1 Introduction

Power systems are undergoing unprecedented changes with the rapidly increasing adoption of renewable sources such as wind and solar energy. A critical aspect of the use of renewable sources is the large amounts of uncertainties and variabilities associated with them. A recent study from the National Renewable Energy Laboratory (NREL) found the errors in solar and wind forecasts can be as high as 25% of the net demand in the system (see Figure 1) even with current levels of adoption [1]. In California particularly, the adoption of rooftop solar generation has created significant variability in the net load (total demand minus distributed generation). The California ISO (Independent System Operator) estimates they already have 5000 MW of distributed solar energy [62], and this number is expected to grow significantly given California's renewable energy and emissions goals [2].

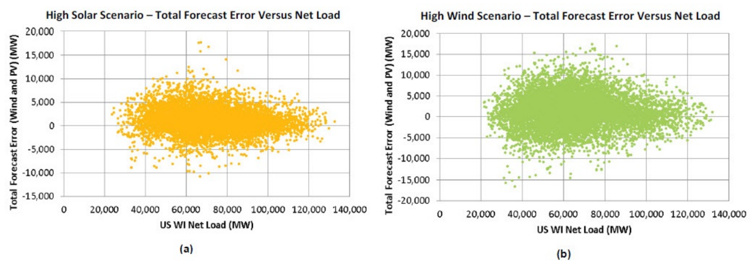

Figure 1: Solar and wind forecast errors in a high penetration scenario on the Western Interconnect [1, Figs. 156,159].

The renewable sources of energy are clean, and have nearly zero operating costs (unlike gas, coal, or nuclear energy). However, since these sources depend on the weather, their energy output is not perfectly controllable (unlike in a gas plant, where more power can be produced by burning additional fuel, for instance). The power output of non-dispatchable sources needs to be taken into the grid as is (although curtailment or throwing away excess power is possible, it is undesirable and has regulatory issues [64]). Further, this output can be forecasted with only limited accuracy. Demand-side flexibility is increasingly adopted as a resource to balance intermittent fluctuations in solar and wind generation [69]. At the same time, the amount of demand-side flexibility can in turn be difficult to predict [59, 66]. The whole situation results in increased uncertainty in *operations*, the process of making sure there is sufficient power generation to meet the demand while accounting for losses, network limits, etc.

Two central computations performed as part of power systems operations are power flow (PF) (or load flow) studies and security-constrained economic dispatch, or optimal power flow (OPF). Power flow studies ensure the power grid state (voltages and flows across the network) will remain within acceptable limits in spite of contingencies (e.g., loss of a generator or transmission line) and other uncertainties (e.g., shifting demand or renewable outputs). OPF seeks further to choose values for controllable assets in the system (e.g., dispatchable generators) so as to meet demand at minimum cost. While most system operators today use a linearized DC power flow model for OPF (due to a variety of factors, including the need to clear electricity markets reliably), it is increasingly acknowledged that the nonlinear factors have a heightened effect as uncertainty increases in the grids [5]. It is therefore of paramount importance to understand and characterize the effects of *uncertainty* on the *nonlinear* PF and OPF problems.

In this project, we propose to take a *robust viewpoint* of the uncertainty, i.e., we aim to quantify the *worst-case* impact of the uncertainty on the PF and OPF problems. More concretely, we will use ideas from algebraic topology and nonlinear analysis—specifically Borsuk's theorem (a generalization of the intermediate value theorem) and topological degree theory—to develop efficient algorithms for robust versions of the PF and OPF problems. We will develop a framework to derive *mathematically rigorous guarantees* for robust feasibility and optimization in *nonlinear* power systems using *scalable algorithms*. The proposed combination of rigorous guarantees, efficient algorithms, and ability to handle nonlinearities is novel and critical for operating modern power systems with significant uncertainty.

The power system can be described by a system of nonlinear equations in a set of variables that capture the state of the power grid (voltages at every point in the power network) and include the controllable inputs (power from dispatchable generators) as well as uncertain inputs (power from non-dispatchable generators). In the main PF problem, we are given a fixed value of the controllable inputs and an uncertainty set for the uncertain inputs. The goal is to characterize whether the system has a solution within specified bounds (capturing engineering limits on voltages, flows, etc.) for *each* choice of the uncertain inputs in the uncertainty set. We call this problem the *robust feasibility problem* (this is the robust version of the standard PF problem). We will also study the *robust optimization problem*, where the goal is to choose the controllable inputs so as to minimize a cost function of these inputs (e.g., fuel costs for a natural gas or coal-fired generator) subject to the robust feasibility constraints which require that the system has a solution within the specified bounds for each choice from the uncertainty set (this is the robust version of the standard OPF problem). In this project, we will use results from algebraic topology and nonlinear analysis (topological degree theory, Borsuk's theorem) to develop tests for the existence of solutions, and combine the tests with ideas from optimization (convex relaxations of quadratic constraints) to develop rigorous and efficient algorithms for robust feasibility and optimization. On the computational side, we will develop efficient implementations of these algorithms capable of scalably solving large instances of PF and OPF problems.

Previous work, and their shortcomings: Robust feasibility and optimization have been well-studied by both the optimization and topology communities. In optimization, the focus has been on robust *convex* optimization where uncertainty sets are specified for the parameters of a convex optimization problem (typically an LP or conic program) [7]. Robust *nonconvex* optimization has received only limited attention (a notable exception is the work of Bertsimas et al. [11]). These approaches *do not provide rigorous guarantees for robust feasibility with nonconvex constraints*.

In algebraic topology, there have been a number of studies on these problems based on several approaches, including ones based on robustness of level sets and persistent homology [10, 31], well groups and diagrams [17, 36, 37], topologiocal degree and robust satisfiability [35, 38], and on Borsuk's theorem and interval arithmetic [42, 41, 40]. While the theory developed by these approaches is fairly complete and applies to general problems, the associated algorithms typically rely on explicit simplicial or cellular decomposition of the set $\mathcal{X}$. But the size of such decompositions typically grows exponentially in the problem dimension, and hence these *algorithms are typically impractical for large-scale applications*.

Looking specifically at power systems, there has been significant interest in solving the non-robust version of the OPF problem to global optimality. The driver has been the development of strong convex relaxations of the nonconvex optimization problems combined with ideas from global optimization (spatial branch-and-cut, bound tightening etc.) [13, 21]. Uncertainty has been handled in a chance-constrained framework. However, this approach has typically been applied only to linear approximations or convex relaxations of the AC power flow equations, and *does not guarantee feasibility with respect to the true nonlinear power flow equations* [12, 63, 68, 49].

1.1 Summary of Proposed Research, and Specific Goals

Our approach will combine research advances made by the optimization and topology communities to derive efficient algorithms that solve the robust feasibility and robust optimization with rigorous guarantees (the solutions found are accompanied by a mathematical certificate of robust feasibility and distance to optimality). To the best of our knowledge, this combination of computational tractability and rigorous bounds on the robustness for nonlinear power system models has not been obtained in prior work.

We will focus on achieving the following **specific goals**:

G1. Develop a framework to provide mathematically rigorous guarantees for robust feasibility using efficient algorithms. We will leverage topological results to obtain existence conditions, which we will pose as nonlinear quadratically constrained programs. We will then use convex relaxations of these optimization problems to derive sufficient conditions for robust feasibility (the relaxations drop some of the nonconvex constraints in the nonlinear programs, and enable us to solve them to global optimality in polynomial time). Applying the convex relaxations to a different problem, we will also obtain necessary conditions for robust feasibility. Combining these results, we obtain *both upper and lower bounds* on the robustness margin. We will use this approach to solve the robust feasibility problem and quantify the conservatism in our estimate of the robustness margin. We will extend these results to the robust optimization problem by iteratively constructing robust feasibility certificates around any feasible solution. In detail, develop theory to guarantee conditions under which we can estimate robustness margins accurately (using results from approximation algorithms for nonconvex quadratic programs), and demonstrate that our algorithms can solve high dimensional robust feasibility problems accurately, efficiently, and in a scalable manner.

G2. Extend the framework for robust feasibility to robust optimization: In order to solve the robust optimization problem, we will construct simple *analytical certificates* that define a nonempty convex region around any strictly feasible point. These certificates can be constructed using the same topological results (such as Borsuk's theorem), but are obtained by analytically bounding terms in the existence tests to obtain a simple sufficient condition on the inputs that guarantees feasibility (for instance, both controllable and uncertain inputs lie in a polytope). We will then leverage standard results on robust convex optimization to solve the resulting problem. At the end, this will lead to a *sequential robust convex optimization* approach to solve the original robust nonconvex optimization problem that is guaranteed to compute a high-quality feasible solution for the robust optimization problem.

G3. Develop strong convex relaxations for the robust optimization problem, and use them to obtain strong/tight lower bounds on its optimal value: The work done under goal G2 will enable us to compute *feasible solutions* for the robust optimization problem, but they may not be globally optimal. In order to quantify the gap between the solution found and the global optimum, we will use *convex relaxations of the robust optimization problem*. We do this by building on convex relaxations of the standard OPF problem, and augmenting them with ideas from robust multistage optimization [44] and cutting plane algorithms.

G4. Evaluate and test the overall framework on large-scale robust OPF problems. Through this testing, we will understand when we are able to compute feasible solutions with costs close to the lower bound (and hence nearly optimal).

Qualification and strength: This project will require both theoretical and computational ideas from various mathematical disciplines including algebraic topology, analysis, and optimization. The PI Krishnamoorthy brings aboard these multiple areas of expertise. He has years of experience working in computational algebraic topology, geometric measure theory, and integer optimization. He has direct industry experience in power systems from consulting for a hydroelectric power company in Spokane, WA, to determine power generation schedules at various time scales for several hydroelectric stations in order to meet demand in a cost-effective manner. He has also collaborated with orthopedic surgeons, molecular biologists, and bioengineers on various applications of optimization, topology, and geometry to problems from biology and medicine.

The project will benefit from the indirect cooperation with many of the collaborators of the PI such as Hongbo mathematicians Dong (WSU), Anil Hirani (Illinois), Kevin Vixie (WSU); computer scientists Tamal Dey (Ohio State U.), Krishnamurthy Dvijotham (Google DeepMind), and Bei Wang (U. Utah).

2 Specific Goals: Methodologies

We describe in detail our approaches to achieve Specific Goals **G1**, **G2**, and **G3** (stated in section 1.1). We present preliminary results on computational testing as part of Specific Goal **G4**.

2.1 Specific Goal G1: Robust feasibility

We develop a general framework that incorporates complicated nonlinear models (coming from, e.g., power systems, gas distribution networks, and other flow models) as well as various types of uncertainties (e.g., uncertainties in network parameters or network topology). We study a general parameterized system of quadratic equations

$$\mathbf{f}\left(\mathbf{x}, \mathbf{u}, \boldsymbol{\omega}\right) \;=\; \underbrace{Q(\mathbf{x}, \mathbf{x}, \mathbf{u}, \boldsymbol{\omega})}_{\text{quadratic terms}} + \underbrace{L(\mathbf{x}, \mathbf{u}, \boldsymbol{\omega})}_{\text{linear terms}} + \underbrace{\mathbf{c}(\mathbf{u}, \boldsymbol{\omega})}_{\text{constant terms}} \;=\; \mathbf{0}, \tag{1}$$

where $Q : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^p \mapsto \mathbb{R}^n$ and $L : \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^p \mapsto \mathbb{R}^n$ are multilinear maps (linear in each argument when other arguments are fixed) that represent the quadratic and linear terms in $\mathbf{x}$, and $\mathbf{c} : \mathbb{R}^k \times \mathbb{R}^p \mapsto \mathbb{R}^n$ represents the constant terms in $\mathbf{x}$. $\mathbf{u}$ represents the controllable decision variables (e.g., dispatchable generator output) and $\boldsymbol{\omega}$ represents uncertain inputs (renewable generation, uncertain estimates of impedances, etc.) into the system. Since quadratic forms can be represented using symmetric matrices, we can assume without loss of generality that Q is homogeneous and symmetric in its first two arguments:

$$Q(\mathbf{x}, \mathbf{y}, \mathbf{u}) = Q(\mathbf{y}, \mathbf{x}, \mathbf{u})\; \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \quad \text{and} \quad Q(\mathbf{x}, \mathbf{0}, \mathbf{u}) = \mathbf{0} = Q(\mathbf{0}, \mathbf{x}, \mathbf{u}) \quad \forall \mathbf{x} \in \mathbb{R}^n.$$

We then formulate two concrete computational problems relating to the impact of the uncertainty $\boldsymbol{\omega}$. The first problem requires checking whether the system of equations has a solution within acceptable limits for all values of the uncertainty, and the second one poses an optimization problem subject to the constraint imposed in the first problem.

Definition 2.1 (Robust feasibility and robustness margin)**.** Let $(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{0})$ satisfy (1). Let $\boldsymbol{\omega}_r = \{\boldsymbol{\omega} : \|\boldsymbol{\omega}\| \leq r\}$ and let $\mathcal{X} = \{\mathbf{x} : A\mathbf{x} \leq \mathbf{b}\}$ be a polytope. Then the *robust optimization problem* is the following problem:

$$\forall \boldsymbol{\omega} \in \boldsymbol{\omega}_r \quad \text{does there exist } \; \mathbf{x} \in \mathcal{X} \; \text{ such that } \; \mathbf{f}\left(\mathbf{x}, \mathbf{u}^\star, \boldsymbol{\omega}\right) = \mathbf{0}\,? \tag{2}$$

If yes, $\boldsymbol{\omega}_r$ is said to be *robust feasible*. The largest r such that $\boldsymbol{\omega}_r$ is robust feasible is the *robustness margin*.

In this formulation, $\mathcal{X}$ denotes the limits on the physical state $\mathbf{x}$ (e.g., voltage and flow limits in the grid) and $\boldsymbol{\omega}_r$ denotes a bounded uncertainty set (e.g., defined by the forecast error bounds on solar and wind generation in the grid). Computing the robustness margin is equivalent to asking the question: how much uncertainty can the system tolerate before a violation of physical constraints will be seen?

2.1.1 Prior work in robust feasibility

Robust feasibility of systems of nonlinear equations has been studied in a series of recent papers [35, 38]. These studies show that the problem is decidable (a terminating algorithm guaranteed to produce an answer exists) as long as the number of variables is not more than twice the number of equations minus 3. In our main application domain, we are primarily concerned with the AC power flow equations where the number of equations is the same as the number of variables, and hence the aforementioned algorithm [35, 38] applies to Problem (2). However, the algorithm requires computing a *simplicial decomposition* of the set $\mathcal{X}$ (a triangulation). The algorithm scales as $\tilde{O}(N^4)$ where N is the number of simplices in the complex. Further, the number of simplices in a simplicial decomposition grows exponentially with the dimension n, severely limiting the scalability of these algorithms to realistically-sized power grids ($n = 1000$ or higher).

Specific to power systems, there has been significant empirical work on solving the PF equations with probabilistic uncertainty [72, 61]. However, these algorithms are based on sampling heuristics and do not offer mathematical guarantees of robust feasibility. There has been recent work on specifying conditions on the power injections over which the power flow equations are guaranteed to have a solution [14, 70, 71]. However, they do not directly address the robust feasibility problem, and using these regions to assess robust feasibility can produce very conservative results. Further, they do not address uncertainty in the quadratic and linear objective terms (they consider uncertainty only the constant term $\mathbf{c}(\mathbf{u}, \boldsymbol{\omega})$).

2.1.2 Proposed research

Since Problem (2) is a generalization of the standard ACOPF problem, it is NP-hard. There are no efficient algorithms for these problems in general, and we must settle for weaker than worst-case guarantees. Our goal is to develop *rigorous and efficient algorithms* for the robust feasibility problem (2). Specifically, we aim to create a mathematical framework that supports algorithms which satisfy the following properties:

1. If the algorithm verifies robust feasibility, it also produces a mathematical certificate of robustness (based on topological degree theory or the Borsuk-Ulam theorem).
2. If the algorithm fails to verify robust feasibility, it produces an $r' \geq r$ such that $\boldsymbol{\omega}_{r'}$ is *not robust feasible* (r' is an upper bound on the robustness margin).
3. The algorithm is guaranteed to run in polynomial time and produce an answer (robust feasibility certificate or upper bound on robustness margin) on all instances.

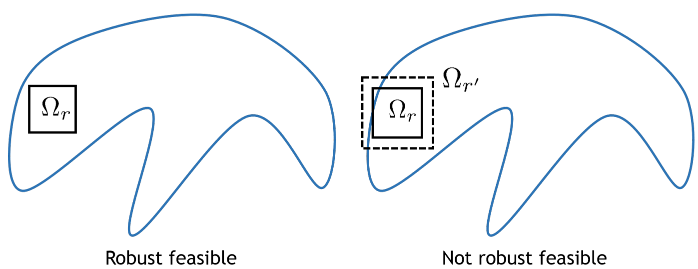

Figure 2: Robust feasibility guarantees

Verifying robust feasibility involves certifying the existence of solutions to the nonlinear system of equations $\mathbf{f}(\mathbf{x}, \mathbf{u}) = \mathbf{0}$. we describe two concrete approaches that combine topological ideas (degree theory and the Borsuk-Ulam theorem) with convex optimization to devise efficient algorithms for robust feasibility.

Approaches based on degree theory: *Degree* is a topological invariant defined on a function $F : \mathbb{R}^n \mapsto \mathbb{R}^n$, a closed set $\mathcal{X}$, and a vector $\mathbf{y} \in \mathbb{R}^n$, as follows [34]:

$$\mathrm{Deg}\left(F, \mathcal{X}, \mathbf{y}\right) = \sum_{\mathbf{x}: F(\mathbf{x})=\mathbf{y}, \mathbf{x} \in \mathcal{X}} \mathrm{sgn}\left(\det\left(\frac{\partial F}{\partial \mathbf{x}}\right)\right) \tag{3}$$

where sgn is the signum function: $\mathrm{sgn}\left(x\right) = \{0 \text{ if } x = 0,\ 1 \text{ if } x > 0,\ -1 \text{ if } x < 0\}$. If there are no solutions to $F\left(\mathbf{x}\right) = \mathbf{y}, \mathbf{x} \in \mathcal{X}$, the degree is defined to be 0. This definition of degree is not useful for analyzing existence of solutions, since computing the degree itself requires computing all solutions within $\mathcal{X}$. However, the following theorem provides a way out.

Theorem 2.2 ([34])**.** *Let $H\left(\mathbf{x}, t\right) : \mathcal{X} \times [0, 1] \mapsto \mathbb{R}^n$ be a continuous mapping such that $H\left(\mathbf{x}, 1\right) = F\left(\mathbf{x}\right)$ $\forall \mathbf{x} \in \mathcal{X}$. Suppose further that*

$$\forall t \in [0, 1] \quad \nexists \mathbf{x} \in \mathrm{Bd}\left(\mathcal{X}\right) \text{ such that } H\left(\mathbf{x}, t\right) = 0.$$

Then

$$\mathrm{Deg}\left(H\left(\cdot, t\right), \mathcal{X}, \mathbf{y}\right) = \mathrm{Deg}\left(F, \mathcal{X}, \mathbf{y}\right) \quad \forall t \in [0, 1].$$

Further, if $\mathrm{Deg}\left(H\left(\cdot, 0\right), \mathcal{X}, \mathbf{y}\right) \neq 0$*, then* $\exists \mathbf{x} \in \mathcal{X}$ *such that* $F\left(\mathbf{x}\right) = \mathbf{y}$.

Using this result, we can establish the existence of solutions of $F(\mathbf{x}) = \mathbf{y}$ by studying the existence of solutions for a "simpler" map $H(\mathbf{x}, 0) = \mathbf{y}$. In previous applications of this result (see [6], for example), $H(\mathbf{x}, 0)$ is assumed to be a linear map (often the identity) so that the degree of $H(\cdot, 0)$ is computed easily. Further, the condition of non-existence of boundary solutions is verified using interval arithmetic. This approach is conservative, and does not easily extend to the robust feasibility problem (2).

Instead of using interval arithmetic and homotopies starting at a linear problem, we pose the verification of the conditions of Theorem 2.2 as *optimization problems*. By relaxing nonconvexity in these optimization problems, we obtain sufficient conditions for existence of solutions which require solving only a set of convex optimization problems.

Theorem 2.3. *Let* $(\mathbf{x}^\star, \mathbf{u}^\star)$ *satisfy* $\mathbf{f}(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{0}) = 0$ *and* $\boldsymbol{\omega}_r, \mathcal{X}$ *be as specified in Definition 2.1. Let*

$$\mathrm{OPT}(i) = \min_{\mathbf{x}: A\mathbf{x} \leq \mathbf{b}, [A\mathbf{x}]_i = b_i, \boldsymbol{\omega} \in \boldsymbol{\omega}_r} \|\mathbf{f}(\mathbf{x}, \mathbf{u}^\star, \boldsymbol{\omega})\|, \quad \textit{and} \tag{4a}$$

$$\mathrm{OPT} = \min_{\mathbf{x} \in \mathcal{X}} \left(\|\mathbf{f}(\mathbf{x}, \mathbf{u}^\star, \mathbf{0})\| - \epsilon\|\mathbf{x} - \mathbf{x}^\star\|\right) \tag{4b}$$

for some $\epsilon > 0$*. Suppose that* $\mathrm{OPT}(i) > 0$ *for each* $i = 1, \ldots, m$ *(where* m *is the number of rows of* A*) and* $\mathrm{OPT} \geq 0$*. Then* $\boldsymbol{\omega}_r$ *is robust feasible.*

Proof. The first set of conditions ensures that the homotopy $H(\mathbf{x}, t) = \mathbf{f}(\mathbf{x}, \mathbf{u}^\star, t\boldsymbol{\omega})$ does not have solutions on the boundary $\mathrm{Bd}(\mathcal{X})$ for any $t \in [0, 1], \boldsymbol{\omega} \in \boldsymbol{\omega}_r$. The second condition ensures that $\mathbf{f}(\mathbf{x}, \mathbf{u}^\star, \mathbf{0}) = \mathbf{0}$ has a unique solution in $\mathcal{X}$, and hence that $\mathrm{Deg}(\mathbf{f}(\cdot, \mathbf{u}^\star, \mathbf{0}), \mathcal{X}, \mathbf{0}) \neq 0$ (since degree is non-zero if there is an odd number of solutions). □

The optimization problems (4a) and (4b) are nonlinear and nonconvex. Hence it may be difficult to solve them in general. However, since $\mathbf{f}$ is quadratic in $\mathbf{x}$, they are quadratically constrained quadratic programs (QCQPs), and we can use well known *semidefinite programming relaxations for QCQPs* [27] to obtain lower bounds on the optimal values. To illustrate the use of convex relaxations, consider the simpler case where $\mathbf{f}(\mathbf{x}, \mathbf{u}, \boldsymbol{\omega}) = Q(\mathbf{x}, \mathbf{x}, \mathbf{u}) + L(\mathbf{x}, \mathbf{u}) + \boldsymbol{\omega} = \mathbf{0}$ and $\mathcal{X} = \{\mathbf{x} : -\mathbf{1} \leq \mathbf{x} \leq \mathbf{1}\}$ ($\mathbf{1}$ is the vector of ones), and suppose that $\boldsymbol{\omega}_r = \{\boldsymbol{\omega} : \|\boldsymbol{\omega}\|_\infty \leq r\}$. Since Q is quadratic, it can also be written as a linear function of $\mathbf{x}\mathbf{x}^T$. More concretely, we can write

$$[Q(\mathbf{x}, \mathbf{x}, \mathbf{u})]_i = \mathbf{x}^T Q_i(\mathbf{u})\,\mathbf{x} = \mathrm{Trace}\left(Q_i(\mathbf{u})\,\mathbf{x}\mathbf{x}^T\right)$$

where Q_i is an affine function of $\mathbf{u}$ taking values in the space of symmetric $n \times n$ matrices. For brevity, we write $Q(\mathbf{x}, \mathbf{x}, \mathbf{u}) = \mathcal{L}\left(\mathbf{x}\mathbf{x}^T, \mathbf{u}\right)$ where $\mathcal{L}$ is a linear map. We can construct the following relaxation of the Problem (4a):

$$\mathrm{OPTRELAX}(i) = \min_{\mathbf{x}, \boldsymbol{\omega} \in \boldsymbol{\omega}_r} \|\mathcal{L}(X, \mathbf{u}^\star) + L(\mathbf{x}, \mathbf{u}^\star) + \boldsymbol{\omega}\| \tag{5a}$$

$$\text{subject to } \begin{pmatrix} 1 & \mathbf{x}^T \\ \mathbf{x} & X \end{pmatrix} \succeq 0 \tag{5b}$$

$$\begin{pmatrix} X & -X \\ -X & X \end{pmatrix} + \begin{pmatrix} \mathbf{x} \\ -\mathbf{x} \end{pmatrix}\begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix}^T + \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix}\begin{pmatrix} \mathbf{x} \\ -\mathbf{x} \end{pmatrix}^T + \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix}\begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix}^T \geq O \tag{5c}$$

where O denotes the $2n \times 2n$ matrix of zeros, and the constraints are relaxations of the nonconvex constraint $X = \mathbf{x}\mathbf{x}^T, -\mathbf{1} \leq \mathbf{x} \leq \mathbf{1}$. This is a convex optimization problem (a semidefinite program, in fact) and can be solved efficiently. Since this is a relaxation of Problem (4a), $\mathrm{OPTRELAX}(i) \leq \mathrm{OPT}(i)$. Hence, if $\mathrm{OPTRELAX}(i) > 0$ for each i, the condition in Theorem 2.3 (for Equation (4a)) is satisfied. A similar relaxation can be constructed for Problem (4b).

Lower bounds on robustness margin: We can look a slight variant of the system in Equation (5) to obtain a lower bound on the robustness margin:

$$\mathrm{LB}(i) = \min_{\mathbf{x}} \ \|\mathcal{L}(X, \mathbf{u}^\star) + L(\mathbf{x}, \mathbf{u}^\star)\| \tag{6a}$$

$$\text{subject to } \begin{pmatrix} 1 & \mathbf{x}^T \\ \mathbf{x} & X \end{pmatrix} \succeq 0 \tag{6b}$$

$$\begin{pmatrix} X & -X \\ -X & X \end{pmatrix} + \begin{pmatrix} \mathbf{x} \\ -\mathbf{x} \end{pmatrix} \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix}^T + \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ -\mathbf{x} \end{pmatrix}^T + \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix}^T \geq O \tag{6c}$$

$$X_{ii} = 1, \ x_i = 1 \tag{6d}$$

If the optimal value of this problem is r for each i, the robustness margin is at least r, since for any $\boldsymbol{\omega} \in \boldsymbol{\omega}_{r'}, r' < r$, we will have

$$\|\mathcal{L}(X, \mathbf{u}^\star) + L(\mathbf{x}, \mathbf{u}^\star) + \boldsymbol{\omega}\| \geq \|\mathcal{L}(X, \mathbf{u}^\star) + L(\mathbf{x}, \mathbf{u}^\star)\| - \|\boldsymbol{\omega}\| > 0.$$

Thus, Problem (6) provides a lower bound on the robustness margin, and this lower bound will get tighter as we tighten the relaxation used for Problem (6) (we present more details of this step in the following section).

Approaches based on Borsuk's theorem: Borsuk's theorem provides an alternative route to certify existence of solutions to nonlinear systems. Compared to degree theory,

1. the test is explicit and does not rely on constructing a homotopy, and
2. there is no need to compute the degree of the mapping (i.e., the analog of Problem (4b) is absent).

However, the price to pay is that we have to restrict $\mathcal{X}$ to be a symmetric convex set. In power systems applications, though, this might not be a huge restriction, as most engineering limits are simple bound constraints that define symmetric convex sets (symmetric about the mid-point of the bounds).

Theorem 2.4 (Borsuk's theorem [42])**.** *Let $\mathcal{X}$ be a symmetric convex set centered at $\mathbf{x}^\star$ and let $F : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a continuous function. Then F has a zero in $\mathcal{X}$ if*

$$\forall \lambda \in (0,1), \ \mathbf{y} \in \mathrm{Bd}(\mathcal{X}) - \mathbf{x}^\star, \qquad F(\mathbf{x}^\star + \mathbf{y}) \neq \lambda F(\mathbf{x}^\star - \mathbf{y}).$$

As in the previous section, we pose testing this condition as an optimization problem:

Theorem 2.5. *Let $\mathcal{X} = \{\mathbf{x} : A\mathbf{x} \leq \mathbf{b}\}$ be a symmetric convex set centered at $\mathbf{x}^\star$ and let $\mathbf{f}(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{0}) = \mathbf{0}$. Define*

$$\mathrm{OPT}(i) = \min_{\mathbf{y}, \lambda, \boldsymbol{\omega}} \quad \|\mathbf{f}(\mathbf{x}^\star + \mathbf{y}, \mathbf{u}^\star, \boldsymbol{\omega}) - \lambda f(\mathbf{x}^\star - \mathbf{y}, \mathbf{u}^\star, \boldsymbol{\omega})\| \tag{7a}$$

$$\text{subject to } \ A(\mathbf{x}^\star + \mathbf{y}) = \mathbf{b}, \ [A(\mathbf{x}^\star + \mathbf{y})]_i = b_i, \boldsymbol{\omega} \in \boldsymbol{\omega}_r, 0 \leq \lambda \leq 1. \tag{7b}$$

If $\mathrm{OPT}(i) > 0$ for each $i = 1, \ldots, n$, then $\boldsymbol{\omega}_r$ is robust feasible.

We can compute lower bounds for Problem (7) using convex relaxations similar to that in System (5), hence obtaining computationally tractable *sufficient conditions* for robust feasibility and lower bounds on the robustness margin.

Another advantage of Borsuk's theorem is that one can derive *analytical feasibility certificates* that do not require solving an optimization problem. We will exploit this advantage in our approach to solve the robust optimization problem (Section 2.2).

Approaches based on topological persistence: We previously (in Section 2.1.1) mentioned the algorithm of Franek et al. [38], which estimates the robustness margin by computing cohomological obstructions to extendability of the function, and their persistence. This work improves the computational aspects of previous results by the same authors [35, 37], and also generalizes related approaches based on well groups [17, 36] and robustness of level sets using persistence [10, 31]. But the requirement of a simplicial decomposition of $\mathcal{X}$ rules out this approach for realistic instances, e.g., from power systems. Instead, we will explore a *cellular* version of this approach. When $\mathcal{X}$ is specified by only bounds (i.e., is a box), a cellular subdivision of $\mathcal{X}$ into smaller boxes (rather than simplices) could be more manageable in size than a simplicial complex. In order to use this approach, we will first re-derive the theoretical results [38] in the setting of *cubical* or *cellular* (instead of simplicial) complexes. This derivation would be of independent theoretical as well as computational interest. In practice, we will consider also implementing *adaptive* meshing, where the sizes of the component boxes could be varied across the domain. Through computational testing, we will explore the limits on sizes of systems that could be handled by this approach (as part of Specific Goal **G4**).

Upper bounds on the robustness margin We have presented computationally tractable sufficient conditions for robust feasibility as well as lower bounds on the robustness margin. However, in order to estimate how far these tractable sufficient conditions are from being necessary, it is of interest to establish *necessary conditions for robust feasibility* as well as upper bounds on the robustness margin. We now present a general approach for these objectives.

Theorem 2.6. *Let* $\mathbf{f}(\mathbf{x},\mathbf{u},\boldsymbol{\omega}) = \mathbf{f}(\mathbf{x},\mathbf{u}) + \boldsymbol{\omega}$ *and* $\boldsymbol{\omega}_r = \{\boldsymbol{\omega} : \|\boldsymbol{\omega}\|_\infty \leq r\}$. *Then, the value of the optimization problem*

$$\min_{\|\boldsymbol{\lambda}\|_1=1} \max_{\mathbf{x}\in\mathcal{X}} \boldsymbol{\lambda}^T \mathbf{f}(\mathbf{x},\mathbf{u}^\star) \tag{8}$$

is an upper bound on the robustness margin.

Proof. Let r' be larger than the optimal value of Problem (8), and let $\boldsymbol{\lambda}^\star$ be a corresponding optimal solution. Then, there exists $\boldsymbol{\omega}^\star \in \boldsymbol{\omega}_{r'}$ such that ${\boldsymbol{\lambda}^\star}^T\boldsymbol{\omega}^\star = -r'$ (simply choose $\boldsymbol{\omega}^\star_i = -\mathrm{sign}(\lambda_i)\, r'$). But, from the definition of r', we know that

$$\forall \mathbf{x} \in \mathcal{X},\ \boldsymbol{\lambda}^T \mathbf{f}(\mathbf{x},\mathbf{u}^\star) + \boldsymbol{\omega}^\star \, < \, r' - r' \, < 0$$

so that there is no $\mathbf{x} \in \mathcal{X}$ such that $\boldsymbol{\lambda}^T\mathbf{f}(\mathbf{x},\mathbf{u}^\star) + \boldsymbol{\omega}^\star = 0$, which means that $\boldsymbol{\omega}_{r'}$ is not robust feasible. □

While the optimization problem (8) is itself nonconvex due to the constraint $\|\boldsymbol{\lambda}\|_1 = 1$, we can pose it as a convex mixed-integer program (see System (9)). This problem can be solved using a branch-and-cut algorithm. Although the global optimum can be hard to find, the optimization can be stopped at any time to find an upper bound on the robustness margin. The inner maximization problem $\left(\max_{\mathbf{x}\in\mathcal{X}} \boldsymbol{\lambda}^T\mathbf{f}(\mathbf{x},\mathbf{u}^\star)\right)$ might itself be hard, but this can be handled if a convex relaxation of $\mathbf{f}$ is available and techniques similar to ones in the previous section can be invoked.

$$\min_{\boldsymbol{\lambda}} \left(\max_{x\in\mathcal{X}} \boldsymbol{\lambda}^T f(\mathbf{x},\mathbf{u}^\star)\right) \tag{9a}$$

$$\text{subject to } \lambda_i + \lambda'_i \geq 0,\ \ \lambda_i - \lambda'_i \leq 0 \tag{9b}$$

$$\lambda_i - \lambda'_i \,\geq\, \sigma_i - 1 \tag{9c}$$

$$\lambda_i + \lambda'_i \,\leq\, \sigma_i + 1 \tag{9d}$$

$$\sigma_i = \pm 1 \tag{9e}$$

$$\sum\nolimits_i \lambda'_i = 1, \boldsymbol{\lambda}' \geq \mathbf{0} \tag{9f}$$

Computational issues (Specific Goal G4): The relaxation presented here is mainly for illustrative purposes, and is the simplest approach to certifying robust feasibility. But one can do much better both in terms of tightness and computational efficiency by leveraging the excellent work done in constructing tight convex relaxations of the ACOPF problem [57, 58, 20, 49]. Any relaxation of the ACOPF problem contains a method to relax the AC power flow equations, and hence a method to relax the problems (4), (7), and (9).

Although the relaxations are convex optimization problems, one has to solve several relaxations (one for each $i = 1, \ldots, m$) which could be computationally intensive. However, since these problems are independent, they can be solved in parallel. Further, there have been recent developments in ideas around "bound-tightening", the idea that relaxations of the AC Power Flow equations can be enhanced if tight bounds on the variables (voltage magnitudes and phases) are known, and tighter bounds can be inferred from relaxations [21] or using analytical techniques [18].

Finally, network structure can be often successfully exploited to develop efficient algorithms to solve relaxations. Several works exploiting network structure to decompose semidefinite programming relaxations of ACOPF have appeared in the literature [46, 4, 60]. Finally, for low-treewidth networks, alternative approaches based on dynamic programming can be used to solve the problem to a desired tolerance [29].

We will investigate the application of all these ideas to the problems (4), (7), and (9).

2.2 Specific Goal G2: Robust optimization

The natural extension of robust feasibility is robust optimization. Previously, $\mathbf{u}^\star$ is fixed. Here we have a set of controllable decision variables $\mathbf{u}$ (e.g., power outputs and voltage setpoints of dispatchable generators) which are to be chosen so that the system is robust feasible, while also minimizing a cost function $\ell(\mathbf{u})$ (e.g., fuel costs for traditional generators). Robust feasibility of the system will now depend on the choice of $\mathbf{u}$.

Definition 2.7 (Robust optimization)**.** Let $\mathcal{U} \subseteq \mathbb{R}^p$ be a convex set and $\ell : \mathcal{U} \mapsto \mathbb{R}$ be a convex function. Let $\boldsymbol{\omega}_r, \mathcal{X}$ be as in Definition 2.1. Then, the *robust optimization problem* is defined as follows.

$$\min_{\mathbf{u} \in \mathcal{U}} \quad \ell(\mathbf{u}) \tag{10a}$$

$$\text{subject to} \quad \forall \boldsymbol{\omega} \in \boldsymbol{\omega}_r \quad \exists \mathbf{x} \in \mathcal{X} \text{ such that } \mathbf{f}(\mathbf{x}, \mathbf{u}, \boldsymbol{\omega}) = \mathbf{0}. \tag{10b}$$

2.2.1 Prior work in robust optimization

Robust optimization has been a topic of significant interest in the optimization community in the last two decades. Much of the research has focused on convex optimization problems (specifically, linear and conic optimization problems) [7]. Robust convex optimization problems can be hard in general [8], and much of the work on robust convex optimization has focused on identifying tractable cases and developing *safe tractable approximations* for more general problems. A safe tractable approximation is an inner approximation of the robust feasible set, or convex conditions on the decision variables which guarantee feasibility for all values of the uncertainty. Researchers have also worked on finding lower bounds. These approaches do not directly extend to nonconvex optimization problems, however.

Robust nonconvex optimization has received far less attention; a notable work from the optimization community is that of Bertsimas et al. [11]. However, this work does not truly guarantee robust feasibility or optimality, since it relies on a local optimizer to find the worst case disturbance for the objective and constraints in the problem. Further, since we are concerned with the existence of a solution in Equation (10), our problem is more like a 2-stage robust optimization problem where the choice of $\mathbf{x}$ is made after the realization of the uncertainty $\boldsymbol{\omega}$. Lasserre [53] has developed a general approach for polynomial inequality constraints and quantifiers (such as $\forall, \exists$). However, the approach requires $\mathcal{U}$ to be a simple set (box constraints or ellipsoid), and requires special assumptions to handle the case with two quantifiers. Franek et al. [39] developed an approach that would apply to Equation (10). However, their algorithm requires a simplicial decomposition of $\mathcal{X}$, the size of which grows exponentially in the dimension n.

Louca and Bitar [56] have recently studied a problem similar to ours—a stochastic/robust optimal power flow problem. However, they make the assumption that all nodes have a generator as their approach cannot handle nonlinear equality constraints directly.

2.2.2 Proposed research

We propose to develop an algorithm to solve Problem (10) that satisfies the following desiderata:

1. is computationally efficient: polynomial time and scalable to systems with $n, p, k \geq 1000$;
2. is rigorous: the solution computed for Problem (10) is always accompanied by a certificate of robust feasibility (as described in Section 2.1); and
3. offers optimality guarantees: the gap between the objective of Problem (10) at the solution computed by the algorithm and the true global optimum of (10) can be quantified.

We now focus on the aims 1 and 2 listed above. The third aim will be addressed in Section 2.3.

In section 2.1, we discussed techniques to compute upper and lower bounds on the robustness margin, hence providing a rigorous solution to the robust feasibility problem combined with a tightness guarantee. In principle, these algorithms can be used as a subroutine within the robust optimization problem (10) to ensure a candidate solution is robust feasible. However, this approach can make the overall procedure computationally intensive, since in each iteration of the optimization algorithm we need to run a subproblem to guarantee robust feasibility.

Instead, we take a different approach akin to the idea of trust regions in nonlinear programming. Given a feasible solution $\mathbf{u}^\star$ to Equation (10), we construct a convex region around $\mathbf{u}^\star$ that is guaranteed to remain feasible. This region is analogous to a trust region and one can compute the next iterate by optimizing over this set. A series of recent papers has studied the problem of constructing simple convex inner approximations of the feasible set [70, 71, 73, 14]. But most of the previous approaches have limited applicability and only apply to specific formulations of the power flow equations.

Recently, the PI's collaborator Dvijotham and coworkers [28] have generalized the previous results based on Borsuk's theorem. These results apply to the general formulation from Section 2.1 (the Jacobian of the system in Equation (1) is defined via its action on a vector $\mathbf{y}$):

Theorem 2.8. *Let $(\mathbf{x}^\star, \mathbf{u}^\star)$ be such that $\mathbf{f}(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{0}) = \mathbf{0}$ and define*

$$J(\mathbf{x}, \mathbf{u}, \boldsymbol{\omega}) = \frac{\partial \mathbf{f}(\mathbf{x}, \mathbf{u}, \boldsymbol{\omega})}{\partial \mathbf{x}}\Big|_{\mathbf{y}} = 2Q(\mathbf{x}, \mathbf{y}, \mathbf{u}, \boldsymbol{\omega}) + L(\mathbf{y}, \mathbf{u}),\ J^\star = J(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{0}),$$

$$\Delta J(\mathbf{u}, \boldsymbol{\omega}) = (J^\star)^{-1}\left(J(\mathbf{x}^\star, \mathbf{u}, \boldsymbol{\omega}) - J(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{0})\right),\ \mathbf{e}(\mathbf{u}, \boldsymbol{\omega}) = \mathbf{f}(\mathbf{x}^\star, \mathbf{u}, \boldsymbol{\omega}) - \mathbf{f}(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{0}) \ \textit{and}$$

$$h(\mathbf{u}, \boldsymbol{\omega}) = \max_{\|d\mathbf{x}\| \leq 1} \|(J^\star)^{-1} Q(d\mathbf{x}, d\mathbf{x}, \mathbf{u}, \boldsymbol{\omega})\|.$$

Then, $\mathbf{f}(\mathbf{x}, \mathbf{u}, \boldsymbol{\omega}) = \mathbf{0}$ has a solution within the set $\left\{\mathbf{x} \,\middle|\, \|\mathbf{x} - \mathbf{x}^\star\| \leq \sqrt{\frac{\|(J^\star)^{-1}\mathbf{e}(\mathbf{u},\boldsymbol{\omega})\|}{h(\mathbf{u},\boldsymbol{\omega})}}\right\}$ provided that

$$\|\Delta J(\mathbf{u}, \boldsymbol{\omega})\| + 2\sqrt{h(\mathbf{u}, \boldsymbol{\omega})\, \|(J^\star)^{-1}\mathbf{e}(\mathbf{u}, \boldsymbol{\omega})\|} \ \leq\ 1. \tag{11}$$

This theorem guarantees an explicit region in the space of $(\mathbf{u}, \boldsymbol{\omega})$ around any given nominal solution with a non-singular Jacobian $J^\star$ and $\mathbf{x}^\star \in \mathrm{Int}(\mathcal{X})$, since for any interior point one can find a radius $t > 0$ such that $\mathcal{X} \supseteq \{\mathbf{x} : \|\mathbf{x} - \mathbf{x}^\star\| \leq t\}$. Further, it is possible to express (11) as a convex condition in $(\mathbf{u}, \boldsymbol{\omega})$ by fixing any $t_1 > 0, t_2 > 0$ and setting

$$h(\mathbf{u}, \boldsymbol{\omega}) \leq t_1,\ \ \|(J^\star)^{-1}\mathbf{e}(\mathbf{u}, \boldsymbol{\omega})\| \ \leq\ t_2,\ \ \ t_2 \ \leq\ r^2 t_1,\ \ \|\Delta J(\mathbf{u}, \boldsymbol{\omega})\| + 2\sqrt{t_1 t_2} \ \leq 1. \tag{12}$$

One can then perform *robust convex optimization* over this set (akin to a trust-region method):

$$u'(\mathbf{u}^\star) = \operatorname*{argmin}_{\mathbf{u} \in \mathbf{u}} \quad \ell(\mathbf{u}) \tag{13a}$$

$$\text{subject to}\ \ h(\mathbf{u}, \boldsymbol{\omega}) \leq t_1, \|(J^\star)^{-1}\mathbf{e}(\mathbf{u}, \boldsymbol{\omega})\| \leq t_2, t_2 \leq r^2 t_1, \|\Delta J(\mathbf{u}, \boldsymbol{\omega})\| + 2\sqrt{t_1 t_2} \leq 1\, \forall \boldsymbol{\omega} \in \boldsymbol{\omega}_r. \tag{13b}$$

Although robust convex optimization problems can be hard in general, significant work has been done to identify *safe tractable approximations* (which guarantee robust feasibility) of robust convex optimization problems [7]. We will explore using these approaches to solve Problem (13). This idea is illustrated in Figure 3. Starting at an initial feasible point $\mathbf{u}^0$, we can apply the mapping $\mathbf{u}^{i+1} \leftarrow u'\left(\mathbf{u}^i\right)$ and iteratively find feasible points with decreasing objective values. We will investigate conditions under which this algorithm is guaranteed to converge, and what claims one can make regarding the point of convergence. One objective of the theoretical work will be to show that this algorithm converges (with a certain guaranteed convergence rate) to a robust local minimum of Problem (10).

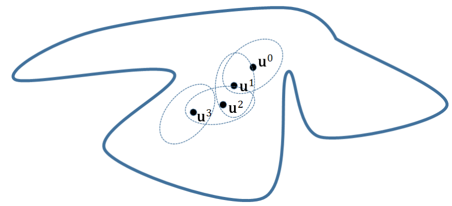

Figure 3: Robust convex optimization with convex "trust regions".

2.3 Specific Goal G3: Computing lower bounds on the robust optimization problem

In Section 2.2, we described algorithms capable of producing feasible but possibly suboptimal solutions to robust optimization problem in Equation (10). In order to assess how far these solutions are from global optimality, it is important to be able to compute *lower bounds* on the robust optimization problem.

Definition 2.9 (Lower bounds on robust optimization)**.** Any number $\ell^\star$ that satisfies

$$\ell\left(\mathbf{u}\right) \geq \ell^\star \qquad \forall \mathbf{u} \in \mathcal{U} \text{ such that } \qquad \left(\forall \boldsymbol{\omega} \in \boldsymbol{\omega}_r,\ \exists \mathbf{x} \in \mathcal{X} \text{ such that } \mathbf{f}(\mathbf{x}, \mathbf{u}, \boldsymbol{\omega}) = \mathbf{0}\right) \tag{14}$$

is a lower bound on the value of the robust optimization problem in Equation (10).

2.3.1 Prior work on lower bounds

A standard approach to computing lower bounds on nonconvex optimization problems is using convex relaxations. Previously, convex relaxations have been applied to the non-robust ACOPF problem [57, 57], and have even shown to be tight under special assumptions [43]. However, these approaches do not apply directly to the robust ACOPF problem.

The robust optimization literature [7] typically studies optimization without quantifiers. Further, intractable cases are typically dealt with using safe tractable approximations (inner approximations) which do not provide lower bounds on the optimal value. Problem (10) is more similar to a multistage robust optimization problem, for which techniques have recently been developed which provide both upper and lower bounds [44]. However, these techniques have been developed primarily for convex problems, and need to be extended to nonconvex problems that arise in Equation (14).

2.3.2 Proposed research

While obtaining lower bounds on Equation (14) directly could be hard, we can obtain lower bounds by considering a relaxation of $\mathbf{f}\left(\mathbf{x}, \mathbf{u}, \boldsymbol{\omega}\right) = \mathbf{0}$ with respect to $\mathbf{x}$ (making it easier to satisfy the robust feasibility constraint, and hence guaranteeing a lower bound). Similar to Section 2.1.2, we relax the condition $\exists \mathbf{x} \in \mathcal{X}$ such that $\mathbf{f}\left(\mathbf{x}, \mathbf{u}, \boldsymbol{\omega}\right) = \mathbf{0}$ to

$$\exists \mathbf{x}, X \text{ such that } \qquad \mathcal{L}\left(X, \mathbf{u}, \boldsymbol{\omega}\right) + L\left(\mathbf{x}, \mathbf{u}, \boldsymbol{\omega}\right) + \mathbf{c}\left(\mathbf{u}, \boldsymbol{\omega}\right) = \mathbf{0}, \tag{15a}$$

$$\begin{pmatrix} \mathbf{1} & \mathbf{x}^T \\ \mathbf{x} & X \end{pmatrix} \succeq 0, \begin{pmatrix} X & -X \\ -X & X \end{pmatrix} + \begin{pmatrix} \mathbf{x} \\ -\mathbf{x} \end{pmatrix} \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix}^T + \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ -\mathbf{x} \end{pmatrix}^T + \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix}^T \geq O. \tag{15b}$$

We can apply techniques given in [44] to this formulation to obtain lower bounds on the robust optimization problem (10). This requires assuming stochastic policies $X\left(\boldsymbol{\omega}\right), \mathbf{x}\left(\boldsymbol{\omega}\right)$ and computing lower bounds that hold for all possible such policies (which can be done using duality and moment conditions [44]).

Alternatively, one can formulate a cutting plane algorithm which tries to find choices of $\boldsymbol{\omega}$ that violate the robust feasibility "the most". To quantify the degree of violation, we consider the problem

$$\begin{aligned} \min_{\mathbf{x},X} \quad & \textstyle\sum_i |\delta_i| \\ \text{subject to} \quad & \mathcal{L}(\mathbf{x},\mathbf{u},\boldsymbol{\omega}) + L(\mathbf{x},\mathbf{u},\boldsymbol{\omega}) + \mathbf{c}(\mathbf{u},\boldsymbol{\omega}) = \boldsymbol{\delta}, \text{constraints in (15b)} \end{aligned}$$

and call its optimal value $e(\mathbf{u},\boldsymbol{\omega})$. Here $\boldsymbol{\delta}$ is a measure of violation of the first constraint (we could choose $\mathbf{x} = \mathbf{1}, X = \mathbf{x}\mathbf{x}^T$ to satisfy other constraints). We then compute $\max_{\boldsymbol{\omega}\in\boldsymbol{\omega}_r} e(\mathbf{u},\boldsymbol{\omega})$, which can be written as a convex mixed-integer program. If the optimal value is 0, then $\mathbf{u}$ is robust feasible with respect to the convex relaxation. If not, it computes the worst case violation $\boldsymbol{\omega}$ of robust feasibility, and adds a cutting plane to refine the constraint (see Figure 4). While solving this problem to optimality can be difficult (except if $\boldsymbol{\omega}_r$ is an ℓ_1-ball), one can exploit modern branch-and-cut solvers with a time limit to ensure the subproblem does not take too long to solve.

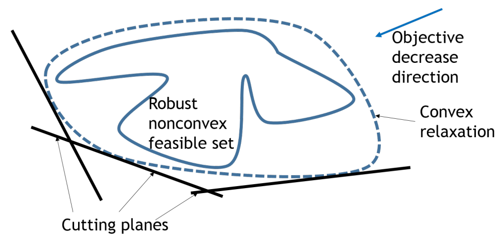

Figure 4: Cutting planes for (14).

2.4 Specific Goal G4: Computational Evaluation on Power System Instances

The relaxations presented in the previous sections can be made computationally and numerically better using special convex relaxations tailored to the AC power flow equations [20]. Extending these relaxations to a multistage robust setting is a major computational challenge which we will focus on. With the cutting plane algorithms, there are several tradeoffs to be explored. For instance, do we add a single or multiple cutting planes in each iteration? Do we solve the worst case violation based optimization to global optimality, or impose a time limit on a branch-and-bound algorithm? The first option will reduce the number of iterations of the cutting plane algorithm, but the second option would make each iteration cheaper.

3 Broader Impacts

We need fundamental concepts from different areas of mathematics and computation to tackle the problems posed. Recent work on various aspects of topology as well as optimization has already catalyzed a synergy between mathematics and computation, as seen in the work of several mathematicians [42, 41, 35, 38, 3, 15, 16, 22, 23, 47, 74] and computational scientists [13, 21, 7, 9, 19, 30, 33]. Our research efforts toward achieving the specific goals described in this project will continue to feed the synergy between mathematics and computation. At the same time, our proposed work is pragmatically targeted at applications—to the modern electrical power grid, and to other similar grids such as gas distribution networks.

Mathematics: Our work will generate new theory and fundamental results of interest to mathematicians. It will also open new avenues for applying sophisticated mathematical ideas to computational problems. The fields of computational topology and optimization have strongly stimulated the branches of mathematics in which they are grounded. In particular, areas such as geometric approximation, homological algebra, topological algorithms, nonlinear analysis, and robust optimization have undergone extensive development when employed to solve important computational problems. Our proposal is very much in that tradition, with its emphasis on fundamental questions that cannot be answered easily with current tools.

On the other hand, our work is part of a major trend of demand for certain areas of mathematics in communities which traditionally have shown little serious interest in them. The growth of computational topology and robust optimization has directly stimulated the study and teaching of differential geometry, analysis, algebraic topology, and homological algebra in engineering and computer science departments.

Our project should also motivate topologists, electrical engineers, and practitioners to learn the mathematical foundations of robust and nonconvex optimization. When engineers and practitioners clamor to learn deep results in topology, analysis, and optimization, the practical impact on mathematics would be considerable.

Computation and Applications: Our project takes a novel approach by employing results from algebraic topology to problems in robust optimization that are currently unsolved from the perspective of critical applications from the modern power grid. These problems include identification of feasible and optimal solutions to systems of nonlinear inequalities under uncertainty, which show up also in many other applications from science and engineering. On the other hand, we will take motivation from robust optimization to push for new results in algebraic topology. This project fuses approaches from pure and applied mathematics as well as electrical engineering, and holds promise for cross-fertilization opportunities with many collaborators of the PI in the future.

The direct impact of our proposed research on solving state-of-the-art problems arising in the modern power grid is evident. We will demonstrate the effectiveness of our approaches on real instances of the power grid, which will motivate the adoption of these techniques in their routine operations. This step could in turn lead to increased operational efficiencies of the grid, at both local and global scales. These successes should motivate the study of robust optimization problems problems arising in other fields of applications.

Education and Outreach: The direct educational impact of this proposal will be in the training of students getting exposed to ideas from topology, analysis, and optimization used in project. PhD student(s) will participate intimately in both the theoretical as well as implementation aspects of the problems, thus gaining valuable practical experience. The broader research community as well as the public will benefit by learning the material through seminars, lecture notes and videos, as well as popular articles (e.g., in blogs).

Curriculum and Program Development: The Mathematics Program in PI's institution is brand new, and will benefit immensely from students' activities in this project (see the RUI Impact Statement in Supplementary Documents for details). Undergraduate students doing research will motivate other students, and will also encourage other high quality students to study at the institution. The PI has a proven track record of mentoring students from underrepresented groups (including females, Hispanics, and disabled students). The PI's institution has a high percentage (25%) of minority students, and he will make efforts to recruit qualified students from such groups for the project.

Project Management: In *Year 1*, the PI will focus on literature survey for the theoretical and computational domains, recruit students and train them, and commence work on theoretical aspects of the Specific Goals. In *Year 2*, the substantial emphasis will be on the development of theory and mathematics, as well as on initial implementations of algorithms. In *Year 3*, we will work vigorously on testing the developed methods on real power grid instances, while fine tuning theory using the resulting feedback, and work on software development. We plan to publish papers throughout the project timeline, as we obtain results. We will also publish expository and popular articles throughout the project timeline.

Results from prior NSF support: Krishnamoorthy was a co-PI in grant DMS-1029482, *Collaborative Research: UBM-Institutional: UI-WSU Program in Undergraduate Mathematics and Biology*, 2010–2015. **Intellectual Merit:** The PI mentored six undergraduate students on research projects involving applications of geometry, topology, and optimization to problems from biochemistry, ecology, and bioengineering. This work resulted in four publications [24, 67, 65, 55]. **Broader Impacts:** Three of the six students are female, and one is disabled.

Krishnamoorthy was the PI in grant CCF-1064600, *AF: Medium: Collaborative Research: Optimality in Homology - Algorithms and Applications*, 2011–2015. **Intellectual Merit:** New connections were discovered between optimization and algebraic topology, resulting in efficient solutions of several problems from homology and geometric measure theory. Techniques from optimization, machine learning, and statistics were used to discover structure in biomedical data. This work has resulted in eight publications [25, 45, 26, 52, 32, 51, 54, 50]. **Broader Impacts:** Mentored five PhD students, two of whom are female, and one is disabled. Also guided four undergraduate students on research. Two of these four students are Hispanic. The biomedical data analysis provided new insights into how to increase availability and effectiveness of certain medical procedures.

Krishnamoorthy was the PI in grant CCF-1654106, *Student Travel Grant: International Workshop on Topological Data Analysis in Biomedicine, Seattle, October 2, 2016*, 2016–2017. **Intellectual Merit:** Nothing significant to report. **Broader Impacts:** Project supported travel of eight graduate students to attend the workshop coorganized by the PI. Five of the students are female, and one is disabled.

Krishnamoorthy is a co-PI in the current grant ABI-1661348, *Collaborative Research: ABI Innovation: A Scalable Framework for Visual Exploration and Hypotheses Extraction of Phenomics Data using Topological Analytics*, 2017–2020. **Intellectual Merit:** The team is working on the development of a new scalable visual analytics platform for hypothesis extraction from complex phenomics data sets (resulting in one publication [48]). **Broader Impacts:** Tools from algebraic topology, optimization, and algorithms are combined to discover new insights from maize phenomics data. The PI is mentoring two PhD students.

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