1. We have the following three equations:

$$\begin{aligned}\hat{y}_t &= -\sigma[\hat{i}_t - \mathbb{E}_t\{\hat{\pi}_{t+1}\}] + \mathbb{E}_t\{\hat{y}_{t+1}\}\\ \hat{\pi}_t &= \kappa(\hat{y}_t - \hat{y}_t^{flex}) + \beta\mathbb{E}_t\{\hat{\pi}_{t+1}\}\\ \hat{i}_t &= \phi_\pi\hat{\pi}_t + v_t\end{aligned} \tag{1}$$

with $\hat{y}_t^{flex} = \frac{1+\varphi}{\gamma+\varphi}\hat{a}_t$.

(a) Suppose that

$$\begin{aligned}\hat{y}_t &= \eta_{ya}\hat{a}_t\\ \hat{\pi}_t &= \eta_{\pi a}\hat{a}_t\\ \hat{i}_t &= \eta_{ia}\hat{a}_t\end{aligned} \tag{2}$$

Also note that $\hat{a}_t = \rho_a\hat{a}_{t-1} + \varepsilon_t$. Thus, we must have $\mathbb{E}_t\{\hat{a}_{t+1}\} = \rho_a\hat{a}_t$. Hence

$$\begin{aligned}\mathbb{E}_t\{\hat{y}_{t+1}\} &= \rho_a\eta_{ya}\hat{a}_t\\ \mathbb{E}_t\{\hat{\pi}_{t+1}\} &= \rho_a\eta_{\pi a}\hat{a}_t\\ \mathbb{E}_t\{\hat{i}_{t+1}\} &= \rho_a\eta_{ia}\hat{a}_t\end{aligned} \tag{3}$$

Substitute (2) and (3) into (1), and then divide $\hat{a}_t$ from each equatio, we have

$$\begin{aligned}\eta_{ya} &= -\sigma[\eta_{ia} - \rho_a\eta_{\pi a}] + \rho_a\eta_{ya}\\ \eta_{\pi a} &= \kappa\left(\eta_{ya} - \frac{1+\varphi}{\gamma+\varphi}\right) + \beta\rho_a\eta_{\pi a}\\ \eta_{ia} &= \phi_\pi\eta_{\pi a}\end{aligned} \tag{4}$$

Solving this linear system gives us

$$\begin{aligned}\eta_{ya} &= \frac{\kappa\sigma(\phi_\pi - \rho_a)(1+\varphi)}{[(1-\rho_a)(1-\beta\rho_a) + \kappa\sigma(\phi_\pi - \rho_a)](\gamma+\varphi)}\\ \eta_{\pi a} &= -\frac{\kappa(1+\varphi)(1-\rho_a)}{[(1-\rho_a)(1-\beta\rho_a) + \kappa\sigma(\phi_\pi - \rho_a)](\gamma+\varphi)}\\ \eta_{ia} &= -\frac{\kappa\phi_\pi(1+\varphi)(1-\rho_a)}{[(1-\rho_a)(1-\beta\rho_a) + \kappa\sigma(\phi_\pi - \rho_a)](\gamma+\varphi)}\end{aligned} \tag{5}$$

Hence, we get

$$\begin{aligned}\hat{y}_t &= \frac{\kappa\sigma(\phi_\pi - \rho_a)(1+\varphi)}{[(1-\rho_a)(1-\beta\rho_a) + \kappa\sigma(\phi_\pi - \rho_a)](\gamma+\varphi)}\hat{a}_t\\ \hat{\pi}_t &= -\frac{\kappa(1+\varphi)(1-\rho_a)}{[(1-\rho_a)(1-\beta\rho_a) + \kappa\sigma(\phi_\pi - \rho_a)](\gamma+\varphi)}\hat{a}_t\\ \hat{i}_t &= -\frac{\kappa\phi_\pi(1+\varphi)(1-\rho_a)}{[(1-\rho_a)(1-\beta\rho_a) + \kappa\sigma(\phi_\pi - \rho_a)](\gamma+\varphi)}\hat{a}_t\end{aligned} \tag{6}$$

(b) Let unknown $\boldsymbol{U} = \{\pi\}$. Let a and v be exogeneous. Note that here y^{flex} is the only variable that explicitly contains a_t. We only need to consider $Z = \{y^{flex}\}$. After calculating irf, we can multiply all result by $\frac{1+\varphi}{\gamma+\varphi}$. Since here $v = 0$, then we can ignore this variable. Other variables have the order y, c, r, i. See the code for details.

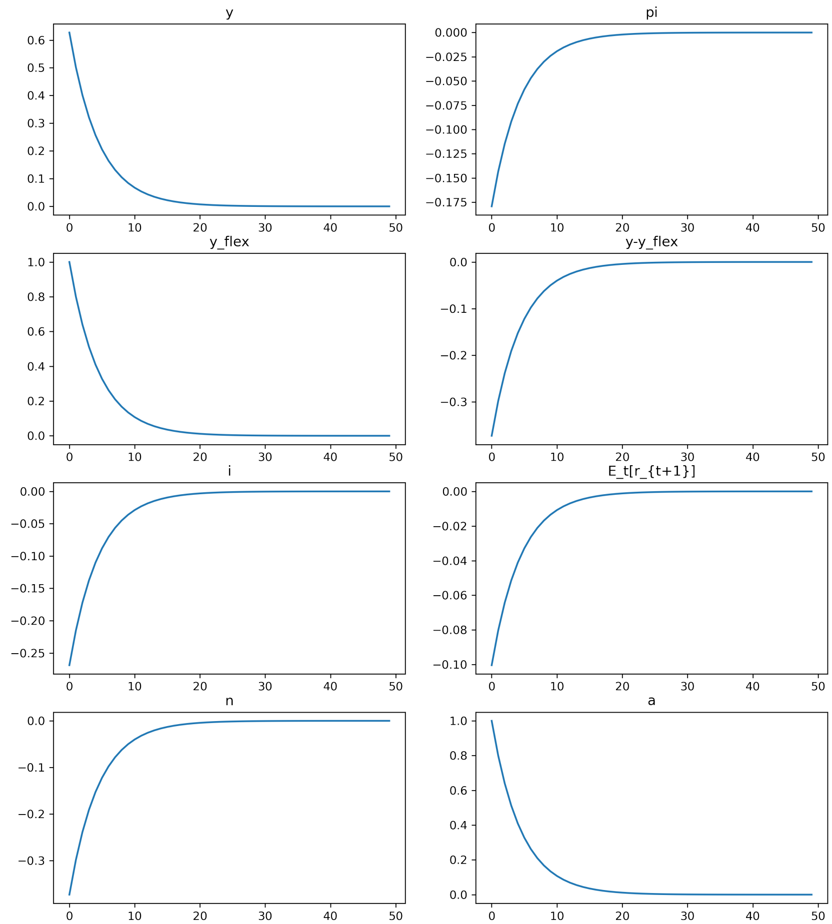

Figure 1: Sequence Space Method IRFs

An alternative way, which is much simplier, is to use the results from (a). We have

$$\hat{r}_t = \hat{i}_t - \mathbb{E}_t[\hat{\pi}_{t+1}] = (\eta_{ia} - \rho_a \eta_{\pi a})\hat{a}_t \tag{7}$$

Hence

$$\mathbb{E}_t[\hat{r}_{t+1}] = (\eta_{ia} - \rho_a \eta_{\pi a})\rho_a \hat{a}_t \tag{8}$$

Also

$$\hat{n}_t = \hat{y}_t - \hat{a}_t = (\eta_{ya} - 1)\hat{a}_t \tag{9}$$

Then we can easily plot those irfs.

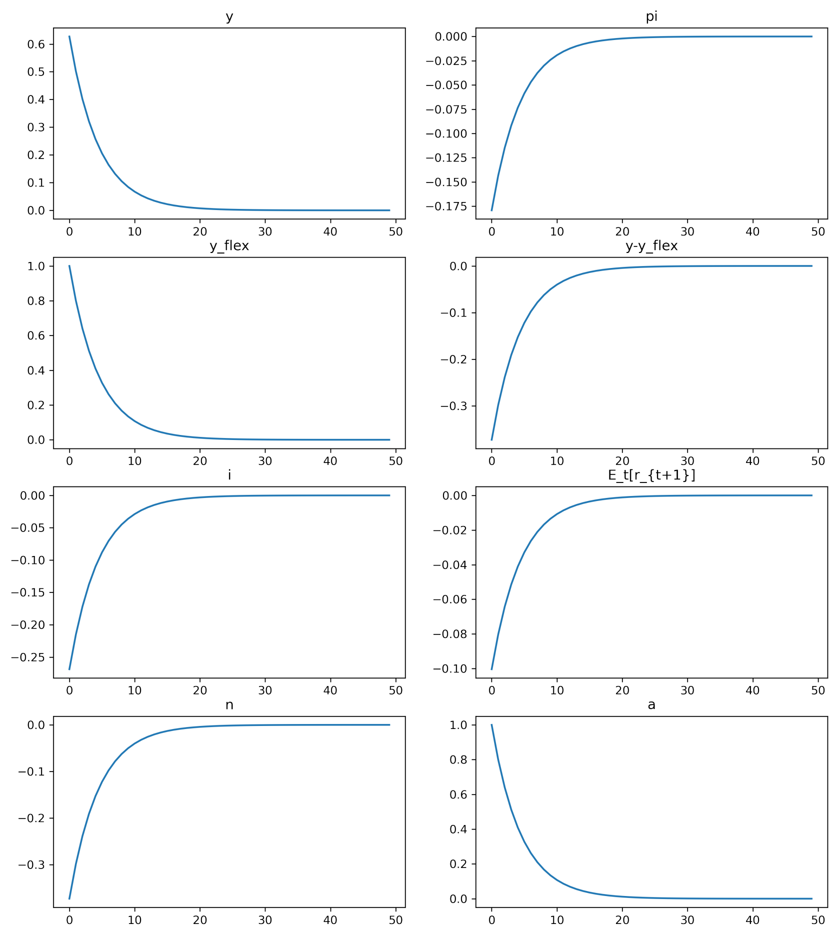

Figure 2: Explicit IRFs

We can see that those two figures are exactly the same.

(c) For a unit shock of a, we can see that the output will increase by 0.6 units while y^{flex} will increase 1 unit. Hence $y - y^{flex}$ is actually decreasing, by 0.4 units approximately. This is because not all firm can adjust prices when the TFP increases, although some of them can adjust. Therefore the overall output is increasing, but less than y^{flex}.

Also, when TFP is increasing, the firm will have less marginal cost and then they can receive the same profit even lowering the price. Hence there is deflation, and π_t is decreasing.

From $y_t = a_t + n_t$ we can easily see that as y_t increasing less than a_t, then n_t is actually decreasing.

The Taylor rule tells us the nominal interest rate $i_t = \phi_\pi \pi_t$ and then i_t is also decreasing.

Finally, we know that $r_t = i_t - \mathbb{E}_t[\pi_{t+1}]$, note that ϕ_π is greater than 1, we know that r_t is also decreasing.

(d) The verification has already been shown in 1(b).

2. (a) Actually, we can see that there is no P_t^* at right hand side equation. So we cannot use P_t^* to represent P_{t+1}^*.

In the model, the firm only considers the state that P_t^* does not change forever. Therefore, when a firm resets its price, the history prices have no effects. Hence, future prices are not a function of current price. We then cannot get recursive expression.

(b) We have

$$F_{2t} = \Lambda_{t,t} Y_t + \sum_{k=1}^{\infty} \theta^k \Lambda_{t,t+k} (P_{t+k}/P_t)^{\varepsilon-1} = Y_t + \sum_{k=1}^{\infty} \theta^k \Lambda_{t,t+1} \Lambda_{t+1,t+k} (P_{t+k}/P_t)^{\varepsilon-1} \tag{10}$$

Note that

$$\Lambda_{t,t+1} = \beta \frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}} \tag{11}$$

Therefore,

$$\begin{aligned} F_{2t} &= Y_t + \theta \Lambda_{t,t+1} \sum_{k=0}^{\infty} \theta^k \Lambda_{t+1,t+k+1} (P_{t+k+1}/P_t)^{\varepsilon-1} (P_{t+1}/P_{t+1})^{\varepsilon-1} \\ &= Y_t + \beta\theta \frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}} \left(\frac{P_{t+1}}{P_t}\right)^{\varepsilon-1} F_{2,t+1} \\ &= Y_t + \beta\theta \Pi_{t+1}^{\varepsilon-1} \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} F_{2,t+1} \end{aligned} \tag{12}$$

(c) We have

$$F_{1t} = (1+\mu) Y_t \frac{W_t}{P_t A_t} + (1+\mu) \sum_{s=1}^{\infty} \theta^s \Lambda_{t,t+1} \Lambda_{t+1,t+s} (P_{t+s}/P_t)^{\varepsilon-1} \frac{W_{t+s}/P_t}{A_{t+s}} \tag{13}$$

Then

$$F_{1t} = (1+\mu)Y_t \frac{W_t/P_t}{A_t} + (1+\mu)\theta\Lambda_{t,t+1} \sum_{s=0}^{\infty} \Lambda_{t+1,t+s+1} \left(\frac{P_{t+s+1}}{P_{t+1}}\right)^{\varepsilon-1} \frac{W_{t+s+1}/P_{t+1}}{A_{t+s+1}} \left(\frac{P_{t+1}}{P_t}\right)^{\varepsilon}$$
$$= (1+\mu)Y_t \frac{W_t/P_t}{A_t} + \beta\theta\Pi_{t+1}^{\varepsilon} \frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}} F_{1,t+1} \tag{14}$$

(d) The gross inflation is

$$\Pi_t = \frac{P_t}{P_{t-1}} \tag{15}$$

Note that we also have

$$P_t = \left[\theta P_{t-1}^{1-\varepsilon} + (1-\theta)(P^*)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}} \tag{16}$$

Now we divide P_t from BHS, we have

$$1 = \left[\theta\Pi_t^{\varepsilon-1} + (1-\theta)\left(\frac{P^*}{P_t}\right)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}} \tag{17}$$

Therefore

$$1 = \theta\Pi_t^{\varepsilon-1} + (1-\theta)(p^*)^{1-\varepsilon} \tag{18}$$

(e) The equation above tells us the convex combination of $\Pi_t^{\varepsilon-1}$ and $(p^*)^{1-\varepsilon}$ equals to 1. So if p^* is greater than one, we must have $\Pi_t^{\varepsilon-1} < 1$. Therefore, $\Pi_t > 1$. Regarding the intuition, if p^* is high, then P^* is also high. Let P_{t-1} being fixed, then note that P_t is the geometric average of P_t^* and P_{t-1}. So P_t is high. Thus we must have the inflation Π_t high.

(f) See code

(g) See figure (3). Alternatively, we actually don't have to do log linearization before applying sequence space method. The solution is shown in figure (4). We can see that those two figures are almost the same. The nonlinear solution should be more precise.

(h) We can see that when θ is small, almost all firms can adjust their prices in response to a productivity shock. Consequently, consumption and output will rise following a positive productivity shock. However, it is important to note that the increase in output is less than the shock itself, leading to a decrease in the employment rate. This change is smaller compared to when θ is large. Additionally, we observe that the change in inflation is greater when θ is small, due to the flexibility in price adjustments. The markup is the reciprocal of marginal costs. When θ is higher, firms will have lower marginal costs due to TFP shocks, resulting in higher markups. Since $\phi_\pi > 1$, we know that both the nominal and real interest rates are declining. When θ is lower, prices will change more, leading to greater disinflation. Consequently, the nominal and real interest rates decline more compared to the case when θ is large.

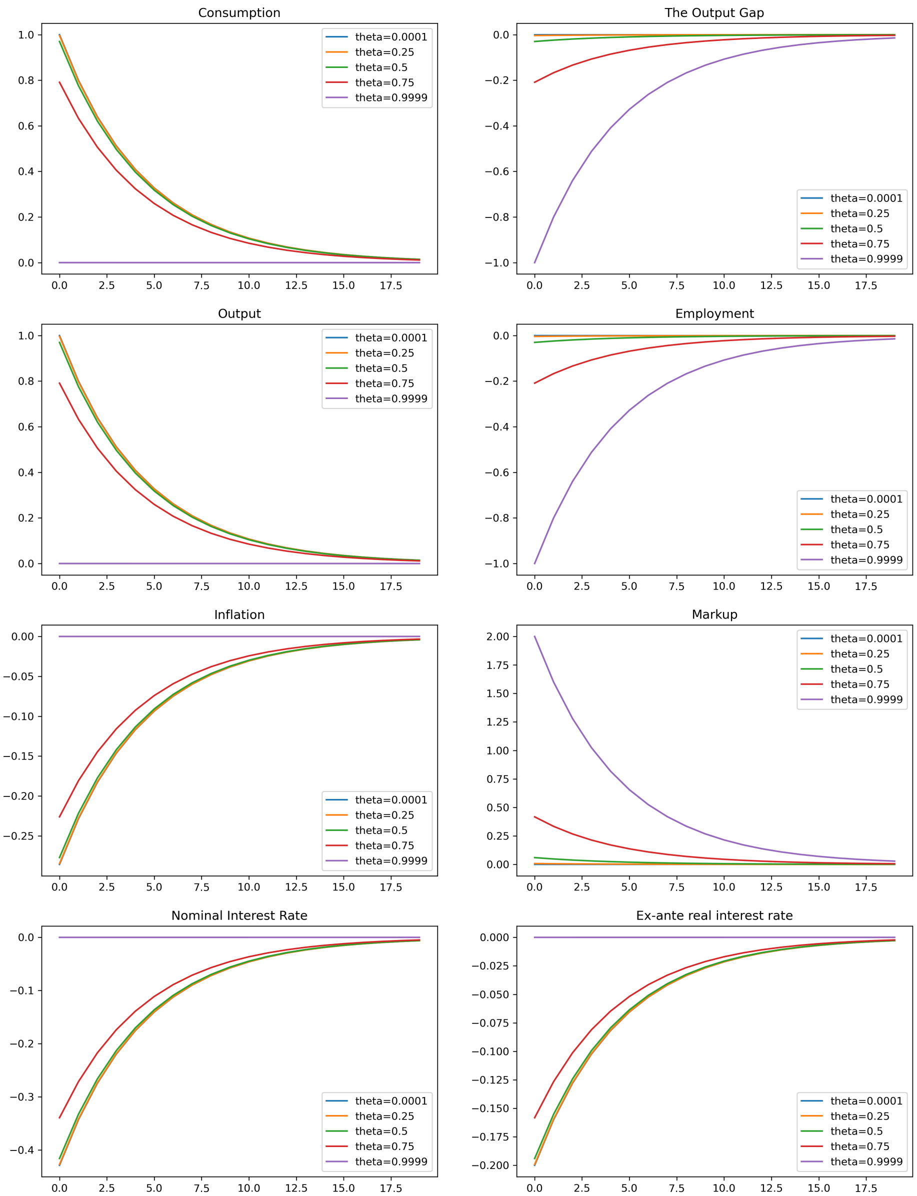

Figure 3: IRFs, log-linearization

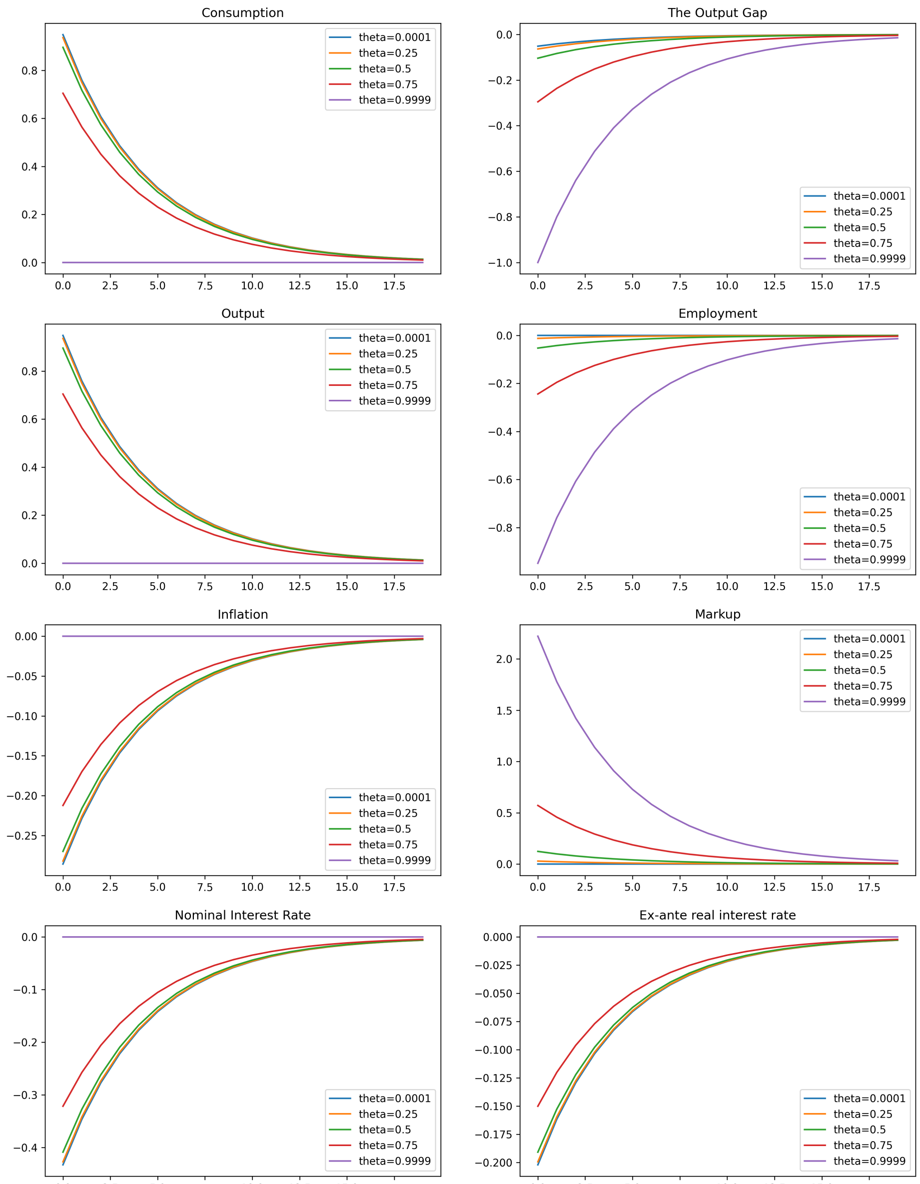

Figure 4: IRFs, nonlinear