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Bk1139

HW 8

Question 7:

Solve the following questions from the Discrete Math zyBook

a. Exercise 6.1.5, sections b-d
   1. Exercise 6.1.5, section b
      i. **(C(13,1) * C(4,3) * C(12,2) * C(4,1) * C(4,1)) / C(52,5) ≈ 0.0211 ≈ 2.11%**
         1. There are **C(13,1) * C(4,3) * C(12,2) * C(4,1) * C(4,1) = 54,912** different possible hands that will result in three of a kind
         2. This is because we have 13 different ranks from which to choose the rank for the three of a kind and then 4 choose 3 different ways to combine the 4 suits of that rank
         3. Then we select two differing ranks from the remaining 12 ranks, and then for each of those 2 ranks there are 4 choose 1 options for the suit of that rank

      ii. There are **C(52,5) = 2,598,960** different possible 5-card hands that can be drawn

      iii. Therefore, the probability that a hand is a three of a kind is **54,912/2,598,960 ≈ 0.0211 ≈ 2.11%**
   2. Exercise 6.1.5, section c
      i. (C(4,1) * C(13,5)) / C(52,5) = 5,148/2,598,960 ≈ 0.00198 ≈ 0.198%
         1. There are **C(4,1) * C(13,5) = 5,148** different possible hands where all 5 cards have the same suit
            a. This is because there are 4 unique suits, from which we want to select 1 suit
            b. Then, since each suit has 13 cards, we want to determine all the different combinations for choosing 5 cards
         2. There are **C(52,5) = 2,598,960** different possible 5-card hands that can be drawn
         3. Therefore, the probability that a hand has all 5 cards with the same suit is **C(4,1) * C(13,5) / C(52,5) = 5,148/2,598,960 ≈ 0.00198 ≈ 0.198%**
   3. Exercise 6.1.5, section d
      i. **C(13,1) * C(4,2) * C(12,3) * $(C(4,1))^3$ / C(52,5) = 1,098,240/2,598,960 ≈ 0.4226**
         1. There are **C(13,1) * C(4,2) * C(12,3) * C(4,1) * C(4,1) * C(4,1) = 1,098,240** different possible hands that will result in two of a kind
            a. This is because we have 13 different ranks from which to choose the rank for the two of a kind and then 4 choose 2 different ways to combine the rank with the 4 suits
            b. Then we need to select three differing ranks from the remaining 12 ranks, and then for each of those 3 ranks there are 4 choose 1 options for the suit of that rank
         2. There are **C(52,5) = 2,598,960** different possible 5-card hands that can be drawn
         3. Therefore, the probability that a hand is a two of a kind is **1,098,240/2,598,960 ≈ 0.4226 ≈ 42.26%**

b. Exercise 6.2.4, sections a-d
   1. Exercise 6.2.4, section a
      i. 1 - C(39, 5) / C(52, 5) = (C(52, 5) – C(39, 5)) / C(52, 5)
         1. 2,023,203 / 2,598,960 ≈ 0.7785 ≈ 77.85%
            a. C(52, 5) – C(39, 5) = 2,023,203
               i. The total number of possible 5-card hands is C(52,5)
               ii. The complement of a 5-card hand with at least one club, is all hands without a club
               iii. total number of possible 5-card hands where there isn't a club is C(39,5)
                  1. There are 39 cards that aren't clubs and we want all combinations of 5-cards from that set
            b. **C(52, 5) = 2,598,960**
   2. Exercise 6.2.4, section b
      i. 1 – (C(13, 5) * 4^5 / C(52, 5)) = (C(52, 5) – C(13, 5) * 4^5) / C(52, 5)
         1. 1,281,072 / 2,598,960 ≈ 0.4929 ≈ 49.29%
            a. C(52, 5) – C(13, 5) * 4^5 = 1,281,072
               i. The total number of possible 5-card hands is C(52,5)
               ii. The complement of a 5-card hand with at least two cards with the same rank is the set of all 5-card hands where less than two (1) card has the same rank, or rather no two cards have the same rank
                  1. C(13, 5) * 4^5 = 1,317,888
                     a. If a five-card hand does not have any two cards of the same rank, it must hold that all cards in the hand have different ranks, and therefore there are 5 ranks.
                     b. First, we determine that there are C(13,5) ways to choose the 5 different ranks of the cards from the 13 existing ranks
                     c. Next, we know that for each rank there are four possible suits that can be chosen.
                     d. Therefore, there will be $4^{\text{\# of different ranks}}$ possibilities for the suit and rank of each card the hand
            b. **C(52, 5) = 2,598,960**

3. Exercise 6.2.4, section c
    i. **(2 * (C(13,1) * C(39,4))) – (C(13,1) * C(13,1) * C(26,3))**
        1. **(2 * 1,069,263 - 439,400) / 2,598,960 ≈ 0.6538 ≈ 65.38%**
        2. p(1-club ∪ 1-spade) = p(1-club) + p(1-spade) − p(1-club ∩ 1-spade)
            a. **p(1-club) = C(13,1) * C(39,4) / C(52,5) = 1,069,263 / 2,598,960**
                i. There are 13 different cards that are clubs, of which we want to choose only 1
                ii. Then there are 39 cards that are not clubs, of which we want to choose 4
            b. **p(1-spade) = C(13,1) * C(39,4) / C(52,5) = 1,069,263 / 2,598,960**
                i. There are 13 different cards that are spades, of which we want to choose only 1
                ii. Then there are 39 cards that are not spades, of which we want to choose 4
            c. **p(1-club ∩ 1-spade) = 2 * C(13,1) * C(26,3) / C(52,5) = 439,400 / 2,598,960**
                i. Determine the hands where there is exactly one club and one spade
                ii. There are 13 different cards that are clubs, of which we want to choose only 1
                iii. There are 13 different cards that are spades, of which we want to choose only 1
                iv. Then there are 26 cards which are not clubs or spades, of which we want to choose 3
4. Exercise 6.2.4, section d
    i. **1- C(26,5) / C(52,5)**
        1. **0.9747 ≈ 97.47%**
        2. p(≥1-club ∪ ≥1-spade) = 1 - p(¬(≥1-club ∪ ≥1-spade))
            a. 1 - p(0-club ∩ 0-spade)
                i. Use De Morgan’s law to apply the negation to the union
                    1. Negation of at least one = 0
                    2. Negation of the union is the intersection
                ii. The intersection of 0 clubs and 0 spades is all 5 card hands made entirely of hearts and diamonds
                iii. There are 26 cards that are hearts or diamonds, of which we want to choose 5
                iv. C(26,5) / C(52,5) = 65,780 / 2,598,960 ≈ 0.02531
            b. 1 - p(0-club ∩ 0-spade) = 1 - 0.02531 ≈ 0.9747

<u>Question 8:</u>

Solve the following questions from the Discrete Math zyBook

a. Exercise 6.3.2, sections a-e
    1. Exercise 6.3.2, section a
        i. There are P(7,7) = **5,040** different possible strings that can be generated from the set {a, b, c, d, e, f, g}
            1. $P(A) = 720 / 5{,}040 = 1/7 \approx .1429 \approx 14.29\%$
                a. If b falls in the middle, that means there are 6 possible options for 3 locations to the left of b and then there will be 3 possible options for 3 locations to the right of b
                b. The different permutations can be determined by calculating **$P(6,3) * 1 * P(3,3) = 6! = 720$**
            2. $P(B) = 1/2 = .5 = 50\%$
                a. There are 7 positions which b can occupy in the different permutations of the set
                b. In each of the 7 positions there are 6! different combinations of strings
                c. 7 positions * 6! = 7! which covers all possible options
                d. If we start from the left and work our way right, we will see that for the 6 remaining letters the probability that c is to the right decreases from 6/6, 5/6, 4/6, 3/6, 2/6, 1/6
                e. If we multiply each probability by 6! And then use the sum rule to add the different combinations we see there are 2,520 results or **1/2 *** (5,040) options
            3. $P(C) = 120/ 5{,}040 \approx 0.0238 \approx 2.38\%$
                a. Let us bundle "def" into one possible choice called k
                b. Therefore, there are 5 possible choices for the permutation of the set {a,b,c,**k**,g}
                c. $P(5,5) = 120$
    2. Exercise 6.3.2, section b
        i. $p(A|C) = p(A \cap C) / p(C) = (12 / 5{,}040) / (120 / 5{,}040) = 1/10$
            1. $p(A \cap C)$
                a. If b falls in the middle and "def" occurs together, then there are two possible segments in which it occurs
                    i. <u>d</u> <u>e</u> <u>f</u> <u>b</u> _ _ _
                    ii. _ _ _ <u>b</u> <u>d</u> <u>e</u> <u>f</u>
                b. For the remaining positions there are P(3,3) = 6 different permutations
                c. Using the sum rule, we see there are 12 different permutations in $A \cap C$
            2. If there are 12 different strings out of 5,040, then **$p(A \cap C) = 12 / 5{,}040$**

3. Exercise 6.3.2, section c
    - **i. $p(B|C) = p(B \cap C) / p(C) = (60 / 5{,}040) / (120 / 5{,}040) = 1/2 = .5$**
        - **1. $p(B \cap C)$ = 60 options / 5,040**
            - a. C
                - i. Let us bundle “def” into one possible choice called k
                - ii. Therefore, there are 5 possible choices for the permutation of the set {a,b,c,**k**,g}
                - **iii. P(5,5) = 120**
            - b. B
                - i. There are 5 positions which b can occupy in the different permutations of the set
                - ii. In each of the 5 positions there are 4! different combinations of strings
                - iii. 5 positions * 4! = 5! which covers all possible options
                - iv. If we start from the left and work our way right, we will see that for the 4 remaining letters the probability that c is to the right decreases from 4/4, 3/4, 2/4, 1/4
                - v. If we multiply each probability by 4! and then use the sum rule to add the different combinations we see there are 60 results or
                - **vi. 1/2 * (120) options**
4. Exercise 6.3.2, section d
    - **i. $p(A|B) = p(A \cap B) / p(B) = (360 / 5{,}040) / (1/2) = 720 / 5{,}040 = 1/7$**
        - 1. $p(A \cap B)$ = 360 options / 5,040
            - a. A
                - i. We know there are 720 possible permutations where b is in the middle of the string (1/7) * 5,040
            - b. B
                - i. Of the 720 permutations where b is in the middle, there is a 1/6 chance that c occupies any given position
                - ii. Therefore, there is a 3/6 chance c is to the left of b and 3/6 it is to the right
                - iii. 1/2 * 720 = 360 options
5. Exercise 6.3.2, section e
    - i. A and C are not independent. $p(A|C) = 1/10 \neq 1/7 = p(A)$
    - ii. B and C are independent. $p(B|C) = p(B) = 1/2$.
    - iii. A and B are independent. $p(A|B) = p(A) = 1/7$.

b. Exercise 6.3.6, sections b, c
    1. Exercise 6.3.6, section b
        i. p(The first 5 flips come up heads. The last 5 flips come up tails.) = **$(1/3)^5$ * $(2/3)^5$**
            1. Since the outcome of each flip is mutually independent, the probability of any flip being heads is 1/3 and tails 2/3
            2. Since we are trying to determine a sequence of heads and tails flips, we will use the product rule to determine the number of possible outcomes and probability
            3. Flipping heads 5 times is equal to $(1/3)^5$
            4. Similarly, flipping tails 5 times in a row is equal to $(2/3)^5$
    2. Exercise 6.3.6, section c
        i. p(The first flip comes up heads. The rest of the flips come up tails.) = **(1/3) * $(2/3)^9$**
            1. Since the outcome of each flip is mutually independent, the probability of any flip being heads is 1/3 and tails 2/3
            2. Since we are trying to determine a sequence of heads and tails flips, we will use the product rule to determine the number of possible outcomes and probability
            3. Flipping heads 1 time is equal to (1/3)
            4. Similarly, flipping tails 9 times in a row is equal to $(2/3)^9$

c. Exercise 6.4.2, section a
    1. Exercise 6.4.2, section a
        i. **≈ 0.4038**
            1. Fair die has a 1/6 probability of landing on any given number 1 – 6
            2. Biased die has a 0.15 probability of landing on each of 1, 2, 3, 4, 5 and .25 of landing on 6
            3. Let us define the sequence of rolls as the event X
            4. The probability of X given the fair die was chosen P(X|F) = (1/6) for each roll and since there are 6 rolls
                a. **P(X|F) = $(1/6)^6$**
            5. The probability of X given the fair die was not chosen $P(X|\overline{F})$ = .15 for each roll (1-5) and .25 for each roll of 6
                a. **$P(X|\overline{F})$ = .15 * .15 * .25 * .25 * .15 * .15 = $(.15)^4$ * $(.25)^2$**
            6. Since there is an equally likely chance of selecting either die at random
            7. $p(F) = p(\overline{F}) = .5$
            8. To determine p(F|X) we can plug the known values into Bayes' theorem formula
            9. **Numerator = $(1/6)^6$ * .5**
            10. **Denominator = $(1/6)^6$ * .5 + $(.15)^4$ * $(.25)^2$ * .5**
            11. **≈ 0.00001071673 / 0.00002653704 ≈ 0.4038**

Question 9:

Solve the following questions from the Discrete Math zyBook

a. Exercise 6.5.2, sections a, b
   1. Exercise 6.5.2, section a
      - **i. {0, 1, 2, 3, 4}**
        1. In any given 5 card hand there could be no aces, 1 ace, 2 aces, 3 aces, or 4 aces.
   2. Exercise 6.5.2, section b
      - i. A random variable X is a function from the sample space S of an experiment to the real numbers. X(S) denotes the range of the function X.
      - ii. The distribution of a random variable is the set of all pairs (r, p(X = r)) such that r ∈ X(S).
      - iii. Distribution for each r
        1. **0**
           - a. If there are no aces in the hand, we want to know all combinations of the remaining 48 cards in a hand from which we can choose 5
           - b. C(48,5) = 1,712,304
           - c. C(52,5) = 2,598, 960
           - **d. (0, (C(48,5)/C(52,5))** ≈ (0, 0.6588) ≈ 65.88 %
        2. **1**
           - a. If there is 1 ace in the hand, there are C(4,1) = 4 ways to choose the ace, and we want to know all combinations of the remaining 48 cards in a hand from which we can choose 4
           - b. C(4,1) = 4
           - c. C(48,4) = 194,580
           - d. C(4,1) * C(48,4) = 778,320
           - e. C(52,5) = 2,598, 960
           - **f. (1, (C(4,1) * C(48,4) / C(52,5))** ≈ (0, 0.2995) ≈ 29.95 %
        3. **2**
           - a. If there are 2 aces in the hand, there are C(4,2) = 6 ways to choose the aces, and we want to know all combinations of the remaining 48 cards in a hand from which we can choose 3
           - b. C(4,2) = 6
           - c. C(48,3) = 17,296
           - d. C(4,2) * C(48,3) = 103,776
           - e. C(52,5) = 2,598,960
           - **f. (2, (C(4,2) * C(48,3) / C(52,5))** ≈ (0, 0.0399) ≈ 3.99 %
        4. **3**
           - a. If there are 3 aces in the hand, there are C(4,3) = 4 ways to choose the aces, and we want to know all combinations of the remaining 48 cards in a hand from which we can choose 2
           - b. C(4,3) = 4
           - c. C(48,2) = 1,128
           - d. C(4,3) * C(48,2) = 4,512
           - e. C(52,5) = 2,598,960
           - **f. (3, (C(4,3) * C(48,2) / C(52,5))** ≈ (0, 0.0017) ≈ 0.17 %

5. **4**
    a. If there are 4 aces in the hand, there are C(4,4) = 1 way to choose the aces, and we want to know all combinations of the remaining 48 cards in a hand from which we can choose 1
    b. C(4,4) = 1
    c. C(48,1) = 48
    d. C(4,4) * C(48,1) = 48
    e. C(52,5) = 2,598,960
    f. **(4, (C(4,4) * C(48,1) / C(52,5))** ≈ (0, 0.000018) ≈ 0.0018 %

b. Exercise 6.6.1, section a
    1. Exercise 6.6.1, section a
        i. **E[G] = 0 * (3/45) + 1 * (21/45) + 2 * (21/45) = 1.4**
            1. There are 10 students in total, of which we want to determine all combinations of selecting two for the student council
                a. **C(10,2) = 45**
            2. The range of G = number of girls chosen = **{0, 1, 2}**
                a. 2 boys (0 girls) could be selected as student council representatives
                b. 1 girl could be selected as student council representative
                c. 2 girls could be selected as student council representatives
            3. The distribution for each possible outcome is as follows:
                a. **0**
                    i. We will segment the potential outcomes to ways to choose 2 boys and 0 girls and then use the product rule to determine the number of combinations
                    ii. There are C(3,2) ways to select 2 boys from 3
                    iii. There are C(7,0) ways to select 0 girls from 7
                    iv. C(3,2) = 3
                    v. C(7,0) = 1
                    vi. There is a **3/45** probability that two boys are selected
                b. **1**
                    i. We will segment the potential outcomes to ways to choose 1 girl and ways to choose 1 boy and then use the product rule to determine the number of combinations
                    ii. C(7,1) = 7 ways to choose 1 girl from 7
                    iii. C(3,1) = 3 ways to choose 1 boy from 3
                    iv. C(7,1) * C(3,1) = 21
                    v. There is a **21/45** probability that one girl and one boy are selected
                c. **2**
                    i. We will segment the potential outcomes to ways to choose 0 boys and 2 girls and then use the product rule to determine the number of combinations
                    ii. There are C(3,0) ways to select 0 boys from 3
                    iii. There are C(7,2) ways to select 2 girls from 7
                    iv. C(3,0) = 1
                    v. C(7,2) = 21
                    vi. C(7,2) * C(3,0) = 21
                    vii. There is a **21/45** probability that two girls are selected

c. Exercise 6.6.4, sections a, b
    1. Exercise 6.6.4, section a
        i. E[X] = 91 * (1/6) = 91/6 ≈ 15.166
            1. Range
                a. A fair die has a range of {1, 2, 3, 4, 5, 6}
                b. Therefore, the range of X = {1^2, 2^2, 3^2, 4^2, 5^2, 6^2} ={1, 4, 9, 16, 25, 36}
            2. Since it is a fair die, there is a 1/6 chance that the die lands on any value in the range
            3. $\sum\{1, 4, 9, 16, 25, 36\}$ = 91
            4. 91 * (1/6) = 91/6 ≈ 15.166
    2. Exercise 6.6.4, section b
        i. E[Y] = 24/8 = 3
            1. Range
                a. A fair coin has a range of {H,T}
                b. The range of the number of heads from a fair coin tossed 3 times = {0, 1, 2, 3}
                    i. All three flips could result in tails, 1 head, 2 heads, or 3 heads
                c. Therefore, the range of Y = {0^2, 1^2, 2^2, 3^2} = {0, 1, 4, 9}
            2. Possible outcomes
                a. There are 2 possible choices for any given coin flip, and since there are 3 flips there are **2^3 = 8 possible sequences** that are generated from the flips
                b. 0 Heads
                    i. There is only **1 possible way** to flip all tails
                c. 1 Heads
                    i. There are **3 different ways** to position the 1 heads flip in the sequence
                        1. 1st, 2nd, or 3rd
                d. 2 Heads
                    i. There are **3 different ways** to position the 1 tails flip in the sequence with the heads flips occupying the other two positions
                        1. 1st, 2nd, or 3rd
                e. 3 Heads
                    i. There is only **1 possible way** to flip all heads
            3. To calculate E[Y] we must sum the probability of each Y times the value of Y
            4. E[Y] = 1/8 * 0 + 3/8 * 1 + 3/8 * 4 + 1/8 * 9 = 24/8 = 3

d. Exercise 6.7.4, section a
    1. Exercise 6.7.4, section a
        i. **E[X] = 1**
            1. There are 10 students and therefore 10 coats
            2. Define E[X], as the expected number of children who get their own coats
            3. Define X as the random variable denoting whether a child receives his or her own coat
            4. If a child receives his or her own coat the expected value will be 1 and if not, it will be 0
            5. Since the coats are chosen uniformly at random to be given to the children, each X_i has the same distribution
            6. $E[X] = E[X_1]+\cdots+E[X_m]=mE[X_1]$
            7. $10E[X_1]$
            8. $X_1 = 1/ 10$ because there 10 possible coats for 1 person
            9. 10 * 1/10 = 1

Question 10:

Solve the following questions from the Discrete Math zyBook

a. Exercise 6.8.1, sections a-d
   1. Exercise 6.8.1, section a
      i. **$C(100,98) * .99^{98} * .01^{2} \approx .1849$**
         1. We know there is a sequence of 100 independent Bernoulli trials (100 circuit boards) = n
            a. n = 100
         2. There is a 99% chance of success and 1% failure
            a. p = .99
            b. q = .01
         3. If there are 2 defects, we want to know the likelihood of 98 successes
         4. $C(100,98) * .99^{98} * .01^{2}$
   2. Exercise 6.8.1, section b
      i. **$1 – (C(100,100) * .99^{100} * .01^{0} + C(100,99) * .99^{99} * .01^{1}) \approx .264$**
         1. To determine the probability that there are at least two defects, we will determine the probability that there are no defects or exactly one defect and subtract the sum of those two from 1 to determine the probability there are at least 2 defects
            a. No defects
               i. **$C(100,100) * .99^{100} * .01^{0} \approx .366$**
            b. 1 defect
               i. **$C(100,99) * .99^{99} * .01^{1} \approx .3697$**
         2. The sum of these two scenarios is .7357
         3. 1 - .7357 = .264
   3. Exercise 6.8.1, section c
      i. **100 – (100*.99) = 1**
         1. $E[K]=E[X_1+X_2+\cdots+X_n] =E[X_1]+E[X_2]+\cdots+E[X_n]=np$
         2. n = 100
         3. p = .99
         4. The expected number of successfully built circuit boards is 99
         5. Therefore, the expected number of circuit boards built with defects is 100-99 = 1
   4. Exercise 6.8.1, section d
      i. The probability that out of 100 circuit boards at least 2 have defects is **≈ .395**
         1. We will evaluate **50 batches,** and want to determine the likelihood that all batches are successfully made and subtract that from 1
            a. If there is a defect in a batch there will be at least 2 defects to the 100 circuit boards, so if the production does not result in at least 2 defects it must be such that there are no defects
            b. **$1 – (C(50,50) * .99^{50} * .01^{0}) \approx 1 - .605 \approx .395$**
      ii. The expected number of circuit boards with defects out of the 100 made is **1**
         1. $E[K]=E[X_1+X_2+\cdots+X_n] =E[X_1]+E[X_2]+\cdots+E[X_n]=np$
         2. n = 50
         3. p = .99
         4. The expected number of successfully built circuit boards is 50*.99 * 2 = 99
         5. Therefore, the expected number of circuit boards built with defects is **100-99 = 1**

b. Exercise 6.8.3, section b
    1. Exercise 6.8.3, section b
        i. The probability that you reach an incorrect conclusion if the coin is biased is ≈ .351
            1. If the coin is biased, the incorrect conclusion will be reached when it is determined that the coin is a fair coin (negation of a biased coin)
            2. So, we need to determine when it will be concluded that the coin is fair when it is in fact biased
            3. The coin will be determined to be fair when there are at least 4 heads
                a. Range = {4, 5, 6, 7, 8, 9, 10}
            4. The probability that a biased coin flipped 10 times results in at least 4 heads is equivalent to 1 – the sum of the probabilities that it is less than 4 heads or {0, 1, 2, 3} heads
                a. 0 Heads
                    i. $C(10,0) * .3^0 * .7^{10} \approx$ **.028**
                b. 1 Heads
                    i. $C(10,1) * .3^1 * .7^9 \approx$ **.121**
                c. 2 Heads
                    i. $C(10,2) * .3^2 * .7^8 \approx$ **.233**
                d. 3 Heads
                    i. $C(10,3) * .3^3 * .7^7 \approx$ **.267**
                e. Sum ≈ .649
            5. 1 - .649 ≈ .351