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Bk1139

HW 11

Question 5:

a. Use mathematical induction to prove that for any positive integer n, $(n^3 + 2n) = 3m$, for some integer m
   1. **Theorem:** $(n^3 + 2n) = 3m$, for some integer m
   2. **Proof.**
      i. By induction on n
   3. **Base case:** $n = 1$
      i. When $n = 1$ the result equates to $1^3 + 2 = 3$
      ii. $3 = 3m$
      iii. $m = 1$
      iv. Therefore, when $n = 1$, $(n^3 + 2n) = 3m$, for some integer m is true and $(n^3 + 2n)$ is divisible by 3
   4. **Inductive step:** Suppose that for any positive integer k , $(k^3 + 2k) = 3m$, for some integer m, and is therefore divisible by 3, then we will show that $((k+1)^3 + 2(k+1)) = 3j$, for some integer j and is therefore divisible by 3
      i. $((k+1)^3 + 2(k+1)) = k^3 + 3k^2 + 3k + 1 + 2k + 2$
      ii. $= k^3 + 3k^2 + 5k + 3$
      iii. $= (k^3 + 2k) + (3k^2 + 3k + 3)$
      iv. $= 3m + (3k^2 + 3k + 3)$
         1. By the inductive hypothesis
      v. $= 3m + 3(k^2 + k + 1)$
         1. Since k is an integer, $(k^2 + k + 1)$ is also an integer which we can represent using another integer l
      vi. $= 3m + 3l$
      vii. $= 3(m + l)$
         1. Since m and l are both integers, m + l must also be an integer which we can represent using j
      viii. $= 3j$
      ix. Therefore, $((k+1)^3 + 2(k+1)) = 3j$ for some integer j, proving $((k+1)^3 + 2(k+1))$ is divisible by 3
      x. ■

b. Use strong induction to prove that any positive integer n ($n \geq 2$) can be written as a product of primes
    1. **Theorem:** For any positive integer n where ($n \geq 2$), n can be written as a product of primes
    2. **Proof.**
        i. By strong induction on n
    3. **Base case:** $n = 2$
        i. When $n = 2$ it is a product of itself as a prime number
    4. **Inductive step:** Assume that for the integer $k \geq 2$, any integer j within the range from 2 to k can be written as a product of primes. We will show that $k + 1$ can be written as a product of primes as well
        i. There are two potential cases for any given integer k+1, so we will address both with a proof by cases
        ii. k + 1 is not prime
            1. If $k + 1$ is not prime, and given that k is an integer, and therefore $k + 1$ is also an integer, it must be that $k + 1$ is the product of two integers a and b
            2. Before analyzing a and b, we will confirm that both are greater than or equal to the base case value of 2 and less than $k + 1$
                a. A prime number is only divisible by 1 and itself
                b. If $k + 1$ is not prime, then it must be divisible by at least two positive integers, a and b, other than 1 and itself by definition
                c. If a and b are both positive and not 1, they must both be greater than 1 or $\geq$ 2
                d. Since neither a nor b equals $k + 1$, and an integer cannot be divisible by a number larger than itself and result in an integer, which a and b both are, it must hold that a and b are both less than k+1
                e. If a and b are both less than k+ 1 and integers, it must hold that they are both $\leq k + 1 - 1$ which is the same as $\leq k$
                    i. The largest integer less than $k + 1 = k+1 -1$
                f. We have now shown that a and b are within the range 2 to k and therefore the inductive hypothesis can be applied
            3. a and b can then be further analyzed where either a or b, both, or none are themselves prime
            4. If either is prime it can be represented as a product of itself
            5. If either or both are not prime, then they would again be products of some other integers a′ and b′ each of which would be evaluated from the top ii. as its own non-prime number
            6. Step 5 will repeat as needed, and eventually there will be a sequence of prime numbers whose product equates to the product of a and b
        iii. k + 1 is prime
            1. If $k + 1$ is prime, then it is a product of itself
        iv. Therefore, we have shown that in both cases above, k+1 can be written as a product of primes
        v. ■

Question 6:

Solve the following questions from the Discrete Math zyBook:

a. Exercise 7.4.1, sections a – g
   1. Exercise 7.4.1, section a
      i. P(3) = $1^2 + 2^2 + 3^2$ = **14**
      ii. P(3) = (3(3+1)(2*3 + 1)) / 6 = (3*4*7)/6 = **14**
      iii. Both sides of the equation evaluate to 14
   2. Exercise 7.4.1, section b
      i. $\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$
   3. Exercise 7.4.1, section c
      i. $\sum_{j=1}^{k+1} j^2 = \frac{k+1((k+1)+1)(2(k+1)+1)}{6}$
      ii. Simplified below
      iii. $\sum_{j=1}^{k+1} j^2 = \frac{k+1(k+2)(2k+3)}{6}$
   4. Exercise 7.4.1, section d
      i. In an inductive proof that for every positive integer n,
      ii. $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$
      iii. The following must be proven in the base case:
         1. That both sides of the equation equal each other for the base case of the first positive number n = 1
            a. Left side of the equation
               i. When n = 1, the sum of j^2 from 1 to 1 is $1^2 = 1$
            b. Right side of the equation
               i. When n = 1, $\sum_{j=1}^{1} j^2 = \frac{1(1+1)(2(1)+1)}{6} = 1$
         2. Therefore, the base case for n = 1 is valid
   5. Exercise 7.4.1, section e
      i. In an inductive proof that for every positive integer n,
      ii. $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$
      iii. The following must be proven in the inductive step:
         1. $\sum_{j=1}^{k+1} j^2 = \frac{k+1((k+1)+1)(2(k+1)+1)}{6}$
         2. $\sum_{j=1}^{k+1} j^2 = \frac{k+1(k+2)(2k+3)}{6}$
   6. Exercise 7.4.1, section f
      i. The inductive hypothesis would be as follows:
         1. Assume that for a positive integer k, $\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$
         2. It would then be used in the below when proving for k+1
            a. $\sum_{j=1}^{k+1} j^2 = \sum_{j=1}^{k} j^2 + (k+1)^2$
            b. $= \boldsymbol{\frac{k(k+1)(2k+1)}{6} + (k+1)^2}$

7. Exercise 7.4.1, section g

   i. **Theorem:** For any positive integer n

      1. $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$

   ii. **Proof.**

      1. By induction on n

   iii. **Base case:** n = 1

      1. Left side of the equation

         a. When n = 1, the sum of j^2 from 1 to 1 is $1^2 = 1$

      2. Right side of the equation

         a. When n = 1, $\sum_{j=1}^{1} j^2 = \frac{1(1+1)(2(1)+1)}{6} = 1$

      3. Therefore, the base case for n = 1 is valid

   iv. **Inductive step:** Suppose that for positive integer k, $\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$, then we will show that $\sum_{j=1}^{k+1} j^2 = \frac{k+1((k+1)+1)(2(k+1)+1)}{6}$, which simplifies to $\sum_{j=1}^{k+1} j^2 = \frac{k+1(k+2)(2k+3)}{6}$

      1. $\sum_{j=1}^{k+1} j^2 = \sum_{j=1}^{k} j^2 + (k+1)^2$
      2. $= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$
      3. $= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)(k+1)}{6}$
      4. $= \frac{(k+1)[k(2k+1)+6(k+1)]}{6}$
      5. $= \frac{(k+1)[2k^2+k+6k+6]}{6}$
      6. $= \frac{(k+1)[2k^2+7k+6]}{6}$
      7. $= \frac{(k+1)[(2k+3)(k+2)]}{6}$
      8. $= \frac{(k+1)(k+2)(2k+3)}{6}$
      9. Therefore, $\sum_{j=1}^{k+1} j^2 = \frac{k+1(k+2)(2k+3)}{6}$
      10. ■

b. Exercise 7.4.3, section c

1. **Theorem:** For n ≥ 1
    i. $\sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$
2. **Proof.**
    i. By induction on n
3. **Base case:** n = 1
    i. Left side of the equation
        1. When n = 1, the sum of j^2 from 1 to 1 is $\frac{1}{1^2}$ = 1
    ii. Right side of the equation
        1. When n = 1, $2 - \frac{1}{1}$ = 2 – 1 = 1
    iii. Therefore, the base case for n = 1 is valid
4. **Inductive step:** Suppose that for k ≥ 1, $\sum_{j=1}^{k} \frac{1}{j^2} \leq 2 - \frac{1}{k}$, then we will show that $\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 - \frac{1}{k+1}$
    i. $\sum_{j=1}^{k+1} \frac{1}{j^2} = (\sum_{j=1}^{k} \frac{1}{j^2}) + (\frac{1}{(k+1)^2})$
    ii. $\leq 2 - \frac{1}{k} + (\frac{1}{(k+1)^2})$
    iii. $\leq 2 - \frac{1}{k} + (\frac{1}{k(k+1)})$
    iv. $\leq 2 + (\frac{1}{k(k+1)}) - \frac{1}{k}$
    v. $\leq 2 + (\frac{1}{k(k+1)}) - \frac{1(k+1)}{k(k+1)}$
    vi. $\leq 2 + (\frac{1}{k(k+1)}) + \frac{-1(k+1)}{k(k+1)}$
    vii. $\leq 2 + \frac{1}{k(k+1)} + \frac{-k-1}{k(k+1)}$
    viii. $\leq 2 + \frac{1+-k-1}{k(k+1)}$
    ix. $\leq 2 + \frac{-k}{k(k+1)}$
    x. $\leq 2 + \frac{k(-1)}{k(k+1)}$
    xi. $\leq 2 + \frac{(-1)}{(k+1)}$
    xii. $\leq 2 - \frac{1}{(k+1)}$
    xiii. Therefore, $\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 - \frac{1}{k+1}$
    xiv. ■

c. Exercise 7.5.1, section a

1. **Theorem:** $(3^{2n} - 1) = 4m$, for some integer m, and is therefore divisible by 4
2. **Proof.**
    i. By induction on n
3. **Base case:** $n = 1$
    i. When $n = 1$, $3^2 - 1 = 8$
    ii. $8 = 4m$
    iii. $m = 2$
    iv. Therefore, when $n = 1$, $(3^{2n} - 1) = 4m$, for some integer m is true and $(3^{2n} - 1)$ is divisible by 4
4. **Inductive step:** Suppose that for any positive integer $k \geq 1$, if $(3^{2k} - 1) = 4m$, for some integer m, and is therefore divisible by 4, then we will show that $(3^{2(k+1)} - 1) = 4j$, for some integer j and therefore divisible by 4
    i. $(3^{2(k+1)} - 1) = (3^{2k+2} - 1)$
    ii. $= (3^{2k} * 3^2 - 1)$
    iii. $= (3^{2k} * 9 - 1)$
    iv. $= 9 *(4m+1) – 1$
        1. By the inductive hypothesis
    v. $= 36m + 9 – 1$
    vi. $= 36m + 8$
    vii. $= 4(9m + 2)$
        1. Since m is an integer, $(9m + 2)$ is also an integer which we can represent using another integer j
    viii. $= 4j$
    ix. Therefore, $(3^{2(k+1)} - 1) = 4j$ for some integer j, proving $(3^{2(k+1)} - 1)$ is divisible by 4
    x. ■