Problem Set 1

Ben Kaiser-Bulmash

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Exercise 1.17

1 For the set $\mathscr{U} = \{a, b, \{c, d\}\}$, which of the following are true:

i. $a \in \mathscr{U}$ is true because a is the first element in the set $\mathscr{U}$.

ii. $\{a\} \in \mathscr{U}$ is false because while $a \in \mathscr{U}$, the set containing a, $\{a\}$, is not.

iii. $a \subset \mathscr{U}$ is false because a is not a set, thus cannot be a subset of anything.

iv. $\{a\} \subset \mathscr{U}$ is true because $\{a\}$ contains only one element, a, and $a \in \mathscr{U}$.

v. $\{a, b\} \in \mathscr{U}$ is false because $\mathscr{U}$ has no set containing both a and b. Both a and b are in $\mathscr{U}$, but there is not a set containing both.

vi. $\{a, b\} \subset \mathscr{U}$ is true because all elements, a and b, are in $\mathscr{U}$.

vii. $\{\{a, b\}\} \in \mathscr{U}$ is false because while both a and $b \in \mathscr{U}$, similarly to v, there is not a set containing a set with a and b inside.

viii. $\{\{a, b\}\} \subset \mathscr{U}$ is false because $\mathscr{U}$ doesn't contain $\{a, b\}$. $\mathscr{U}$ contains both a, and b, but not the set containing the two.

ix. $\{a, b, c, d\} \subset \mathscr{U}$ is false because while a and b are in $\mathscr{U}$, c and d are not. There is a set containing c and $d \in \mathscr{U}$ but not c and d on their own.

3 For the set $\mathscr{U} = \{a, b, \{a, b\}\}$ which of the following are true:

i. $a \in \mathscr{U}$ is true because a is the first element in the set $\mathscr{U}$.

ii. $\{a\} \in \mathscr{U}$ is false because while $a \in \mathscr{U}$, the set containing a, $\{a\}$, is not.

iii. $a \subset \mathscr{U}$ is false because a is not a set, thus cannot be a subset of anything.

iv. $\{a\} \subset \mathscr{U}$ is true because $\{a\}$ contains only one element, a, and $a \in \mathscr{U}$.

v. $\{a, b\} \in \mathscr{U}$ is true because the third item in $\mathscr{U}$ is exactly $\{a, b\}$.

vi. $\{a, b\} \subset \mathscr{U}$ is true because all elements, a and b, are in $\mathscr{U}$.

vii. $\{\{a, b\}\} \in \mathscr{U}$ is false because while both a and $b \in \mathscr{U}$, and a set containing both $(\{a, b\}) \in \mathscr{U}$, there is not a set containing a set with a and b inside $(\{\{a, b\}\})$.

viii. $\{\{a, b\}\} \subset \mathscr{U}$ is true because $\{a, b\} \in \mathscr{U}$ and $\{a, b\}$ is the only element of $\{\{a, b\}\}$.

ix. $\{a, b, c, d\} \subset \mathscr{U}$ is false because while a and b are in $\mathscr{U}$, c and d are not.

5 For the set $\mathscr{U} = \{\{a, b\}\}$ which of the following are true:

i. $a \in \mathscr{U}$ is false because a on its own $\notin \mathscr{U}$, a is within a set $\in \mathscr{U}$.

ii. $\{a\} \in \mathscr{U}$ is false because the set containing only a, $\{a\}$, is not $\in \mathscr{U}$.

iii. $a \subset \mathscr{U}$ is false because a is not a set, thus cannot be a subset of anything.

iv. $\{a\} \subset \mathscr{U}$ is false because a on its own $\notin \mathscr{U}$, a is within a set $\in \mathscr{U}$, but not on its own $\in \mathscr{U}$.

v. $\{a, b\} \in \mathscr{U}$ is true because the the only item in $\mathscr{U}$ is exactly $\{a, b\}$.

vi. $\{a, b\} \subset \mathscr{U}$ is false because the elements, a and b, are not in $\mathscr{U}$. $\mathscr{U}$ only contains a set with a and b in it, but $\mathscr{U}$ doesn't contain either a, or b.

vii. $\{\{a, b\}\} \in \mathscr{U}$ is false because while a set containing both $(\{a, b\}) \in \mathscr{U}$, there is not a set containing a set with a and b inside $(\{\{a, b\}\})$.

viii. $\{\{a, b\}\} \subset \mathscr{U}$ is true because $\{a, b\} \in \mathscr{U}$ and $\{a, b\}$ is the only element of $\{\{a, b\}\}$.

ix. $\{a, b, c, d\} \subset \mathscr{U}$ is false because c and $d \notin \mathscr{U}$.

7 For the set $\mathscr{U} = \{\{\{a\}\}, \{b\}, \{\{a\}, b\}\}$ which of the following are true:

i. $a \in \mathscr{U}$ is false because a on its own $\notin \mathscr{U}$, a is within a set $\in \mathscr{U}$.

ii. $\{a\} \in \mathscr{U}$ is false because $\{a\}$ on its own $\notin \mathscr{U}$, $\{a\}$ is within a set $\in \mathscr{U}$.

iii. $a \subset \mathscr{U}$ is false because a is not a set, thus cannot be a subset of anything.

iv. $\{a\} \subset \mathscr{U}$ is false because $\{a\}$ contains only one element, a, and that element, $a \notin \mathscr{U}$ alone.

v. $\{a, b\} \in \mathscr{U}$ is false because $\{a, b\}$ is not in $\mathscr{U}$.

vi. $\{a, b\} \subset \mathscr{U}$ is false because neither of the elements, a or b, are in $\mathscr{U}$.

vii. $\{\{a, b\}\} \in \mathscr{U}$ is false because $\{a, b\} \notin \mathscr{U}$ In other words, there is not a set containing a set with a and b inside ($\{\{a, b\}\}$) in $\mathscr{U}$.

viii. $\{\{a, b\}\} \subset \mathscr{U}$ is false because $\{a, b\} \notin \mathscr{U}$.

ix. $\{a, b, c, d\} \subset \mathscr{U}$ is false because c and d are not in $\mathscr{U}$.

Problem 2 from section 1.7

Revision

2. Let $\mathscr{X}$ be the set of pairs of real numbers (x, y) that are solutions to both the equation $x^2 + y^2 = 1$ and the equation $x^2 - y^2 = 1$. Prove that $(1, 0) \in \mathscr{X}$ and $(-1, 0) \in \mathscr{X}$.

   Starting with $(1, 0)$ we can plug that into both equations and see $1^2 + 0^2 = 1 + 0$ which equals $= 1$, moving to the other equation, we can see similarly $1^2 - 0^2 = 1 - 0$ which, also, equals $= 1$. Thus, $(1, 0)$ satisfies both entrance criteria and is thus $\in \mathscr{X}$. As for the second pair, $(-1, 0)$, we can see that in both equations we square x thus resulting in 1 and -1 appearing almost identical. $-1^2 + 0^2 = 1 + 0$ which equals $= 1$, and in the second equation $-1^2 - 0^2 = 1 - 0$ which, also equals $= 1$. Thus both $(1, 0)$ and $(-1, 0) \in \mathscr{X}$.

Problem 3 from section 1.7

3. Let $\mathscr{X}$ be the set of pairs of real numbers (x, y) that are solutions to both equations $x^2 + y^2 = 1$ and $x^2 - y^2 = 1$. Prove that any $(x, y) \in \mathscr{X}$ is either $(1, 0)$ or $(-1, 0)$. Combined with the previous exercise, this shows that $(x, y) \in \mathscr{X}$ if and only if $(x, y) = (1, 0)$ or $(x, y) = (-1, 0)$.
   Take an arbitrary $(x, y) \in \mathscr{X}$. That (x, y) must be a solution to the system of equations, $x^2+y^2 = 1$ and $x^2 - y^2 = 1$, in order to be $\in mathscrX$. We can solve the system by adding the two equations together which would yield $2x^2 = 2$, which can be simplified to $x^2 = 1$ this means that $x = \pm 1$. If $x^2 = 1$, then in both equations y^2 must $= 0$, or else the equations wouldn't hold. Thus $\forall (x, y) \in mathscrX, x = \pm 1, y = 0$. This proves $(x, y) = (1, 0)$ or $(x, y) = (-1, 0)$.

Problem 4 from section 1.7

4. It is a fact that if is an odd integer, then there exists an integer such that $x = 2n+1$. Let $\mathscr{O}$ denote the set of odd integers. Use the fact to prove that if $x \in \mathscr{O}$, then x^2 is one more than a multiple of 4.

   Given $x = 2n + 1$, then $x^2 = (2n + 1)^2$. If we expand that out with the distributive property, we get $x^2 = 4n^2 + 4n + 1$. Now taking this we can see three distinct terms making up x^2. We have $4n^2$, lets call this a, we have $4n$, lets call this b, and finally 1, we'll let this be called term c. We know that 4 times any number will be a multiple of 4, because that is almost exactly the definition of a multiple of 4. Thus, term a, and b will always yield multiples of 4. Although these are being added together. Thus, we need to prove multiples of 4 being added will result in a multiple of 4. To do that, lets let $g = 4h$, and $f = 4j$ where h and $j \in \mathbb{R}$. Now if we say that $t = g + f$, that is equivalent to, $t = 4h + 4j$. We can factor that to $t = 4(h + j)$. From this we can see that t by definition is a multiple of 4 (it is a number (h+j) multiplied by 4). Thus, two multiples of 4 added together make another multiple of 4. Returning to our earlier point. If we know a and b are both multiples of 4, and adding them together is still a multiple of 4, then we have a multiple of 4, plus c. Since $c = 1$, we have a multiple of 4, plus 1. Thus any $x^2 \in \mathscr{O}$ is one more than a multiple of 4.

Citation: Used ChatGPT for Latex code help.