Seminar

(Course Support)

Contents

Chapter 1

Revisions and complements on complex numbers

1.1 Definitions

Definition 1
We call a complex number any number of the form $a + ib$ where $(a, b) \in \mathbb{R}^2$ and $i^2 = -1$. The ensemble of complex numbers is denoted $\mathbb{C}$.

If $z = a + ib \in \mathbb{C}$, a is called the real part of z (noted $\mathrm{Re}(z)$) and b the imaginary part of z (noted $\mathrm{Im}(z)$).

Remarks

1. The rules on operations are the same as for $\mathbb{R}$ with the supplementary condition $i^2 = -1$.

   For example if $z_1 = 1 + 2i$ et $z_2 = 4 - 3i$ then $z_1 + z_2 = 5 - i$ and $z_1 z_2 = 10 + 5i$.

2. $z_1 = z_2 \Longleftrightarrow \mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ et $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$.

   In particular $a + ib = 0 \Longleftrightarrow a = 0$ et $b = 0$.

Definition 2
Let $z = a + ib \in \mathbb{C}$. We call the conjugate of z the complex number denoted $\overline{z}$ defined by $\overline{z} = a - ib$.

Proposition 1
Let $(z, z') \in \mathbb{C}^2$. then

1. $\mathrm{Re}(z) = \dfrac{z + \overline{z}}{2}$ and $\mathrm{Im}(z) = \dfrac{z + \overline{z}}{2i}$

2. $z \in \mathbb{R} \Longleftrightarrow z = \overline{z}$ and $z \in i\mathbb{R} \Longleftrightarrow \overline{z} = -z$

3. $\overline{z + z'} = \overline{z} + \overline{z'}$

4. $\overline{zz'} = \overline{z}\,\overline{z'}$

5. If $z \neq 0$, the conjugate of $\dfrac{z'}{z}$ is $\dfrac{\overline{z'}}{\overline{z}}$

1.2 Trigonometric and exponential form

Let $(O, \overrightarrow{u}, \overrightarrow{v})$ be an orthonormal space.

For each complex $z = a + ib$, we associate the point M of coordinates (a, b) in $(O, \overrightarrow{u}, \overrightarrow{v})$.

OM is called the modulus of z and is denoted $|z|$.

A measure of the angle $\theta = (\overrightarrow{u}, \overrightarrow{OM})$ is called an argument of z noted $\text{Arg}(z)$. It is defined up to 2π.
We then write $\text{Arg}(z) \equiv \theta\,[2\pi]$.

Proposition 2
Let $(z, z') \in \mathbb{C}^2$. Then

1. $|z|^2 = z\overline{z}$
2. $|z| = 0 \Longleftrightarrow z = 0$
3. $\big|\text{Re}(z)\big| \leqslant |z| \quad \text{et} \quad \big|\text{Im}(z)\big| \leqslant |z|$
4. $|zz'| = |z||z'|$
5. if $z' \neq 0$, $\left|\dfrac{z}{z'}\right| = \dfrac{|z|}{|z'|}$

Notation

Let $\theta \in \mathbb{R}$. We note $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.
We have in particular $\big(e^{i\theta}\big)^n = e^{in\theta}$ so that $\big(\cos(\theta) + i\sin(\theta)\big)^n = \cos(n\theta) + i\sin(n\theta)$.
Similarly $\cos(\theta) = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ and $\sin(\theta) = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

Proposition 3
Any complex number z can be written as

$$z = |z|\big(\cos(\theta) + i\sin(\theta)\big)$$

Remark
If $z' \neq 0$, $\text{Arg}\Big(\dfrac{z}{z'}\Big) = \text{Arg}(z) - \text{Arg}(z')$.

1.3 Quadratic equations with complex coefficients

1.4 Square roots of a complex number

We look for a square root of $u + iv \in \mathbb{C}$. Therefore, we look for $z = a + ib$ such that $z^2 = u + iv$, that is $\begin{cases} a^2 - b^2 = u \\ a^2 + b^2 = \sqrt{u^2 + v^2} \\ 2ab = v \end{cases}$
The third equation allows us to know whether a and b are of the same sign or of the opposite sign and the first two equations allow us to determine a and b.

1.5 Solving quadratic equations with complex coefficients

Let $az^2 + bz + c = 0$ where $(a, b, c) \in \mathbb{C}^3$ and $a \neq 0$.

Let $\Delta = b^2 - 4ac$ and δ a complex root of Δ. Then the roots of the equation are $\dfrac{-b \pm \delta}{2a}$

Example

We solve in $\mathbb{C}$ the equation $z^2 + z + 1 - i = 0$.

$\Delta = 1 - 4(1 - i) = -3 + 4i$. We determine a root of Δ. We look for $z = a + ib$ such that $z^2 = -3 + 4i$.

Hence, $\begin{cases} a^2 - b^2 = -3 \\ a^2 + b^2 = \sqrt{(-3)^2 + 4^2} \\ 2ab = 4 \end{cases}$ let $\begin{cases} a^2 - b^2 = -3 \\ a^2 + b^2 = 5 \\ ab > 0 \end{cases}$

Hence, $z = 1 + 2i$ is a square root of $-3 + 4i$.

Then $z = \dfrac{1}{2}(-1 + 1 + 2i)$ or $z = \dfrac{1}{2}(-1 - 1 - 2i)$ that is $z = i$ or $z = -1 - i$.

1.6 n^{th} roots

We look for the n n^{th} roots of $re^{i\phi}$.

Therefore, we look for $z = \rho e^{i\theta}$ such that $z^n = re^{i\phi}$, that is $\rho^n = r$ and $n\theta \equiv \phi[2\pi]$.

Hence, the n n^{th} roots of $re^{i\phi}$ are the $\sqrt[n]{r}e^{i(\phi/n + 2k\pi/n)}$ for $k \in \{0, 1, 2, \cdots, n-1\}$.

Chapter 2

Revisions and complements on integration

2.1 Preliminaries

2.1.1 Composed function

Let I and J be two intervals of $\mathbb{R}$.

Definition 3
Let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ such that, for all $x \in I$, $f(x) \in J$ (i.e $f(I) \subset J$). We define $g \circ f : I \to \mathbb{R}$ by

$$g \circ f(x) = g\left(f(x)\right)$$

Example

Let $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto x - 1 \end{cases}$ and g: $\begin{cases} [1; +\infty[\to \mathbb{R} \\ x \mapsto \sqrt{x} \end{cases}$

We look for $x \in \mathbb{R}$ such that $f(x) \in [0; +\infty[$. We find $x \in [1; +\infty[$. Thus, $g \circ f$ is defined on $[1; +\infty[$ by

$$g \circ f(x) = \sqrt{x - 1}$$

2.1.2 Reciprocal function

Let I and J be two intervals of $\mathbb{R}$.

Definition 4
Let $f : I \to J$.

1. We say that f is surjective from I to J if

$$\forall\, y \in J\ \exists x \in I\ y = f(x)$$

2. We say that f is injective from I to J if

$$\forall (x, x') \in I^2, \quad f(x) = f(x') \ \Rightarrow\ x = x'$$

Remarks

1. f surjective from I to J means that the equation $y = f(x)$, with unknown variable $x \in I$, admits at least one solution.
2. f injective from I to J means that the equation $y = f(x)$, with unknown variable $x \in I$, admits at most one solution.

Examples

1. f: $\begin{cases} \mathbb{N} \to \mathbb{R} \\ x \mapsto x^2 \end{cases}$ is not surjective but is injective.
2. g: $\begin{cases} [-\frac{\pi}{2}; \frac{\pi}{2}] \to [-1; 1] \\ x \mapsto \cos(x) \end{cases}$ is surjective but not injective.

Definition 5

Let $f : I \to J$. We say that f is bijective from I to J if f is surjective and injective from I to J. This is equivalent to say that

$$\forall\, y \in J\ \exists!\, x \in I\ y = f(x)$$

Examples

1. $x \mapsto \ln x$ is bijective from $]0; +\infty[$ to $\mathbb{R}$.
2. $x \mapsto e^x$ is bijective from $\mathbb{R}$ to $]0; +\infty[$.
3. $x \mapsto \cos x$ is bijective from $[0; \pi]$ to $[-1; 1]$.
4. $x \mapsto \sin x$ is bijective from $[-\frac{\pi}{2}; \frac{\pi}{2}]$ to $[-1; 1]$.
5. $x \mapsto \tan x$ is bijective from $]-\frac{\pi}{2}; \frac{\pi}{2}[$ to $\mathbb{R}$.

Proposition 4

Let $f : I \to J$. Then,
f is bijective from I to J if and only if there exists a unique function $g : J \to I$ such that

$$f \circ g = Id_J \quad \text{and} \quad g \circ f = Id_I$$

If g exists then g is unique. We note $g = f^{-1}$.
Hence, if f is bijective from I to J, $f^{-1} : J \to I$ verifies

$$\begin{cases} x = f^{-1}(y) \\ y \in J \end{cases} \iff \begin{cases} y = f(x) \\ x \in I \end{cases}$$

Example 1

For all $x \in \mathbb{R}$, $\ln(e^x) = x$ and for all $x \in]0; +\infty[$, $e^{\ln x} = x$. Hence, the logarithm and exponential functions are reciprocal.

Example 2

The function tan: $\begin{cases}]-\frac{\pi}{2};\frac{\pi}{2}[\to\mathbb{R} \\ x\mapsto\tan(x)\end{cases}$ is bijective. We denote $\tan^{-1}=\arctan$ its reciprocal bijection.

Thus, arctan: $\begin{cases} \mathbb{R}\to]-\frac{\pi}{2};\frac{\pi}{2}[\\ x\mapsto\arctan(x)\end{cases}$ verifies

$$\begin{cases} x=\arctan(y) \\ y\in\mathbb{R}\end{cases} \iff \begin{cases} y=\tan x \\ x\in]-\frac{\pi}{2};\frac{\pi}{2}[\end{cases}$$

From this, we deduce, for example that $\arctan(0)=0$, $\arctan(1)=\frac{\pi}{4}$ and $\arctan\sqrt{3}=\frac{\pi}{3}$.

2.1.3 Operations on derivatives

We recall the following results:

Proposition 5

1. Let f, g be two differentiable functions on I and $\lambda\in\mathbb{R}$. Then
   a. $(f+g)'=f'+g'$
   b. $(\lambda f)'=\lambda f'$
   c. $(fg)'=f'g+fg'$
   d. If g does not nullify on I, $\left(\frac{f}{g}\right)'=\frac{f'g-fg'}{g^2}$
2. Let $f:I\to J\subset\mathbb{R}$ and $g:J\to\mathbb{R}$ respectively differentiable on I and J. Then
$$(g\circ f)'=(g'\circ f).f'$$

Remark

Part 2. of the above proposition means that for all $x\in I$,

$$(g\circ f)'(x)=(g'\circ f)(x)\times f'(x)$$

that is

$$(g\circ f)'(x)=g'\big(f(x)\big)\times f'(x)$$

Example

Let $f:x\mapsto\sin\big(\ln(x^2+1)\big)$. Then for all $x\in\mathbb{R}$

$$f'(x)=\cos\big(\ln(x^2+1)\big)\times\frac{1}{x^2+1}\times 2x$$

Proposition 6

Let f be a differentiable function at $x_0\in I$ such that $f'(x_0)\neq 0$. Then, f^{-1} is differentiable at $y_0=f(x_0)$ and

$$\left(f^{-1}\right)'(y_0)=\frac{1}{f'\big(f^{-1}(y_0)\big)}$$

Application

$x \mapsto \arctan x$ is differentiable on $\mathbb{R}$ and, for all $x \in \mathbb{R}$,

$$(\arctan x)' = \frac{1}{1+x^2}$$

2.2 Primitive of a continuous function

In the rest of this course, I is an interval of $\mathbb{R}$ and all the functions are real-valued.

2.2.1 definition

Definition 6

Let f be a continuous function on I. We call primitive of f on I any function F of I to $\mathbb{R}$, differentiable on I such that $F' = f$. We then write for all $t \in I$,

$$F(t) = \int f(t)\,\mathrm{d}t$$

Observation

Do not mix up the concept of primitive and the concept of integral (studied below). We notice that there is no boundary in the notation of the above definition.

Example

Let $f : \begin{cases} \mathbb{R}_*^+ \to \mathbb{R} \\ t \longmapsto \dfrac{1}{t} \end{cases}$

then $F : t \mapsto \ln(t)$ is a primitive of f on $\mathbb{R}_*^+$ car $F' = f$ i.e. for all $t \in \mathbb{R}_*^+$, $F'(t) = f(t)$. One can also write that for all $t \in \mathbb{R}_*^+$,

$$\ln(t) = \int \frac{1}{t}\,\mathrm{d}t$$

2.2.2 Properties

Proposition 7

Let f be a continuous function on I and F a primitive of f on I. Then any primitive of f on I is under the form $F + \lambda$ where $\lambda \in \mathbb{R}$.

Example

Using the previous example, a primitive of f on $\mathbb{R}_*^+$ is $t \mapsto \ln(t)$ and the primitives of f on $\mathbb{R}_*^+$ are the functions $t \mapsto \ln(t) + \lambda$ where $\lambda \in \mathbb{R}$.

Classical primitives

We recall the primitives (up to a constant) of elementary functions :

1. for all $\alpha \in \mathbb{R} - \{-1\}$, $\displaystyle\int t^\alpha \, \mathrm{d}t = \frac{1}{\alpha+1} t^{\alpha+1}$

   and $\displaystyle\int t^{-1} \, \mathrm{d}t = \ln(t)$

2. $\displaystyle\int e^t \, \mathrm{d}t = e^t$

3. $\displaystyle\int \sin(t) \, \mathrm{d}t = -\cos(t)$

4. $\displaystyle\int \cos(t) \, \mathrm{d}t = \sin(t)$

5. $\displaystyle\int \frac{1}{1+t^2} \, \mathrm{d}t = \arctan(t).$

2.2.3 Integral of a continuous function

Definition 7

Let f be a continuous function on I and F a primitive of f on I. We call integral of f between a and b, denoted $\displaystyle\int_a^b f(t) \, \mathrm{d}t$, the real number defined by

$$\int_a^b f(t) \, \mathrm{d}t = F(b) - F(a)$$

Observations

1. We sometimes note $F(b) - F(a)$ as $[F(t)]_a^b$.

2. Let us also recall that the integration variable is « mute » i.e. that

$$\int_a^b f(t) \, \mathrm{d}t = \int_a^b f(x) \, \mathrm{d}x = \int_a^b f(u) \, \mathrm{d}u$$

Example

Let us compute $\displaystyle\int_0^1 t^2\,\mathrm{d}t$. A primitive de $t \mapsto t^2$ is $t \mapsto \dfrac{t^3}{3}$. Hence,

$$\begin{aligned}\int_0^1 t^2\,\mathrm{d}t &= \left[\frac{t^3}{3}\right]_0^1 \\ &= \frac{1}{3}\end{aligned}$$

Properties 1

Let f and g be continuous on $[a, b]$ with $a < b$ and $\lambda \in \mathbb{R}$. Then

1. $\displaystyle\int_a^b (f + \lambda g)(t)\,\mathrm{d}t = \int_a^b f(t)\,\mathrm{d}t + \lambda \int_a^b g(t)\,\mathrm{d}t$

2. For all $c \in [a, b]$, $\displaystyle\int_a^b f(t)\,\mathrm{d}t = \int_a^c f(t)\,\mathrm{d}t + \int_c^b f(t)\,\mathrm{d}t$

3. $f \geqslant 0 \Rightarrow \displaystyle\int_a^b f(t)\,\mathrm{d}t \geqslant 0$

4. $f \leqslant g \Rightarrow \displaystyle\int_a^b f(t)\,\mathrm{d}t \leqslant \int_a^b g(t)\,\mathrm{d}t$

5. $\displaystyle\left|\int_a^b f(t)\,\mathrm{d}t\right| \leqslant \int_a^b |f(t)|\,\mathrm{d}t$

6. let $a \in \mathbb{R}$.

   If f is even, $\displaystyle\int_{-a}^a f(t)\,\mathrm{d}t = 2\int_0^a f(t)\,\mathrm{d}t$

   If f is odd, $\displaystyle\int_{-a}^a f(t)\,\mathrm{d}t = 0$

2.2.4 Geometric interpretation

Definition 8

In the plan $(0, \vec{i}, \vec{j})$, we call area unit, the area of the rectangle defined by $\vec{i}$ and $\vec{j}$.

Proposition 8

Let f be continuous and positive on $[a, b]$ with $a \neq b$. Then $\displaystyle\int_a^b f(t)\,\mathrm{d}t$ is the area, in area unit, of the part of the plan bounded by the axis $0x$, the graph of f and the lines of equations $x = a$ and $x = b$.

2.3 Computational methods for primitives or integrals

2.3.1 Integration by parts

Proposition 9 (Integration by parts)
Let f and g be two functions of class C^1 on $[a, b]$ (i.e. f and g differentiable on I and their derivative is continuous on $[a, b]$). Then

$$\int_a^b f(t)g'(t)\,\mathrm{d}t = \big[f(t)g(t)\big]_a^b - \int_a^b f'(t)g(t)\,\mathrm{d}t$$

Observation

The assumption « of class C^1 » is here only to say that f' are g' are continuous on $[a, b]$ so that it is possible to consider the integral from a to b of $f'g$ and fg'.

Example 1

Let us determine $I = \displaystyle\int_0^1 te^t\,\mathrm{d}t$.

We set $f(t) = t \Rightarrow f'(t) = 1$ and $g'(t) = e^t \Rightarrow g(t) = e^t$. We have then

$$\begin{aligned}
I &= \int_0^1 f(t)g'(t)\,\mathrm{d}t \\
&= \big[f(t)g(t)\big]_0^1 - \int_0^1 f'(t)g(t)\,\mathrm{d}t \\
&= \big[te^t\big]_0^1 - \int_0^1 e^t\,\mathrm{d}t \\
&= e - \big[e^t\big]_0^1 \\
&= e - (e - 1) \\
&= 1
\end{aligned}$$

Example 2

Let us determine $I = \displaystyle\int_0^{\frac{\pi}{4}} t\cos(2t)\,\mathrm{d}t$.

We set $f(t) = t \Rightarrow f'(t) = 1$ and $g'(t) = \cos(2t) \Rightarrow g(t) = \frac{1}{2}\sin(2t)$. We have then

$$\begin{aligned}
I &= \int_0^{\frac{\pi}{4}} f(t)g'(t)\,\mathrm{d}t \\
&= \big[f(t)g(t)\big]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} f'(t)g(t)\,\mathrm{d}t \\
&= \left[\frac{t}{2}\sin(2t)\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \frac{1}{2}\sin(2t)\,\mathrm{d}t \\
&= \frac{\pi}{8} - \left[-\frac{1}{4}\cos(2t)\right]_0^{\frac{\pi}{4}} \\
&= \frac{\pi}{8} - \frac{1}{4}
\end{aligned}$$

2.3.2 Integration by substitution

The following proposition is not to be memorized but you have to know how to use it.

Proposition 10
Let I and J be two intervals of $\mathbb{R}$, $(\alpha, \beta) \in J^2$, f continuous on I and φ of class C^1 from J to I. Then

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(t)\,\mathrm{d}t = \int_\alpha^\beta f\big(\varphi(u)\big)\varphi'(u)\,\mathrm{d}u$$

Remark

When you have to use an integration by substitution, the change of variable will always be told to you. Three steps are necessary to do a change of variable :

- Determine the new « $\mathrm{d}t$ » if t is the new variable.
- Change the integration bounds.
- Make explicit the function of the old variable by a function of the new variable.

Example

Let us compute $I = \displaystyle\int_e^3 \frac{1}{x(\ln(x))^3}\,\mathrm{d}x$ using the change of variable $t = \ln(x)$.

We have then $x = e^t$.

The derivative of x with respect to t is e^t, which we write under the « physicist » form $\dfrac{\mathrm{d}x}{\mathrm{d}t} = e^t$.

Hence $\mathrm{d}x = e^t\,\mathrm{d}t$.

We now change the boundaries : When x is equal to e, then $t\,(= \ln(x))$ is equal to $\ln(e)$ i.e. 1.

When x is equal to 3, $t\,(=\ln(x))$ is equal to $\ln(3)$.

Finally $\dfrac{1}{x(\ln(x))^3} = \dfrac{1}{e^t\,t^3}$

$$
\begin{aligned}
\text{Then}\quad I &= \int_1^{\ln 3} \frac{1}{e^t\,t^3}\,e^t\,\mathrm{d}t \\
&= \int_1^{\ln 3} \frac{1}{t^3}\,\mathrm{d}t \\
&= \left[-\frac{1}{2t^2}\right]_1^{\ln 3} \\
&= -\frac{1}{2(\ln 3)^2} + \frac{1}{2}
\end{aligned}
$$

Chapter 3

Functions of a real variable

3.1 Definitions

Until today, you have worked a lot with functions from $\mathbb{R}$ to $\mathbb{R}$. But do you know the definition of a function?

3.1.1 Cartesian product

Definition 9
Let E and F be two ensembles. We call the cartesian product of E by F denoted $E \times F$ the ensemble of couples (x, y) with $x \in E$ and $y \in F$ i.e.

$$E \times F = \{(x, y);\, x \in E,\ y \in F\}$$

Example

$u \in \mathbb{N}^2 \times \mathbb{R}$ means that $u = ((n, p)\ ,\ x)$ where $(n, p) \in \mathbb{N}^2$ and $x \in \mathbb{R}$ i.e. $n \in \mathbb{N}$, $p \in \mathbb{N}$ and $x \in \mathbb{R}$.

3.1.2 Graph

Definition 10
Let E and F be two ensembles. We call a graph of E to F any part of $E \times F$.

Example

If $E = F = \mathbb{R}$, a graph of E to F is any given part of the plan, for example a circle, a triangle or a line.

3.1.3 Function

Throughout all the course on functions, we will make no distinction between the words «functions» and «applications».

Definition 11
We call function (defined) of E to F, any triplet $f = (E, F, \Gamma)$ where Γ is a graph from E to F such that for all $x \in E$, there exists a unique $y \in F$ avec $(x, y) \in \Gamma$.

Remarks

1. If f is a function from E to F, E is called the starting domain (or domain of definition or domain of source) of f, F is called co-domain of f.
   A function f from E to F will be denoted in the usual way under the form $f \in F^E$ or $f : E \to F$ or $f : \begin{cases} E \to F \\ x \mapsto f(x) \end{cases}$ and the graph Γ of f will then be the ensemble of $(x, f(x))$ for x covering E i.e. the graph of f models what you used to call « representative curve » of f.

2. If f is a function (defined) from E to F, the domain of definition of f, $\mathscr{D}_f$, is equal to E. This is the reason why E is also called domain of definition of f.

3. In particular f function from $\mathbb{R}$ to $\mathbb{R}$ means that any vertical line (i.e. parallel to the y-axis) crosses the graph of f at exactly one point.

For the rest of the course, any function $f = (E, F, \Gamma)$ will be noted $f \in F^E$ or f: $E \to F$. The two notations will be used to get you used to them.

3.2 Concepts of limits

In the rest of this chapter, all the functions will be defined on a part I of $\mathbb{R}$ i.e.
$f : I \subset \mathbb{R} \to \mathbb{R}$. i.e. f is defined at $a \in \mathbb{R}$ means that $a \in I$.

3.2.1 Neighborhood of a real number

Definition 12
Let $a \in \mathbb{R}$. We call a neighborhood of a any interval of the form $]a - h, a + h[$ where $h > 0$.

Remark

A neighborhood of $a \in \mathbb{R}$ is simply an open interval centered in a.

3.2.2 Function defined in a neighborhood of a real number or the infinity

Definition 13
We say that f is defined in the neighborhood of $a \in \mathbb{R}$ if for all $h > 0$, $]a - h, a + h[$ meets I i.e. if

$$\forall h > 0,\ ]a - h, a + h[\cap I \neq \emptyset$$

We say that f is defined in the neighborhood of $+\infty$ (resp. $-\infty$) if for all $A \in \mathbb{R}$, $]A, +\infty[$ meets I (resp. $]-\infty, A[$ meets I) i.e. if

$$\forall A \in \mathbb{R},\ ]A, +\infty[\cap I \neq \emptyset$$

$$(\text{resp. } \forall A \in \mathbb{R},\ ]-\infty, A[\cap I \neq \emptyset)$$

Examples

1. $f : \begin{cases} \mathbb{R}^+ \to \mathbb{R} \\ x \mapsto \sqrt{x} \end{cases}$ is defined in the neighborhood of 0. Indeed any open interval (even very small) centered at 0 meets $\mathbb{R}^+$. More precisely for all $h > 0$, we have

$$]-h, h[\, \cap \mathbb{R}^+ = [0, h[$$

hence

$$]-h, h[\, \cap \mathbb{R}^+ \neq \emptyset$$

2. $g : \begin{cases} [1, +\infty[\to \mathbb{R} \\ x \mapsto \sqrt{x-1} \end{cases}$ is not defined in the neighborhood of 0 as for example

$$\left] -\frac{1}{2}, \frac{1}{2} \right[\cap [1, +\infty[= \emptyset$$

3. $h : \begin{cases} \mathbb{R}^+ \to \mathbb{R} \\ x \mapsto \sqrt{x} \end{cases}$ is defined in the neighborhood of $+\infty$ because any interval $]A, +\infty[$ meets $\mathbb{R}^+$.

Indeed for all $A \in \mathbb{R}$,

$$]A, +\infty[\, \cap \mathbb{R}^+ = \begin{cases}]A, +\infty[\text{ if } A \geqslant 0 \\ \mathbb{R}^+ \text{ if } A < 0 \end{cases}$$

so that we have for all $A \in \mathbb{R}$

$$]A, +\infty[\, \cap \mathbb{R}^+ \neq \emptyset$$

3.2.3 Finite limit of a function at a point

Definition 14
f has a limit $l \in \mathbb{R}$ at $a \in \mathbb{R}$ if f is defined in the neighborhood of a and

$$\forall \varepsilon > 0, \exists \eta > 0, \forall x \in I, |x - a| < \eta \Rightarrow |f(x) - l| < \varepsilon$$

Remark

Saying that f has a limit l at $a \in \mathbb{R}$ simply means that the gap between $f(x)$ and l is as small as we want, provided that x is sufficiently close to a.

Proposition 11
If f has a limit $l \in \mathbb{R}$ at $a \in \mathbb{R}$, then l is unique and we note

$$l = \lim_{x \to a} f(x) \quad \text{or} \quad l = \lim_{a} f$$

Example

Let $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^2 \end{cases}$. We prove that $\lim\limits_{x \to 0} x^2 = 0$. This result is natural but we have to prove it here using quantifiers.

Let $\varepsilon > 0$. We look for $\eta > 0$ such that for all $x \in \mathbb{R}$, $|x - 0| < \eta \Rightarrow |x^2 - 0| < \varepsilon$ i.e.

$$|x| < \eta \Rightarrow x^2 < \varepsilon$$

It is sufficient to choose $\eta = \sqrt{\varepsilon}$. Indeed

$$|x| < \sqrt{\varepsilon} \Rightarrow x^2 < \varepsilon$$

Remarks

1. If f is defined at $a \in \mathbb{R}$ (and not defined only *in a neighborhood* of a) and f admits a limit $l \in \mathbb{R}$ at a then $l = f(a)$.

2. However, the definition of limit still makes sense even if f is not defined at a but only defined in the neighborhood of a as the following example shows it.

   Let $f : \begin{cases} \mathbb{R} - \{1\} \to \mathbb{R} \\ x \mapsto \dfrac{x^3 - 1}{x - 1} \end{cases}$.

   then f is defined in the neighborhood of 1 (but not defined at 1). The limit of f at 1 is nevertheless computable. We have

   $$\lim_{x \to 1} f(x) = 3$$

   Indeed,

   $$f(x) = \frac{x^3 - 1}{x - 1} = \frac{(x - 1)(x^2 + x + 1)}{x - 1} = x^2 + x + 1.$$

   Hence

   $$\lim_{x \to 1} f(x) = 1 + 1 + 1 = 3.$$

Proposition 12

If f has a finite limit at a, then f is bounded in the neighborhood of a.

Proposition 13

If $\lim\limits_a f = l$ and $\lim\limits_a g = m$ then $\lim\limits_a(\lambda f + \mu g) = \lambda l + \mu m$ and $\lim\limits_a fg = lm$ where λ and μ are two real numbers.

Proposition 14

If $\lim\limits_a f = l \neq 0$, then f does not nullify in the neighborhood of a and $\lim\limits_a \dfrac{1}{f} = \dfrac{1}{l}$

Proposition 15

let I and J be two intervals of $\mathbb{R}$, a, b and l three real numbers and $f : I \longrightarrow J$ and $g : J \longrightarrow \mathbb{R}$ such that $\lim\limits_a f = b$ and $\lim\limits_b g = l$ then $\lim\limits_a(g \circ f) = l$

3.2.4 Other types of limit

Definition 15

1. We say that f admits a limit $l \in \mathbb{R}$ at $+\infty$ (and we note $\lim\limits_{x \to +\infty} f(x) = l$) if f is defined in the neighborhood of $+\infty$ and

$$\forall \varepsilon > 0, \exists A \in \mathbb{R}, \forall x \in I, x > A \Rightarrow |f(x) - l| < \varepsilon$$

2. We say that f tends to $+\infty$ at $a \in \mathbb{R}$ (and we note $f(x) \xrightarrow[x \to a]{} +\infty$) if f is defined in the neighborhood of a and

$$\forall A \in \mathbb{R}, \exists \eta > 0, \forall x \in I, |x - a| < \eta \Rightarrow f(x) > A$$

3. We say that f tends to $+\infty$ at $+\infty$ (and we note $f(x) \xrightarrow[x \to +\infty]{} +\infty$) if f is defined in the neighborhood of $+\infty$ and

$$\forall A \in \mathbb{R}, \exists B \in \mathbb{R}, \forall x \in I, x > B \Rightarrow f(x) > A$$

Example

We prove that $x^3 - 1 \xrightarrow[x \to +\infty]{} +\infty$.

Let $A \in \mathbb{R}$. We look for $B \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, $x > B \Rightarrow x^3 - 1 > A$. Since $x^3 - 1 > A \Leftrightarrow x > \sqrt[3]{A+1}$, it is sufficient to choose $B = \sqrt[3]{A+1}$. We have then, for all $x \in \mathbb{R}$,

$$x > B = \sqrt[3]{A+1} \Rightarrow x^3 - 1 > A$$

3.3 Continuity

Until today, your definition of continuity of a function f was maybe like : «f is continuous if its graph can be drawn without lifting your pencil from the paper ». One of the goals of this paragraph is to define the continuity of a function f on an interval using quantifiers.

Definition 16

We say that f is continuous at $a \in I$ if

$$\forall \varepsilon > 0, \exists \eta > 0, \forall x \in I, |x - a| < \eta \Rightarrow |f(x) - f(a)| < \varepsilon$$

We say that f is continuous on I if f is continuous at every point of I.

Remark

f is continuous on I simply means that for all $a \in I$, $\lim\limits_{x \to a} f(x) = f(a)$.

3.3.1 Intermediate value theorem

One of the key theorems on continuity is the intermediate value theorem.

Theorem 1 (Intermediate value theorem)
Let f continuous on an interval I of $\mathbb{R}$ and $(a, b) \in I^2$. If $f(a)f(b) < 0$, then there exists (at least one) $c \in]a, b[$ such that $f(c) = 0$.

Remark

The assumption $f(a)f(b) < 0$ simply means that $f(a)$ and $f(b)$ are of opposite sign.

Example

We prove that the equation $x^2 \cos(x) + x \sin(x) + 1 = 0$ admits at least one solution $x \in \mathbb{R}$.
Let $f : x \mapsto x^2 \cos(x) + x \sin(x) + 1$. Then f is continuous on $\mathbb{R}$, $f(0) = 1 > 0$ and $f(\pi) = 1 - \pi^2 < 0$. Using the intermediate value theorem, there exists at least one $x \in]0, \pi[$ such that $f(x) = 0$ i.e. such that $x^2 \cos(x) + x \sin(x) + 1 = 0$.

3.3.2 Image of an interval by a continuous function

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ and $A \subset I$. We recall that the image of f by A, denoted $f(A)$ is defined by

$$f(A) = \{f(x); x \in A\}$$

Hence $y \in f(A) \Leftrightarrow$ there exists $x \in A$ such that $y = f(x)$.

Example : Let us take $f : x \mapsto x^2$. Then $f([-1, 2]) = [0, 4]$

Proposition 16
The image of an interval by a continuous function is an interval.

3.3.3 Image of a segment by a continuous function

Proposition 17
The image of a segment $[a, b]$ by a continuous function is a segment.

Remark

The assumption « segment » is fundamental, as the following counter-example shows it:

$f : \begin{cases}]0, 1] \to \mathbb{R} \\ x \mapsto \dfrac{1}{x} \end{cases}$. Then $f(]0, 1]) = [1, +\infty[$. But $]0, 1]$ is not a segment !

Corollary 1
Let f be a continuous function on a segment $[a, b]$. Then

$$f([a, b]) = [m, M]$$

where m (resp. M) is the minimum (resp. maximum) of f on $[a, b]$.

Remark

In particular, we have for all $x \in [a, b]$, $m \leqslant f(x) \leqslant M$. We say that f is bounded and reaches its boundaries.

3.4 Differentiability

All the functions of this chapter are of the form $f : I \to \mathbb{R}$ where I is an interval of $\mathbb{R}$ containing at least two points.

3.4.1 Definitions

Definition 17

We say that f is differentiable on a if the increasing rate $\tau_a : x \mapsto \dfrac{f(x) - f(a)}{x - a}$ has a finite limit at a. If this is the case, we note this limit $f'(a)$ (called derivative number of f at a) i.e.

$$f'(a) = \lim_{x \to a} \tau_a(x)$$

If f is differentiable at any point of I, we say that f is differentiable on I and the function $x \mapsto f'(x)$ is called derivative of f.

Remarks

1. Setting $h = x - a$, f differentiable at a is equivalent to $h \mapsto \dfrac{f(a + h) - f(a)}{h}$ has a finite limit at 0. If this is the case
$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$
2. f differentiable at a if and only if the graph of f admits a non vertical tangent at $A(a, f(a))$. In this case, $f'(a)$ represents the slope of the tangent of the graph of f at a.
3. If $\tau_a(x) \xrightarrow[x \to a]{} +\infty$ or $\tau_a(x) \xrightarrow[x \to a]{} -\infty$, then the graph of f admits a vertical tangent at $A(a, f(a))$.

3.4.2 Differentiability and continuity

Is there a link between differentiability and continuity? This section answers the question.

Proposition 18

Let f be differentiable en a. Then f is continuous en a.

Remark

The reciprocal is false, as the following counter-example shows it. Let us consider the function $f : x \mapsto \sqrt{x}$. Then f is continuous on $\mathbb{R}^+$, and in particular at 0 but is not differentiable at 0. Indeed

$$\frac{f(x) - f(0)}{x - 0} = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}} \xrightarrow[x \to 0]{} +\infty$$

3.4.3 Local extremum

Definition 18

We say that f admits a local maximum (resp. minimum) at a if $f(x) \leqslant f(a)$ (resp. $f(x) \geqslant f(a)$) provided that x be sufficiently close a i.e. if

$$\exists \eta > 0, \forall x \in I, |x-a| < \eta \Rightarrow f(x) \leqslant f(a) \quad (\text{resp. } f(x) \geqslant f(a))$$

We say that f admits a local extremum at a if f admits a local mimimum or a local maximum at a.

Proposition 19

We assume that a is not a boundary of the interval I, that f is differentiable at a and that f presents a local extremum at a. Then $f'(a) = 0$.

Remarks

This proposition has to be carefully used, as the following remarks show :

1. If a is a boundary of the interval I then the proposition is false, as the following counter-example shows :

   let $f : \begin{cases} [0,1] \to \mathbb{R} \\ x \mapsto x \end{cases}$. Then f is differentiable on $[0,1]$, in particular at 0 and 1, f admits a local minimum at 0 and a local maximum at 1 and yet $(f)'(0) \neq 0$ and $(f)'(1) \neq 0$ as for all $x \in [0,1]$, $f'(x) = 1$.

2. A function can have one extremum at a without being differentiable at a. For example, the function $x \mapsto \sqrt{x}$ admits a minimum at 0 but is not differentiable at 0 (cf. remark in previous section).

3. The reciprocal of the proposition is false, as the following counter-example shows :

   Let $f : \begin{cases} [-2,2] \to \mathbb{R} \\ x \mapsto x^3 \end{cases}$ Then f admits no extremum and yet $f'(0) = 0$ because for all $x \in [-2,2]$, $f'(x) = 3x^2$.

3.5 Classical theorems

3.5.1 Rolle's theorem

Theorem 2 (Rolle)

Let a, b be two real distinct numbers, f continuous on $[a,b]$, differentiable on $]a,b[$ such that $f(a) = f(b)$. Then there exists (at least one) $c \in]a,b[$ such that $f'(c) = 0$.

Example

Let $f : I \to \mathbb{R}$ two times differentiable (i.e. f' and f'' exist) admitting three zeros x_0, x_1 and x_2 (i.e. $f(x_0) = f(x_1) = f(x_2) = 0$). Then f'' admits at least one zero. Indeed, it is sufficient to apply three times the Rolle's as follows :

f is continuous, differentiable on I and $f(x_0) = f(x_1)\,(= 0)$ so that using Rolle's theorem, there exists $y_1 \in]x_0, x_1[$ such that $f'(y_1) = 0$. Similarly $f(x_1) = f(x_2)$ so there exists again $y_2 \in]x_1, x_2[$ such that $f'(y_2) = 0$. Now we have a function f' continuous and differentiable on I such that $f'(y_1) = f'(y_2)\,(= 0)$. Using Rolle's theorem for the last time , we conclude that there exists $z \in]y_1, y_2[$ such that $(f')'(z) = 0$ i.e such that $f''(z) = 0$.

3.5.2 Mean value theorem

What happens if we remove $f(a) = f(b)$ from the assumptions of Rolle's theorem ? The following theorem gives the answer.

Theorem 3 (Mean value theorem)
Let a, b be two distinct real numbers, f continuous on $[a, b]$ and differentiable on $]a, b[$. Then there exists (at least one) $c \in]a, b[$ such that $f(b) - f(a) = (b - a)f'(c)$.

Remark

The previous theorem is often used with $a = 0$ and $b = x$, as the following example shows it.

Example

We want to prove that for all $x \in \mathbb{R}_*^+$, $\dfrac{x}{x+1} < \ln(1 + x) < x$.

Let us set $f : x \mapsto \ln(1 + x)$. let $x > 0$. Then f is continuous and differentiable on $[0, x]$.
Using the Mean value theorem on $[0, x]$, there exists $c \in]0, x[$ such that

$$f(x) - f(0) = (x - 0)f'(c)$$

Yet $f(0) = 0$ and for all $x \in \mathbb{R}_*^+$, $f'(x) = \dfrac{1}{1+x}$. Hence, there exists $c \in]0, x[$ such that

$$\ln(1 + x) = x.\frac{1}{1+c} = \frac{x}{1+c}$$

Yet

$$\begin{aligned} 0 < c < x \quad &\Rightarrow \quad 1 < 1 + c < 1 + x \\ &\Rightarrow \quad \frac{1}{1+x} < \frac{1}{1+c} < 1 \\ &\Rightarrow \quad \frac{x}{1+x} < \frac{x}{1+c} < x \end{aligned}$$

Hence, for all $x > 0$,

$$\frac{x}{x+1} < \ln(1 + x) < x$$