Multinomial Kelly Bet Sizing - The Dice Game

Ben Reilly

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Abstract

The impetus for our examination of multinomial Kelly Bet Sizing begins with a simple question of expected value. The game begins as such: you are playing a dice game. You will be rewarded the amount of dollars equivalent to the face of the die you land on. For example, rolling a 6 would yield 6 dollars. Theoretically, you would pay up to \$3.50 to play this game as the expected value of the game is \$3.50.

Now we assume that you have the optionality to re-roll once if you would like to. That is, if your initial roll was unsatisfactory, you can roll once more. The expected value of this game is \$4.25, as for half of the rolls you would re-roll in the event you did not receive a 4, 5, or 6, and for half of the rolls you would not.

Splitting by case and summing, we see that

$$4.25 = .5 * E[4,5,6] + .5 * E[1,2,3,4,5,6] \tag{1}$$

$$4.25 = .5 * (4 + 5 + 6/3) + .5 * (1 + 2 + 3 + 4 + 5 + 6/6) = 2.5 + 1.75 \tag{2}$$

To prompt our use of the Kelly Criterion, we assume that we have mispriced the game at \$4.20 to play. You have \$100, and can size your bets however you see fit - Betting \$42 would yield \$60 for rolling a 6.

What is the optimal betting strategy for this game? That is, what percentage of the portfolio should be bet in each trial to maximize returns?

1 Possible Outcomes

We begin with articulating the probabilities of possible outcomes for the game.

The ultimate result of a single round of our game must be an integer 1-6. The probabilities of each game are defined as follows:

For integers 1, 2, and 3, the probability is

$$\frac{1}{2} * \frac{1}{6} = \frac{1}{12} \tag{3}$$

For integers 4, 5, and 6, the probability is

$$\frac{(1 - 3 * \frac{1}{12})}{3} = \frac{1}{4} \tag{4}$$

Therefore P(4) = P(5) = P(6) = $\frac{1}{4}$ and P(1) = P(2) = P(3) = $\frac{1}{12}$.

Given the cost to play of \$4.20, the mean returns of our game are equal to

$$u = \sum_{i=0}^{m} (n - C) p_i \tag{5}$$

where n is equal to the resultant payout and C is the cost.
Evaluating this term with our calculated probabilities we receive

$$u = 0.05 \tag{6}$$

The variance can be calculated similarly using the equation

$$s^2 = |\sum_{i=0}^{m}(n - C)p_i(1 - p_i)| \tag{7}$$

and is equal to

$$s^2 = 0.054 \tag{8}$$

2 Kelly Criterion

The Kelly Criterion is defined as the following equation. We are interested in the fraction of our balance f_* that maximizes B_m in the equation

$$B_m = B_o \prod_{i=0}^{m}(1 + f_* X_i) \tag{9}$$

where B_m is your final balance, B_o is your initial balance, m is the total number of trials, and X_i is the percent return of the game for the i-th trial.
X_i can take on a finite number of values: 1 through 6. We defined the respective probabilities of receiving each of these values in section 1.

3 Growth Rate

We examine the growth rate in order to optimize the Kelly Criterion. The growth rate is equivalent to the following equation:

$$G = \frac{1}{m}(\log B_m - \log B_o) \tag{10}$$

Substituting B_m for it's equivalent terms defined previously:

$$G = \frac{1}{m}(\log B_o \prod_{i=0}^{m}(1 + f_* X_i) - \log B_o) \tag{11}$$

Distributing the logarithm and simplifying:

$$G = \frac{1}{m}\sum_{i=0}^{m}\log(1 + f_* X_i) \tag{12}$$

Now we must determine our expected growth rate, as we are not privy to the actual results of the random variable X_i but only it's probabilistic distribution.

$$E[G] = \frac{1}{m}\sum_{i=0}^{m}E[\log(1 + f_* X_i)] \tag{13}$$

To estimate $E[\log(1 + f_* X_i)]$ we use the Taylor expansion.

$$\log(1 + x) \approx x - \frac{1}{2}x^2 \tag{14}$$

$$E[\log(1 + f_* X_i)] \approx f_* E[X_i] - \frac{1}{2} f_*^2 E[X_i^2] \tag{15}$$

Substituting the mean u and variance s^2 for the expected values respectively:

$$E[\log(1 + f_* X_i)] \approx f_* u - \frac{1}{2} f_*^2 s^2 \tag{16}$$

Solving for f_* to maximize G, we take the derivative with respect to f_* and set equal to 0.

$$0 = u - f_* s^2 \tag{17}$$

$$f_* = \frac{u}{s^2} \tag{18}$$

Inputting our previously calculated u and s^2 to our equation we see that

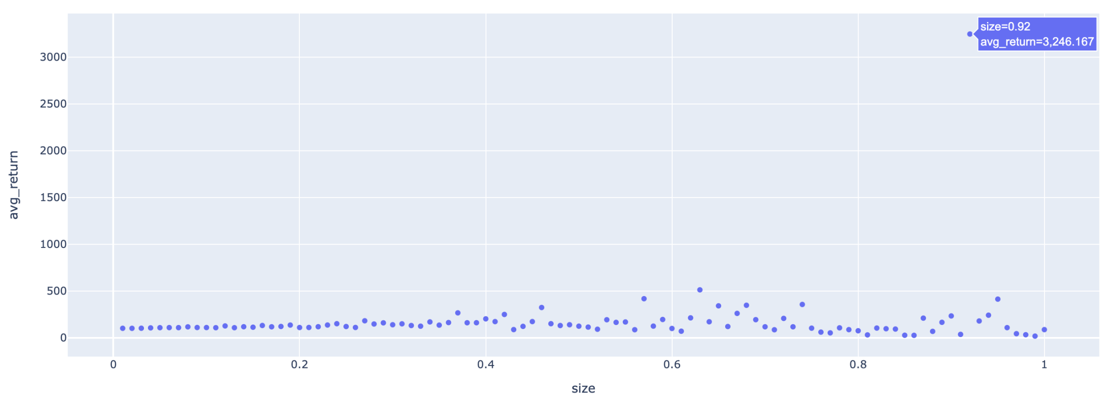
$$f_* = \frac{0.05}{0.054} \tag{19}$$

$$f_* = 0.923 \tag{20}$$

Therefore, the optimal bet sizing for this game is to bet 92% of your balance for each trial.

4 Simulation

In the attached python notebook, a simulated environment is demonstrated where this game takes place. With admitted drawbacks due to hardware restraints, the simulation is able to yield a very similar result to our calculated ideal bet size.



Because of the similarity between simulation and derivation, it can be stated that the optimal bet sizing for this scenario is $\approx$ 92% of the total portfolio size.