Assignment 1: Communication Theory

5ETB0 - Communication Theory

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Contents

1 | Problem 1: 4-PAM Modulation

The symbols in this problem are labeled as s_1 (at -3), s_2 (at -1), s_3 (at 1) and s_4 (at 3)

1.1 | Q1.1

A set of 4 symbols can represent 2 bits of information per symbol.

1.2 | Q1.2

For equiprobable symbols, the error is minimized when the decision boundary is at their midpoint. The decision rules (for estimating the sent symbol) are then

$$f(r) = \hat{s_m} = \begin{cases} -3 & \text{for } r \leq -2 \\ -1 & \text{for } -2 < r \leq 0 \\ 1 & \text{for } 0 < r \leq 2 \\ 3 & \text{for } 2 \leq r \end{cases} \tag{1.1}$$

1.3 | Q1.3

There are 3 decision boundaries to be computed, the points at which they lie will be labeled b_1 (between s_1 and s_2), b_2 (between s_2 and s_3), and b_3 (between s_3 and s_4).

The decision bounds are found by maximizing the decision variables

$$\begin{aligned} f(r) &= \text{argmax } P_r(S = s_m) p_R(r|S = s_m) \\ f(r) &= \text{argmax } \frac{P(S = s_m)}{\sqrt{2\pi\sigma^2}} exp(-\frac{(||r - s_m||)^2}{2\sigma^2}) \\ f(r) &= \text{argmax } 2\sigma^2 ln(P(S = s_m)) - (||r - s_m||)^2) \end{aligned} \tag{1.2}$$

filling in values

$$\begin{aligned} f(r) = \text{argmax } &(-3.79\sigma^2 - (||r+3||)^2, -2.1\sigma^2 - (||r+1||)^2, -2.1\sigma^2 - (||r-1||)^2, -3.79\sigma^2 - (||r-3||)^2) \\ = \text{argmax } &(-3.79\sigma^2 - 6r - 9, -2.1\sigma^2 - 2r - 1, -2.1\sigma^2 + 2r - 1, -3.79\sigma^2 + 6r - 9,) \\ = \text{argmax } &(-1.69\sigma^2 - 6r - 8, -2r, 2r, -1.69\sigma^2 + 6r - 8) \end{aligned} \tag{1.3}$$

Each decision boundary is computed using an inequality equation.

$$\begin{aligned} (-1.69\sigma^2 - 6r - 8 \geq -2r) \Rightarrow (r \leq -2 - 0.42\sigma^2) \Rightarrow b_1 = -2 - 0.42\sigma^2 \\ (-2r \geq 2r) \Rightarrow (r \leq 0) \Rightarrow b_2 = 0 \\ (-1.69\sigma^2 + 6r - 8 \geq +2r) \Rightarrow (r \leq 2 + 0.42\sigma^2) \Rightarrow b_3 = 2 + 0.42\sigma^2 \end{aligned} \tag{1.4}$$

1.4 | Q1.4

Using $\sigma = 1$, $b_1 = -2.42$, $b_2 = 0$, $b_3 = 2.42$. x_{ij} is the distance from the i-th symbol to the j-th decision bound. The error probability is then computed as

$$
\begin{aligned}
P_e = \sum_{m=1}^{4} P(s_m \neq f(r)|S = s_m)P(S = s_m) \\
P(s_1 \neq f(r)|S = s_1) = Q(\frac{x_{11}}{\sigma}) \\
P(s_2 \neq f(r)|S = s_2) = Q(\frac{x_{21}}{\sigma}) + Q(\frac{x_{22}}{\sigma}) \\
P(s_3 \neq f(r)|S = s_3) = Q(\frac{x_{32}}{\sigma}) + Q(\frac{x_{33}}{\sigma}) \\
P(s_1 \neq f(r)|S = s_1) = Q(\frac{x_{43}}{\sigma})
\end{aligned} \tag{1.5}
$$

Where $x_{11} = 0.58$, $x_{21} = 1.42$, $x_{22} = 1$, $x_{32} = 1$, $x_{33} = 1.42$ and $x_{43} = 0.58$. Plugging in values

$$
\begin{aligned}
P_e = (0.15 * 0.28) + (0.35 * (0.077 + 0.16)) + (0.35 * (0.077 + 0.16)) + (0.15 * 0.28) \\
P_e = 0.25
\end{aligned} \tag{1.6}
$$

1.5 | Q1.5

$$
\begin{aligned}
E_s = \frac{1}{4} \sum_{m=1}^{4} P(S = s_m)(s_m)^2 \\
E_s = \frac{1}{4}[(0.15 * 9) + (0.35 * 1) + (0.35 * 1) + (0.15 * 9)] = 0.85
\end{aligned} \tag{1.7}
$$

2 | Problem 2: 8-PSK Modulation and Repetition Coding

The signal space has 8 symbols with equal energy but distinguished by an angle $\theta_i = i\pi/4$. This is a 2-dimensional signal space.

2.1 | Q2.1

With all symbols equally likely, the decision rule is that the estimate $\hat{s_m}$ is equal to the closest s_m by euclidean distance. $x \cdot y$ refers to the dot product between x and y and $|x|$ is the magnitude of x.

$$\begin{aligned} s_{1i} &= cos(i\pi/4) \\ s_{2i} &= sin(i\pi/4) \\ \hat{s_m} &= \text{argmin } (r_1 - s_{1i})^2 + (r_2 - s_{2i})^2 \\ \hat{s_m} &= \text{argmin } r_1^2 - 2r_1 cos(i\pi/4) + cos(i\pi/4)^2 + r_2^2 - 2r_2 sin(i\pi/4) + sin(i\pi/4)^2 \\ \hat{s_m} &= \text{argmin } (|r|)^2 - 2(r \cdot s_i) + 1 \\ \hat{s_m} &= \text{argmax } (r \cdot s_i) = \text{argmax } |r||s_i| cos(\theta_{rs}) \end{aligned} \tag{2.1}$$

$$\hat{s_m} = \text{argmax } cos(\theta_{rsi}) \tag{2.2}$$

Where θ_{rsi} is the angle between r and s_i. Equation 2.2 shows that the optimum receiver chooses the s_i closest in angle to r. Figure 2.1 shows the signal constellation, with the decision regions. Due to symmetry arguments, regions on opposite sides will have the same symbol error rate. However, symbols that are not opposing might have different error rates if the variance of the noise on the s_1 channel is different to the variance of the noise on the s_2 channel.

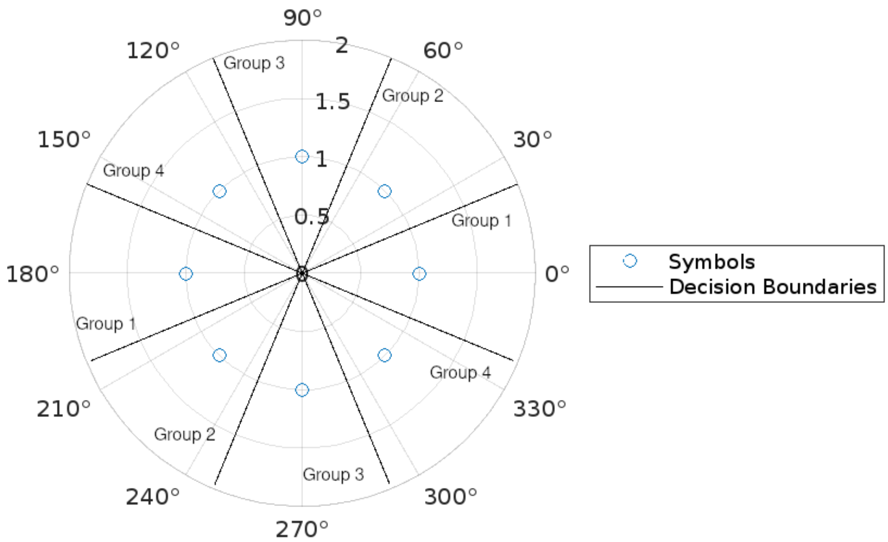

Figure 2.1: 8-PSK constellation

2.2 | Q2.2

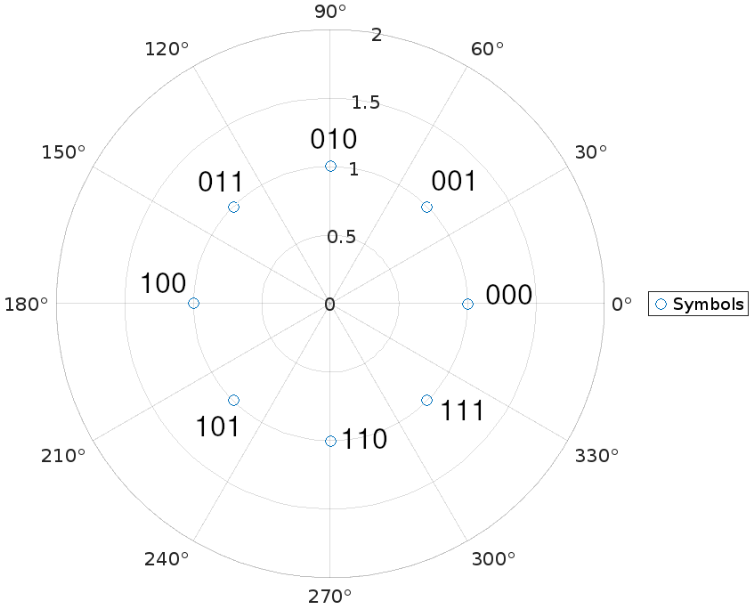

Figure 2.2: 8-PSK binary encoding

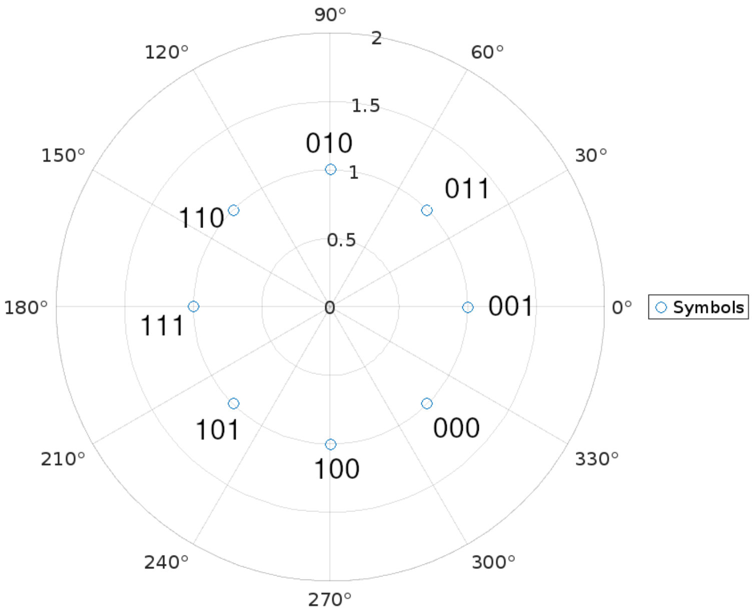

Figure 2.3: 8-PSK gray encoding

2.3 | Q2.3

The gray encoding yields a lower bit error probability. The most likely type of error occurs when the estimated symbol is adjacent to the sent signal. In binary encoding, this could cause between 1 to 3 bits of error (from 000 to 001, or 000 to 111). In gray encoding, the bit error between adjacent symbols is always causes only 1 bit by definition.

2.4 | Q2.4

Assuming that the errors in each bit are independent of each other, the per bit error probability is b_e, then the probability of a 000 corrupting into a sequence with a ones is $b_e^a * (1 - b_e)^{3-a}$ while the probability that a 111 sequence had transformed is $b_e^{3-a} * (1 - b_e)^a$. These can be taken as the a-priori probabilities $P(R = r|S = s_m)$.

The decision rule becomes (based on maximising the propabilities)

$$\hat{m} = \begin{cases} 0 \text{ if } b_e \leq 0.5, a = 0, 1 \\ 0 \text{ if } b_e \geq 0.5, a = 2, 3 \\ 1 \text{ if } b_e \leq 0.5, a = 2, 3 \\ 1 \text{ if } b_e \geq 0.5, a = 0, 1 \end{cases} \tag{2.3}$$

2.5 | Q2.5

With the proposed decision rule, errors occur when $m = 0$ but $a = 2$ or $a = 3$, or when $m = 1$ but $a = 0$ or $a = 1$. With $b_e = 0.25$ These events have probabilities

$$\begin{aligned} P(a = 2|m = 0) &= 0.25^2 * 0.75^1 = \frac{3}{64} \\ P(a = 3|m = 0) &= 0.25^3 * 0.75^0 = \frac{1}{64} \\ P(a = 0|m = 1) &= 0.25^3 * 0.75^0 = \frac{1}{64} \\ P(a = 2|m = 1) &= 0.25^2 * 0.75^1 = \frac{3}{64} \end{aligned} \tag{2.4}$$

The total error probability is

$$P_e = \frac{1}{2}(\frac{3}{64} + \frac{1}{64}) + \frac{1}{2}(\frac{3}{64} + \frac{1}{64}) = \frac{1}{16} \tag{2.5}$$

3 | Problem 3: Error Probability of 16-QAM

3.1 | Q3.3

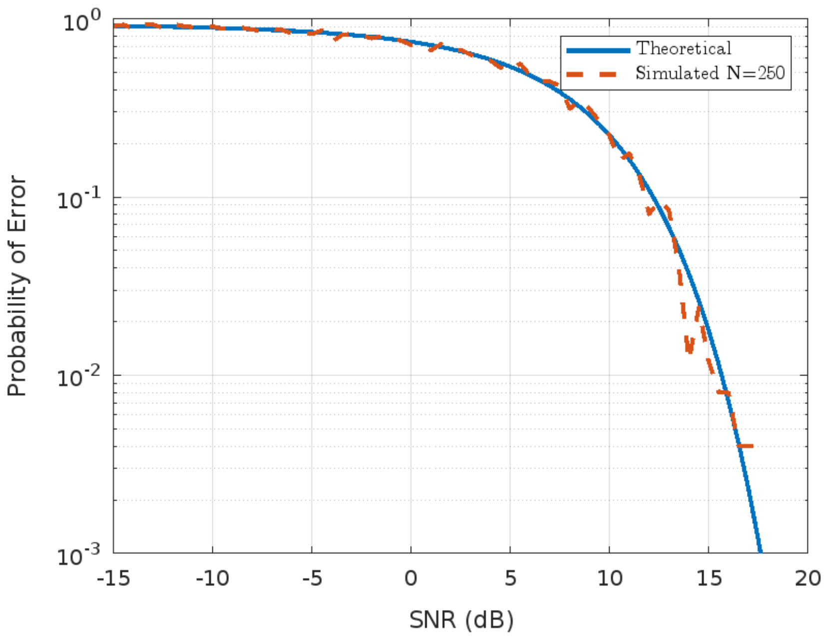

Figure 3.1: Error simulation results for 250 symbols

The simulation with 250 symbols took 0.533227 seconds and the one with 50,000 symbols took 10.016098 seconds.

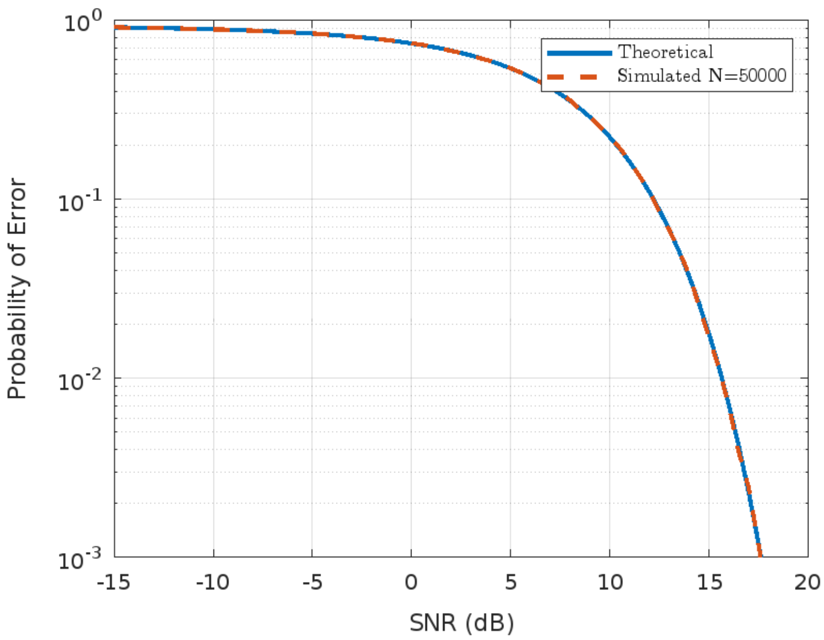

Figure 3.2: Error simulation results for 50000 symbols

3.2 | Q3.4

The data in the $N = 50000$ case follows the theoretical predictions much more closely than in the $N = 250$ case, thus assuming the theoretical result is accurate the $N = 50000$ simulation produced more accurate results, as is expected. The trade off is that the simulation took about 20 times more time (0.5 seconds vs 10 seconds). If these numbers are representative, it takes about 0.19 milliseconds to do 1 symbol simulation on average.