1. Write the following linear programming problem in standard form:

$$\begin{cases} \arg\min\limits_{x_1,x_2} \quad x_2 \\ x_1 + x_2 \geq 1 \\ x_1 - x_2 \leq 0 \end{cases} \tag{1}$$

Solution: First, we introduce non-negative variables y_1, y_2, z_1, z_2 such that

$$x_1 = y_1 - z_1, \quad x_2 = y_2 - z_2 \tag{2}$$

Then the optimization problem (1) becomes

$$\begin{cases} \arg\min\limits_{y_1,y_2,z_1,z_2} \quad y_2 - z_2 \\ y_1 - z_1 + y_2 - z_2 \geq 1 \\ y_1 - z_1 - y_2 + z_2 \leq 0 \\ y_1, y_2, z_1, z_2 \geq 0 \end{cases} \tag{3}$$

Now we just have to convert inequality constraints to equalities. Let $w_1, w_2 \geq 0$, we can rewrite (3) as

$$\begin{cases} \arg\min\limits_{y_1,y_2,z_1,z_2,w_1,w_2} \quad y_2 - z_2 \\ y_1 - z_1 + y_2 - z_2 - w_1 = 1 \\ y_1 - z_1 - y_2 + z_2 + w_2 = 0 \\ y_1, y_2, z_1, z_2, w_1, w_2 \geq 0 \end{cases} \tag{4}$$

This is a standard form LP. ■

2. Is the following optimization problem LP? Then, whether or not it is, draw the feasible region and determine the optimal solution, if it exists, through a visualization.

$$\begin{cases} \arg\min\limits_{x_1,x_2} \quad (-x_1 - x_2) \\ -3x_1 + x_2 \leq 0 \\ x_1^2 + x_2^2 \leq 1 \\ x_1 \geq 0, x_2 \geq 0 \end{cases} \tag{5}$$

Solution: Obviously, it is **NOT** a LP, since $x_1^2 + x_2^2 \leq 1$ this constraint is not linear. To make geometric illustration more intuitive, we can convert the minimization into maximization by adding a negative sign to the objective function. That is

$$\begin{cases} \arg\max\limits_{x_1,x_2} \quad x_1 + x_2 \\ -3x_1 + x_2 \leq 0 \\ x_1^2 + x_2^2 \leq 1 \\ x_1 \geq 0, x_2 \geq 0 \end{cases} \tag{6}$$

It is equivalent to the original problem. Then we plot the feasible region. See the shading region in the figure (2) below:

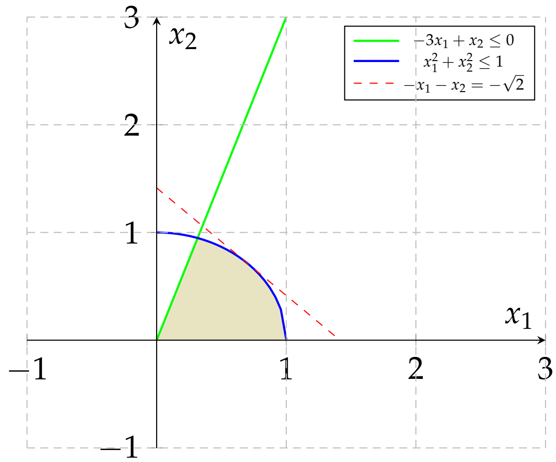

Figure 1: Visualization of optimization problem

The geometric interpretation of the optimization problem (6) is to find the maximum b such that the line $x_1 + x_2 = b$ has at least one point in the feasible set. Obviously, it is achieved at the corner of the circle that $x_1 + x_2 = b$ is tangent to the circle. Hence,

$$x^* = (x_1^*, x_2^*) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \tag{7}$$

is the solution to the optimization problem (5). ■

Remark: Alternatively, we can use KKT to solve the optimization problem(5). The Lagrangian is

$$\mathcal{L} = -x_1 - x_2 + \lambda_1(-3x_1 + x_2) + \lambda_2(x_1^2 + x_2^2 - 1) - \lambda_3 x_1 - \lambda_4 x_2$$

Then

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial x_1} = -1 - 3\lambda_1 + 2\lambda_2 x_1 - \lambda_3 = 0 \\ \dfrac{\partial \mathcal{L}}{\partial x_2} = -1 + \lambda_1 + 2\lambda_2 x_2 - \lambda_4 = 0 \\ -3x_1 + x_2 \leq 0, \quad x_1^2 + x_2^2 \leq 1, \quad x_1, x_2 \geq 0 \\ \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0 \\ \lambda_1(-3x_1 + x_2) = 0, \quad \lambda_2(x_1^2 + x_2^2 - 1) = 0, \quad \lambda_3 x_1 = 0, \quad \lambda_4 x_2 = 0 \end{cases}$$

Discussion:

(a) If $x_1 = 0$, then x_2 must be 0 since $-3x_1 + x_2 \leq 0$. In this case $-x_1 - x_2 = 0$.

(b) If $x_2 = 0$ but $x_1 \neq 0$. Then $\lambda_1 = \lambda_3 = 0$ due to complementary slackness condiions.Hence $\lambda_4 = 1$. We have $2\lambda_2 x_1 = 1$, then $\lambda_2 > 0$. Hence $x_1^2 + x_2^2 \leq 1$ must be bind, i.e. $x_1^2 + x_2^2 = 1$. Therefore, $x_1 = 1$. In this case, $-x_1 - x_2 = -1$.

(c) If $x_1, x_2 > 0$, then $\lambda_3 = \lambda_4 = 0$ due to complementary slackness conditions. First order conditions tell us $\lambda_2 \neq 0$, otherwise $-1 - 3\lambda_1 = -1 + \lambda_1 = 0$ has no solution. Hence $x_1^2 + x_2^2 \leq 1$ must be bind, i.e. $x_1^2 + x_2^2 = 1$. Then solving this system gives us $x_1 = x_2 = \frac{\sqrt{2}}{2}$. In this case, $-x_1 - x_2 = \sqrt{2}$.

In conclusion, $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ solves the optimization problem. ■

3. A company has the following short term financing problem (numbers in thousands of dollars):]

Month	Jan	Feb	Mar	Apr	May	Jun
Cash Flow	-150	-100	200	-200	50	300

The company has the following sources of funds:

- A line of credit of up to 100 with an interest rate of 1% per month;
- In any one of the first 3 months, can issue 60-day commercial paper bearing a total interest of 2% for the 2-month period;
- Excess funds can be invested at an interest rate of 0.3% per month.

Formulate an LP that maximizes the company's wealth at the end of June.

Solution: Notations(the same as the lecture slides):

(a) v: net worth after June

(b) x_i: total amount drawn from line of credit at month $i \leq 5$

(c) y_j: amount of commercial paper to issue in month $j \leq 3$.

(d) z_i: excess funds after month $i \leq 5$.

We must have $x_i, y_j, z_i \geq 0$ and $x_i \leq 100$. So we can write the optimization problem as:

$$\begin{cases} \underset{x_1,x_2,x_3,x_4,x_5,y_1,y_2,y_3,z_1,z_2,z_3,z_4,z_5,v}{\arg\max} \quad v \\ x_1 + y_1 - z_1 = 150 \\ x_2 - 1.01x_1 + y_2 + 1.003z_1 - z_2 = 100 \\ x_3 - 1.01x_2 + y_3 - 1.02y_1 + 1.003z_2 - z_3 = -200 \\ x_4 - 1.01x_3 - 1.02y_2 + 1.003z_3 - z_4 = 200 \\ x_5 - 1.01x_4 - 1.02y_3 + 1.003z_4 - z_5 = -50 \\ -1.01x_5 + 1.003z_5 - v = -300 \\ x_1, x_2, x_3, x_4, x_5 \leq 100 \\ x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, z_1, z_2, z_3, z_4, z_5 \geq 0 \end{cases} \tag{8}$$

■

4. Assume a bank receives the following liability schedule (numbers in dollars):

Year 1	Year 2	Year 3	Year 4	Year 5	Year 6	Year 7	Year 8
12,000	18,000	20,000	20,000	16,000	15,000	12,000	10,000

The bonds available for purchase today (Year 0), all with face values of 100, are given in the following table (coupon figures are annual):

Bond	1	2	3	4	5	6	7	8	9	10
Price	102	99	101	98	98	104	100	101	102	94
Coupon	5	3.5	5	3.5	4	9	6	8	9	7
Maturity Year	1	2	2	3	4	5	5	6	7	8

Assuming all these bonds are widely available and can be purchased in any quantities at the stated price, formulating an LP that minimizes the cost of portfolios bought today to meet the oblibgations over the next eight years.

Solution: Let x_i be the amount of bond i, z_t be the surplus at the end opf year t. The cost (objective function) is

$$f(x) = 102x_1 + 99x_2 + 101x_3 + 98x_4 + 98x_5 + 104x_6 + 100x_7 + 101x_8 + 102x_9 + 94x_{10} \tag{9}$$

For year 1, we have

$$z_1 = (100+5)x_1 + 3.5x_2 + 5x_3 + 3.5x_4 + 4x_5 + 9x_6 + 6x_7 + 8x_8 + 9x_9 + 7x_{10} - 12000$$

Similarly, we have

$$\begin{cases} z_2 = z_1 + 100 + 3.5)x_2 + (100+5)x_3 + 3.5x_4 + 4x_5 + 9x_6 + 6x_7 + 8x_8 + 9x_9 + 7x_{10} - 18000 \\ z_3 = z_2 + (100+3.5)x_4 + 4x_5 + 9x_6 + 6x_7 + 8x_8 + 9x_9 + 7x_{10} - 20000 \\ z_4 = z_3 + (100+4)x_5 + 9x_6 + 6x_7 + 8x_8 + 9x_9 + 7x_{10} - 20000 \\ z_5 = z_4 + (100+9)x_6 + (100+6)x_7 + 8x_8 + 9x_9 + 7x_{10} - 16000 \\ z_6 = z_5 + (100+8)x_8 + 9x_9 + 7x_{10} - 15000 \\ z_7 = z_6 + (100+9)x_9 + 7x_{10} - 12000 \\ z_8 = z_7 + (100+7)x_{10} - 10000 \end{cases}$$

So the optimization problem is just:

$$
\begin{cases}
\underset{\{x_i\}_{i=1}^{10},\{z_j\}_{j=1}^{8}}{\arg\min} \quad 102x_1 + 99x_2 + 101x_3 + 98x_4 + 98x_5 \\
+ 104x_6 + 100x_7 + 101x_8 + 102x_9 + 94x_{10} \\
z_1 = (100+5)x_1 + 3.5x_2 + 5x_3 + 3.5x_4 + 4x_5 + 9x_6 + 6x_7 + 8x_8 + 9x_9 + 7x_{10} - 12000 \\
z_2 = z_1 + (100+3.5)x_2 + (100+5)x_3 + 3.5x_4 + 4x_5 + 9x_6 + 6x_7 + 8x_8 + 9x_9 + 7x_{10} - 18000 \\
z_3 = z_2 + (100+3.5)x_4 + 4x_5 + 9x_6 + 6x_7 + 8x_8 + 9x_9 + 7x_{10} - 20000 \\
z_4 = z_3 + (100+4)x_5 + 9x_6 + 6x_7 + 8x_8 + 9x_9 + 7x_{10} - 20000 \\
z_5 = z_4 + (100+9)x_6 + (100+6)x_7 + 8x_8 + 9x_9 + 7x_{10} - 16000 \\
z_6 = z_5 + (100+8)x_8 + 9x_9 + 7x_{10} - 15000 \\
z_7 = z_6 + (100+9)x_9 + 7x_{10} - 12000 \\
z_8 = z_7 + (100+7)x_{10} - 10000 \\
x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} \geq 0 \\
z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8 \geq 0
\end{cases}
\tag{10}
$$

■

Remark: As hinted in discussion session, we can also use matrix form to present the optimization problem.

- $\boldsymbol{x}$: The column vector of x_i (amount of bond), for $i = 1, \cdots, 10$
- $\boldsymbol{z}$: The column vector of z_t (surplus cash in year t) for $t = 1, \cdots, 8$
- $\tilde{\boldsymbol{z}} := [0, z_1, \cdots, z_7]^T$
- $\boldsymbol{p}$: Bond price vector
- $\boldsymbol{b}$: Liability schedule vector

Then the LP can be written as

$$
\begin{aligned}
\min_{\boldsymbol{x},\boldsymbol{z}} \quad & \boldsymbol{p}^T\boldsymbol{x} \\
\text{s.t.} \quad & A\boldsymbol{x} + \tilde{\boldsymbol{z}} = \boldsymbol{b} + \boldsymbol{z} \\
& \boldsymbol{x}, \boldsymbol{z} \geq 0
\end{aligned}
\tag{11}
$$

We can use MATLAB to solve this LP[1]. The optimal objective value, i.e. minimum cost is 94, 050. ■

5. Consider the following table of exchange rates for four currencies C_1, C_2, C_3, C_4:

[1]For mathematical problems, such as numerical analysis, solving PDE, optimization, I will use MATLAB. For data related problem, I **may** use Python. cvx is a very powerful MATLAB package to solve optimization problem

	C_1	C_2	C_3	C_4
C_1	1	0.8706	0.0075	1.4279
C_2	1.1486	1	0.00861	1.6401
C_3	133.33	116.14	1	190.48
C_4	0.7003	0.6097	0.00525	1

where the number in the ith row and jth column of the table stands for how much C_i is exchange from 1 of C_j. Formulate an LP for an arbitrage opportunity with these currencies.

Solution: Let r_{ij} be how much of C_i is exchanged form one C_j. Let $R = (r_{ij})$ be the matrix of table above. Let x_{ij} be how much of C_i is exchenged into C_j for $i \neq j$, f_i be the net amount of C_i after all exchanges. Therefore

$$\begin{aligned}
f_1 &= -x_{12} - x_{13} - x_{14} + r_{12}x_{21} + r_{13}x_{31} + r_{14}x_{41} \\
f_2 &= -x_{21} - x_{23} - x_{24} + r_{21}x_{12} + r_{23}x_{32} + r_{24}x_{42} \\
f_3 &= -x_{31} - x_{32} - x_{34} + r_{31}x_{13} + r_{32}x_{23} + r_{34}x_{43} \\
f_4 &= -x_{41} - x_{42} - x_{43} + r_{41}x_{14} + r_{42}x_{24} + r_{43}x_{34}
\end{aligned} \tag{12}$$

So the optimization problem is just

$$\begin{cases}
\underset{x_{12},x_{13},x_{14},x_{21},x_{23},x_{24},x_{31},x_{32},x_{34},x_{41},x_{42},x_{43},f_1,f_2,f_3,f_4}{\arg\max} \quad f_1 \\
f_1 = -x_{12} - x_{13} - x_{14} + r_{12}x_{21} + r_{13}x_{31} + r_{14}x_{41} \\
f_2 = -x_{21} - x_{23} - x_{24} + r_{21}x_{12} + r_{23}x_{32} + r_{24}x_{42} \\
f_3 = -x_{31} - x_{32} - x_{34} + r_{31}x_{13} + r_{32}x_{23} + r_{34}x_{43} \\
f_4 = -x_{41} - x_{42} - x_{43} + r_{41}x_{14} + r_{42}x_{24} + r_{43}x_{34} \\
x_{12}, x_{13}, x_{14}, x_{21}, x_{23}, x_{24}, x_{31}, x_{32}, x_{34}, x_{41}, x_{42}, x_{43} \geq 0 \\
f_1 \leq 1
\end{cases} \tag{13}$$

■

Appendix

Codes can be found on my website