EEE3069W Formula Sheet

Benjamin Scholtz, 2015

General

$s^2 x + 2\zeta\omega_0 sx + \omega_0^2 x$

$g_{CL} = \frac{L}{1+L}$

$ZOH = (1 - z^{-1}) = \frac{1 - e^{-sT}}{s}$

$t_{2\%} \simeq 4\tau$

$\zeta = cos(\theta)$

$\tau = \frac{-1}{Re\{s\}} = \frac{1}{\zeta\omega_n}$

$z = e^{sT} = e^{(\sigma + j\omega)T} = e^{\sigma T}e^{j\omega T} = \rho e^{\varphi}$

$t_{2\%} = b : R\{z\} < e^{\frac{-4T}{b}}$

Jury Stability Test

$A(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n = 0$

$A(1) > 0$, $A(-1) > 0$ for n even, $A(-1) < 0$ for n odd.

$|a_0| < |a_n|$, $|b_0| > |b_{n-1}|$, $|c_0| > |c_{n-2}|$, ...

where $b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}$, ...

Z-Transform

$$Z[x(kT)] = \sum_{k=0}^{\infty} x(kT)z^{-k}$$

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$$

$e^{j\theta} = cos(\theta) + jsin(\theta)$

$sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$

Sampling

$g^*(s) = g(z)$

$g^*(t) = g(nT)$

$y(s) = g(s)u(s); y^*(s) = gu^*(s);$

$y(z) = gu(z)$

$\frac{y(z)}{r(z)}$: pulse transfer function.

$u^*(s + jm\Omega) = u^*(s); \Omega = \frac{2\pi}{T};$

$\Omega/2 > BW$

Controllability and Observability

$\frac{d}{dt} x(t) = Ax(t) + bu(t)$ with n states.

$y(t) = C^T x(t) + du(t)$

C: $Rank[b|Ab|A^2 b|...|A^{n-1}b] = n$

O: $Rank[C^T|C^T A|C^T A^2|...|C^T A^{n-1}]^T = n$

where R(rows) = R(columns) = LI columns / no. of pivots / no. zero rows in RRE form.

Can you 'see' control action (u(t)) and output?

$x = P\bar{x}; P^{-1}AP = D$

Discrete Controller Implementation

Difference equation:

$k(z) = \frac{u(z)}{e(z)}; u(z)z^{-a} = u(n-a)$

Direct: powers of z^{-1}; introduce internal state variable x(z); equate denominator then numerator; no. past values = no. poles

Cascade: split into first order blocks connected in cascade; apply direct; use internal variables to equate

Parallel: partial fraction expansion; apply direct; use internal variables to equate

$e(z) \Rightarrow k(z) \Rightarrow u(z)$

State Feedback Control

$u(t) = -k^T x(t) + r(t)$

Replace u(t) in basic state space model and take Laplace:

$g(s) = \frac{y}{r} = C^T \frac{(sI - A + bk^T)b}{det(sI - A + bk^T)}$

$\phi_c = det(sI - A + bk^T)$

$det(sI - A + bk^T) = 0$

for closed-loop poles = eigenvalues of ($\bar{A} = A - bk^T$).

Set-point tracking: *add K/s, block diagram manipulation, state space for integrator:*

$\frac{d}{dt}x_a = r(t) - C^T x(t) + du(t)$

$z = -k_a x_a$

Integrator output becomes a state and augmented matrix formed:

$\frac{d}{dt}\begin{bmatrix} x \\ x_a \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C^T & 0 \end{bmatrix}\begin{bmatrix} x \\ x_a \end{bmatrix} + \begin{bmatrix} b \\ -d \end{bmatrix}u + \begin{bmatrix} 0 \\ 1 \end{bmatrix}r$

$u(t) = -\begin{bmatrix} k^T & k_a \end{bmatrix}\begin{bmatrix} x \\ x_a \end{bmatrix}$ *subbing this in gives:*

$\frac{d}{dt}\begin{bmatrix} x \\ x_a \end{bmatrix} = (\bar{A} - \bar{b}\bar{k}^T)\begin{bmatrix} x \\ x_a \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}r$

$\phi_c = |sI - \bar{A} + \bar{b}\bar{k}^T|$

Observer: *the observer state is added and error poles can be designed separately from CL poles:*

$\frac{d}{dt}\tilde{x}(t) = A\tilde{x}(t) + bu(t) + p[y(t) - \tilde{y}(t)]$

$\tilde{y}(t) = C^T\tilde{x}(t)$

state observer is a mathematical model of plant with error term. $\tilde{x}(t)$ is an estimated state.

Introducing error state:

$\frac{d}{dt}e = x - \bar{x} = (A - pC^T)e$

$|sI - A + bk^T||sI - A + pC^T| = 0$

Modified Z-Transform

$Z_m[gh(s)] = Z[gh(s)e^{-s\theta}]$

$m = 1 - \frac{\theta}{T} = 1 - \Delta$

Other Transformations

$z = \frac{1+w}{1-w}$

$w = \frac{z-1}{z+1}$

$z = \frac{1 + w'\frac{T}{2}}{1 - w'\frac{T}{2}}$

$w' = \frac{2(z-1)}{T(z+1)}$

Used for: Nyquist of gh(z)

$z = e^{j\omega T}$

$w = \frac{e^{j\omega T} - 1}{e^{j\omega T} + 1} = jtan(wT/2) \simeq j\omega T/2 \simeq j\Omega$

Root Locus

$n_\infty = n_p - n_z$

$angle = \frac{180^0 - 360^0 l}{n_\infty}$

$center = \frac{sum of poles - sum of zeros}{n_\infty}$

Symmetric about real axis.

Left of odd number of poles and zeros.

$q(s) = \gamma\frac{N_q(s)}{D_q(s)}$

$\frac{d}{ds}\gamma(s) = N_q(s)\frac{d}{ds}D_q(s) - D_q(s)\frac{d}{ds}N_q(s) = 0$

Between open loop poles - breakaway point.

Controller Synthesis

$k(z) = \frac{1}{gh(z)} \cdot \frac{\frac{y(z)}{r(z)}}{1 - \frac{y(z)}{r(z)}}$

Dalin: $\frac{z^{-N}(1 - e^{\frac{-T}{\tau}})}{z - e^{\frac{-T}{\tau}}}$

Table 2.1: Summary of Laplace Transform Properties

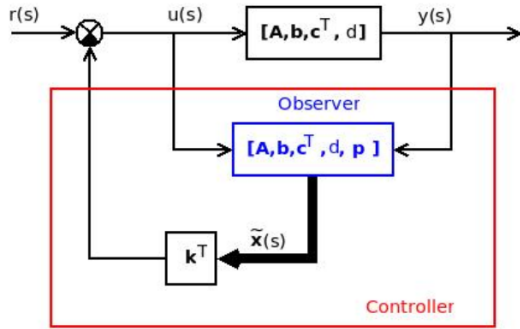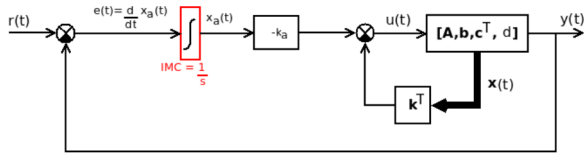| Description	Time Domain	Laplace Domain
Linearity	$\mathscr{L}[(\gamma f(t) + \beta g(t))]$	$\gamma F(s) + \beta G(s)$
Differentiation	$\mathscr{L}[\frac{d^n}{dt^n} f(t)]$	$s^n F(s)$
Integration	$\mathscr{L}[\int_1 \dots \int_n f(t)dt]$	$\frac{F(s)}{s^n}$
Final Value Theorem	$\lim_{t\to\infty} f(t)$	$\lim_{s\to 0} sF(s)$
Initial Value Theorem	$\lim_{t\to 0} f(t)$	$\lim_{s\to\infty} sF(s)$
Time Delay	$\mathscr{L}[f(t - d)]$	$e^{-sd}F(s)$
Time Scaling	$\mathscr{L}[f(\frac{t}{\alpha})]$	$\alpha F(\alpha s)$
Frequency Scaling	$\alpha f(\alpha t)$	$\mathscr{L}^{-1}[F(\frac{s}{\alpha})]$
Complex Translation	$\mathscr{L}[e^{-\alpha t} f(t)]$	$F(s + \alpha)$
Multiplication	$\mathscr{L}[f(t) \times g(t)]$	$F(s) * G(s)$
Convolution	$\mathscr{L}[f(t) * g(t)]$	$F(s) \times G(s)$

Table of Laplace and Z-transforms

	$X(s)$	$x(t)$	$x(kT)$ or $x(k)$	$X(z)$
1.	–	–	Kronecker delta $\delta_0(k)$ $1 \qquad k=0$ $0 \qquad k \neq 0$	1
2.	–	–	$\delta_0(n\text{-}k)$ $1 \qquad n=k$ $0 \qquad n \neq k$	z^{-k}
3.	$\dfrac{1}{s}$	$1(t)$	$1(k)$	$\dfrac{1}{1-z^{-1}}$
4.	$\dfrac{1}{s+a}$	e^{-at}	e^{-akT}	$\dfrac{1}{1-e^{-aT}z^{-1}}$
5.	$\dfrac{1}{s^2}$	t	kT	$\dfrac{Tz^{-1}}{\left(1-z^{-1}\right)^2}$
6.	$\dfrac{2}{s^3}$	t^2	$(kT)^2$	$\dfrac{T^2 z^{-1}\left(1+z^{-1}\right)}{\left(1-z^{-1}\right)^3}$
7.	$\dfrac{6}{s^4}$	t^3	$(kT)^3$	$\dfrac{T^3 z^{-1}\left(1+4z^{-1}+z^{-2}\right)}{\left(1-z^{-1}\right)^4}$
8.	$\dfrac{a}{s(s+a)}$	$1-e^{-at}$	$1-e^{-akT}$	$\dfrac{\left(1-e^{-aT}\right)z^{-1}}{\left(1-z^{-1}\right)\left(1-e^{-aT}z^{-1}\right)}$
9.	$\dfrac{b-a}{(s+a)(s+b)}$	$e^{-at}-e^{-bt}$	$e^{-akT}-e^{-bkT}$	$\dfrac{\left(e^{-aT}-e^{-bT}\right)z^{-1}}{\left(1-e^{-aT}z^{-1}\right)\left(1-e^{-bT}z^{-1}\right)}$
10.	$\dfrac{1}{(s+a)^2}$	te^{-at}	kTe^{-akT}	$\dfrac{Te^{-aT}z^{-1}}{\left(1-e^{-aT}z^{-1}\right)^2}$
11.	$\dfrac{s}{(s+a)^2}$	$(1-at)e^{-at}$	$(1-akT)e^{-akT}$	$\dfrac{1-\left(1+aT\right)e^{-aT}z^{-1}}{\left(1-e^{-aT}z^{-1}\right)^2}$
12.	$\dfrac{2}{(s+a)^3}$	$t^2 e^{-at}$	$(kT)^2 e^{-akT}$	$\dfrac{T^2 e^{-aT}\left(1+e^{-aT}z^{-1}\right)z^{-1}}{\left(1-e^{-aT}z^{-1}\right)^3}$
13.	$\dfrac{a^2}{s^2(s+a)}$	$at-1+e^{-at}$	$akT-1+e^{-akT}$	$\dfrac{\left[\left(aT-1+e^{-aT}\right)+\left(1-e^{-aT}-aTe^{-aT}\right)z^{-1}\right]z^{-1}}{\left(1-z^{-1}\right)^2\left(1-e^{-aT}z^{-1}\right)}$
14.	$\dfrac{\omega}{s^2+\omega^2}$	$\sin \omega t$	$\sin \omega kT$	$\dfrac{z^{-1}\sin \omega T}{1-2z^{-1}\cos \omega T+z^{-2}}$
15.	$\dfrac{s}{s^2+\omega^2}$	$\cos \omega t$	$\cos \omega kT$	$\dfrac{1-z^{-1}\cos \omega T}{1-2z^{-1}\cos \omega T+z^{-2}}$
16.	$\dfrac{\omega}{(s+a)^2+\omega^2}$	$e^{-at}\sin \omega t$	$e^{-akT}\sin \omega kT$	$\dfrac{e^{-aT}z^{-1}\sin \omega T}{1-2e^{-aT}z^{-1}\cos \omega T+e^{-2aT}z^{-2}}$
17.	$\dfrac{s+a}{(s+a)^2+\omega^2}$	$e^{-at}\cos \omega t$	$e^{-akT}\cos \omega kT$	$\dfrac{1-e^{-aT}z^{-1}\cos \omega T}{1-2e^{-aT}z^{-1}\cos \omega T+e^{-2aT}z^{-2}}$
18.	–	–	a^k	$\dfrac{1}{1-az^{-1}}$
19.	–	–	a^{k-1} $k=1,2,3,\dots$	$\dfrac{z^{-1}}{1-az^{-1}}$
20.	–	–	ka^{k-1}	$\dfrac{z^{-1}}{\left(1-az^{-1}\right)^2}$
21.	–	–	$k^2 a^{k-1}$	$\dfrac{z^{-1}\left(1+az^{-1}\right)}{\left(1-az^{-1}\right)^3}$
22.	–	–	$k^3 a^{k-1}$	$\dfrac{z^{-1}\left(1+4az^{-1}+a^2 z^{-2}\right)}{\left(1-az^{-1}\right)^4}$
23.	–	–	$k^4 a^{k-1}$	$\dfrac{z^{-1}\left(1+11az^{-1}+11a^2 z^{-2}+a^3 z^{-3}\right)}{\left(1-az^{-1}\right)^5}$
24.	–	–	$a^k \cos k\pi$	$\dfrac{1}{1+az^{-1}}$

$x(t) = 0 \qquad$ for $t < 0$
$x(kT) = x(k) = 0 \quad$ for $k < 0$
Unless otherwise noted, $k = 0, 1, 2, 3, \dots$

Definition of the Z-transform

$$\mathscr{Z}\{x(k)\} = X(z) = \sum_{k=0}^{\infty} x(k)z^{-k}$$

Important properties and theorems of the Z-transform

	$x(t)$ or $x(k)$	$Z\{x(t)\}$ or $Z\{x(k)\}$
1.	$ax(t)$	$aX(z)$
2.	$ax_1(t)+bx_2(t)$	$aX_1(z)+bX_2(z)$
3.	$x(t+T)$ or $x(k+1)$	$zX(z)-zx(0)$
4.	$x(t+2T)$	$z^2X(z)-z^2x(0)-zx(T)$
5.	$x(k+2)$	$z^2X(z)-z^2x(0)-zx(1)$
6.	$x(t+kT)$	$z^kX(z)-z^kx(0)-z^{k-1}x(T)-...-zx(kT-T)$
7.	$x(t-kT)$	$z^{-k}X(z)$
8.	$x(n+k)$	$z^kX(z)-z^kx(0)-z^{k-1}x(1)-...-zx(k1-1)$
9.	$x(n-k)$	$z^{-k}X(z)$
10.	$tx(t)$	$-Tz\dfrac{d}{dz}X(z)$
11.	$kx(k)$	$-z\dfrac{d}{dz}X(z)$
12.	$e^{-at}x(t)$	$X(ze^{aT})$
13.	$e^{-ak}x(k)$	$X(ze^a)$
14.	$a^kx(k)$	$X\left(\dfrac{z}{a}\right)$
15.	$ka^kx(k)$	$-z\dfrac{d}{dz}X\left(\dfrac{z}{a}\right)$
16.	$x(0)$	$\lim\limits_{z\to\infty} X(z)$ if the limit exists
17.	$x(\infty)$	$\lim\limits_{z\to1}\left[(1-z^{-1})X(z)\right]$ if $(1-z^{-1})X(z)$ is analytic on and outside the unit circle
18.	$\nabla x(k)=x(k)-x(k-1)$	$(1-z^{-1})X(z)$
19.	$\Delta x(k)=x(k+1)-x(k)$	$(z-1)X(z)-zx(0)$
20.	$\sum\limits_{k=0}^{n}x(k)$	$\dfrac{1}{1-z^{-1}}X(z)$
21.	$\dfrac{\partial}{\partial a}x(t,a)$	$\dfrac{\partial}{\partial a}X(z,a)$
22.	$k^mx(k)$	$\left(-z\dfrac{d}{dz}\right)^m X(z)$
23.	$\sum\limits_{k=0}^{n}x(kT)y(nT-kT)$	$X(z)Y(z)$
24.	$\sum\limits_{k=0}^{\infty}x(k)$	$X(1)$

References

[1] Dipartimento di Ingegneria dellinformazione e scienze matematiche Dipartimento di Ingegneria dellinformazione e scienze matematiche. (2016, May) Table of Laplace and Z-transforms . (Accessed May, 2016). [Online]. Available: http://www.dii.unisi.it/~control/sdc/altro/TabellaTrasformataZ.pdf