Q1 Consider the function $f : [-5, 5] \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} x^2 + x & \text{for } -5 \le x \le 2, \\ x^3 & \text{for } 5 \ge x > 2. \end{cases}$$

(a) Is f piecewise smooth? Justify fully. **[3 marks]**

(b) Rewrite f with the help of the unit step function defined as **[3 marks]**

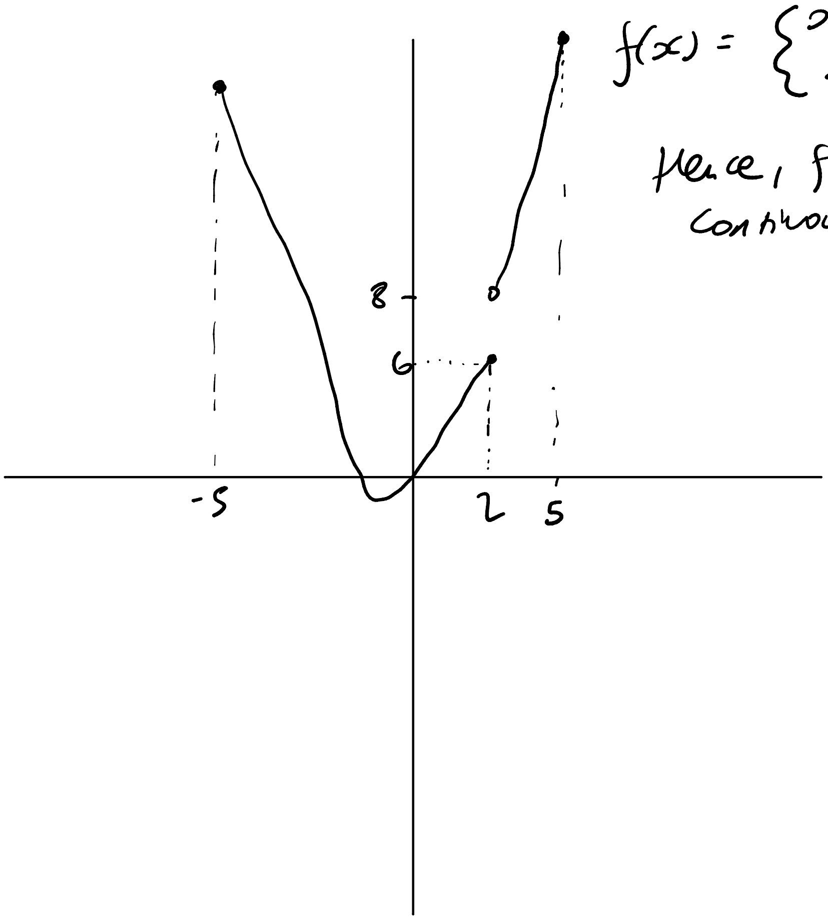
$$\Theta(x) = \begin{cases} 1 & \text{for } x > 0, \\ 0 & \text{for } x \le 0. \end{cases}$$

(c) Calculate the derivative of f in the sense of distributions. **[4 marks]**

a) Check: ① f is piecewise continuous
② f' is piecewise continuous

① Let $(a,b) = [-5,5]$, and subintervals

$[-5,2], (2,5]$.

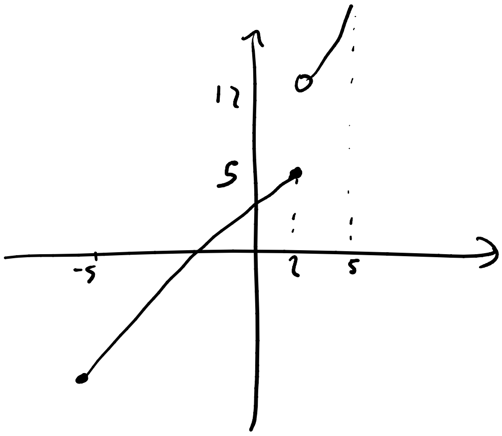


$f(x) = \begin{cases} x^2+x, & -5 \le x \le 2 \\ x^3, & 2 < x \le 5 \end{cases}$

Hence, $f(x)$ is piecewise continuous

② $f'(x) = \begin{cases} 2x+1, & -5 \le x \le 2 \\ 3x^2, & 2 < x \le 5 \end{cases}$, also piecewise

(i) $f(x) = \begin{cases} \ldots \\ 3x^2, & 2 < x \leq 5 \end{cases}$, also piecewise Continuous



Hence $f(x)$ is piecewise smooth.

b) $$f(x) = \begin{cases} x^2 + x, & -5 \leq x \leq 2 \\ x^3, & 2 < x \leq 5 \end{cases}$$

$$\check{f}(x) = (x^2+x) + (x^3 - x^2 - x)\,\theta(x-2)$$

c) $$f'(x) = (2x+1) + (3x^2 - 2x - 1)\,\theta(x-2) + (x^3 - x^2 - x)\,\delta(x-2)$$

$$\left(\phi(x)\delta(x-2) = \phi(2)\delta(x-2) \right)$$

$$= (2x+1) + (3x^2 - 2x - 1)\,\theta(x-2) + (2^3 - 2^2 - 2)\,\delta(x-2)$$

$$= 2x + 1 + 2\delta(x-2) + \begin{cases} 0, & -5 \leq x \leq 2 \\ 3x^2 - 2x - 1, & 2 < x \leq 5 \end{cases}$$

Q2 (a) Consider the function $f : \mathbb{R} \to \mathbb{R} : x \mapsto f(x) = x(x^2 + 3x + 2)$ and calculate, for any test function $\psi \in \mathcal{D}(\mathbb{R})$, the integral **[4 marks]**

$$\int_{\mathbb{R}} \delta(f(x))\, \psi(x)\, dx,$$

where $\delta \in \mathcal{D}'(\mathbb{R})$ is the Dirac delta distribution.

(b) Consider the distribution $(x^2 - 1)\, \delta_{(\alpha)} \in \mathcal{D}'(\mathbb{R})$, where $\delta_{(\alpha)}$ is the Dirac delta distribution dilated by a factor $\alpha \in \mathbb{R} - \{0\}$.

(i) For which value of B and C do we have the following equalities of distributions? Justify your answers by integrating both sides against an arbitrary test function $\psi \in \mathcal{D}(\mathbb{R})$.

(1) $(x^2 - 1)\, \delta_{(2)} = B\delta$ **[2 marks]**

(2) $(x^2 - 1)\, \delta_{(-2)} = C\delta$ **[2 marks]**

(ii) Can you infer a basic identity satisfied by $\delta(x)$ 'in the sense of distributions' from the results you obtained in part (i)(1) and part (i)(2)? **[2 marks]**

a) $f(x) = x(x^2+3x+2) = x(x+2)(x+1)$

$f(x) = 0 \Rightarrow x = 0, -2, -1$

$f'(x) = 3x^2 + 6x + 2 \Rightarrow f'(0) = 2,\ f'(-2) = 2,\ f(-1) = -1$

$$I = \int_{\mathbb{R}} \delta(f(x))\, \psi(x)\, dx$$

$$= \frac{1}{|2|}\psi(0) + \frac{1}{|2|}\psi(-2) + \frac{1}{|-1|}\psi(-1)$$

$$= \frac{1}{2}\psi(0) + \frac{1}{2}\psi(-2) + \psi(-1)$$

b)i) $$I = \int_{\mathbb{R}} (x^2-1)\, \delta_{(2)}(x)\, \psi(x)\, dx$$

$$= \int_{\mathbb{R}} (x^2-1)\, \delta(2x)\, \psi(x)\, dx$$

Let $y = 2x \Rightarrow dy = |2|\, dx = 2dx$

and $x^2 - 1 = y^2$,

and $x^2-1 = \frac{y^2}{4}-1$

So $I = \frac{1}{2}\int_{\mathbb{R}}\left(\frac{y^2}{4}-1\right)\delta(y)\psi(y/2)\,dy$

$= \frac{1}{2}\left(\frac{0}{4}-1\right)\psi(0) = -\frac{1}{2}\psi(0) = -\frac{1}{2}\delta$

so $B = -\frac{1}{2}$

ii) $I = \int_{\mathbb{R}} (x^2-1)\,\delta_{(-2)}(x)\,\psi(x)\,dx$

$= \int_{\mathbb{R}} (x^2-1)\,\delta(-2x)\,\psi(x)\,dx$

Let $y = -2x \Rightarrow dy = |-2|\,dx = 2dx$

and $x^2-1 = \frac{y^2}{4}-1$

So $I = \frac{1}{2}\int_{\mathbb{R}}\left(\frac{y^2}{4}-1\right)\delta(y)\psi(-y/2)\,dy$

$= \frac{1}{2}\left(\frac{0}{4}-1\right)\psi(0) = -\frac{1}{2}\psi(0) = -\frac{1}{2}\delta$

so $C = -\frac{1}{2}$

iii) $(x^2-1)\delta = \frac{-1}{1}\delta$.

iii) $\quad (x^2-1)\,\delta_{(\alpha)} = \frac{-1}{|\alpha|}\,\delta\,.$