Unit 3

Lesson 11

1. a) The total cost of manufacturing 1000 items is
$C(1000) = 0.02(1000)^2 + 40(1000) + 5000 = 65\ 000$ dollars.

The average cost of manufacturing 1000 items is
$\frac{65\,000}{1000} = 65$ dollars.

b) To find the marginal cost, compute the derivative of the total-cost function:

$C'(x) = 0.04x + 40$

The marginal cost of manufacturing the 1000th item is
$C'(1000) = 0.04(1000) + 40 = 80$ dollars.

2. If you increase the price by \$0.10 x times, the number of sales will be reduced by $5x$. In other words, if the price of a T-shirt is $12 + 0.10x$ dollars, the number of T-shirts sold will be $200 - 5x$ per month. The vendor's total monthly revenue will be $(12 + 0.10x)(200 - 5x)$ dollars per month. The monthly revenue can be expressed as

$$\begin{aligned} R(x) &= (12 + 0.10x)(200 - 5x) \\ &= 2400 - 60x + 20x - 0.5x^2 \\ &= 2400 - 40x - 0.5x^2 \end{aligned}$$

The derivative of the total revenue function is $R'(x) = -40 - 1.0x$.

$R'(x) = 0$ when $x = -40$. Also, $R'(x)$ is positive when $x < -40$ and $R'(x)$ is negative when $x > -40$. Therefore, $R(x)$ has a maximum when $x = -40$.

A negative value of x corresponds with a decrease in the price of T-shirts. Assuming that the trend described in the problem continues for negative values of x, the price the vendor should charge for a T-shirt is $12 + 0.10(-40) = 8$ dollars. The number of T-shirts she should expect to sell per month is $200 - 5(-40) = 400$.

3. The amount of Cesium-147 after t days is given by

$$C(t) = 500\left(\frac{1}{2}\right)^{\frac{t}{30}}.$$

If you're interested in the rate of decay, you need the derivative of $C(t)$, which is $C'(t) = 500\left(\frac{1}{2}\right)^{\frac{t}{30}} \cdot \ln\left(\frac{1}{2}\right) \cdot \left(\frac{1}{30}\right)$.

The rate of decay after 40 days is given by

$$C'(40) = 500\left(\frac{1}{2}\right)^{\frac{40}{30}} \cdot \ln\left(\frac{1}{2}\right) \cdot \left(\frac{1}{30}\right) = -4.58$$

After 40 days, the Cesium-147 is decaying at a rate of 4.58 g per day.

4. If you're interested in the rate of growth of the population, you need the derivative of $P(t)$. Rewrite $P(t)$ as a product and differentiate using the chain rule:

$$P(t) = 6000(1 + 49(0.6)^t)^{-1}$$

$$P'(t) = 6000(1 + 49(0.6)^t)'(-1)(1 + 49(0.6)^t)^{-2}$$

$$P'(t) = 6000(49)\ln(0.6)(0.6)^t(-1)(1 + 49(0.6)^t)^{-2}$$

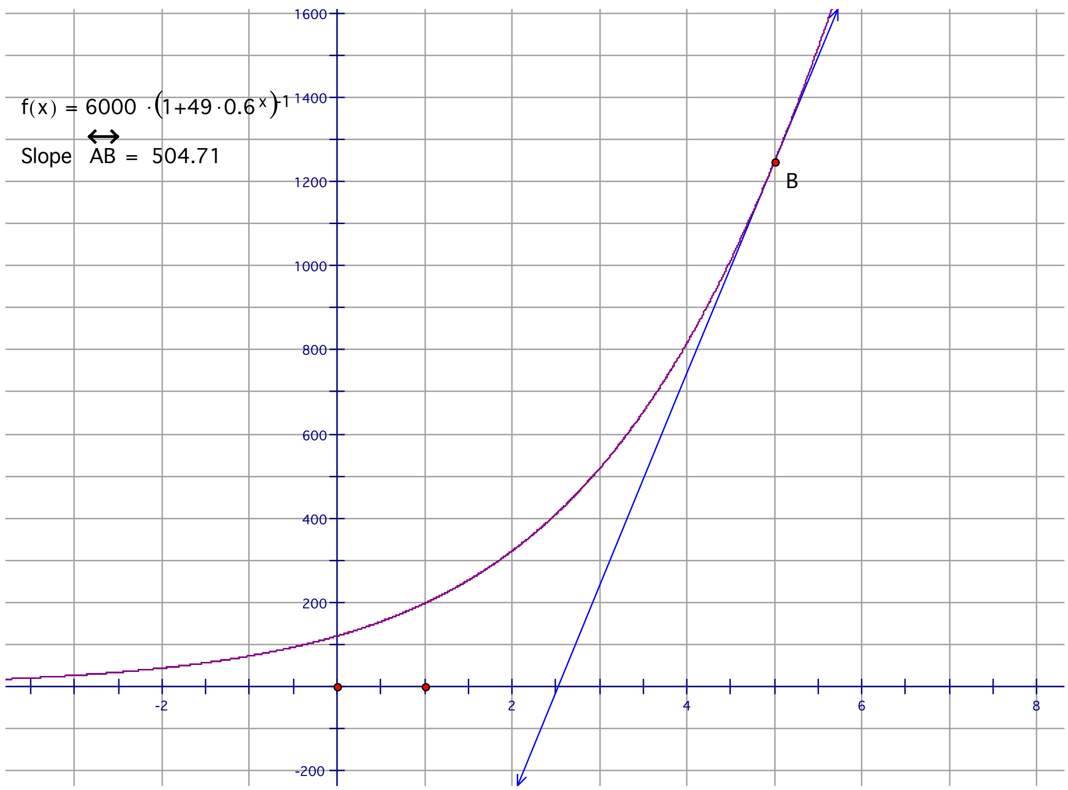
The rate of growth after 5 days is given by

$$P'(5) = 6000(49)\ln(0.6)(0.6)^5(-1)(1 + 49(0.6)^5)^{-2}$$

$$P'(5) = 504.71$$

After 5 days, the population of butterflies is increasing at a rate of 504.71 per day.

Graph $P(t)$ and find the slope of the tangent at $t = 5$.



The slope of the tangent at $t = 5$ is 504.71.

Lesson 12

5. Let x be the amount of wire used for the circle and $50 - x$ the amount used for the square.

The perimeter of a circle is $2\pi \cdot r$, where r is the radius, so $2\pi r = x$. The perimeter of a square is $4s$, where s is the length of a side, so $4s = 50 - x$.

Rearrange equations to solve for r and s in terms of x. This gives $r = \dfrac{1}{2\pi}x$ and $s = \dfrac{50 - x}{4}$.

The total area of the square and the circle is $\pi r^2 + s^2$, so the total area of the two shapes is given by

$$A = \pi\left(\frac{1}{2\pi}x\right)^2 + \left(\frac{50-x}{4}\right)^2$$

$$A = \frac{\pi}{4\pi^2}x^2 + \frac{(50-x)^2}{16}$$

$$A = \frac{1}{4\pi}x^2 + \frac{2500-100x+x^2}{16}$$

$$A = \left(\frac{1}{4\pi} + \frac{1}{16}\right)x^2 - \frac{100}{16}x + \frac{2500}{16}$$

$$A = \left(\frac{1}{4\pi} + \frac{1}{16}\right)x^2 - \frac{25}{4}x + \frac{625}{4}$$

After simplifying, it's easy to take the derivative of A:

$$A' = \left(\frac{1}{2\pi} + \frac{1}{8}\right)x - \frac{25}{4}$$

Set the derivative equal to zero:

$$\left(\frac{1}{2\pi} + \frac{1}{8}\right)x - \frac{25}{4} = 0$$

$$\left(\frac{1}{2\pi} + \frac{1}{8}\right)x = \frac{25}{4}$$

$$\left(\frac{4}{8\pi} + \frac{\pi}{8\pi}\right)x = \frac{25}{4}$$

$$x = \frac{25}{4}\cdot\frac{8\pi}{4+\pi} = \frac{50\pi}{4+\pi} = 21.995$$

Since A' is a linear function, it's not hard to see that A' is negative when $x < 21.995$ and A' is positive when $x > 21.995$. Therefore, the function A has a minimum when $x = 21.995$.

In order to minimize the total area of the two shapes, 21.995 cm of wire should be used to make the circle and 28.005 cm of wire to make the square. The total area obtained is $A = \pi\left(\frac{1}{2\pi}(21.995)\right)^2 + \left(\frac{50-21.995}{4}\right)^2 = 87.516\text{ cm}^2$.

6. The volume of a cylinder is $\pi r^2 h$, so $\pi r^2 h = 20\pi$, or equivalently, $r^2 h = 20$.

The area of the top of the cylinder is πr^2 and the area of the bottom of the cylinder is also πr^2, for a total area of $2\pi r^2$ for the top and bottom. The area of the side of the cylinder is $2\pi rh$.

For the top and bottom, $2\pi r^2$ m^2 of material at a cost of \$10 per m^2, whereas for the side, $2\pi rh$ m^2 of material at a cost of \$8 per m^2. This gives a total cost, in dollars, of $C = 10(2\pi r^2) + 8(2\pi rh)$.

Express this as a function of just one variable by eliminating the variable h.

$r^2 h = 20$, so $h = \dfrac{20}{r^2}$. Using this, rewrite function as

$$C = 10(2\pi r^2) + 8\left(\frac{2\pi r 20}{r^2}\right)$$

$$C = 20\pi r^2 + 320\pi r^{-1}$$

Now that you've expressed C as a function of just one variable, take the derivative:

$C' = 40\pi r - 320\pi r^{-2}$

Next, to find out where C' is zero, positive, or negative, solve the equation $C' = 0$:

$40\pi r - 320\pi r^{-2} = 0$

$40\pi r = 320\pi r^{-2}$

$40\pi(r) = 40\pi(8r^{-2})$

$r = 8r^{-2}$

$r^3 = 8$

$r = 2$

$C' = 0$ when $r = 2$. By plugging $r = 1$ and $r = 3$ into the formula for C', you can see that C' is negative when $r < 2$, and C' is positive when $r > 2$. As a result, the function C has a minimum at $r = 2$.

Therefore, the radius of the cylinder should be 2 m, and the height should be $\dfrac{20}{2^2} = 5$ m.

Suggested Answers to Support Questions

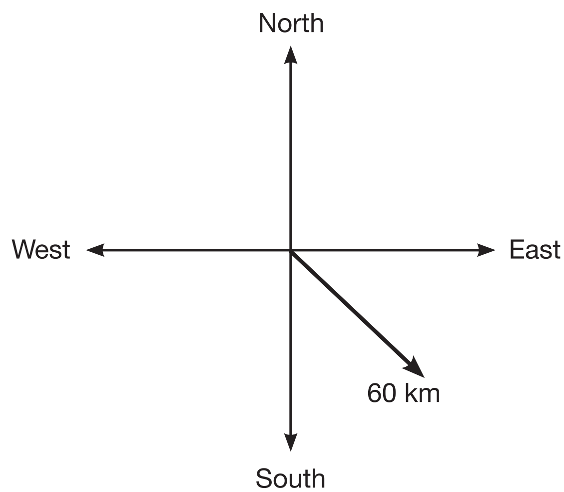
Lesson 13

7. a) No, height is a scalar because it has only a magnitude.

   b) Current in a river is a vector because it has a magnitude and a direction.

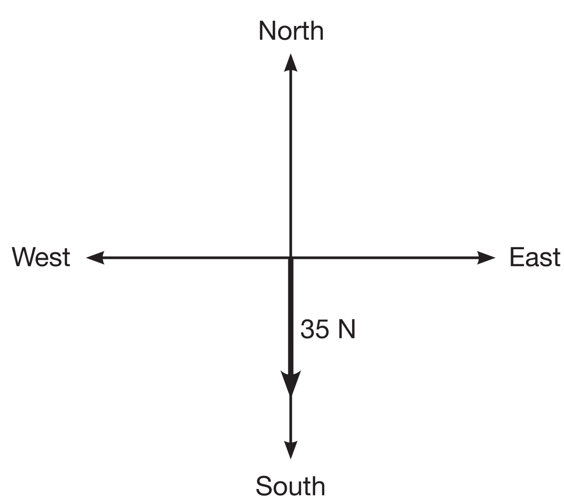
   c) Velocity of an airplane is a vector because it has a magnitude and a direction.

8. a)



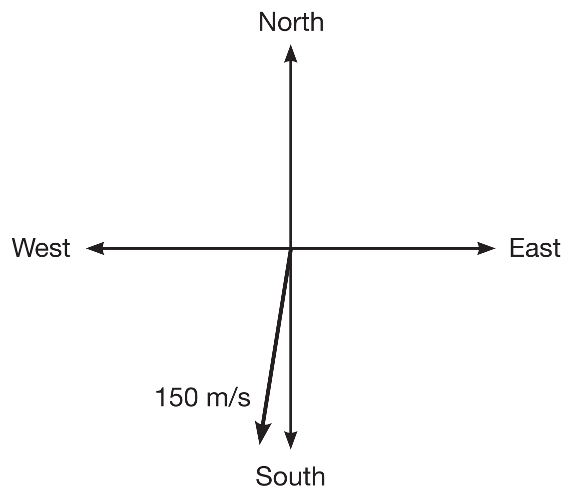
   Alternative notation: S45°E

   b)



   Alternative notation: Bearing of 180°

c)



Alternative notation: S8°W

9. a) $\vec{KS} = \vec{SL} = \vec{TU}$

$\vec{MT} = \vec{TK} = \vec{US}$

$\vec{ST} = \vec{LU} = \vec{UM}$

b) $\vec{AF} = \vec{CD} = \vec{BG} = \vec{GE}$

$\vec{AB} = \vec{FG} = \vec{GC} = \vec{ED}$

$\vec{CB} = \vec{DG} = \vec{GA} = \vec{EF}$

10. a) (1, –2)

$$r = \sqrt{(1)^2 + (-2)^2} = \sqrt{5}$$

$$\theta = \tan^{-1}\left(\frac{-2}{1}\right)$$

Using a scientific calculator: $\theta \cong -63.4°$

Since (1, –2) is in the fourth quadrant, the vector makes an angle of 360 – 63.4 = 296.6° with the positive x-axis, measured counter-clockwise.

Suggested Answers to Support Questions

b) $(-1, 6)$

Let $\vec{r} = (-1, 6)$

$$\left|\vec{r}\right| = \sqrt{(-1)^2 + (6)^2} = \sqrt{37}$$

$$\theta = \tan^{-1}\left(\frac{6}{-1}\right)$$

Using a scientific calculator: $\theta \cong -80.5°$

Since the terminal point of the vector $(-1, 6)$, is in the second quadrant, the vector makes an angle of $180 - 80.5 = 99.5°$ with the positive x-axis, measured counter-clockwise.

c) $\left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$

$$\sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}}$$

$$= 1$$

$$\theta = \tan^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{\frac{-1}{2}}\right)$$

$$= \tan^{-1}\left(-\sqrt{3}\right)$$

Using a scientific calculator: $\theta \cong -60°$

Since $\left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$ is in the second quadrant, the vector makes an angle of $180 - 60 = 120°$ with the positive x-axis, measured counter-clockwise.

11. a) $|\vec{u}| = 10$ and $\theta = 30°$

$$x = 10\cos(30)$$
$$= 5\sqrt{3}$$
$$y = 10\sin(30)$$
$$= 5$$
$$\vec{u} = (5\sqrt{3}, 5)$$

b) $|\vec{v}| = 12$ and $\theta = 240°$

$$x = 12\cos(240)$$
$$= -6$$
$$y = 12\sin(240)$$
$$= -6\sqrt{3}$$
$$\vec{v} = (-6, -6\sqrt{3})$$

c) $|\vec{w}| = 8$ and $\theta = 330°$

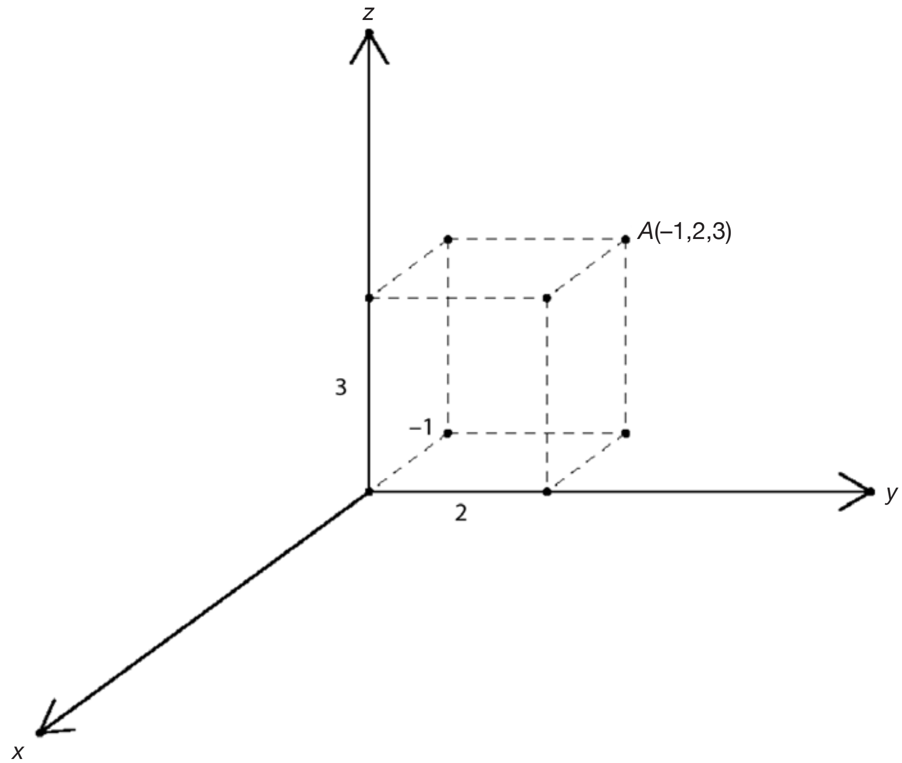
$$x = 8\cos(330)$$
$$= 4\sqrt{3}$$
$$y = 8\sin(330)$$
$$= -4$$
$$\vec{w} = (4\sqrt{3}, -4)$$

d) $|\vec{x}| = 16$ and $\theta = 135°$

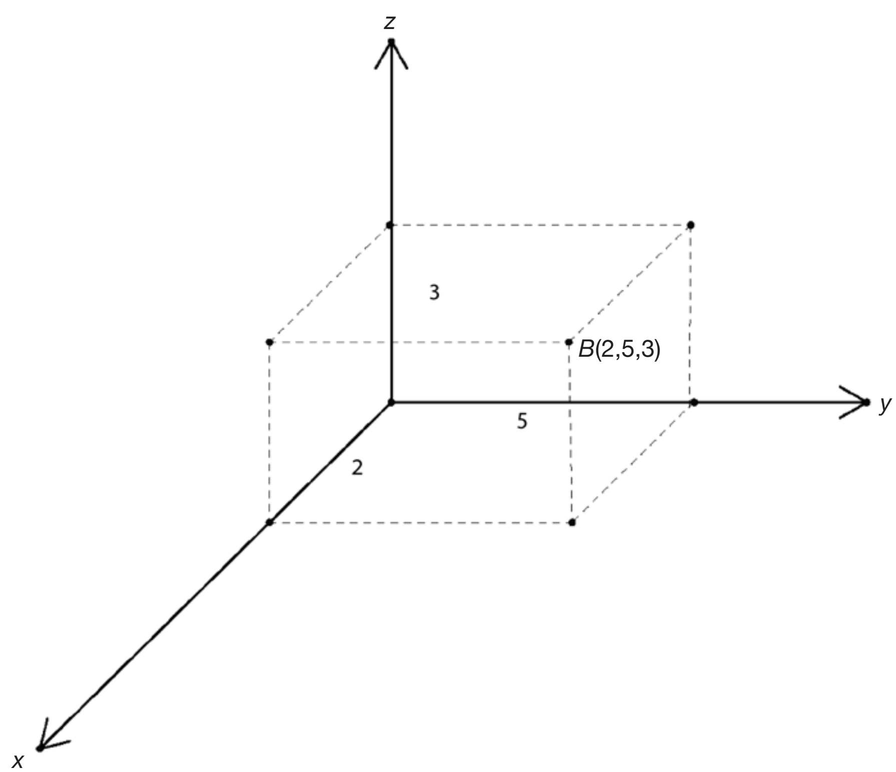
$$x = 16\cos(135)$$
$$= -8\sqrt{2}$$
$$y = 16\sin(135)$$
$$= 8\sqrt{2}$$
$$\vec{x} = (-8\sqrt{2}, 8\sqrt{2})$$

Lesson 14

12. a)



b)



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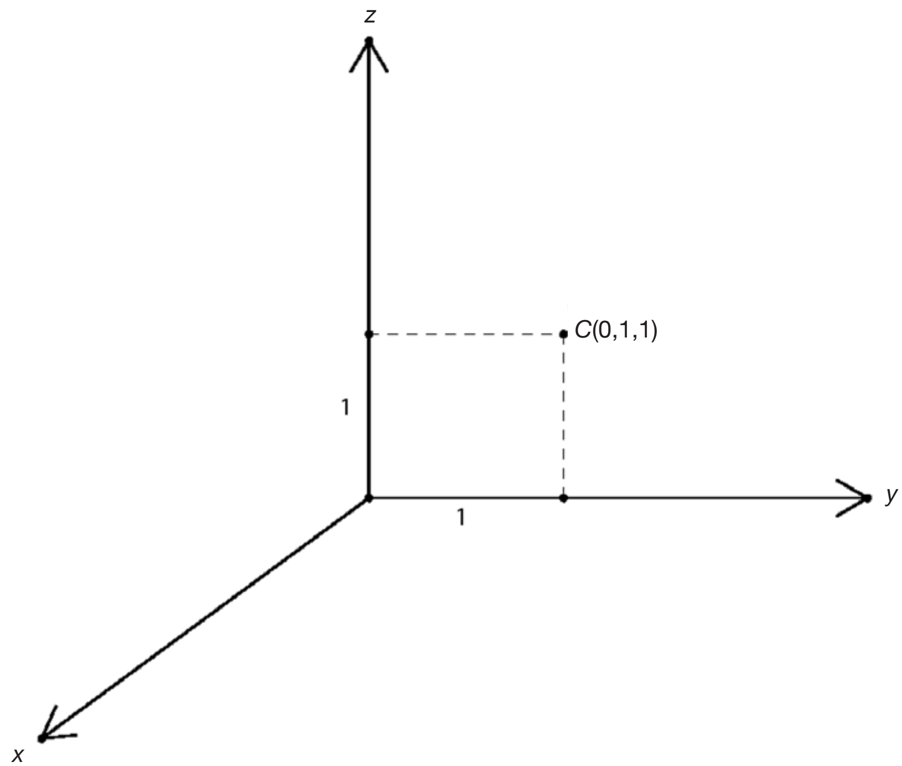


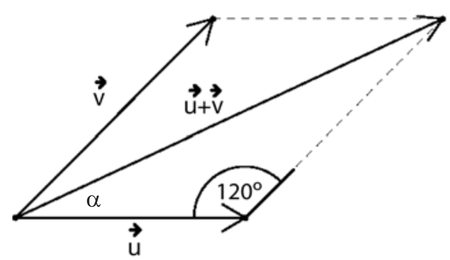
13. a) $A(a, 0, 0)$ The point A is on the x-axis, a units from the origin.

b) $B(m, 0, n)$ The point B is in the x-z plane, m units from the origin along the x-axis and n units from the origin along the z-axis.

c) $C(0, 0, n)$ The point C is on the z-axis, n units from the origin.

d) $D(0, s, t)$ The point D is in the y-z plane, s units from the origin along the y-axis and t units from the origin along the z-axis.

14.



$\theta = 180° - 60° = 120°$

$$\left|\vec{u}+\vec{v}\right|^2 = \left|\vec{u}\right|^2 + \left|\vec{v}\right|^2 - 2\left|\vec{u}\right|\left|\vec{v}\right|\cos(\theta)$$

$$\begin{aligned}\left|\vec{u}+\vec{v}\right|^2 &= (6)^2 + (8)^2 - 2(6)(8)\cos(120)\\ &= 36 + 64 - 96\cos(120)\\ &= 148\end{aligned}$$

$$\left|\vec{u}+\vec{v}\right| = \sqrt{148}$$

$$\left|\vec{u}+\vec{v}\right| \approx 12.17$$

Use the sine law to find the direction of the magnitude:

$$\frac{\sin(120)}{\left|\vec{u}+\vec{v}\right|} = \frac{\sin(\alpha)}{\left|\vec{v}\right|}$$

$$\frac{\sin(120)}{12.17} = \frac{\sin(\alpha)}{8}$$

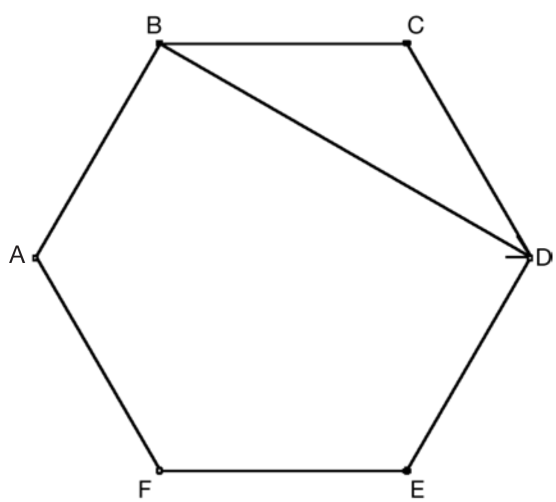
$$\sin(\alpha) = \frac{8\sin(120)}{12.17}$$

$$\sin(\alpha) \approx 0.57$$

$$\alpha \approx 34.7°$$

The resultant of $\vec{u}$ and $\vec{v}$ is approximately 34.7° from $\vec{u}$.

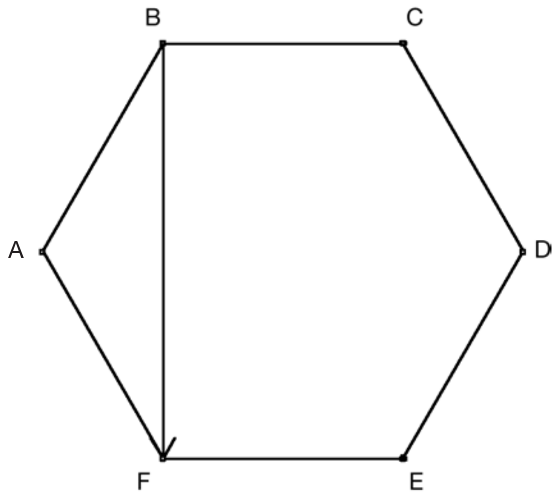
15. a) Using the triangle law of addition: $\vec{BC} + \vec{CD} = \vec{BD}$.



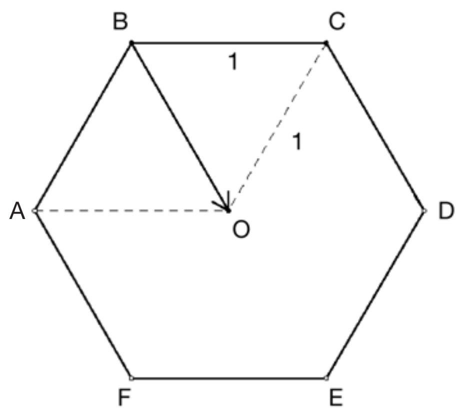
b) $\vec{BA} - \vec{DC} = \vec{BA} + (-\vec{DC})$

But $-\vec{DC} = \vec{CD} = \vec{AF}$

$\vec{BA} - \vec{DC} = \vec{BA} + (-\vec{DC}) = \vec{BA} + \vec{AF} = \vec{BF}$



16.



$\vec{BA} + \vec{BC} + \vec{BE} = (\vec{BA} + \vec{BC}) + \vec{BE}$

$\vec{BA} + \vec{BC} = \vec{BO}$

Observe that the triangle OBC is an equilateral triangle. All sides are equal and hence $|\vec{BO}| = 1$

To find $\vec{BO} + \vec{BE}$, observe that the vectors $\vec{BO}$ and $\vec{BE}$ have the same direction. To find the magnitude of the resultant, add the magnitudes:

$|\vec{BA} + \vec{BC} + \vec{BE}| = |\vec{BO}| + |\vec{BE}| = 1 + 2 = 3$

The direction of the resultant is the same as the direction of $\vec{BE}$.

Suggested Answers to Support Questions

17. a) $(1, -2, 3)$

$$\sqrt{(1)^2 + (-2)^2 + 3^2} = \sqrt{1+4+9}$$
$$= \sqrt{14}$$

b) $\left(\sqrt{5}, \sqrt{14}, \sqrt{6}\right)$

$$\sqrt{5+14+6} = \sqrt{25}$$
$$= 5$$

c) $(3, 0, 0)$

$$\sqrt{3^2} = 3$$

18. a) $A(-1, 2), B(-4, 2)$

Start by finding the vector $\vec{AB}$, then calculating its magnitude:

$$\vec{AB} = \vec{OB} - \vec{OA}$$
$$= (-4-(-1), 2-2)$$
$$= (-3, 0)$$

$$|\vec{AB}| = \sqrt{(-3)^2}$$
$$= 3$$

b) $P(-2, 3, 0), N(1, 2, 6)$

$$\vec{PN} = \vec{ON} - \vec{OP}$$
$$= (1, 2, 6) - (-2, 3, 0)$$
$$= (3, -1, 6)$$

$$|\vec{PN}| = \sqrt{(3)^2 + (-1)^2 + (6)^2}$$
$$= \sqrt{9+1+36}$$
$$= \sqrt{46}$$

19. a) $A(-3, 1, 2)$, $B(1, -3, -1)$, and $C(3, -1, -1)$

Calculate the magnitude of each side:

$$\begin{aligned}\vec{AB} &= (1-(-3), -3-1, -1-2)\\ &= (4, -4, -3)\end{aligned}$$

$$\begin{aligned}|\vec{AB}| &= \sqrt{16+16+9}\\ &= \sqrt{41}\end{aligned}$$

$$\begin{aligned}\vec{AC} &= (3-(-3), -1-1, -1-2)\\ &= (6, -2, -3)\end{aligned}$$

$$\begin{aligned}|\vec{AC}| &= \sqrt{36+4+9}\\ &= \sqrt{49}\\ &= 7\end{aligned}$$

$$\begin{aligned}\vec{BC} &= (3-1, -1-(-3), -1-(-1))\\ &= (2, 2, 0)\end{aligned}$$

$$\begin{aligned}|\vec{BC}| &= \sqrt{4+4+0}\\ &= \sqrt{8}\end{aligned}$$

$\vec{AC}$ has the largest magnitude, so check if

$$|\vec{AC}|^2 = |\vec{AB}|^2 + |\vec{BC}|^2$$

$$7^2 = \left(\sqrt{41}\right)^2 + \left(\sqrt{8}\right)^2$$

$$49 = 49$$

Since $|\vec{AC}|^2 = |\vec{AB}|^2 + |\vec{BC}|^2$, the triangle ABC is a right-angle triangle.

Suggested Answers to Support Questions

b) If $ABCD$ is a rectangle, then $\vec{AD} = \vec{BC}$.

As calculated in a), $\vec{BC} = (2, 2, 0)$.

$\vec{AD} = (x + 3, y - 1, z - 2) = (2, 2, 0)$

$x + 3 = 2, x = -1$

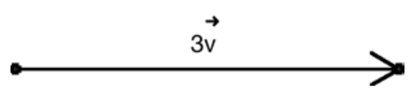
$y - 1 = 2, y = 3$

$z - 2 = 0, z = 2$

The coordinates of D are $(-1, 3, 2)$.

Lesson 15

20. a)



b)

21. a) $(1, 2) + 3(-2, 1)$

$= (1, 2) + (-6, 3)$

$= (-5, 5)$

b) $(1, -2) - 2(3, -1)$

$= (1, -2) + (-6, 2)$

$= (-5, 0)$

c) $(1, 2, -3) + 2(1, -1, 2)$

$= (1, 2, -3) + (2, -2, 4)$

$= (3, 0, 1)$

22. a) $\vec{x} - 3(\vec{x} + 4\vec{y})$

$= \vec{x} - 3\vec{x} - 12\vec{y}$

$= -2\vec{x} - 12\vec{y}$

b) $-3\vec{x} + 2(5\vec{x} - \vec{y} + 3\vec{x}) + 7\vec{x}$

$= -3\vec{x} + 10\vec{x} - 2\vec{y} + 6\vec{x} + 7\vec{x}$

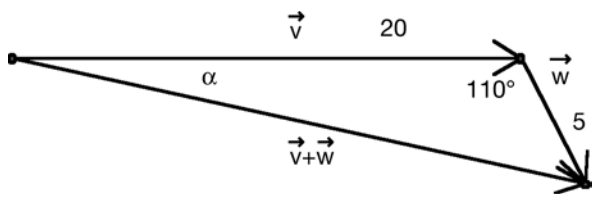
$= 20\vec{x} - 2\vec{y}$

c) $5(2\vec{x} - 3\vec{y}) + 4(3\vec{x} - 2\vec{y}) - 6(2\vec{x} - 3\vec{y})$

$= 10\vec{x} - 15\vec{y} + 12\vec{x} - 8\vec{y} - 12\vec{x} + 18\vec{y}$

$= 10\vec{x} - 5\vec{y}$

23. Let $\vec{v}$ represent the velocity the boat is steering, and let $\vec{w}$ represent the velocity of the water current. The resultant velocity of the boat is the vector $\vec{v} + \vec{w}$.



Find the magnitude of $\vec{v} + \vec{w}$ using the cosine law:

$|\vec{v} + \vec{w}|^2 = |\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}||\vec{w}|\cos 110°$

$|\vec{v} + \vec{w}|^2 = 20^2 + 5^2 - 2 \times 20 \times 5 \times \cos 110°$

$|\vec{v} + \vec{w}|^2 = 400 + 25 + 68.4$

$|\vec{v} + \vec{w}|^2 = 493.4$

$|\vec{v} + \vec{w}| = \sqrt{493.4}$

$|\vec{v} + \vec{w}| = 22.2$

Find the direction of $\vec{v} + \vec{w}$ in relation to $\vec{v}$ using the sine law:

$$\frac{\sin\alpha}{|\vec{w}|} = \frac{\sin 110°}{|\vec{v} + \vec{w}|}$$

$$\frac{\sin\alpha}{5} = \frac{\sin 110°}{22.2}$$

$$\sin\alpha = \frac{\sin 110°}{22.2} \times 5$$

$$\sin\alpha = \frac{0.94}{22.2} \times 5$$

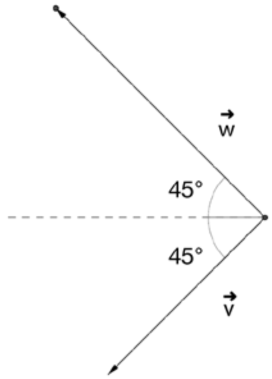
$$\alpha = \sin^{-1}(0.21)$$

$$\alpha \approx 12°$$

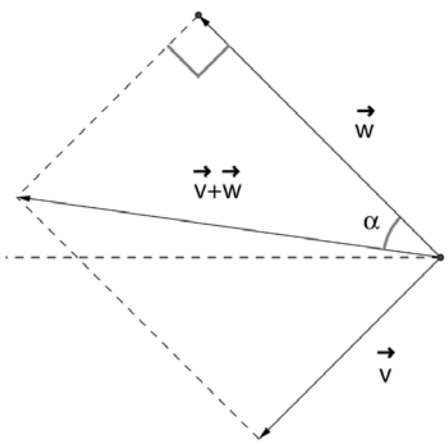
The resultant direction of the boat is approximately E12°S.

Suggested Answers to Support Questions

24. Let vector $\vec{v}$ represent the first force, and let vector $\vec{w}$ represent the second force:



The three vectors $\vec{v}$, $\vec{w}$, and $\vec{v} + \vec{w}$ form a triangle.



Since the angle opposite the side $\vec{v} + \vec{w}$ is a right angle, you can find the magnitude of $\vec{v} + \vec{w}$ using the Pythagorean theorem.

$$|\vec{v} + \vec{w}|^2 = |\vec{v}|^2 + |\vec{w}|^2$$

$$|\vec{v} + \vec{w}|^2 = 6^2 + 8^2 = 36 + 64$$

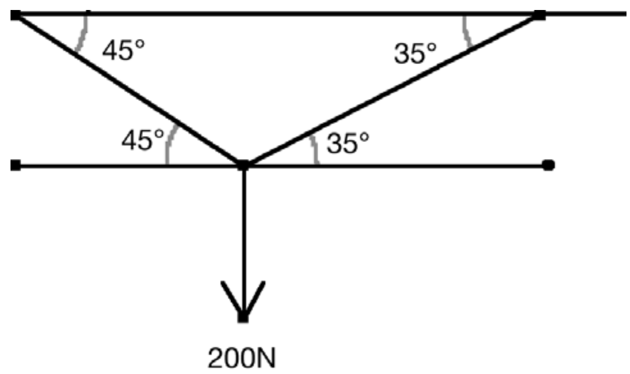
$$|\vec{v} + \vec{w}|^2 = 100$$

$$|\vec{v} + \vec{w}| = 10$$

Find the angle α using the definition of any of the trigonometric functions.

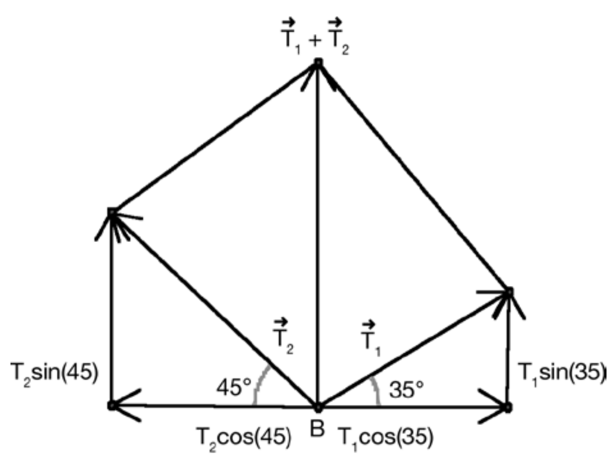
For example, $\tan\alpha = \dfrac{6}{8}$, so $\alpha = 36.87°$. This value is subtracted from 45°. The resultant force $\vec{v} + \vec{w}$ acts in the direction W8.13°N.

25. Start with a diagram of the ropes:



Draw a diagram of the forces acting on the object. T_1 and T_2 represent the tension (the force) on the ropes that keep the object from falling.

The resultant of T_1 and T_2 must be equal and opposite to the weight of the hanging object. The horizontal components of T_1 and T_2 must be opposite and equal and the sum of their vertical components must be equal to 200 N.



Horizontal components:

$$
\begin{aligned}
T_1 \cos(35) &= T_2 \cos(45) \\
T_1 &= \frac{T_2 \cos(45)}{\cos(35)} \\
T_1 &= 0.86 T_2
\end{aligned}
$$

Vertical components:

$T_1 \sin(35) + T_2 \sin(45) = 200$

Substitute $T_1 = 0.86T_2$ into $T_1 \sin(35) + T_2 \sin(45) = 200$.

$$\begin{aligned} 0.86T_2 \sin(35) + T_2 \sin(45) &= 200 \\ 0.49T_2 + 0.71T_2 &= 200 \\ 1.2T_2 &= 200 \\ T_2 &\approx 166.67 \text{ N} \end{aligned}$$

$$\begin{aligned} T_1 &= 0.86T_2 \\ T_1 &= 0.86(166.67) \\ T_1 &\approx 143.34 \text{ N} \end{aligned}$$

The tensions in the ropes are approximately 166.67 N and 143.34 N.