Exponential Loss:
Output for models $M_0, \ldots, M_N$:

$$f(x) = \begin{bmatrix} 0 \\ \log(K_{0,1}) \\ \log(K_{0,2}) \\ \vdots \\ \log(K_{0,N}) \end{bmatrix}, \quad K_{0,k} = \frac{p(x|M_0)}{p(x|M_k)},$$

Find any log Bayes factor by computing

$$\log(K_{i,j}) = \log(K_{0,j}) - \log(K_{0,i}) = f_j(x) - f_i(x)$$

The generalized exponential loss is

$$\mathscr{L}(f(x), m) = \sum_{i=0,\ i\neq m}^{N} e^{-\frac{1}{2}(f_i(x) - f_m(x))}$$

Where m is the index of the correct model.

Other thoughts:

- Can add square root inside on each exponential term, or outside of the entire sum. Beta distribution tests show better results with square root on inside terms (better spread of Bayes values, square root on outside would often return correct prediction but underestimate).

Logistic Loss:
Output for models $M_0, \ldots, M_N$:

$$f(x) = \begin{bmatrix} 0 \\ \log(K_{0,1}) \\ \log(K_{0,2}) \\ \vdots \\ \log(K_{0,N}) \end{bmatrix}, \quad K_{0,k} = \frac{p(x|M_0)}{p(x|M_k)},$$

Find any log Bayes factor by computing

$$\log(K_{i,j}) = \log(K_{0,j}) - \log(K_{0,i}) = f_j(x) - f_i(x)$$

The generalized exponential loss is

$$\mathscr{L}(f(x), m) = \log\left(1 + \sum_{i=0,\, i\neq m}^{N} e^{f_m(x) - f_i(x)}\right)$$

Where m is the index of the correct model.

Other thoughts:

- Transformation of Exponential loss so it is dependent on what generalized version you use.
- Essentially replaces $\sqrt{x}$ with $\log(1 + x)$, so might be worth having log on each term in the sum instead since the analogous exponential version worked better.

l-POP-Exponential Loss:
Output for models $M_0, \dots, M_N$:

$$f(x) = \begin{bmatrix} 0 \\ \mathcal{J}_\alpha^{-1}\left(\log(K_{0,1})\right) \\ \mathcal{J}_\alpha^{-1}\left(\log(K_{0,2})\right) \\ \vdots \\ \mathcal{J}_\alpha^{-1}\left(\log(K_{0,N})\right) \end{bmatrix}, \quad K_{0,k} = \frac{p(x|M_0)}{p(x|M_k)},$$

Find any log Bayes factor by computing

$$\log(K_{i,j}) = \log(K_{0,j}) - \log(K_{0,i}) = \mathcal{J}_\alpha(f_j(x)) - \mathcal{J}_\alpha(f_i(x))$$

The generalized exponential loss is

$$\mathscr{L}(f(x), m) = \sum_{i=0,\, i\neq m}^{N} e^{-\frac{1}{2}(\mathcal{J}_\alpha(f_i(x)) - \mathcal{J}_\alpha(f_m(x)))}$$

Where m is the index of the correct model.

Other thoughts:

- Transformation of Exponential loss so it is dependent on what generalized version you use.

Exponential Loss Justification:

$$I[f] = \sum_{m=0}^{N} \int \mathscr{L}(f(x), m) p(x, m) dx$$

$$= \int \sum_{m=0}^{N} \sum_{i=0,\ i\neq m}^{N} e^{-\frac{1}{2}(f_i(x) - f_m(x))} p(m|x) p(x) dx$$

Optimized by Euler-Lagrange:

$$\frac{d}{df_j} \left(\sum_{m=0}^{N} \sum_{i=0,\ i\neq m}^{N} e^{-\frac{1}{2}(f_i(x) - f_m(x))} p(m|x) p(x) \right) = 0 \text{ for } j = 0, 1, \ldots, N$$

$$\implies \sum_{i=0, i\neq j}^{N} \frac{1}{2} e^{-\frac{1}{2}(f_i(x) - f_j(x))} p(j|x) p(x) - \sum_{m=0, m\neq j}^{N} \frac{1}{2} e^{-\frac{1}{2}(f_j(x) - f_m(x))} p(m|x) p(x) = 0$$

$$\implies \sum_{i=0, i\neq j}^{N} e^{-\frac{1}{2}(f_i(x) - f_j(x))} p(j|x) = \sum_{m=0, m\neq j}^{N} e^{-\frac{1}{2}(f_j(x) - f_m(x))} p(m|x)$$

$$\implies p(j|x) e^{f_j(x)/2} \sum_{i=0, i\neq j}^{N} e^{-\frac{1}{2} f_i(x)} = e^{-f_j(x)/2} \sum_{m=0, m\neq j}^{N} e^{\frac{1}{2} f_m(x)} p(m|x)$$

$$f_j(x) = \log \left(\frac{\sum_{m=0, m\neq j}^{N} e^{\frac{1}{2} f_m(x)} p(m|x)}{p(j|x) \sum_{i=0, i\neq j}^{N} e^{-\frac{1}{2} f_i(x)}} \right) = \log \left(\frac{p(0|x)}{p(j|x)} \frac{1 + \sum_{m=1, m\neq j}^{N} e^{\frac{1}{2} f_m(x)} \frac{p(m|x)}{p(0|x)}}{1 + \sum_{i=1, i\neq j}^{N} e^{-\frac{1}{2} f_i(x)}} \right)$$

$$= \log(K_{0,j}) + \log \left(\frac{1 + \sum_{m=1, m\neq j}^{N} e^{\frac{1}{2} f_m(x)} \frac{p(m|x)}{p(0|x)}}{1 + \sum_{i=1, i\neq j}^{N} e^{-\frac{1}{2} f_i(x)}} \right)$$

Checking the solution that $f_j(x) = \log(K_{0,j})$ for $j = 0, 1, \ldots, N$, the second term goes to 0, giving us a minimum. The loss function is convex, so since we found this local minimum, it must be the unique global minimum.

Logistic Loss Justification: The transformation function is monotonic increasing, so it achieves the same minimizer as exponential loss.

l-POP Loss Justification: Just change of variables