

EEE3069W Formula Sheet

General

$$s^2x + 2\zeta\omega_0sx + \omega_0^2x$$

$$g_{CL} = \frac{L}{1+L}$$

$$ZOH = (1 - z^{-1}) = \frac{1 - e^{-sT}}{s}$$

$$t_{2\%} \simeq 4\tau$$

$$\zeta = \cos(\theta)$$

$$\tau = \frac{-1}{\operatorname{Re}\{s\}} = \frac{1}{\zeta\omega_n}$$

$$z = e^{sT} = e^{(\sigma+j\omega)T} = e^{\sigma T} e^{j\omega T} = \rho e^{j\varphi}$$

$$t_{2\%} = b : R\{z\} < e^{-\frac{4T}{b}}$$

Jury Stability Test

$$A(z) = a_0 + a_1z + a_2z^2 + \dots + a_nz^n = 0$$

$$A(1) > 0, A(-1) > 0 \text{ for } n \text{ even, } A(-1) < 0 \text{ for } n \text{ odd.}$$

$$|a_0| < |a_n|, |b_0| > |b_{n-1}|, |c_0| > |c_{n-2}|, \dots$$

$$\text{where } b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}, \dots$$

Z-Transform

$$Z[x(kT)] = \sum_{k=0}^{\infty} x(kT)z^{-k}$$

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$$

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Sampling

$$g^*(s) = g(z)$$

$$g^*(t) = g(nT)$$

$$y(s) = g(s)u(s); y^*(s) = gu^*(s);$$

$$y(z) = gu(z)$$

$$\frac{y(z)}{r(z)}: \text{pulse transfer function.}$$

$$u^*(s + jm\Omega) = u^*(s); \Omega = \frac{2\pi}{T};$$

$$\Omega/2 > BW$$

Controllability and Observability

$$\frac{d}{dt}x(t) = Ax(t) + bu(t) \text{ with } n \text{ states.}$$

$$y(t) = C^T x(t) + du(t)$$

$$C: \operatorname{Rank}[b|Ab|A^2b|\dots|A^{n-1}b] = n$$

$$O: \operatorname{Rank}[C^T|C^T A|C^T A^2|\dots|C^T A^{n-1}]^T = n$$

where $R(\text{rows}) = R(\text{columns}) = LI \text{ columns}$
/ no. of pivots / no. zero rows in RRE form.

Can you 'see' control action ($u(t)$) and output?

$$x = P\bar{x}; P^{-1}AP = D$$

Discrete Controller Implementation

Difference equation:

$$k(z) = \frac{u(z)}{e(z)}; u(z)z^{-a} = u(n-a)$$

Direct: powers of z^{-1} ; introduce internal state variable $x(z)$; equate denominator then numerator; no. past values = no. poles

Cascade: split into first order blocks connected in cascade; apply direct; use internal variables to equate

Parallel: partial fraction expansion; apply direct; use internal variables to equate $e(z) \Rightarrow k(z) \Rightarrow u(z)$

State Feedback Control

$$u(t) = -k^T x(t) + r(t)$$

Replace $u(t)$ in basic state space model and take Laplace:

$$g(s) = \frac{y}{r} = C^T \frac{(sI - A + bk^T)b}{\det(sI - A + bk^T)}$$

$$\phi_c = \det(sI - A + bk^T)$$

$$\det(sI - A + bk^T) = 0$$

for closed-loop poles = eigenvalues of $(\bar{A} = A - bk^T)$.

Set-point tracking: add K/s , block diagram manipulation, state space for integrator:

$$\frac{d}{dt}x_a = r(t) - C^T x(t) + du(t)$$

$$z = -k_a x_a$$

Integrator output becomes a state and augmented matrix formed:

$$\frac{d}{dt} \begin{bmatrix} x \\ x_a \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C^T & 0 \end{bmatrix} \begin{bmatrix} x \\ x_a \end{bmatrix} + \begin{bmatrix} b \\ -d \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

$$u(t) = -[k^T \quad k_a] \begin{bmatrix} x \\ x_a \end{bmatrix} \text{ subbing this in gives:}$$

$$\frac{d}{dt} \begin{bmatrix} x \\ x_a \end{bmatrix} = (\bar{A} - \bar{b}\bar{k}^T) \begin{bmatrix} x \\ x_a \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

$$\phi_c = |sI - \bar{A} + \bar{b}\bar{k}^T|$$

Observer: the observer state is added and error poles can be designed separately from CL poles:

$$\frac{d}{dt}\tilde{x}(t) = A\tilde{x}(t) + bu(t) + p[y(t) - \tilde{y}(t)]$$

$$\tilde{y}(t) = C^T \tilde{x}(t)$$

state observer is a mathematical model of plant with error term. $\tilde{x}(t)$ is an estimated state.

Introducing error state:

$$\frac{d}{dt}e = x - \bar{x} = (A - pC^T)e$$

$$\left| sI - A + bk^T \right| \left| sI - A + pC^T \right| = 0$$

Modified Z-Transform

$$Z_m[gh(s)] = Z[gh(s)e^{-s\theta}]$$

$$m = 1 - \frac{\theta}{T} = 1 - \Delta$$

Other Transformations

$$z = \frac{1+w}{1-w}$$

$$w = \frac{z-1}{z+1}$$

$$z = \frac{1+w'\frac{T}{2}}{1-w'\frac{T}{2}}$$

$$w' = \frac{2(z-1)}{T(z+1)}$$

Used for: Nyquist of $gh(z)$

$$z = e^{j\omega T}$$

$$w = \frac{e^{j\omega T} - 1}{e^{j\omega T} + 1} = j \tan(\omega T/2) \simeq j\omega T/2 \simeq j\Omega$$

Root Locus

$$n_{\infty} = n_p - n_z$$

$$\text{angle} = \frac{180^\circ - 360^\circ l}{n_{\infty}}$$

$$\text{center} = \frac{\text{sum of poles} - \text{sum of zeros}}{n_{\infty}}$$

Symmetric about real axis.

Left of odd number of poles and zeros.

$$q(s) = \gamma \frac{N_q(s)}{D_q(s)}$$

$$\frac{d}{ds}\gamma(s) = N_q(s)\frac{d}{ds}D_q(s) - D_q(s)\frac{d}{ds}N_q(s) = 0$$

Between open loop poles - breakaway point.

Controller Synthesis

$$k(z) = \frac{1}{gh(z)} \cdot \frac{\frac{y(z)}{r(z)}}{1 - \frac{y(z)}{r(z)}}$$

$$\text{Dalin: } \frac{z^{-N}(1 - e^{-\frac{T}{\tau}})}{z - e^{-\frac{T}{\tau}}}$$

Table 2.1: Summary of Laplace Transform Properties

Description	Time Domain	Laplace Domain
Linearity	$\mathcal{L}[(\gamma f(t) + \beta g(t))]$	$\gamma F(s) + \beta G(s)$
Differentiation	$\mathcal{L}[\frac{d^n}{dt^n} f(t)]$	$s^n F(s)$
Integration	$\mathcal{L}[\int_1 \dots \int_n f(t) dt]$	$\frac{F(s)}{s^n}$
Final Value Theorem	$\lim_{t \rightarrow \infty} f(t)$	$\lim_{s \rightarrow 0} sF(s)$
Initial Value Theorem	$\lim_{t \rightarrow 0} f(t)$	$\lim_{s \rightarrow \infty} sF(s)$
Time Delay	$\mathcal{L}[f(t - d)]$	$e^{-sd} F(s)$
Time Scaling	$\mathcal{L}[f(\frac{t}{\alpha})]$	$\alpha F(\alpha s)$
Frequency Scaling	$\alpha f(\alpha t)$	$\mathcal{L}^{-1}[F(\frac{s}{\alpha})]$
Complex Translation	$\mathcal{L}[e^{-\alpha t} f(t)]$	$F(s + \alpha)$
Multiplication	$\mathcal{L}[f(t) \times g(t)]$	$F(s) * G(s)$
Convolution	$\mathcal{L}[f(t) * g(t)]$	$F(s) \times G(s)$

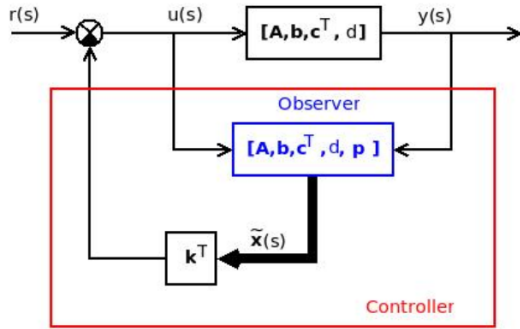
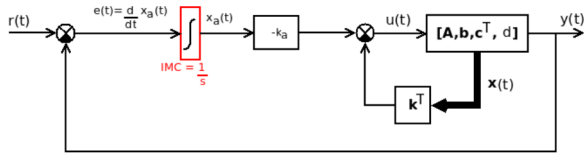


Table of Laplace and Z-transforms

	$X(s)$	$x(t)$	$x(kT)$ or $x(k)$	$X(z)$
1.	–	–	Kronecker delta $\delta_0(k)$ 1 $k = 0$ 0 $k \neq 0$	1
2.	–	–	$\delta_0(n-k)$ 1 $n = k$ 0 $n \neq k$	z^{-k}
3.	$\frac{1}{s}$	$1(t)$	$1(k)$	$\frac{1}{1-z^{-1}}$
4.	$\frac{1}{s+a}$	e^{-at}	e^{-akT}	$\frac{1}{1-e^{-aT}z^{-1}}$
5.	$\frac{1}{s^2}$	t	kT	$\frac{Tz^{-1}}{(1-z^{-1})^2}$
6.	$\frac{2}{s^3}$	t^2	$(kT)^2$	$\frac{T^2 z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$
7.	$\frac{6}{s^4}$	t^3	$(kT)^3$	$\frac{T^3 z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4}$
8.	$\frac{a}{s(s+a)}$	$1-e^{-at}$	$1-e^{-akT}$	$\frac{(1-e^{-aT})z^{-1}}{(1-z^{-1})(1-e^{-aT}z^{-1})}$
9.	$\frac{b-a}{(s+a)(s+b)}$	$e^{-at}-e^{-bt}$	$e^{-akT}-e^{-bkT}$	$\frac{(e^{-aT}-e^{-bT})z^{-1}}{(1-e^{-aT}z^{-1})(1-e^{-bT}z^{-1})}$
10.	$\frac{1}{(s+a)^2}$	te^{-at}	kTe^{-akT}	$\frac{Te^{-aT}z^{-1}}{(1-e^{-aT}z^{-1})^2}$
11.	$\frac{s}{(s+a)^2}$	$(1-at)e^{-at}$	$(1-akT)e^{-akT}$	$\frac{1-(1+aT)e^{-aT}z^{-1}}{(1-e^{-aT}z^{-1})^2}$
12.	$\frac{2}{(s+a)^3}$	$t^2 e^{-at}$	$(kT)^2 e^{-akT}$	$\frac{T^2 e^{-aT}(1+e^{-aT}z^{-1})z^{-1}}{(1-e^{-aT}z^{-1})^3}$
13.	$\frac{a^2}{s^2(s+a)}$	$at-1+e^{-at}$	$akT-1+e^{-akT}$	$\frac{[(aT-1+e^{-aT})+(1-e^{-aT}-aTe^{-aT})z^{-1}]z^{-1}}{(1-z^{-1})^2(1-e^{-aT}z^{-1})}$
14.	$\frac{\omega}{s^2+\omega^2}$	$\sin \omega t$	$\sin \omega kT$	$\frac{z^{-1} \sin \omega T}{1-2z^{-1} \cos \omega T + z^{-2}}$
15.	$\frac{s}{s^2+\omega^2}$	$\cos \omega t$	$\cos \omega kT$	$\frac{1-z^{-1} \cos \omega T}{1-2z^{-1} \cos \omega T + z^{-2}}$
16.	$\frac{\omega}{(s+a)^2+\omega^2}$	$e^{-at} \sin \omega t$	$e^{-akT} \sin \omega kT$	$\frac{e^{-aT} z^{-1} \sin \omega T}{1-2e^{-aT} z^{-1} \cos \omega T + e^{-2aT} z^{-2}}$
17.	$\frac{s+a}{(s+a)^2+\omega^2}$	$e^{-at} \cos \omega t$	$e^{-akT} \cos \omega kT$	$\frac{1-e^{-aT} z^{-1} \cos \omega T}{1-2e^{-aT} z^{-1} \cos \omega T + e^{-2aT} z^{-2}}$
18.	–	–	a^k	$\frac{1}{1-az^{-1}}$
19.	–	–	a^{k-1} $k = 1, 2, 3, \dots$	$\frac{z^{-1}}{1-az^{-1}}$
20.	–	–	ka^{k-1}	$\frac{z^{-1}}{(1-az^{-1})^2}$
21.	–	–	$k^2 a^{k-1}$	$\frac{z^{-1}(1+az^{-1})}{(1-az^{-1})^3}$
22.	–	–	$k^3 a^{k-1}$	$\frac{z^{-1}(1+4az^{-1}+a^2 z^{-2})}{(1-az^{-1})^4}$
23.	–	–	$k^4 a^{k-1}$	$\frac{z^{-1}(1+11az^{-1}+11a^2 z^{-2}+a^3 z^{-3})}{(1-az^{-1})^5}$
24.	–	–	$a^k \cos k\pi$	$\frac{1}{1+az^{-1}}$

$x(t) = 0$ for $t < 0$

$x(kT) = x(k) = 0$ for $k < 0$

Unless otherwise noted, $k = 0, 1, 2, 3, \dots$

Definition of the Z-transform

$$\mathcal{X}\{x(k)\} = X(z) = \sum_{k=0}^{\infty} x(k)z^{-k}$$

Important properties and theorems of the Z-transform

	$x(t)$ or $x(k)$	$Z\{x(t)\}$ or $Z\{x(k)\}$
1.	$ax(t)$	$aX(z)$
2.	$ax_1(t) + bx_2(t)$	$aX_1(z) + bX_2(z)$
3.	$x(t+T)$ or $x(k+1)$	$zX(z) - zx(0)$
4.	$x(t+2T)$	$z^2X(z) - z^2x(0) - zx(T)$
5.	$x(k+2)$	$z^2X(z) - z^2x(0) - zx(1)$
6.	$x(t+kT)$	$z^kX(z) - z^kx(0) - z^{k-1}x(T) - \dots - zx(kT-T)$
7.	$x(t-kT)$	$z^{-k}X(z)$
8.	$x(n+k)$	$z^kX(z) - z^kx(0) - z^{k-1}x(1) - \dots - zx(k-1)$
9.	$x(n-k)$	$z^{-k}X(z)$
10.	$tx(t)$	$-Tz \frac{d}{dz} X(z)$
11.	$kx(k)$	$-z \frac{d}{dz} X(z)$
12.	$e^{-at}x(t)$	$X(ze^{aT})$
13.	$e^{-ak}x(k)$	$X(ze^a)$
14.	$a^kx(k)$	$X\left(\frac{z}{a}\right)$
15.	$ka^kx(k)$	$-z \frac{d}{dz} X\left(\frac{z}{a}\right)$
16.	$x(0)$	$\lim_{z \rightarrow \infty} X(z)$ if the limit exists
17.	$x(\infty)$	$\lim_{z \rightarrow 1} \left[(1-z^{-1})X(z) \right]$ if $(1-z^{-1})X(z)$ is analytic on and outside the unit circle
18.	$\nabla x(k) = x(k) - x(k-1)$	$(1-z^{-1})X(z)$
19.	$\Delta x(k) = x(k+1) - x(k)$	$(z-1)X(z) - zx(0)$
20.	$\sum_{k=0}^n x(k)$	$\frac{1}{1-z^{-1}} X(z)$
21.	$\frac{\partial}{\partial a} x(t, a)$	$\frac{\partial}{\partial a} X(z, a)$
22.	$k^m x(k)$	$\left(-z \frac{d}{dz}\right)^m X(z)$
23.	$\sum_{k=0}^n x(kT)y(nT-kT)$	$X(z)Y(z)$
24.	$\sum_{k=0}^{\infty} x(k)$	$X(1)$

References

- [1] Dipartimento di Ingegneria dell'informazione e scienze matematiche Dipartimento di Ingegneria dell'informazione e scienze matematiche. (2016, May) Table of Laplace and Z-transforms . (Accessed May, 2016). [Online]. Available: <http://www.dii.unisi.it/~control/sdc/altro/TabellaTrasformataZ.pdf>

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