Proof

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1 Factorial Algorithm / Exponential Algorithm Proof

Let a directed graph with N nodes, E edges, and Σ edge labels.

For the exponential algorithm, let S_E be the **set** of all $\langle I, O, L\rangle$ triples (3-tuple) that exponential algorithm is going to enumerate. $|S_E| = 2^{2(N+E)\cdot E\cdot log(\Sigma)}$. (In this case, since the enumeration guarantees no repeat, I can confirm S_E is a set with the size $|S_E| = 2^{2(N+E)\cdot E\cdot log(\Sigma)}$)

For the factorial (permutation) algorithm, the brute force approach is to try $N!$ orderings without skipping any repeat cases.
Let S_{F_O} be the **set** of all node permutation orderings, and $S_{F_{IOL}}$ be the **set** of all $\langle I, O, L\rangle$ triples (3-tuple) yielded by each ordering in S_{F_O}. Let the function $f_{mapping} : S_{F_O} \rightarrow S_{F_{IOL}}$.

If there is no any pair of triples in $S_{F_{IOL}}$ that are the same, then $|S_{F_{IOL}}| = N!$. In this case, $f_{mapping}$ is an injective surjective function (bijective, one-to-one), and it is the worst case scenario for the factorial algorithm since all triples (encodings) in $|S_{F_{IOL}}|$ have to be checked in order to confirm if a graph is a wheeler graph.

If there is any two node orderings yielding the same triple, then $|S_{F_{IOL}}| < N!$. In this case, $f_{mapping}$ is a non-injective surjective function. For an unoptimized factorial algorithm without skipping any cases, it still takes $N!$ times to recognize a wheeler graph; however, we can carefully implement our algorithm and make the number of checking less than $N!$.

Therefore, $|S_{F_{IOL}}| \leq N!$

To compare the exponential algorithm to my factorial algorithm, I take the worst case of my factorial algorithm. Here, I want to show that even the worst case of my exponential algorithm ($|S_{F_{IOL}}| = N!$) is smaller than all encoding enumarations in the exponential algorithm ($|S_E| = 2^{2(N+E) \cdot E \cdot log(\Sigma)}$).

Proof. $S_{F_{IOL}} \subset S_E$ $\square$

$\forall x \in S_{F_{IOL}}$, $x = \langle I_x, O_x, L_x \rangle$

$\because |I_x| = N + E$, $|O_x| = N + E$, and $|L_x| = E \cdot log(\Sigma)$

$\therefore x \in S_E$

$\Rightarrow S_{F_{IOL}} \subseteq S_E$

To further prove $S_{F_{IOL}}$ is a proper subset,

take $y \in S_E$ and $y = \langle I, O, L \rangle$ with

$I = 000..000$ ($|I| = E + N$),

$O = 000..000$ ($|O| = E + N$), and

$L = 000..000$ ($|L| = E \cdot log(\Sigma)$)

$\because y \notin S_{F_{IOL}}$

$\Rightarrow S_{F_{IOL}} \subset S_E$

(**Therefore, my claim is that my factorial algorithm is faster!!!**)