Now consider the sequence space method. We also impose the assumption that $A = 1$ into the equation. Hence we know that

- Firm Block:
$$\begin{aligned} \hat{y}_t &= \hat{n}_t \\ \hat{w}_t - \hat{p}_t &= 0 \end{aligned} \tag{1}$$

- Households Block:
$$\begin{aligned} \hat{w}_t - \hat{p}_t &= \gamma \hat{c}_t + \varphi \hat{n}_t - (\gamma - \nu)\Omega\eta\hat{q}_t \\ \hat{m}_t - \hat{p}_t &= \hat{c}_t - \eta\hat{q}_t \end{aligned} \tag{2}$$

- Market Clearing:
$$\begin{aligned} \hat{y}_t &= \hat{c}_t \\ 0 &= \mathbb{E}[\hat{q}_t + \hat{p}_t - \hat{p}_{t+1} - \nu(\hat{c}_{t+1} - \hat{c}_t) + (-\gamma + \nu)(\hat{x}_{t+1} - \hat{x}_t)] \end{aligned} \tag{3}$$

- Money Process:
$$\hat{m}_t = \rho_m \hat{m}_{t-1} + \varepsilon_t^m \tag{4}$$

- Expression of X:
$$\hat{x}_t = (1-\Omega)\hat{c}_t + \Omega(\hat{m}_t - \hat{p}_t) \tag{5}$$

There are 8 variables in total: y, n, c, m, q, p, w, x and 8 equations in total.

Let

$$\boldsymbol{H}(\boldsymbol{Y}, \varepsilon_m) = \begin{pmatrix} \hat{n}_t - \hat{y}_t \\ \hat{w}_t - \hat{p}_t \\ \gamma\hat{c}_t + \varphi\hat{n}_t - (\gamma-\nu)\Omega\eta\hat{q}_t - (\hat{w}_t - \hat{p}_t) \\ \hat{c}_t - \eta\hat{q}_t - \hat{m}_t + \hat{p}_t \\ \hat{c}_t - \hat{y}_t \\ \hat{q}_t + \hat{p}_t - \hat{p}_{t+1} - \nu(\hat{c}_{t+1} - \hat{c}_t) + (-\gamma+\nu)(\hat{x}_{t+1} - \hat{x}_t) \\ \rho_m\hat{m}_{t-1} + \varepsilon_t^m - \hat{m}_t \\ (1-\Omega)\hat{c}_t + \Omega(\hat{m}_t - \hat{p}_t) - \hat{x}_t \end{pmatrix} \tag{6}$$

I set $\boldsymbol{U} = \{\hat{n}, \hat{p}\}$. Note that the Euler equation contains $\hat{p}$, hence our $\frac{\partial H}{\partial U}$ will have an additional term (direct effect) compared to the lecture slides. The calculations are really tedious, I have done them on my scratch paper and you can find those formulas in my code.

The DAG is shown in figure (1).

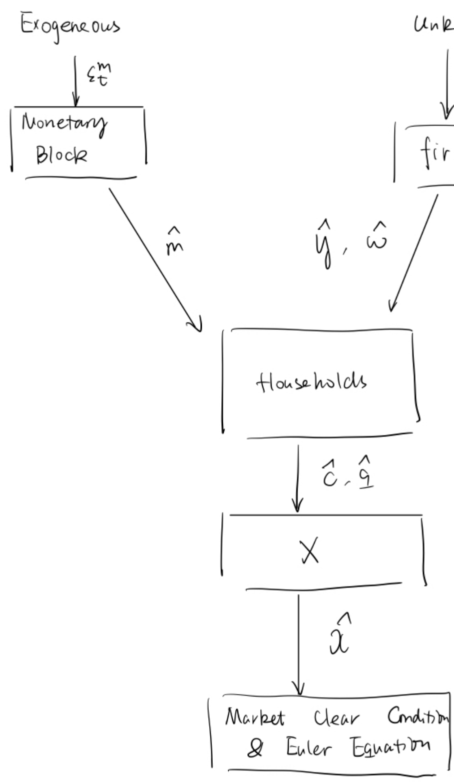

Figure 1: DAG

Here we define

$$\boldsymbol{Y} = \{\hat{c}, \hat{q}, \hat{x}, \hat{y}, \hat{w}\}, \quad \boldsymbol{U} = \{\hat{n}, \hat{p}\} \tag{7}$$

Note that

$$\mathrm{d}\boldsymbol{Y} = \left(\frac{\partial \boldsymbol{Y}}{\partial \boldsymbol{U}} \boldsymbol{H}_{\boldsymbol{U}}^{-1} \boldsymbol{H}_{\boldsymbol{Z}} + \frac{\partial \boldsymbol{Y}}{\partial \boldsymbol{Z}}\right) \mathrm{d}\boldsymbol{Z} \tag{8}$$

Note that $\boldsymbol{H}$ contains two equation, we have

$$\frac{\partial \boldsymbol{H}}{\partial \boldsymbol{Y}} = \begin{pmatrix} \Phi_{gm,c} & \Phi_{gm,q} & \Phi_{gm,x} & \Phi_{gm,y} & \Phi_{gm,w} \\ \Phi_{eu,c} & \Phi_{eu,q} & \Phi_{eu,x} & \Phi_{eu,y} & \Phi_{eu,w} \end{pmatrix} \tag{9}$$

From the market clearing block, we can see that

$$\Phi_{gm,c} = I, \quad \Phi_{gm,y} = -I, \quad \Phi_{eu,c} = \nu I - \nu I_p, \quad \Phi_{eu,q} = I, \quad \Phi_{eu,x} = (\gamma - \nu) I - (\gamma - \nu) I_p \tag{10}$$

where I_p is

$$I_p = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

All other matrices are zero. Note that we know have $\frac{\partial \boldsymbol{H}}{\partial \boldsymbol{U}}$ term.

$$\frac{\partial \boldsymbol{H}}{\partial \boldsymbol{U}} = \begin{pmatrix} \Phi_{gm,n} & \Phi_{gm,p} \\ \Phi_{eu,n} & \Phi_{eu,p} \end{pmatrix} \tag{11}$$

where

$$\Phi_{eu,p} = I - I_p \tag{12}$$

all other matrices are 0. Now we consider $\frac{\partial \boldsymbol{Y}}{\partial \boldsymbol{U}}$. First, in the firm block, we have

$$\Phi_{y,n} = I, \quad ,\Phi_{w,p} = I \tag{13}$$

Others are 0. In the households block, we do $\Phi_{c,n}$ and $\Phi_{q,n}$ as an example: From the following equations

$$\begin{aligned} &\hat{c}_t - \eta \hat{q}_t - \hat{m}_t + \hat{p}_t = 0 \\ &\gamma \hat{c}_t + \varphi \hat{n}_t - (\gamma - \nu)\Omega \eta \hat{q}_t = 0 \end{aligned} \tag{14}$$

Taking partial derivative gives us

$$\begin{aligned} &\Phi_{c,n} - \eta \Phi_{q,n} = 0 \\ &\gamma \Phi_{c,n} + \varphi - (\gamma - \nu)\Omega \eta \Phi_{q,n} = 0 \end{aligned} \tag{15}$$

Therefore, we get

$$\Phi_{c,n} = -\frac{\eta \varphi}{\gamma \eta - (\gamma - \nu)\Omega \eta} I, \quad \Phi_{q,n} = -\frac{\varphi}{\gamma \eta - (\gamma - \nu)\Omega \eta} I \tag{16}$$

Similarly, we can get all other partial derivatives. They are in the codes. Therefore, we get

$$\frac{\partial \boldsymbol{Y}}{\partial \boldsymbol{U}} = \begin{pmatrix} \Phi_{c,n} & \Phi_{c,p} \\ \Phi_{q,n} & \Phi_{q,p} \\ \Phi_{x,n} & \Phi_{x,p} \\ \Phi_{y,n} & \Phi_{y,p} \\ \Phi_{w,n} & \Phi_{w,p} \end{pmatrix} \tag{17}$$

Hence, combining (9), (17) and (11) we get

$$\boldsymbol{H}_{\boldsymbol{U}} = \frac{\partial \boldsymbol{H}}{\partial \boldsymbol{Y}} \frac{\partial \boldsymbol{Y}}{\partial \boldsymbol{U}} + \frac{\partial \boldsymbol{H}}{\partial \boldsymbol{U}} \tag{18}$$

On the other hand,

$$\frac{\partial \boldsymbol{Y}}{\partial \boldsymbol{Z}} = \begin{pmatrix} \Phi_{c,m} \\ \Phi_{q,m} \\ \Phi_{x,m} \\ \Phi_{y,m} \\ \Phi_{w,m} \end{pmatrix} \tag{19}$$

Those calculation are similar to (17). Then we have

$$\boldsymbol{H}_{\boldsymbol{Z}} = \frac{\partial \boldsymbol{H}}{\partial \boldsymbol{Y}} \frac{\partial \boldsymbol{Y}}{\partial \boldsymbol{Z}} \tag{20}$$

Hence we can compute

$$\mathrm{d}\boldsymbol{Y} = \left(\frac{\partial \boldsymbol{Y}}{\partial \boldsymbol{U}} \boldsymbol{H}_{\boldsymbol{U}}^{-1} \boldsymbol{H}_{\boldsymbol{Z}} + \frac{\partial \boldsymbol{Y}}{\partial \boldsymbol{Z}} \right) \mathrm{d}\boldsymbol{Z} \tag{21}$$