

## 9 — MORPHOLOGICAL PROCESSING

---

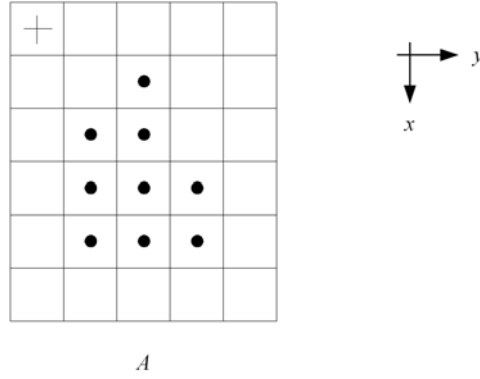
*Morphology* is a branch of biology that deals with the form and structure of animals and plants. In the image processing context, *mathematical morphology* is a tool

- for extracting image components that are useful in representation and description of region shapes, such as boundaries, skeletons and the convex hull
- for processing techniques such as morphological filtering, thinning and pruning.

The language of mathematical morphology is *set theory*. In this context, *sampling* partitions the  $xy$  plane into a grid, with the coordinates of the centre of each grid being a pair of elements from the 2D integer space  $Z \times Z$  (or  $Z^2$ ), which is the set of all ordered pairs of elements  $(a, b)$ ,  $a, b \in Z$ .

Sets in mathematical morphology represents the shapes of objects in an image. For example, the set of all black pixels in a binary image is a complete description of the image. In binary images, the sets in question are members  $Z^2$ , where each element of a set is a 2-tuple (or 2-D vector) whose coordinates are the  $x, y$  coordinates of a black (by convention) pixel in the image.

## Example



(+ denotes the origin)

$$A = \{(1, 2), (2, 1), (2, 2), (3, 1), (3, 2), (3, 3), (4, 1), (4, 2), (4, 3)\}$$

## PRELIMINARIES

Let  $A$  be a set in  $Z^2$ , with components

$$a = (a_1, a_2)$$

The empty set is denoted by  $\emptyset$ . If  $a$  is an element of set  $A$ , we write

$$a \in A \tag{1}$$

If every element of set  $A$  is also an element of another set  $B$ , then  $A$  is said to be a subset of  $B$ , denoted by

$$A \subseteq B \tag{2}$$

The *union* of two sets  $A$  and  $B$ , denoted by

$$C = A \cup B \tag{3}$$

is the set of all elements belonging to either  $A$ ,  $B$ , or both.

The *intersection* of two sets  $A$  and  $B$ , denoted by

$$D = A \cap B \quad (4)$$

is the set of all elements belonging to both  $A$  and  $B$ .

Two sets  $A$  and  $B$  are said to be *disjoint* or mutually exclusive if they have no common elements. In this case,

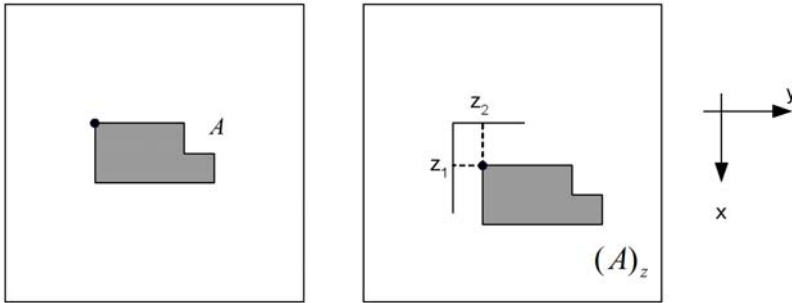
$$A \cap B = \emptyset \quad (5)$$

The *complement* of set  $A$  is the set of elements not contained in  $A$ :

$$A^c \equiv \{x | x \notin A\} \quad (6)$$

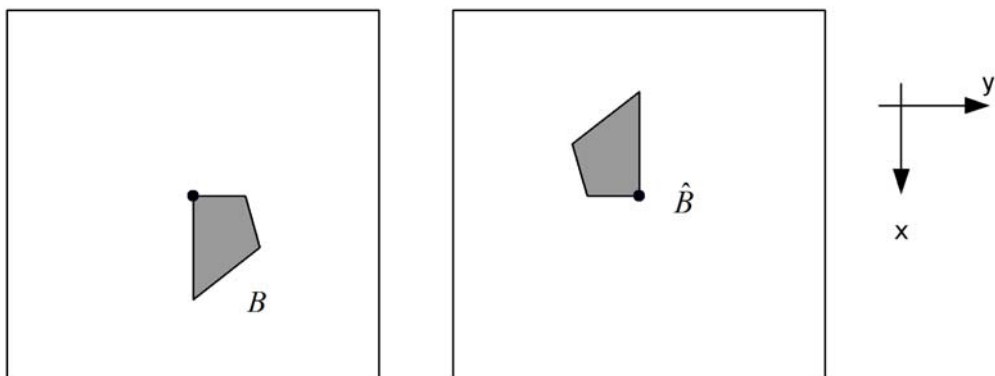
The *translation* of  $A$  by  $z = (z_1, z_2)$ , denoted  $(A)_z$ , is

$$(A)_z \equiv \{x | x = a + z, \text{ for } a \in A\} \quad (7)$$



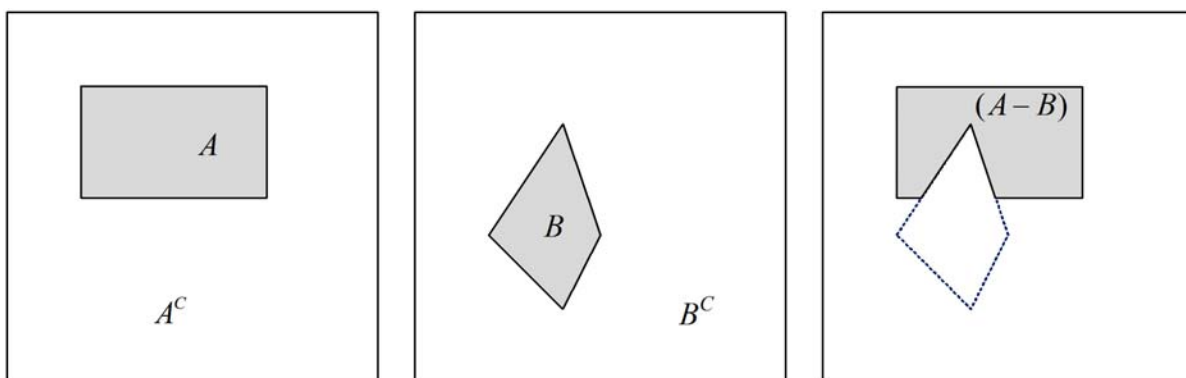
The *reflection* of  $B$ , denoted  $\hat{B}$ , is

$$\hat{B} \equiv \{x|x = -b, \text{ for } b \in B\} \quad (8)$$



The *difference* of two sets  $A$  and  $B$  is

$$A - B = \{x|x \in A, x \notin B\} = A \cap B^c \quad (9)$$



# DILATION AND EROSION

*Dilation* and *erosion* are the bases for many morphological operations. Dilation expands an image and erosion shrinks it.

## Dilation

With  $A$  and  $B$  as sets in  $Z^2$  and  $\emptyset$  denoting the empty set, a definition of the *dilation* of  $A$  by  $B$  is

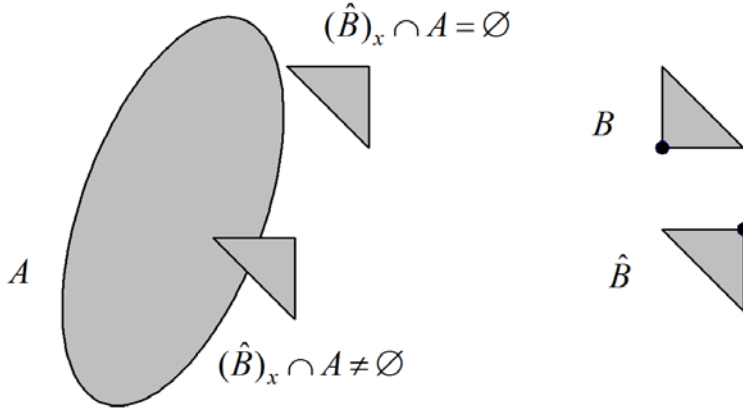
$$A \oplus B = \{x | (\hat{B})_x \cap A \neq \emptyset\} \quad (10)$$

i.e., we first obtain the reflection of  $B$  about its origin and then shift this reflection by  $x$ . The dilation of  $A$  by  $B$  is then the set of all  $x$  displacements such that  $\hat{B}$  and  $A$  overlap by at least one non-zero element (i.e., they have a non-empty intersection).

This equation may be rewritten as

$$A \oplus B = \{x | [(\hat{B})_x \cap A] \subseteq A\} \quad (11)$$

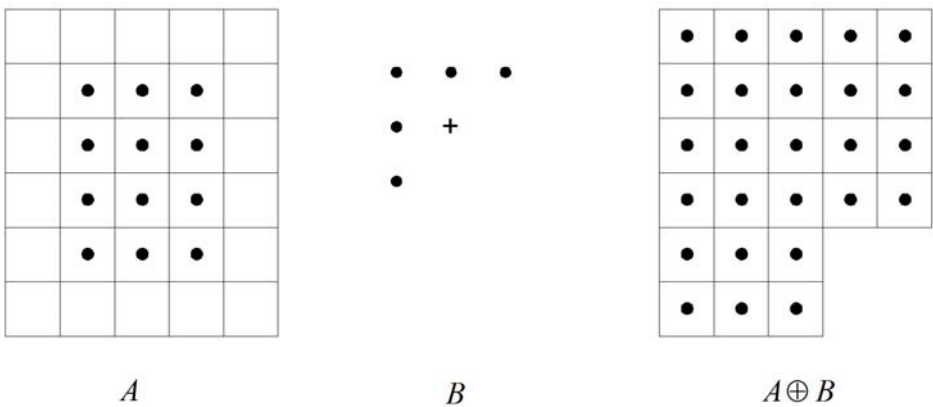
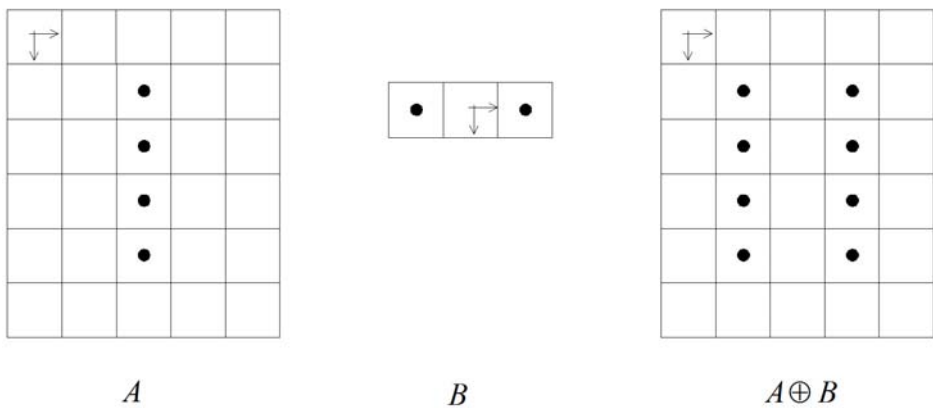
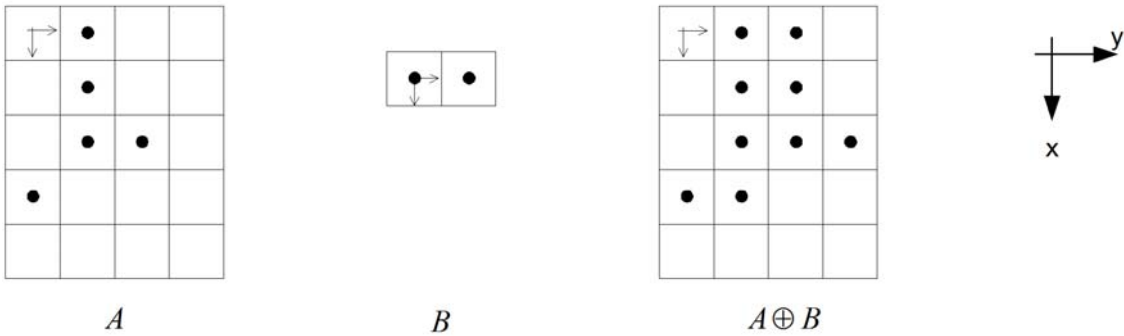
Set  $B$  is commonly referred to as the *structuring element*.

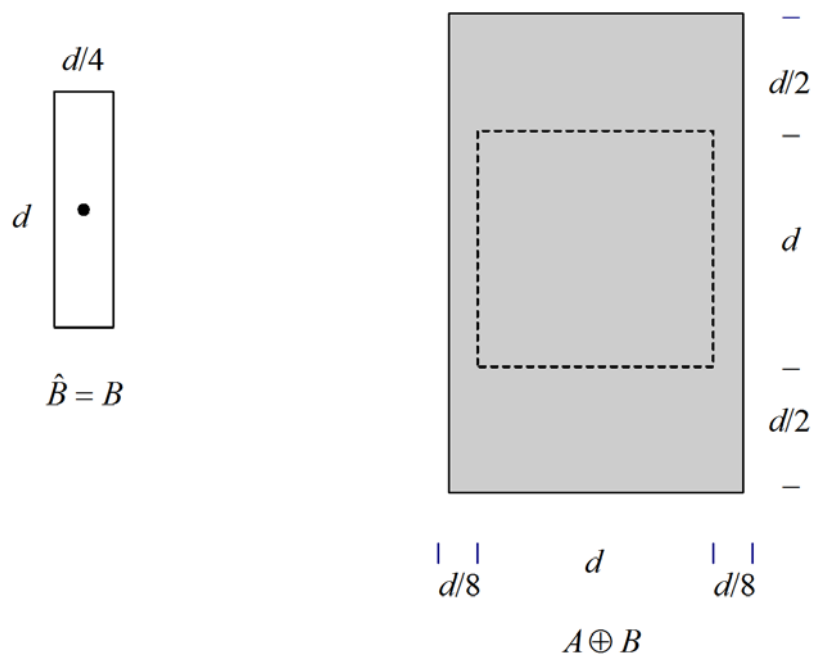
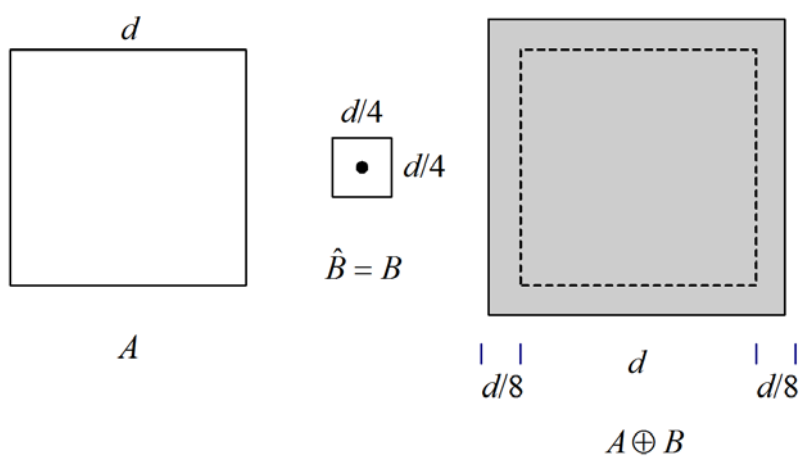
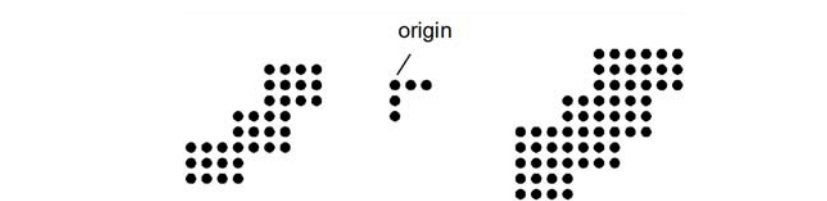


# Example

$$A = \{(0,1), (1,1), (2,1), (2,2), (3,0)\} \qquad B = \{(0,0), (0,1)\}$$

$$A \oplus B = \{(0,1), (1,1), (2,1), (2,2), (3,0) \\ (0,2), (1,2), (2,2), (2,3), (3,1)\}$$



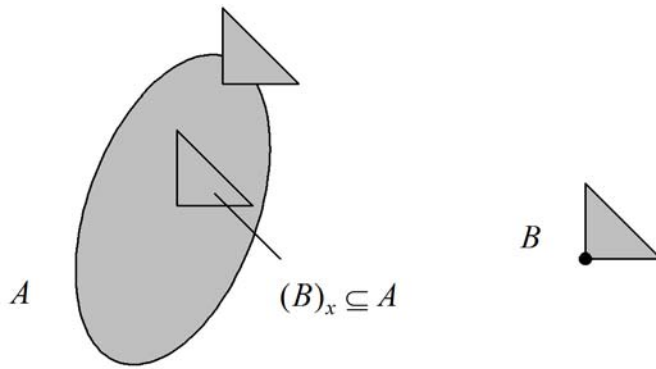


# Erosion

For sets  $A$  and  $B$  in  $Z^2$ , the erosion of  $A$  by  $B$  may be defined as

$$A \ominus B = \{x | (B)_x \subseteq A\} \quad (12)$$

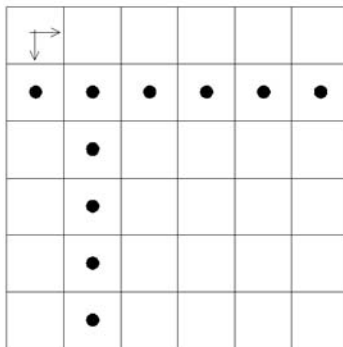
which says that the erosion of  $A$  by  $B$  is the set of all points  $x$  such that  $B$ , translated by  $x$ , is contained in  $A$ .



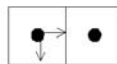
$$A = \{(1,0), (1,1), (1,2), (1,3), (1,4), (1,5), (2,1), (3,1), (4,1), (5,1)\}$$

$$B = \{(0,0), (0,1)\}$$

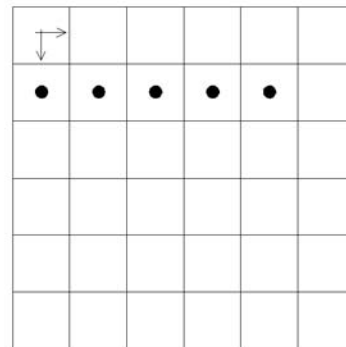
$$A \ominus B = \{(1,0), (1,1), (1,2), (1,3), (1,4)\}$$



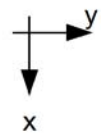
$A$



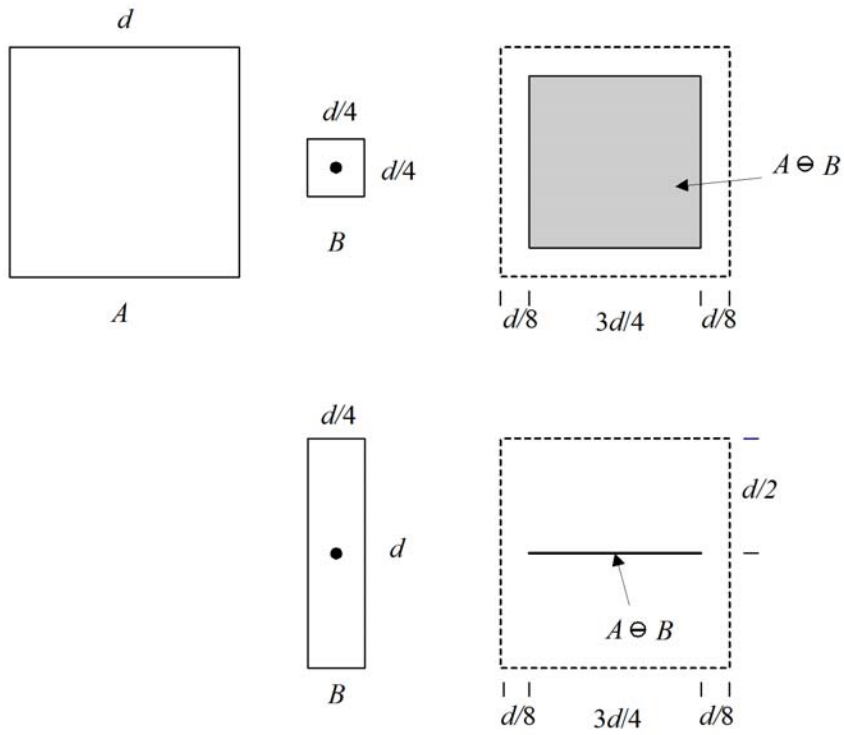
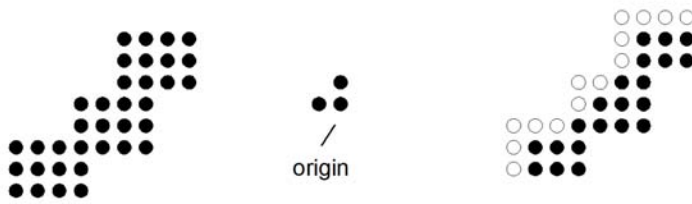
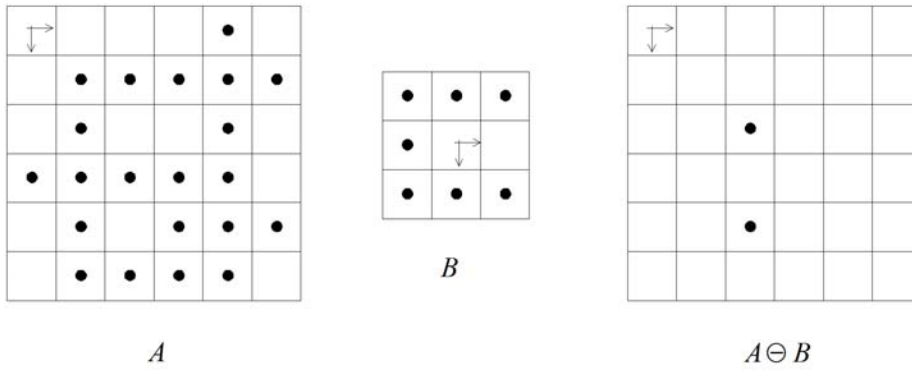
$B$



$A \ominus B$

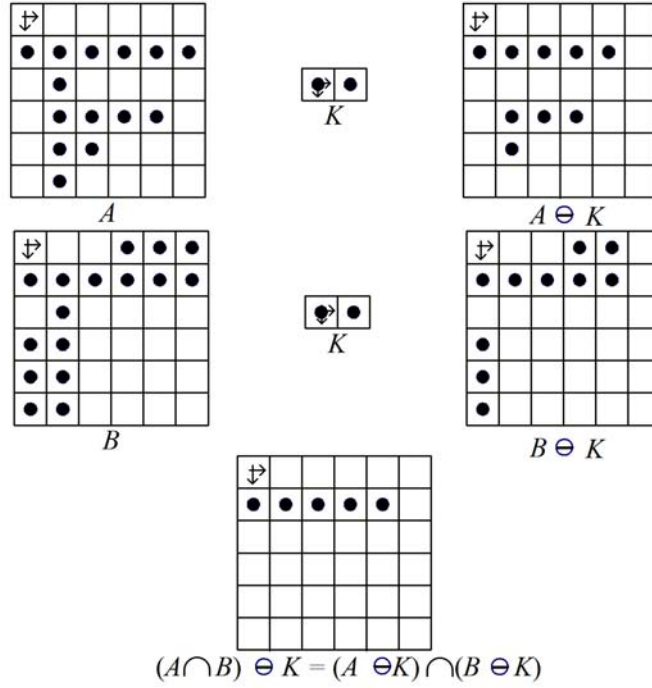
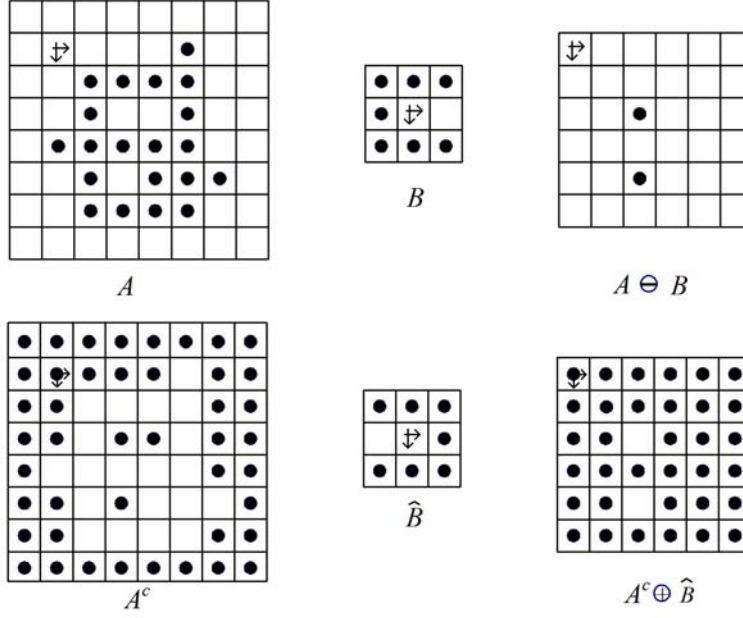






Dilation and erosion are duals of each other with respect to set complementation and reflection. That is,

$$(A \ominus B)^c = A^c \oplus \hat{B} \quad (13)$$



## OPENING AND CLOSING

*Opening* basically opens up strips; it generally smooths the contours of an image, breaks narrow isthmuses, and eliminates thin protrusions.

*Closing* basically closes gaps; it tends to smooth contours, but as opposed to closing, it generally fuses narrow breaks and long thin gulfs, eliminates small holes, and fill gaps in the contour.

The opening of set  $A$  by structuring element  $B$ , is

$$A \circ B = (A \ominus B) \oplus B \quad (14)$$

which says that the opening of  $A$  by  $B$  is simply the erosion of  $A$  by  $B$ , followed by a dilation of the result by  $B$ .

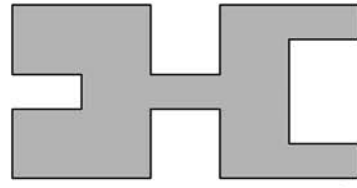
The closing of set  $A$  by structuring element  $B$  is

$$A \bullet B = (A \oplus B) \ominus B \quad (15)$$

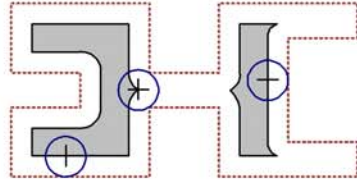
i.e., the dilation of  $A$  by  $B$ , followed by the erosion of the result by  $B$ .

Opening and closing are duals with respect to set complementation and reflection:

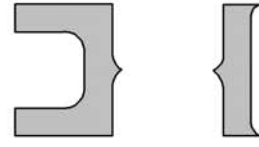
$$(A \bullet B)^c = (A^c \circ \hat{B}) \quad (16)$$



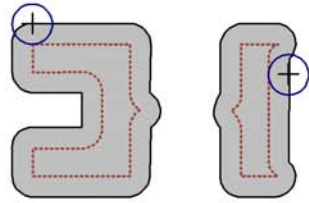
(a) A



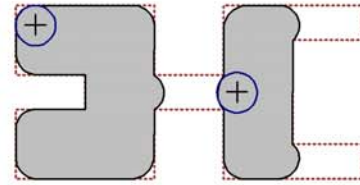
(b)



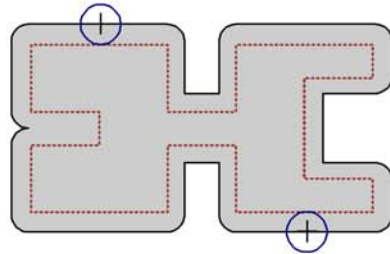
(c)  $A \ominus B$



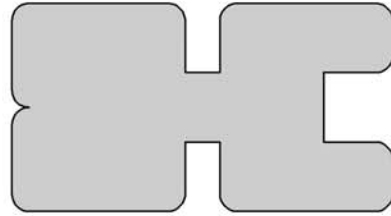
(d)



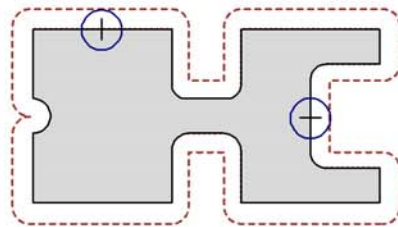
(e)  $A \circ B = (A \ominus B) \oplus B$



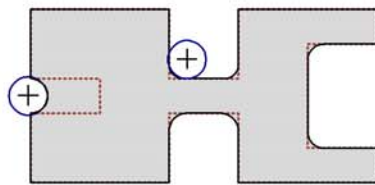
(f)



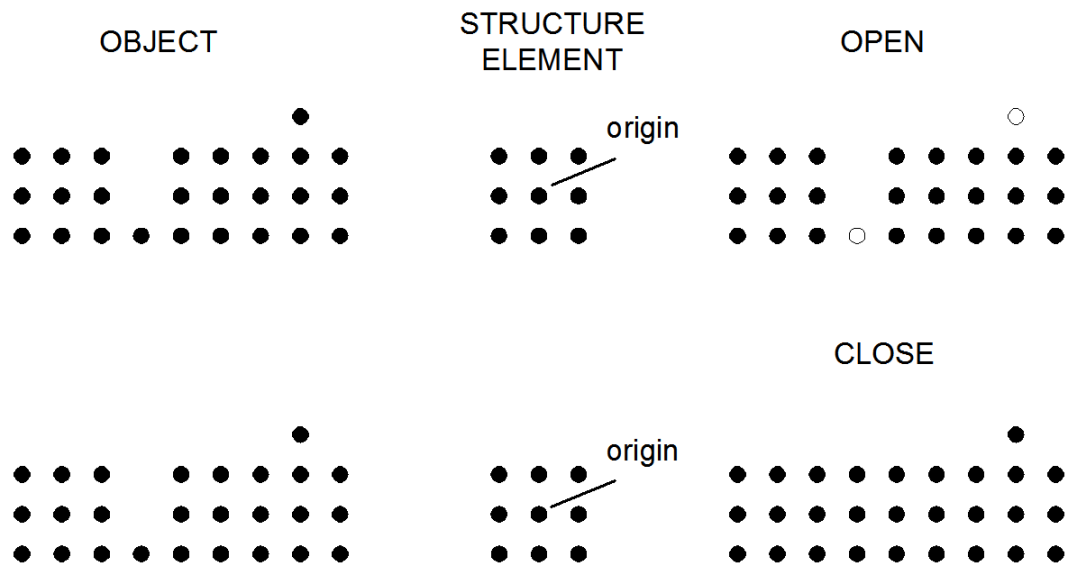
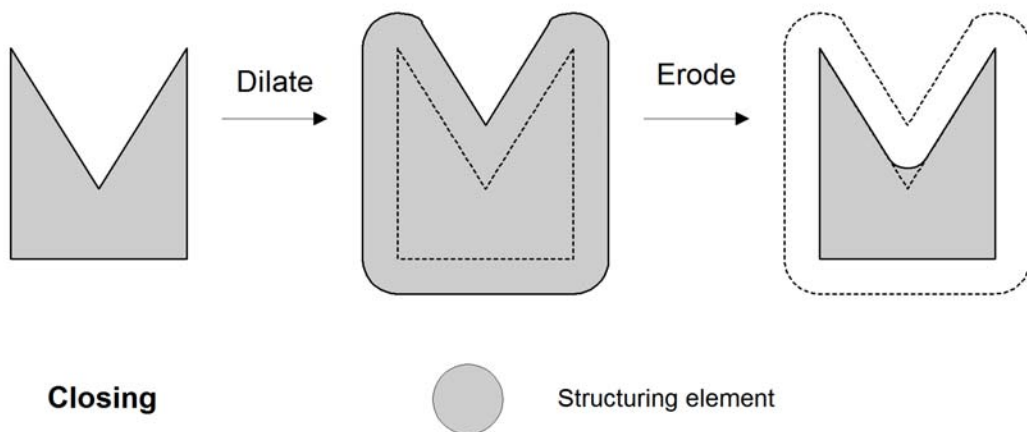
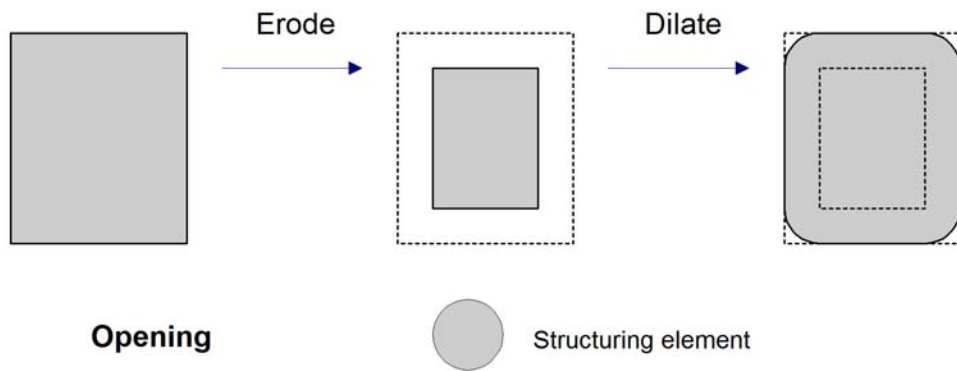
(g)  $A + B$

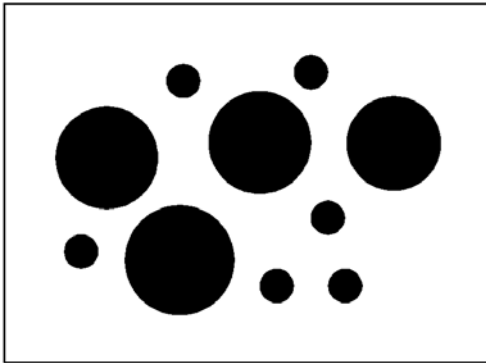


(h)



(i)  $A \bullet B = (A \oplus B) \ominus B$

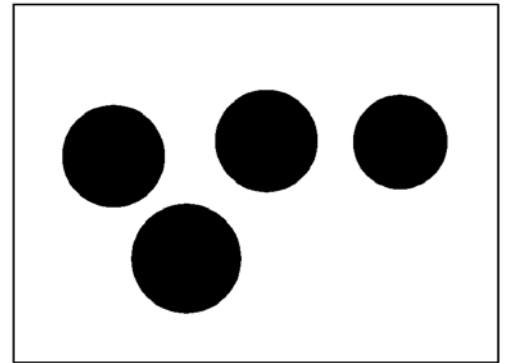




A

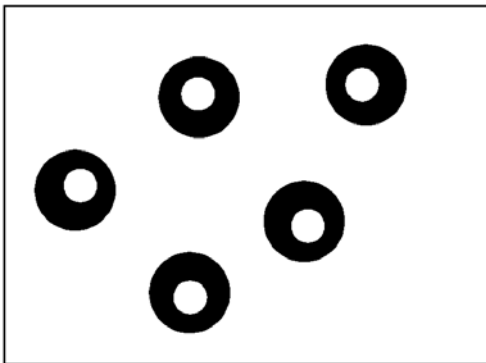


B



$A \circ B$

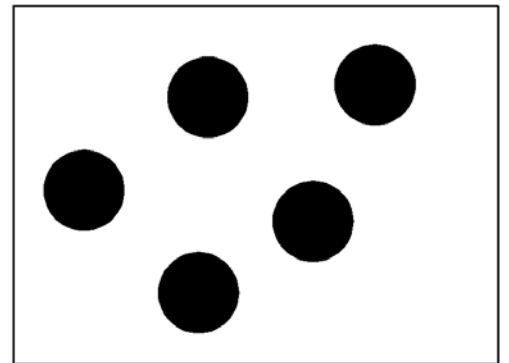
Using opening to separate objects of different sizes.



A



B

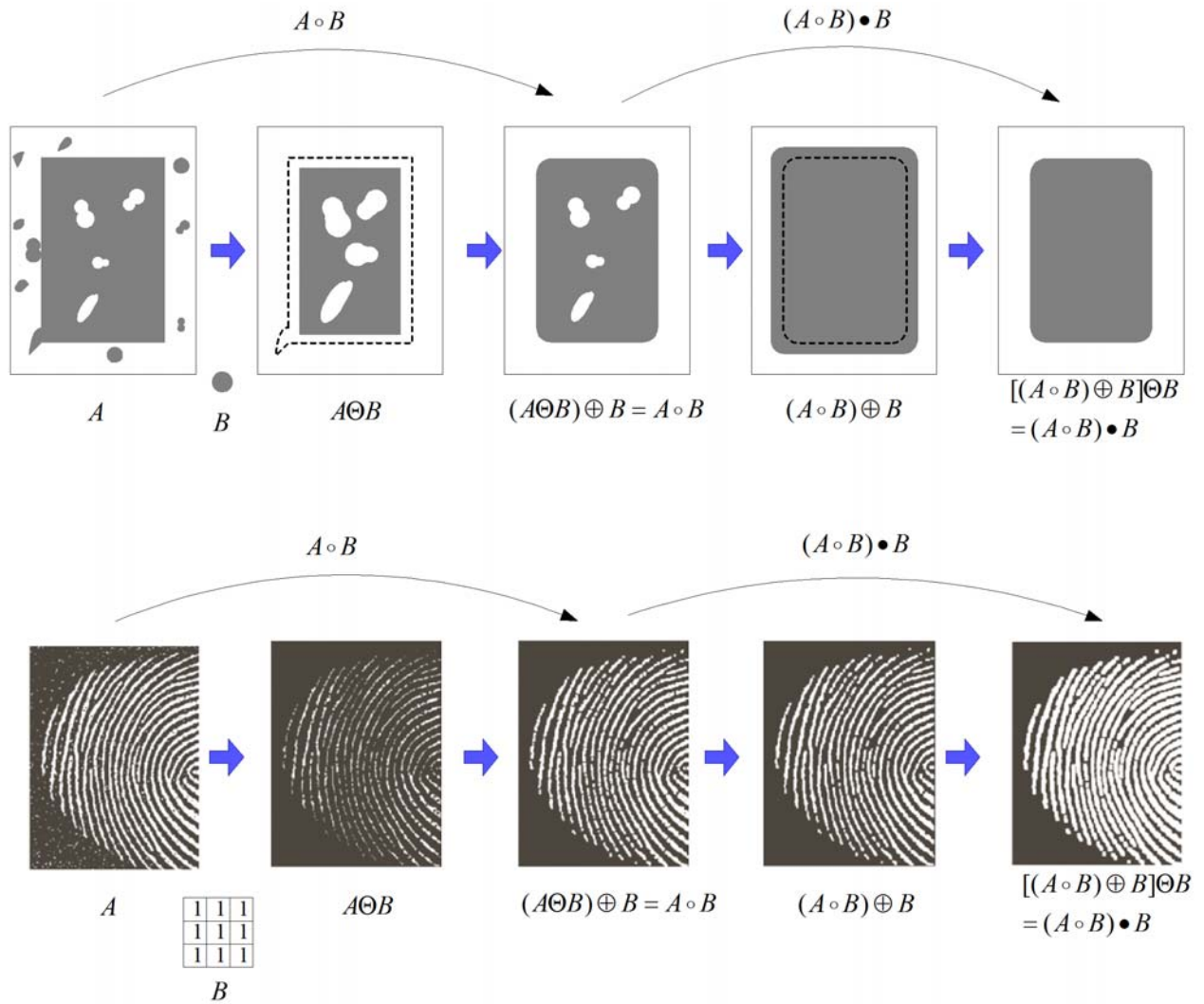


$A \bullet B$

Using closing to fill in holes within objects.

## Examples

Using the morphological filter  $(A \circ B) \bullet B$  to remove noise.



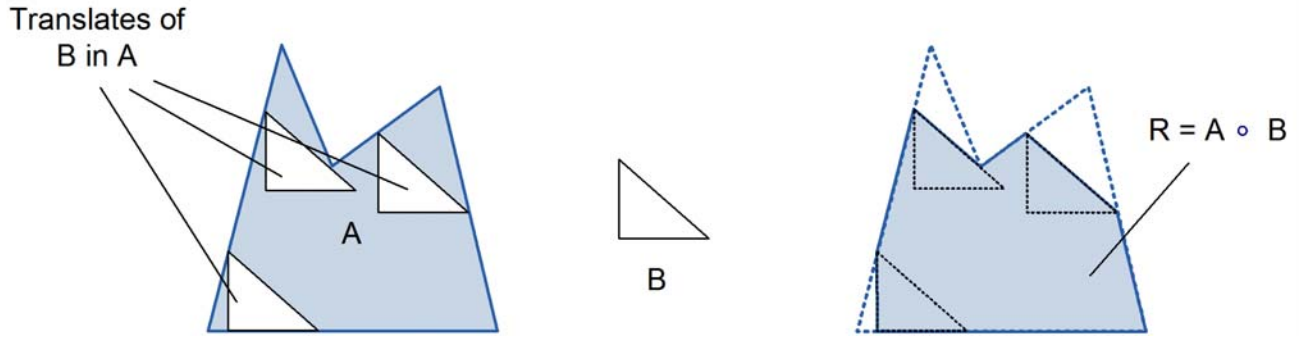
Opening and closing have a simple geometric interpretation.

The boundary of  $R = A \circ B$  is given by the points on the boundary of  $B$  that reach the farthest towards the boundary of  $A$  as  $B$  is translated around the inside of this boundary.

Outward pointing corners become rounded whereas inward pointing corners are not affected. Protruding elements where  $B$  does not fit are eliminated.

In other words, the opening of  $A$  by  $B$  can be obtained by taking the union of all translates of  $B$  that fit into  $A$ , i.e.,

$$R = A \circ B = \cup \{(B)_x \mid (B)_x \subseteq A\} \quad (17)$$

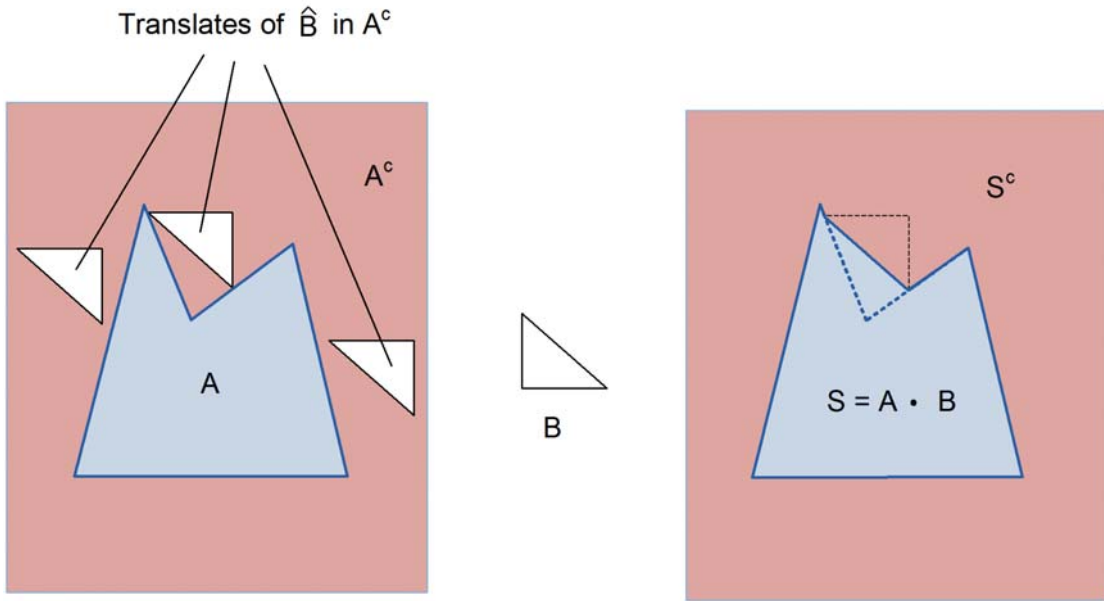




Closing has a similar geometric interpretation, except that now we translate  $\hat{B}$  on the outside of the boundary. The boundary of  $S = A \bullet B$  is then given by the points on the boundary of  $\hat{B}$  that reach the farthest towards the boundary of  $A$ . Mathematically, we have

$$S^C = (A \bullet B)^c = \cup \{(\hat{B})_x \mid (\hat{B})_x \subseteq A^c\} \quad (18)$$

Inward pointing corners become rounded whereas outward pointing corners remain unchanged. Intrusions may be reduced in size if the structuring element does not fit there.



# MORPHOLOGICAL ALGORITHMS

One application of morphology is extracting image components that are useful in the representation and description of shape, e.g.,

- boundaries
- connected components
- convex hull
- skeleton of a region

Another application is in the implementation of processing techniques such as

- region filling
- thinning
- thickening
- pruning

In the following binary images, 1's are shown shaded and 0's shown in white.

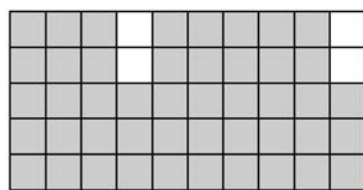
## Boundary Extraction

The boundary of a set  $A$ , denoted by  $\beta(A)$ , can be obtained by

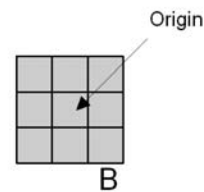
$$\beta(A) = A - (A \ominus B) \quad (19)$$

i.e., first eroding  $A$  by a suitable structuring element  $B$ , and then performing the set difference between  $A$  and its erosion.

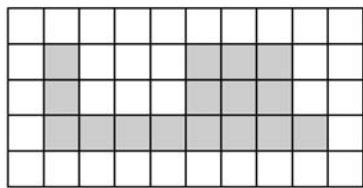
The  $3 \times 3$  structuring element is commonly used. Other structuring elements may also be employed; e.g., with a  $5 \times 5$  structuring element, the boundary would be between 2 and 3 pixels thick.



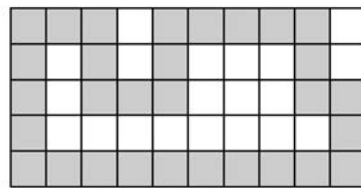
(a)  $A$



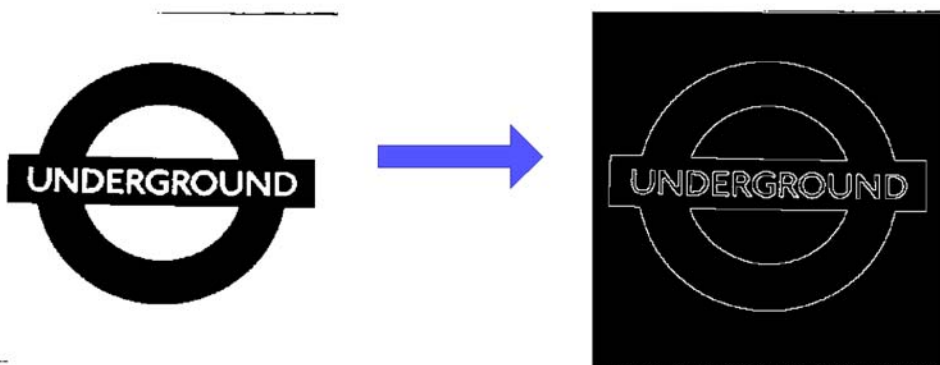
(b)  $B$



(c)  $A \ominus B$



(d)  $\beta(A)$



## Region Filling

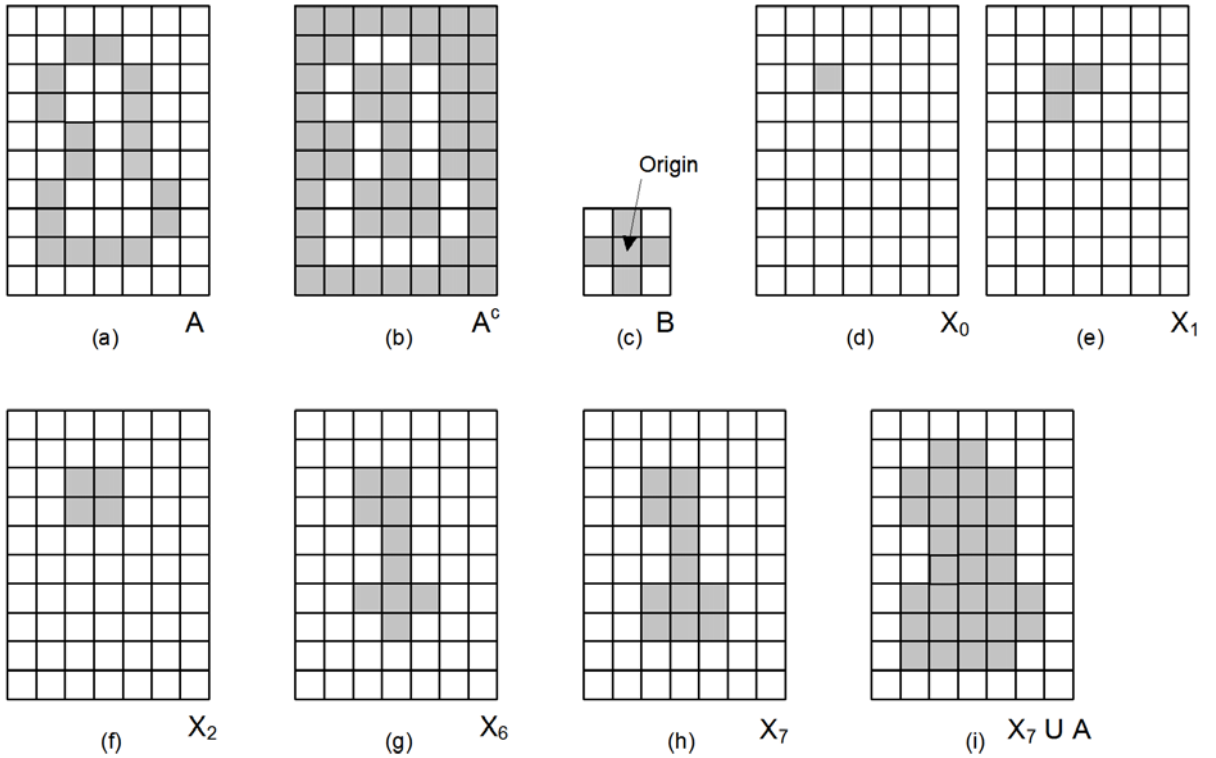
In (a),  $A$  denotes a set containing a subset whose elements are 8-connected boundary points of a region. Beginning with a point  $p$  inside the boundary, the objective is to fill the entire region with 1's.

Since all non-boundary points are labelled 0, we first assign a value of 1 to  $p$ . The following procedure then fills the region with 1's:

$$X_k = (X_{k-1} \oplus B) \cap A^c \quad k = 1, 2, 3, \dots \quad (20)$$

where  $X_0 = p$ , and  $B$  is the symmetric structuring element shown in (c).

The algorithm terminates at iteration step  $k$  if  $X_k = X_{k-1}$ . The set union of  $X_k$  and  $A$  contains the filled set and its boundary. The intersection at each step with  $A^c$  limits the result to the inside of the region of interest.



## Extraction of Connected Components

Let  $Y$  represent a connected component. Assume that a point  $p$  of a connected component  $Y$  is known. All the elements of  $Y$  are obtained by

$$X_k = (X_{k-1} \oplus B) \cap Y \quad k = 1, 2, 3, \dots \quad (21)$$

where  $X_0 = p$ , and  $B$  is a suitable structuring element. If  $X_k = X_{k-1}$ , the algorithm has converged and we let  $Y = X_k$ .

The intersection with  $Y$  at each iterative step eliminates dilations centred on elements labelled 0.

