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Bk1139

HW 5

<u>Question 3:</u>

Solve the following questions from the Discrete Math zyBook:

a) Exercise 4.1.3, sections b, c
   1. Exercise 4.1.3, section b
      i. **f(x) = 1/(x^2−4) is not well defined for x = 2 and x = -2, therefore f(x) = 1/(x^2−4) is not a function from R to R**
   2. Exercise 4.1.3, section c
      i. $f(x) = \sqrt{x^2}$ **is a function from R to R. Its range is all real numbers greater than or equal to 0.**
         1. **Range = {x ∈ R: x ≥ 0}**

b) Exercise 4.1.5, sections b, d, h, i, l
   1. Exercise 4.1.5, section b
      i. **{4, 9, 16, 25}**
         1. {2^2, 3^2, 4^2, 5^2}
   2. Exercise 4.1.5, section d
      i. **{0, 1, 2, 3, 4, 5}**
         1. $\{0,1\}^5$ can have any range of 1's from 0 to 5
            a. 00000 = 0
            b. 00001 = 1
            c. 00011 = 2
            d. 00111 = 3
            e. 01111 = 4
            f. 11111 = 5
   3. Exercise 4.1.5, section h
      i. **{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}**
         1. A x A = {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}
            a. f(1, 1) = (1, 1)
            b. f(1, 2) = (2, 1)
            c. f(1, 3) = (3, 1)
            d. f(2, 1) = (1, 2)
            e. f(2, 2) = (2, 2)
            f. f(2, 3) = (3, 2)
            g. f(3, 1) = (1, 3)
            h. f(3, 2) = (2, 3)
            i. f(3, 3) = (3, 3)

4. Exercise 4.1.5, section i
    i. **{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)}**
        1. A x A = {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}
            a. f(1, 1) = (1, 2)
            b. f(1, 2) = (1, 3)
            c. f(1, 3) = (1, 4)
            d. f(2, 1) = (2, 2)
            e. f(2, 2) = (2, 3)
            f. f(2, 3) = (2, 4)
            g. f(3, 1) = (3, 2)
            h. f(3, 2) = (3, 3)
            i. f(3, 3) = (3, 4)
5. Exercise 4.1.5, section l
    i. **{∅, {2}, {3}, {2, 3}}**
        1. P(A) = {∅, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}}
            a. f(∅) = ∅
            b. f({1}) = ∅
            c. f({2}) = {2}
            d. f({3}) = {3}
            e. f({1, 2}) = {2}
            f. f({1, 3}) = {3}
            g. f({2, 3}) = {2, 3}
            h. f({1, 2, 3}) = {2, 3}

Question 4:

I. Solve the following questions from the Discrete Math zyBook:

a) Exercise 4.2.2, sections c, g, k
   1. Exercise 4.2.2, section c
      i. h: Z → Z. $h(x) = x^3$
         1. **The function is one-to-one**
            a. If we assume $f(x_1) = f(x_2)$ we can show that $x_1 = x_2$
            b. $f(x_1) = f(x_2)$ is equivalent to $(x_1)^3 = (x_2)^3$
            c. If we take the cubed root of both we prove $x_1 = x_2$ therefore the function is one-to-one.
         2. **The function is not onto**
            a. The cubed root of y = x
            b. However, some y values will result in x values that aren't in the domain as they are not integers
            c. For example, the cubed root of 3 is not an integer.
            d. Conversely there is no integer x such that $x^3 = 3$, which is an integer within the target.
   2. Exercise 4.2.2, section g
      i. f: Z × Z → Z × Z, f(x, y) = (x+1, 2y)
         1. **The function is one-to-one**
            a. If we look at the function f(x, y), we see that it results in an ordered pair comprised of two functions which are one-to-one
               i. x + 1
                  1. If we assume $f(x_1) = f(x_2)$ we can show that $x_1 = x_2$
                  2. $f(x_1) = f(x_2)$ is equivalent to $(x_1 + 1) = (x_2 + 1)$
                  3. If we subtract 1 from both sides, we prove $x_1 = x_2$ therefore the function is one-to-one.
               ii. 2y
                  1. If we assume $f(y_1) = f(y_2)$ we can show that $y_1 = y_2$
                  2. $f(y_1) = f(y_2)$ is equivalent to $2y_1 = 2y_2$
                  3. If we divide both sides by 2, we prove $y_1 = y_2$ therefore the function is one-to-one.
            b. Therefore, every unique combination of x, y will result in a unique ordered pair for f(x, y)
         2. **The function is not onto**
            a. The y value in the target's ordered pair is determined by the function 2y, which means each y value in the range will be even
            b. In this case x = y/2 for some x in the y coordinate of the domain
            c. However, some y values will result in x values that aren't in the domain as they are not integers
            d. For example, if y = 1, 1/2 is not an integer.
            e. Conversely there is no integer x such that 2x = 1.
            f. Furthermore, there is no combination of x, y such that f(x,y) = f(1,1)
            g. Therefore, f(x,y) will not have y values in the ordered pair that are odd and so the target of all integers does not equal the range of this function.

3. Exercise 4.2.2, section k
    i. f: $Z^+ \times Z^+ \rightarrow Z^+$, $f(x, y) = 2^x + y$.
        1. **The function is not one-to-one**
            a. The function is not one-to-one because different ordered pairs in the domain result in the same target value
                i. For example, f(1, 4) and f(2, 2) both result in 6.
        2. **The function is not onto**
            a. The range of the function f does not equal the target of the function
            b. The domain is that of positive integers (≥ 1), therefore the smallest values that can be entered into the function f(x,y) are 1 and 1
            c. This results in $2^1 + 1 = 3$
            d. The range of the function is every integer $\geq$ to 3
            e. However, the target of the function is every integer ≥ 1
            f. Therefore, there are elements in the target (1 and 2) that are not in the range of the function, and therefore it is not onto.

b) Exercise 4.2.4, sections b, c, d, g
1. Exercise 4.2.4, section b
    i. f: $\{0, 1\}^3 \rightarrow \{0, 1\}^3$
        1. **The function is not one-to-one**
            a. f(000) and f(100) = 100
        2. **The function is not onto**
            a. There is no x value in $\{0, 1\}^3$ such that f(x) = 000
2. Exercise 4.2.4, section c
    i. $\{0, 1\}^3 \rightarrow \{0, 1\}^3$
        1. $\{0, 1\}^3$ = {000, 001, 010, 011, 100, 101, 110, 111}
            a. f(000) = 000
            b. f(001) = 100
            c. f(010) = 010
            d. f(011) = 110
            e. f(100) = 001
            f. f(101) = 101
            g. f(110) = 011
            h. f(111) = 111
        2. **The function is one-to-one**
            a. For each unique value x in $\{0, 1\}^3$, f(x) results in a unique y
        3. **The function is onto**
            a. The range and target of the function are equal as can be seen in bullet 1

3. Exercise 4.2.4, section d
    i. $\{0, 1\}^3 \rightarrow \{0, 1\}^4$
        1. $\{0, 1\}^3$ = {000, 001, 010, 011, 100, 101, 110, 111}
            a. f(000) = 0000
            b. f(001) = 0010
            c. f(010) = 0100
            d. f(011) = 0110
            e. f(100) = 1001
            f. f(101) = 1011
            g. f(110) = 1101
            h. f(111) = 1111
        2. **The function is one-to-one**
            a. For each unique value x in $\{0, 1\}^3$, f(x) results in a unique y
        3. **The function is not onto**
            a. The range and target of the function are not equal
            b. There is no x value in $\{0, 1\}^3$ such that f(x), results in 1000
4. Exercise 4.2.4, section g
    i. A = {1, 2, 3, 4, 5, 6, 7, 8} and let B = {1}. f: P(A) → P(A). For X ⊆ A, f(X) = X – B
        1. **The function is not one-to-one**
            a. f({1,2}) and f({2}) = {2} so $f(x_1) = f(x_2)$, but x_1 and x_2 are not equal
        2. **The function is not onto**
            a. {1} is not in the range of f but is part of the target P(A)

II. Give an example of a function from the set of integers to the set of positive integers that is:

a) One-to-one, but not onto
    1. **Piecewise function: f(x) =**
        i. **2|x|, for x < 0**
        ii. **2x + 3, for x ≥ 0**
    2. One-to-one
        i. If the number is negative it will map to a unique even number in Z^+
        ii. If the number is greater than or equal to zero it will map to a unique odd number
        iii. Therefore, no two x's map to the same y
    3. Not Onto
        i. The range of the function will include all positive integers except for 1, which is in the target of Z^+
b) Onto, but not one-to-one
    1. **y = |x| + 1**
        i. Onto
            1. Absolute value of x will reflect the positive of any integer thus mapping it from Z to Z^+
            2. The +1 is to ensure 0 also falls within Z^+
        ii. Not one-to-one because x = -1 and x = 1 both result in 2

c) One-to-one and onto

   1. **Piecewise function: $f(x) =$**
      i. **$2|x| + 1$, for $x \leq 0$**
      ii. **$2x$, for $x > 0$**
   2. One-to-one
      i. If the number is negative or 0 it will map to a unique odd number in Z^+
      ii. If the number is positive it will map to a unique even number
      iii. Therefore, no two x's map to the same y
   3. Onto
      i. The range of the function will include all positive even and odd integers and therefore equals the target of Z^+

d) Neither one-to-one nor onto

   1. **$2x^2 + 1$**
      i. Not one-to-one
         1. $x = -1$ and $x = 1$ both result in 3
      ii. Not onto
         1. Since x is an integer, x^2 is an integer which we can represent as k
         2. Therefore, the function is of the form $2k + 1$ which will always result in an odd integer
         3. Therefore, there is no integer x, such that $2x^2 + 1 = 2$

<u>Question 5:</u>

Solve the following questions from the Discrete Math zyBook:

a) Exercise 4.3.2, sections c, d, g, i
   1. Exercise 4.3.2, section c
      - **i. The function has a well-defined inverse**
         1. The function is a bijection, so it must have a well-defined inverse
            - a. One-to-one
               - I. Assume $f(x_1) = f(x_2)$ and prove $x_1 = x_2$
               - II. $2x_1 + 3 = 2x_2 + 3$
               - III. Subtract three from both sides and then divide by 2
               - IV. $x_1 = x_2$
            - b. Onto
               - I. $y = 2x + 3$
               - II. $x = (y-3)/2$
               - III. Since all real numbers are mapped to all real numbers, for each y in the target there exists an x such that $x = (y-3)/2$
         2. **$f^{-1}(x) = (x-3)/2$**
   2. Exercise 4.3.2, section d
      - **i. The function does not have a well-defined inverse, because the function is not one-to-one and therefore not a bijection**
         1. **f({1}) and f({2}) both equal 1.**
   3. Exercise 4.3.2, section g
      - **i. The function has a well-defined inverse**
         1. f is one-to-one and onto, and therefore it is a bijection
            - a. One-to-one
               - I. $\{0, 1\}^3$ = {000, 001, 010, 011, 100, 101, 110, 111}
                  1. f(000) = 000
                  2. f(001) = 100
                  3. f(010) = 010
                  4. f(011) = 110
                  5. f(100) = 001
                  6. f(101) = 101
                  7. f(110) = 011
                  8. f(111) = 111
               - II. For each unique value x in $\{0, 1\}^3$, f(x) results in a unique y
            - b. Onto
               - I. The range of the function equals the target as can be seen above, therefore the function is onto
         2. f is a bijection, therefore it has a well-defined inverse
         3. **The output of f^{-1} is obtained by taking the input string and reversing the bits. For example, $f^{-1}(011) = 110$**

4. Exercise 4.3.2, section i
    i. **The function has a well-defined inverse**
        1. f(x, y) is a bijection, therefore it has a well-defined inverse
            a. One-to-one
                I. x+5
                    1. Assume $f(x_1) = f(x_2)$, prove $x_1 = x_2$
                    2. $x_1 + 5 = x_2 + 5$
                    3. Subtract 5 from both sides
                    4. $x_1 = x_2$
                II. y-2
                    1. Assume $f(y_1) = f(y_2)$, prove $y_1 = y_2$
                    2. $y_1 - 2 = y_2 - 2$
                    3. Add 2 to both sides
                    4. $y_1 = y_2$
            b. Onto
                I. x+5
                    1. y = x + 5
                    2. x = y – 5
                    3. For any given y in the target, there exists an x in the domain such that x = y – 5
                II. y-2 (shown as x-2)
                    1. y = x - 2
                    2. x = y + 2
                    3. For any given y in the target, there exists an x in the domain such that x = y + 2
        2. **$f^{-1}(x, y) = (x-5, y+2)$**

b) Exercise 4.4.8, sections c, d
    1. Exercise 4.4.8, section c
        i. **f o h(x) = $2x^2 + 5$**
            1. $f(x^2 + 1)$
            2. $2(x^2 + 1) + 3$
            3. $2x^2 + 5$
    2. Exercise 4.4.8, section d
        i. **h o f(x) = $4x^2 + 12x + 10$**
            1. h(2x + 3)
            2. $(2x + 3)^2 + 1$
            3. $(4x^2 + 12x + 9) + 1$

c) Exercise 4.4.2, sections b-d
    1. Exercise 4.4.2, section b
        i. **f o h(52) = 121**
            1. h(52) = 11
            2. f(11) = 121
    2. Exercise 4.4.2, section c
        i. **g o h o f(4) = 16**
            1. f(4) = 16
            2. h(16) = 4
            3. g(4) = 16

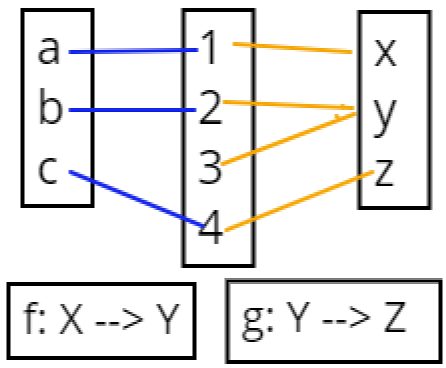
3. Exercise 4.4.2, section d
    - i. **$\lceil x^2/5 \rceil$**

d) Exercise 4.4.6, sections c-e

1. Exercise 4.4.6, section c
    - i. **h o f(010) = 111**
        1. f(010) = 110
        2. h(110) = 111
2. Exercise 4.4.6, section d
    - i. **The range of h o f = {101, 111}**
        1. $\{0, 1\}^3$ = {000, 001, 010, 011, 100, 101, 110, 111}
            - a. f(000) = 100
            - b. f(001) = 101
            - c. f(010) = 110
            - d. f(011) = 111
            - e. f(100) = 100
            - f. f(101) = 101
            - g. f(110) = 110
            - h. f(111) = 111
        2. Range of f(x) is {100, 101, 110, 111}
        3. h(x) for the strings in the set above is:
            - a. h(100) = 101
            - b. h(101) = 101
            - c. h(110) = 111
            - d. h(111) = 111
3. Exercise 4.4.6, section e
    - i. **The range of g o f = {001, 101, 011, 111}**
        1. $\{0, 1\}^3$ = {000, 001, 010, 011, 100, 101, 110, 111}
            - a. f(000) = 100
            - b. f(001) = 101
            - c. f(010) = 110
            - d. f(011) = 111
            - e. f(100) = 100
            - f. f(101) = 101
            - g. f(110) = 110
            - h. f(111) = 111
        2. Range of f(x) is {100, 101, 110, 111}
        3. g(x) for the strings in the set above is:
            - a. g(100) = 001
            - b. g(101) = 101
            - c. g(110) = 011
            - d. g(111) = 111

e) Extra credit: Exercise 4.4.4, sections c, d

1. Exercise 4.4.4, section c
    - i. Is it possible that f is not one-to-one and g o f is one-to-one?
        1. **No, it is not possible.**
            - a. **If f is not one-to-one, then, by definition, there exist elements x_1 and x_2 in the domain of f, such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$**
            - b. **If $f(x_1) = f(x_2)$ then let us represent both as y.**
            - c. **If we examine x_1 and x_2 in the following $g(f(x_1))$ and $g(f(x_2))$ and substitute from bullet b we see that for both x_1 and x_2 the result is g(y) and g(y)**
            - d. **Since $x_1 \neq x_2$, but $g(f(x_1)) = g(f(x_2))$ g o f cannot be one to one**
2. Exercise 4.4.4, section d
    - i. Is it possible that g is not one-to-one and g o f is one-to-one?
        1. **Yes, it is possible.**



        2.
            - a. g(f(a)) = x
            - b. g(f(b)) = y
            - c. g(f(c)) = z
        3. Notice that g is not one to one because g(2) = g(3) =y