NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2004-2005

MA1506 MATHEMATICS II

April/May 2005 Time allowed: 2 hours 30 minutes

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided below.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **TEN (10)** questions and comprises **FORTY ONE (41)** printed pages.
3. Answer **ALL** questions. Write your answers and working in the spaces provided inside the booklet following each question.
4. The marks for each question are indicated at the beginning of the question.
5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Matriculation Number:

For official use only. Do not write below this line.

Question	1	2	3	4	5	6	7	8	9	10
Marks										

Answer **all** *the questions.*

Question 1 [10 marks]

Let S be the surface given by the equation

$$\frac{e^{xy}}{1+z^2} = \frac{1}{2}$$

and T_P the tangent plane of S at the point $P(0,2,1)$.

(i) Find a cartesian equation of T_P.

(ii) If the curve C is the intersection of T_P and the cylinder $x^2+y^2=1$, find a vector equation

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

for C.

Show your working below and on the next three pages.

(Working spaces for Question 1 - Indicate your parts clearly)

(More working spaces for Question 1)

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Question 2 [**10 marks**]

A chemical plant uses two types of chemicals, chemical **X** and chemical **Y**. The number of kilogrammes of toxic waste produced by the plant in a year is given by

$$W(x, y) = x^2 + 2y^2 - xy - 250000,$$

where x is the amount of chemical **X** in thousands of kilogrammes used annually and y is the amount of chemical **Y** in thousands of kilogrammes used annually.
If the plant uses a combined amount of 1600000 kilogrammes of chemicals **X** and **Y** in a year, use the method of **Lagrange multipliers** to determine the amount (in kilogrammes) of each chemical that should be used annually in order to minimize the amount of toxic waste produced.

Show your working below and on the next three pages.

(Working spaces for Question 2 - Indicate your parts clearly)

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Question 3 **[10 marks]**

(a) Let a and b be positive constants such that $a \geq b$. Let D be the solid region within the cylinder $x^2 + y^2 = b^2$ between the planes $y + z = a$ and $z = 0$. Express the volume of D as a triple integral and find the volume by evaluating this integral.

(Leave your answer in terms of a and b.)

(b) Evaluate the iterated integral $\int_0^6 \int_{x/3}^2 x\sqrt{y^3+1}\, dydx$.

Show your working below and on the next three pages.

(Working spaces for Question 3 - Indicate your parts clearly)

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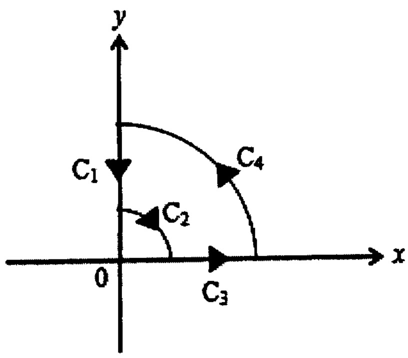
Question 4 [10 marks]

A closed curve with positive orientation is made up of four curves C_1, C_2, C_3 and C_4 as shown in the diagram below.



C_2 is the portion of the unit circle $x^2 + y^2 = 1$ that lies in the first quadrant. C_4 is the portion of the circle $x^2 + y^2 = 9$ that lies in the first quadrant. Evaluate the following line integrals:

(a) $\displaystyle\oint_{C_1+C_2+C_3+C_4} \left(\frac{-y}{x^2+y^2} + y^2\right)\, dx + \left(\frac{x}{x^2+y^2} - x^2\right)\, dy,$

(b) $\displaystyle\int_{C_1+C_2+C_3} \left(\frac{-y}{x^2+y^2} + y^2\right)\, dx + \left(\frac{x}{x^2+y^2} - x^2\right)\, dy.$

Show your working below and on the next three pages.

(Working spaces for Question 4 - Indicate your parts clearly)

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Question 5 [10 marks]

Let S be the portion of the unit sphere $x^2 + y^2 + z^2 = 1$ in the first octant and let C be the boundary of S. The orientation of C is counterclockwise when looking down at the surface S. Find a vector field $\mathbf{G}(x, y, z)$ such that

$$\oint_C x^2\, dx + 2xy\, dy + xz\, dz = \iint_S \mathbf{G}(x, y, z) \bullet d\mathbf{S},$$

and evaluate the surface integral $\iint_S \mathbf{G}(x, y, z) \bullet d\mathbf{S}$ directly.

Show your working below and on the next three pages.

(Working spaces for Question 5 - Indicate your parts clearly)

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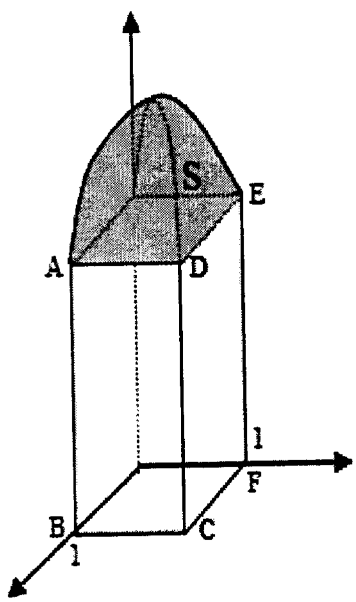
Question 6 [10 marks]

Let Ω be a solid region with the top bounded by some smooth surface S as shown in the diagram below. The base of Ω is the unit square $(0 \le x \le 1,\ \ 0 \le y \le 1)$ in the xy-plane.



Let $\mathbf{F}$ be the vector field given by

$$\mathbf{F} = x\mathbf{i} - 4y\mathbf{j} + (3z + 7)\mathbf{k}.$$

Suppose the outward flux, i.e. surface integral of $\mathbf{F}$ with normal vector pointing away from Ω, through the side $ABCD$ parallel to the yz-plane is 1, and the outward flux through the side $CDEF$ parallel to the xz-plane is -5. What is the outward flux through the top surface S?

Show your working below and on the next three pages.

(Working spaces for Question 6 - Indicate your parts clearly)

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Question 7 **[10 marks]**

Let R be a two-dimensional rectangular metal plate placed on the xy-plane such that

$$0 \leq x \leq \pi, \quad 0 \leq y \leq 2.$$

Let $u(x, y)$ be the steady-state temperature of R such that

$$u(x,0) = \sin x, \quad u(x,2) = \sin x, \quad u(0,y) = 0, \quad u(\pi, y) = 0.$$

(i) Find $u(x, y)$ by solving the Dirichlet boundary problem.

[You may use any known formulas from the lecture notes.]

(ii) Find the stationary point(s) of $u(x, y)$ in R.

[Recall that $\sinh x = \frac{1}{2}(e^x - e^{-x})$.]

Show your working below and on the next three pages.

(Working spaces for Question 7 - Indicate your parts clearly)

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Question 8 [10 marks]

Let $L(w) = W(x, s)$ be the Laplace transform of $w(x, t)$ $(t \geq 0)$ with respect to t.

(a) Suppose $W(x, s)$ satisfies the ODE (by regarding s as a constant)

$$W_x + sW = 1 \quad \text{with} \quad W(0, s) = 0.$$

Find $w(x, t)$.

[Recall the formulas

$L(1) = \dfrac{1}{s}$ and $L(u(t-a)) = \dfrac{e^{-as}}{s}$ where $u(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$.]

(b) Suppose $w(x, 0) = x^2$, $w_t(x, 0) = x + x^2$, $w_{tt}(x, 0) = x^2$.

Find $L(w_{ttt})$ in terms of W, x and s.

[Note: Part (b) is independent of part (a).
You may use the formula $L(w_t) = sL(w) - w(x, 0)$ without deriving it.]

Show your working below and on the next three pages.

(Working spaces for Question 8 - Indicate your parts clearly)

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Question 9 **[10 marks]**

(a) The following is the reduced row echelon form of the augmented matrix of a linear system which has infinitely many solutions.

$$\left[\begin{array}{ccccc|c} 1 & 2 & a & 3 & 0 & 2 \\ 0 & 0 & b & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & c \end{array}\right]$$

(i) Find the values of a, b and c.

(ii) Find the solution set of the linear system.

(b) Consider the following set of vectors in $\mathbf{R}^4$:

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 3 \\ a \end{pmatrix} \right\}.$$

Find all possible values of a so that the set is linearly independent.

Show your working below and on the next three pages.

(Working spaces for Question 9 - Indicate your parts clearly)

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Question 10 **[10 marks]**

(i) Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 8 \\ 5 \\ 2 \end{bmatrix}$$

and

$$A = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{bmatrix}.$$

By left multiplying A to each of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, show that the three vectors are eigenvectors of A.

What are the corresponding eigenvalues for the three eigenvectors?

(ii) Use the findings in part (i) to solve the linear system

$$y_1' = 2y_1 + 2y_2 + 3y_3, \qquad y_2' = y_1 + 2y_2 + y_3, \qquad y_3' = 2y_1 - 2y_2 + y_3$$

given that $y_1(0) = 9, \quad y_2(0) = 2, \quad y_3(0) = 1.$

Show your working below and on the next three pages.

(Working spaces for Question 10 - Indicate your parts clearly)

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