Variance swaps

Section 5, *Stochastic Volatility Modelling*, L. Bergomi

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Contents

Understanding Variance Swaps: Synthesis, Replication, and Implications

Payoff :

$$\frac{1}{T-t}\sum_{i=0}^{N-1} \ln^2\left(\frac{S_{i+1}}{S_i}\right) - \widehat{\sigma}^2_{\mathrm{VS},T}(t)$$

with:

- t: initial time
- T: contract's maturity
- S_i asset closed price observed at time t_i

$\widehat{\sigma}^2_{\mathrm{VS},T}(t)$ is chosen such that the value of the swap at the initial time is zero (realized volatility estimator)

Synthesis of Variance Swaps

Definition: Variance swaps → financial derivatives to trade volatility directly, no directional risk.

Payoff = (realized - implied)*volatility over a specified period.

Synthesis of VS with European payoffs using Carr-Madan formula on log contracts. Replicate the payoff of a variance swap by constructing a portfolio of European options with different strikes & weights.

Replication -> exposure to volatility and managing risk more effectively exposure to volatility.

$$\widehat{\sigma}^2_{\mathrm{VS},T} = \frac{e^{rT}}{T}\int_0^\infty \frac{2}{K^2}\left(P^{KT}_{\mathrm{market}} - P^{KT}_{\widehat{\sigma}=0}\right)dK \qquad P^{KT}_{\widehat{\sigma}=0} = e^{-rT}\left(Se^{(r-q)T} - K\right)^+$$

Discrete Forward Variance

Strategy considered at time t :

- Purchase of (T2 - t) VS of maturity T2
- Sell of $(T_1 - t)\, e^{-r(T_2 - T_1)}$ VS of maturity T1

Payoff in T2:

$$\sum_{T_1}^{T_2} \ln^2\left(\frac{S_{i+1}}{S_i}\right) - \left((T_2 - t)\,\widehat{\sigma}^2_{\mathrm{VS},T_2}(t) - (T_1 - t)\,\widehat{\sigma}^2_{\mathrm{VS},T_1}(t)\right)$$

$$= \sum_{T_1}^{T_2} \ln^2\left(\frac{S_{i+1}}{S_i}\right) - (T_2 - T_1)\,\widehat{\sigma}^2_{\mathrm{VS},T_1 T_2}(t)$$

Continuous and Discrete Forward Variance

Discrete Forward Variance :

$$\widehat{\sigma}^2_{\mathrm{VS},T_1T_2}(t) = \frac{(T_2 - t)\,\widehat{\sigma}^2_{\mathrm{VS},T_2}(t) - (T_1 - t)\,\widehat{\sigma}^2_{\mathrm{VS},T_1}(t)}{T_2 - T_1}$$

→ Always positive by construction (well defined)

Continuous Forward Variance :

$$\frac{d}{dT}\left((T - t)\,\widehat{\sigma}^2_{\mathrm{VS},T}(t)\right)$$

More interesting strategy

Two steps strategy :

- Same strategy at time t
- Opposite strategy at time t'

Payoff in T2 :

$$(T_2 - T_1)\left(\widehat{\sigma}^2_{\mathrm{VS},T_1 T_2}(t') - \widehat{\sigma}^2_{\mathrm{VS},T_1 T_2}(t)\right)$$

→ Initial stake of 0 (implies P&L with no dependence in r)

→ No More dependence on the realized variance of S

→ Linear P&L in the variation of $\widehat{\sigma}^2_{\mathrm{VS},T_1 T_2}$

Relationship between Variance Swap and log-contract

Reminder : payoff of variance swap : $\frac{1}{T-t}\sum_{i=0}^{N-1} \ln^2\left(\frac{S_{i+1}}{S_i}\right) - \widehat{\sigma}^2_{\mathrm{VS},T}(t)$

First observation:

$$e^{-rT}\sum_{i=0}^{N-1} ln^2(\frac{S_{i+1}}{S_i}) = e^{-rT}\sum_{i=0}^{N-1} ln^2(1+\frac{S_{i+1}-S_i}{S_i}) \simeq e^{-rT}\sum_{i=0}^{N-1}(\frac{S_{i+1}-S_i}{S_i})^2 = e^{-rT}\sum_{i=0}^{N-1}(\frac{\delta S_i}{S_i})^2$$

$$= \sum_{i=0}^{N-1} e^{-rt_i} e^{-r(T-t_i)}\left(\frac{\delta S_i}{S_i}\right)^2$$

Reminder: P&L theta gamma :

$$P\&L = -\sum_i e^{-rt_i}\frac{S_i^2}{2}\frac{d^2P_{\widehat{\sigma}}}{dS^2}(t_i,S_i)\left(r_i^2 - \widehat{\sigma}^2\delta t\right) \quad \text{avec} \quad r_i = (\frac{\delta S_i}{S_i})^2$$

Relationship between Variance Swap and log-contract

→ Same formula with $\frac{1}{2}S^2\frac{d^2P_{\widehat{\sigma}=0}}{dS^2} = e^{-r(T-t)}$

→ Condition satisfied by the log-contract $Q^T(t,S) = -2e^{-r(T-t)}\left(\ln S + (r-q)(T-t) - \frac{\widehat{\sigma}^2}{2}(T-t)\right)$

with $\widehat{\sigma} = 0$:

$$Q^T_{\hat{\sigma}=0}(t,S) = -2e^{-r(T-t)}(lnS + (r-q)(T-t))$$

because $\frac{dQ^T_{\widehat{\sigma}=0}}{dS} = -e^{-r(T-t)}\frac{2}{S}$ and $\frac{1}{2}S^2\frac{d^2Q^T_{\widehat{\sigma}=0}}{dS^2} = e^{-r(T-t)}$

Let's go a step further...

→ Buying $Q^T_{\widehat{\sigma}=0}$ generates a mark-to-market of the form $-(Q^T_{\text{market}} - Q^T_{\widehat{\sigma}=0})$

therefore :

$$\widehat{\sigma}^2_{\text{VS},T} = \frac{e^{rT}}{T}\left(Q^T_{\text{market}} - Q^T_{\widehat{\sigma}=0}\right)$$

Note the $\widehat{\sigma}_T$ implicit volatility of the log-contract :

$$Q^T_{market} = -2e^{-r(T-t)}(lnS + (r-q)(T-t) - \frac{\hat{\sigma}^2_T}{2}(T-t))$$

This gives: with the previous formula $\widehat{\sigma}_{\text{VS},T} = \widehat{\sigma}_T$

Let's go a step further...

Now, as the log-contract is replicated by :

$$-2\ln S = -2\ln S_0 - \frac{2}{S_0}(S - S_0) + \int_0^{S_0} \frac{2}{K^2}(K-S)^+ \, dK + \int_{S_0}^{\infty} \frac{2}{K^2}(S-K)^+ \, dK$$

(comes from section 3 of the book where Bergomi shown that

$$f(S) = f(K_0) + \left.\frac{df}{dK}\right|_{K_0}(S - K_0) + \int_0^{K_0} \frac{d^2 f}{dK^2}(K-S)^+ \, dK + \int_{K_0}^{\infty} \frac{d^2 f}{dK^2}(S-K)^+ \, dK$$

with $f(S) = -2lnS$)

→ Payoff replicated (order 2) with delta-hedge of this portfolio (with zero implied vol)

Impact of Large Returns

→ We have done a Taylor expansion of order 2 in $\frac{\delta S}{S}$

→ Remember that we have : $\frac{1}{2}S^2\frac{d^2P_{\widehat{\sigma}=0}}{dS^2} = e^{-r(T-t)}$

→ Suppose that S follows : $dS_t = (r-q)S_t dt + \overline{\sigma}_t S_t dW_t$

→ But one knows that, in a local volatility model : $P_{\overline{\sigma}}(t=0) = P_\sigma(0,S_0) + E_{\overline{\sigma}}\left[\int_0^T \frac{1}{2}e^{-rt}S_t^2\frac{d^2P_\sigma}{dS^2}\left(\overline{\sigma}_t^2 - \sigma(t,S_t)^2\right)dt\right]$

Robustesse de la formule de Black-Scholes

Ref : El Karoui, Jeanblanc, Shreve '98

Théorème. Supposons qu'un trader couvre un call avec un modèle calibré au marché à $t=0$. Alors son P&L en T est

$$P\&L_T = V_T - (S_T - K)_+ = e^{rT}\int_0^T e^{-rt}\frac{1}{2}S_t^2\partial_x^2 u^{\mathtt{Model}}(t,S_t)((\sigma^{\mathtt{Model}})^2(t,S_t) - \sigma_t^2)\mathrm{d}t$$

avec σ_t la vol. instantanée de S et $u^{\mathtt{Model}}$ la fonction de pricing utilisée.

— Moyenne trajectorielle entre la différence des variances calibrées et réalisées pondérée par les gammas

— Intérêt d'un marché de variance swaps pour couvrir le risque de modèle

Remarque (Propagation de convexité). Dans un modèle à volatilité locale, u hérite de la convexité en x du payoff ➠ $\partial_x^2 u(t,S_t) \geq 0$

Impact of Large Returns

By setting $\sigma(t, S) \equiv 0$ we have :

$$P_{\overline{\sigma}}^{T} = P_{\widehat{\sigma}=0}^{T} + e^{-rT} E_{\overline{\sigma}}\left[\int_0^T \overline{\sigma}_t^2 dt\right]$$

Let's go back to our variance swap… We have :

$$d \ln S_t = (r - q - \frac{1}{2}\overline{\sigma}_t^2)dt + \overline{\sigma}_t dW_t$$

So $\lim_{dt \to 0} \frac{1}{dt} (d \ln S_t)^2 = \overline{\sigma}_t^2$ and $\lim_{\Delta t \to 0} \sum_{i=0}^{N-1} \ln^2\left(\frac{S_{i+1}}{S_i}\right) = \int_0^T \overline{\sigma}_t^2 dt$

Impact of Large Returns

As a consequence, we have :

$$\widehat{\sigma}^2_{\mathrm{VS},T} = \frac{1}{T}E\Big[\lim_{\Delta t\to 0}\sum_{i=0}^{N-1}\ln^2\Big(\frac{S_{i+1}}{S_i}\Big)\Big] = E\Big[\frac{1}{T}\int_0^T \overline{\sigma}_t^2\,dt\Big]$$

Assuming that our model is calibrated to the market smile $P^T_{\overline{\sigma}} = P^T_{\mathrm{Market}}$, we get :

$$\widehat{\sigma}^2_{\mathrm{VS},T} = E_{\overline{\sigma}}\Big[\frac{1}{T}\int_0^T \overline{\sigma}_t^2\,dt\Big] = \frac{e^{rT}}{T}\left(P^T_{\mathrm{Market}} - P^T_{\widehat{\sigma}=0}\right)$$

→ All diffusive models price variance swaps identically

Impact of Large Returns

Now remember that :

$$P_{\overline{\sigma}}\,(t=0) \;=\; P_\sigma\,(0, S_0) \;+\; E_{\overline{\sigma}}\left[\int_0^T \frac{1}{2} e^{-rt} S_t^2 \frac{d^2 P_\sigma}{dS^2}\left(\overline{\sigma}_t^2 - \sigma(t, S_t)^2\right) dt\right]$$

By setting $\sigma\,(t, S) \equiv 0$ we get :

$$P^T_{\mathrm{Market}} \;=\; P^T_{\widehat{\sigma}} \;=\; P^T_{\widehat{\sigma}=0} + e^{-rT} T \widehat{\sigma}_T^2$$

With the previous expression $\widehat{\sigma}^2_{\mathrm{VS},T} \;=\; E_{\overline{\sigma}}\Big[\frac{1}{T}\int_0^T \overline{\sigma}_t^2 dt\Big] \;=\; \frac{e^{rT}}{T}\left(P^T_{\mathrm{Market}} - P^T_{\widehat{\sigma}=0}\right)$ we have :

$$\widehat{\sigma}_{\mathrm{VS},T} \;=\; \widehat{\sigma}_T$$

Impact of daily-return skewness

→ Short variance swap & long delta-hedged log-contrat with zero interest rate and repo (for simplicity)

→ Remember that : $Q^T(t,S) = -2e^{-r(T-t)}\left(\ln S + (r-q)(T-t) - \frac{\hat{\sigma}^2}{2}(T-t)\right)$

so we have : $Q^T(t,S) = -2lnS + \hat{\sigma}^2(T-t)$

Then, we get : $Q^T(t_{i+1}, S_{i+1}) - Q^T(t_i, S_i) = 2(e^{r_i} - 1) - 2r_i - \hat{\sigma}_T^2 \Delta t$ $\quad r_i = \ln(\frac{S_{i+1}}{S_i})$

Impact of daily-return skewness

→ Total P&L (short VS / long delta-hedged log-contract) :

$$\begin{aligned} P\&L \; = \; & \left(Q^T(t_{i+1}, S_{i+1}) - Q^T(t_i, S_i)\right) - \frac{dQ^T}{dS}(t_i, S_i)\,(S_{i+1} - S_i) \\ & - (r_i^2 - \widehat{\sigma}^2_{\mathrm{VS},T}\Delta t) \\ = \; & \left(2(e^{r_i} - 1) - 2r_i - \widehat{\sigma}^2_T \Delta t\right) \; - \; \left(r_i^2 - \widehat{\sigma}^2_{\mathrm{VS},T}\Delta t\right) \\ = \; & \left(2(e^{r_i} - 1) - 2r_i - r_i^2\right) \; - \; \left(\widehat{\sigma}^2_T - \widehat{\sigma}^2_{\mathrm{VS},T}\right)\Delta t \end{aligned}$$

with $r_i = \ln(\frac{S_{i+1}}{S_i})$

Impact of daily-return skewness

Then $P\&L \simeq \frac{r_i^3}{3} - \left(\widehat{\sigma}_T^2 - \widehat{\sigma}_{\mathrm{VS},T}^2\right)\Delta t$ (Taylor expansion)

By imposing $\mathbb{E}(P\&L) = 0$ we obtain :

$$\widehat{\sigma}_{\mathrm{VS},T}^2 - \widehat{\sigma}_T^2 \simeq -\frac{\langle r^3 \rangle}{3\Delta t} \simeq -\frac{s_{\Delta t}}{3}\widehat{\sigma}_T^3\sqrt{\Delta t}$$

with $s_{\Delta t} = \langle r^3 \rangle / \langle r^2 \rangle^{\frac{3}{2}}$.

and $\langle r^2 \rangle = \widehat{\sigma}_T^2 \Delta t$

Impact of daily-return skewness

Writing: $\hat{\sigma_{VS,T}}^2 - \hat{\sigma_T}^2 \simeq 2\hat{\sigma_T}(\hat{\sigma_{VS,T}} - \hat{\sigma_T})$

We obtain:

$$\frac{\widehat{\sigma}_{\text{VS},T}}{\widehat{\sigma}_T} - 1 \simeq -\frac{s_{\Delta t}}{6}\widehat{\sigma}_T\sqrt{\Delta t}$$

$$\widehat{\sigma}_{\text{VS},T} \simeq \widehat{\sigma}_T\left(1 - \frac{s_{\Delta t}}{6}\widehat{\sigma}_T\sqrt{\Delta t}\right)$$

→ The difference between $\widehat{\sigma}_T$ and $\widehat{\sigma}_{\text{VS},T}$ depends on the non-Gaussian character of the daily log-returns (skewness)

Impact of strike discreteness

- Difference between VS payoff & P&L from delta-hedging a log contract.
- Neither the log nor the delta-hedged log contract is traded, and cannot be perfectly synthesized from vanilla options due to discrete strikes.
- Practical Replication: Replication of a 1-year VS using daily closing quotes of the S&P 500. Log contract ~ vanilla options with specified strike intervals.
- Coarser discretization implies less accurate replication of VS payoff
- Impact on Volatility Adjustment: Coarser discretization implies larger adjustment in volatility.

Conclusion

→ Variance Swaps are replicated by delta-hedging a log contract

→ The difference between $\widehat{\sigma}_T$ and $\widehat{\sigma}_{\mathrm{VS},T}$ is a measure of the implied skewness of daily returns

→ Variance Swaps are highly liquid, unlike options that are very out-of-the-money: market makers extrapolate implied volatility with VS, which reinforces $\widehat{\sigma}_T = \widehat{\sigma}_{\mathrm{VS},T}$

→ In reality, strikes are not continuous so there is a discretization (and a mismatch)

Conclusion

Market Liquidity Impact: Despite the absence of actively traded log contracts, variance swaps (VS) are more liquid than far out-of-the-money vanilla options in many cases, influencing their pricing dynamics.

Model Dependence: The choice of model for pricing VS, whether jump-diffusion or other, significantly affects the implied volatilities and the spread between $\widehat{\sigma}_{\mathrm{VS},T}$ and $\widehat{\sigma}_T$

Practical Adjustments: Equations such as (5.42) provide practical adjustments to dissociate $\widehat{\sigma}_{\mathrm{VS},T}$ from $\widehat{\sigma}_T$ allowing for accurate pricing of VS even without direct market trading of log contracts.

Risk Management: Automatic adjustment mechanisms in pricing libraries ensure consistency and facilitate risk monitoring, especially concerning interest rate volatility and dividend models for longer-dated VSs.

Continuous Evaluation: Ongoing evaluation and adjustment of market practices are necessary to adapt to changing market conditions and improve the accuracy of VS pricing and risk management strategies.