Inference about Mean Vectors

Motivating Example: Iris Data

This data set consists of the measurements of the variables sepal length and width, and petal length and width (in centimeters), respectively, for 50 flowers from each of 3 species (setosa, versicolor and virginica) of iris.

- Source: Fisher, R. A. (1936) The use of multiple measurements in taxonomic problems. Annals of Eugenics, 7, Part II, 179-188.
- Dataset: Available in R as 'iris.'

Motivating Example: Iris Data

The first few rows are given below

Sepal.Length	Sepal.Width	Petal.Length	Petal.Width	Species
5.1	3.5	1.4	0.2	setosa
4.9	3.0	1.4	0.2	setosa
4.7	3.2	1.3	0.2	setosa
4.6	3.1	1.5	0.2	setosa
5.0	3.6	1.4	0.2	setosa
5.4	3.9	1.7	0.4	setosa

- Q1: Estimate the mean of (SL, SW, PL, PW) for setosa flowers.
- Q2: Provide confidence intervals.
- Q3: Compare the mean of these characteristics across species.

Inference about Mean Vectors

Let $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ be two population mean vectors.

- How to carry out inference about $\boldsymbol{\mu}_1$ (hypothesis testing & confidence intervals/regions)
- How to formally assess a hypothesis of the form $H_0 : \boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$.
- How to formally assess a hypothesis of the form

  H_0 : the change in the components of $\boldsymbol{\mu}_1 =$ the change in the component of $\boldsymbol{\mu}_2$

- How to construct confidence intervals/regions for the mean difference $\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2$.

Inferences about a Vector of Population Means

- We will generalize t-tests to test hypotheses about a $p \times 1$ vector of population means $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_p)'$.

- We could consider p hypothesis tests (one for each μ_j in $\boldsymbol{\mu}$) but that would not take advantage of the correlations between the variables X_1, X_2, ..., X_p.

- We will consider applications to repeated measures (longitudinal) studies.

- We will also generalize the two-sample t-test to test the null hypothesis that vectors of population means are the same for two populations.

Review of Univariate Hypothesis Testing

- Is μ_0 a plausible value for the population mean μ?

- We formulate the question as a hypothesis testing exercise. The competing hypotheses are

$$H_0 : \mu = \mu_0 \quad \text{and} \quad H_a : \mu \neq \mu_0.$$

- Given a sample $x_1, ..., x_n$ from a normal population, we compute the *test statistic*

$$t = \frac{(\bar{x} - \mu_0)}{s/\sqrt{n}}.$$

- If t is 'small', then $\bar{x}$ is close enough to μ_0 to suggest that μ_0 is plausible and we fail to reject H_0.

- If t is "large" then $\bar{x}$ is too far from μ_0 to believe that μ_0 is plausible and we reject H_0.

Univariate Hypothesis Testing

- When H_0 is true, the statistic t has a student t distribution with $n-1$ degrees of freedom. We reject the null hypothesis at significance level α when $|t| > t_{(n-1),1-\alpha/2}$.
- Notice that rejecting H_0 when t is large is equivalent to rejecting H_0 when the squared standardized distance

$$t^2 = \frac{(\bar{x} - \mu_0)^2}{s^2/n} = n(\bar{x} - \mu_0)(s^2)^{-1}(\bar{x} - \mu_0)$$

is large.

- We reject H_0 when

$$n(\bar{x} - \mu_0)(s^2)^{-1}(\bar{x} - \mu_0) > t^2_{(n-1),1-\alpha/2}$$

i.e., the squared standardized distance exceeds the $1-\alpha$ percentile of a central F-distribution with $(1,\ n-1)$ df.

Univariate Hypothesis Testing

- If we fail to reject H_0, we conclude that μ_0 is close (in units of standard deviations of $\bar{x}$) to $\bar{x}$, and thus is a plausible value for μ.
- The *set of plausible values* for μ is the set of all values that lie in the $100(1-\alpha)\%$ confidence interval for μ:

$$\bar{x} - t_{(n-1),1-\alpha/2}\frac{s}{\sqrt{n}} \leq \mu_0 \leq \bar{x} + t_{(n-1),1-\alpha/2}\frac{s}{\sqrt{n}}.$$

- Given the value of $\bar{x}$, a $(1-\alpha) * 100\%$ confidence interval consists of all the μ_0 values that would not be rejected by the α level test of $H_0 : \mu = \mu_0$.
- How to interpret confidence intervals?

General Approach to Multivariate Inference

- Let $\mathbf{x}_i' = (x_{i1}, x_{i2}, \ldots, x_{ip})$ denote the ith observation.

- Compute summary statistics:

$$\bar{\mathbf{x}} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i, \qquad \text{and} \qquad S = \frac{1}{n-1}\sum_{i=1}^{n} \mathbf{x}_i - \bar{\mathbf{x}})(\mathbf{x}_i - \bar{\mathbf{x}})'.$$

- When $\bar{\mathbf{x}}$ is too "far away" from the hypothesized vector of population means, $\boldsymbol{\mu}_0$, there is evidence against the null hypothesis.

- Otherwise, we will conclude that $\boldsymbol{\mu}_0$ is plausible.

Problem with the Univariate Approach

- A naive approach for testing a multivariate hypothesis is to compute the t-test statistics for each individual variables, i.e.,

$$t_j = \frac{\bar{x}_j - \mu_{0j}}{\sqrt{s_j^2/n}}.$$

- We could reject $H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_0$ if $|t_j| > t_{n-1,\alpha/2}$ for at least one variable $j \in \{1, 2, \ldots, n\}$.
- *Family-wise error rate*: Pr(rejecting at least one $H_0 : \mu_j = \mu_{0j}$ | all H_0's are true).
- The naive approach implies that the family-wide error rate is

$$1 - (1 - \alpha)^p > \alpha.$$

- This approach is very conservative (i.e., tends to reject more null hypotheses than we should).

Hotelling's T^2 Statistic

- Consider the question of whether the $p \times 1$ vector $\boldsymbol{\mu}_0 = (\mu_{01}, \mu_{02}, ..., \mu_{0p})'$ is plausible for the unknown vector of population means $\boldsymbol{\mu} = (\mu_1, \mu_2, ..., \mu_p)'$.

- The squared statistical distance between the vector of sample means and the hypothesized mean vector is

$$T^2 = (\bar{\mathbf{x}} - \boldsymbol{\mu}_0)' \left(\frac{1}{n} S\right)^{-1} (\bar{\mathbf{x}} - \boldsymbol{\mu}_0) = n(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)' S^{-1} (\bar{\mathbf{x}} - \boldsymbol{\mu}_0)$$

  is called the one-sample *Hotelling's T^2* statistic.

- In the expression above,

$$\bar{\mathbf{x}} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i, \qquad S = \frac{1}{n-1} \sum_{i=1}^{n} (\mathbf{x}_i - \bar{\mathbf{x}})(\mathbf{x}_i - \bar{\mathbf{x}})'.$$

Hotelling's T^2 Statistic

- If the observed T^2 value is 'large' we reject $H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_0$.
- To decide how large is too large, we need the sampling distribution of T^2 when the hypothesized mean vector is correct:

$$T^2 \sim \frac{(n-1)p}{(n-p)} F_{p,n-p} \quad \text{or} \quad \frac{n-p}{p(n-1)} T^2 \sim F_{p,n-p}$$

- We reject the null hypothesis $H_0 : \mu = \mu_0$ for the p-dimensional vector μ at level α when

$$\frac{n-p}{p(n-1)} T^2 > F_{(p,n-p),1-\alpha}$$

where $F_{(p,n-p),1-\alpha}$ is the upper $1-\alpha$ percentile of the central F distribution with p and $n-p$ degrees of freedom.

Hotelling's T^2 Statistic

This result depends on the following conditions

- The vectors of observations $\mathbf{x}_1$, $\mathbf{x}_2$, ...$\mathbf{x}_n$ are independent, each randomly drawn from the same population (a simple random sample).

- Each $\mathbf{x}_i$ is randomly selected from a population with a multivariate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix Σ.

When only $p = 1$ response is obtained from each subject

$$\begin{aligned}
\frac{n-p}{p(n-1)}T^2 &= \frac{n-1}{(1)(n-1)}T^2 = T^2 \\
&= (\bar{x}-\mu_0)\left(\frac{n}{s^2}\right)(\bar{x}-\mu_0) = \left(\frac{\bar{x}-\mu_o}{s/\sqrt{n}}\right)^2 = t^2
\end{aligned}$$

Example: Female Sweat Data (J&W)

- Perspiration from a sample of 20 healthy females was analyzed. Three variables were measured for each woman:

  X_1 =sweat rate

  X_2 =sodium concentration

  X_3 =potassium concentration

- The question is whether $\boldsymbol{\mu}_0 = [4, 50, 10]'$ is plausible for the population mean vector.
- At level $\alpha = 0.10$, we reject the null hypothesis if

$$T^2 = 20(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)' S^{-1} (\bar{\mathbf{x}} - \boldsymbol{\mu}_0) > \frac{(n-1)p}{(n-p)} F_{(p,n-p),0.9} = \frac{19(3)}{17} F_{(3,17),0.9} = 8.18.$$

- The vector of sample means is

$$\bar{\mathbf{x}} = \begin{bmatrix} 4.64 \\ 45.4 \\ 9.96 \end{bmatrix} \quad \text{and} \quad \bar{\mathbf{x}} - \boldsymbol{\mu}_0 = \begin{bmatrix} 4.64 - 4 \\ 45.4 - 50 \\ 9.96 - 10 \end{bmatrix} = \begin{bmatrix} 0.64 \\ -4.6 \\ -0.04 \end{bmatrix}.$$

Example: Sweat Data

- After computing the inverse of the 3×3 sample covariance matrix S^{-1} we can compute the value of the T^2 statistic as

$$\begin{aligned} T^2 &= 20[\ 0.64 \quad -4.6 \quad -0.04\ ] \begin{bmatrix} 0.586 & -0.022 & 0.258 \\ -0.022 & 0.006 & -0.002 \\ 0.258 & -0.002 & 0.402 \end{bmatrix} \begin{bmatrix} 0.64 \\ -4.60 \\ -0.04 \end{bmatrix} \\ &= 9.74. \end{aligned}$$

- Because $9.74 > 8.18$, we reject H_0 and conclude that $\boldsymbol{\mu}_0$ is not a plausible value for $\boldsymbol{\mu}$ at the 10% level.

- At this point, we do not know which of the three hypothesized mean values is not supported by the data.

See R code for illustrations

Confidence Region for a Vector of Population Means

A $(1-\alpha)\times 100\%$ confidence region for the vector of population means $\boldsymbol{\mu}$ is the ellipsoid determined by all $\boldsymbol{\mu}$ such that

$$n(\bar{\mathbf{x}}-\boldsymbol{\mu})'S^{-1}(\bar{\mathbf{x}}-\boldsymbol{\mu}) \leq \frac{(n-1)p}{(n-p)}F_{(p,n-p),1-\alpha}$$

This is a p-dimensional ellipsoid centered at $\bar{\mathbf{x}}$

- The axes of the confidence ellipsoid are parallel to the eigenvectors of the sample covariance matrix S.
- The lengths of the axes are proportional to the eigenvalues of S.

Confidence Regions

- The distance from the center of the confidence ellipsoid at $\bar{\mathbf{x}}$ to the edge of the ellipsoid along the i-th axis is

$$\pm\sqrt{\lambda_i}\sqrt{\frac{(n-1)p}{n(n-p)}F_{(p,n-p),1-\alpha}}.$$

- The directions of the the eigenvectors and sizes of the eigenvalues depend
  - The relative sizes of the variances of the measured variables
  - The sizes of the correlations between pairs of variables

Example: Microwave Ovens

- Data was collected on two measurements of radiation emitted from $n = 42$ microwave ovens (J & W, Tables 4.1 and 4.5) Here, x_j denotes the transformed radiation measurements (fourth root)

- Sample statistics for the transformed data are:

$$\bar{\mathbf{x}} = \begin{bmatrix} 0.564 \\ 0.603 \end{bmatrix}, \quad S = \begin{bmatrix} 0.014 & 0.012 \\ 0.012 & 0.015 \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} 203.02 & -163.39 \\ -163.39 & 200.23 \end{bmatrix}.$$

- Eigenvalue and eigenvector pairs for S are

$$\lambda_1 = 0.026 \qquad \mathbf{e}_1' = [0.704,\ 0.710]$$

$$\lambda_2 = 0.002 \qquad \mathbf{e}_2' = [-0.71,\ 0.704]$$

Example: Microwave Ovens

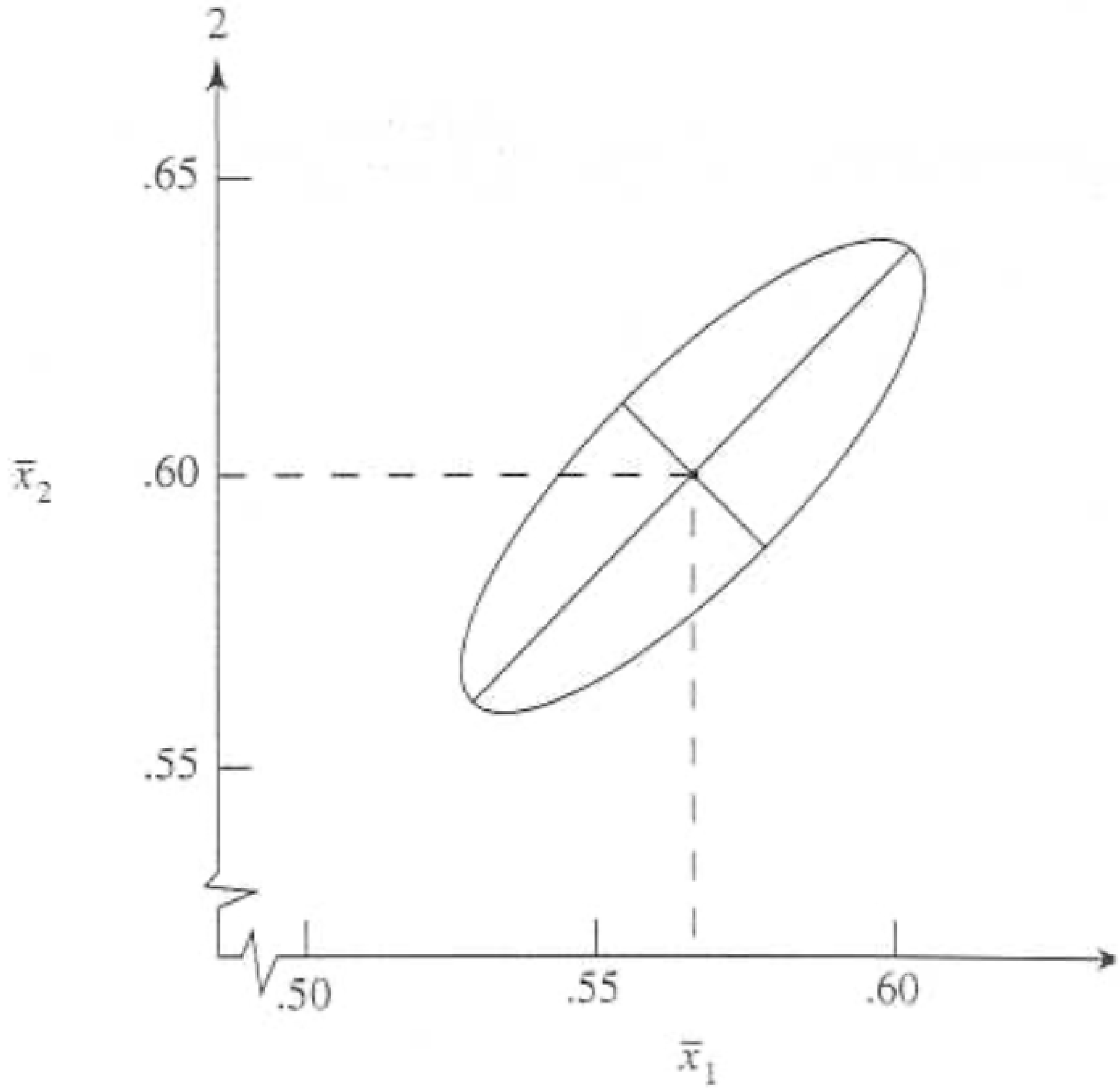
- The 95% CR for the population mean vector is given by all vectors $\boldsymbol{\mu} = (\mu_1, \mu_2)'$ that satisfy:

$$42[0.564-\mu_1 \quad 0.603-\mu_2] \begin{bmatrix} 203.02 & -163.39 \\ -163.39 & 200.23 \end{bmatrix} \begin{bmatrix} 0.564 - \mu_1 \\ 0.603 - \mu_2 \end{bmatrix} \leq 6.62,$$

where

$$\frac{2(41)}{40} F_{(2,40),0.95} = \frac{2(41)}{40} 3.23 = 6.62.$$

- Is $\boldsymbol{\mu} = [0.562 \quad 0.589]'$ a plausible value for the population mean vector? To check, plug $\boldsymbol{\mu} = [0.562 \quad 0.589]'$ into the expression above and see if it satisfies the inequality. In this case, we get 1.30 which is less than 6.62, and conclude that this is plausible at the 95% level.


2
.65
$\bar{x}_2$
.60
.55
.50
.55
.60
$\bar{x}_1$

Example: Microwave Ovens

- The joint confidence ellipsoid is centered at $\bar{\mathbf{x}} = [0.564 \;\; 0.603]'$ and the half lengths of the two axes are

$$\sqrt{0.026}\sqrt{\frac{2(41)}{42(40)}3.23} = 0.064, \quad \sqrt{0.002}\sqrt{\frac{2(41)}{42(40)}3.23} = 0.018.$$

- The axes are parallel to the two eigenvectors of S.

- The ratio of the square roots of the eigenvalues

$$\frac{\sqrt{0.026}}{\sqrt{0.002}} = 3.6$$

indicates that the major axis is 3.6 times longer than the minor axis.

Simultaneous Confidence Statements

- Often we are interested in drawing inference about each μ_j.
- One possibility is to construct ordinary confidence intervals

$$\bar{x}_j \pm t_{(n-1),1-\frac{\alpha}{2}}\sqrt{\frac{s_{jj}}{n}},$$

for each μ_j. One problem is that the combined set of individual intervals result in a *simultaneous confidence level* that is less than the nominal $1-\alpha$.

- There are various ways of constructing a collection of individual confidence intervals so that the joint confidence level for the family of parameters remains at $1-\alpha$
- Intuitively, CI's that protect against erosion of the confidence level will be wider than the individual $(1-\alpha) \times 100\%$ CI's.

Simultaneous Confidence Statements

- The following set of confidence intervals enclose the confidence ellipsoid and have at least probability $1-\alpha$ of simultaneously containing all p of the population means.

$$\bar{x}_j \pm \sqrt{\frac{p(n-1)}{(n-p)} F_{(p,n-p),1-\alpha}} \sqrt{\frac{S_{jj}}{n}} \qquad\qquad j = 1, 2, ..., p$$

- These are called T^2 confidence intervals.

Example: Microwave Ovens

- Compute simultaneous 95% T^2 confidence intervals for μ_1 and μ_2, the means of the fourth root of the amount of radiation with door closed and door open.

- First note that

$$\sqrt{\frac{p(n-1)}{n(n-p)}F_{(p,n-p),0.95}} = \sqrt{\frac{2(41)}{42(40)}3.23} = 0.397.$$

is common to both intervals.

Example: Microwave Ovens

- For μ_1 and μ_2:

$$\bar{x}_1 \pm 0.397\sqrt{s_{11}} \;\Rightarrow\; 0.564 \pm (0.397 \times 0.12) \Rightarrow 0.564 \pm 0.0476$$
$$\bar{x}_2 \pm 0.397\sqrt{s_{22}} \;\Rightarrow\; 0.603 \pm (0.397 \times 0.121) \Rightarrow 0.603 \pm 0.048.$$

- For the difference between the means for the door closed and open:

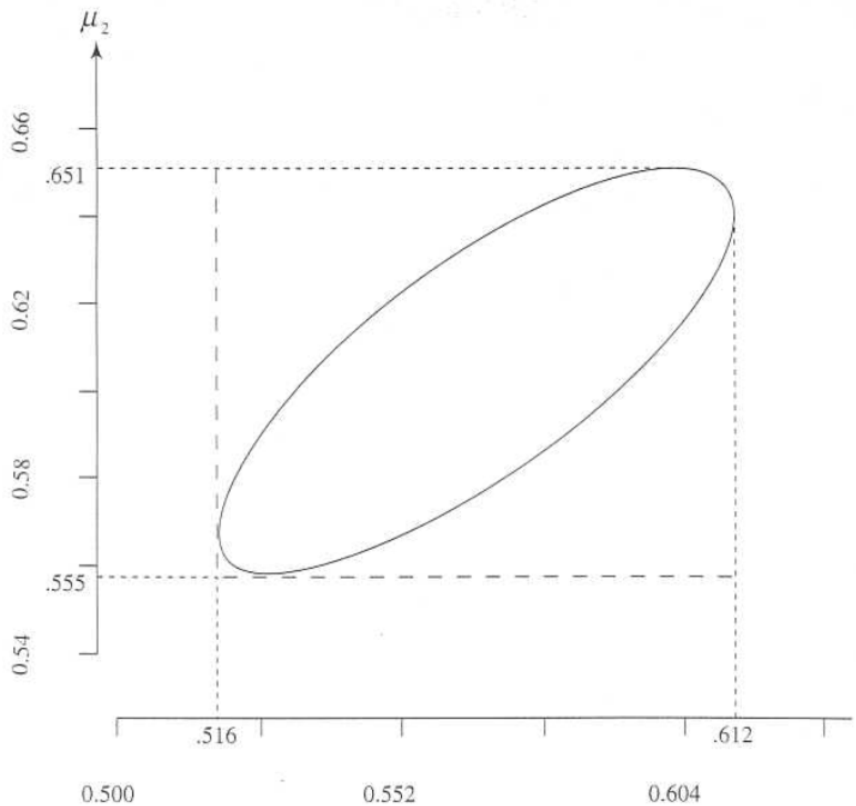
$$\begin{aligned}\bar{x}_1 - \bar{x}_2 \pm 0.397\sqrt{s_{11} - 2s_{12} + s_{22}} &\Rightarrow -0.039 \pm (0.397 \times 0.0748)\\ &\Rightarrow [-0.069, \;\; -0.009],\end{aligned}$$

suggesting that closing the door significantly reduces mean (fourth root) radiation emitted by the ovens.

- The T^2 intervals are shadows or projections of the confidence ellipse onto the component axes.

Example: Microwave Ovens

The T^2 intervals are projections of the confidence ellipse onto the component axes.



The Bonferroni method for Simultaneous Confidence Intervals

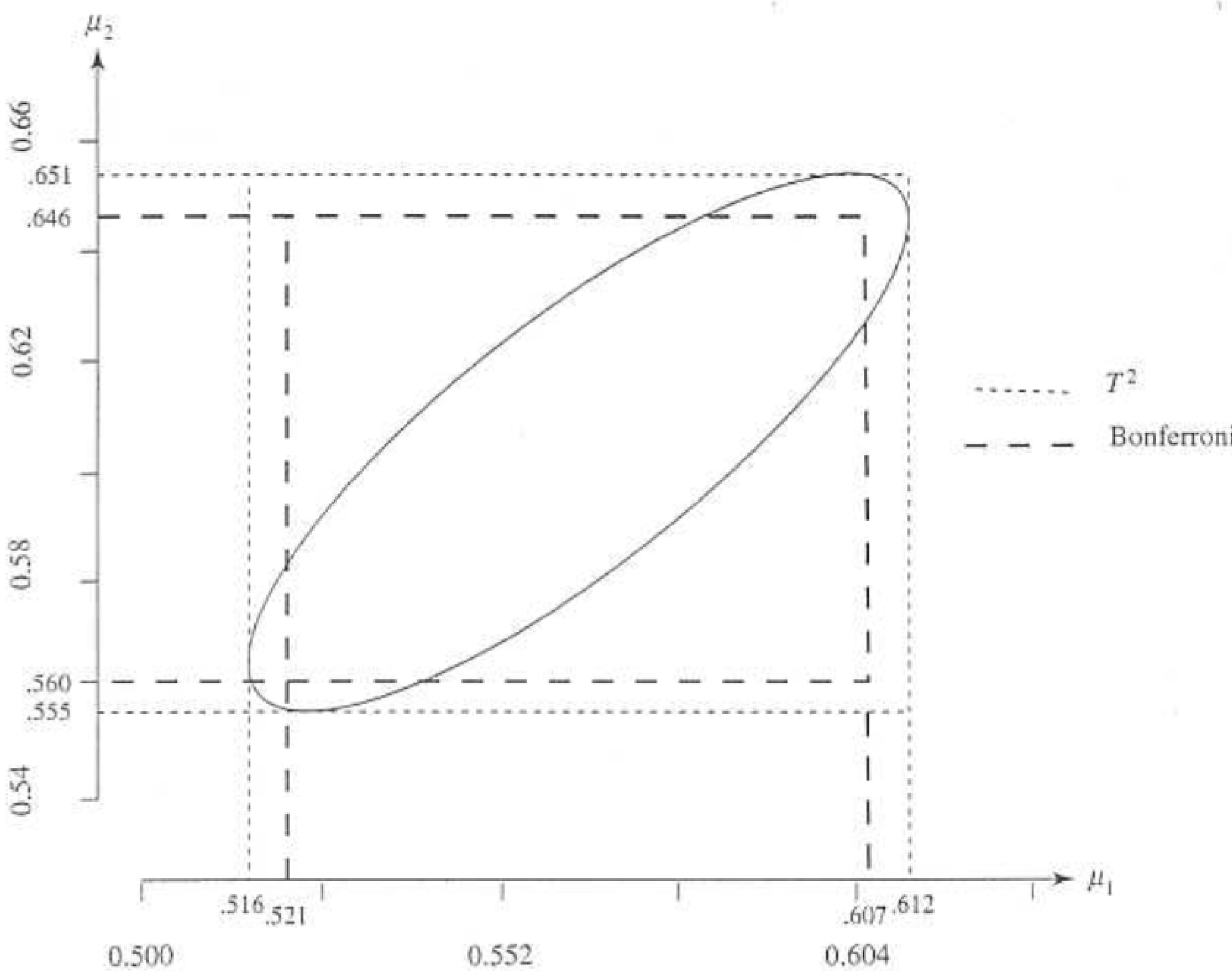
- Simultaneous $(1-\alpha)\times 100\%$ Bonferroni confidence intervals for p population means are

$$\bar{X}_i \pm t_{(n-1),1-\frac{\alpha}{2p}}\sqrt{\frac{s_{ii}}{n}}$$

- The probability that all p of the confidence intervals simultaneously contain the corresponding population means is at least $1-\alpha$

- Microwave ovens: see T^2 and Bonferroni intervals in the next figure.

T^2 and Bonferroni Confidence Intervals



T^2 and Bonferroni Confidence Intervals

Simultaneous confidence intervals can be made for any linear combination of the population means, say $\mathbf{c}'_k\boldsymbol{\mu} = \sum_{j=1}^{p} c_{jk}\mu_j$ for k=1,2,...,m combinations. The k-th vector of coefficients is $\mathbf{c}_k = (c_{1k}, c_{2k}, ...c_{pk})$.

- T^2 method

$$\left(\sum_{j=1}^{p} c_{jk}\bar{x}_j\right) \pm \sqrt{\frac{p(n-1)}{(n-p)}F_{(p,n-p),1-\alpha}}\sqrt{\frac{1}{n}\mathbf{c}'_k S\mathbf{c}_k} \qquad k = 1, 2, ..., m.$$

- Bonferroni method (these get wider as the number of combinations m gets bigger)

$$\left(\sum_{j=1}^{p} c_{jk}\bar{x}_j\right) \pm t_{(n-1),1-\frac{\alpha}{2m}}\sqrt{\frac{1}{n}\mathbf{c}'_k S\mathbf{c}_k} \qquad k = 1, 2, ..., m.$$

Repeated Measures Studies

- Hotelling's T-squared test for a single vector of population means has useful applications for studies with repeated measurements on each of the subjects in the study.

  1. Two treatments are applied to each sample unit and p variables are measured on each unit under each treatment.

  2. Studies in which the same variable is measured at a set of p time points on each subject in the study.

Repeated Measures Studies

- Examples:
  - We measure volumes of sales of a certain product in a certain market before and after an advertising campaign
  - We measure blood pressure on a sample of individuals before receiving some drug and at monthly time intervals for one year after receiving the drug.

Repeated Measures Studies

- One advantage of this type of study is that the differences in the outcome measures under one treatment or the other reflect only the effect of treatment because everything else about the subject receiving the treatments is identical.

- Using the T^2 method allows us to account for correlations between responses provided by the same subject.

- Other things to worry about: In drug trials there may be carry-over effects to worry about and in 'before and after' experiments because conditions may change.

Repeated Measures Studies

- Suppose now that p treatments are applied to each of the n sample units. Then

$$\mathbf{x}_j = [x_{j1}, X_{j2}, ..., x_{jp}]', \quad j = 1, ..., n.$$

- Of interest are contrasts of the elements of $\boldsymbol{\mu} = E(\mathbf{x}_j)$ such as

$$\begin{bmatrix} \mu_1 - \mu_2 \\ \mu_1 - \mu_3 \\ \vdots \\ \mu_1 - \mu_p \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & -1 \end{bmatrix} \boldsymbol{\mu} = C\boldsymbol{\mu}.$$

- The *contrast matrix* C has $p - 1$ independent rows.
- If the null hypothesis that all treatments provide the same mean response is true, then $C\boldsymbol{\mu} = \mathbf{0}$.

Testing Hypotheses in Repeated Measures Studies

- To test $H_0 : C\boldsymbol{\mu} = \mathbf{0}$ we again use T^2:

$$T^2 = n(C\bar{\mathbf{x}} - \mathbf{0})'(CSC')^{-1}(C\bar{\mathbf{x}} - \mathbf{0})$$

- The null hypothesis is rejected if

$$T^2 \geq \frac{(n-1)(p-1)}{(n-p+1)} F_{(p-1,n-p+1),1-\alpha}.$$

- Note that numerator degrees of freedom are $p-1$ instead of p because the null hypothesis puts only $p-1$ constraints on the means responses

Example: Dog Anesthetics (J&W)

- A sample of 19 dogs were administered four treatments each: (1) high CO_2 pressure, (2) low CO_2 pressure, (3) high pressure + halothane, (4) low pressure + halothane.

- Outcome variable was milliseconds between heartbeats.

- Three possible contrasts that could be of interest:

$$
\begin{array}{lcl}
(\mu_3 + \mu_4) - (\mu_1 + \mu_2) & : & \text{Effect of halothane} \\
 & & \text{averaging across } CO_2 \text{ pressure levels} \\
(\mu_1 + \mu_3) - (\mu_2 + \mu_4) & : & \text{Effect of } CO_2 \text{ pressure} \\
 & & \text{averaging across levels of halothane} \\
(\mu_1 - \mu_2) - (\mu_3 - \mu_4) & : & \text{Interaction between halothane and } CO_2.
\end{array}
$$

Example: Dog Anesthetics

- In class exercise: The 3×4 contrast matrix C is

- The test of $H_0 : C\boldsymbol{\mu} = 0$ rejects the null if

$$T^2 = n(C\bar{\mathbf{x}})'(CSC')^{-1}(C\bar{\mathbf{x}}) = 116 \geq \frac{(18)(3)}{(16)} F_{(3,16),0.95} = 10.94.$$

- We clearly reject the null, so the question now is whether there is a difference between CO_2 pressure, between halothane levels or perhaps there is no main effect of treatment but there is still an interaction.

Example: Dog Anesthetics

- Three simultaneous confidence intervals (one for each row of C). First consider the halothane effect, summing across CO_2 pressure levels: $(\mu_3 + \mu_4) - (\mu_1 + \mu_2)$

$$
\begin{aligned}
(\bar{x}_3 + \bar{x}_4) - (\bar{x}_1 + \bar{x}_2) \;&\pm\; \sqrt{\frac{18(3)}{16} F_{(3,16),0.95}} \sqrt{\frac{c_1' S c_1}{19}} \\
\Longrightarrow \quad 209.31 \;&\pm\; 73.70.
\end{aligned}
$$

- For the effect of CO_2 pressure and the interaction, we find respectively: $-60.05 \pm 54.70, \quad -12.79 \pm 65.97$
- Use of halothane produces longer times between heartbeats. This effect is similar at *both* high and low CO_2 pressure levels because the interaction is not significant (third contrast). Higher CO_2 pressure produces shorter times between heartbeats.

Example: Dog Anesthetics

- The same null hypothesis can be represented with different contrast matrices
- For example, the null hypothesis that all four treatments have the same mean is also tested with $H_0 : C\boldsymbol{\mu} = 0$, where

$$C = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \quad \text{or} \quad C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

- The value of T^2 is unchanged

$$T^2 = n(C\bar{\mathbf{x}})'(CSC')^{-1}(C\bar{\mathbf{x}}) = 116$$

- The value of T^2 will be the same for any set of contrasts that implies the same null hypothesis.

- R code demonstration