The Monty Hall problem and its generalisations

Benjamin Lowery | 201700585

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Abstract

The Monty Hall problem provides an interesting look into how a simple game of seemingly random choice, inspired by an innocuous 60's TV show can result in an intriguing investigation into both the simplicity and deceitful nature of probability. Whilst also investigating generalisations and adaptations of the problem. With the help of Bayes' Theorem and support from numerical simulations, we can have a discussion into the surprising, misleading and sometimes trivial nature possessed by Monty and his doors.

1 Introduction

In a 1975 letter to the American Statistician Columnist Marilyn vos Savant, an American professor, Steve Selvin, posed a problem loosely based off the 1960's American TV show Let's Make a Deal. The game consisted of three doors, two of which had goats behind them, and a third door containing a dream car. The host of the show, Monty Hall, asks the contestant to select a door. Monty then chooses a remaining door to reveal a goat. He then asks the contestant if they would like to switch to the one remaining unrevealed door, or stay with their initial choice. In her response, vos Savant postulated that it will be in the best interest for the contestant to switch doors [1]. This idea caused uproar amongst many readers, amateur Mathematicians and even those possessing PhD's in Maths based disciplines. With the conundrum still causing issues with people many years later [2]. So are Slevin and vos Savant, who argued that the probability of winning when switching is double that of if you choose to stay, right? And is there extensions and more generalised way of looking at such a problem?

2 Original Monty Hall Problem

Let's pose the problem in a more formal sense by having a run through. We have three doors that can be labelled 1,2,3. Our esteemed host, Monty Hall, asks the contestant to pick a door. Say they pick door 1. Monty, who knows what's behind each door, opens door 2 and reveals a goat. He then poses the question "would you like to stick with your choice or switch to door three?". To help quell the contestants dilemma, we can first think about the odds of initially picking the correct door. With three doors, the contestant has a $\frac{1}{3}$ chance of picking correct straight away. Then if we remove a door, does that change our odds when switching? One naïve way to think about it is that, if each door has equal probability, surely switching makes no difference? Since removing a door will leave us with two options, we must have a $\frac{1}{2}$ chance of winning either way? The problem with this approach is we do not account for the host knowing what is behind each door. Given that the first door possesses a probability of $\frac{1}{3}$ of containing our Car. The other two doors must attain a $\frac{2}{3}$ probability it lies behind either one; and behind one of these doors, must lie a goat. Since Monty knows that, in this scenario, door 2 has a goat, this door is revealed and we are left with the option of switching to door 3. Yet we still have a $\frac{2}{3}$ probability that the Car doesn't lie behind the first door, so this probability carries over to represent just door 3. Hence we have probabilities of $\frac{1}{3}$ for door 1, 0 for door 2, and $\frac{2}{3}$ for door 3. Concluding that switching will be the correct decision to this dilemma.

2.1 Theoretical understanding using Bayes' Theorem

Since we have discussed that it may be in our best interests to switch when given the opportunity on Monty's Show, it would be ideal to have a way to show this with actual Maths and not just wordy anecdotes. One way to achieve this is using Bayes' Theorem and the idea that, given two events (A and B); we know that if we are given information for one of these events, A say, then we can calculate the probability of event B happening given the information of A. We denote this $Pr(B|A)$. Bayes' theorem then states the following

$$Pr(B|A) = \frac{Pr(B)Pr(A|B)}{Pr(A)}.$$

Appendix A provides a quick proof into why this rule holds.

It now worth for one to turn attention to how we can apply this to our Monty Hall Problem. In this problem we have three doors 1,2,3 and we have the events that the prize lies between each respective door, denoted D_1, D_2, D_3. Lets suppose we select door 1 initially, and our esteemed host Monty opens door 3 to show a goat. So it follows that $Pr(D_3) = 0$. Now let G be the information that there is a goat behind door 3. We can use Bayes' Theorem to formulate the following probabilities for finding the Car behind doors 1 or 2, given the information of G, as:

$$Pr(D_1|G) = \frac{Pr(D_1)Pr(G|D_1)}{Pr(G)} \tag{1}$$

$$Pr(D_2|G) = \frac{Pr(D_2)Pr(G|D_2)}{Pr(G)} \tag{2}$$

We now need to calculate these probabilities of the RHS of equations (1) and (2). i.e.

$Pr(D_1) = Pr(D_2) = 1/3$

$Pr(G|D_1) = 1/2$ (As if the prize was behind door 1, we can choose either 2 or 3 to open)

$Pr(G|D_2) = 1$ (As we are just restricted to opening door 3 if the prize is behind 2)

$Pr(G|D_3) = 0$ (If the prize lies here, we can't open it).

Since D_1, D_2 and D_3 are all mutually exclusive events, then,

$$Pr(G) = \sum_{i=1}^{3} Pr(D_i)Pr(G|D_i) = (1/3)(1/2) + (1/3)(1) + (1/3)(0) = 1/2.$$

Finally, putting these values back into (1) and (2) yields,

$$Pr(D_1|G) = \frac{Pr(D_1)Pr(G|D_1)}{Pr(G)} = \frac{(1/3) \cdot (1/2)}{1/2} = \frac{1}{3} \tag{3}$$

$$Pr(D_2|G) = \frac{Pr(D_2)Pr(G|D_2)}{Pr(G)} = \frac{(1/3) \cdot (1)}{1/2} = \frac{2}{3} \tag{4}$$

Hence, if we switch to door 2, it gives us a $\frac{2}{3}$ chance of winning and staying with door 1 produces a $\frac{1}{3}$ chance [3].

2.2 Numerical simulation for original Monty Hall problem

Now we have created a strong theoretical footing, it should now be time that we test Bayes' Theorem on the situations of switching and staying. This can be achieved by running a large number of simulations and logging the results. Why a large number? We can use the idea of the Law Of Large Numbers [4] to help us determine that with more samples and runs of the simulation, the performance shown by said simulations should improve. Ideally producing results that reflect the idea of our theoretical results gathered from Bayes'

Theorem. We can use MATLAB to create a function that runs a Monty Hall Simulation, say 1000 times, and provide information regarding the winning chance as we go through more iterations[1]. Alongside a graph to show this progression and back up our calculations from using Bayes' Theorem.

Below is the graph after running said simulation,

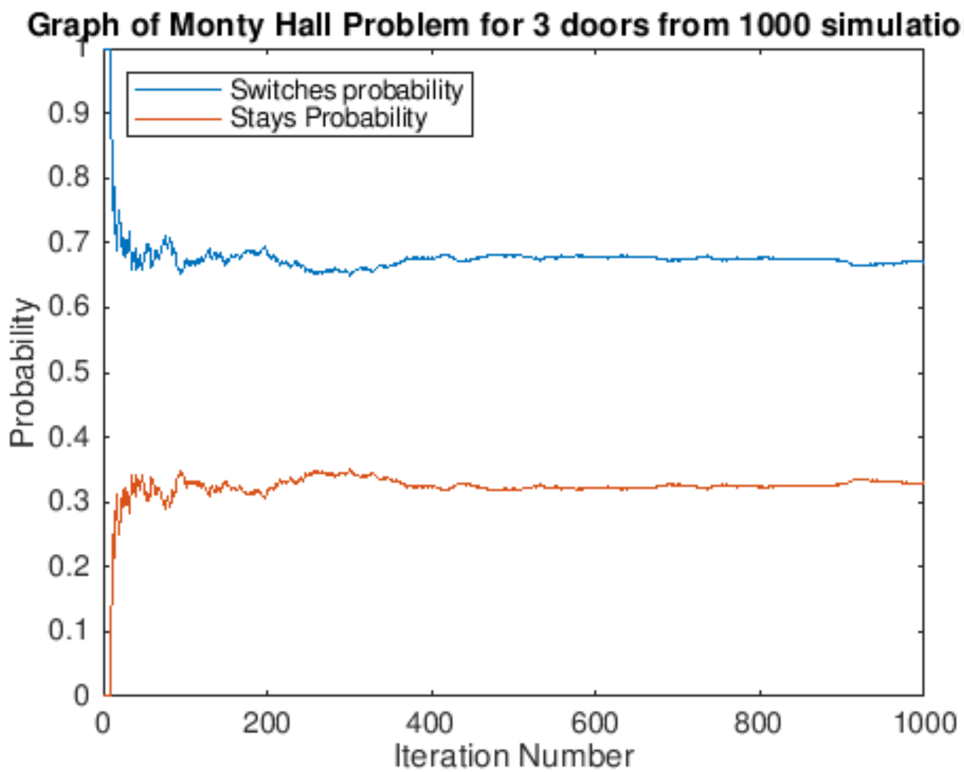

Figure 1: Graph showing the likelihood of winning if you stayed or switched in the Original Monty Hall Problem. With the graph showing how the probability of success from each option changes with more simulations of the scenario, leading to final probabilities of winning as 0.691 and 0.309 for switching and staying respectively.

As we can see, after some initial noise and wild variations in the first `~100` simulations, the two lines eventually smooth out with more games being played. Resulting in the final approximations of:

Switching: 0.691 Staying: 0.309 Error:[2] 0.02433

With this data being provided by the corresponding `MHdata.csv` file that is outputted by the MATLAB program. These final probabilities are slightly off, albeit not by much, from our Bayes' Prediction; and it is evident that their is a substantial difference in the odds of winning if the player stays behind his door or switches for the chance to win the Car. If the reader desires numerical simulations to more aptly reflect the Bayes' Theory we used in the previous section, the following is respective outputs for 5,10 and 15 thousand simulations.

5000 simualtions:	Switching: 0.673	Staying: 0.327	Error: 0.00633
10000 simualtions:	Switching: 0.671	Staying: 0.329	Error: 0.00433
15000 simualtions:	Switching: 0.665	Staying: 0.335	Error: 0.00016

And it can clearly be noted the error between the Bayes' theory and numerical simulation gradually diminishes (due to the Law of Large Numbers referenced earlier).

[1]See Appendix B for how the MATLAB function works.
[2]Error from theoretical value

3 Generalising Monty Hall

3.1 Expanding to d doors

Let's envision the following fictitious scenario; in which after the success of his three door final showdown, Monty and his team have been gifted a bigger budget to make a more elaborate show. Here, Monty utilises this increased budget to order his producers to purchase more doors. Now that he possesses a studio filled to the brim of disused doors. Monty again places one car behind a door and keeps count of where it is placed. Whilst a flood of goats trundle in and hide behind the rest, he asks a contestant, now slightly more intimidated than their predecessors, to pick a door. The contestant hesitantly chooses, giving way to our host opening every door but the contestants and one final door. Now we again pose the question, stick or switch? Given the information we attained from the original Monty Hall situation, it makes sense to have an intuitive guess that switching will be in the contestants best interest. And we can test this again by using Bayes' Theorem and some basic Probability theory.

3.1.1 Bayes' Theorem for the generalised problem

Like in the previous chapter, we can define a few events and variables for a generalised version. Instead of 3 doors, we now possess d different doors. With each of these doors needing an event that the door may possess the prize behind it. Thus we can define these as the events $D_1, D_2, ..., D_d$. Whilst also allowing G to be, again, the event we open all but doors 1 and d to reveal a Goat. So with this in mind, we can formulate the following equations:

$$Pr(D_1|G) = \frac{Pr(D_1)Pr(G|D_1)}{Pr(G)} \tag{5}$$

$$Pr(D_d|G) = \frac{Pr(D_d)Pr(G|D_d)}{Pr(G)}. \tag{6}$$

Then calculating the following probabilities needed for the RHS,

$Pr(D_1) = ... = Pr(D_d) = 1/d$

$Pr(G|D_1) = \frac{1}{d-1}$ (As if the prize was behind door 1, we can choose any of the reamining doors to open)

$Pr(G|D_d) = 1$ (As we are just restricted to opening every other door if the prize is here)

$Pr(G|D_2) = ... = Pr(G|D_{d-1}) = 0$ (If the prize lies in all these doors we want to open, we clearly can't open them).

And given events D_1 to D_d are all mutually exclusive it follows that:

$$\begin{aligned}
Pr(G) = \sum_{i=1}^{d} Pr(D_i)Pr(G|D_i) = Pr(D_1)Pr(G|D_1) + \sum_{i=2}^{d} Pr(D_i)Pr(G|D_i) \\
= Pr(D_1)Pr(G|D_1) + Pr(D_d)Pr(G|D_d) + \sum_{i=2}^{d-1} Pr(D_i)Pr(G|D_i) \\
= \frac{1}{d} \cdot \frac{1}{d-1} + \frac{1}{d} \cdot 1 + 0 \\
= \frac{1}{d}\left(\frac{1}{d-1} + 1\right) = \frac{1}{d}\left(\frac{d}{d-1}\right).
\end{aligned}$$

Finally given us the following generalised Monty hall results by substituting these equations back into (5) and (6)

$$Pr(D_1|G) = \frac{Pr(D_1)Pr(G|D_1)}{Pr(G)} = \frac{(1/d)\cdot(1/d-1)}{(1/d)\cdot(d/d-1)} = \frac{1}{d} \tag{7}$$

$$Pr(D_d|G) = \frac{Pr(D_d)Pr(G|D_d)}{Pr(G)} = \frac{(1/d)\cdot(1)}{(1/d)(d/d-1)} = \frac{d-1}{d}. \tag{8}$$

As required. Harking back to our original Monty Hall Problem It is clear to see that substituting 3 doors gives us probabilities of switching and staying as $\frac{2}{3}$ and $\frac{1}{3}$ respectively. So now we have a rather straightforward method to show, no matter how many doors we have, if we open all but one door and the original door, it is in our best interest to switch. In addition to this, although trivial to point out, we see that with more doors, our likelihood of winning when switching increases [5].

3.1.2 Numerical simulation for d doors

Akin to the analysis of the original Monty Hall problem, it should be worth testing the theory with some numerical computations. If we adapt our original MATLAB function to allow for the generalisation to d doors. We can then run a simulation for 7 doors, again computing 1000 simulations. Here we should expect the switch probability to be

$$\frac{6}{7} \approx 0.857$$

this is by subbing $d = 7$ into equation (7). We can see if this is the case by analysing figure 2 below.

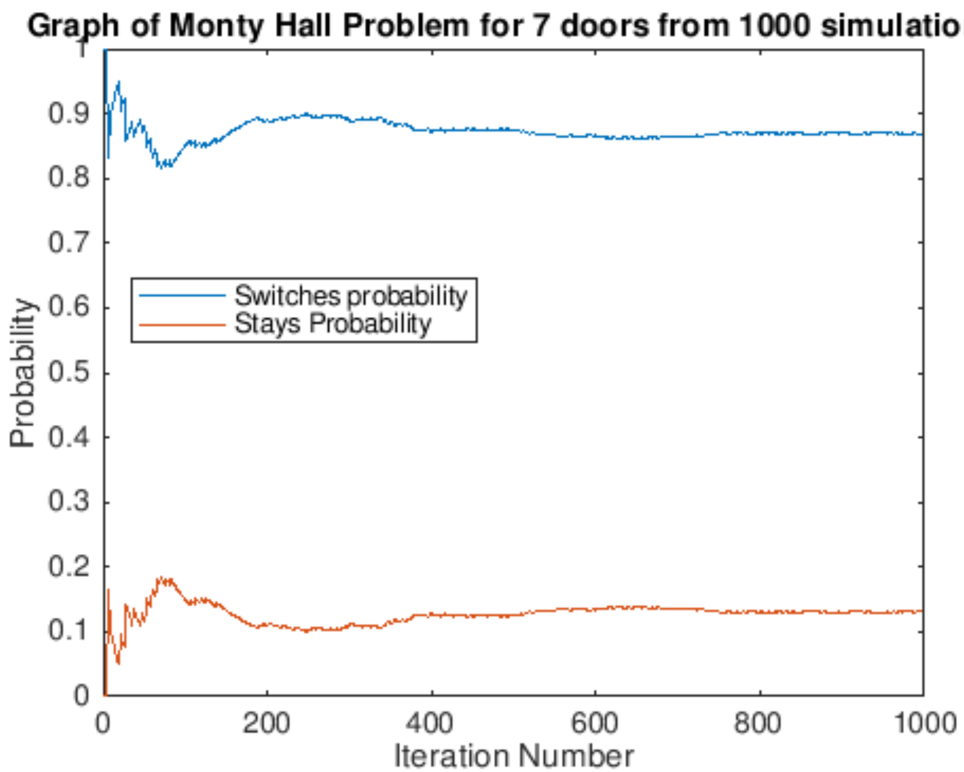

Figure 2: Graph showing the likelihood of winning if you stayed or switched in the Monty Hall Problem with 7 doors. With the graph showing how the probability of success from each option changes with more simulations of the scenario.

As is evident, there is some fluctuations in the first `~100` or so simulations. This then eventually evens out to a more stable estimation with the more simulations we have, which, as discussed earlier, comes from the Law of Large Numbers. It is also worth pointing out the following final probabilities for simulation 1000,

Switching: 0.872 Staying: 0.128 Error: 0.01486

With the relatively small error, we can say that the simulations do reflect the idea from our theoretical findings. This is the idea that, no matter what value of doors we have, the outcome when switching is always far more likely to win than staying. It is also the case, from prior discussion, that if we have 7 doors, the chance of winning when switching is higher than that of the original 3 door problem. For more clarification, the following graph, again produced using our MATLAB function (details in Appendix B), shows the outcomes of 3000 simulations of the Monty Hall problem with different door numbers.

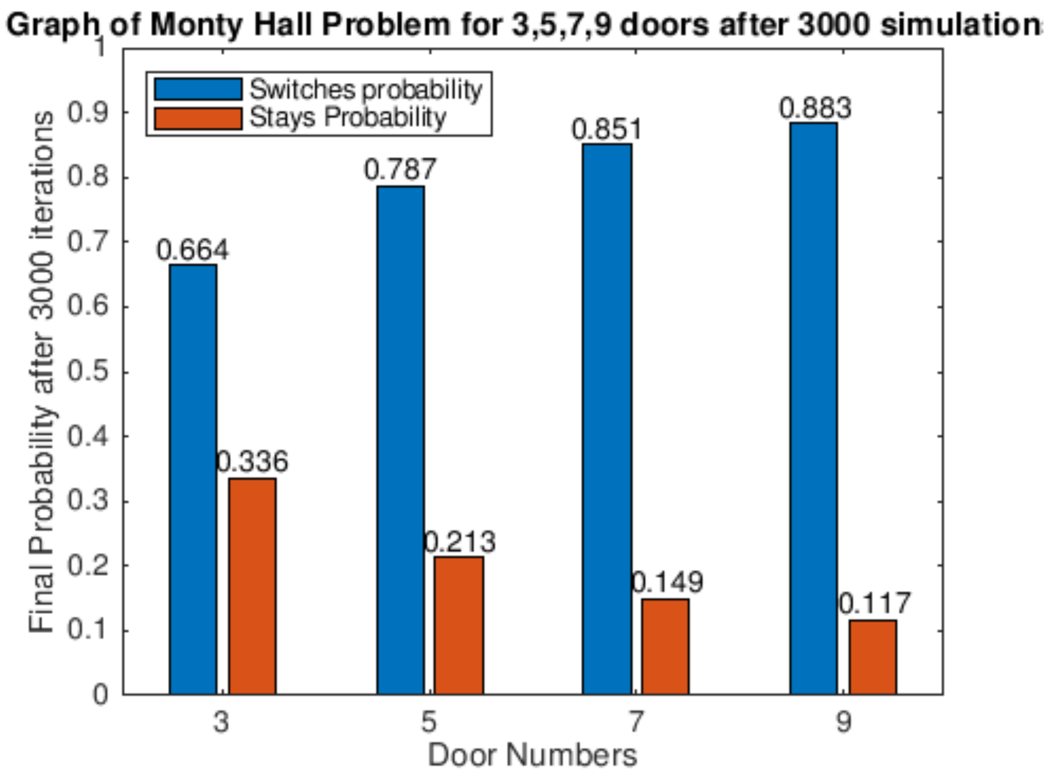

Figure 3: Bar Chart showing the likelihood of winning if you stayed or switched in the Monty Hall Problem with a variety of different door option. i.e. 3,5,7 and 9 doors. Unlike earlier figures, this one only shows the final probability of staying or switching with respect to the doors. This is done after running through 3000 simulations.

This highlights the clear correlation that, when opening all but one door to reveal a goat, our odds of winning when switching is greater with the more doors being used in the game. This should not be a surprise given the theoretical footing we made for ourselves. Monty's Kindness has no bounds it seems!

3.2 Opening k Doors

Our second generalisation into Monty Hall's problem is one in which we are looking to try flip the odds back into the favour of the host. Given the generosity of winning when we expand the problem to d doors, it is now worth seeing if limiting the number of doors Monty opens can make it more, or less likely for the contestant to win when switching. If we think about this in a logical manner, it should be the case that now we have the option of which door we can switch too, we are less likely to get the prize than in the previous scenario if we switch. But it is worth calculating how much of a detriment this new rule is to our contestant. And analysing how drastically our odds can change by opening less and less doors.

3.2.1 Bayes' Theorem for opening k doors

Since we are now veterans when calculating theoretical probabilities using Bayes', it may be worth taking an more direct and simpler approach to calculate the probability of winning if staying, or switching. For staying, this probability is rather trivial, being $\frac{1}{d}$[3] . To calculate the chance of winning when switching, we

[3]Since their are d doors. Picking one is a $\frac{1}{d}$ chance.

can define two events. Event A being that of not picking the Car in our first choice, and event B denoting we will now pick a door containing a car, given we are now opening k doors (where $0 \leq k \leq d-2$) rather than all but one. Then we can create the following probabilities from these events:

$$Pr(A) = \frac{d-1}{d}$$

$$Pr(B|A) = \frac{1}{d-k-1} \quad \text{(Since we have opened k doors and we cant change to the door we chose initially).}$$

Hence calculating that events A and B **both** happen, will give us the probability of winning when switching, and using the fact $Pr(A \cap B) = Pr(A)Pr(B|A)$ [4]. Thus we can get the following probability for winning when switching

$$Pr(A \cap B) = Pr(A)Pr(B|A) = \left(\frac{d-1}{d}\right) \cdot \left(\frac{1}{d-k-1}\right) \quad [6]. \tag{9}$$

We can see here that the probability of winning in this case when switching will decrease as k decreases.

3.2.2 Numerical simulation for opening k doors

So how much can we bend the odds of winning into Monty's favour? We can first start by looking at simple simulation in which we are given 20 doors, and we would only want to open half of these doors. We can see below the results when running 3000 iterations of the Monty Hall game

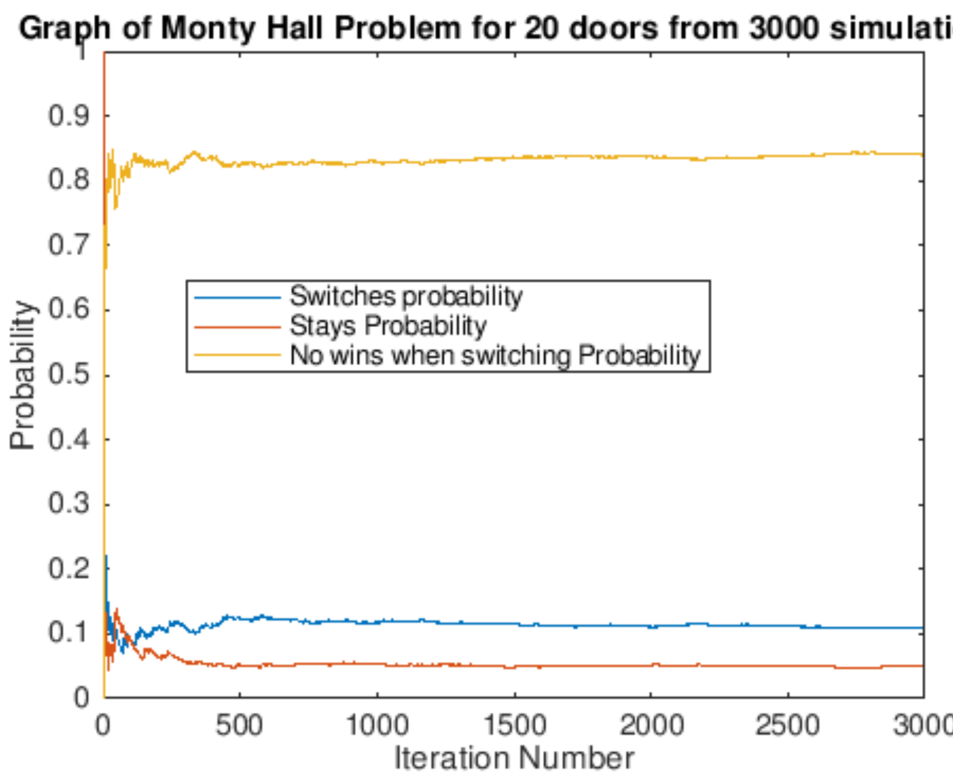

Figure 4: Graph showing the results of a Monty Hall game consisting of 20 doors, in which 10 doors are opened to reveal goats. The graph shows the progressive probability of winning if staying, winning if switching, and not winning if switching after each simulation. and ends after running 3000 simulations

Clearly, the mostly likely outcome is we are going to lose,

Switching: 0.110 Switching and not winning: 0.840 Staying: 0.050 Error: 0.00444

[4]this result is used in the proof of Bayes' Theorem found in Appendix A

Given the relatively small error from the theoretical value, our short number of trials accurately represents the odds of winning the Monty Hall scenario under these constraints are relatively small.

We can also look at how varying the door numbers can incur major changes in the probabilities of winning the game when switching.

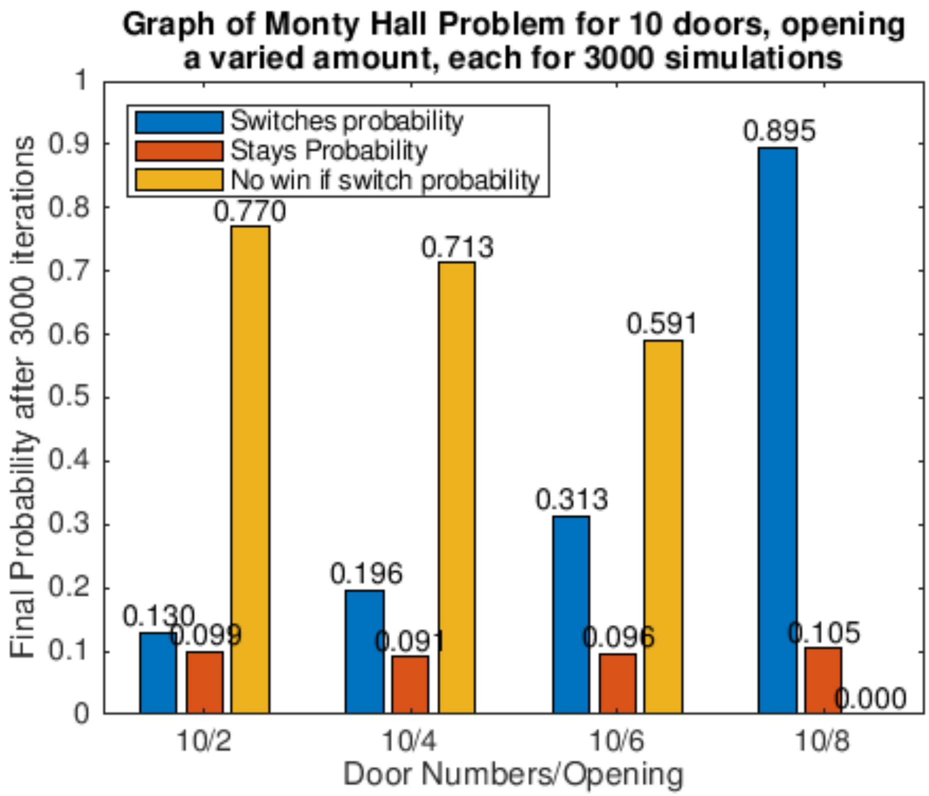

Figure 5: Graph showing the results of a Monty Hall game consisting of 10 doors, in which 2,4,6 and then 8 doors are opened to reveal goats. The graph shows the final chance of winning if staying, winning if switching, and not winning if switching. This is after running 3000 simulations

As we can see in the choice between 10 doors, when opening 8 (which is the max number of doors we can open) and opening 6 (in which we have a choice as to what we can switch to), there exists a significant dip in the probability of winning when switching, with it then being more likely to not win the game whatever we do. This is due to the truly random choice we now have with the selection of the door we might want to switch to. Despite all these changes, the chance of switching consistently gives better odds than staying. And we can see that if we go to the boundaries of our k parameter (0 and $d-2$) and enter this into (9). We can get a upper bound of $\frac{d-1}{d}$ and a lower bound of $\frac{1}{d}$. As the lower bound is always greater than or equal to the probability of staying with our initial choice (again $\frac{1}{d}$). We can say under these circumstances, there is still no reason not to switch doors for the best odds at success.

4 Conclusion

Throughout this paper we have discussed and discovered some interesting results of probability and how, with the aid of computational Mathematics, the Monty Hall problem can be tackled and evaluated with great rigour. Whether this is in its purest form we initially discussed with 3 doors, or the devilish scenario of possibly infinite doors and selecting only a certain number of doors to open; the problem can be discussed, dissected and backed up using Bayes' Theorem and Numerical simulations.

With this being said, as often is the case of applying Mathematics to the real world, our logical reasoning may still not be perfect, nor reveal the true solution to the problem. since arguments could be made in that randomising the choices of contestants in the simulation and using conditional probability detracts from the human element in the game. That in which the host, the atmosphere and the audience play a crucial role in the dilemma posed to the contestant, perhaps leading to a bias in the options available. This is something that probability and random simulations simply cannot account for. Hence it could even be contested that in reality, based on the host's hints and approach towards the contestant, the probability of finding the winning car can range from $\frac{1}{2}$ to 1 [7]. Nevertheless, the numerical computations and Bayes' Theory provides a thorough analysis and the most accurate as currently possible, solution to the Monty Hall problem and certain generalisations.

Further Reading

[1] Saenen L, Heyvaert M, Van Dooren W, Schaeken W, Onghena P. Why Humans Fail in Solving the Monty Hall Dilemma: A Systematic Review., 2018.
[2] J.V. Stone. Bayes' Rule: A Tutorial Introduction to Bayesian Analysis. Sebtel Press, 2013.

References

[1] Steve Selvin, M. Bloxham, A. I. Khuri, Michael Moore, Rodney Coleman, G. Rex Bryce, James A. Hagans, Thomas C. Chalmers, E. A. Maxwell, and Gary N. Smith. Letters to the editor. The American Statistician, 29(1):67–71, 1975.

[2] Keith Devlin. Monty hall, July-August 2003.

[3] Keith Devlin. Monty hall revisited, Dec 2005.

[4] Morris H DeGroot and Mark J Schervish. Probability and statistics. Pearson Education, 2012. Chapter 6.2, Theorem 2.3.1.

[5] Leo Depuydt and Richard D Gill. Higher variations of the monty hall problem (3.0 and 4.0) and empirical definition of the phenomenon of mathematics, in boole's footsteps, as something the brain does. arXiv preprint arXiv:1208.2638, 2012. Section 3.

[6] SQB (https://math.stackexchange.com/users/106234/sqb). Monty hall problem extended. Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/609552 (version: 2017-04-13).

[7] J. P. Morgan, N. R. Chaganty, R. C. Dahiya, and M. J. Doviak. Let's make a deal: The player's dilemma. The American Statistician, 45(4):284–287, 1991.

Appendices

A Proof Of Bayes' Theorem

Theorem 1. *(Bayes' Theorem) Let A and B be events such that $Pr(A) > 0$ and $Pr(B) > 0$. Then*

$$Pr(B|A) = \frac{Pr(B)Pr(A|B)}{Pr(A)}.$$

Proof.
Let A and B be two events. Then the probability of B and A happening, denoted $Pr(B \cap A)$, is the product of the probability of B, $Pr(B)$, and the probability of A given B has occurred, $Pr(A|B)$.

$$Pr(B \cap A) = Pr(B)Pr(A|B)$$

Also, note that $Pr(B \cap A)$ is also equal to the RHS if we switch A and B in this statement, thus it is the case that

$$Pr(B \cap A) = Pr(A)Pr(B|A).$$

By Equating the two we have

$$Pr(A)Pr(B|A) = Pr(B)Pr(A|B).$$

We can rearrange and hence

$$Pr(B|A) = \frac{Pr(B)Pr(A|B)}{Pr(A)}$$

as required.

□

[4]

B How to use MontyHall.m

You can think of `MontyHall.m` as an interface function to access the different types of possibilities discussed in this paper. So a few variables are required to run the program. some may seem a bit tedious, but it allows the creation of many variations between different door numbers and the doors we want to open without the need of multiple MATLAB Programs and files. The MATLAB file consists of two functions. The first, `MontyHall()`, being the interface referenced earlier, taking four arguments. And the other, smaller function being `SimulateMH()`, which allows us to calculate a single simulation of the Monty Hall problem, with the option to generalise to d Doors and opening k Doors. The `MontyHall.m` function has been used to generate all the results and graphs we've seen in this paper. To run the program, we call:

```
MontyHall(N_it,door_Num,d20_Num,verbose)
```

and insert the four required parameters which are:

- **Number Of Simulations** (`N_it`): The number of simulations we want to run.
- **Number of Doors** (`door_Num`): The number of doors we want, can evaluate multiple door options by entering a vector. If multiple entered, a bar chart will be generated, otherwise just a line chart.
- **Number of Doors to open** (`d2o_Num`): Number of doors to open for each door option we enter in previous parameter. For an original Monty hall problem, enter this as two less than the door number entered
- **Write to file** (Verbose): Can be entered as `true` or `false`. If true, a verbose output of the details for each game played a will be written to a comma separated values file called `MHdata.csv`.

For example to run the original Monty Hall problem 3000 times, and say we are not too bothered with writing data to a file, we can enter:

```
>> MontyHall(3000,3,1,false)
```

Alternatively, if we want to replicate something like figure 6 and want an output to a file to view the data, we can input:

```
>> MontyHall(3000,[10 10 10 10],[2 4 6 8],true)
```

B.1 Advantages of writing to a .csv file

Outputting to a format such as `.csv` provides us with the ability to easily extract and analyse the data in other ways from our current MontyHall.m function. For example, if we want to examine the error of say figure 1 compared to the theoretical mean over simulations $500 - 1000$, we can write the following code to extract the data and then plot it.

```
% Load file
data=readmatrix('MHdata.csv');
% Get Error column items (last one)
error=data(1:end,end);
% Plot error against theoretical value
plot(linspace(500,1000,501),error(500:1000));
% Graph details
title('Absolute error betweeen simulations 500 and 1000 from theoretical value');
xlabel('Simulation');
ylabel('Error');
xlim([500 1000]);
```

and this will result in the graph below

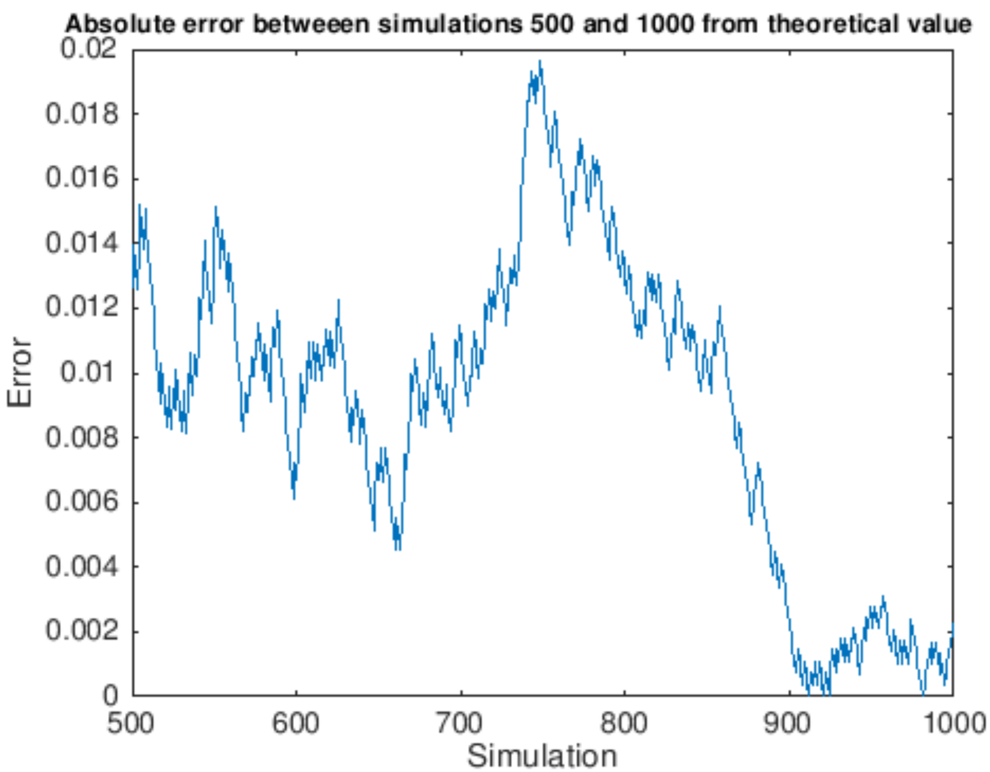

Figure 6: Graph highlighting the absolute error between the simulated and theoretical value of the Monty Hall problem between simulation 500 and simulation 1000.

This is a very neat and simple way to take advantage of our findings in this paper and allow us to utilise this data in many different ways.