Matriculation Number:

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2007-2008

MA1506 MATHEMATICS II

April 2008 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided above.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **EIGHT (8)** questions and comprises **THIRTY THREE (33)** printed pages.
3. Answer **ALL** questions. For each question, write your answer in the box and your working in the space provided inside the booklet following that question.
4. The marks for each question are indicated at the beginning of the question.
5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

For official use only. Do not write below this line.

Question	1	2	3	4	5	6	7	8
Marks								

Question 1 (a) [5 marks]

Let $y > 0$ be a solution of the differential equation

$$\frac{dy}{dx} = \frac{3x^2 + 1}{2y}$$

with the initial condition $y(0) = 1$. Find the value of $y(1)$.

Answer 1(a)	

(Show your working below and on the next page.)

(More working space for Question 1(a))

Question 1 (b) [5 marks]

An old object was dug up near NUS and you were asked to help to find out how old is this object. You carried out an experiment and found that the object contained 95% of the carbon-14 found in a similar present-day sample. You looked up a table from your chemistry book and found that the half-life of carbon-14 is 5730 years (i.e. it takes 5730 years for 50% of an amount of carbon-14 to decay away). Assume that carbon-14 decays at a rate proportional to the amount present, approximately how many years old is this object?

Answer 1(b)	

(Show your working below and on the next page.)

(More working space for Question 1(b))

Question 2 (a) [5 marks]

Solve the differential equation

$$y'' + 4y = \cos^2 x - \sin^2 x$$

with the initial conditions

$$y(0) = 1, \quad y'(0) = 2.$$

Answer 2(a)	

(Show your working below and on the next page.)

(More working space for Question 2(a))

Question 2 (b) [5 marks]

A particle moves along the x-axis in accordance with the equation of motion

$$\ddot{x} + 6\dot{x} - 16x = 0.$$

At $t = 0$ sec, the particle is at $x = 2$ m and moving to the left with a velocity of 10 m/sec. When will the particle change direction and go to the right?

Answer 2(b)	

(Show your working below and on the next page.)

(More working space for Question 2(b))

Question 3 (a) [5 marks]

A psychologist used the equation

$$\frac{dP}{dt} = \frac{1}{1+t^2}(M - 2tP)$$

to model the performance of a certain student. Here, P denotes the student's performance at any time $t \geq 0$ and M denotes a positive constant. Assume that $P = 0$ at time $t = 0$. What is the value of t when P first reaches 40% of M ?

Answer 3(a)	

(Show your working below and on the next page.)

(More working space for Question 3(a))

Question 3 (b) [5 marks]

A certain bird population has a birth rate per capita of 10% per year. They had been protected by law for many years and attained a logistic equilibrium of 100000 birds. The government then allowed people to shoot E birds per year and after a long time, the population settled down to a new equilibrium of 68000 birds. Find the value of E.

Answer 3(b)	

(Show your working below and on the next page.)

(More working space for Question 3(b))

Question 4 (a) [5 marks]

A cantilevered beam of length L, made up of an extremely strong and light material, is horizontal at the end where it is attached to a wall, and carries a load of P Newtons at its end (that is, at $x = L$.) Assuming the weight of the beam is negligible compared to P, the beam has a moment function given by

$$M(x) = -(L - x)P.$$

Find the maximum deflection at $x = L$. (You may wish to recall the formula $EI = M / \frac{d^2y}{dx^2}$.)

Answer 4(a)	

(Show your working below and on the next page.)

(More working space for Question 4(a))

Question 4 (b) [5 marks]

Let $g(t)$ be defined as

$$g(t) = \begin{cases} t^2, & 0 \le t < 2 \\ 4, & 2 \le t < 4 \\ 0, & t \ge 4. \end{cases}$$

Sketch $g(t)$ and compute its Laplace transform, $G(s)$, at $s = 4$.

Answer 4(b)	

(Show your working below and on the next page.)

(More working space for Question 4(b))

Question 5 (a) [5 marks]

Find the inverse Laplace transform of $\dfrac{1}{(s-1)(s^2+9)}$.

Answer 5(a)	

(Show your working below and on the next page.)

(More working space for Question 5(a))

Question 5 (b) [5 marks]

Solve the following initial value problem

$$y'' + 2\pi y' + 4\pi^2 y = \delta(t-1)$$

with initial conditions $y(0) = 1$ and $y'(0) = -\pi$. Evaluate $y(2)$.

Answer 5(b)	

(Show your working below and on the next page.)

(More working space for Question 5(b))

Question 6 (a) [5 marks]

Consider the following system of differential equations.

$$
\begin{aligned}
\frac{dx}{dt} &= 2x + y, \qquad x(0) = 0, \\
\frac{dy}{dt} &= x - 2y, \qquad y(0) = 1.
\end{aligned}
$$

Find the solution for $x(t)$ using the Laplace transform.

Answer 6(a)	

(Show your working below and on the next page.)

(More working space for Question 6(a))

Question 6 (b) [5 marks]

The weather of a typical day in Antarctica can be classified into a normal day, a cold day and an extremely cold day. There is a 10% chance that a normal day turns into a cold day and a 10% chance that a normal day turns into an extremely cold day. The probability that a cold day remains a cold day is 70% and the probability that a cold day turns into a normal day is x%. An extremely cold day has a 90% chance of staying extremely cold and it is impossible for an extremely cold day to turn normal.

If today is extremely cold and the probability that three days from now is still extremely cold is 75.6%, find x.

Answer 6(b)	

(Show your working below and on the next page.)

(More working space for Question 6(b))

Question 7(a) [5 marks]

Given that the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

has three distinct eigenvectors which form a basis in three dimensions, and given that two of its eigenvalues are 0 and $\frac{1}{2}[3\sqrt{33}+15]$, find the third eigenvalue. Find also an eigenvector corresponding to the eigenvalue 0.

Answer 7(a)	

(Show your working below and on the next page.)

(More working space for Question 7(a))

Question 7 (b) [5 marks]

A bioengineer studies the interactions of two kinds of bacteria in a particular culture. Bacterium A feeds on bacterium B and depends on it for its food, while bacterium B depends only on sunlight. The bioengineer checks the numbers of A and B every hour; let A_k and B_k be these numbers, measured in millions, in the k-th hour. His model of the situation is given by the following equations:

$$A_{k+1} = \frac{A_k}{2} + \frac{B_k}{100}, \quad B_{k+1} = -\frac{50A_k}{4} + \frac{5B_k}{4}.$$

Initially there are 50 million of type A and 5000 million of type B. By diagonalizing a matrix, compute how many bacteria of type A the model predicts there will be in four hours. [NOTE: zero marks if you don't diagonalize the matrix.]

Answer 7(b)	

(Show your working below and on the next page.)

(More working space for Question 7(b))

Question 8 (a) [5 marks]

A chemical engineer has two tanks containing 100 litres of water. Tank A initially contains water in which 25 kg of a dangerous chemical are dissolved, and tank B contains x kilograms of this chemical. Pure water is poured into tank A at a constant rate of 4 litres per minute. The thoroughly mixed solution from tank A is constantly pumped into tank B at a rate of 6 litres per minute, while the solution from tank B is pumped back to tank A at a rate of 2 litres per minute. The solution in tank B is also pumped out and discarded at a rate of 4 litres per minute. The engineer wants to choose x in such a way that the ratio of the amount of the chemical in tank A to the amount in tank B is constant. Find x.

Answer 8(a)	

(Show your working below and on the next page.)

(More working space for Question 8(a))

Question 8(b) [5 marks]

Classify the systems of linear ordinary differential equations with the following coefficient matrices:

$$(a)\begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}, (b)\begin{pmatrix} 2 & -2 \\ 7 & 0 \end{pmatrix}, (c)\begin{pmatrix} -2 & -8 \\ 5 & 0 \end{pmatrix}, (d)\begin{pmatrix} -6 & 4 \\ -2 & 1 \end{pmatrix}, (e)\begin{pmatrix} 7 & -4 \\ 2 & -1 \end{pmatrix}.$$

Answer 8(b)	

(Show your working below and on the next page.)

(More working space for Question 8(b))