# Maxwell's equations in 3d FDTD - summerize

## Preliminary considerations

### Normalized fields

$$\begin{aligned} \tilde{E} &= \sqrt{\frac{\varepsilon_0}{\mu_0}} E = \frac{E}{\eta_0} \qquad & \tilde{P} = c_0 P \\ \tilde{D} &= c_0 D & \tilde{J} = c_0 J \end{aligned} \tag{1}$$

### Uniaxial perfectly matched layer (UPML)

$$[S] = \begin{pmatrix} \frac{S_y S_z}{S_x} & 0 & 0 \\ 0 & \frac{S_x S_z}{S_y} & 0 \\ 0 & 0 & \frac{S_x S_y}{S_z} \end{pmatrix} \tag{2}$$

$$\begin{aligned} S_x(x) &= 1 + \frac{\sigma'_x(x)}{j\omega\varepsilon_0} \qquad & \sigma'_x(x) = \frac{\varepsilon_0}{2\Delta \mathrm{t}} \left(\frac{x}{L_x}\right)^3 \\ S_y(y) &= 1 + \frac{\sigma'_y(y)}{j\omega\varepsilon_0} & \sigma'_y(y) = \frac{\varepsilon_0}{2\Delta \mathrm{t}} \left(\frac{y}{L_y}\right)^3 \\ S_z(z) &= 1 + \frac{\sigma'_z(z)}{j\omega\varepsilon_0} & \sigma'_z(z) = \frac{\varepsilon_0}{2\Delta \mathrm{t}} \left(\frac{z}{L_z}\right)^3 \end{aligned} \tag{3}$$

$L_{x,y,z}$ is the length of the PML in the $x, y, z$ directions.

## 3d FDTD

### Evaluate Maxwell's curl equations

$\nabla \times \tilde{E} = -j\frac{\omega}{\eta_0}[S]B \Rightarrow$

$$\begin{aligned} \hat{x} &: \partial_t B_x + \frac{\sigma'_y + \sigma'_z}{\varepsilon_0} B_x + \frac{\sigma'_y \sigma'_z}{(\varepsilon_0)^2} \int_{-\infty}^{t} B_x \, d\tau = -\eta_0 \left[\frac{\partial \tilde{E}_z}{\partial y} - \frac{\partial \tilde{E}_y}{\partial z}\right] - \frac{\eta_0 \sigma'_x}{\varepsilon_0} \int_{-\infty}^{t} \left[\frac{\partial \tilde{E}_z}{\partial y} - \frac{\partial \tilde{E}_y}{\partial z}\right] d\tau \\ \hat{y} &: \partial_t B_y + \frac{\sigma'_x + \sigma'_z}{\varepsilon_0} B_y + \frac{\sigma'_x \sigma'_z}{(\varepsilon_0)^2} \int_{-\infty}^{t} B_y \, d\tau = -\eta_0 \left[\frac{\partial \tilde{E}_x}{\partial z} - \frac{\partial \tilde{E}_z}{\partial x}\right] - \frac{\eta_0 \sigma'_y}{\varepsilon_0 \eta_0} \int_{-\infty}^{t} \left[\frac{\partial \tilde{E}_x}{\partial z} - \frac{\partial \tilde{E}_z}{\partial x}\right] d\tau \\ \hat{z} &: \partial_t B_z + \frac{\sigma'_x + \sigma'_y}{\varepsilon_0} B_z + \frac{\sigma'_x \sigma'_y}{(\varepsilon_0)^2} \int_{-\infty}^{t} B_z \, d\tau = -\eta_0 \left[\frac{\partial \tilde{E}_y}{\partial x} - \frac{\partial \tilde{E}_x}{\partial y}\right] - \frac{\eta_0 \sigma'_z}{\varepsilon_0} \int_{-\infty}^{t} \left[\frac{\partial \tilde{E}_y}{\partial x} - \frac{\partial \tilde{E}_x}{\partial y}\right] d\tau \end{aligned} \tag{4}$$

## Discrete evaluation in Yee grid

Update equation for $B_x$

$$B_x\Big|_{t+\frac{\Delta \mathrm{t}}{2}}^{i,j,k} = m_{B_x1}\Big|^{i,j,k} B_x\Big|_{t-\frac{\Delta \mathrm{t}}{2}}^{i,j,k} + m_{B_x2}\Big|^{i,j,k} C^E{}_X\Big|_{t}^{i,j,k} + m_{B_x3}\Big|^{i,j,k} I_{C^E{}_x}\Big|_{t}^{i,j,k} + m_{B_x4}\Big|^{i,j,k} I_{B_x}\Big|_{t-\frac{\Delta \mathrm{t}}{2}}^{i,j,k} \tag{5}$$

where the coefficients are given by

$$m_{B_x0}\Big|^{i,j,k} = \left(\frac{1}{\Delta \mathrm{t}} + \frac{\sigma^B{}_y\Big|^{i,j,k} + \sigma^B{}_z\Big|^{i,j,k}}{2\varepsilon_0} + \frac{\sigma^B{}_y\Big|^{i,j,k} \sigma^B{}_z\Big|^{i,j,k} \Delta \mathrm{t}}{4(\varepsilon_0)^2}\right)$$

$$m_{B_x1}\Big|^{i,j,k} = \frac{1}{m_{B_x0}\Big|^{i,j,k}} \left(\frac{1}{\Delta \mathrm{t}} - \frac{\sigma^B{}_y\Big|^{i,j,k} + \sigma^B{}_z\Big|^{i,j,k}}{2\varepsilon_0} - \frac{\sigma^B{}_y\Big|^{i,j,k} \sigma^B{}_z\Big|^{i,j,k} \Delta \mathrm{t}}{4(\varepsilon_0)^2}\right)$$

$$m_{B_x2}\Big|^{i,j,k} = \frac{1}{m_{B_x0}\Big|^{i,j,k}} \left(-\eta_0\right)$$

$$m_{B_x3}\Big|^{i,j,k} = \frac{1}{m_{B_x0}\Big|^{i,j,k}} \left(-\frac{\Delta \mathrm{t}\eta_0 \sigma^B{}_x\Big|^{i,j,k}}{\varepsilon_0}\right)$$

$$m_{B_x4}\Big|^{i,j,k} = \frac{1}{m_{B_x0}\Big|^{i,j,k}} \left(-\frac{\sigma^B{}_y\Big|^{i,j,k} \sigma^B{}_z\Big|^{i,j,k} \Delta \mathrm{t}}{(\varepsilon_0)^2}\right)$$

and

$$C^E{}_X\Big|_{t}^{i,j,k} = \left[\frac{1}{\Delta \mathrm{y}}\left(\tilde{E}_z\Big|_{t}^{i,j+1,k} - \tilde{E}_z\Big|_{t}^{i,j,k}\right) - \frac{1}{\Delta \mathrm{z}}\left(\tilde{E}_y\Big|_{t}^{i,j,k+1} - \tilde{E}_y\Big|_{t}^{i,j,k}\right)\right]$$

$$I_{C^E{}_x}\Big|_{t}^{i,j,k} = \sum_{T=0}^{t}\left[\frac{1}{\Delta \mathrm{y}}\left(\tilde{E}_z\Big|_{T}^{i,j+1,k} - \tilde{E}_z\Big|_{T}^{i,j,k}\right) - \frac{1}{\Delta \mathrm{z}}\left(\tilde{E}_y\Big|_{T}^{i,j,k+1} - \tilde{E}_y\Big|_{T}^{i,j,k}\right)\right]$$

$$I_{B_x}\Big|_{t-\frac{\Delta \mathrm{t}}{2}}^{i,j,k} = \sum_{T=\frac{\Delta \mathrm{t}}{2}}^{t-\frac{\Delta \mathrm{t}}{2}} B_x\Big|_{T}^{i,j,k}$$

Update equation for $B_y$

$$B_y\Big|_{t+\frac{\Delta \mathrm{t}}{2}}^{i,j,k} = m_{B_y1}\Big|^{i,j,k} B_y\Big|_{t-\frac{\Delta \mathrm{t}}{2}}^{i,j,k} + m_{B_y2}\Big|^{i,j,k} C^E{}_y\Big|_{t}^{i,j,k} + m_{B_y3}\Big|^{i,j,k} I_{C^E{}_y}\Big|_{t}^{i,j,k} + m_{B_y4}\Big|^{i,j,k} I_{B_y}\Big|_{t-\frac{\Delta \mathrm{t}}{2}}^{i,j,k} \tag{6}$$

where the coefficients are given by

$$m_{B_y0}\Big|^{i,j,k} = \left(\frac{1}{\Delta \mathrm{t}} + \frac{\sigma^B{}_x\Big|^{i,j,k} + \sigma^B{}_z\Big|^{i,j,k}}{2\varepsilon_0} + \frac{\sigma^B{}_x\Big|^{i,j,k} \sigma^B{}_z\Big|^{i,j,k} \Delta \mathrm{t}}{4(\varepsilon_0)^2}\right)$$

$$m_{B_y1}\Big|^{i,j,k} = \frac{1}{m_{B_y0}\Big|^{i,j,k}} \left(\frac{1}{\Delta \mathrm{t}} - \frac{\sigma^B{}_x\Big|^{i,j,k} + \sigma^B{}_z\Big|^{i,j,k}}{2\varepsilon_0} - \frac{\sigma^B{}_x\Big|^{i,j,k} \sigma^B{}_z\Big|^{i,j,k} \Delta \mathrm{t}}{4(\varepsilon_0)^2}\right)$$

$$m_{B_y2}\Big|^{i,j,k} = \frac{1}{m_{B_y0}\big|^{i,j,k}}\left(-\eta_0\right)$$

$$m_{B_y3}\Big|^{i,j,k} = \frac{1}{m_{B_y0}\big|^{i,j,k}}\left(-\frac{\Delta \mathrm{t}\eta_0{\sigma^B}_y\big|^{i,j,k}}{\varepsilon_0}\right)$$

$$m_{B_y4}\Big|^{i,j,k} = \frac{1}{m_{B_y0}\big|^{i,j,k}}\left(-\frac{{\sigma^B}_x\big|^{i,j,k}{\sigma^B}_z\big|^{i,j,k}\Delta \mathrm{t}}{(\varepsilon_0)^2}\right)$$

and

$${C^E}_y\Big|_t^{i,j,k} = \left[\frac{1}{\Delta \mathrm{z}}\left(\tilde{E}_x\Big|_t^{i,j,k+1} - \tilde{E}_x\Big|_t^{i,j,k}\right) - \frac{1}{\Delta \mathrm{x}}\left(\tilde{E}_z\Big|_t^{i+1,j,k} - \tilde{E}_z\Big|_t^{i,j,k}\right)\right]$$

$$I_{{C^E}_y}\Big|_t^{i,j,k} = \sum_{T=0}^{t}\left[\frac{1}{\Delta \mathrm{z}}\left(\tilde{E}_x\Big|_T^{i,j,k+1} - \tilde{E}_x\Big|_T^{i,j,k}\right) - \frac{1}{\Delta \mathrm{x}}\left(\tilde{E}_z\Big|_T^{i+1,j,k} - \tilde{E}_z\Big|_T^{i,j,k}\right)\right]$$

$$I_{B_y}\Big|_{t-\frac{\Delta \mathrm{t}}{2}}^{i,j,k} = \sum_{T=\frac{\Delta \mathrm{t}}{2}}^{t-\frac{\Delta \mathrm{t}}{2}} B_y\Big|_T^{i,j,k}$$

Update equation for $B_z$

$$B_z\Big|_{t+\frac{\Delta \mathrm{t}}{2}}^{i,j,k} = m_{B_z1}\Big|^{i,j,k} B_z\Big|_{t-\frac{\Delta \mathrm{t}}{2}}^{i,j,k} + m_{B_z2}\Big|^{i,j,k} {C^E}_z\Big|_t^{i,j,k} + m_{B_z3}\Big|^{i,j,k} I_{{C^E}_z}\Big|_t^{i,j,k} + m_{B_z4}\Big|^{i,j,k} I_{B_z}\Big|_{t-\frac{\Delta \mathrm{t}}{2}}^{i,j,k} \tag{7}$$

where the coefficients are given by

$$m_{B_z0}\Big|^{i,j,k} = \left(\frac{1}{\Delta \mathrm{t}} + \frac{{\sigma^B}_x\big|^{i,j,k} + {\sigma^B}_y\big|^{i,j,k}}{2\varepsilon_0} + \frac{{\sigma^B}_x\big|^{i,j,k}{\sigma^B}_y\big|^{i,j,k}\Delta \mathrm{t}}{4(\varepsilon_0)^2}\right)$$

$$m_{B_z1}\Big|^{i,j,k} = \frac{1}{m_{B_z0}\big|^{i,j,k}}\left(\frac{1}{\Delta \mathrm{t}} - \frac{{\sigma^B}_x\big|^{i,j,k} + {\sigma^B}_y\big|^{i,j,k}}{2\varepsilon_0} - \frac{{\sigma^B}_x\big|^{i,j,k}{\sigma^B}_y\big|^{i,j,k}\Delta \mathrm{t}}{4(\varepsilon_0)^2}\right)$$

$$m_{B_z2}\Big|^{i,j,k} = \frac{1}{m_{B_z0}\big|^{i,j,k}}\left(-\eta_0\right)$$

$$m_{B_z3}\Big|^{i,j,k} = \frac{1}{m_{B_z0}\big|^{i,j,k}}\left(-\frac{\Delta \mathrm{t}\eta_0{\sigma^B}_z\big|^{i,j,k}}{\varepsilon_0}\right)$$

$$m_{B_z4}\Big|^{i,j,k} = \frac{1}{m_{B_z0}\big|^{i,j,k}}\left(-\frac{{\sigma^B}_x\big|^{i,j,k}{\sigma^B}_y\big|^{i,j,k}\Delta \mathrm{t}}{(\varepsilon_0)^2}\right)$$

and

$${C^E}_z\Big|_t^{i,j,k} = \left[\frac{1}{\Delta \mathrm{x}}\left(\tilde{E}_y\Big|_T^{i+1,j,k} - \tilde{E}_y\Big|_T^{i,j,k}\right) - \frac{1}{\Delta \mathrm{y}}\left(\tilde{E}_x\Big|_T^{i,j+1,k} - \tilde{E}_x\Big|_T^{i,j,k}\right)\right]$$

$$I_{{C^E}_z}\Big|_t^{i,j,k} = \sum_{T=0}^{t}\left[\frac{1}{\Delta \mathrm{x}}\left(\tilde{E}_y\Big|_T^{i+1,j,k} - \tilde{E}_y\Big|_T^{i,j,k}\right) - \frac{1}{\Delta \mathrm{y}}\left(\tilde{E}_x\Big|_T^{i,j+1,k} - \tilde{E}_x\Big|_T^{i,j,k}\right)\right]$$

$$I_{B_z}\Big|_{t-\frac{\Delta \mathrm{t}}{2}}^{i,j,k} = \sum_{T=\frac{\Delta \mathrm{t}}{2}}^{t-\frac{\Delta \mathrm{t}}{2}} B_z\Big|_T^{i,j,k}$$

$$\nabla \times H = j\frac{\omega}{c_0}[S]\tilde{D} \Rightarrow$$

$$\begin{aligned}
\hat{x} &: \partial_t \tilde{D}_x + \frac{\sigma'_y+\sigma'_z}{\varepsilon_0}\tilde{D}_x + \frac{\sigma'_y\sigma'_z}{(\varepsilon_0)^2}\int_{-\infty}^{t}\tilde{D}_x d\tau = c_0\left[\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right] + \frac{c_0\sigma'_x}{\varepsilon_0}\int_{-\infty}^{t}\left[\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right]d\tau \\
\hat{y} &: \partial_t \tilde{D}_y + \frac{\sigma'_x+\sigma'_z}{\varepsilon_0}\tilde{D}_y + \frac{\sigma'_x\sigma'_z}{(\varepsilon_0)^2}\int_{-\infty}^{t}\tilde{D}_y d\tau = c_0\left[\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right] + \frac{c_0\sigma'_y}{\varepsilon_0}\int_{-\infty}^{t}\left[\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right]d\tau \\
\hat{z} &: \partial_t \tilde{D}_z + \frac{\sigma'_x+\sigma'_y}{\varepsilon_0}\tilde{D}_z + \frac{\sigma'_x\sigma'_y}{(\varepsilon_0)^2}\int_{-\infty}^{t}\tilde{D}_z d\tau = c_0\left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right] + \frac{c_0\sigma'_z}{\varepsilon_0}\int_{-\infty}^{t}\left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right]d\tau
\end{aligned} \tag{8}$$

**Discrete evaluation in Yee grid**

Update equation for $\tilde{D}_x$

$$\tilde{D}_x\Big|_{t+\Delta t}^{i,j,k} = m_{D_x1}\Big|^{i,j,k}\tilde{D}_x\Big|_{t}^{i,j,k} + m_{D_x2}\Big|^{i,j,k} C^H{}_X\Big|_{t+\frac{\Delta t}{2}}^{i,j,k} + m_{D_x3}\Big|^{i,j,k} I_{C^H{}_x}\Big|_{t-\frac{\Delta t}{2}}^{i,j,k} + m_{D_x4}\Big|^{i,j,k} I_{D_x}\Big|_{t}^{i,j,k} \tag{9}$$

where the coefficients are given by

$$m_{D_x0}\Big|^{i,j,k} = \left(\frac{1}{\Delta t} + \frac{\sigma^D{}_y\Big|^{i,j,k} + \sigma^D{}_z\Big|^{i,j,k}}{2\varepsilon_0} + \frac{\sigma^D{}_y\Big|^{i,j,k}\sigma^D{}_z\Big|^{i,j,k}\Delta t}{4(\varepsilon_0)^2}\right)$$

$$m_{D_x1}\Big|^{i,j,k} = \frac{1}{m_{D_x0}\Big|^{i,j,k}}\left(\frac{1}{\Delta t} - \frac{\sigma^D{}_y\Big|^{i,j,k} + \sigma^D{}_z\Big|^{i,j,k}}{2\varepsilon_0} - \frac{\sigma^D{}_y\Big|^{i,j,k}\sigma^D{}_z\Big|^{i,j,k}\Delta t}{4(\varepsilon_0)^2}\right)$$

$$m_{D_x2}\Big|^{i,j,k} = \frac{1}{m_{D_x0}\Big|^{i,j,k}}(c_0)$$

$$m_{D_x3}\Big|^{i,j,k} = \frac{1}{m_{D_x0}\Big|^{i,j,k}}\left(\frac{c_0\Delta t\sigma^D{}_x\Big|^{i,j,k}}{\varepsilon_0}\right)$$

$$m_{D_x4}\Big|^{i,j,k} = \frac{1}{m_{D_x0}\Big|^{i,j,k}}\left(-\frac{\sigma^D{}_y\Big|^{i,j,k}\sigma^D{}_z\Big|^{i,j,k}\Delta t}{(\varepsilon_0)^2}\right)$$

and

$$C^H{}_X\Big|_{t+\frac{\Delta t}{2}}^{i,j,k} = \left[\frac{1}{\Delta y}\left(H_z\Big|_{t+\frac{\Delta t}{2}}^{i,j,k} - H_z\Big|_{t+\frac{\Delta t}{2}}^{i,j-1,k}\right) - \frac{1}{\Delta z}\left(H_y\Big|_{t+\frac{\Delta t}{2}}^{i,j,k} - H_y\Big|_{t+\frac{\Delta t}{2}}^{i,j,k-1}\right)\right]$$

$$I_{C^H{}_x}\Big|_{t-\frac{\Delta t}{2}}^{i,j,k} = \sum_{T=\frac{\Delta t}{2}}^{t-\frac{\Delta t}{2}}\left[\frac{1}{\Delta y}\left(H_z\Big|_{t+\frac{\Delta t}{2}}^{i,j,k} - H_z\Big|_{t+\frac{\Delta t}{2}}^{i,j-1,k}\right) - \frac{1}{\Delta z}\left(H_y\Big|_{t+\frac{\Delta t}{2}}^{i,j,k} - H_y\Big|_{t+\frac{\Delta t}{2}}^{i,j,k-1}\right)\right]$$

$$I_{D_x}\Big|_{t}^{i,j,k} = \sum_{T=0}^{t}\tilde{D}_x\Big|_{T}^{i,j,k}$$

Update equation for $\tilde{D}_y$

$$\tilde{D}_y\Big|_{t+\Delta t}^{i,j,k} = m_{D_y1}\Big|^{i,j,k}\tilde{D}_y\Big|_{t}^{i,j,k} + m_{D_y2}\Big|^{i,j,k} C^H{}_y\Big|_{t+\frac{\Delta t}{2}}^{i,j,k} + m_{D_y3}\Big|^{i,j,k} I_{C^H{}_y}\Big|_{t-\frac{\Delta t}{2}}^{i,j,k} + m_{D_y4}\Big|^{i,j,k} I_{D_y}\Big|_{t}^{i,j,k} \tag{10}$$

where the coefficients are given by

$$m_{D_y 0}\Big|^{i,j,k} = \left( \frac{1}{\Delta\mathrm{t}} + \frac{{\sigma^{D}}_{x}\Big|^{i,j,k} + {\sigma^{D}}_{z}\Big|^{i,j,k}}{2\varepsilon_0} + \frac{{\sigma^{D}}_{x}\Big|^{i,j,k} {\sigma^{D}}_{z}\Big|^{i,j,k} \Delta\mathrm{t}}{4(\varepsilon_0)^2} \right)$$

$$m_{D_y 1}\Big|^{i,j,k} = \frac{1}{m_{D_y 0}\Big|^{i,j,k}} \left( \frac{1}{\Delta\mathrm{t}} - \frac{{\sigma^{D}}_{x}\Big|^{i,j,k} + {\sigma^{D}}_{z}\Big|^{i,j,k}}{2\varepsilon_0} - \frac{{\sigma^{D}}_{x}\Big|^{i,j,k} {\sigma^{D}}_{z}\Big|^{i,j,k} \Delta\mathrm{t}}{4(\varepsilon_0)^2} \right)$$

$$m_{D_y 2}\Big|^{i,j,k} = \frac{1}{m_{D_y 0}\Big|^{i,j,k}} (c_0)$$

$$m_{D_y 3}\Big|^{i,j,k} = \frac{1}{m_{D_y 0}\Big|^{i,j,k}} \left( \frac{c_0 \Delta\mathrm{t} {\sigma^{D}}_{y}\Big|^{i,j,k}}{\varepsilon_0} \right)$$

$$m_{D_y 4}\Big|^{i,j,k} = \frac{1}{m_{D_y 0}\Big|^{i,j,k}} \left( - \frac{{\sigma^{D}}_{x}\Big|^{i,j,k} {\sigma^{D}}_{z}\Big|^{i,j,k} \Delta\mathrm{t}}{(\varepsilon_0)^2} \right)$$

and

$$ {C^{H}}_{y}\Big|^{i,j,k}_{t+\frac{\Delta\mathrm{t}}{2}} = \left[ \frac{1}{\Delta\mathrm{z}} \left( H_x\Big|^{i,j,k}_{t+\frac{\Delta\mathrm{t}}{2}} - H_x\Big|^{i,j,k-1}_{t+\frac{\Delta\mathrm{t}}{2}} \right) - \frac{1}{\Delta\mathrm{x}} \left( H_z\Big|^{i,j,k}_{t+\frac{\Delta\mathrm{t}}{2}} - H_z\Big|^{i-1,j,k}_{t+\frac{\Delta\mathrm{t}}{2}} \right) \right]$$

$$ I_{{C^{H}}_{y}}\Big|^{i,j,k}_{t-\frac{\Delta\mathrm{t}}{2}} = \sum_{T=\frac{\Delta\mathrm{t}}{2}}^{t-\frac{\Delta\mathrm{t}}{2}} \left[ \frac{1}{\Delta\mathrm{z}} \left( H_x\Big|^{i,j,k}_{t+\frac{\Delta\mathrm{t}}{2}} - H_x\Big|^{i,j,k-1}_{t+\frac{\Delta\mathrm{t}}{2}} \right) - \frac{1}{\Delta\mathrm{x}} \left( H_z\Big|^{i,j,k}_{t+\frac{\Delta\mathrm{t}}{2}} - H_z\Big|^{i-1,j,k}_{t+\frac{\Delta\mathrm{t}}{2}} \right) \right]$$

$$ I_{D_y}\Big|^{i,j,k}_{t} = \sum_{T=0}^{t} \tilde{D}_y\Big|^{i,j,k}_{T}$$

Update equation for $\tilde{D}_z$

$$\tilde{D}_z\Big|^{i,j,k}_{t+\Delta\mathrm{t}} = m_{D_z 1}\Big|^{i,j,k} \tilde{D}_z\Big|^{i,j,k}_{t} + m_{D_z 2}\Big|^{i,j,k} {C^{H}}_{z}\Big|^{i,j,k}_{t+\frac{\Delta\mathrm{t}}{2}} + m_{D_z 3}\Big|^{i,j,k} I_{{C^{H}}_{z}}\Big|^{i,j,k}_{t-\frac{\Delta\mathrm{t}}{2}} + m_{D_z 4}\Big|^{i,j,k} I_{D_z}\Big|^{i,j,k}_{t} \tag{11}$$

where the coefficients are given by

$$m_{D_z 0}\Big|^{i,j,k} = \left( \frac{1}{\Delta\mathrm{t}} + \frac{{\sigma^{D}}_{x}\Big|^{i,j,k} + {\sigma^{D}}_{y}\Big|^{i,j,k}}{2\varepsilon_0} + \frac{{\sigma^{D}}_{x}\Big|^{i,j,k} {\sigma^{D}}_{y}\Big|^{i,j,k} \Delta\mathrm{t}}{4(\varepsilon_0)^2} \right)$$

$$m_{D_z 1}\Big|^{i,j,k} = \frac{1}{m_{D_z 0}\Big|^{i,j,k}} \left( \frac{1}{\Delta\mathrm{t}} + \frac{{\sigma^{D}}_{x}\Big|^{i,j,k} + {\sigma^{D}}_{y}\Big|^{i,j,k}}{2\varepsilon_0} + \frac{{\sigma^{D}}_{x}\Big|^{i,j,k} {\sigma^{D}}_{y}\Big|^{i,j,k} \Delta\mathrm{t}}{4(\varepsilon_0)^2} \right)$$

$$m_{D_z 2}\Big|^{i,j,k} = \frac{1}{m_{D_z 0}\Big|^{i,j,k}} (c_0)$$

$$m_{D_z 3}\Big|^{i,j,k} = \frac{1}{m_{D_z 0}\Big|^{i,j,k}} \left( \frac{c_0 \Delta\mathrm{t} {\sigma^{D}}_{z}\Big|^{i,j,k}}{\varepsilon_0} \right)$$

$$m_{D_z 4}\Big|^{i,j,k} = \frac{1}{m_{D_z 0}\Big|^{i,j,k}} \left( - \frac{{\sigma^{D}}_{x}\Big|^{i,j,k} {\sigma^{D}}_{y}\Big|^{i,j,k} \Delta\mathrm{t}}{(\varepsilon_0)^2} \right)$$

and

$$ {C^{H}}_{z}\Big|^{i,j,k}_{t+\frac{\Delta\mathrm{t}}{2}} = \left[ \frac{1}{\Delta\mathrm{x}} \left( H_y\Big|^{i,j,k}_{t+\frac{\Delta\mathrm{t}}{2}} - H_y\Big|^{i-1,j,k}_{t+\frac{\Delta\mathrm{t}}{2}} \right) - \frac{1}{\Delta\mathrm{y}} \left( H_x\Big|^{i,j,k}_{t+\frac{\Delta\mathrm{t}}{2}} - H_x\Big|^{i,j-1,k}_{t+\frac{\Delta\mathrm{t}}{2}} \right) \right]$$

$$I_{C^H{}_z}\Big|_{t-\frac{\Delta t}{2}}^{i,j,k} = \sum_{T=\frac{\Delta t}{2}}^{t-\frac{\Delta t}{2}} \left[\frac{1}{\Delta x}\left(H_y\Big|_{t+\frac{\Delta t}{2}}^{i,j,k} - H_y\Big|_{t+\frac{\Delta t}{2}}^{i-1,j,k}\right) - \frac{1}{\Delta y}\left(H_x\Big|_{t+\frac{\Delta t}{2}}^{i,j,k} - H_x\Big|_{t+\frac{\Delta t}{2}}^{i,j-1,k}\right)\right]$$
$$I_{D_z}\Big|_{t}^{i,j,k} = \sum_{T=0}^{t} \tilde{D}_z\Big|_{T}^{i,j,k}$$

# Constitutive relations

## Dispersive electric materials

$$\tilde{D}(\omega) = \varepsilon_r(\omega)\tilde{E}(\omega) = (\varepsilon_\infty + \chi_e(\omega))\,\tilde{E}(\omega) = \varepsilon_\infty\tilde{E}(\omega) + \tilde{P}(\omega) \tag{12}$$

$$\varepsilon_r(\omega) = \varepsilon_\infty + \frac{{\omega_p}^2}{{\omega_0}^2 - \omega^2 + j\omega\Gamma} \tag{13}$$

$$\chi_e(\omega) = \frac{{\omega_p}^2}{{\omega_0}^2 - \omega^2 + j\omega\Gamma} \tag{14}$$

$$\tilde{P}(\omega) = {\omega_p}^2\tilde{E}(\omega)\sum_{m=1} \frac{f_m}{{\omega_{0,m}}^2 - \omega^2 + j\omega\Gamma_m} \tag{15}$$

For a single resonance

$\mathrm{FT}^{-1}\left\{{\omega_{0,m}}^2\tilde{P}_m + (\mathrm{j}\omega)^2\tilde{P}_m + \mathrm{j}\omega{\Gamma}_m\tilde{P}_m = {\omega_p}^2 f_m\tilde{E}\right\} \Rightarrow$

$${\omega_{0,m}}^2\tilde{P}_m + \partial_t\tilde{J}_m + \Gamma_m\tilde{J}_m = {\omega_p}^2 f_m\tilde{E} \tag{16}$$

$$\tilde{J}_m = \partial_t\tilde{P}_m \tag{17}$$

### Discrete evaluation in Yee grid

Update equation for $\tilde{P}_m$

$$\begin{aligned}
\tilde{P}_{x,m}\Big|_{t+\Delta t}^{i,j,k} &= \tilde{P}_{x,m}\Big|_{t}^{i,j,k} + \Delta t\tilde{J}_{x,m}\Big|_{t+\frac{\Delta t}{2}}^{i,j,k}\\
\tilde{P}_{y,m}\Big|_{t+\Delta t}^{i,j,k} &= \tilde{P}_{y,m}\Big|_{t}^{i,j,k} + \Delta t\tilde{J}_{y,m}\Big|_{t+\frac{\Delta t}{2}}^{i,j,k}\\
\tilde{P}_{z,m}\Big|_{t+\Delta t}^{i,j,k} &= \tilde{P}_{z,m}\Big|_{t}^{i,j,k} + \Delta t\tilde{J}_{z,m}\Big|_{t+\frac{\Delta t}{2}}^{i,j,k}
\end{aligned} \tag{18}$$

Update equation for $\tilde{J}_{x,m}$

$$\tilde{J}_{x,m}\Big|_{t+\frac{\Delta t}{2}}^{i,j,k} = m_{J_{x,m}1}\Big|^{i,j,k}\tilde{J}_{x,m}\Big|_{t-\frac{\Delta t}{2}}^{i,j,k} + m_{J_{x,m}2}\Big|^{i,j,k}\tilde{E}_{x,m}\Big|_{t}^{i,j,k} + m_{J_{x,m}3}\Big|^{i,j,k}\tilde{P}_{x,m}\Big|_{t}^{i,j,k}$$

where the coefficients are given by

$$m_{J_{x,m}0}\Big|^{i,j,k} = \left(\frac{1}{\Delta \mathrm{t}} + \frac{\Gamma_m\big|^{i,j,k}}{2}\right)$$

$$m_{J_{x,m}1}\Big|^{i,j,k} = \frac{1}{m_{J_{x,m}0}\big|^{i,j,k}}\left(\frac{1}{\Delta \mathrm{t}} - \frac{\Gamma_m\big|^{i,j,k}}{2}\right)$$

$$m_{J_{x,m}2}\Big|^{i,j,k} = \frac{1}{m_{J_{x,m}0}\big|^{i,j,k}}\left({\omega_p}^2\Big|^{i,j,k} f_m\Big|^{i,j,k}\right)$$

$$m_{J_{x,m}3}\Big|^{i,j,k} = \frac{1}{m_{J_{x,m}0}\big|^{i,j,k}}\left(-{\omega_{0,m}}^2\Big|^{i,j,k}\right)$$

Update equation for $\tilde{J}_{y,m}$

$$\tilde{J}_{y,m}\Big|^{i,j,k}_{t+\frac{\Delta \mathrm{t}}{2}} = m_{J_{y,m}1}\Big|^{i,j,k} \tilde{J}_{y,m}\Big|^{i,j,k}_{t-\frac{\Delta \mathrm{t}}{2}} + m_{J_{y,m}2}\Big|^{i,j,k} \tilde{E}_{y,m}\Big|^{i,j,k}_{t} + m_{J_{y,m}3}\Big|^{i,j,k} \tilde{P}_{y,m}\Big|^{i,j,k}_{t}$$

where the coefficients are given by

$$m_{J_{y,m}0}\Big|^{i,j,k} = \left(\frac{1}{\Delta \mathrm{t}} + \frac{\Gamma_m\big|^{i,j,k}}{2}\right)$$

$$m_{J_{y,m}1}\Big|^{i,j,k} = \frac{1}{m_{J_{y,m}0}\big|^{i,j,k}}\left(\frac{1}{\Delta \mathrm{t}} - \frac{\Gamma_m\big|^{i,j,k}}{2}\right)$$

$$m_{J_{y,m}2}\Big|^{i,j,k} = \frac{1}{m_{J_{y,m}0}\big|^{i,j,k}}\left({\omega_p}^2\Big|^{i,j,k} f_m\Big|^{i,j,k}\right)$$

$$m_{J_{y,m}3}\Big|^{i,j,k} = \frac{1}{m_{J_{y,m}0}\big|^{i,j,k}}\left(-{\omega_{0,m}}^2\Big|^{i,j,k}\right)$$

Update equation for $\tilde{J}_{z,m}$

$$\tilde{J}_{z,m}\Big|^{i,j,k}_{t+\frac{\Delta \mathrm{t}}{2}} = m_{J_{z,m}1}\Big|^{i,j,k} \tilde{J}_{z,m}\Big|^{i,j,k}_{t-\frac{\Delta \mathrm{t}}{2}} + m_{J_{z,m}2}\Big|^{i,j,k} \tilde{E}_{z,m}\Big|^{i,j,k}_{t} + m_{J_{z,m}3}\Big|^{i,j,k} \tilde{P}_{z,m}\Big|^{i,j,k}_{t}$$

where the coefficients are given by

$$m_{J_{z,m}0}\Big|^{i,j,k} = \left(\frac{1}{\Delta \mathrm{t}} + \frac{\Gamma_m\big|^{i,j,k}}{2}\right)$$

$$m_{J_{z,m}1}\Big|^{i,j,k} = \frac{1}{m_{J_{z,m}0}\big|^{i,j,k}}\left(\frac{1}{\Delta \mathrm{t}} - \frac{\Gamma_m\big|^{i,j,k}}{2}\right)$$

$$m_{J_{z,m}2}\Big|^{i,j,k} = \frac{1}{m_{J_{z,m}0}\big|^{i,j,k}}\left({\omega_p}^2\Big|^{i,j,k} f_m\Big|^{i,j,k}\right)$$

$$m_{J_{z,m}3}\Big|^{i,j,k} = \frac{1}{m_{J_{z,m}0}\big|^{i,j,k}}\left(-{\omega_{0,m}}^2\Big|^{i,j,k}\right)$$

Update equation for $\tilde{E}$

$$
\begin{aligned}
\tilde{E}_x\Big|_{t}^{i,j,k} &= \frac{1}{\varepsilon_\infty\Big|^{i,j,k}}\left(\tilde{D}_x\Big|_{t}^{i,j,k} - \sum_m \tilde{P}_{x,m}\Big|_{t}^{i,j,k}\right)\\
\tilde{E}_y\Big|_{t}^{i,j,k} &= \frac{1}{\varepsilon_\infty\Big|^{i,j,k}}\left(\tilde{D}_y\Big|_{t}^{i,j,k} - \sum_m \tilde{P}_{y,m}\Big|_{t}^{i,j,k}\right)\\
\tilde{E}_z\Big|_{t}^{i,j,k} &= \frac{1}{\varepsilon_\infty\Big|^{i,j,k}}\left(\tilde{D}_z\Big|_{t}^{i,j,k} - \sum_m \tilde{P}_{z,m}\Big|_{t}^{i,j,k}\right)
\end{aligned}
\tag{19}
$$

## Dispersive magnetic materials

$$
B(\omega) = \mu_0\mu_r(\omega)H(\omega) = \mu_0\left(\mu_\infty + \chi_M(\omega)\right)H(\omega) = \mu_0\mu_\infty H(\omega) + M(\omega) \tag{20}
$$

$$
\mu_r(\omega) = \mu_\infty + \frac{{\omega_{\mathrm{Mp}}}^2}{{\omega_{\mathrm{M0}}}^2 - \omega^2 + j\omega\Gamma_M} \tag{21}
$$

$$
\chi_M(\omega) = \frac{{\omega_{\mathrm{Mp}}}^2}{{\omega_{\mathrm{M0}}}^2 - \omega^2 + j\omega\Gamma_M} \tag{22}
$$

$$
M(\omega) = \mu_0{\omega_{\mathrm{Mp}}}^2 H(\omega)\sum_{m=1}\frac{f_{M,m}}{{\omega_{\mathrm{M0},m}}^2 - \omega^2 + j\omega\Gamma_{M,m}} \tag{23}
$$

For a single resonance

$$
\mathrm{FT}^{-1}\left\{{\omega_{\mathrm{M0},m}}^2 M_m + (\mathrm{j}\omega)^2 M_m + \mathrm{j}\omega\Gamma_{M,m}M_m = \mu_0{\omega_{\mathrm{Mp}}}^2 f_{M,m}H\right\} \Rightarrow \tag{24}
$$

$$
{\omega_{\mathrm{M0},m}}^2 M_m + \partial_t J_{M,m} + \Gamma_{M,m}J_{M,m} = {\omega_{\mathrm{Mp}}}^2 f_{M,m}H \tag{25}
$$

$$
J_{M,m} = \partial_t M_m \tag{26}
$$

### Discrete evaluation in Yee grid

Update equation for $M_m$

$$
\begin{aligned}
M_{x,m}\Big|_{t+\frac{\Delta\mathrm{t}}{2}}^{i,j,k} &= M_{x,m}\Big|_{t-\frac{\Delta\mathrm{t}}{2}}^{i,j,k} + \Delta\mathrm{t}\mathrm{J}_{\mathrm{Mx},m}\Big|_{t}^{i,j,k}\\
M_{y,m}\Big|_{t+\frac{\Delta\mathrm{t}}{2}}^{i,j,k} &= M_{y,m}\Big|_{t-\frac{\Delta\mathrm{t}}{2}}^{i,j,k} + \Delta\mathrm{t}\mathrm{J}_{\mathrm{My},m}\Big|_{t}^{i,j,k}\\
M_{z,m}\Big|_{t+\frac{\Delta\mathrm{t}}{2}}^{i,j,k} &= M_{z,m}\Big|_{t-\frac{\Delta\mathrm{t}}{2}}^{i,j,k} + \Delta\mathrm{t}\mathrm{J}_{\mathrm{Mz},m}\Big|_{t}^{i,j,k}
\end{aligned}
\tag{27}
$$

Update equation for $J_{\mathrm{Mx},m}$

$$J_{\mathrm{Mx},m}\Big|_{t+\Delta\mathrm{t}}^{i,j,k} = m_{J_{\mathrm{Mx},m}1}\Big|^{i,j,k} J_{\mathrm{Mx},m}\Big|_{t}^{i,j,k} + m_{J_{\mathrm{Mx},m}2}\Big|^{i,j,k} H_{x,m}\Big|_{t+\frac{\Delta\mathrm{t}}{2}}^{i,j,k} + m_{J_{\mathrm{Mx},m}3}\Big|^{i,j,k} M_{x,m}\Big|_{t+\frac{\Delta\mathrm{t}}{2}}^{i,j,k}$$

where the coefficients are given by

$$m_{J_{\mathrm{Mx},m}0}\Big|^{i,j,k} = \left(\frac{1}{\Delta\mathrm{t}} + \frac{\Gamma_{M,m}\big|^{i,j,k}}{2}\right)$$

$$m_{J_{\mathrm{Mx},m}1}\Big|^{i,j,k} = \frac{1}{m_{J_{\mathrm{Mx},m}0}\big|^{i,j,k}}\left(\frac{1}{\Delta\mathrm{t}} - \frac{\Gamma_{M,m}\big|^{i,j,k}}{2}\right)$$

$$m_{J_{\mathrm{Mx},m}2}\Big|^{i,j,k} = \frac{1}{m_{J_{\mathrm{Mx},m}0}\big|^{i,j,k}}\left({\omega_{\mathrm{Mp}}}^2\Big|^{i,j,k} f_{M,m}\Big|^{i,j,k}\right)$$

$$m_{J_{\mathrm{Mx},m}3}\Big|^{i,j,k} = \frac{1}{m_{J_{\mathrm{Mx},m}0}\big|^{i,j,k}}\left(-{\omega_{0M,m}}^2\Big|^{i,j,k}\right)$$

Update equation for $J_{\mathrm{My},m}$

$$J_{\mathrm{My},m}\Big|_{t+\Delta\mathrm{t}}^{i,j,k} = m_{J_{\mathrm{My},m}1}\Big|^{i,j,k} J_{\mathrm{My},m}\Big|_{t}^{i,j,k} + m_{J_{\mathrm{My},m}2}\Big|^{i,j,k} H_{y,m}\Big|_{t+\frac{\Delta\mathrm{t}}{2}}^{i,j,k} + m_{J_{\mathrm{My},m}3}\Big|^{i,j,k} M_{y,m}\Big|_{t+\frac{\Delta\mathrm{t}}{2}}^{i,j,k}$$

where the coefficients are given by

$$m_{J_{\mathrm{My},m}0}\Big|^{i,j,k} = \left(\frac{1}{\Delta\mathrm{t}} + \frac{\Gamma_{M,m}\big|^{i,j,k}}{2}\right)$$

$$m_{J_{\mathrm{My},m}1}\Big|^{i,j,k} = \frac{1}{m_{J_{\mathrm{My},m}0}\big|^{i,j,k}}\left(\frac{1}{\Delta\mathrm{t}} - \frac{\Gamma_{M,m}\big|^{i,j,k}}{2}\right)$$

$$m_{J_{\mathrm{My},m}2}\Big|^{i,j,k} = \frac{1}{m_{J_{\mathrm{My},m}0}\big|^{i,j,k}}\left({\omega_{\mathrm{Mp}}}^2\Big|^{i,j,k} f_{M,m}\Big|^{i,j,k}\right)$$

$$m_{J_{\mathrm{My},m}3}\Big|^{i,j,k} = \frac{1}{m_{J_{\mathrm{My},m}0}\big|^{i,j,k}}\left(-{\omega_{0M,m}}^2\Big|^{i,j,k}\right)$$

Update equation for $J_{\mathrm{Mz},m}$

$$J_{\mathrm{Mz},m}\Big|_{t+\Delta\mathrm{t}}^{i,j,k} = m_{J_{\mathrm{Mz},m}1}\Big|^{i,j,k} J_{\mathrm{Mz},m}\Big|_{t}^{i,j,k} + m_{J_{\mathrm{Mz},m}2}\Big|^{i,j,k} H_{z,m}\Big|_{t+\frac{\Delta\mathrm{t}}{2}}^{i,j,k} + m_{J_{\mathrm{Mz},m}3}\Big|^{i,j,k} M_{z,m}\Big|_{t+\frac{\Delta\mathrm{t}}{2}}^{i,j,k}$$

where the coefficients are given by

$$m_{J_{\mathrm{Mz},m}0}\Big|^{i,j,k} = \left(\frac{1}{\Delta\mathrm{t}} + \frac{\Gamma_{M,m}\big|^{i,j,k}}{2}\right)$$

$$m_{J_{\mathrm{Mz},m}1}\Big|^{i,j,k} = \frac{1}{m_{J_{\mathrm{Mz},m}0}\big|^{i,j,k}}\left(\frac{1}{\Delta\mathrm{t}} - \frac{\Gamma_{M,m}\big|^{i,j,k}}{2}\right)$$

$$m_{J_{\mathrm{Mz},m}2}\Big|^{i,j,k} = \frac{1}{m_{J_{\mathrm{Mz},m}0}\Big|^{i,j,k}}\left(\omega_{\mathrm{Mp}}{}^{2}\Big|^{i,j,k} f_{M,m}\Big|^{i,j,k}\right)$$
$$m_{J_{\mathrm{Mz},m}3}\Big|^{i,j,k} = \frac{1}{m_{J_{\mathrm{Mz},m}0}\Big|^{i,j,k}}\left(-\omega_{0M,m}{}^{2}\Big|^{i,j,k}\right)$$
Update equation for $H$

$$\begin{aligned}
H_x\Big|^{i,j,k}_{t+\frac{\Delta t}{2}} &= \frac{1}{\mu_0\big|^{i,j,k}\mu_\infty\big|^{i,j,k}}\left(B_x\Big|^{i,j,k}_{t+\frac{\Delta t}{2}} - \textstyle\sum_m M_{x,m}\Big|^{i,j,k}_{t+\frac{\Delta t}{2}}\right)\\
H_y\Big|^{i,j,k}_{t+\frac{\Delta t}{2}} &= \frac{1}{\mu_0\big|^{i,j,k}\mu_\infty\big|^{i,j,k}}\left(B_y\Big|^{i,j,k}_{t+\frac{\Delta t}{2}} - \textstyle\sum_m M_{y,m}\Big|^{i,j,k}_{t+\frac{\Delta t}{2}}\right)\\
H_z\Big|^{i,j,k}_{t+\frac{\Delta t}{2}} &= \frac{1}{\mu_0\big|^{i,j,k}\mu_\infty\big|^{i,j,k}}\left(B_z\Big|^{i,j,k}_{t+\frac{\Delta t}{2}} - \textstyle\sum_m M_{z,m}\Big|^{i,j,k}_{t+\frac{\Delta t}{2}}\right)
\end{aligned} \tag{28}$$