(a) The optimization problem is

$$
\begin{aligned}
\max_{\{C_t,N_t,M_t,B_t\}_{t=0}^{\infty}} \quad & \mathbb{E}\left[\sum_{t=0}^{\infty}\beta^t U(X_t,L_t)\right] \\
\text{s.t.} \quad & P_tC_t + B_t + M_t \leq W_tN_t + Q_{t-1}B_{t-1} + M_{t-1} + P_t(TR_t + PR_t)
\end{aligned} \tag{1}
$$

where

$$
X_t = \left[(1-\theta)C_t^{1-\nu} + \theta\left(\frac{M_t}{P_t}\right)^{1-\nu}\right]^{\frac{1}{1-\nu}} \tag{2}
$$

The Lagrangian is

$$
\begin{aligned}
\mathcal{L} = \mathbb{E}\Bigg\{\sum_{t=0}^{\infty}\beta^t\Bigg[&\left(\frac{X_t^{1-\gamma}-1}{1-\gamma} - \chi\frac{N_t^{1+\varphi}}{1+\varphi}\right) \\
&+\lambda_t(W_tN_t + Q_{t-1}B_{t-1} + M_{t-1} + P_t(TR_t+PR_t) - P_tC_t - B_t - M_t)\Bigg]\Bigg\}
\end{aligned} \tag{3}
$$

The FOCs are

$$
\begin{aligned}
[C_t]: \quad & X_t^{-\gamma+\nu}(1-\theta)C_t^{-\nu} - \lambda_tP_t = 0 \\
[M_t]: \quad & X_t^{-\gamma+\nu}\theta\frac{M_t^{-\nu}}{P_t^{1-\nu}} + \beta\mathbb{E}_t[\lambda_{t+1}] = \lambda_t \\
[B_t]: \quad & \lambda_t = \beta\mathbb{E}_t[\lambda_{t+1}Q_t] \\
[N_t]: \quad & \chi N_t^{\varphi} = \lambda_tW_t
\end{aligned} \tag{4}
$$

The Euler equation is

$$
1 = \beta\mathbb{E}_t\left[Q_t\frac{P_t}{P_{t+1}}\frac{X_{t+1}^{-\gamma+\nu}}{X_t^{-\gamma+\nu}}\frac{C_{t+1}^{-\nu}}{C_t^{-\nu}}\right] \tag{5}
$$

The Labour-Leisure condition is

$$
\frac{W_t}{P_t} = \frac{\chi N_t^{\varphi}}{(1-\theta)C_t^{-\nu}X_t^{-\gamma+\nu}} \tag{6}
$$

Combine $[C_t]$, $[M_t]$ and $[B_t]$, we have

$$
\frac{M_t}{P_t} = \left(1-\frac{1}{Q_t}\right)^{-\frac{1}{\nu}}\left(\frac{\theta}{1-\theta}\right)^{\frac{1}{\nu}}C_t \tag{7}
$$

(b) In slides 1, we know that money neutrality is "Real outcomes are independent of the price level and unaffected by nominal variables".

Here w e first follow the setup of firm problems in the lecture. Firm produces $Y_t = A_tN_t$ and it maximizes profits $\max_{N_t} Y_t - \frac{W_t}{P_t}N_t$. The FOC is

$$
\frac{W_t}{P_t} = A_t \tag{8}
$$

The market clearing conditions in this economy are $Y_t = C_t$, $B_t^D = B_t^S$, $M_t^D = M_t^S$, $N_t^D = N_t^S$. Therefore, we must have

$$\frac{W_t}{P_t} = A_t = \frac{\chi N_t^{\varphi}}{(1-\theta)C_t^{-\nu}X_t^{-\gamma+\nu}} \tag{9}$$

Thus

$$N_t = \left[\frac{1-\theta}{\chi}C_t^{-\nu}X_t^{-\gamma+\nu}A_t\right]^{\frac{1}{\varphi}} \tag{10}$$

Note that there is M_t (included in X_t) in this formula. So if money is neutral, we must have $\nu = \gamma$. Then it is independent of nominal variables. Also, $Y_t = A_t N_t$ does not contain nominal variables and so does C_t.

The equilibrium equations are

$$\begin{cases} Y_t = A_t N_t \\ \dfrac{W_t}{P_t} = A_t \\ \dfrac{W_t}{P_t} = \dfrac{\chi N_t^{\varphi}}{(1-\theta)C_t^{-\nu}X_t^{-\gamma+\nu}} \\ Y_t = C_t \\ \dfrac{M_t}{P_t} = \left(1-\dfrac{1}{Q_t}\right)^{-\frac{1}{\nu}}\left(\dfrac{\theta}{1-\theta}\right)^{\frac{1}{\nu}} C_t \\ 1 = \beta \mathbb{E}_t\left[Q_t \dfrac{P_t}{P_{t+1}} \dfrac{X_{t+1}^{-\gamma+\nu}}{X_t^{-\gamma+\nu}} \dfrac{C_{t+1}^{-\nu}}{C_t^{-\nu}}\right] \end{cases} \tag{11}$$

(c) Now, since $A = 1$, we must have $Y = N$ and hence $C = N$. Also, we assume that in the steady state, the real interest rate is constant, i.e.

$$Q_t \frac{P_t}{P_{t+1}} := Q_t \frac{1}{\pi} \equiv R = \frac{1}{\beta} \tag{12}$$

where π is ratio of P_{t+1}/P_t. Therefore, $Q = \pi/\beta$. Thus, from (7) we can derive that

$$\frac{M}{P} = \left(1-\frac{\beta}{\pi}\right)^{-\frac{1}{\nu}}\left(\frac{\theta}{1-\theta}\right)^{\frac{1}{\nu}} C \tag{13}$$

Hence

$$X = \left[(1-\theta) + \theta\left(1-\frac{\beta}{\pi}\right)^{-\frac{1-\nu}{\nu}}\left(\frac{\theta}{1-\theta}\right)^{\frac{1-\nu}{\nu}}\right]^{\frac{1}{1-\nu}} C \tag{14}$$

Substitute $C = N$ into (10) and combine the equation above, we get

$$C = \left\{\frac{1-\theta}{\chi}\left[(1-\theta) + \theta\left(1-\frac{\beta}{\pi}\right)^{-\frac{1-\nu}{\nu}}\left(\frac{\theta}{1-\theta}\right)^{\frac{1-\nu}{\nu}}\right]^{\frac{\nu-\gamma}{1-\nu}}\right\}^{\frac{1}{\varphi+\gamma}} \tag{15}$$

Also, if we further assume that P_t is also not changing, then $\pi = 1$, $Q = \frac{1}{\beta}$.

(d) Now we take the Q and M_t as given. After specifying the model parameters, we can get π by βQ. Then we can solve C by the equation (15). After that, we can get X and M/P. Since M_t is also given, we can solve the price level P_t.

(e) Since given ν, we can express C as a function of θ. Note that also

$$\frac{M}{P} = \left(1-\frac{\beta}{\pi}\right)^{-\frac{1}{\nu}}\left(\frac{\theta}{1-\theta}\right)^{\frac{1}{\nu}}\left\{\frac{1-\theta}{\chi}\left[(1-\theta)+\theta\left(1-\frac{\beta}{\pi}\right)^{-\frac{1-\nu}{\nu}}\left(\frac{\theta}{1-\theta}\right)^{\frac{1-\nu}{\nu}}\right]^{\frac{\nu-\gamma}{1-\nu}}\right\}^{\frac{1}{\varphi+\gamma}} \tag{16}$$

Then we can use this equation to calibrate θ. If we further assume $\pi = 1$ **and** $A_t = 1$ as in the previous question, then we know that

(f) From the previous question, we have

$$M = \left(1-\frac{\beta}{\pi}\right)^{-\frac{1}{\nu}}\left(\frac{\theta}{1-\theta}\right)^{\frac{1}{\nu}}\left\{\frac{1-\theta}{\chi}\left[(1-\theta)+\theta\left(1-\frac{\beta}{\pi}\right)^{-\frac{1-\nu}{\nu}}\left(\frac{\theta}{1-\theta}\right)^{\frac{1-\nu}{\nu}}\right]^{\frac{\nu-\gamma}{1-\nu}}\right\}^{\frac{1}{\varphi+\gamma}} \tag{17}$$

when $P = 1$. Also, here and thereafter, we can impose $\pi = 1$ and $Q = \frac{1}{\beta}$, i.e.

$$M = (1-\beta)^{-\frac{1}{\nu}}\left(\frac{\theta}{1-\theta}\right)^{\frac{1}{\nu}}\left\{\frac{1-\theta}{\chi}\left[(1-\theta)+\theta\,(1-\beta)^{-\frac{1-\nu}{\nu}}\left(\frac{\theta}{1-\theta}\right)^{\frac{1-\nu}{\nu}}\right]^{\frac{\nu-\gamma}{1-\nu}}\right\}^{\frac{1}{\varphi+\gamma}} \tag{18}$$

(g) The log-linearization of system (11) is

$$\begin{cases} \hat{y}_t = \hat{a}_t + \hat{n}_t \\ \hat{w}_t - \hat{p}_t = \hat{a}_t \\ \hat{w}_t - \hat{p}_t = \gamma\hat{c}_t + \varphi\hat{n}_t + (\gamma-\nu)\Omega[\hat{c}_t - (\hat{m}_t - \hat{p}_t)] \\ \hat{y}_t = \hat{c}_t \\ \hat{m}_t - \hat{p}_t = \hat{c}_t - \eta\hat{q}_t \\ 0 = \mathbb{E}_t[\hat{r}_{t+1} - \nu(\hat{c}_{t+1} - \hat{c}_t) + (-\gamma+\nu)(\hat{x}_{t+1} - \hat{x}_t)] \end{cases} \tag{19}$$

For the third equation, we first consider the right hand side as a function of N_t, C_t and X_t, then

$$\frac{1}{P}W\hat{w}_t - \frac{W}{P^2}P\hat{p}_t = \varphi\frac{W}{PN}N\hat{n}_t + \nu\frac{W}{PC}C\hat{c}_t + (\gamma-\nu)\frac{W}{PX}X\hat{x}_t \tag{20}$$

Hence

$$\hat{w}_t - \hat{p}_t = \nu\hat{c}_t + \varphi\hat{n}_t + (\gamma-\nu)\hat{x}_t \tag{21}$$

Also, we can log-linearize the expression of X_t, we get

$$X\hat{x}_t = (1-\theta)X^{\nu}C^{1-\nu}\hat{c}_t + \theta X^{\nu}\left(\frac{M}{P}\right)^{1-\nu}(\hat{m}_t - \hat{p}_t)$$

Also note that

$$\frac{M}{P} = \left(1-\frac{\beta}{\pi}\right)^{-\frac{1}{\nu}}\left(\frac{\theta}{1-\theta}\right)^{\frac{1}{\nu}} C$$

Thus, we get

$$\begin{aligned}(1-\theta)(\hat{c}_t - \hat{x}_t) &= \theta\left(1-\frac{\beta}{\pi}\right)^{-\frac{1-\nu}{\nu}}\left(\frac{\theta}{1-\theta}\right)^{\frac{1-\nu}{\nu}}[\hat{x}_t - (\hat{m}_t - \hat{p}_t)] \\ &= \theta\left(1-\frac{\beta}{\pi}\right)^{-\frac{1-\nu}{\nu}}\left(\frac{\theta}{1-\theta}\right)^{\frac{1-\nu}{\nu}}[\hat{x}_t - \hat{c}_t + \hat{c}_t - (\hat{m}_t - \hat{p}_t)]\end{aligned}$$

Therefore,

$$\hat{c}_t - \hat{x}_t = \frac{\theta^{\frac{1}{\nu}}\left(1-\frac{\beta}{\pi}\right)^{1-\frac{1}{\nu}}}{(1-\theta)^{\frac{1}{\nu}} + \theta^{\frac{1}{\nu}}\left(1-\frac{\beta}{\pi}\right)^{1-\frac{1}{\nu}}}[\hat{c}_t - (\hat{m}_t - \hat{p}_t)]$$

Substitute this into (21), we get

$$\hat{w}_t - \hat{p}_t = \gamma\hat{c}_t + \varphi\hat{n}_t - (\gamma-\nu)\frac{\theta^{\frac{1}{\nu}}\left(1-\frac{\beta}{\pi}\right)^{1-\frac{1}{\nu}}}{(1-\theta)^{\frac{1}{\nu}} + \theta^{\frac{1}{\nu}}\left(1-\frac{\beta}{\pi}\right)^{1-\frac{1}{\nu}}}[\hat{c}_t - (\hat{m}_t - \hat{p}_t)] \tag{22}$$

For notation simplicity, we can define

$$\Omega := \frac{\theta^{\frac{1}{\nu}}\left(1-\frac{\beta}{\pi}\right)^{1-\frac{1}{\nu}}}{(1-\theta)^{\frac{1}{\nu}} + \theta^{\frac{1}{\nu}}\left(1-\frac{\beta}{\pi}\right)^{1-\frac{1}{\nu}}} \tag{23}$$

Then the equation above becomes

$$\hat{w}_t - \hat{p}_t = \gamma\hat{c}_t + \varphi\hat{n}_t - (\gamma-\nu)\Omega[\hat{c}_t - (\hat{m}_t - \hat{p}_t)] \tag{24}$$

Also, we can deduce that

$$\hat{m}_t - \hat{p}_t = \hat{c}_t - \frac{1}{\nu(Q-1)}\hat{q}_t := \hat{c}_t - \eta\hat{q}_t \tag{25}$$

where $\eta = \frac{1}{\nu(Q-1)}$. Given that $P=1$, we know $\eta = \frac{\beta}{\nu(1-\beta)}$. Hence, we get our system:

$$\begin{cases} \hat{y}_t = \hat{a}_t + \hat{n}_t \\ \hat{w}_t - \hat{p}_t = \hat{a}_t \\ \hat{w}_t - \hat{p}_t = \gamma\hat{c}_t + \varphi\hat{n}_t - (\gamma-\nu)\Omega\eta\hat{q}_t \\ \hat{y}_t = \hat{c}_t \\ 0 = \mathbb{E}_t[\hat{r}_{t+1} - \nu(\hat{c}_{t+1} - \hat{c}_t) + (-\gamma+\nu)(\hat{x}_{t+1} - \hat{x}_t)] \end{cases} \tag{26}$$

(h) Recall that

$$M=(1-\beta)^{-\frac{1}{\nu}}\left(\frac{\theta}{1-\theta}\right)^{\frac{1}{\nu}}\left\{\frac{1-\theta}{\chi}\left[(1-\theta)+\theta\,(1-\beta)^{-\frac{1-\nu}{\nu}}\left(\frac{\theta}{1-\theta}\right)^{\frac{1-\nu}{\nu}}\right]^{\frac{\nu-\gamma}{1-\nu}}\right\}^{\frac{1}{\varphi+\gamma}} \tag{27}$$

Note that in steady state, we assume that $M=1$. Therefore, we can use this equation to solve θ. On the other hand, one can substitute those parameters equal to 1 into the equation W_t/P_t, which yields

$$N_t=(1-\theta)C_t^{-\nu}X_t^{-1+\nu} \tag{28}$$

Note that $N_t=C_t$ and $M/P=1$, then

$$1-\theta=(1-\theta)C_t^2+\theta C_t^{1+\nu} \tag{29}$$

Also note that

$$\frac{M}{P}=\left(1-\frac{\beta}{\pi}\right)^{-\frac{1}{\nu}}\left(\frac{\theta}{1-\theta}\right)^{\frac{1}{\nu}}C \tag{30}$$

Hence

$$1-\theta=1-\theta(1-\beta)^{\frac{2}{\nu}}\left(\frac{\theta}{1-\theta}\right)^{-\frac{2}{\nu}}+\theta(1-\beta)^{\frac{1+\nu}{\nu}}\left(\frac{\theta}{1-\theta}\right)^{-\frac{1+\nu}{\nu}} \tag{31}$$

The solution can be expressed as

$$\left(\frac{\theta}{1-\theta}\right)^{-\frac{1}{\nu}}=\frac{-(1-\beta)^{\frac{1}{\nu}}+\sqrt{(1-\beta)^2+4(1-\beta)}}{2(1-\beta)^{\frac{1}{\nu}}} \tag{32}$$

Now consider the sequence space method. We also impose the assumption that $A=1$ into the equation. Hence we know that

- Firm Block:

$$\begin{aligned}\hat{y}_t&=\hat{n}_t\\ \hat{w}_t-\hat{p}_t&=0\end{aligned} \tag{33}$$

- Households Block:

$$\begin{aligned}\hat{w}_t-\hat{p}_t&=\gamma\hat{c}_t+\varphi\hat{n}_t-(\gamma-\nu)\Omega\eta\hat{q}_t\\ \hat{m}_t-\hat{p}_t&=\hat{c}_t-\eta\hat{q}_t\end{aligned} \tag{34}$$

- Market Clearing:

$$\begin{aligned}\hat{y}_t&=\hat{c}_t\\ 0&=\mathbb{E}[\hat{q}_t+\hat{p}_t-\hat{p}_{t+1}-\nu(\hat{c}_{t+1}-\hat{c}_t)+(-\gamma+\nu)(\hat{x}_{t+1}-\hat{x}_t)]\end{aligned} \tag{35}$$

- Money Process:

$$\hat{m}_t=\rho_m\hat{m}_{t-1}+\varepsilon_t^m \tag{36}$$

- Expression of X:

$$\hat{x}_t = (1-\Omega)\hat{c}_t + \Omega(\hat{m}_t - \hat{p}_t) \tag{37}$$

There are 8 variables in total: y, n, c, m, q, p, w, x and 8 equations in total.

Let

$$\boldsymbol{H}(\boldsymbol{Y}, \varepsilon_m) = \begin{pmatrix} \hat{n}_t - \hat{y}_t \\ \hat{w}_t - \hat{p}_t \\ \gamma\hat{c}_t + \varphi\hat{n}_t - (\gamma - \nu)\Omega\eta\hat{q}_t - (\hat{w}_t - \hat{p}_t) \\ \hat{c}_t - \eta\hat{q}_t - \hat{m}_t + \hat{p}_t \\ \hat{c}_t - \hat{y}_t \\ \hat{q}_t + \hat{p}_t - \hat{p}_{t+1} - \nu(\hat{c}_{t+1} - \hat{c}_t) + (-\gamma + \nu)(\hat{x}_{t+1} - \hat{x}_t) \\ \rho_m\hat{m}_{t-1} + \varepsilon_t^m - \hat{m}_t \\ (1-\Omega)\hat{c}_t + \Omega(\hat{m}_t - \hat{p}_t) - \hat{x}_t \end{pmatrix} \tag{38}$$

I set $\boldsymbol{U} = \{\hat{n}, \hat{p}\}$. Note that the Euler equation contains $\hat{p}$, hence our $\frac{\partial H}{\partial U}$ will have an additional term (direct effect) compared to the lecture slides. The calculations are really tedious, I have done them on my scratch paper and you can find those formulas in my code.

The DAG is shown in figure (1).

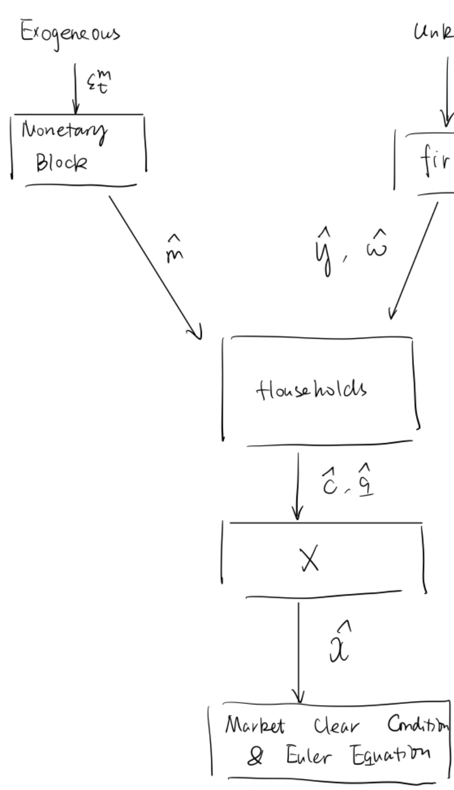

Figure 1: DAG

The IRFs are shown in the figures below:

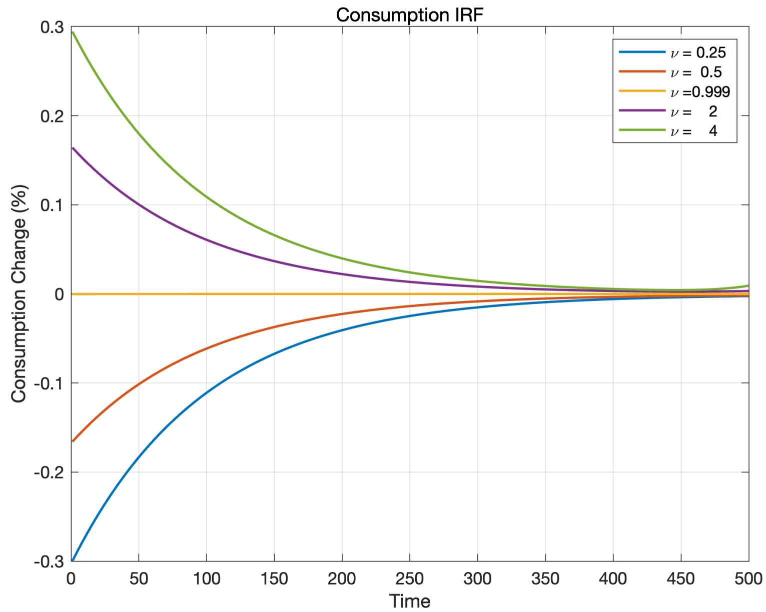

Figure 2: Consumption IRF

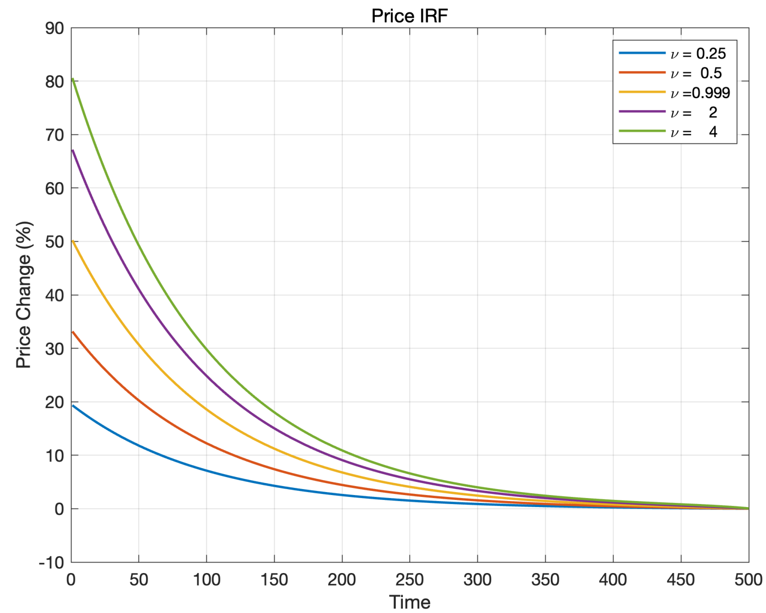

Figure 3: Price IRF

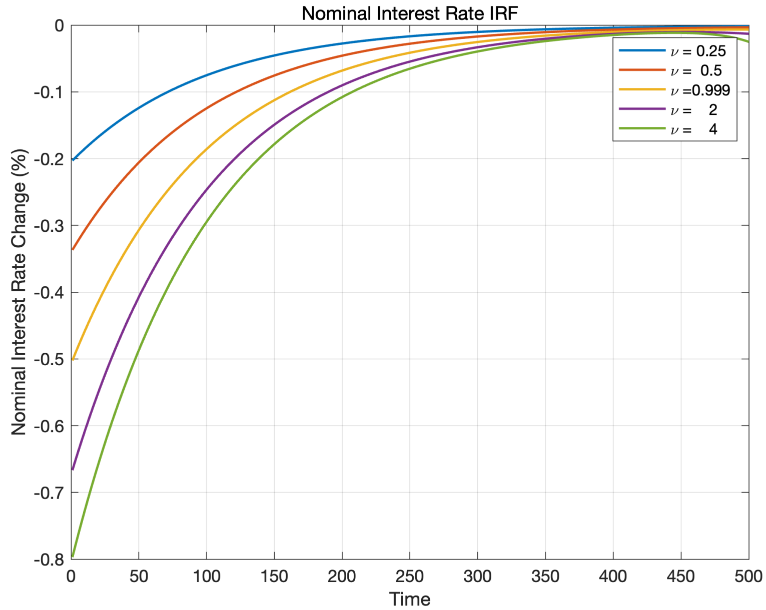

Figure 4: Nominal Interest Rate IRF

(i) We can see that in general, when ν is increasing, it leads to increasing in price change and decreasing in nominal interest rates. Hence the consumption change is decreasing. Note that actually ν is inverse elasticity of substitution between consumption C_T and $\frac{M_t}{P}t$.

When ν is small($\nu < 1$) then the elasticity is large. So increasing in M_t will decrease the marginal utility of consumption. Hence the labor supply will decrease. Consequently, this will cause both production and consumption decreasing. Also, when money supply is increasing, the price also increase. Then the money demand is high, so the bond interest rates must decrease.

When $\nu > 1$, the money supply will increase marginal utility of consumption. Hence there will be more labor supply. And then the production and consumption will increase. But also high ν will lead to intertemporal substitution from money to consumption. Hence the interest rate will decline more and price will increase more.

When $\nu = 1$, increasing in M_t does not affect marginal utility of consumption so there is no change in the consumption.

(j) From the discussion above, we can rule out those $\nu \leq 1$. Since only cases when $\nu > 1$ will have increasing consumption with increasing money supply.

(k) Code is uploaded to Github. 'hw1.m'