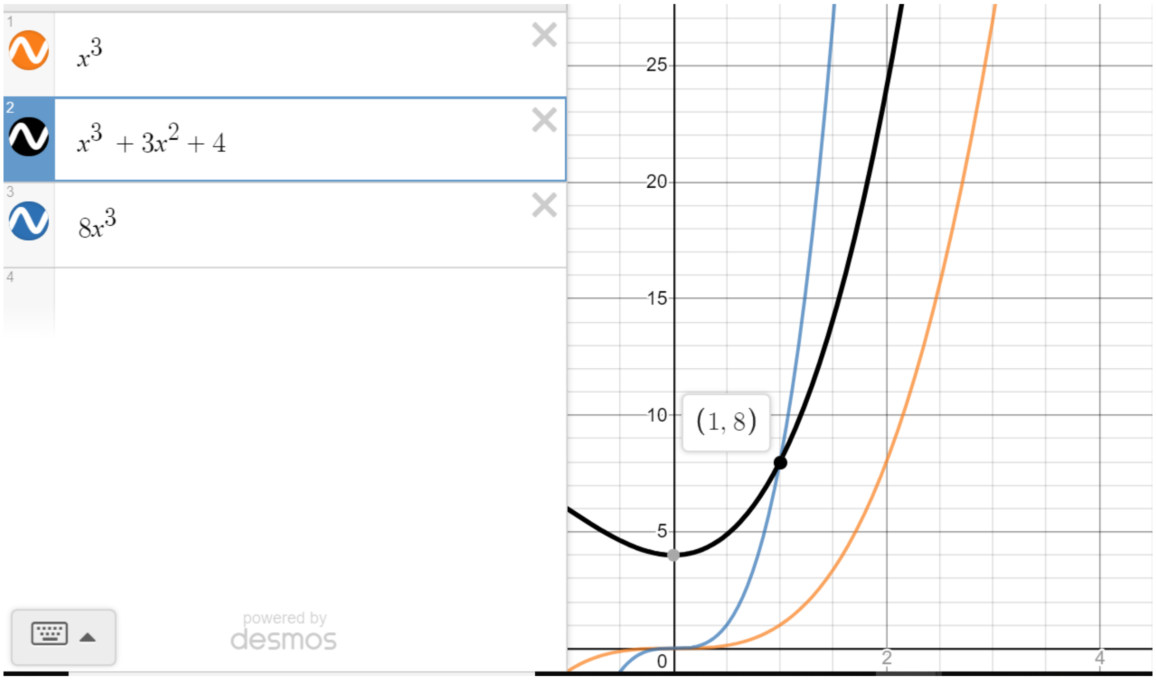
Ben Kadosh

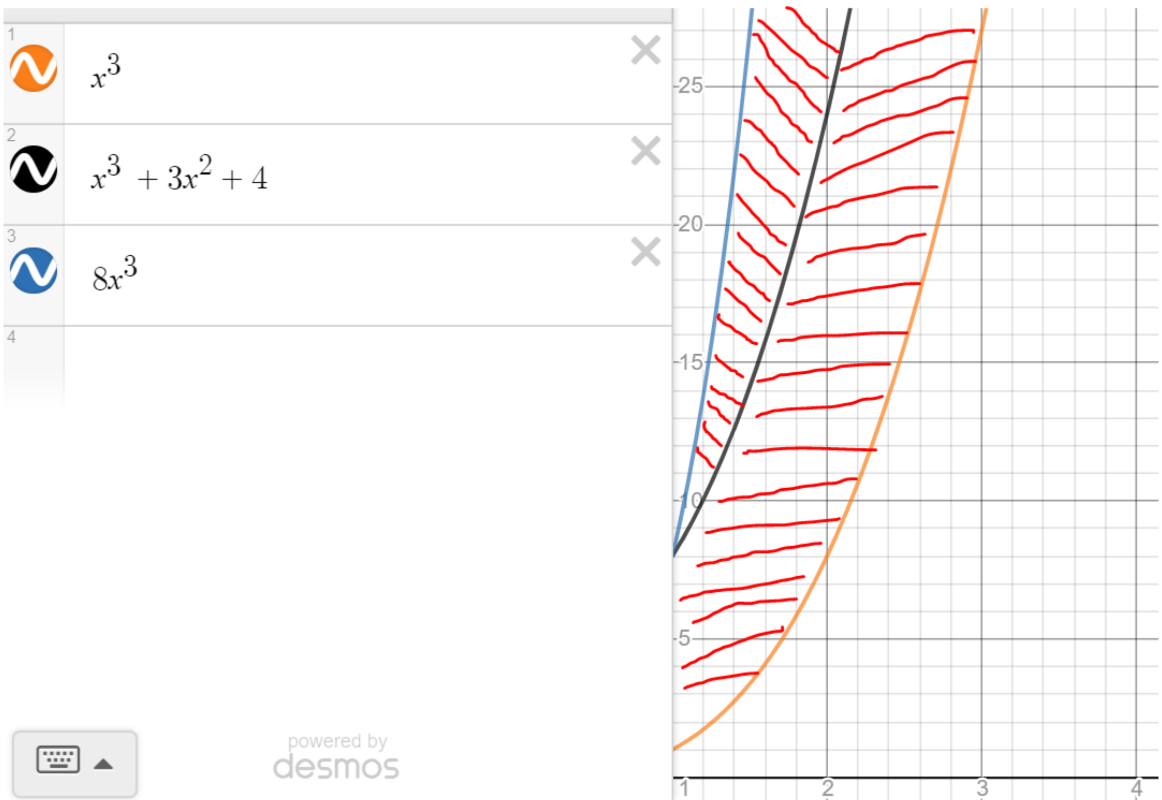
Bk1139

HW 7

Question 3:

a. Solve Exercise 8.2.2, section b from the Discrete Math zyBook
    1. Exercise 8.2.2, section b
        i. $n^3 + 3n^2 + 4 = \Theta(n^3)$
        ii. To prove this, we will show that $n^3 + 3n^2 + 4 = \Theta(n^3)$, if there exist positive real constants c_1, c_2, and a positive integer constant n_0 such that $c_2*n^3 \leq n^3 + 3n^2 + 4 \leq c_1*n^3$ for all $n \geq n_0$
            1. Let's address c_1
                a. $n^3 + 3n^2 + 4 \leq c_1*n^3$
                b. $n^3 + 3n^2 + 4 \leq n^3 + 3n^3 + 4n^3$
                c. $n^3 + 3n^2 + 4 \leq 8n^3$
                d. This inequality will hold true for all $n \geq n_0$ where $n_0 = 1$
            2. Let's address c_2
                a. $c_2*n^3 \leq n^3 + 3n^2 + 4$
                b. $n^3 \leq n^3 + 3n^2 + 4$
                c. This inequality will hold true for all $n \geq n_0$ where $n_0 = 1$
            3. Therefore, there exist positive real constants c_1, c_2, and a positive integer constant n_0 such that $c_2*n^3 \leq n^3 + 3n^2 + 4 \leq c_1*n^3$ for all $n \geq n_0$ where
                a. $c_1 = 8$
                b. $c_2 = 1$
                c. $n_0 = 1$
            4. Graphical support in screenshots below


1
x^3
2
$x^3 + 3x^2 + 4$
3
$8x^3$
4
powered by
desmos
25
20
15
10
5
0
2
4
$(1, 8)$


1
x^3
2
$x^3 + 3x^2 + 4$
3
$8x^3$
4
powered by
desmos
25
20
15
10
5
1
2
3
4

b. Solve Exercise 8.3.5, sections a-e from the Discrete Math zyBook
    1. Exercise 8.3.5, section a
        i. **The algorithm re-arranges the sequence of numbers such that:**
            1. **All numbers in the sequence from a_1 to $a_{i-1} < p$**
            2. **All numbers in the sequence from a_i to $a_j \geq p$**
        ii. For example (8, 1, 2, 3, 4) and p = 5
            1. Returns (1, 2, 3, 4, 8)
            2. i = 5
            3. j = 5
            4. $a_i = 8$
            5. $a_j = 8$
            6. a_1 to $a_4 < 5$
            7. a_5 to $a_5 \geq 5$
        iii. Please note the example above is not meant to prove that the statement holds for all sequences but rather to show that it holds for one as additional support*
    2. Exercise 8.3.5, section b
        i. **The total number of times that the lines "i := i + 1" or "j := j - 1" are executed on a sequence of length n, only depends on the length of the sequence**
            1. **The total number of executions = n-1**
            2. n-1 is the difference between j and i's initial values
            3. We want the difference because the difference is the number of times i and/or j will need to increment/decrement respectively to satisfy the requirement that i is not less than j
    3. Exercise 8.3.5, section c
        i. **The total number of times that the swap operation is executed depends on the actual values of the numbers in the sequence, the length of the sequence, and the value of p**
            1. **Input that minimizes swaps: the swap operation can occur zero times** in sequences where values are already listed in sequential order
                a. (-3, -2, -1, 0, 1, 2, 3)
                b. p = 0
            2. **Input that maximizes swaps: the swap operation can occur a max of n/2** times in a sequence where the right half of the sequence is less than p and the left half is greater than or equal to p
                a. (1, 2, 3, 4, -1, -2, -3, -4)
                b. p = 0
                c. Would require n/2 = 4 swap operations
                    i. (n-1)/2 for sequences with an odd length
                d. Each swap accounts for 2 numbers in the sequence, so 2 * (n/2) swaps ensures we cover all n numbers in the sequence
    4. Exercise 8.3.5, section d
        i. **The lower bound is Ω (n)**
        ii. In examining the worst-case input to determine an asymptotic lower bound on the time complexity of the algorithm, the number of swaps that will occur is n/2 times
        iii. If a swap occurs, then i will increment and j will decrement at least once after the swap
        iv. If a swap occurs n/2 times, then i will increment at least n/2 times and j will decrement at least n/2 times

v. $n/2$ increments for i and $n/2$ decrements for j results in $(n/2 + n/2) = 2n/2 = n$, which means n numbers in the sequence have been traversed through such that i will no longer be less than j, but rather equal j
vi. With a worse case input, we see that at a minimum of $n/2$ operations occur in each of the following sections:
   1. While ($i < j$ and $a_i < p$)
   2. While ($i < j$ and $a_j \geq p$)
   3. If ($i < j$), swap a_i and a_j
vii. This results in $3n/2$ operations, with the rest of the operations in the program being declarations of variables and return values, such that they are a constant value
viii. Therefore, the lower bound is **Ω (n)**

5. Exercise 8.3.5, section e
   i. The upper bound is O(n)
   ii. Similarly, in examining the worst-case input to determine an asymptotic upper bound on the time complexity of the algorithm, the number of swaps that will occur is $n/2$ times
   iii. We can see that $3n/2$ operations occur within the outer while loop and there are additional operations performed outside the loop that are at most a constant d number of operations
   iv. Therefore, we can say that the algorithm is of the form $cn + d$ where c and d are constants in a linear function
   v. Therefore, the upper bound is **O(n)**

Question 4:

Solve the following questions from the Discrete Math zyBook

a. Exercise 5.1.1, sections b, c
    1. Exercise 5.1.1, section b
        i. **$40^7 + 40^8 + 40^9$**
            1. Let D be the set of digits, L the set of letters, and S the set of special characters. The three sets are mutually disjoint, so the total number of characters is $|D \cup L \cup S| = |D| + |L| + |S| = 10 + 26 + 4 = 40$
            2. Each of the characters in the string can be any of the 40 characters, so there is a total of 40^n strings of length n for each n.
    2. Exercise 5.1.1, section c
        i. **$(14*40^6) + (14*40^7) + (14*40^8)$**
            1. Let D be the set of digits, L the set of letters, and S the set of special characters. The three sets are mutually disjoint, so the total number of characters is $|D \cup L \cup S| = |D| + |L| + |S| = 10 + 26 + 4 = 40$
            2. Each of the characters in the string other than the first can be any of the 40 characters, so there are a total of 40^{n-1} strings of length n-1.
            3. If a character cannot be a letter, then the total number of characters is $|D \cup S| = |D| + |S| = 10 + 4 = 14$
            4. The first character in the string can be any of the 14 characters
            5. So, for length n strings, there are $14*40^{n-1}$ different possible passwords

b. Exercise 5.3.2, section a
    1. Exercise 5.3.2, section a
        i. The number of strings over the set {a, b, c} that have length 10 in which no two consecutive characters are the same is:
            1. **$3*2^9$**
                a. There are three possible choices for the first character and each character thereafter from 2-10 only has two possible choices dictated by the previous character

c. Exercise 5.3.3, sections b, c
    1. Exercise 5.3.3, section b
        i. **$(10 * 9 * 8) * 26^4 = 720 * 26^4$**
            1. There are 10 ways to fill the first digit location, 9 ways to fill the second digit location, and 8 ways to fill the third digit location if no digit can appear more than once
            2. There are 26 ways to fill each of the 4 letter locations
    2. Exercise 5.3.3, section c
        i. **(10 * 9 * 8) * (26 * 25 * 24 * 23)**
            1. There are 10 ways to fill the first digit location, 9 ways to fill the second digit location, and 8 ways to fill the third digit location if no digit can appear more than once
            2. There are 26 ways to fill the first letter location, 25 ways to fill the second letter location, and 24 ways to fill the third letter location, and 23 ways to fill the fourth letter location if no letter can appear more than once

d. Exercise 5.2.3, sections a, b

1. Exercise 5.2.3, section a

   i. **Define the function f: $B^9 \rightarrow E_{10}$ such that if $x \in B^9$, then f(x) is obtained by adding a 0 to the end of the binary string, x, if there is an even number of 1's in x, and adding a 1 to the end of the binary string, x, if there is an odd number of 1's in x.**

   ii. To show it is a bijection we can see that it **has a well-defined inverse** such that dropping the last character in each string of the target maps to a unique element in the domain

      1. The inverse of a function f that maps set B^9 to set E_{10} is a function g that maps B^9 to E_{10} such that for every $s \in B^9$ and every $t \in E_{10}$, f(s) = t, if and only if g(t) = s.

   iii. In addition, to show it is a bijection we **will prove that it is both one-to-one and onto**

      1. **One-to-one**

         a. Since a set is a collection of unique elements, and B is a set, B must only contain unique elements, which we know to be true as {0, 1}

         b. since B^9 is the cartesian product of a set of unique elements, the resulting binary strings will all be unique by definition

         c. Since each element, x, in B^9 is unique, adding a 0 or 1 to each of them in f(x), would not result in the creation of any duplicate elements in the target

         d. More formally, if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$ since each unique x maps to a unique f(x)

      2. **Onto**

         a. In order to prove that the function is onto, we will first show that the cardinality of the target E_{10} is equal to that of B^9

         b. E_{10} is a subset of B^{10}, since B^{10} is the set containing all binary strings of length 10 and E_{10} is a set containing all binary strings of length 10 where there is an even number of 1's

         c. For the number of 1's to be even in E_{10} each element must have either 0, 2, 4, 6, or 8 1's.

         d. That means E_{10} contains half the elements of B^{10} with the rest of the elements being those that have an odd number of 1's

         e. If the cardinality of $B^{10} = 2^{10}$, and E_{10} has half the number of elements, E_{10}'s cardinality must be $2^{10}/2 = 2^9$

         f. $|B^9| = 2^9$

         g. Since $|E_{10}| = |B^9|$ and we proved that the function is one-to-one, it must be such that the function is onto, and that for each y in the target there exists an x in the domain that maps to it

2. Exercise 5.2.3, section b

   i. **$|E_{10}| = |B^9| = 2^9$**

      1. We have shown there is a bijection between B^9 and E_{10}, therefore the cardinalities of both sets must be equal as a result of the definition of a bijection.

Question 5:

Solve the following questions from the Discrete Math zyBook

a. Exercise 5.4.2, sections a, b
   1. Exercise 5.4.2, section a
      i. **$2*(10^4) = 20,000$**
         1. Since the first 3 digits are already determined there are 4 digits that can have any of 10 possible digits
         2. Therefore, there are 10^4 possible endings to the phone numbers
         3. Since there are 10^4 options for both the numbers starting in 824 and 825, we must multiply the possible numbers 10^4 by 2
   2. Exercise 5.4.2, section b
      i. **$2* P(10,4) = 10,080$**
         1. If the last four digits are all different, then the number of possible choices decreases by one after each choice
         2. To determine the possible outcomes, we will calculate the number of permutations $P(10,4) = 5,040$
         3. Since there are 5,040 options for both the numbers starting in 824 and 825, we must multiply the possible numbers by 2

b. Exercise 5.5.3, sections a-g
   1. Exercise 5.5.3, section a
      i. **$2^{10} = 1,024$**
         1. Each bit has 2 choices (0, 1) therefore there are 2 choices for each of the 10 bits, therefore there 2^{10}
   2. Exercise 5.5.3, section b
      i. **$2^7 = 128$**
         1. The first three bits of the string are already defined, therefore each of the remaining 7 bits has 2 choices (0, 1)
   3. Exercise 5.5.3, section c
      i. **$2^7 + 2^8 = 384$**
         1. 001
            a. The first three bits of the string are already defined, therefore each of the remaining 7 bits has 2 choices (0, 1)
         2. 10
            a. The first two bits of the string are already defined, therefore each of the remaining 8 bits has 2 choices (0, 1)
         3. Since the two beginnings to the string differ, the set containing all possible choices for each is disjointed from the other, so we add the possibilities from both

4. Exercise 5.5.3, section d
    - **i. $2^2 * 2^6 = 2^8 = 256$**
        1. There are two choices for each of the first two bits, so there are $2^2 = 4$ options for the starting bits
            a. 00
            b. 01
            c. 10
            d. 11
        2. Since in each case the first two and last two bits are defined, there are six bits that each have two choices (0,1) = 2^6 possible strings
        3. Therefore, there are 2^6 possible strings for each of the 4 options noted above, which equals $2^2 * 2^6 = 2^8$
5. Exercise 5.5.3, section e
    - **i. C(10, 6) = 210**
        1. Choose 6 locations for 0's from 10 locations
6. Exercise 5.5.3, section f
    - **i. C(9,6) = 84**
        1. Since the first bit is a 1, there are 9 locations remaining, of which we want to fill exactly 6 with 0's
7. Exercise 5.5.3, section g
    - **i. C(5,1) * C(5,3) = 5 * 10 = 50**
        1. If we cut the string in half, then there are 5 bits in each half
        2. We want to choose 1 location for a 1 from 5 locations
        3. We want to choose 3 locations for a 1 from 5 locations
        4. We then multiply both to bring it back to the 10-bit string

c. Exercise 5.5.5, section a
1. Exercise 5.5.5, section a
    - i. **C(30,10) * C(35,10)**
        1. There are **C(30,10)** "30 choose 10" different combinations to select a subset of 10 boys for the orchestra
        2. There are **C(35,10)** "35 choose 10" different combinations to select a subset of 10 girls for the orchestra
        3. Since the chorus will contain both girls and boys, we must piece together the segmented subsets for boys and girls above to combine them
        4. We will combine the different combinations using the product rule, which will result in **C(30,10) * C(35,10)** different ways to select the members of the chorus

d. Exercise 5.5.8, sections c - f
    1. Exercise 5.5.8 section c
        i. C(26,5) = 65,780
            1. There are 13 cards for each respective suit, so 26 cards in total that are hearts or diamonds
            2. Since we are looking for a five-card hand we are looking to fill all possible combinations of 5 cards from 26 cards
    2. Exercise 5.5.8, section d
        i. C(13,1) * C(48,1) = 624
            1. Let us segment this calculation into two parts
                a. Determining the number of possible combinations for four cards of the same rank
                    i. There are **13 unique ranks** in a 52-card deck, each of which has 4 cards of the same rank
                        1. In order to have four cards of the same rank we want to know how many different combinations exist for selecting 1 rank from the 13 that exist
                        2. C(13,1)
                b. Determining the number of possible combinations for choosing the 5^{th} card in the hand
                    i. Since we are looking for all combinations of five-card hands with four cards of the same rank, it must be such that one of the cards is not of the same rank, therefore there are 52 – 4 = 48 cards of a different rank that could be the 5^{th} card in the hand
                    ii. So we want to determine the number of different possible outcomes for selection 1 card from 48
                    iii. **C(48,1)**
            2. Since both segments above belong to one calculation, the product rule is used to determine the full number of combinations
    3. Exercise 5.5.8, section e
        i. **C(13,1) * C(12,1) * C(4,3) * C(4,2) = 3,744**
            1. Let us segment this calculation into two parts
                a. Determining the number of possible combinations for selecting the ranks
                    i. There are C(13,1) ways to choose the first rank
                    ii. Since the second rank will be different from the first, there are C(12,1) ways to choose the second rank
                    iii. Therefore, there are C(13,1) * C(12,1) different combinations for determining the two different ranks in the full house
                b. Determining the number of possible combinations for selecting the suits of each rank selected
                    i. There are C(4,3) ways to choose three suits for four cards of the same rank
                    ii. There are C(4,2) ways to choose two suits for four cards of the same rank
                    iii. Therefore, there are C(4,3) * C(4,2) different combinations for determining the suits of the cards in the hand with two different ranks

4. Exercise 5.5.8, section f
    i. **$C(13, 5) * C(4,1)^5$ = 1,317,888**
        1. If a five-card hand does not have any two cards of the same rank, it must hold that al of the cards have different ranks, and therefore there are 5 ranks.
        2. First, we determine that there are C(13,5) ways to choose the 5 different ranks of the cards from the 13 existing ranks
        3. Next, we know that for each rank there are four possible suits that can be chosen C(4,1).
        4. Therefore, there will be $4^{\#\text{ of different ranks}}$ possibilities for the suit of all cards in the hand

e. Exercise 5.6.6, sections a, b
    1. Exercise 5.6.6, section a
        i. C(56, 5) * C(44, 5)
            1. To ensure there is the same number of Demonstrators as Repudiators in the committee, we must divide the committee in half such that 5 members from each party are selected
            2. We must then choose 5 members from the 56 repudiators and 5 members from the 44 demonstrators
    2. Exercise 5.6.6, section b
        i. 44 * 43 * 56 * 55 = 5,827,360
            1. For the Demonstrator party there would be P(44,2) = 44 * 43 possible choices for the speaker and vice speaker
                a. 44 for all members who could be the speaker and then 1 less for the vice speaker once the speaker is selected
            2. For the Repudiator party there would be P(56,2) = 56 * 55 possible choices for the speaker and vice speaker
                a. 56 for all members who could be the speaker and then 1 less for the vice speaker once the speaker is selected

Question 6:

Solve the following questions from the Discrete Math zyBook

a. Exercise 5.7.2, sections a, b
    1. Exercise 5.7.2, section a
        i. **C(52, 5) – C(39, 5) = 2,023,203**
            1. The total number of possible 5-card hands is C(52,5)
            2. The complement of a 5-card hand with at least one club, is all hands without a club
            3. total number of possible 5-card hands where there isn't a club is C(39,5)
                a. There are 39 cards that aren't clubs and we want all combinations of 5-cards from that set
    2. Exercise 5.7.2, section b
        i. **C(52, 5) – C(13, 5) * 4^5 = 1,281,072**
            1. The total number of possible 5-card hands is C(52,5)
            2. The complement of a 5-card hand with at least two cards with the same rank is the set of all 5-card hands where less than two (1) card has the same rank, or rather no two cards have the same rank
                a. C(13, 5) * 4^5 = 1,317,888
                    i. If a five-card hand does not have any two cards of the same rank, it must hold that al of the cards have different ranks, and therefore there are 5 ranks.
                    ii. First, we determine that there are C(13,5) ways to choose the 5 different ranks of the cards from the 13 existing ranks
                    iii. Next, we know that for each rank there are four possible suits that can be chosen.
                    iv. Therefore, there will be $4^{\text{\# of different ranks}}$ possibilities for the suit and rank of each card the hand

b. Exercise 5.8.4, sections a, b
    1. Exercise 5.8.4, section a
        i. **5^{20}**
            1. If there are no restrictions, then for each of the 20 books there are 5 possible owners and if we use the product rule for each comic book, we will get 5*5*5*5*5…….
    2. Exercise 5.8.4, section b
        i. **20! ÷ $(4!)^5$**
            1. There are 20 choices (comic books) for which the first kid can receive 4
            2. There are 16 choices for which the second kid can receive 4
            3. There are 12 choices for which the third kid can receive 4
            4. There are 8 choices for which the fourth kid can receive 4
            5. There are 4 choices for which the fifth kid can receive 4
            6. The 4! repeats 5 times
            7. 20! = n! due to the original number of choices (comic books)

<u>Question 7:</u>

How many one-to-one functions are there from a set with five elements to sets with the following number of elements?

a. 4
    1. **0 one-to-one functions**
        i. It is impossible to have a one-to-one function where the cardinality of the domain is larger than that of the target
        ii. There must exist an element y in the target such that $x_1 \neq x_2$, but $f(x_1) = f(x_2) = y$

b. 5
    1. **5! = 120 one-to-one functions**
        i. No two elements from the domain can be mapped to the same element in the target
        ii. Since the cardinalities of both sets match, for the functions to be one-to-one each element in the domain must be mapped to a unique element in the target
            1. The first element in the domain can be mapped to any of the 5 elements in the target, so there are **5 choices**
            2. To maintain the function's one-to-one status:
                a. The second element in the domain can be mapped to any of the 4 remaining elements in the target, so there are **4 choices**
                b. The third element in the domain can be mapped to any of the 3 remaining elements in the target, so there are **3 choices**
                c. The fourth element in the domain can be mapped to any of the 2 remaining elements in the target, so there are **2 choices**
                d. The fifth element in the domain can be mapped to any of the 1 remaining elements in the target, so there is **1 choice**

c. 6
    1. **6 * 5 * 4 * 3 * 2 = 720 one-to-one functions**
        i. No two elements from the domain can be mapped to the same element in the target
            1. The first element in the domain can be mapped to any of the 6 elements in the target, so there are **6 choices**
            2. To maintain the function's one-to-one status:
                a. The second element in the domain can be mapped to any of the 5 remaining elements in the target, so there are **5 choices**
                b. The third element in the domain can be mapped to any of the 4 remaining elements in the target, so there are **4 choices**
                c. The fourth element in the domain can be mapped to any of the 3 remaining elements in the target, so there are **3 choices**
                d. The fifth element in the domain can be mapped to any of the 2 remaining elements in the target, so there are **2 choices**

d. 7

    1. **7* 6 * 5 * 4 * 3 = 2,520 one-to-one functions**
        i. No two elements from the domain can be mapped to the same element in the target
            1. The first element in the domain can be mapped to any of the 7 elements in the target, so there are **7 choices**
            2. To maintain the function's one-to-one status:
                a. The second element in the domain can be mapped to any of the 6 remaining elements in the target, so there are **6 choices**
            3. The third element in the domain can be mapped to any of the 5 remaining elements in the target, so there are **5 choices**
            4. The fourth element in the domain can be mapped to any of the 4 remaining elements in the target, so there are **4 choices**
            5. The fifth element in the domain can be mapped to any of the 3 remaining elements in the target, so there are **3 choices**