

Numerical Methods of Differential Equations.

1. Calculating First Derivatives.

$$f'(x) \equiv \lim_{\Delta h \rightarrow 0} \frac{f(x + \Delta h) - f(x)}{\Delta h}$$

(i.e. the fundamental theorem of calculus)

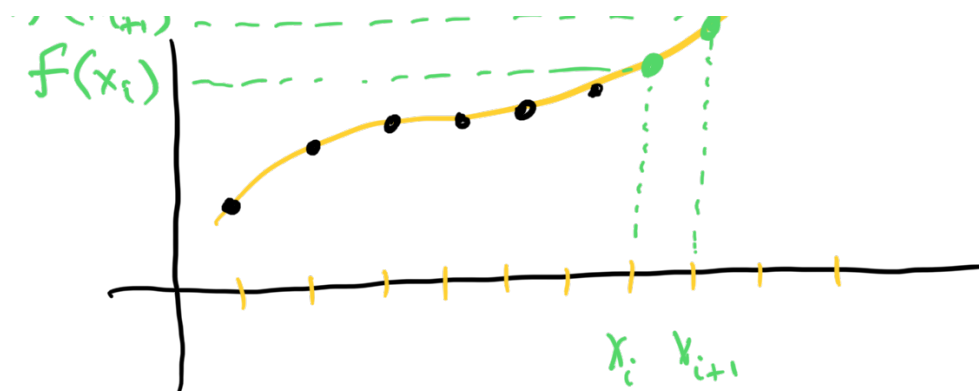
(i) Forward Difference Method

$$f'(x) \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

for "small" Δx

$f(x)$





$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{\Delta x}$$

Notation :

$$f(x_i) \rightarrow f_i$$

$$f(x_{i+1}) \rightarrow f_{i+1}$$

$$f'(x_i) \rightarrow f'_i$$

$$f'_i = \frac{f_{i+1} - f_i}{\Delta x}$$

Note:

... is a discrete array

If the x-axis is x , which runs from index 0 to index N , then f' will only be defined from index 0 to index $N-1$! (since we need f_{i+1} to calculate it)

(i:) Backward Difference.

$$f'_i = \frac{f_i - f_{i-1}}{\Delta x}$$

→ Here, f' will only be defined for index 1 to index N

Example 1.

A traveling wave (i.e. pulse) in one dimension.

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0$$

- u is the wave function
- x is position
- t is time

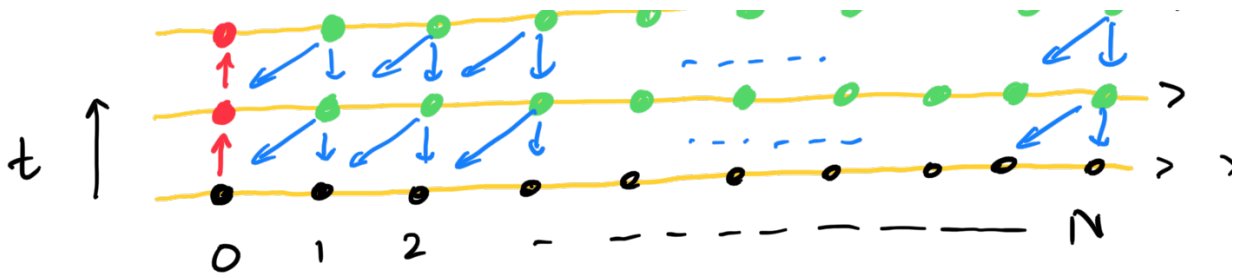
Method: Use the forward difference for time derivative, and the backward difference for the spatial derivative.

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + c \left(\frac{u_i^n - u_{i-1}^n}{\Delta x} \right) = 0$$

↓

$$u_i^{n+1} = u_i^n - \frac{c \Delta t}{\Delta x} (u_i^n - u_{i-1}^n)$$





(i.e. f_0 never changes)

Important → need

$$\frac{c \Delta t}{\Delta x} \leq 1$$

∴ there is a trade-off ...
cannot make Δt and Δx arbitrarily small.

Let $x = [0, 20 \text{ m}]$
 $c = 25 \text{ m/s}$

for a pulse traveling to the right,
it should take $t = \frac{20\text{m}}{25\text{m/s}} \sim 0.8\text{s}$
to traverse the distance.

$$\text{Let } t = [0, 0.625\text{s}]$$

Divide the x -axis into 400
slices. $\therefore i = 0, \dots, 400$

$$N_x = 401$$

$$\text{Let } \frac{c \Delta t}{\Delta x} = 1 \quad \Delta t = \frac{\Delta x}{c}$$
$$\Delta t = \frac{20\text{m}/400}{25\text{m/s}}$$

$$\boxed{\Delta t = 0.002\text{s}}$$

Call this Δt_{max}

\rightarrow we must choose a value of Δt

that is smaller than this.

→ let's choose $\Delta t = 0.00025 \text{ s}$

$$\text{then } N_t = \frac{0.625 \text{ s}}{0.00025 \text{ s}} + 1 = \underline{\underline{2500}} + 1 = 2501$$

So, we will discretize in the following way.

$$x = [0, 20 \text{ m}] \quad N_x = 401$$

$$t = [0, 0.625 \text{ s}] \quad N_t = 2501$$

$$c = 25 \text{ m/s}$$

with $\boxed{\frac{c \Delta t}{\Delta x} = 0.125}$

We see that in some situations, this
... work!

Does not work.

Let's try $C \frac{\Delta t}{\Delta x} = 1$

let $t = [0, 0.620 \text{ s}]$

$$C \frac{\Delta t}{\Delta x} < 1 \quad \Delta t = \frac{\Delta x}{C} = \frac{20\text{m}/400}{25\text{m/s}} = 0.002\text{s}$$

$$N_t = \frac{0.620\text{s}}{0.002\text{s}} + 1 = 310 + 1 = \boxed{311}$$

Now, it all works perfectly, and we simulate the traveling wave well.

Lessons Learned:

- the translation from continuous to discrete functions, necessary for computational calculus, is not always as simple as it seems.

filled with pits of viper and
dark monsters and nastiness!

What weapons can we manufacture
to help us fight the monsters?

The reason that in the method
above that we unlocked the monster
is rooted in how we calculated
the derivatives!

Math to the rescue!

Taylor series expansion:

$$f(x) = f(x_0) + f'(x)(x-x_0) + \frac{f''(x)(x-x_0)^2}{2} \\ + \frac{f'''(x)(x-x_0)^3}{6} + \dots$$

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2$$

$$\begin{aligned}
 f(x-h) &= f(x) - f'(x)h + \frac{f''(x)}{2}h^2 - \frac{f'''(x)}{6}h^3 + \dots \\
 &\approx f(x) - f'(x)h + \frac{f''(x)}{2}h^2 - \frac{f'''(x)}{6}h^3 + \dots
 \end{aligned}$$

[Before : $f'(x) = \frac{f(x+h) - f(x)}{h} + O(h)$

or $f'(x) = \frac{f(x) - f(x+h)}{h} + O(h)$

i.e. ignore terms of $O(h^2)$ (and higher)

Now :

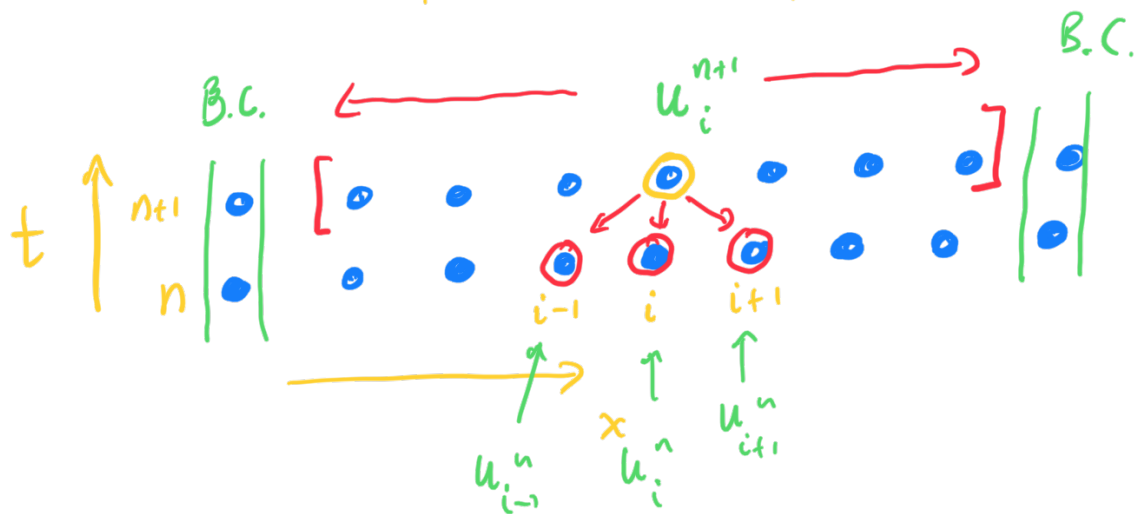
$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2)$$

Write $\frac{du}{dx} = \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + c \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = 0$$

$$u_i^{n+1} = u_i^n - \frac{c\Delta t}{2\Delta x} (u_{i+1}^n - u_{i-1}^n)$$

Centered Difference Solution to The
Advection Equation.



Lax Method :

→ Replace u_i^n with average of
 u_{i+1}^n and u_{i-1}^n

$$u_i^{n+1} = \frac{1}{2} (u_{i+1}^n + u_{i-1}^n) - \frac{c \Delta t}{2 \Delta x} (u_{i+1}^n - u_{i-1}^n)$$

Lax - Wendroff Method:

- Add a second order term:

$$u_i^{n+1} = u_i^n - \frac{c \Delta t}{2 \Delta x} (u_{i+1}^n - u_{i-1}^n) + \frac{c^2 \Delta t^2}{2 \Delta x^2} (u_{i+1}^n + u_{i-1}^n - 2u_i^n)$$

related to the 2nd Derivative.

$$f(x) = f(x_0) + f'(x_0) \Delta x + \frac{f''(x_0)}{2} \Delta x^2$$

$$f^+ = f + f' \Delta x + \frac{f''}{2} \Delta x^2$$

$$f^- = f - f' \Delta x + \frac{f''}{2} \Delta x^2$$

$$\begin{aligned} f^+ + f^- - 2f &= f + f' \Delta x + \frac{f''}{2} \Delta x^2 \\ &\quad + f - f' \Delta x + \frac{f''}{2} \Delta x^2 \\ &\quad - 2f \\ &= f'' \Delta x^2 \end{aligned}$$

$$\therefore f'' = \frac{f^+ + f^- - 2f}{\Delta x^2}$$