Pushing C-Bounds for inner approximation

Given the initial equation

$$x^T Q x + q^t x + c = 0$$

with the constraint that $Ax \leq b$ we wish to discover the widest bounds on c such that we can gaurantee a solution to the equation with the constraint. To begin we verify that a solution exists for a given b^*. Then we search for the largest set of bounds on c for various $b \leq b^*$ using the following model for each new bound choice on c at each dimension, i, of b. First note that $Ax \leq b \Rightarrow (b - Ax)(b - Ax)^T \geq 0 \Rightarrow bb^T - Axb^T - b(Ax)^T + A(xx^T)A^T \geq 0$, which can be relaxed linearly by allowing a symmetric positive semidefinite matrix $\mathcal{X}$ to replace xx^T.

$$\max A[i,:]x$$

Subject To:

$$\begin{aligned}
&(1)\ Tr\,(Q\mathcal{X}) + q^t x + c = 0\\
&(2)\ Ax \leq b\\
&(3)\ bb^T - Axb^T - b(Ax)^T + A\mathcal{X}A^T \geq 0\\
&(4)\ \mathcal{X} \text{ is symmtric, positive semidefinite}
\end{aligned}$$

Deriving the constraints for coding we see that for $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$ we have that

$$bb^T - Axb^T - b(Ax)^T + A\mathcal{X}A^T \geq 0$$

implies

$$\begin{bmatrix} b_1b_1 & b_1b_2 & \dots & b_1b_m \\ b_2b_1 & b_2b_2 & \dots & b_2b_m \\ \vdots & \vdots & \ddots & \vdots \\ b_mb_1 & b_mb_2 & \dots & b_mb_m \end{bmatrix} - \begin{bmatrix} b_1\sum_{i=1}^n A_{1i}x_i & b_2\sum_{i=1}^n A_{1i}x_i & \dots & b_m\sum_{i=1}^n A_{1i}x_i \\ b_1\sum_{i=1}^n A_{2i}x_i & b_2\sum_{i=1}^n A_{2i}x_i & \dots & b_m\sum_{i=1}^n A_{2i}x_i \\ \vdots & \vdots & \ddots & \vdots \\ b_1\sum_{i=1}^n A_{mi}x_i & b_2\sum_{i=1}^n A_{mi}x_i & \dots & b_m\sum_{i=1}^n A_{mi}x_i \end{bmatrix}$$

$$- \begin{bmatrix} b_1\sum_{i=1}^n A_{1i}x_i & b_1\sum_{i=1}^n A_{2i}x_i & \dots & b_1\sum_{i=1}^n A_{mi}x_i \\ b_2\sum_{i=1}^n A_{1i}x_i & b_2\sum_{i=1}^n A_{2i}x_i & \dots & b_2\sum_{i=1}^n A_{mi}x_i \\ \vdots & \vdots & \ddots & \vdots \\ b_m\sum_{i=1}^n A_{m1i}x_i & b_m\sum_{i=1}^n A_{2i}x_i & \dots & b_m\sum_{i=1}^n A_{mi}x_i \end{bmatrix} + \begin{bmatrix} \sum_{j=1}^n \left[\sum_{i=1}^n \mathcal{X}_{ij}a_{1i}\right] a_{1j} & \sum_{j=1}^n \left[\sum_{i=1}^n \mathcal{X}_{ij}a_{1i}\right] a_{2j} & \dots & \sum_{j=1}^n \left[\sum_{i=1}^n \mathcal{X}_{ij}a_{1i}\right] a_{mj} \\ \sum_{j=1}^n \left[\sum_{i=1}^n \mathcal{X}_{ij}a_{2i}\right] a_{1j} & \sum_{j=1}^n \left[\sum_{i=1}^n \mathcal{X}_{ij}a_{2i}\right] a_{2j} & \dots & \sum_{j=1}^n \left[\sum_{i=1}^n \mathcal{X}_{ij}a_{2i}\right] a_{mj} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^n \left[\sum_{i=1}^n \mathcal{X}_{ij}a_{mi}\right] a_{1j} & \sum_{j=1}^n \left[\sum_{i=1}^n \mathcal{X}_{ij}a_{mi}\right] a_{2j} & \dots & \sum_{j=1}^n \left[\sum_{i=1}^n \mathcal{X}_{ij}a_{mi}\right] a_{mj} \end{bmatrix} \geq 0$$

Entry-wise we have for $i, j \in [1, m]$

$$b_ib_j - b_j\sum_{k=1}^n A_{ik}x_k - b_i\sum_{k=1}^n A_{jk}x_k + \sum_{K=1}^n \left[\sum_{L=1}^n \mathcal{X}_{LK}A_{iL}\right] A_{jK} \geq 0$$

Outer C Bound Approximation

$$\min_{||\lambda||=1} \max_x \lambda^T \left[Tr\left(Q\mathcal{X}\right) + x^T q + c \right]$$

subject to constraints (2-3), and relaxing (4) to symmetric only. Constructing the dual of the inner maximal objective we must first write constraints (2-3) as $M\hat{x} \geq B$ and the objective as $c(\lambda)^T\hat{x}$. Where $\hat{x}^T = [x^T, \hat{\mathcal{X}}^T]$ ($\hat{\mathcal{X}}$ the vector form of the upper triangular (including diagonal) portion of $\mathcal{X}$). We now have

$$\min_{||\lambda||=1,y} B^T y$$

Subject To:

$M^T y = c(\lambda), \quad y \leq 0$

It follows then that $M \in \mathbb{R}^{\left((m^2+m)\times(n+(n+n^2)/2\right)}$. We can write $M\hat{x} \geq B$ starting with constraint (2) then constraint (3) as follows

$$\begin{bmatrix} -A_{11} & -A_{12} & \dots & -A_{1n} & 0 & 0 & \dots & 0 \\ -A_{21} & -A_{22} & \dots & -A_{2n} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -A_{m1} & -A_{m2} & \dots & -A_{mn} & 0 & 0 & \dots & 0 \\ -b_1A_{11} - b_1A_{11} & -b_1A_{12} - b_1A_{12} & \dots & -b_1A_{1n} - b_1A_{1n} & A_{11}A_{11} & 2A_{12}A_{11} & \dots & A_{1n}A_{1n} \\ -b_2A_{11} - b_1A_{21} & -b_2A_{12} - b_1A_{22} & \dots & -b_2A_{1n} - b_1A_{2n} & A_{21}A_{11} & 2A_{22}A_{11} & \dots & A_{2n}A_{1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -b_mA_{m1} - b_mA_{m1} & -b_mA_{m2} - b_mA_{m2} & \dots & -b_mA_{mn} - b_mA_{mn} & A_{m1}A_{m1} & 2A_{m2}A_{m1} & \dots & A_{mn}A_{mn} \end{bmatrix} \hat{x} \geq \begin{bmatrix} -b_1 \\ -b_2 \\ \vdots \\ -b_m \\ -b_1b_1 \\ -b_1b_2 \\ \vdots \\ -b_mb_m \end{bmatrix}$$

We can write the inner objective function $\max_x \lambda^T \left[Tr\left(Q\mathcal{X}\right) + x^T q + c \right]$ and thus $c(\lambda)^T$ as

$$\max_x \left[\sum_{j=1}^n q_1^j \lambda_j, \sum_{j=1}^n q_2^j \lambda_j, \dots, \sum_{j=1}^n q_n^j \lambda_j, \sum_{j=1}^n \lambda_j Q_{11}^j, 2\sum_{j=1}^n \lambda_j Q_{12}^j, \dots, \sum_{j=1}^n \lambda_j Q_{nn}^j \right] \hat{x}$$