

Using Linear Programming to Find Shortest Path

Darice Guittet and Benjamin Shoeman

8 November 2017

1 Abstract

2 Introduction

The problem of optimization arises in many contexts, from logistical considerations such as scheduling appointments, to financial concerns such as maximizing return from investments, to technical questions regarding control of interconnected systems, such as a power grid network. Linear programming is a group of techniques for tackling such problems when the system and its constraints can be represented linearly. Specifically, we investigate the Simplex Algorithm, which is a simple approach which can be proven via the Duality Theorem to return the optimal value, provided such a feasible solution exists. We apply our Simplex solver on the shortest-path problem on a graph by formulating it as a minimum flow problem. While the formulation can be generalized to more complex flow problems, we turn to the problem of finding the shortest path to reach one location on campus from another. When a student has only ten minutes between classes to get to another, an optimal path is desirable.

2.1 Is there an Optimal Solution?

Linear programming is the process of optimizing a system by maximizing or minimizing an objective function given inequalities as constraints. Given a linear program, it can be: feasible bounded, if a solution exists and the objective function is bounded; feasible unbounded, if a solution exists but the objective function can assume arbitrarily large values; or infeasible. In this section, we are interested in determining whether a linear program has an optimal, a feasible vector which gives the objective function its highest value.

Suppose we want to find the values of $x_1, x_2, \dots, x_n \geq 0$ that maximize the objective function:

$$c_1x_1 + \dots + c_nx_n = \mathbf{c}^T\mathbf{x} = z(\mathbf{x})$$

subject to the constraints:

$$\begin{aligned}
x_1 a_{1,1} + \cdots + x_i a_{1,i} + \cdots + x_n a_{1,n} &\leq b_1 \\
&\vdots \\
x_1 a_{m,1} + \cdots + x_i a_{m,i} + \cdots + x_n a_{m,n} &\leq b_m \\
x_1, x_2, \cdots, x_n &\geq 0
\end{aligned}$$

In the maximum problem, the coefficients of the objective function, c_i , represent the cost or value of each element x_i , and the vector \mathbf{b} represents requirements on properties of x , tabulated by the matrix A . By requiring nonnegativity of x_i in the last constraint, we put the problem into standard form. All problems can be put into standard form [2], in which all constraints are inequalities and $\mathbf{x} \geq 0$, so the following theory regarding standard problems apply to general problems. The standard maximum problem can be written as:

$$\begin{aligned}
&\text{maximize } z(\mathbf{x}) = \mathbf{c}^T \mathbf{x} \\
&\text{subject to } A\mathbf{x} \leq \mathbf{b} \\
&\text{where } \mathbf{x} \geq \mathbf{0}
\end{aligned} \tag{1}$$

The dual of this problem is the minimum problem. The maximum problem can be converted into its dual using the fact that $(a \leq b \wedge c \leq d) \implies y_1 a + y_2 c \leq y_1 b + y_2 d$, for some scaling factors $y_1, y_2 \in \mathbb{R}$.

For each constraint in the maximum problem, we can derive a new inequality by scaling each and looking at the sum:

$$\begin{aligned}
&y_1 (x_1 a_{1,1} + \cdots + x_i a_{1,i} + \cdots + x_n a_{1,n}) + \\
&\quad \cdots \\
&+ y_m (x_1 a_{m,1} + \cdots + x_i a_{m,i} + \cdots + x_n a_{m,n}) \\
&\leq \\
&y_1 b_1 + \cdots + y_m b_m
\end{aligned}$$

We rewrite the inequality by gathering x_i terms and find that for every feasible \mathbf{x} , a linear combination of the x_i must be less than an upper bound $\mathbf{y}^T \mathbf{b}$.

$$\begin{aligned}
&x_1 (y_1 a_{1,1} + \cdots + y_i a_{i,1} + \cdots + y_m a_{m,1}) + \\
&\quad \cdots \\
&+ x_n (y_1 a_{1,n} + \cdots + y_i a_{i,n} + \cdots + y_m a_{m,n}) \\
&\leq \\
&y_1 b_1 + \cdots + y_m b_m
\end{aligned}$$

If we choose y_i such that they satisfy the following constraints, where c_i are the coefficients of the objective function,

$$\begin{aligned}
c_1 &\leq y_1 a_{1,1} + \cdots + y_i a_{i,1} + \cdots + y_m a_{m,1} \\
&\vdots \\
c_n &\leq y_1 a_{1,n} + \cdots + y_i a_{i,n} + \cdots + y_m a_{m,n}
\end{aligned}$$

and multiply the i inequality by x_i while summing each inequality as we did before, we arrive at a lower bound for the same linear combination of the x_i for which we found an upper bound.

$$\begin{aligned}
&c_1 x_1 + \cdots + c_n x_n \\
&\leq \\
&x_1 (y_1 a_{1,1} + \cdots + y_i a_{i,1} + \cdots + y_m a_{m,1}) + \\
&\quad \cdots \\
&+ x_n (y_1 a_{1,n} + \cdots + y_i a_{i,n} + \cdots + y_m a_{m,n}) \\
&\leq \\
&y_1 b_1 + \cdots + y_m b_m
\end{aligned}$$

The dual of the standard maximum problem (1), the standard minimum problem, comes from solving for the scaling factors y_i for which we get as tight of an upper bound as possible.

$$\begin{aligned}
&\text{minimize } z'(\mathbf{y}) = \mathbf{y}^T \mathbf{b} \\
&\text{subject to } A^T \mathbf{y} \geq \mathbf{c} \\
&\text{where } \mathbf{y} \geq \mathbf{0}
\end{aligned} \tag{2}$$

In fact, the dual of the dual of the linear program is the linear program itself, called the *primal* [4], which can be formed by taking the negative of (2) to make it a maximum problem and applying the above process to find the dual.

$$\begin{array}{ll}
\text{maximize } -z'(\mathbf{y}) = -\mathbf{y}^T \mathbf{b} & \text{minimize } -z(\mathbf{y}) = -\mathbf{c}^T \mathbf{x} \\
\text{subject to } -A^T \mathbf{y} \leq -\mathbf{c} & \implies \text{subject to } -A \mathbf{x} \geq -\mathbf{b} \\
\text{where } \mathbf{y} \geq \mathbf{0} & \text{where } \mathbf{x} \geq \mathbf{0}
\end{array}$$

Then retaking the negative to find the maximum, we get the primal problem (1) back. We see from the inequality above that:

$$\mathbf{c}^T \mathbf{x} \leq \mathbf{y}^T A \mathbf{x} \leq \mathbf{y}^T \mathbf{b} \implies \text{opt}(\text{primal}) \leq \text{opt}(\text{dual}) \tag{3}$$

This leads to several results. One is that if a standard problem and its dual are both feasible, then both are bounded feasible because $\mathbf{y}^T \mathbf{b}$ is an upper bound for x for the maximum problem, and vice versa. By a similar reasoning, if the primal is unbounded then its dual is infeasible, and vice versa.

In the case of feasible problems, if there exists feasible x^* and y^* for the standard maximum problem and its dual such that $\mathbf{c}^T \mathbf{x}^* = \mathbf{y}^{*T} \mathbf{b}$ then both are optimal for

their respective problems because for any feasible x , it must satisfy (3) $\mathbf{c}^T \mathbf{x}^* \leq \mathbf{y}^{*T} \mathbf{b} = \mathbf{c}^T \mathbf{x}^*$. Therefore x attains its optimum at x^* , and the symmetric argument applies for y^* .

The Duality Theorem tells us that such an optimal solution always exists if one of the problems is bounded feasible.

Duality Theorem *If the primal or its dual is feasible and bounded, then so is the other, and $\text{opt}(\text{primal}) = \text{opt}(\text{dual})$.*

We shall see that the Simplex method is a constructive proof of this theorem.

3 Mathematical Formulation of Simplex Algorithm

References

- [1] Chandrasekaran, R. *Shortest Path*. <https://www.utdallas.edu/~chandra/documents/networks/net3.pdf>
- [2] Ferguson, T. S. *Linear Programming*. <https://www.math.ucla.edu/~tom/LP.pdf>
- [3] Gale, D. *Linear Programming and the Simplex Method*. Notices of the AMS. Volume 54, Number 3. <https://www.ams.org/notices/200703/fea-gale.pdf>
- [4] Trevisan, L. *Lecture 6 Notes*. <https://people.eecs.berkeley.edu/~luca/cs261/lecture06.pdf>