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I pledge my honor that I have abided by the Steven Honor System.

1 Problem

Let A be that matrix. If A has a null space in R^4 that spans two dimensions, I know A must 4 columns, two of which are free. An $m = 2, n = 4$ sized matrix will satisfy these conditions.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}$$

I will first generate my two independent columns in the first two position using a $2x2$ identity matrix. I am allowd to do this becuase I know two columns are independent, and I will be able to generate specific null space vectors if I set my dependent columns to have the correct coeficients.

$$\begin{bmatrix} 1 & 0 & a_{13} & a_{14} \\ 0 & 1 & a_{23} & a_{24} \end{bmatrix}$$

Let u and v represent each null space vector. After multiplying $Au = 0$ and $Av = 0$, I get the following relations.

$2 + a_{13} = 0$
$2 + a_{23} = 0$
$3 + a_{14} = 0$
$1 + a_{24} = 0$

I notice that all the variables are independent in the equations above, this means I will not have to do any substitution and can just skip to filling in the correct values for each a_{ij}

$a_{13} = -2$
$a_{23} = -2$
$a_{14} = -4$
$a_{24} = -1$

$$A = \begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 1 & -2 & -1 \end{bmatrix}$$

2 Problem

Let matrix A be the one I am constructing. For the first two vectors I can just add them as columns in my matrix.

$$\begin{bmatrix} 1 & 0 \\ 1 & 3 \\ 5 & 1 \end{bmatrix}$$

Next, becuase the null space is in R^3 I will add an additional column.

$$\begin{bmatrix} 1 & 0 & a_{13} \\ 1 & 3 & a_{23} \\ 5 & 1 & a_{33} \end{bmatrix}$$

Now I will multiply A by the null space.

$$\begin{bmatrix} 1 & 0 & a_{13} \\ 1 & 3 & a_{23} \\ 5 & 1 & a_{33} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which gives me the following equations.

$1 + 2a_{13} = 0$
$1 + 3 + 2a_{23} = 0$
$5 + 1 + 2a_{33} = 0$
Simplifying I get
$a_{13} = -\frac{1}{2}$
$a_{23} = -2$
$a_{33} = -3$
Now I can substitute these values back into my matrix

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{bmatrix}$$

3 Problem

I will be using the fact that a matrix which is invertable is indepent columns. By creating matrix A such that the columns of A are u_1 throuch u_3 I get.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

I notice that simply subtracting $R2$ from $R1$ and subtracting $R3$ from $R2$, I get the identity matrix. And since row subtraction can be represented with a matrix E I know that this matrix is invertable and therefore has independent columns. Now to determine if u_1, u_2, u_3, u_4 is invertable, I will construct another matrix.

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

Using the fundamental theorem of linear algebra, I know $C(A)+N(A) = n$ where n is the number of columns of this matrix. I also know that $C(A) = C(A^T)$ and becuase $C(A^T) \leq n$, where n is the number of rows, I can assume $C(A) \leq 3$, when $n = 3$. Furthermore, I can show $N(A)$ must be at least 1 in order for $C(A) + N(A) = 4$. Therefore there is at least one vector in the null space of this matrix. Which leads me to the conclusion that while the previous matrix was independent, adding u_4 made them dependent. Also $u_4 = 4u_3 - u_2 - u_1$.

4 Problem

The number of pivots of a matrix after elimination is equivalent to the rank. So I will put A into the form of U and count the number of pivots. If they equal 2 then A will have rank 2.

$$\begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix}$$

Currently I have 3 pivots $1, c, d$, but I will be able to use the fact that a value of 0 can never be used as a pivot. So I will set $c = 0$.

$$\begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix}$$

Finnaly if I set $d = 2$, I will be able to get rid of the unwanted pivot by subtracting $R2$ from $R3$.

$$U = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Position U_{11} and U_{42} are the only valid positions for my pivots. So U has two pivots, and as stated above, confirms that A has rank 2.

5 Problem

A basis is a set of a subspace. One basis for $C(A)$ is:

$$ColumnSpace \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$$

I found this basis by scanning the columns from left to right and selecting non-trivial columns that were independent of any previous columns I selected.

Col 1 is trivial, Skip

Col 2 is non-trivial and not a linear combinations of previously choosen columns, Add

Col 3 is 2 times Col 2, Skip

Col 4 is non-trivial a not a linear combination of Col 2, Add

Col 5 Can be made from 2 of Col 4 and -1 of Col 2, Skip

For the null space, I have already found the free columns as shown above, which are, $Col 1, 3, 5$. I will use each one of these to generate a vetor in the null space. By setting $Ax = 0$.

$$NullSpace \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

For the row space I will transpose the matrix and find the column space in the same manner as I did before.

$$A^T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \\ 3 & 4 & 1 \\ 4 & 6 & 2 \end{bmatrix}$$

Col 1 is non trivial. Add
Col 2 is non trial and is not a linear combination of Col 1. Add
Col 3 is Col 2 - Col 1. Skip

$$RowSpace \begin{bmatrix} 0 \\ 1 \\ 2 \\ 4 \\ 6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

And finnaly for the Left Null space I will set $A^T x = 0$. Setting the free column of AT to 1.

$$LeftNullSpace \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

6 Problem

In order to find the two vectors that span the orthogonal complement $S^{\perp}$. I must satisfy the requirements of being a complement vector. Any orthogonal vector must have the property $s_1 \cdot v = 0$, and $s_2 \cdot v = 0$. This relation can be shown in matrix form like so

$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

I will now put the matrix in rref because this will not change the null space.

First subtract R1 from R2.

$$\begin{bmatrix}1 & 2 & 2 & 3\\ 0 & 1 & 1 & -1\end{bmatrix}\cdot\begin{bmatrix}v_1\\ v_2\\ v_3\\ v_4\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

Finnaly subtract 2 R2 from R1.

$$\begin{bmatrix}1 & 0 & 0 & 5\\ 0 & 1 & 1 & -1\end{bmatrix}\cdot\begin{bmatrix}v_1\\ v_2\\ v_3\\ v_4\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

Although v will not the be same vector, it will still span the same space.

$$\begin{bmatrix}1 & 0 & 0 & 5\\ 0 & 1 & 1 & -1\end{bmatrix}\cdot\begin{bmatrix}5\\ -1\\ 0\\ 1\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0 & 5\\ 0 & 1 & 1 & -1\end{bmatrix}\cdot\begin{bmatrix}0\\ -1\\ 1\\ 0\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

These are my two solutions to $rref(A)x = 0$. Now I can use them to form $S^{\perp}$

$$\begin{bmatrix}0\\ -1\\ 1\\ 0\end{bmatrix}\begin{bmatrix}5\\ -1\\ 0\\ 1\end{bmatrix}$$

7 Problem

From what is given about P. The following equation must be true

$$\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}\cdot\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}=\begin{bmatrix}0\end{bmatrix}$$

Also I know the relation $P^{\perp} = N(P^T)$. Given both of these facts I can generate:

$$\begin{bmatrix}x_1 & x_2 & x_3 & x_4\end{bmatrix}\cdot\begin{bmatrix}1\\ 1\\ 1\\ 1\end{bmatrix}=\begin{bmatrix}0\end{bmatrix}$$

Therefore I can set the bases fro $P^{\perp}$ to be

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$