EEE3069W Formula Sheet

General

$s^2x + 2\zeta\omega_0 sx + \omega_0^2 x$
$g_{CL} = \frac{L}{1+L}$
$ZOH = (1 - z^{-1}) = \frac{1-e^{-sT}}{s}$
$t_{2\%} \simeq 4\tau$
$\zeta = cos(\theta)$
$\tau = \frac{-1}{Re\{s\}} = \frac{1}{\zeta\omega_n}$
$z = e^{sT} = e^{(\sigma+j\omega)T} = e^{\sigma T}e^{j\omega T} = \rho e^{\varphi}$
$t_{2\%} = b : R\{z\} < e^{\frac{-4T}{b}}$

Jury Stability Test

$A(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n = 0$
$A(1) > 0$, $A(-1) > 0$ for n even, $A(-1) < 0$ for n odd.
$|a_0| < |a_n|$, $|b_0| > |b_{n-1}|$, $|c_0| > |c_{n-2}|$, ...
where $b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}$, ...

Z-Transform

$$Z[x(kT)] = \sum_{k=0}^{\infty} x(kT)z^{-k}$$

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$$

$e^{j\theta} = cos(\theta) + jsin(\theta)$
$sin(\theta) = \frac{e^{j\theta}-e^{-j\theta}}{2j}$

Sampling

$g^*(s) = g(z)$
$g^*(t) = g(nT)$
$y(s) = g(s)u(s); y^*(s) = gu^*(s);$
$y(z) = gu(z)$
$\frac{y(z)}{r(z)}$: pulse transfer function.
$u^*(s + jm\Omega) = u^*(s); \Omega = \frac{2\pi}{T};$
$\Omega/2 > BW$

Controllability and Observability

$\frac{d}{dt}x(t) = Ax(t) + bu(t)$ *with n states.*
$y(t) = C^T x(t) + du(t)$
C: $Rank[b|Ab|A^2b|...|A^{n-1}b] = n$
O: $Rank[C^T|C^TA|C^TA^2|...|C^TA^{n-1}]^T = n$
where R(rows) = R(columns) = LI columns / no. of pivots / no. zero rows in RRE form.
Can you 'see' control action (u(t)) and output?
$x = P\bar{x}; P^{-1}AP = D$

Discrete Controller Implementation

Difference equation:
$k(z) = \frac{u(z)}{e(z)}; u(z)z^{-a} = u(n-a)$
Direct: powers of z^{-1}; introduce internal state variable x(z); equate denominator then numerator; no. past values = no. poles
Cascade: split into first order blocks connected in cascade; apply direct; use internal variables to equate
Parallel: partial fraction expansion; apply direct; use internal variables to equate
$e(z) \Rightarrow k(z) \Rightarrow u(z)$

State Feedback Control

$u(t) = -k^T x(t) + r(t)$
Replace u(t) in basic state space model and take Laplace:
$g(s) = \frac{y}{r} = C^T \frac{(sI-A+bk^T)b}{det(sI-A+bk^T)}$
$\phi_c = det(sI - A + bk^T)$
$det(sI - A + bk^T) = 0$
for closed-loop poles = eigenvalues of $(\bar{A} = A - bk^T)$.

Set-point tracking: *add K/s, block diagram manipulation, state space for integrator:*
$\frac{d}{dt}x_a = r(t) - C^T x(t) + du(t)$
$z = -k_a x_a$
Integrator output becomes a state and augmented matrix formed:
$\frac{d}{dt}\begin{bmatrix} x \\ x_a \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C^T & 0 \end{bmatrix}\begin{bmatrix} x \\ x_a \end{bmatrix} + \begin{bmatrix} b \\ -d \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$
$u(t) = -\begin{bmatrix} k^T & k_a \end{bmatrix}\begin{bmatrix} x \\ x_a \end{bmatrix}$ *subbing this in gives:*
$\frac{d}{dt}\begin{bmatrix} x \\ x_a \end{bmatrix} = (\bar{A} - \bar{b}\bar{k}^T)\begin{bmatrix} x \\ x_a \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$
$\phi_c = |sI - \bar{A} + \bar{b}\bar{k}^T|$

Observer: *the observer state is added and error poles can be designed separately from CL poles:*
$\frac{d}{dt}\tilde{x}(t) = A\tilde{x}(t) + bu(t) + p[y(t) - \tilde{y}(t)]$
$\tilde{y}(t) = C^T\tilde{x}(t)$
state observer is a mathematical model of plant with error term. $\tilde{x}(t)$ *is an estimated state.*
Introducing error state:
$\frac{d}{dt}e = x - \bar{x} = (A - pC^T)e$
$|sI - A + bk^T|\,|sI - A + pC^T| = 0$

Modified Z-Transform

$Z_m[gh(s)] = Z[gh(s)e^{-s\theta}]$
$m = 1 - \frac{\theta}{T} = 1 - \Delta$

Other Transformations

$z = \frac{1+w}{1-w}$
$w = \frac{z-1}{z+1}$

$z = \frac{1+w'\frac{T}{2}}{1-w'\frac{T}{2}}$
$w' = \frac{2(z-1)}{T(z+1)}$
Used for: Nyquist of gh(z)

$z = e^{j\omega T}$
$w = \frac{e^{j\omega T}-1}{e^{j\omega T}+1} = jtan(wT/2) \simeq j\omega T/2 \simeq j\Omega$

Root Locus

$n_\infty = n_p - n_z$
$angle = \frac{180^0 - 360^0 l}{n_\infty}$
$center = \frac{sumofpoles - sumofzeros}{n_\infty}$
Symmetric about real axis.
Left of odd number of poles and zeros.
$q(s) = \gamma\frac{N_q(s)}{D_q(s)}$
$\frac{d}{ds}\gamma(s) = N_q(s)\frac{d}{ds}D_q(s) - D_q(s)\frac{d}{ds}N_q(s) = 0$
Between open loop poles - breakaway point.

Controller Synthesis

$k(z) = \frac{1}{gh(z)} \cdot \frac{\frac{y(z)}{r(z)}}{1-\frac{y(z)}{r(z)}}$
Dalin: $\frac{z^{-N}(1-e^{\frac{-T}{\tau}})}{z-e^{\frac{-T}{\tau}}}$

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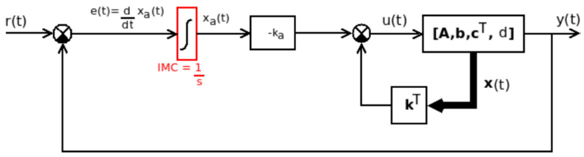
Table 2.1: Summary of Laplace Transform Properties

Description	Time Domain	Laplace Domain
Linearity	$\mathscr{L}[(\gamma f(t) + \beta g(t))]$	$\gamma F(s) + \beta G(s)$
Differentiation	$\mathscr{L}[\frac{d^n}{dt^n} f(t)]$	$s^n F(s)$
Integration	$\mathscr{L}[\int_1 \dots \int_n f(t)dt]$	$\frac{F(s)}{s^n}$
Final Value Theorem	$\lim_{t\to\infty} f(t)$	$\lim_{s\to 0} sF(s)$
Initial Value Theorem	$\lim_{t\to 0} f(t)$	$\lim_{s\to\infty} sF(s)$
Time Delay	$\mathscr{L}[f(t-d)]$	$e^{-sd}F(s)$
Time Scaling	$\mathscr{L}[f(\frac{t}{\alpha})]$	$\alpha F(\alpha s)$
Frequency Scaling	$\alpha f(\alpha t)$	$\mathscr{L}^{-1}[F(\frac{s}{\alpha})]$
Complex Translation	$\mathscr{L}[e^{-\alpha t} f(t)]$	$F(s+\alpha)$
Multiplication	$\mathscr{L}[f(t) \times g(t)]$	$F(s) * G(s)$
Convolution	$\mathscr{L}[f(t) * g(t)]$	$F(s) \times G(s)$



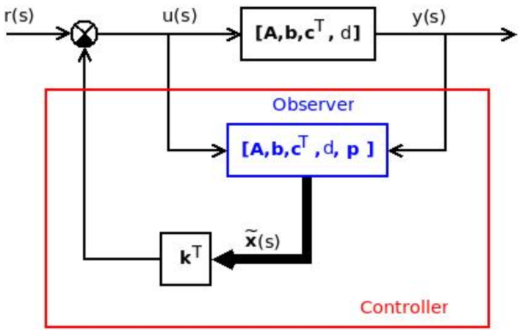


Table of Laplace and Z-transforms

	$X(s)$	$x(t)$	$x(kT)$ or $x(k)$	$X(z)$
1.	–	–	Kronecker delta $\delta_0(k)$ 1 $\quad k = 0$ 0 $\quad k \neq 0$	1
2.	–	–	$\delta_0(n-k)$ 1 $\quad n = k$ 0 $\quad n \neq k$	z^{-k}
3.	$\frac{1}{s}$	$1(t)$	$1(k)$	$\frac{1}{1-z^{-1}}$
4.	$\frac{1}{s+a}$	e^{-at}	e^{-akT}	$\frac{1}{1-e^{-aT}z^{-1}}$
5.	$\frac{1}{s^2}$	t	kT	$\frac{Tz^{-1}}{\left(1-z^{-1}\right)^2}$
6.	$\frac{2}{s^3}$	t^2	$(kT)^2$	$\frac{T^2z^{-1}\left(1+z^{-1}\right)}{\left(1-z^{-1}\right)^3}$
7.	$\frac{6}{s^4}$	t^3	$(kT)^3$	$\frac{T^3z^{-1}\left(1+4z^{-1}+z^{-2}\right)}{\left(1-z^{-1}\right)^4}$
8.	$\frac{a}{s(s+a)}$	$1 - e^{-at}$	$1 - e^{-akT}$	$\frac{\left(1-e^{-aT}\right)z^{-1}}{\left(1-z^{-1}\right)\left(1-e^{-aT}z^{-1}\right)}$
9.	$\frac{b-a}{(s+a)(s+b)}$	$e^{-at} - e^{-bt}$	$e^{-akT} - e^{-bkT}$	$\frac{\left(e^{-aT}-e^{-bT}\right)z^{-1}}{\left(1-e^{-aT}z^{-1}\right)\left(1-e^{-bT}z^{-1}\right)}$
10.	$\frac{1}{(s+a)^2}$	te^{-at}	kTe^{-akT}	$\frac{Te^{-aT}z^{-1}}{\left(1-e^{-aT}z^{-1}\right)^2}$
11.	$\frac{s}{(s+a)^2}$	$(1 - at)e^{-at}$	$(1 - akT)e^{-akT}$	$\frac{1-(1+aT)e^{-aT}z^{-1}}{\left(1-e^{-aT}z^{-1}\right)^2}$
12.	$\frac{2}{(s+a)^3}$	t^2e^{-at}	$(kT)^2e^{-akT}$	$\frac{T^2e^{-aT}\left(1+e^{-aT}z^{-1}\right)z^{-1}}{\left(1-e^{-aT}z^{-1}\right)^3}$
13.	$\frac{a^2}{s^2(s+a)}$	$at - 1 + e^{-at}$	$akT - 1 + e^{-akT}$	$\frac{\left[\left(aT-1+e^{-aT}\right)+\left(1-e^{-aT}-aTe^{-aT}\right)z^{-1}\right]z^{-1}}{\left(1-z^{-1}\right)^2\left(1-e^{-aT}z^{-1}\right)}$
14.	$\frac{\omega}{s^2+\omega^2}$	$\sin \omega t$	$\sin \omega kT$	$\frac{z^{-1}\sin\omega T}{1-2z^{-1}\cos\omega T+z^{-2}}$
15.	$\frac{s}{s^2+\omega^2}$	$\cos \omega t$	$\cos \omega kT$	$\frac{1-z^{-1}\cos\omega T}{1-2z^{-1}\cos\omega T+z^{-2}}$
16.	$\frac{\omega}{(s+a)^2+\omega^2}$	$e^{-at}\sin \omega t$	$e^{-akT}\sin \omega kT$	$\frac{e^{-aT}z^{-1}\sin\omega T}{1-2e^{-aT}z^{-1}\cos\omega T+e^{-2aT}z^{-2}}$
17.	$\frac{s+a}{(s+a)^2+\omega^2}$	$e^{-at}\cos \omega t$	$e^{-akT}\cos \omega kT$	$\frac{1-e^{-aT}z^{-1}\cos\omega T}{1-2e^{-aT}z^{-1}\cos\omega T+e^{-2aT}z^{-2}}$
18.	–	–	a^k	$\frac{1}{1-az^{-1}}$
19.	–	–	a^{k-1} $k = 1, 2, 3, \ldots$	$\frac{z^{-1}}{1-az^{-1}}$
20.	–	–	ka^{k-1}	$\frac{z^{-1}}{\left(1-az^{-1}\right)^2}$
21.	–	–	k^2a^{k-1}	$\frac{z^{-1}\left(1+az^{-1}\right)}{\left(1-az^{-1}\right)^3}$
22.	–	–	k^3a^{k-1}	$\frac{z^{-1}\left(1+4az^{-1}+a^2z^{-2}\right)}{\left(1-az^{-1}\right)^4}$
23.	–	–	k^4a^{k-1}	$\frac{z^{-1}\left(1+11az^{-1}+11a^2z^{-2}+a^3z^{-3}\right)}{\left(1-az^{-1}\right)^5}$
24.	–	–	$a^k \cos k\pi$	$\frac{1}{1+az^{-1}}$

$x(t) = 0$ for $t < 0$
$x(kT) = x(k) = 0$ for $k < 0$
Unless otherwise noted, $k = 0, 1, 2, 3, \ldots$

Definition of the Z-transform

$$\mathscr{Z}\{x(k)\} = X(z) = \sum_{k=0}^{\infty} x(k) z^{-k}$$

Important properties and theorems of the Z-transform

	$x(t)$ or $x(k)$	$\mathcal{Z}\{x(t)\}$ or $\mathcal{Z}\{x(k)\}$
1.	$ax(t)$	$aX(z)$
2.	$ax_1(t)+bx_2(t)$	$aX_1(z)+bX_2(z)$
3.	$x(t+T)$ or $x(k+1)$	$zX(z)-zx(0)$
4.	$x(t+2T)$	$z^2X(z)-z^2x(0)-zx(T)$
5.	$x(k+2)$	$z^2X(z)-z^2x(0)-zx(1)$
6.	$x(t+kT)$	$z^kX(z)-z^kx(0)-z^{k-1}x(T)-\ldots-zx(kT-T)$
7.	$x(t-kT)$	$z^{-k}X(z)$
8.	$x(n+k)$	$z^kX(z)-z^kx(0)-z^{k-1}x(1)-\ldots-zx(k1-1)$
9.	$x(n-k)$	$z^{-k}X(z)$
10.	$tx(t)$	$-Tz\dfrac{d}{dz}X(z)$
11.	$kx(k)$	$-z\dfrac{d}{dz}X(z)$
12.	$e^{-at}x(t)$	$X(ze^{aT})$
13.	$e^{-ak}x(k)$	$X(ze^{a})$
14.	$a^kx(k)$	$X\left(\dfrac{z}{a}\right)$
15.	$ka^kx(k)$	$-z\dfrac{d}{dz}X\left(\dfrac{z}{a}\right)$
16.	$x(0)$	$\lim_{z\to\infty} X(z)$ if the limit exists
17.	$x(\infty)$	$\lim_{z\to1}\left[\left(1-z^{-1}\right)X(z)\right]$ if $\left(1-z^{-1}\right)X(z)$ is analytic on and outside the unit circle
18.	$\nabla x(k)=x(k)-x(k-1)$	$\left(1-z^{-1}\right)X(z)$
19.	$\Delta x(k)=x(k+1)-x(k)$	$(z-1)X(z)-zx(0)$
20.	$\sum_{k=0}^{n}x(k)$	$\dfrac{1}{1-z^{-1}}X(z)$
21.	$\dfrac{\partial}{\partial a}x(t,a)$	$\dfrac{\partial}{\partial a}X(z,a)$
22.	$k^mx(k)$	$\left(-z\dfrac{d}{dz}\right)^m X(z)$
23.	$\sum_{k=0}^{n}x(kT)y(nT-kT)$	$X(z)Y(z)$
24.	$\sum_{k=0}^{\infty}x(k)$	$X(1)$

References

[1] Dipartimento di Ingegneria dellinformazione e scienze matematiche Dipartimento di Ingegneria dellinformazione e scienze matematiche. (2016, May) Table of Laplace and Z-transforms . (Accessed May, 2016). [Online]. Available: http://www.dii.unisi.it/~control/sdc/altro/TabellaTrasformataZ.pdf

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