MCV4U-A

14

Vectors in Three-Space, Addition and Subtraction

Introduction

Can you imagine the latest video game being designed using only two-dimensional space?

In Lesson 13, you learned about vectors in two-dimensional space (two-space) and ways to represent such vectors. In this lesson, you will learn to represent vectors in three-dimensional space (three-space).

Once you are comfortable with vectors in three-space, you will begin to learn about different calculations that can be done using vectors. You will find the distance between two points and the magnitude of a vector using its Cartesian representation. In addition, you will also learn about adding and subtracting vectors.

Estimated Hours for Completing This Lesson	
Vectors in Three-Space	0.5
Addition and Subtraction of Vectors	2
Adding and Subtracting Vectors in Cartesian Form	1
Key Questions	1.5

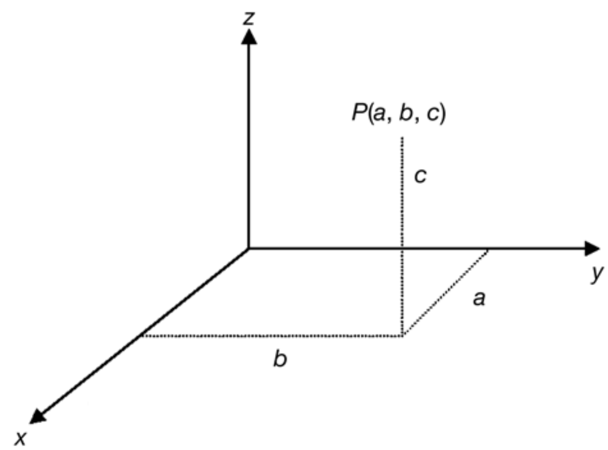
What You Will Learn

After completing this lesson, you will be able to

- represent vectors in three-space in Cartesian form
- perform the operation of addition and subtraction on vectors represented as directed line segments in two-space, and on vectors in Cartesian form in two-space and three-space
- find the magnitude of a vector and the distance between two points

Vectors in Three-Space

A three-dimensional Cartesian system can be formed when you draw three perpendicular axes that intersect at an origin as shown in the following diagram. Each point in space has three coordinates, referred to as an ordered triplet and written (a, b, c), where a is the x-coordinate, b is the y-coordinate, and c is the z-coordinate.



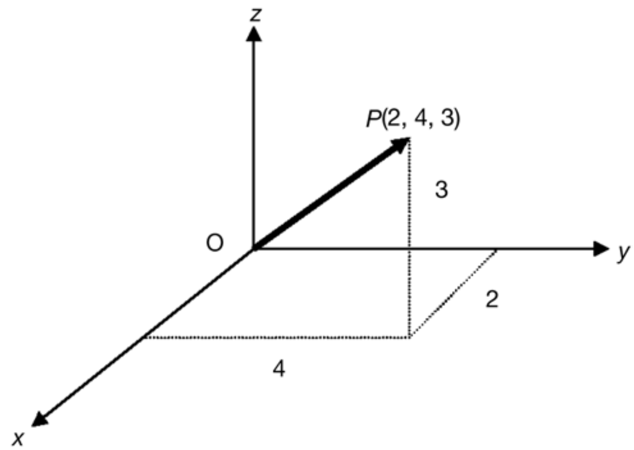
To draw the point $P(a, b, c)$ in three-space, start at the origin, move a units in the x-axis direction, b units in the y direction, and c units in the z direction. Just as in the two-dimensional example, a vector can be moved so that its tail is at the origin and its tip is at a point P with coordinates (a, b, c).

You can represent any vector $\vec{u}$ in three-space as an ordered triplet (a, b, c), and its magnitude is given by $|\vec{u}| = \sqrt{a^2 + b^2 + c^2}$.

Example

Draw the vector $\vec{OP} = (2, 4, 3)$ and find its magnitude.

Solution



The magnitude is calculated as follows:

$$\begin{aligned}\vec{OP} &= \sqrt{(2)^2 + (4)^2 + (3)^2}\\ &= \sqrt{29}\end{aligned}$$

In the next example, you will look at the location of points in three-space given three coordinates.

Example

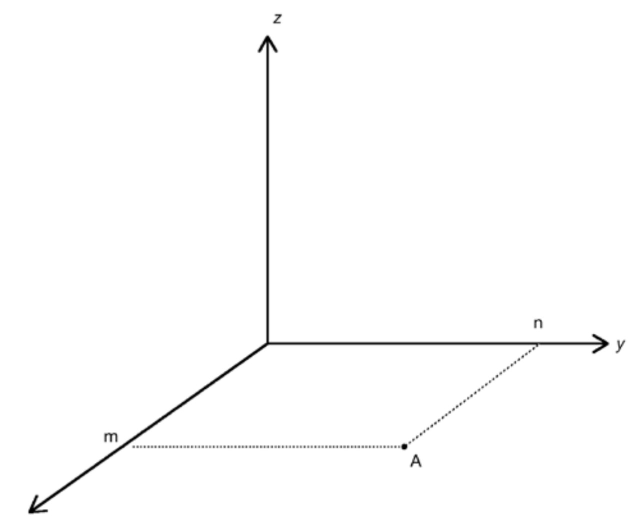
Describe where the following points are located:

a) $A(m, n, 0)$

b) $B(m, 0, 1)$

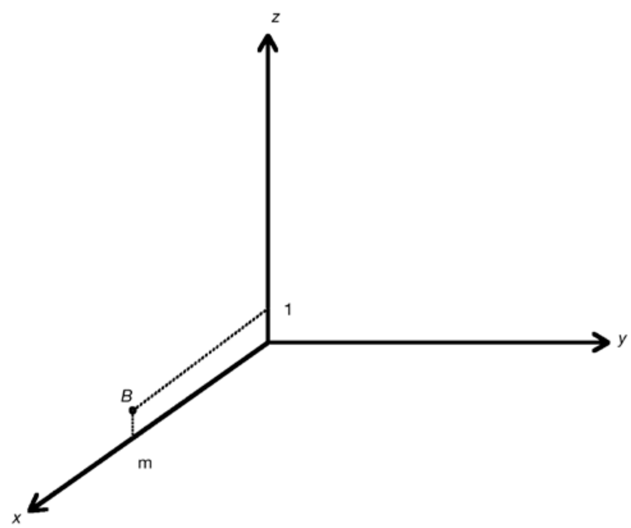
Solution

a)



As shown in the diagram, the point A is in the x-y plane, m units from the origin along the x-axis and n units from the origin along the y-axis.

b)



As shown in the diagram, the point B is in the x-z plane, m units from the origin along the x-axis and 1 unit from the origin along the z-axis.

Support Questions
(do not send in for evaluation)

12. Draw the following points on a three-dimensional grid:

    a) $A(-1, 2, 3)$

    b) $B(2, 5, 3)$

    c) $C(0, 1, 1)$

13. Describe where the following points are located:

    a) $A(a, 0, 0)$

    b) $B(m, 0, n)$

    c) $C(0, 0, n)$

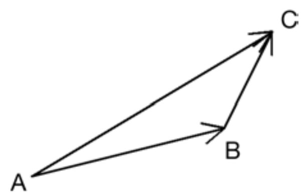
    d) $D(0, s, t)$

There are Suggested Answers to Support Questions at the end of this unit.

Addition and Subtraction of Vectors

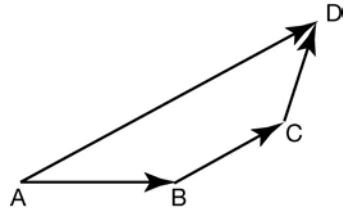
In many situations, it is important to find the combined effect or sum of two vectors. For example, consider the effect of two forces on an object, or the path of a swimmer crossing a river with a current.

Suppose $\vec{AB}$ represents the displacement from city A to city B and $\vec{BC}$ represents the displacement from city B to city C. Travelling from city A to city B and then from city B to city C has the same effect as travelling from city A to city C.



$\vec{AC}$ is called the resultant of $\vec{AB}$ and $\vec{BC}$. This is written: $\vec{AC} = \vec{AB} + \vec{BC}$.

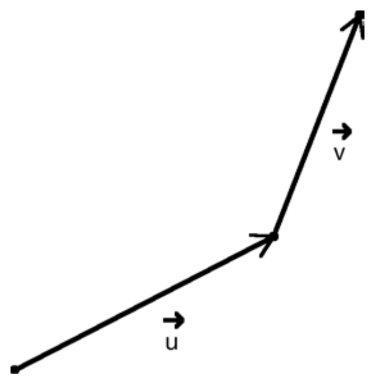
When adding three vectors, find the sum of the first two vectors and add the answer to the third. Building on the previous example, suppose you want to travel to city D instead of city C. In the following diagram, $\vec{AB}$ represents the displacement from city A to city B, $\vec{BC}$ represents the displacement from city B to city C, and $\vec{CD}$ represents the displacement from city C to city D. Travelling from city A to city B, from city B to city C, and then from city C to city D has the same effect as travelling from city A to city D.

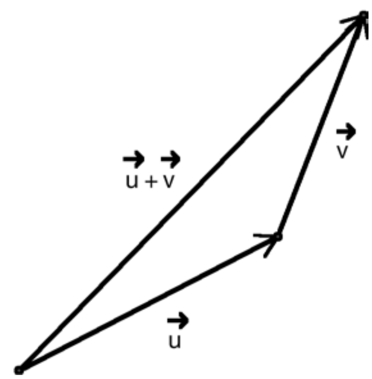


$\vec{AD}$ is the resultant of $\vec{AB}$, $\vec{BC}$, and $\vec{CD}$. It is written: $\vec{AD} = \vec{AB} + \vec{BC} + \vec{CD}$.

Triangle Law of Vector Addition

If two vectors $\vec{u}$ and $\vec{v}$ are drawn tip to tail, the third side that completes the triangle is the sum of the two vectors. The sum $\vec{u} + \vec{v}$ or the resultant of $\vec{u}$ and $\vec{v}$ is the vector from the tail of $\vec{u}$ to the tip of $\vec{v}$.



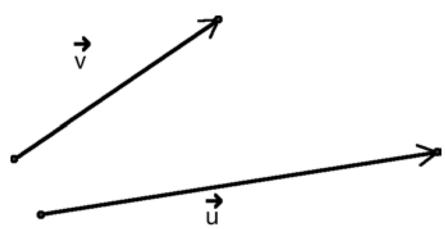


Examples

Given the following two vectors, draw:

a) $\vec{u} + \vec{v}$

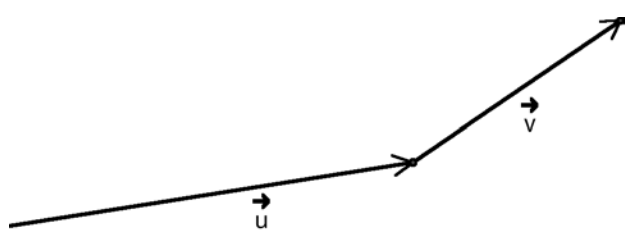
b) $\vec{v} + \vec{u}$



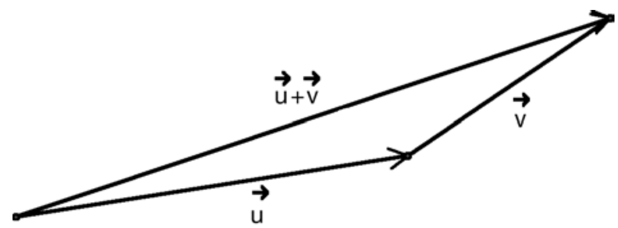
Solutions

a) $\vec{u} + \vec{v}$

Start by translating (moving) vector $\vec{v}$ so its tail is at the tip of $\vec{u}$:

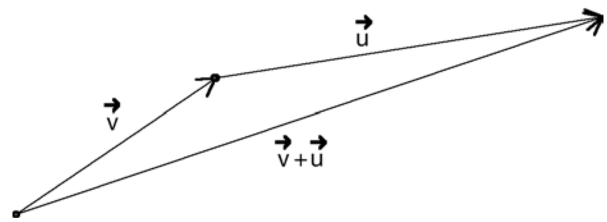


Complete the triangle by drawing a vector from the tail of $\vec{u}$ to the tip of $\vec{v}$. This is the vector $\vec{u} + \vec{v}$:

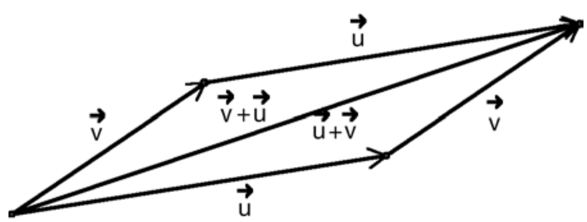


b) $\vec{v} + \vec{u}$

Translate $\vec{u}$ so its tail is at the tip of $\vec{v}$, then complete the triangle by drawing a vector from the tail of $\vec{v}$ to the tip of $\vec{u}$.

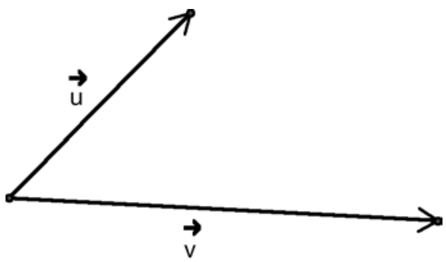


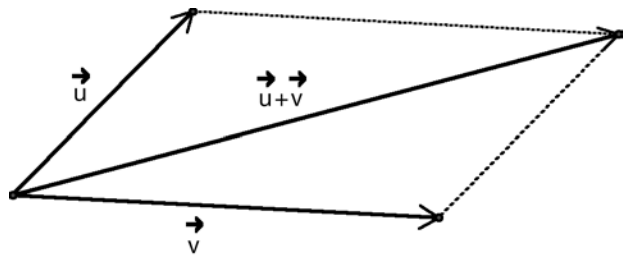
Is there a difference between the two resultants $\vec{u} + \vec{v}$ and $\vec{v} + \vec{u}$? The following diagram illustrates that the two sums are equivalent and represent the diagonal of a parallelogram. This property is referred to as the commutative property of vector addition. This means that the order in which vectors are added is irrelevant; the sum is the same.



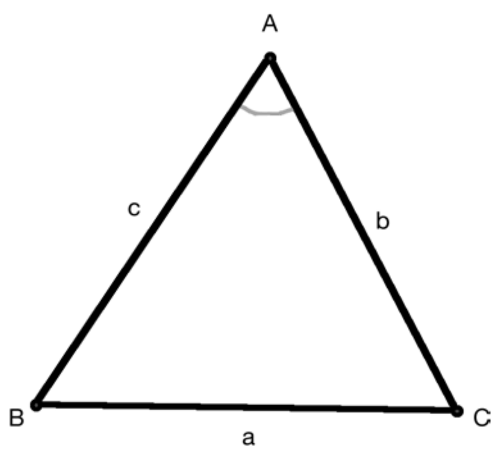
Parallelogram Law of Vector Addition

The parallelogram method can be used when the vectors to be added are arranged so the tails of the two vectors coincide. If you think of the vectors as two adjacent sides of a parallelogram, then the diagonal of the parallelogram through the common point represents the sum of the two vectors in both magnitude and direction.





You need to use the cosine law to solve for the magnitude and the sine law to solve for the direction of the sum of two vectors. Here is an example. Recall that given a triangle ABC with sides a, b, and c and interior angles A, B, and C the cosine and sine laws state the following:



Cosine Law:

$a^2 = b^2 + c^2 - 2bc\cos(A)$

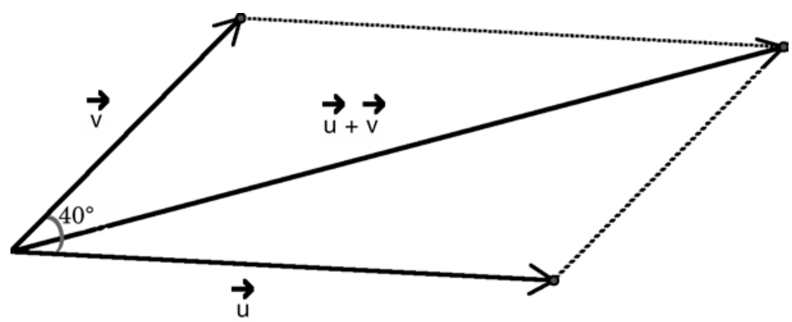
Sine Law:

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$$

Example

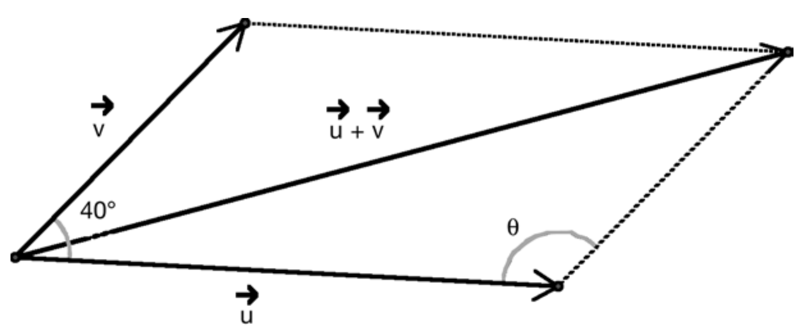
Find the magnitude and direction of the resultant of two vectors $\vec{u}$ and $\vec{v}$ with magnitudes 6 and 2 and an angle of 40° between them.

Solution

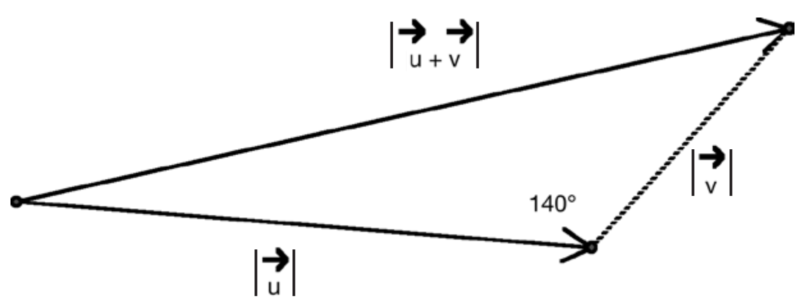


Step 1: Start with a diagram of the two vectors with an angle of 40° between them. Using the parallelogram law of vector addition, complete the parallelogram and draw the diagonal that is the resultant of $\vec{u}$ and $\vec{v}$.

Step 2: To find the magnitude of $\vec{u} + \vec{v}$, calculate the magnitude of the diagonal. To do this, use the cosine law.

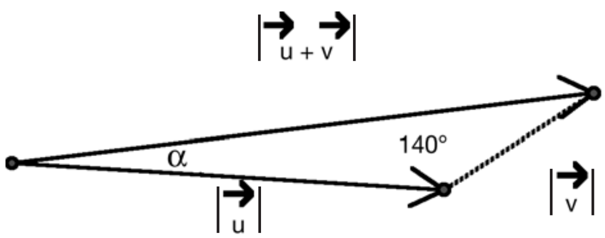


$\theta = 180 - 40 = 140°$



$$
\begin{aligned}
|\vec{u}+\vec{v}|^2 &= |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos(\theta) \\
|\vec{u}+\vec{v}|^2 &= (6)^2 + (2)^2 - 2|6||2|\cos(140) \\
|\vec{u}+\vec{v}|^2 &= 36 + 4 - 24\cos(140) \\
|\vec{u}+\vec{v}|^2 &\approx 58.38 \\
|\vec{u}+\vec{v}| &= \sqrt{58.38} \\
|\vec{u}+\vec{v}| &\approx 7.64
\end{aligned}
$$

Step 3: To find the direction of the resultant, find the angle between it and one of the vectors given (in this case, the angle separating it from $\vec{u}$). Use the sine law:



$$
\begin{aligned}
\frac{\sin\alpha}{|\vec{v}|} &= \frac{\sin 140}{|\vec{u}+\vec{v}|} \\
\frac{\sin\alpha}{2} &= \frac{\sin 140}{7.64} \\
\sin\alpha &= \frac{2\sin 140}{7.64} \\
\sin\alpha &\approx 0.168 \\
\alpha &= \sin^{-1}(0.168)
\end{aligned}
$$

Using a calculator, you will find $\alpha \approx 9.69°$.

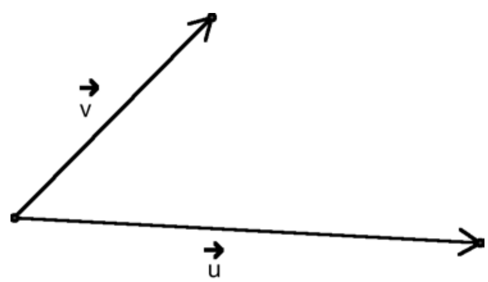
The magnitude of $\vec{u} + \vec{v}$ is 7.64 and it makes an angle of 9.69° with $\vec{u}$.

Subtracting Vectors

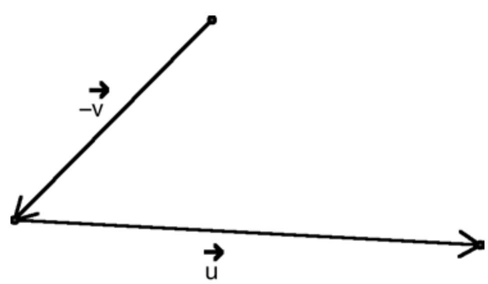
In arithmetic, you can think of subtraction as the addition of the negative of the number being subtracted. For example, 5 – 3 = 5 + (–3). The same idea is used to subtract two vectors: to find $\vec{u} - \vec{v}$, add the opposite of vector $\vec{v}$ to $\vec{u}$:

$$\begin{aligned}\vec{u} - \vec{v} &= \vec{u} + (-\vec{v}) \\ &= (-\vec{v}) + \vec{u}\end{aligned}$$

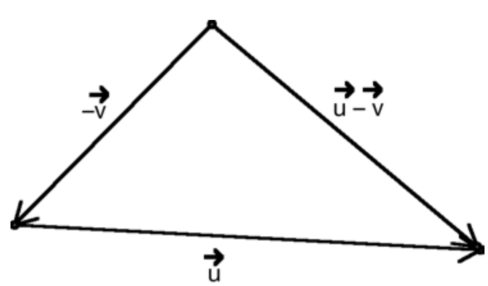
Subtract two vectors using the following diagram:



To find $\vec{u} - \vec{v}$, draw the vector $-\vec{v}$ by changing the direction of $\vec{v}$:



Then use the triangle law of addition to draw $\vec{u} + (-\vec{v})$:

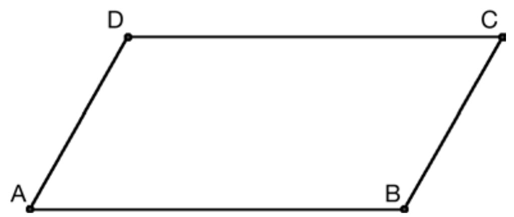


Examples

In the parallelogram $ABCD$, find a vector that represents each of the following:

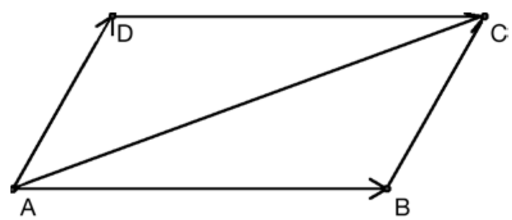
a) $\vec{AB} + \vec{BC}$

b) $\vec{AB} - \vec{BC}$



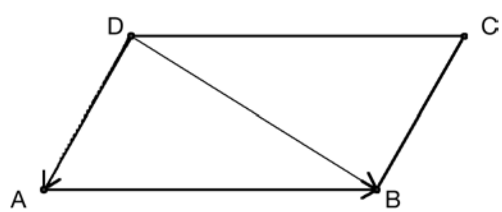
Solutions

a) The vectors $\vec{AB}$ and $\vec{BC}$ are tip to tail. Using the triangle law of addition, $\vec{AC}$ represents $\vec{AB} + \vec{BC}$.



b) $\vec{AB} - \vec{BC} = \vec{AB} + (-\vec{BC})$

$\vec{DA} = -\vec{BC}$, so $\vec{AB} - \vec{BC} = \vec{AB} + \vec{DA}$. Since the vectors $\vec{DA}$ and $\vec{AB}$ are tip to tail, you use the triangle law to conclude that $\vec{DB} = \vec{AB} - \vec{BC}$.



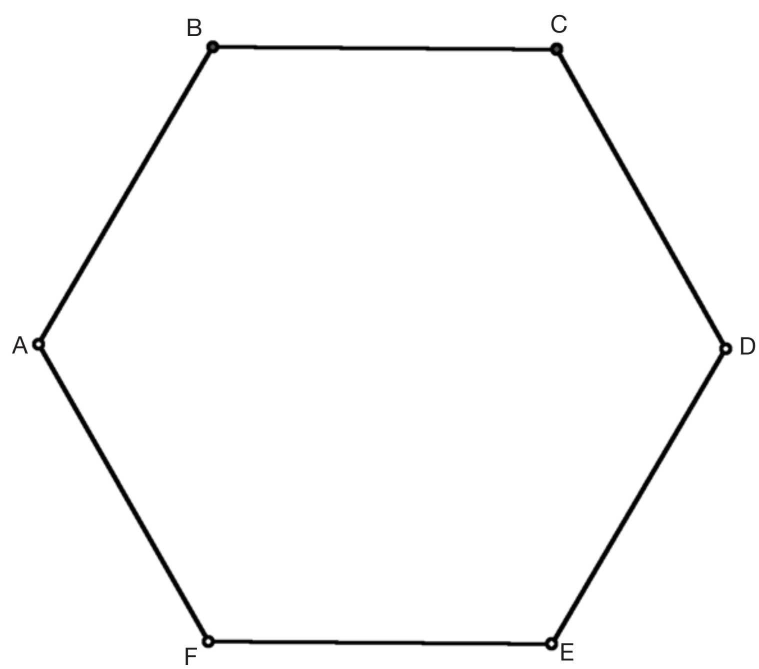
Support Questions
(do not send in for evaluation)

14. Find the magnitude and direction of the resultant of two vectors $\vec{u}$ and $\vec{v}$ with magnitudes 6 and 8 and an angle of 60° between them.

15. In the regular hexagon *ABCDEF*, determine and draw a vector equivalent to each of the following:

    a) $\vec{BC} + \vec{CD}$

    b) $\vec{BA} - \vec{DC}$



16. Suppose each side of the hexagon is 1. Find the magnitude and direction of $\vec{BA} + \vec{BC} + \vec{BE}$. (**Hint:** Each angle of the hexagon is 120°.)

Adding and Subtracting Vectors in Cartesian Form

Open "Lesson 14 Activity 1" on your course page at *ilc.org*.

You will use this applet to gain an understanding of vector addition and subtraction in Cartesian form.

To begin, observe that $OC = OA + OB$.

You should note that the coordinates of A, B, and C represent OA, OB, and OC respectively in Cartesian form.

Now see what happens if you click and drag point A. You should observe that as A moves, its coordinates change, as does point C.

Now do the same for point B. Click and drag point B and notice that as it moves its coordinates change, as does point C. Based on your observations, what is the relationship between OC in Cartesian form, and OA and OB?

The next sections provide an answer to this question.

Addition in Cartesian Form

Observe that $\vec{OC} = \vec{OA} + \vec{OB}$ (parallelogram law). Using the initial values of the vectors shown in the graph:

$$\begin{aligned}\vec{OA} + \vec{OB} &= (10, 3) + (4, 5)\\ &= (14, 8)\\ &= \vec{OC}\end{aligned}$$

Subtraction in Cartesian Form

Using the same example:

$$\begin{aligned}\vec{OA} - \vec{OB} &= \vec{OA} + (-\vec{OB})\\ &= (10, 3) + (-4, -5)\\ &= (6, -2)\end{aligned}$$

The following table summarizes addition and subtraction of Cartesian vectors:

Adding Cartesian vectors in two-space $\vec{u} = (a_1, b_1)$ and $\vec{v} = (a_2, b_2)$ $\vec{u} + \vec{v} = (a_1 + a_2, b_1 + b_2)$
Adding Cartesian vectors in three-space $\vec{u} = (a_1, b_1, c_1)$ and $\vec{v} = (a_2, b_2, c_2)$ $\vec{u} + \vec{v} = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$
Subtracting Cartesian vectors in two-space $\vec{u} = (a_1, b_1)$ and $\vec{v} = (a_2, b_2)$ $\vec{u} - \vec{v} = (a_1 - a_2, b_1 - b_2)$
Subtracting Cartesian vectors in three-space $\vec{u} = (a_1, b_1, c_1)$ and $\vec{v} = (a_2, b_2, c_2)$ $\vec{u} - \vec{v} = (a_1 - a_2, b_1 - b_2, c_1 - c_2)$

An Important Fact

Given the coordinates of two points A and B, you can find the Cartesian vector $\vec{AB}$ by subtracting the components of the vectors $\vec{OA}$ and $\vec{OB}$:

$$\begin{aligned}\vec{AB} &= \vec{AO} + \vec{OB}\\ &= -\vec{OA} + \vec{OB}\\ &= \vec{OB} - \vec{OA}\end{aligned}$$

This is useful when calculating the distance between two points.

Examples

Calculate the distance between each pair of points:

a) $A(-1, 3)$ and $B(3, 4)$

b) $M(-1, 0, 3)$ and $N(-2, 3, 4)$

Solutions

a) To calculate the distance between the two points A and B, find the Cartesian vector $\vec{AB}$ and calculate its magnitude:

$$\begin{aligned}\vec{AB} &= \vec{OB} - \vec{OA}\\ &= (3 - (-1), 4 - 3)\\ &= (4, 1)\end{aligned}$$

$$\begin{aligned}|\vec{AB}| &= \sqrt{4^2 + 1^2}\\ &= \sqrt{17}\end{aligned}$$

b) $$\begin{aligned}\vec{MN} &= \vec{ON} - \vec{OM}\\ &= (-2 - (-1), 3 - 0, 4 - 3)\\ &= (-1, 3, 1)\end{aligned}$$

$$\begin{aligned}|\vec{MN}| &= \sqrt{(-1)^2 + (3)^2 + 1^2}\\ &= \sqrt{11}\end{aligned}$$

Support Questions
(do not send in for evaluation)

17. Find the magnitude of the following vectors:

    a) (1, –2, 3)

    b) $\left(\sqrt{5}, \sqrt{14}, \sqrt{6}\right)$

    c) (3, 0, 0)

18. Find the vector between each pair of points in Cartesian form and calculate the distance between them:

    a) $A(-1, 2)$, $B(-4, 2)$

    b) $P(-2, 3, 0)$, $N(1, 2, 6)$

19. The vertices of a triangle are given by $A(-3, 1, 2)$, $B(1, -3, -1)$, and $C(3, -1, -1)$.

    a) Verify that triangle ABC is a right-angle triangle. (**Hint**: Use the Pythagorean theorem.)

    b) What are the coordinates of $D(x, y, z)$, such that $ABCD$ is a rectangle?

Conclusion

In this lesson, you learned about addition and subtraction of vectors. In the next lesson, you will learn about properties of these operations. You will use them to solve problems arising from real-world situations.

Key Questions

Save your answers to the Key Questions. When you have completed the unit, submit them to ILC for marking.

(12 marks)

37. Find the vector between each pair of points in Cartesian form and calculate the distance. **(2 marks: 1 mark each)**

    a) $A(1, 2), B(-2, 3)$

    b) $M(-3, 3, 4), N(-3, 4, 6)$

38. Find the magnitude and direction of the resultant of two vectors $\vec{u}$ and $\vec{v}$ with magnitudes 6 and 2 and an angle of 140° between them. **(4 marks)**

39. $ABCD$ is a parallelogram where A is $(4, 2)$, B is $(-6, 1)$, and D is $(-3, -4)$. Find the coordinates of C. **(2 marks)**

40. The vertices of a figure are given by $A(-8, 4, -2)$, $B(-6, 3, 5)$, and $C(-10, 5, -9)$. What type of figure is ABC? Justify your answer. (**Hint**: Calculate the magnitudes of the sides of the figure.) **(4 marks)**

Now go on to Lesson 15. Do not submit your coursework to ILC until you have completed Unit 3 (Lessons 11 to 15).

