Electrical & Computer Engineering Department
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EE2011 Engineering Electromagnetics
Tutorial 3: Field Operators

Q1(a) Find the spherical coordinates of the point P specified by (4, 120°, 3) in the cylindrical coordinate system.

only need to determine 2 coordinates when converting from (r,ϕ,z) to (R,θ,ϕ)

- $R = \left|\overrightarrow{OP}\right| = \sqrt{(r\cos\phi)^2 + (r\sin\phi)^2 + z^2} = \sqrt{2^2 + \left(2\sqrt{3}\right)^2 + 3^2} = 5$
- $\cos\theta = \sqrt{\frac{z}{R}} = \sqrt{\frac{3}{5}} \quad \Rightarrow \quad \theta = 53.1^o$
- change position of (unchanged) azimuthal coordinate

$\therefore$ spherical coordinates of P given by (5, 53.1°, 120°)

Q1(b) Determine the angle between the vectors $\vec{E}$ and $\vec{B}$ at the point P where

$\vec{E} = \frac{25}{R^2}\hat{u}_R$	expressed in spherical coordinates
$\vec{B} = 2\hat{u}_x - 2\hat{u}_y + \hat{u}_z$	expressed in Cartesian coordinates
$\overrightarrow{OP} = -3\hat{u}_x + 4\hat{u}_y - 5\hat{u}_z$	expressed in Cartesian coordinates.

need first to find denominator of $\vec{E}$ at P (–3, 4, –5)

$$R^2 = x^2 + y^2 + z^2 = (-3)^2 + 4^2 + (-5)^2 = 50$$

more convenient to re-write $\vec{E} = \frac{1}{2}\hat{u}_R$ in Cartesian coordinate format

$\therefore$ have to find direction of $\vec{E}$ which is parallel to $\overrightarrow{OP} = x\hat{u}_x + y\hat{u}_y + z\hat{u}_z$

- find angle θ between z-axis and $x\hat{u}_x + y\hat{u}_y + z\hat{u}_z$

$$\cos\theta = \frac{z}{R} = \frac{-5}{5\sqrt{2}} = \frac{-1}{\sqrt{2}} \quad \Rightarrow \quad \theta = 135^o$$

- find angle ϕ between x-axis and $x\hat{u}_x + y\hat{u}_y$

$$\cos\phi = \frac{x}{\sqrt{x^2+y^2}} = \frac{-3}{5} \quad \Rightarrow \quad \phi = 127^o$$

can now decompose $\vec{E} = \frac{1}{2}\,\hat{u}_R$ into components along x, y and z directions

$$E_x = E_R \sin\theta \cos\phi = \tfrac{1}{2}\tfrac{1}{\sqrt{2}}\left(-\tfrac{3}{5}\right) = -\tfrac{3}{10\sqrt{2}}$$

$$E_y = E_R \sin\theta \sin\phi = \tfrac{1}{2}\tfrac{1}{\sqrt{2}}\tfrac{4}{5} = \tfrac{2}{5\sqrt{2}}$$

$$E_z = E_R \cos\theta = \tfrac{1}{2}\left(-\tfrac{1}{\sqrt{2}}\right) = -\tfrac{1}{2\sqrt{2}}$$

finally use dot product to find angle ψ between vectors $\vec{E}$ and $\vec{B}$

$$\cos\psi = \frac{\vec{E}}{|\vec{E}|} \bullet \frac{\vec{B}}{|\vec{B}|} = \frac{E_x B_x + E_y B_y + E_z B_z}{\sqrt{E_x^2+E_y^2+E_z^2}\,\sqrt{B_x^2+B_y^2+B_z^2}} = -0.8957$$

$$\Rightarrow \quad \psi = 153.6^{\circ}$$

Q1(c) For the vector function $\vec{E} = y\hat{u}_x + x\hat{u}_y$, evaluate the scalar line integral $\int_P^Q \vec{E}\bullet d\vec{s}$ from P (2, 1, –1) to Q (8, 2, –1) along the parabolic contour $x = 2\,y^2$ on the $z = -1$ plane.

reduce integral to one variable along parabolic contour

- need to substitute $x = 2\,y^2$
- need to substitute $dx = 4\,y\,dy$

$$\begin{aligned}
\int_P^Q \vec{E}\bullet d\vec{s} &= \int_P^Q \left(y\hat{u}_x + x\hat{u}_y\right)\bullet\left(dx\,\hat{u}_x + dy\,\hat{u}_y\right) \\
&= \int_{y=1}^{y=2} \left(y\hat{u}_x + 2y^2\hat{u}_y\right)\bullet\left(4y\,dy\,\hat{u}_x + dy\,\hat{u}_y\right) \\
&= \int_{y=1}^{y=2} 6y^2\,dy \\
&= 14
\end{aligned}$$

Q2 Determine the following for the scalar function $V = \sin\left(\frac{\pi}{2}x\right)\sin\left(\frac{\pi}{3}y\right)e^{-z}$ **:**

(a) grad V at the point P $(1, 2, 3)$

(b) rate of increase of V at P in the direction of $\overrightarrow{PO}$ (*i.e.* towards the origin).

find components of grad V**:**

$$\frac{\partial V}{\partial x} = \frac{\pi}{2}\cos\left(\frac{\pi}{2}x\right)\sin\left(\frac{\pi}{3}y\right)e^{-z}$$

$$\frac{\partial V}{\partial y} = \frac{\pi}{3}\sin\left(\frac{\pi}{2}x\right)\cos\left(\frac{\pi}{3}y\right)e^{-z}$$

$$\frac{\partial V}{\partial x} = -\sin\left(\frac{\pi}{2}x\right)\sin\left(\frac{\pi}{3}y\right)e^{-z}$$

combining components and subsituting coordinate values**:**

$$\nabla V = \begin{pmatrix} \frac{\pi}{2}\cos\frac{\pi}{2}\sin\frac{2\pi}{3} \\ \frac{\pi}{3}\sin\frac{\pi}{2}\cos\frac{2\pi}{3} \\ \sin\frac{\pi}{2}\sin\frac{2\pi}{3} \end{pmatrix} e^{-3} = \begin{pmatrix} 0 \\ -0.026 \\ -0.043 \end{pmatrix}$$

need to find decompose ∇V to find component in direction of $\overrightarrow{PO}$ **:**

$$\text{rate of change towards origin } = \nabla V \bullet \frac{\overrightarrow{PO}}{\left|\overrightarrow{PO}\right|}$$

$$= \begin{pmatrix} 0 \\ -0.026 \\ -0.043 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} \frac{1}{\sqrt{1^2+2^2+3^2}}$$

$$= 0.049 \ \text{m}^{-1}$$

Q3 For the vector function $\vec{E} = y^2 z\,\hat{u}_x + y^3\hat{u}_y + xz\,\hat{u}_z$, verify that the Divergence Theorem holds for the cube enclosed by the plane surfaces S1 (where x = 1), S2 (where x = –1), S3 (where y = 1), S4 (where y = –1), S5 (where z = 2) and S6 (where z = 0).

evaluate contour integral in LHS**:**

$$\oiint \vec{E} \bullet d\vec{A} = \iint_{S1} \vec{E}\bullet\left(dy\,dz\,\hat{u}_x\right) + \iint_{S2} \vec{E}\bullet\left(-dy\,dz\,\hat{u}_x\right) +$$
$$\iint_{S3} \vec{E}\bullet\left(dx\,dz\,\hat{u}_y\right) + \iint_{S4} \vec{E}\bullet\left(-dx\,dz\,\hat{u}_y\right) +$$
$$\iint_{S5} \vec{E}\bullet\left(dx\,dy\,\hat{u}_z\right) + \iint_{S6} \vec{E}\bullet\left(-dx\,dy\,\hat{u}_z\right)$$

$$= \iint_{S1} y^2 z\,dy\,dz\Big|_{x=1} - \iint_{S2} y^2 z\,dy\,dz\Big|_{x=-1} +$$
$$\iint_{S3} y^3\,dx\,dz\Big|_{y=1} - \iint_{S4} y^3\,dx\,dz\Big|_{y=-1} +$$
$$\iint_{S5} xz\,dx\,dy\Big|_{z=2} - \iint_{S6} xz\,dx\,dy\Big|_{z=0}$$

$$= \int_{y=-1}^{y=1} y^2\,dy \int_{z=0}^{z=2} z\,dz - \int_{y=-1}^{y=1} y^2\,dy \int_{z=0}^{z=2} z\,dz +$$
$$\int_{x=-1}^{x=1} dx \int_{z=0}^{z=2} dz - (-1)^3 \int_{x=-1}^{x=1} dx \int_{z=0}^{z=2} dz +$$
$$2\int_{x=-1}^{x=1} x\,dx \int_{y=-1}^{y=1} dy + 0 \text{ (from S6 because of multiplication by } z=0\text{)}$$

$$= \tfrac{4}{3} - \tfrac{4}{3} + 4 + 4 + 0 + 0 = 8$$

evaluate area integral in RHS**:**

$$\iiint \nabla\bullet\vec{E}\,dV = \iiint\left(0 + 3y^2 + x\right)dx\,dy\,dz$$
$$= \int_{y=-1}^{y=1}\int_{x=-1}^{x=1}\left(x + 3y^2\right)dx\,dy \int_{z=0}^{z=2} dz$$
$$= 2\int_{y=-1}^{y=1} \left[\tfrac{1}{2}x^2 + 3y^2 x\right]_{x=-1}^{x=1} dy$$
$$= 2\int_{y=-1}^{y=1} \left(6y^2\right)dy = 8$$

Q4 For the vector function $\vec{B} = 3x^2y^3\hat{u}_x - x^3y^2\hat{u}_y$, verify that Stoke's Theorem holds for the triangular contour PQR where the Cartesian coordinates of the three vertices are given by P (2, 2, 0), Q (2, 1, 0) and R (1, 1, 0).

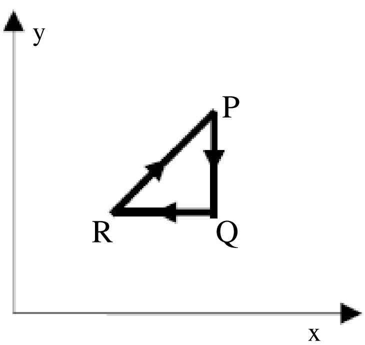
evaluate contour integral in LHS**:**

$$\oint \vec{B} \bullet d\vec{s} = \int_P^Q \vec{B} \bullet dy\,\hat{u}_y \Big|_{x=2} + \int_Q^R \vec{B} \bullet dx\,\hat{u}_x \Big|_{y=1} + \int_R^P \vec{B} \bullet \left(dx\,\hat{u}_x + dy\,\hat{u}_y\right)\Big|_{x=y}$$

$$= \int_P^Q \left(-x^3y^2\right)dy\Big|_{x=2} + \int_Q^R \left(3x^2y^3\right)dx\Big|_{y=1} + \int_R^P \begin{pmatrix} -x^3y^2 \\ 3x^2y^3 \end{pmatrix} \bullet \begin{pmatrix} dx \\ dy \end{pmatrix}\Bigg|_{x=y}$$

$$= -2^3\int_P^Q y^2\,dy + 3\int_Q^R x^2\,dx + 2\int_R^P x^5\,dx$$

$$= 32\tfrac{2}{3}$$



evaluate area integral in RHS**:**

$$\iint \nabla\times\vec{B} \bullet d\vec{A} = \iint \begin{vmatrix} \hat{u}_x & \hat{u}_y & \hat{u}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & 0 \end{vmatrix} \bullet \left(-dx\,dy\,\hat{u}_z\right)$$

$$= \iint \left\{\tfrac{\partial}{\partial x}\left(-x^3y^2\right) - \tfrac{\partial}{\partial y}\left(3x^2y^3\right)\right\}(-dx\,dy)$$

$$= \iint 12x^2y^2\,dx\,dy$$

$$= 12\int_{y=1}^{y=2} y^2 \left(\int_{x=y}^{x=2} x^2\,dx\right) dy$$

$$= 12\int_{y=1}^{y=2} y^2 \left[\tfrac{1}{3}x^3\right]_{x=y}^{x=2} dy$$

$$= 4\int_{y=1}^{y=2} y^2 \left[8 - y^3\right] dy$$

$$= 4\left[\tfrac{8}{3}y^3 - \tfrac{1}{6}y^6\right]_{y=1}^{y=2}$$

$$= 32\tfrac{2}{3}$$