THE UNIVERSITY of EDINBURGH

SCHOOL OF ENGINEERING

COMPUTATIONAL METHODS AND MODELLING 3

MECE09033

Exam Date: **09/12/2022** From and To: **13:00-1500** Exam Diet: **Dec 2022**

Please read full instructions before commencing writing

Exam paper information

- This exam has specific technical instructions, listed below. Please read these carefully before starting your exam.
- This exam paper consists of FOUR questions.
- Candidates should attempt ALL FOUR questions.

Special instructions

- This is an open book exam. This means you can freely access any printed or online materials to aid you in your answers. Online materials can include text, images, videos and data. You may NOT engage in interactions or discussions relating to the exam questions or examined subject matter in any form. Sharing the answers to this exam in any way, by any means and in any form is STRICTLY NOT allowed.
- Students should assume reasonable values for any data not given in a question nor available on a datasheet, and should make any such assumptions clear on their script.
- Students in any doubt as to the interpretation of the wording of a question, should make their own decision, and should state it clearly on their script.
- This examination will be marked anonymously.
- Use of an approved calculator and dictionary is permitted.

Technical instructions

- All Python programme files should be named with your exam number as follows, e.g., 'MECE09033-B123456-Q1a', clearly indicating the question number after the second hyphen in the name. If you choose one code to answer all questions, then please use the following format: e.g. 'MECE09033-B123456-CODES'.
- A cover report should be written giving the answers to each section with a short description of the solution strategy used. Each answer should be labelled with the relevant question number, e.g. Q1a. The cover report should be a PDF/word document.
- Please place your cover report (labelled with your exam number e.g. MECE09033-B123456.pdf) plus all your programme files into one zipped folder also labelled with your exam number (e.g. MECE09033-B123456) and upload the zip folder to the dropbox in the EXAM area for this course on LEARN.

Special items

- None

Convenor of Board of Examiners: **Dr D Yang**
External Examiner: **Professor A Morina**

Question 1.

The following series converges to the well know mathematical constant π:

$$\sqrt{6\sum_{i=1}^{\infty}\frac{1}{i^2}} = \sqrt{6\left(\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\frac{1}{25}+\cdots\right)} = \pi. \qquad \textbf{Equation Q1.1}$$

Therefore, if it is truncated to a certain specified number of terms N, it provides an approximation of π:

$$\sqrt{6\sum_{i=1}^{N}\frac{1}{i^2}} = \sqrt{6\left(\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\frac{1}{25}+\cdots+\frac{1}{N}\right)} \approx \pi. \qquad \textbf{Equation Q1.2}$$

a) Write python code to evaluate the truncated series in Equation Q1.2 with a specified number of terms N. Compute and report the approximation of π obtained with exactly 10, 100, and 1000 terms ($N = 10$, $N = 100$ and $N = 1000$). **(5)**

b) For the same number of terms ($N = 10$, $N = 100$ and $N = 1000$), compute and report the error in the approximation of π using the two following definitions:
- a true error, appropriately defined assuming that the true solution π is known.
- an estimated error computed assuming that the true solution is not known.

Include in the cover report an explanation of the definition you used for the true and estimated error. **(5)**

Question 2.

The following ordinary differential equation (ODE):

$$dy/dt = 10y^2 - y^3 \qquad \textbf{Equation Q2.1}$$

with initial condition $y(0) = \delta$ describes the propagation of a combustion front. It can be used, for example, as a model for the ignition of a match (neglecting gravity). The unknown $y(t)$ represents the time evolution of the radius (in mm) of the spherical flame; it starts from the small (imposed) value δ and increases, rather slowly in the early phase and very fast after, to finally converge to a constant value when the two terms on the right-hand side of the equation balance each other. A sketch of the typical solution is shown in Figure Q2.1.

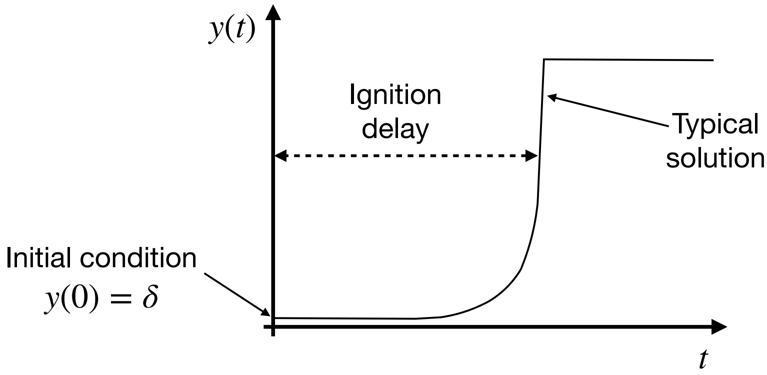

Figure Q2.1: Typical solution of Equation Q2.1.

An engineer needs to perform an analysis of this problem based on numerical simulations. To this end, python code which employs the Euler method for the solution of ODEs needs to be written.

a) Write python code to solve Equation Q2.1 with the Euler method and report the value of the solutions y at $t = 4$, $t = 5$ and $t = 10$, starting from the following initial condition at $t = 0$:

$$y(0) = 0.02$$

Report also the value of the step, h, that you decided to use in the numerical method implemented and explain how this was selected. **(5)**

b) A sketch of the typical solution of this problem is shown in figure Q2.1. The radius increases slowly initially and then very fast. The time between the initial state (at $t = 0$) and the moment when the sudden increase is observed is called the "ignition delay". Compute and report the value of the ignition delay for the three following values of the initial condition:

$- y(0) = 0.02$

$- y(0) = 0.01$

$- y(0) = 0.005$ **(5)**

c) Equation Q2.1 is stiff and requires care when solved with an explicit method like the Euler approach. In particular, the step h must be chosen appropriately to ensure stability. By experimenting with different values of h, estimate and report a threshold value for h above which the method becomes unstable. For this task, use the initial condition:

$$y(0) = 0.02$$

(5)

d) Briefly explain the concept of stability of a numerical method for the solution of ODEs. Also discuss the difference between the explicit and implicit Euler methods and the implications for stability. Your answer should include the rational for the selection of the step h. **(5)**

Question 3.

Equation Q3.1 below is an example of an iterative equation.

$$\boldsymbol{x_{n+1} = \frac{1}{sin(x_n)} + \frac{1}{4}} \qquad \textbf{Equation Q3.1}$$

a) For Equation Q3.1, use an appropriate numerical root finding technique to determine any value of x in the domain $0 < x < 4$, for which: $x_n = x_{n+1}$. **(5)**

b) Use any closed interval root finding technique to determine a root of $f(x)$ in the domain $0 < x < 4$, where $f(x)$ is defined by placing all terms of Equation Q3.1 on the right-hand side. **(5)**

c) Use any open root finding technique to find the same root of $f(x)$ in the same domain of $0 < x < 4$. **(5)**

d) Explain why there is a difference in answers obtained for each of the above three techniques by referring to:
i) the absolute error of the computation **(2)**
ii) the number of iterations required by each technique to reach the final answer. **(3)**
You should use a table to illustrate your answer.

Question 4

Equation Q4.1 is the equation in time (t) that describes the relationship between displacement (x), mass (m), system stiffness (k), and frictional damping coefficient (b) of a damped harmonic oscillator (i.e., a system with an oscillation amplitude that decays in time).

The general integral solution of Equation Q4.1 is given by Equation Q4.2, where ω is defined by Equation Q4.3. In the latter, A_0 is the initial displacement (= 0.05 m), ω_0 is the initial oscillation frequency (5 Hz), and φ is the phase lag angle of the oscillation (= 0 radians).

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0.$$ **Equation Q4.1**

$$x(t) = A_0 e^{-\frac{b}{2m}t}\cos(\omega t + \varphi).$$ **Equation Q4.2**

$$\omega = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}.$$ **Equation Q4.3**

a) Use an appropriate python algebra package to show that Equation Q4.2 is the general integral solution to Equation Q4.1. **(5)**

b) By making an appropriate substitution in Equation Q4.1, show that the instantaneous frequency of a damped oscillator can be described by Equation Q4.3. **(5)**

c) Calculate the time taken for a pendulum of mass, 1 kg, with an initial oscillation frequency, 5 Hz, to reach an oscillation amplitude of 1% of the initial value A_0. Assume that the damping coefficient, b, has a value of 0.1 s^{-1} and φ = 0. **(5)**

d) What dumping coeficient of the pendulum would be required to halve the time to reach 1% of A_0 calculated for the system in part c) of this question? **(5)**

END OF PAPER