Introduction to Random Matrix Theory

Global and Local Behaviour

Ben Rickard

Durham University
Department of Mathematics

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Introduction to Random Matrices

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- Matrices whose elements are randomly sampled from probability distributions
- First studied by Wishart in the 1920s, who was concerned with sample covariance matrices $\frac{1}{N}XX^T$
- More intensive study began in the 1950s by physicist Wigner, motivated by nuclear physics applications
- The theory now branches over several areas of mathematics and physics
- How do their eigenvalues behave?

Defining Wigner Matrices

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Definition

- An $N \times N$ matrix X_N is a **Wigner matrix** if it has entries

$$X_N(i,j) = X_N(j,i) = \begin{cases} \frac{Z_{i,j}}{\sqrt{N}}, & \text{if } i < j, \\ \frac{Y_i}{\sqrt{N}}, & \text{if } i = j, \end{cases}$$

  where there are two families of independent and identically distributed zero-mean real random variables $\{Z_{i,j}\}_{1 \le i < j}$ and $\{Y_i\}_{1 \le i}$, such that $\mathbb{E}[Z_{1,2}^2] = 1$
- Also require finite moments i.e. $r_k := \max\{\mathbb{E}[|Z_{1,2}|^k], \mathbb{E}[|Y_1|^k]\} < \infty$

Example of a Wigner Matrix

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- Taking $Z_{i,j} \sim N(0,1)$ and $Y_i \sim N(0,2)$, the corresponding Wigner matrix X_3 would look like:

$$\frac{1}{\sqrt{3}}\begin{pmatrix} y_1 & z_{1,2} & z_{1,3} \\ z_{1,2} & y_2 & z_{2,3} \\ z_{1,3} & z_{2,3} & y_3 \end{pmatrix}$$

- A realisation of such a Wigner matrix could for example be:

$$\frac{1}{\sqrt{3}}\begin{pmatrix} 0.546 & 1.585 & -0.473 \\ 1.585 & 1.126 & 0.831 \\ -0.473 & 0.831 & -1.129 \end{pmatrix}$$

Key Probability Distributions

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Definition

We define the **empirical distribution** of eigenvalues as the probability measure on $\mathbb{R}$
$L_N = \frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i^N}$, where λ_i^N are such that $\lambda_1^N < \ldots < \lambda_N^N$ are the real ordered eigenvalues

Definition

The **semicircle distribution** is the probability distribution $\sigma(x)\,dx$ on $\mathbb{R}$ with density
$\sigma(x) = \frac{1}{2\pi}\sqrt{4-x^2}\,\mathbf{1}_{\{|x|\leq 2\}}$

Eigenvalue Distribution of Wigner Matrices

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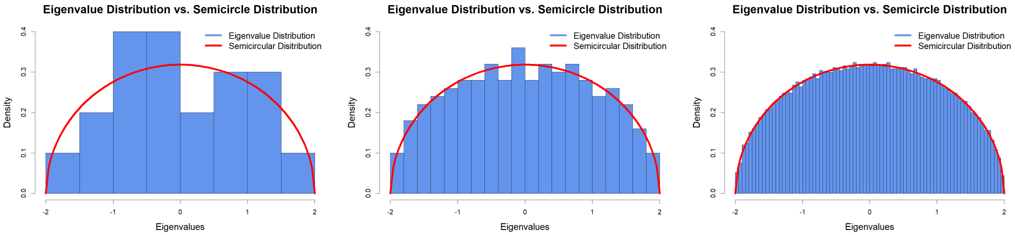
Wigner's Semicircle Theorem

For any Wigner matrix X_N, the empirical distribution L_N converges weakly in probability to the semicircle distribution σ, i.e. for all $f \in C_b(\mathbb{R})$ and for all $\epsilon > 0$,

$$\lim_{N\to\infty} \mathbb{P}\left(\left|\int_{\mathbb{R}} f dL_N - \int_{\mathbb{R}} f d\sigma\right| > \epsilon\right) = 0$$

Eigenvalue Distribution of Wigner Matrices



(a) N=20 (b) N=250 (c) N=5000

Figure 1: Empirical Eigenvalue Distribution Converging to the Semicircle Distribution as N Increases

Largest Eigenvalue

- By considering the empirical eigenvalue density and semicircle distribution for $N = 5000$, we may believe the largest eigenvalue of Wigner matrices concentrates around 2

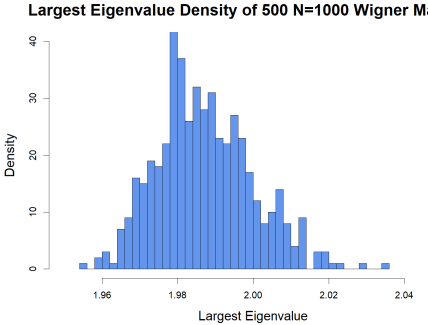


Figure 2: Largest Eigenvalue Density of 500 N=1000 Wigner Matrices Showing Concentration Around 2

Largest Eigenvalue Distribution

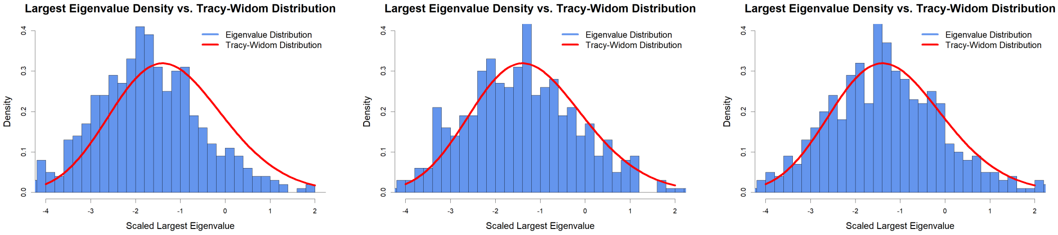
Tracy-Widom Distribution

For all $-\infty < t \leq \infty$,

$$\lim_{N\to\infty} \mathbb{P}(N^{\frac{2}{3}}(\lambda_N^N - 2) \leq t) = \exp\left(-\int_t^{\infty}(x-t)q(x)^2\,dx\right) =: \mathscr{F}(t)$$

- $\mathscr{F}(t)$ is called the Tracy-Widom Distribution
- $q(x)$ is a polynomial which satisfies the ODE $q'' = tq + 2q^3$, and also has certain asymptotic properties

Convergence of Largest Eigenvalue



(a) N=20 (b) N=250 (c) N=1000

Figure 3: Largest Eigenvalue Distribution Converging to the Tracy-Widom Distribution as N Increases

- Note that the Tracy-Widom distribution is asymmetric and skewed to the right

Conclusion

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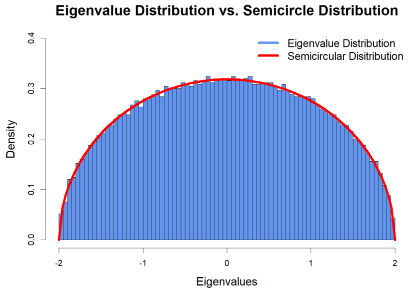
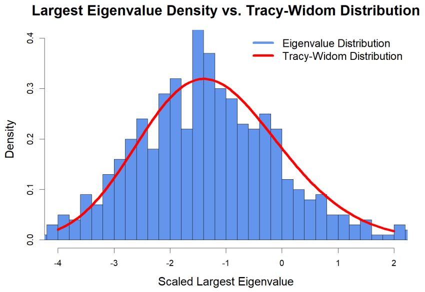
To summarise what we have seen:

- Defined a class of random matrices called the Wigner matrices
- Explored both global and local behaviour of eigenvalues
  - Global: Semicircle Law - one of the most renowned RMT results
  - Local: Tracy-Widom distribution for largest eigenvalue



(a) Semicircle Distribution for $N = 5000$



(b) Tracy-Widom Distribution for $N = 1000$