sure that the original answer is properly erased.
9. For each answer, **do not shade more than one circle**. The answer for a question with more than one circle shaded will be marked wrong.
10. **Do not fold** FORM CC1.
11. Submit FORM CC1 before you leave the test hall.

# Formulae Sheet

1. Integrating factor for $y'+Py=Q$ is given by

$$R=\exp\left(\int P\,dx\right).$$

2. The variation of parameters formulae for $y''+py'+qy=r$:

$$u=\int\frac{-ry_2}{y_1y_2'-y_2y_1'}dx$$

$$v=\int\frac{ry_1}{y_1y_2'-y_2y_1'}dx.$$

1. Let $f(x)$ be a solution of the differential equation $f'(x) = x + \sqrt{x}$ such that $f(1) = 1$. Then $f(4) =$

**(A)** $\frac{79}{6}$

**(B)** $\frac{46}{3}$

**(C)** $\frac{25}{2}$

**(D)** $13$

**(E)** $\frac{40}{3}$

2. Let $y$ be a solution of the differential equation

$$xy' = y + xe^{\frac{y}{x}},\ -e < x < 0,$$

such that

$$y(-1) = 0.$$

Then $y\,(-2) =$

**(A)** $\ln(1 - \ln 2)$

**(B)** $\ln(1 + \ln 2)$

**(C)** $2\ln(1 + \ln 2)$

**(D)** $2\ln(1 - \ln 2)$

**(E)** $2\ln(\ln 2)$

3. Let $y$ be a solution of the differential equation

$$xy' + 3y = \frac{\sin x}{x^2},\ x > 0,$$

such that

$$y(\frac{\pi}{2}) = 1.$$

Then $y\,(\pi) =$

**(A)** $\frac{1}{3} - \frac{1}{\pi^2}$

**(B)** $\frac{1}{\pi} + \frac{1}{\pi^3}$

**(C)** $\frac{1}{8} + \frac{1}{\pi^3}$

**(D)** $\frac{1}{3} + \frac{1}{\pi^3}$

**(E)** $\frac{1}{8} - \frac{1}{\pi^2}$

4. Let $y$ be a solution of the differential equation

$$2xyy' = y^2 - x^3,\ x > 0,$$

such that

$$y(1) = 0.$$

If $y\left(\frac{1}{2}\right) = a$, then $a$ satisfies the equation

**(A)** $a^2 = \frac{3}{16}$

**(B)** $a^2 = \frac{5}{16}$

**(C)** $a^2 = \frac{5}{8}$

**(D)** $a^2 = \frac{3}{8}$

**(E)** $a^2 = \frac{15}{16}$

5. At time $t = 0$ a piece of ice which has a shape of a perfect cube starts to melt. Suppose that the rate of reduction of the volume of this ice cube is proportional to its surface area. Suppose also that this ice cube retains its cubical shape as it melts. At $t = 1$ you observe that 25% of the ice cube's original volume is gone. You keep watching it and observe that at $t = T$ the ice cube disappears. What is the approximate value of $T$?

**(A)** 4

**(B)** 9

**(C)** 11

**(D)** 16

**(E)** 6

6. At time $t = 0$ hour, you started an experiment with $c$ gm of mold. At time $t = 6$ hours you found that you had 600 gm of mold. At time $t = 24$ hours you found that you had 1000 gm of mold. Assume that mold grows at a rate proportional to the amount present, what is the approximate value of $c$ ?

**(A)** 506

**(B)** 479

**(C)** 495

**(D)** 487

**(E)** 518

7. The solution of $y'' - 25y = 0$ with $y(0) = 1$ and $y'(0) = 0$ is

**(A)** $y = 2e^{5x} - e^{-5x}$

**(B)** $y = \frac{1}{2}e^{5x} + \frac{1}{2}e^{-5x}$

**(C)** $y = 1 + \sinh(5x)$

**(D)** $y = e^{5x} + \frac{1}{2}xe^{5x}$

**(E)** $y = \cos(5x) + \sin(5x)$

8. Let $y(t)$ be a solution of the differential equation

$$0.5y''(t) + 2.5y'(t) + 3y(t) = 0$$

such that

$$y(0) = 0, \quad y(-1) = -1.$$

Then $\lim_{t \to +\infty} y(t) =$

**(A)** $e^3$

**(B)** $\infty$

**(C)** $-\infty$

**(D)** $e^3 + e^2$

**(E)** $0$

9. Let $y(x)$ be a solution of $y'' - y' - 2y = xe^{2x}$, such that $y(0) = 2$, $y'(0) = \frac{8}{9}$. Then $y(1) =$

**(A)** $\frac{19}{18}e^2 + \frac{1}{e^2}$

**(B)** $\frac{19}{18}e^2 + \frac{1}{e}$

**(C)** $\frac{19}{18}e + \frac{1}{e}$

**(D)** $\frac{5}{8}e^2 - \frac{1}{e^2}$

**(E)** $\frac{9}{16}e + \frac{1}{e}$

10. Let $y(x)$ be a solution of the differential equation

$$y''(x) + 4y'(x) + 4y\,(x) = 1 + e^x$$

such that

$$y(0) = \frac{49}{36}, \quad y'(0) = \frac{1}{9}.$$

Then $y(-1) =$

**(A)** $\frac{1}{4} - e^2 + \frac{1}{9e}$

**(B)** $\frac{1}{4} - e^2 - \frac{1}{9e}$

**(C)** $\frac{1}{4} + e^2 + \frac{1}{9e}$

**(D)** $-\frac{1}{4} + e^2 - \frac{1}{9e}$

**(E)** $\frac{1}{4} + e^2 - \frac{1}{9e}$

END OF PAPER

Answers to mid term test for session 2

1. A
2. D
3. C
4. A
5. C
6. A
7. B
8. E
9. B
10. A

# Session 2

1). $f'(x) = x + \sqrt{x} \Rightarrow f(x) = \frac{1}{2}x^2 + \frac{2}{3}x^{3/2} + C$

$f(1) = 1 \Rightarrow 1 = \frac{1}{2} + \frac{2}{3} + C \Rightarrow C = -\frac{1}{6}$

$\Rightarrow f(x) = \frac{1}{2}x^2 + \frac{2}{3}x^{3/2} - \frac{1}{6}$

$f(4) = 8 + \frac{16}{3} - \frac{1}{6} = \underline{\underline{\frac{79}{6}}}$

2). $y = vx \Rightarrow y' = v'x + v$

$v'x + v = v + e^v$

$v'x = e^v$

$e^{-v}dv = \frac{dx}{x}$

$-e^{-v} = \ln|x| + C$

$e^{-\frac{y}{x}} = C - \ln|x|$

$y(-1) = 0 \Rightarrow 1 = C$

$\Rightarrow e^{-\frac{y}{x}} = 1 - \ln|x|$

$\Rightarrow -\frac{y}{x} = \ln\left(1 - \ln|x|\right)$

$\Rightarrow y = -x\ln(1 - \ln|x|)$

$\therefore y(-2) = \underline{\underline{2\ln(1 - \ln 2)}}$

3). $xy' + 3y = \frac{\sin x}{x^2} \Rightarrow y' + \frac{3}{x} y = \frac{\sin x}{x^3}$

Integrating factor $= e^{\int \frac{3}{x} dx} = e^{3 \ln x} = x^3$

$\therefore y = \frac{1}{x^3} \int x^3 \frac{\sin x}{x^3} dx = \frac{1}{x^3} (-\cos x + C)$

$y(\frac{\pi}{2}) = 1 \Rightarrow 1 = \frac{8}{\pi^3} (0 + C) \Rightarrow C = \frac{\pi^3}{8}$

$\therefore y = \frac{1}{x^3} \left( \frac{\pi^3}{8} - \cos x \right)$

$\therefore y(\pi) = \frac{1}{\pi^3} \left( \frac{\pi^3}{8} + 1 \right) = \underline{\underline{\frac{1}{8} + \frac{1}{\pi^3}}}$

4). $y' - \frac{y}{2x} = -\frac{1}{2} x^2 y^{-1}$

Let $z = y^{1-(-1)} = y^2 \Rightarrow dz = 2y\,dy$

$\Rightarrow \frac{dz}{2y\,dx} - \frac{1}{2x} y = -\frac{1}{2} x^2 y^{-1} \Rightarrow \frac{dz}{dx} - \frac{1}{x} z = -x^2$

Integrating factor $= e^{\int -\frac{1}{x} dx} = e^{-\ln x} = \frac{1}{x}$

$\therefore z = x \int \frac{1}{x} (-x^2) dx = x \left( -\frac{1}{2} x^2 + C \right)$

$\therefore y^2 = -\frac{1}{2} x^3 + Cx$

$y(1) = 0 \Rightarrow 0 = -\frac{1}{2} + C \Rightarrow C = \frac{1}{2}$

$\therefore y^2 = -\frac{1}{2} x^3 + \frac{1}{2} x$

$y(\frac{1}{2}) = a \Rightarrow a^2 = -\frac{1}{2} \left( \frac{1}{8} \right) + \frac{1}{2} \left( \frac{1}{2} \right) = \underline{\underline{\frac{3}{16}}}$

5). Let $x$ = length of one side at time $t$.

$\therefore V = x^3, \quad A = 6x^2$

$$\frac{dV}{dt} = 3x^2\frac{dx}{dt} = kA = 6kx^2 \Rightarrow \frac{dx}{dt} = 2k \Rightarrow x = 2kt + C$$

Let $x(0) = a \Rightarrow a = C \Rightarrow x = 2kt + a$.

at $t=1$, $V = 75\%$ of $a^3 = \frac{3}{4}a^3 \Rightarrow x = \left(\frac{3}{4}\right)^{1/3} a$

$$\therefore \left(\frac{3}{4}\right)^{1/3} a = 2k + a \Rightarrow k = \frac{1}{2}\left\{\left(\frac{3}{4}\right)^{1/3} a - a\right\}$$

$$\therefore x(T) = 0 \Rightarrow 0 = 2kT + a \Rightarrow T = -\frac{a}{2k}$$

$$\therefore T = \frac{1}{1-\left(\frac{3}{4}\right)^{1/3}} \approx \underline{\underline{11}}$$

6). $\frac{dM}{dt} = kM \Rightarrow M = Ae^{kt}$

$M(0) = C \Rightarrow C = A \Rightarrow M = Ce^{kt}$

$M(6) = 600 \Rightarrow 600 = Ce^{6k} \Rightarrow (600)^4 = C^4 e^{24k}$

$M(24) = 1000 \Rightarrow 1000 = Ce^{24k}$

$$\therefore C^3 = \frac{(600)^4}{1000} \approx \underline{\underline{506}}$$

7). $y'' - 25y = 0 \Rightarrow \lambda^2 - 25 = 0 \Rightarrow \lambda = \pm 5$

$\therefore y = Ae^{5x} + Be^{-5x}$

$\therefore y' = 5Ae^{5x} - 5Be^{-5x}$

$$\left.\begin{array}{l} y(0) = 1 \Rightarrow 1 = A + B \\ y'(0) = 0 \Rightarrow 0 = A - B \end{array}\right\} \Rightarrow A = \tfrac{1}{2},\ B = \tfrac{1}{2}$$

$\therefore \underline{\underline{y = \tfrac{1}{2}e^{5x} + \tfrac{1}{2}e^{-5x}}}$

8). $0.5y'' + 2.5y' + 3y = 0 \Rightarrow 0.5\lambda^2 + 2.5\lambda + 3 = 0$

$\Rightarrow \lambda^2 + 5\lambda + 6 = 0 \Rightarrow \lambda = -2, -3.$

$\therefore y = Ae^{-2t} + Be^{-3t}$

$\therefore \lim_{t \to \infty} y = \underline{\underline{0}}$

9). $y''-y'-2y=0 \Rightarrow \lambda^2-\lambda-2=0 \Rightarrow \lambda=-1,\ 2.$

For $y''-y'-2y=xe^{2x}$

let $y=ue^{2x}$

$y'=u'e^{2x}+2ue^{2x}$

$y''=u''e^{2x}+4u'e^{2x}+4ue^{2x}$

$\therefore u''e^{2x}+4u'e^{2x}+4ue^{2x}-u'e^{2x}-2ue^{2x}-2ue^{2x}=xe^{2x}$

$u''+3u'=x$

let $u=Ax^2+Bx+C$

$u'=2Ax+B$

$u''=2A$

$\therefore 2A+6Ax+3B=x \Rightarrow A=\frac{1}{6},\ B=-\frac{1}{9}$

$\therefore$ General solution of $y''-y'-2y=xe^{2x}$ is

$y=Ae^{-x}+Be^{2x}+(\frac{1}{6}x^2-\frac{1}{9}x)e^{2x}$

$\therefore y'=-Ae^{-x}+2Be^{2x}+(\frac{1}{3}x-\frac{1}{9})e^{2x}+2(\frac{1}{6}x^2-\frac{1}{9}x)e^{2x}$

$$\therefore \left.\begin{aligned} &y(0)=2 \Rightarrow A+B=2 \\ &y'(0)=\frac{8}{9} \Rightarrow -A+2B-\frac{1}{9}=\frac{8}{9} \end{aligned}\right\} \Rightarrow A=1,\ B=1$$

$\therefore y=e^{-x}+e^{2x}+(\frac{1}{6}x^2-\frac{1}{9}x)e^{2x}$

$\therefore y(1)=e^{-1}+e^2+\frac{1}{18}e^2=\underline{\underline{\frac{19}{18}e^2+\frac{1}{e}}}$

10). $y''+4y'+4y=0 \Rightarrow \lambda^2+4\lambda+4=0 \Rightarrow \lambda=-2$ double root.

For $y''+4y'+4y=1+e^x$ ------ ①

let $y=A+Be^x$

$y'=Be^x$

$y''=Be^x$

$\therefore Be^x+4Be^x+4A+4Be^x=1+e^x \Rightarrow A=\frac{1}{4},\ B=\frac{1}{9}$

$\therefore$ General solution of ① is

$$y=Ae^{-2x}+Bxe^{-2x}+\frac{1}{4}+\frac{1}{9}e^x.$$

$\therefore y'=-2Ae^{-2x}+Be^{-2x}-2Bxe^{-2x}+\frac{1}{9}e^x$

$y(0)=\frac{49}{36} \Rightarrow A+\frac{1}{4}+\frac{1}{9}=\frac{49}{36} \Rightarrow A=1$

$y'(0)=\frac{1}{9} \Rightarrow -2A+B+\frac{1}{9}=\frac{1}{9} \Rightarrow B=2A=2$

$\therefore y=e^{-2x}+2xe^{-2x}+\frac{1}{4}+\frac{1}{9}e^x$

$\therefore y(-1)=e^2-2e^2+\frac{1}{4}+\frac{1}{9}e^{-1}$

$$=\frac{1}{4}-e^2+\frac{1}{9e}$$