To find: vol. of intersection of two cylinders

$$x^2+y^2=a^2$$

$$y^2+z^2=a^2$$

using calculation by polar coordinates.

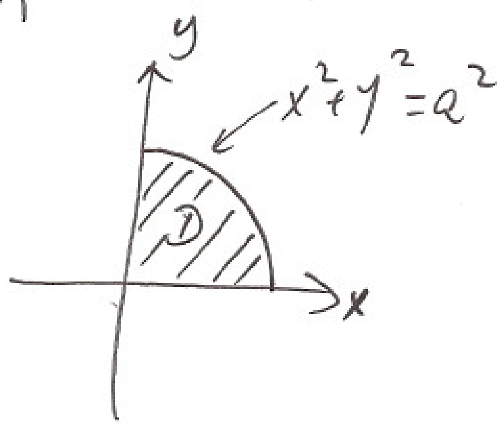
(compare: Tutorial 7, Q.7)

(Note: In Tutorial 7, Q.7, the cylinders are given as $x^2+y^2=r^2$, $y^2+z^2=r^2$. Here I use a instead of r so that it won't mix up in the polar coordinates.)

Solution: As in Tutorial 7, Q.7, the vol. is equal to 8 times the integral

$$\iint_D \sqrt{a^2-y^2}\, dA$$

where D is:



Now we change to polar coordinates:

$$\iint_D \sqrt{a^2-y^2}\,dA = \int_0^{\frac{\pi}{2}}\left\{\int_0^a \sqrt{a^2-r^2\sin^2\theta}\; r\,dr\right\}d\theta$$

$$= \int_0^{\frac{\pi}{2}}\left\{\int_0^a \frac{1}{-2\sin^2\theta}\sqrt{a^2-r^2\sin^2\theta}\; d(a^2-r^2\sin^2\theta)\right\}d\theta$$

$$= \int_0^{\frac{\pi}{2}}\left[-\frac{1}{2\sin^2\theta}\,\frac{2}{3}\left(a^2-r^2\sin^2\theta\right)^{3/2}\right]_{r=0}^{r=a} d\theta$$

$$= \int_0^{\frac{\pi}{2}} -\frac{1}{3\sin^2\theta}\left(a^3\cos^3\theta - a^3\right)d\theta$$

$$= \frac{a^3}{3}\int_0^{\frac{\pi}{2}} \frac{1-\cos^3\theta}{\sin^2\theta}\,d\theta$$

$$= \frac{a^3}{3}\int_0^{\frac{\pi}{2}} \frac{(1-\cos\theta)(1+\cos\theta+\cos^2\theta)}{(1-\cos\theta)(1+\cos\theta)}\,d\theta$$

$$= \frac{a^3}{3}\int_0^{\frac{\pi}{2}} \frac{1+\cos\theta+\cos^2\theta}{1+\cos\theta}\,d\theta$$

$$= \frac{a^3}{3}\int_0^{\frac{\pi}{2}} \frac{1+\cos\theta(1+\cos\theta)}{1+\cos\theta}\,d\theta$$

$$= \frac{a^3}{3}\int_0^{\frac{\pi}{2}} \left(\frac{1}{1+\cos\theta} + \cos\theta\right) d\theta$$

$$= \frac{a^3}{3}\int_0^{\frac{\pi}{2}} \left(\frac{1}{2\cos^2\frac{\theta}{2}} + \cos\theta\right) d\theta$$

$$= \frac{a^3}{3}\int_0^{\frac{\pi}{2}} \left(\frac{1}{2}\sec^2\frac{\theta}{2} + \cos\theta\right) d\theta$$

$$= \frac{a^3}{3}\left[\tan\frac{\theta}{2} + \sin\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{2a^3}{3}$$

$$\text{vol.} = 8\left(\frac{2a^3}{3}\right) = \underline{\underline{\frac{16a^3}{3}}}$$