# Optimal Portfolio Choice under Prospect Theory

Kai Li[1], Jun Liu[2] and Jiahui Shui[2]

[1]Macquarie Business School, Macquarie University, NSW 2109, Australia

[2]Rady School of Management, University of California San Diego, La Jolla, California 92093

kai.li@mq.edu.au, junliu@ucsd.edu, jishui@ucsd.edu

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**Abstract**

This paper provides a comprehensive analysis of portfolio choice under prospect theory. We show that there is no optimal solution for half of the parameter space in which loss aversion is low. However, there always exist optimal solutions for the other half of the space. The lower bound of loss aversion above which optimal solutions exist increases without bound when asset returns become more skewed, the number of states increases, or the investor uses the cumulative probability weighting of prospect theory. When the optimal solutions exist, their properties depend crucially on the level of the reference point. For example, an investor with a high reference point seeks negative skewness and may long (short) an asset with a negative (positive) risk premium. The optimal holdings in stocks decrease with the degree of loss aversion if the reference point is high but are independent of loss aversion if the reference point is low.

*Key words*: Prospect theory, loss aversion, optimal portfolio choice, solution existence.

*JEL Classification*: C61, G11

# 1 Introduction

Prospect theory (Kahneman and Tversky, 1979; 1992) is one of the most important topics in behavioral finance. It has been documented to successfully explain some investor behaviors, such as the disposition effect, preference for lottery-like assets, and low participation rates in equity markets (Gomes, 2005; Barberis and Huang, 2008), and financial market anomalies, such as the equity premium puzzle and the value effect (Benartzi and Thaler, 1995; Barberis and Huang, 2001). In this literature, the agents' optimal portfolio weights are usually derived under certain restrictions and/or using numerical methods. Surprisingly, decision making under the original prospect theory utility function of Tversky and Kahneman (1992) is largely overlooked, and the optimal policy is not rigorously understood. In this paper, we provide a comprehensive analysis of portfolio choice under prospect theory.

Prospect theory incorporates two complementary components: a loss aversion utility function and a probability weighting function. Loss aversion refers to the observation that people are more sensitive to losses than to equivalent gains. It is characterized by an $S$-shaped utility function that is concave for gains and convex for losses. The probability weighting function determines the decision weights that are different from physical probabilities. In a portfolio choice problem, the decision weights depend on portfolio weights and hence are endogenous. The interaction of the two components leads to complexity. To understand their effects, we first study the optimal portfolio under loss aversion and then examine the impacts of probability weighting on the optimal portfolio. Our method allows for closed-form solutions.

Although loss aversion has been extensively studied in the portfolio choice and asset pricing literature, few papers discuss the existence of optimal solutions.[1] In this paper, we analytically derive the existence conditions. We show that there is no optimal solution for half of the parameter space in which loss aversion is low. In this case, low loss aversion leads to insufficient penalty for huge losses. As a result, an investor can achieve arbitrarily large expected utility by increasing terminal wealth in one state and at the same time decreasing in another under the budget constraint.

However, there always exist optimal solutions for the other half of the parameter space with sufficiently high loss aversion, even though the investor exhibits risk seeking behavior. The high loss aversion imposes sufficiently heavy penalty for large losses, while the utility gains are relatively smaller, preventing the expected utility from approaching infinity. These results show that the nonexistence of optimal solution is essentially due to insufficient loss penalty rather than risk seeking preferences.

Furthermore, we show that the lower bound of loss aversion above which optimal solutions exist increases without bound as asset returns are more skewed (either left or right) or as

[1] He and Zhou (2011) is one exception.

the number of states of the world increases. The literature typically requires loss aversion parameter to be greater than 1 (investors are more sensitive to losses than to gains). This is consistent with the estimate in Tversky and Kahneman (1992); however, our results show that it is far from sufficient to guarantee the existence of solutions. The lack of definition over a substantial fraction of the parameter space represents a limitation of prospect theory.

When imposing positive wealth constraint, which is sometimes assumed in the literature, we show that optimal solutions always exist; however, they are often given by corner solutions (extreme solutions). We also find that wealth constraint can still have effects in the case when the unconstrained choice has optimal solutions. Imposing constraints can qualitatively change the optimal strategy. Notably, imposing wealth constraint redefines the utility function: it sets the utility to be minus infinity for wealth level beyond the constraint.

When the optimal solutions exist, their properties depend crucially on the ratio of the reference point to the investor's initial wealth level. Hereafter, we simply call this ratio the reference point since we can normalize initial wealth to one. When the reference point is lower than the gross return of the riskless asset ("riskless rate"), the optimal terminal wealth is greater than the reference point in all states. In fact, within the "risk-averse region", the expected utility has a unique local minimum, which is also the global maximum. In this case, the expected utility is always positive.

When the reference point is higher than the riskless rate, the optimal wealth is lower than the reference point in the worst state, against which it is most expensive to protect, but higher in all the other states. Intuitively, it is not optimal to set wealth below the reference point in all states because the investor would exhibit risk seeking. Therefore, when the reference point is high (or the initial wealth is low), the investor pursuits negative skewness by leaving all pains in a single state. Optimal wealth's skewness can be much lower than the skewness of all individual assets. This differs from and complements the literature that typically finds that due to the probability weighting that overweighs the tails of outcome distribution, prospect theory investors dislike negative skewness.[2] Furthermore, the pain in this single state dominates the pleasure in all other states, and as a consequence the value function is always negative.

In this case with a reference point higher than the riskless rate, the expected utility function is nonconvex. This leads to multiple local maximums, each for one partition of states, and the greatest of them are the global maximums. There can be multiple global maximums at the same time; thus, the investor can achieve the same highest expected utility by either longing or shorting the risky asset, leading to jumps in the optimal portfolio.

[2]Investors' strategy return distribution with negative skewness is observed from both the lab and field. For example, Heimer, Iliewa, Imas and Weber (2023) find that traders follow a "gain-exit" strategy that leads to a realized return distribution that is negatively skewed. They also find evidence suggesting that traders tend to take extreme positions if without constraints.

When the reference point equals the riskless rate, which is the case assumed in many papers (e.g., Barberis and Huang, 2008), the investor does not participate into the equity market, and hence her terminal wealth equals the reference point in all states. In fact, under loss aversion preferences, first-order risk aversion (Segal and Spivak, 1990) applies at the reference point. As a result, the investor is reluctant to take on small risks when her wealth is at the reference point. This result is consistent with the literature that uses loss aversion to explain the low participation rates in the equity markets. However, second-order risk aversion applies at all wealth levels greater than the reference point. Therefore, outside this breakpoint, the investor always participates into the equity markets. Particularly, when the reference point is higher than the riskless rate, the investor tends to take a large position, either long or short, in the risky asset.[3]

When the reference point is lower than the riskless rate, the investor always longs/shorts a risky asset with a positive/negative risk premium, similar to standard expected utility investors. However, when the reference point is higher than the riskless rate, the sign of the optimal portfolio weight is not determined by the risk premium but by the relative levels of pricing kernel across states, which determine the tradeoff between losses in one state and gains in another. As a result, the optimal portfolio weight can be positive (negative) even if the asset has a negative (positive) risk premium. The investor can long (short) an asset with an arbitrarily low (high) risk premium.

Furthermore, when the reference point is lower than the riskless rate, the optimal demand is independent of the degree of loss aversion. However, when the reference point is higher than the riskless rate, as the degree of loss aversion increases, the optimal holdings of stocks always decrease due to the aversion to big losses.

We extend our analyses to different situations. In the case with different curvature parameters over risk-averse and risk-seeking regions, as generally allowed in Tversky and Kahneman (1992),[4] there exist optimal solutions only when the utility function has higher curvature over the risk-averse region than the risk-seeking region, independent of the level of loss aversion. In addition, under incomplete markets, the infinitely many state prices lead to infinitely many constraints. We show that the portfolio choice problem in this case is equivalent to one with finitely many constraints, and the optimal portfolio tends to exhibit similar patterns to that under complete markets.

In addition to the $S$-shaped loss aversion utility as studied above, probability weighting

[3]When an investor has disappointment aversion preferences, which implies aversion to losses but first-order risk aversion at every level, Ang, Bekaert and Liu (2005) show that the investor holds no equity in a non-participation region, instead of a single point.

[4]The literature has different estimates of two coefficients. For example, Tversky and Kahneman (1992) estimates the same values of them, Wu and Gonzalez (1996) find higher curvature over the risk-seeking region, and Abdellaoui (2000) estimate a higher curvature over the risk-averse region.

is another major component of cumulative prospect theory. With probability weighting, portfolio choice becomes a fixed-point problem, since the decision weights used by a prospect theory investor depend on portfolio values, which are functions of decision weights. We find that with probability weighting, the conditions for the existence of optimal solutions become stricter. Intuitively, when the decision weights are endogenous, the investor needs to compare all possible orderings of portfolio values across states; thus, the existence condition that should allow solutions for all possible orderings is stricter than the case without probability weighting. We also show that the value function tends to be higher than the case without probability weighting since probability weighting allows the investor to compare different orderings of portfolio values.

Our paper highlights the role of the reference point. In a static setting, the reference point is exogenous. A strand of literature assumes a reference point equal to the riskless rate (e.g., Barberis and Huang, 2008; He and Zhou, 2011). Our results show that a prospect theory investor in this case does not invest in the risky asset regardless of its return distribution. Our paper further contributes to this literature by providing a comprehensive analysis of the effects of the reference point. In a dynamic model, the reference point can be time-varying and endogenous. Setting the reference point as the riskless rate as sometimes did in the literature leads to zero stock holdings over the last time period but leaves the portfolio demand in all prior periods undetermined and irrelevant. On the other hand, an exogenous (time-varying) reference point in a dynamic setting, e.g., in Berkelaar, Kouwenberg and Post (2004), is similar to a constant reference point in a static setting.[5]

Berkelaar et al. (2004) is closely related to our paper. They study optimal portfolio choice under loss aversion where the stock price follows a geometric Brownian motion. Berkelaar et al. (2004) derive their results by (implicitly) imposing positive terminal wealth constraint, which is, however, not an implication of the martingale approach as they argued. Under this constraint, they show that optimal solution always exists. However, our results suggest that it is unlikely to exist an internal solution in this setting. Furthermore, Berkelaar et al. (2004) find that the optimal wealth is either above the reference point or equal to zero. However, we show that this finding may not hold true either in the unconstrained case or with a sufficiently low level of initial wealth.

He and Zhou (2011) is amongst the few studies that analytically examine the existence of optimal solutions under prospect theory. They demonstrate that a prospect theory model with general functional forms of utility and probability weighting, which include those proposed in Tversky and Kahneman (1992) and following variations, is "easily ill-posed" (i.e., optimal solutions do not exist). This finding is in sharp contrast with the standard expected

[5]Another strand of literature also studies dynamic portfolio choice under preferences with endogenous references, such as investor's consumption in Dybvig and Rogers (2013), Dybvig, Jang and Koo (2013), and Choi, Jeon and Koo (2022), that are axiomatically motivated.

utility and is consistent with our results. While He and Zhou (2011) focus on the case with a single risky asset with continuously distributed returns, in which the market is incomplete, we study multiple assets in both complete and incomplete markets. While He and Zhou (2011) solve for the optimal portfolio for the case with the reference point being the riskless rate, our analytical results generally apply to any reference point.

To allow optimal portfolios under prospect theory, a strand of literature imposes certain constraints, such as short-sales constraints in Aït-Sahalia and Brandt (2001), binary stock holdings in Li and Yang (2013), or maximum tolerable loss (similar to wealth constraint) in Ingersoll (2016). Another strand of literature studies variations of the utility function of Tversky and Kahneman (1992). For example, Gomes (2005) imposes concavity when wealth is low. Barberis and Huang (2001) and Barberis, Huang and Santos (2001) consider that investors' utility function consists of a standard expected utility component, in addition to the loss aversion utility. Easley and Yang (2015) examine a combination of loss aversion with Epstein-Zin preferences. Different from these studies, we focus on the loss aversion utility as proposed in the original paper Tversky and Kahneman (1992). Given the central role of the prospect theory in behavioural economics, a rigorous understanding of its implications for decision making helps further explore other economic implications.

Benartzi and Thaler (1995) find that loss aversion helps explain the equity premium puzzle due to the reluctance of investors to invest in stocks. Barberis and Huang (2001) find that loss aversion helps produce a substantial value premium in the cross section. Both studies use numerical methods to solve for the portfolio weights. Gomes (2005) use loss aversion to explain the disposition effect and low equity market participation rates. Easley and Yang (2015) show that loss aversion affects agents' survival mainly through its effect on the agents' portfolio holdings. Grinblatt and Han (2005) find that prospect theory helps explain the momentum effect as investors tend to hold on to their losing stocks. Li and Yang (2013) find that prospect theory helps to simultaneously explain the disposition effect, momentum, and the equity premium puzzle.

The literature also documents challenges in the applications of prospect theory in financial markets. Azevedo and Gottlieb (2012) show that a risk-neutral firm can extract arbitrarily high expected values from consumers with prospect theory preferences since such consumers accept gambles with arbitrarily large negative expected values. As a result, there is no solution to the firm's problem. If the consumers are subject to wealth constraints, the firm can extract all the consumers' wealth with probability approaching one. Ingersoll (2016) shows that under prospect theory, a complete market is not sufficient to guarantee that the market portfolio is efficient or that the standard representative-agent analysis is valid. Our paper adds to this literature by pointing out a fundamental issue that the optimal choice problem under prospect theory often has no solution.

The paper is organized as follows. Section 2 discusses the prospect theory preferences and the general setup. We first study portfolio choice under loss aversion. Section 3 studies a two-state static model with complete markets, and Section 4 examines different extensions, e.g., with multiple states, in a dynamic setting, and in incomplete markets. Section 5 studies the effects of probability weighting, another major component of prospect theory. Section 6 concludes. Calculation details are included in the appendices.

# 2 The Setup

We study the optimal portfolio selection problem for an investor who has prospect theory preferences. Following Tversky and Kahneman (1992), the utility function is defined over gains and losses relative to a reference point $\theta$:

$$u(w) = \begin{cases} -A\frac{1}{1-\gamma_-}(\theta - w)^{1-\gamma_-} & \text{if } w \leq \theta; \\ \frac{1}{1-\gamma_+}(w-\theta)^{1-\gamma_+} & \text{if } w \geq \theta, \end{cases} \tag{1}$$

where $w$ is the investor's wealth, $\theta$ is the reference point, $\gamma_\pm \in [0, 1)$ controls the curvature,[6] and $A > 0$ determines the degree of loss aversion. With a single period, the reference point $\theta$ is a constant. The utility function $u(w)$ is increasing and is concave for $w \geq \theta$ and convex for $w \leq \theta$, as illustrated in Figure 1.

Under loss aversion (1), first-order risk aversion (for large $A$) or first-order risk seeking (for small $A$) applies at the reference point $\theta$. For the other points, second-order risk aversion ($w > \theta$) or risk seeking ($w < \theta$) applies. As a result, an investor with loss aversion is reluctant to take on small risks when her wealth is close to the reference point $\theta$, but always accepts a gamble with a positive mean when her wealth is above $\theta$. This is different from Knightian uncertainty and disappointment aversion (Gul, 1991), with which the risk aversion is first-order at every level.

In addition to the $S$-shaped utility function (1) that characterizes loss aversion, another component of prospect theory is probability weighting, which is a form of subjective probabilities. We will specify and study probability weighting in Section 5.

The investment opportunity set of an investor with an initial wealth $w_0$ contains $N + 1$ assets, including $N$ risky assets and 1 riskless asset. Denote by $\mathbf{R}$ and $r$ the gross returns of the risky assets and the riskless asset, respectively. The optimization problem for the investor is given by

$$\max_{\boldsymbol{x}} \mathbb{E}\big[u(w_T)\big], \tag{2}$$

[6]If $\gamma_\pm \geq 1$, the utility approaches minus infinity when $w$ approaches $\theta$ from above, and hence the utility function is not increasing.

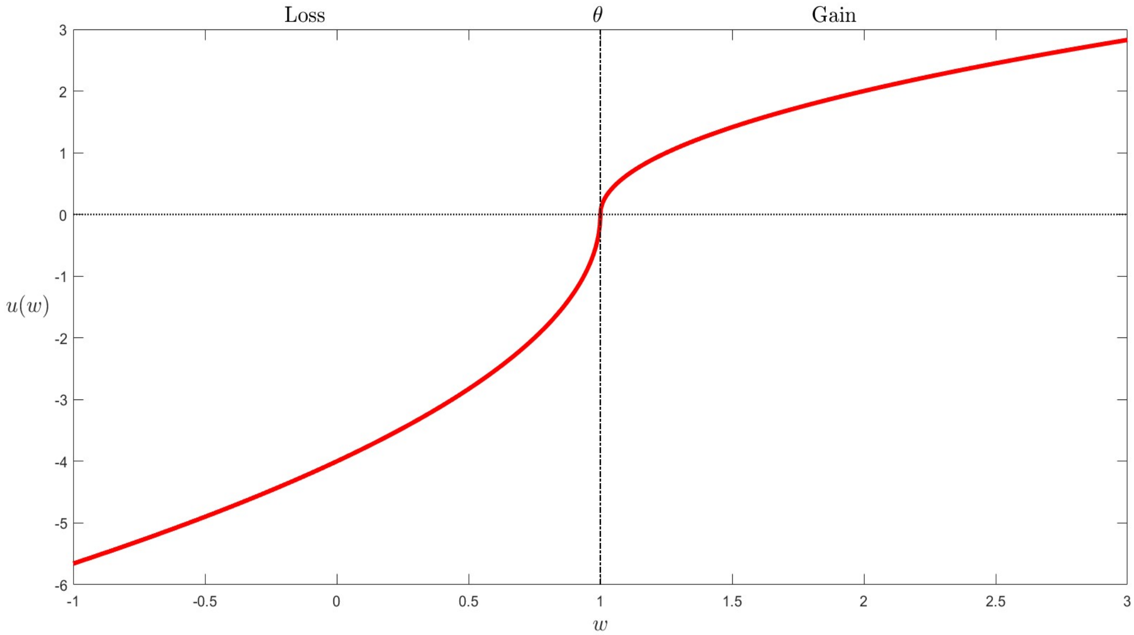


Figure 1: This figure illustrates the loss aversion utility function. Here, $A = 2$, $\gamma_{\pm} = 0.5$, and $\theta = 1$.

subject to the budget constraint $w_{t+1} = w_t r + \boldsymbol{x}_t'(\mathbf{R}_{t+1} - r)$, where $\boldsymbol{x}$ is a vector of the values of the investor's holdings of the risky assets, $u(\cdot)$ is the utility function given by (1), and $w_T$ is the investor's terminal wealth at $T$. The optimization problem (2) has no portfolio constraints. We will also study the case with wealth constraints in Section 3.5. Note that the loss aversion utility (1) generally allows any wealth levels. Accordingly, our results developed in this paper also hold for any initial wealth $w_0$.

When $\gamma_+ = \gamma_-$, which is estimated by Tversky and Kahneman (1992) and widely assumed in the literature (e.g., Berkelaar et al., 2004; Barberis and Huang, 2001), the utility (1) becomes

$$u(w) = \begin{cases} -A\frac{1}{1-\gamma}(\theta - w)^{1-\gamma} & \text{if } w \le \theta; \\ \frac{1}{1-\gamma}(w - \theta)^{1-\gamma} & \text{if } w \ge \theta. \end{cases} \tag{3}$$

In this paper, we focus mainly on this case. We will also study $\gamma_+ \neq \gamma_-$ in Section 3.4.

In the following analysis, we first study portfolio choice under loss aversion in Sections 3–4. We start with a single-period binomial model in Section 3 and then extend to multiple states and a dynamic setting in Section 4. We study both complete markets and incomplete markets and with and without wealth constraints. In Section 5, we will discuss the impacts of probability weighting.

# 3 Binomial Model

To provide clear economic insights, in this section we study a single-period two-state economy. There are two assets: a riskless asset with a constant gross return $r$ and a risky asset. The gross return of the risky asset is either $u$, with probability $p$, or $d$ $(< u)$, with probability $1 - p$. We choose $d < r < u$ to guarantee no arbitrage. The probabilities ($p$ and $1 - p$) are exogenous and can be either physical probabilities or subjective probabilities. To focus on loss aversion, we first consider predetermined probabilities in the current section and in Section 4. We study probability weighting in Section 5 that leads to endogenous probabilities. In the following analysis in this section, we will first study the optimal wealth by using the martingale approach and then resolve the optimal trading strategy.

With two assets and two states, the markets are complete. There exist unique strictly positive state prices. The state prices for state $u$ and $d$ are given, respectively, by $p\xi_u$ and $(1 - p)\xi_d$, where the pricing kernel $\xi$ satisfies $\xi_u = \frac{r-d}{p(u-d)r}$ and $\xi_d = \frac{u-r}{(1-p)(u-d)r}$. As a result, the optimization problem (2) can be rewritten in terms of state prices:

$$\max_{w_u, w_d} pu(w_u) + (1 - p)u(w_d), \tag{4}$$

subject to the budget constraint

$$p\xi_u w_u + (1 - p)\xi_d w_d = w_0, \tag{5}$$

where $w_u$ and $w_d$ are the investor's portfolio wealth in states $u$ and $d$, respectively.

Berkelaar et al. (2004) consider the dynamic portfolio problem under loss aversion. The dynamic portfolio choice problem, i.e., Equation (4) of Berkelaar et al. (2004), is equivalent to a static problem (10) in their paper, in which there are infinitely many states, indexed by a Brownian motion. The problem studied in this section can be viewed as a single-period two-state version of Berkelaar et al. (2004).

## 3.1 Existence of Optimal Solutions

We solve for $w_d$ in terms of $w_u$ from the budget constraint (5), $w_d = \frac{w_0 - p\xi_u w_u}{(1-p)\xi_d}$. This allows us to rewrite the expected utility function $EU$ in terms of $w_u$:

$$EU(w_u) = pu(w_u) + (1 - p)u\Big(\frac{w_0 - p\xi_u w_u}{(1 - p)\xi_d}\Big). \tag{6}$$

The following lemma states the asymptotic behavior of the expected utility (6).

**Lemma 1.** *(Asymptotic behavior.) When $w_u \to +\infty$, the $EU$ satisfies*

$$EU \to \begin{cases} +\infty, & \text{if} \quad A < \big(\frac{p}{1-p}\big)^{\gamma}\big(\frac{\xi_d}{\xi_u}\big)^{1-\gamma}; \\ 0, & \text{if} \quad A = \big(\frac{p}{1-p}\big)^{\gamma}\big(\frac{\xi_d}{\xi_u}\big)^{1-\gamma}; \\ -\infty, & \text{if} \quad A > \big(\frac{p}{1-p}\big)^{\gamma}\big(\frac{\xi_d}{\xi_u}\big)^{1-\gamma}. \end{cases} \tag{7}$$

*When $w_u \to -\infty$, the EU satisfies*

$$EU \to \begin{cases} +\infty, & \text{if} \quad A < \left(\frac{1-p}{p}\right)^{\gamma}\left(\frac{\xi_u}{\xi_d}\right)^{1-\gamma}; \\ 0, & \text{if} \quad A = \left(\frac{1-p}{p}\right)^{\gamma}\left(\frac{\xi_u}{\xi_d}\right)^{1-\gamma}; \\ -\infty, & \text{if} \quad A > \left(\frac{1-p}{p}\right)^{\gamma}\left(\frac{\xi_u}{\xi_d}\right)^{1-\gamma}. \end{cases} \tag{8}$$

Both (7) and (8) show that under the budget constraint, the expected utility can be unbounded from above if loss aversion $A$ is sufficiently low. In this case, the low loss aversion leads to insufficient penalty for extremely negative wealth. As a result, the investor can achieve arbitrarily large expected utility by allocating her terminal wealth to be positive infinity in one state and negative infinity in the other. This is different from standard expected utility functions that are uniformly concave. They impose sufficiently heavy penalty for strongly low level of wealth, while the utility gains in good states are relatively smaller, preventing the $EU$ from approaching positive infinity.

In order to have solutions for the optimal portfolio choice problem (4), the $EU$ should be bounded from above. Therefore, Lemma 1 further leads to the conditions of the existence of optimal solutions.

**Proposition 1.** *(Existence of solutions.) Define*

$$\underline{A} = \max\left\{\left(\frac{p}{1-p}\right)^{\gamma}\left(\frac{\xi_d}{\xi_u}\right)^{1-\gamma}, \quad \left(\frac{1-p}{p}\right)^{\gamma}\left(\frac{\xi_u}{\xi_d}\right)^{1-\gamma}\right\}. \tag{9}$$

1. *When $A > \underline{A}$, the EU has global maximums.*
2. *When $A < \underline{A}$, the EU has no global maximum.*
3. *When $A = \underline{A}$, the EU has global maximums for $w_0 r \geq \theta$ and has no global maximum for $w_0 r < \theta$.*

Proposition 1 shows that there is a lower bound $\underline{A}$ for the loss aversion parameter $A$. The optimal portfolio choice problem (4) has no solution when $A$ is lower than this bound ($A < \underline{A}$). This is essentially because the penalty of loss is too light. Consider a budget preserving change

$$\Delta w = \left(\frac{\Delta}{p\xi_u}, \ \frac{-\Delta}{(1-p)\xi_d}\right).$$

It says that if the investor wants to increase wealth in, e.g., state $u$ by $\frac{\Delta}{p\xi_u}$ , she has to give up $\frac{-\Delta}{(1-p)\xi_d}$ in the other state, $d$. When $A < \underline{A}$, the utility gain in one state (with $w > \theta$) is always higher than the utility loss in the other state. As a result, the investor can always increase her expected utility without bound by allocating more wealth in the first state and at the same time less wealth in the second state. Therefore, the expected utility has no upper bound, and there is no bounded optimal portfolio weight.

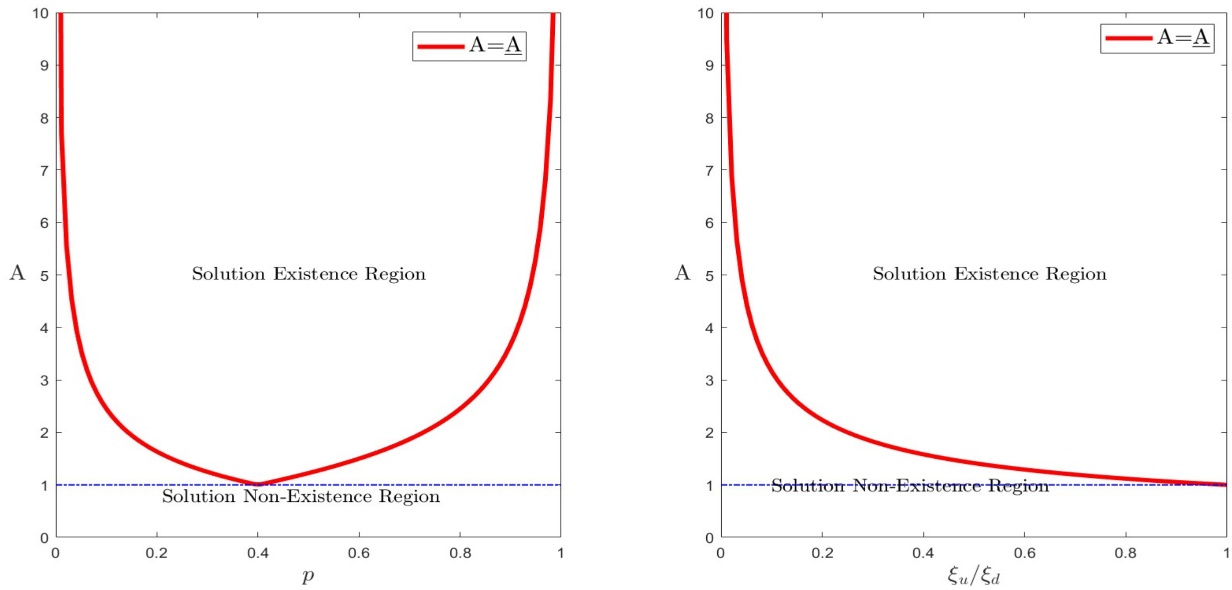


Figure 2: The left panel plots the lower bound $\underline{A}$ against probability $p$, and the right panel plots $\underline{A}$ against the pricing kernel ratio $\xi_u/\xi_d$ (we assume $\xi_u \leq \xi_d$ without loss of generality). Here, $\gamma = 0.5$, $\xi_u = 1$ and $\xi_d = 1.5$ for the left panel and $\gamma = 0.5$ and $p = 0.5$ for the right panel.

However, there always exist optimal solutions for $A > \underline{A}$, even though the investor exhibits risk seeking behavior. With a high level of loss aversion $A$, the utility gain when wealth exceeds the reference point is lower than the utility loss when wealth is lower than the reference point by the same amount. In this case, extreme allocations of wealth with high wealth level in one state and low in another would suffer huge penalty and cannot be optimal.

The above results therefore suggest that the main driver of the nonexistence of optimum is insufficient loss penalty rather than risk seeking preferences. With sufficient penalty on losses (large loss aversion $A$), there always exist global optimum even under risk seeking.

Furthermore, it follows from Condition (9) that $\underline{A} \geq 1$. Thus, to have optimal solutions, loss aversion $A$ must be great than 1. Most papers generally impose $A > 1$ (investors are more sensitive to losses than to equivalent gains), e,g., Benartzi and Thaler (1995), Berkelaar et al. (2004), and Gomes (2005), among many others. However, our results suggest than this is not sufficient to guarantee interior solutions, depending on the market parameters $p$, $\xi_u$, and $\xi_d$, as well as utility parameter $\gamma$. For example, if $\xi_d/\xi_u$ is large, the lower bound for $A$ can be very large. We will show shortly in Section 4.1 that a large number of states of the world will lead to an extremely strict $\underline{A}$.

Figure 2 plots the lower bound $\underline{A}$ against $p$ (the left panel) and against pricing kernel ratio $\xi_u/\xi_d$ (the right panel). It shows that $\underline{A}$ is low when asset return distribution is symmetric.

When the probability is close to zero (positively skewed return) or 1 (negatively skewed return),[7] or the pricing kernel ratio is close to zero, the lower bound $\underline{A}$ approaches infinity, and it is unlikely to have bounded optimal solutions.

## 3.2 Optimal Wealth

We have derived the conditions under which the optimal solutions exist. Now we study the optimal wealth under these conditions. The following proposition describes the results.

**Proposition 2.** *(Optimal wealth and value function.) Suppose $A > \underline{A}$.*

1. *When $\theta \le w_0 r$, the optimal wealth in the two states is given by*

$$\begin{aligned} w_u^* - \theta &= \left(w_0 - r^{-1}\theta\right)\left[p\xi_u^{1-\frac{1}{\gamma}} + (1-p)\xi_d^{1-\frac{1}{\gamma}}\right]^{-1}\xi_u^{-\frac{1}{\gamma}} \ge 0,\\ w_d^* - \theta &= \left(w_0 - r^{-1}\theta\right)\left[p\xi_u^{1-\frac{1}{\gamma}} + (1-p)\xi_d^{1-\frac{1}{\gamma}}\right]^{-1}\xi_d^{-\frac{1}{\gamma}} \ge 0, \end{aligned} \tag{10}$$

   *and the value function is given by $J = \frac{(w_0 - r^{-1}\theta)^{1-\gamma}}{1-\gamma}[p\xi_u^{1-\frac{1}{\gamma}} + (1-p)\xi_d^{1-\frac{1}{\gamma}}]^{\gamma}$. The equalities in (10) hold for $w_0 r = \theta$.*

2. *When $\theta > w_0 r$, the EU has two local maximums, and the global maximum is the greater of them. Specifically, one local maximum occurs for $w_d < \theta < w_u$, at which the portfolio wealth is*

$$\begin{aligned} w_u^* - \theta &= \left(r^{-1}\theta - w_0\right)\left[-p\xi_u^{1-\frac{1}{\gamma}} + (1-p)A^{\frac{1}{\gamma}}\xi_d^{1-\frac{1}{\gamma}}\right]^{-1}\xi_u^{-\frac{1}{\gamma}} > 0,\\ \theta - w_d^* &= \left(r^{-1}\theta - w_0\right)\left[-p\xi_u^{1-\frac{1}{\gamma}} + (1-p)A^{\frac{1}{\gamma}}\xi_d^{1-\frac{1}{\gamma}}\right]^{-1}A^{\frac{1}{\gamma}}\xi_d^{-\frac{1}{\gamma}} > 0, \end{aligned} \tag{11}$$

   *and the EU is given by $EU_+ = -\frac{(r^{-1}\theta - w_0)^{1-\gamma}}{1-\gamma}[-p\xi_u^{1-\frac{1}{\gamma}} + (1-p)A^{\frac{1}{\gamma}}\xi_d^{1-\frac{1}{\gamma}}]^{\gamma}$; another local maximum occurs for $w_u < \theta < w_d$, at which the wealth is*

$$\begin{aligned} \theta - w_u^{**} &= \left(r^{-1}\theta - w_0\right)\left[pA^{\frac{1}{\gamma}}\xi_u^{1-\frac{1}{\gamma}} - (1-p)\xi_d^{1-\frac{1}{\gamma}}\right]^{-1}A^{\frac{1}{\gamma}}\xi_u^{-\frac{1}{\gamma}} > 0,\\ w_d^{**} - \theta &= \left(r^{-1}\theta - w_0\right)\left[pA^{\frac{1}{\gamma}}\xi_u^{1-\frac{1}{\gamma}} - (1-p)\xi_d^{1-\frac{1}{\gamma}}\right]^{-1}\xi_d^{-\frac{1}{\gamma}} > 0, \end{aligned} \tag{12}$$

   *and the EU is $EU_- = -\frac{(r^{-1}\theta - w_0)^{1-\gamma}}{1-\gamma}[pA^{\frac{1}{\gamma}}\xi_u^{1-\frac{1}{\gamma}} - (1-p)\xi_d^{1-\frac{1}{\gamma}}]^{\gamma}$.*

Although the reference point $\theta$ does not affect the existence of solutions in Proposition 1, it significantly affects the properties of optimal portfolio (if exists). Proposition 2 shows that the optimal solution exhibits starkly different properties for $\theta \le w_0 r$ and $\theta > w_0 r$. We first consider the case $\theta < w_0 r$, that is, $\mathbb{E}[\xi]\theta < w_0$. Together with the budget constraint

[7] Return skewness is $p(1-p)(1-2p)(\frac{u-d}{\sigma})^3$, which is positive for $p < 0.5$ and negative for $p > 0.5$.

$\mathbb{E}[\xi w] = w_0$, we have $\mathbb{E}[\xi(w - \theta)] > 0$, showing that the terminal wealth $w$ must be greater than the reference point $\theta$ in at least one of the two states $u$ and $d$. Proposition 2 shows that it is not optimal to have the terminal wealth higher than the reference point in one state but lower in the other state, and the optimal wealth is greater than the reference point in both states. In fact, in this "risk-averse region" (i.e., $w > \theta$), the $EU$ has a local maximum, which is also the global maximum. Furthermore, the expected utility is always positive in this case.

When $\theta = w_0 r$, it is optimal to allocate wealth in both states to be at the reference point, $w^*_{u,d} = \theta$. In this case, the value function is zero. Intuitively, consider a budget preserving change starting from $w_u = w_d = \theta$. If wealth in one state is greater than the reference point, the wealth in the other state must be lower than the reference point. Due to the strong loss penalty (for $A > \underline{A}$), the utility gain in the former state is lower than the utility loss in the latter, leading to negative expected utility.

When $\theta > w_0 r$, due to the budget constraint, the investor can either choose her terminal wealth to be lower than the reference point at one state and greater at the other state, or choose her terminal wealth to be lower than the reference point at both states. Proposition 2 shows that the optimal solution occurs only in the former case: either $w^*_d < \theta < w^*_u$ or $w^*_u < \theta < w^*_d$. In fact, if the investor's terminal wealth is lower than the reference point in both states, she is risk seeking. She can always increase $EU$ by choosing a more extreme wealth allocation across states, eventually pushing wealth above the reference point in one of the two states. The results on the relative levels of optimal terminal wealth to the reference point are summarized in the following corollary.

**Corollary 1.** *Suppose $A > \underline{A}$.*

1. *When $\theta \leq w_0 r$, the optimal wealth is greater than the reference point at both states ($w^*_{u,d} > \theta$).*

2. *When $\theta > w_0 r$, the optimal wealth is greater than the reference point at one state and lower than the reference point at the other state.*

Figure 3 plots the $EU$ against $w_u$. The left panel is for the case $\theta \leq w_0 r$. We divide the interval of $w_u$ into three subintervals, $w_u \in (-\infty, \theta)$, $[\theta, \hat{w}_u]$, and $(\hat{w}_u, +\infty)$, where $\hat{w}_u = \frac{w_0 - (1-p)\xi_d \theta}{p \xi_u}$ corresponds to the case with the wealth in state $d$ being at the referent point ($\hat{w}_d = \theta$). First, for $w_u < \theta$, which implies $w_d > \theta$ due to the budget constraint, the $EU$ increases with $w_u$. Second, for $w_u \in [\theta, \hat{w}_u]$, the wealth at both states is higher than the reference point ($w_{u,d} \geq \theta$), and the $EU$ is concave. Therefore, there is a local maximum in this interval.[8] Third, for $w_u > \hat{w}_u$, which corresponds to $w_d < \theta < w_u$, the $EU$ decreases

[8]In particular, if $\gamma = 0$, the $EU$ is linear not concave in the interval $w_u \in [\theta, \hat{w}_u]$, and the optimum occurs at the boundary.

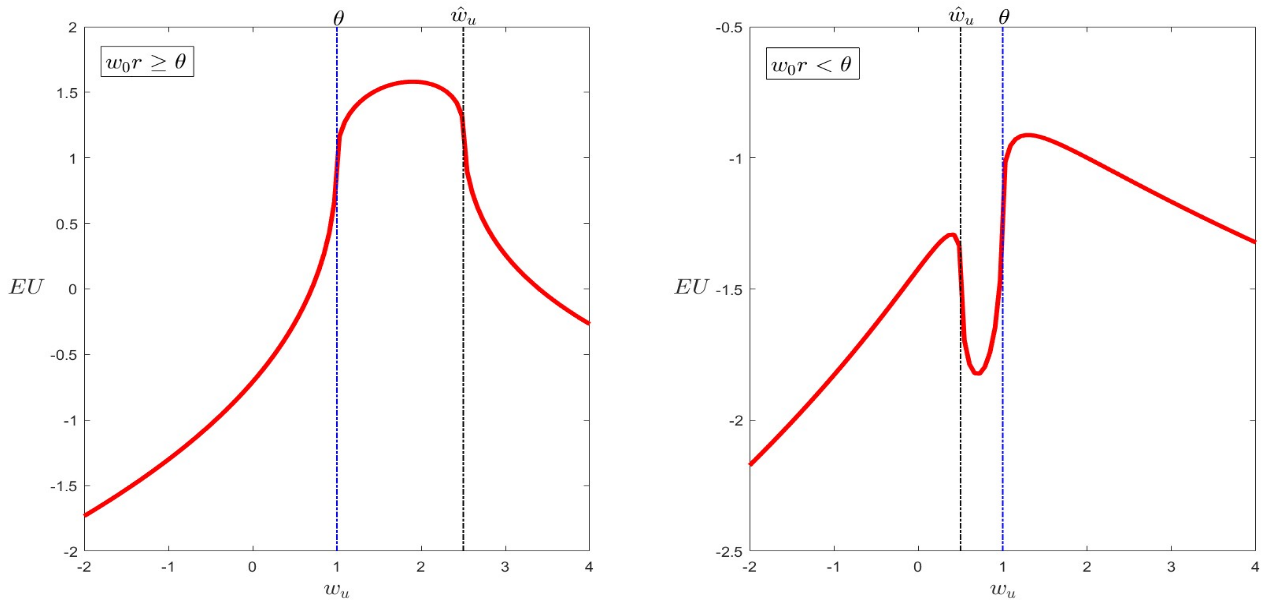


Figure 3: The figure plots the expected utility function $EU$ against wealth $w_u$ for $\theta \leq w_0r$ (the left panel) and $\theta > w_0r$ (the right panel). Here, $A = 2$, $\gamma = 0.5$, $\xi_u = 1$, $\xi_d = 1.5$, and $p = 0.5$. We also set $w_0/\theta = 2$ ($\geq r$) for the left panel and $w_0/\theta = 1$ ($< r$) for the right panel.

with $w_u$. These results thus show that the local maximum over the interval $w_u \in [\theta, \hat{w}_u]$ is the global maximum.

The right panel illustrates the case $\theta > w_0r$. It shows that the expected utility function is nonconvex and hence has multiple local maximums. We still discuss the results in terms of three subintervals of $w_u$. For $w_u \in (-\infty, \hat{w}_u)$ or $w_u \in (\theta, +\infty)$, which implies $w_u < \theta < w_d$ or $w_d < \theta < w_u$, respectively, the $EU$ is concave and has a local maximum in each of the two intervals. For $w_u \in [\hat{w}_u, \theta]$, the wealth at both states is lower than the reference point ($w_{u,d} \leq \theta$), and the $EU$ is convex. Thus, there is no local maximum in this interval. Therefore, there are two local maximums, one in $(-\infty, \hat{w}_u)$ and one in $(\theta, +\infty)$, and one of them (the first one under our parameters) is the global maximum. The following corollary further provides the conditions that determine which one is the global maximum.

**Corollary 2.** *Suppose $A > \underline{A}$ and $\theta > w_0r$.*

1. *When $(1-p)\xi_d^{1-\frac{1}{\gamma}} < p\xi_u^{1-\frac{1}{\gamma}}$, the global maximum occurs for $w_d^* < \theta < w_u^*$.*
2. *When $(1-p)\xi_d^{1-\frac{1}{\gamma}} > p\xi_u^{1-\frac{1}{\gamma}}$, the global maximum occurs for $w_u^{**} < \theta < w_d^{**}$.*
3. *When $(1-p)\xi_d^{1-\frac{1}{\gamma}} = p\xi_u^{1-\frac{1}{\gamma}}$, there are two global maximums, one for $w_d^* < \theta < w_u^*$ and one for $w_u^{**} < \theta < w_d^{**}$.*

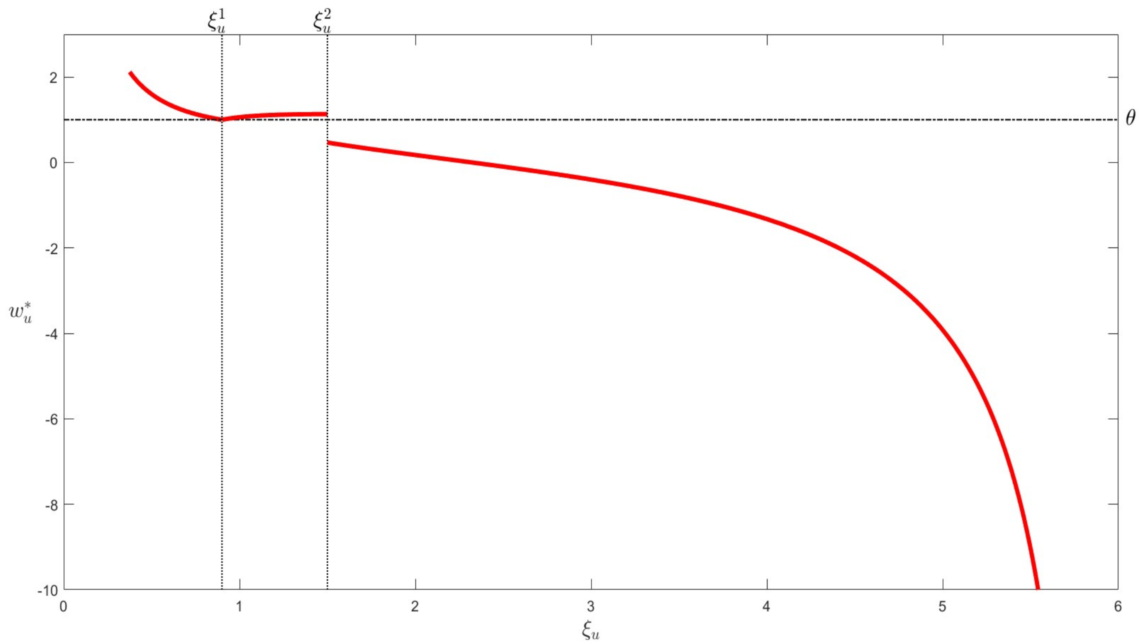


Figure 4: This figure illustrates the optimal wealth $w_u^*$ against pricing kernel $\xi_u$. Here, $\xi_d = 0.5$, $\gamma = 0.5$, $A = 2$, $w_0 = 1.2$, $\theta = 1$, and $p = 0.5$.

When $(1-p)\xi_d^{1-\frac{1}{\gamma}} < p\xi_u^{1-\frac{1}{\gamma}}$, $EU_+$ is the global maximum (i.e., $EU_- < EU_+$). In this case, Proposition 2 shows that the global maximum occurs for $w_d^* < \theta < w_u^*$. To understand this result, we consider a case with $p = 0.5$ and $\xi_u < \xi_d$, and the investor searches the optimal wealth allocation starting from, e.g., $w_u = w_d = rw_0$. Because a change in $w_d$ is associated with a larger change in $w_u$ under the budget constraint, the investor's expected utility increases with $w_u$. Therefore, the global maximum must occur for $w_d^* < \theta < w_u^*$. In this case, the optimal wealth is lower than the reference point in the bad state $d$, against which it is more expensive to protect, but higher in the good state $u$.

The result is the opposite for $(1-p)\xi_d^{1-\frac{1}{\gamma}} > p\xi_u^{1-\frac{1}{\gamma}}$. In this case, the global maximum that occurs for $w_u^* < \theta < w_d^*$, and the value function is given by $EU_-$. Particularly, when $(1-p)\xi_d^{1-\frac{1}{\gamma}} = p\xi_u^{1-\frac{1}{\gamma}}$, we have $EU_- = EU_+$, and both are global maximums.

Figure 4 illustrates the optimal wealth $w_u^*$ as a function of the pricing kernel $\xi_u$. When the pricing kernel is too small or too large, there is no optimal solution as shown in Proposition 1. Within the interval in which there exist optimal solutions, there are two thresholds, $\xi_u^1$ and $\xi_u^2$, for $\xi_u$. The first threshold $\xi_u^1$ is such that $w_0 r = \theta$, and the second $\xi_u^2$ is such that $(1-p)\xi_d^{1-\frac{1}{\gamma}} = p\xi_u^{1-\frac{1}{\gamma}}$. When $\xi_u < \xi_u^1$, which corresponds to $\theta \le w_0 r$, the optimal wealth $w_u^*$ is given by (10) and is positive. When $\xi_u > \xi_u^1$, we have $\theta > w_0 r$. In this case. if $\xi_u^1 < \xi_u < \xi_u^2$, the optimal wealth $w_u^*$ is given by (11) and is also positive; if $\xi_u > \xi_u^2$, the optimal wealth $w_u^*$ is given by (12) and is negative.

In particular, at $\xi_u = \xi_u^2$, there are two different optimal wealth $w_u^*$, one is higher than the reference point and one is lower.

Proposition 2 also shows that when $\theta > w_0 r$, the expected utility functions at both local maximums are negative. Therefore, the value function, which is the greater of the two expected utility $J = \max\{EU_+, EU_-\}$, is also negative. The sign of value function is summarized in the following corollary. It shows that it is determined by the initial wealth level.

**Corollary 3.** *The value function is positive for $\theta < w_0 r$, negative for $\theta > w_0 r$, and equal to zero for $\theta = w_0 r$.*

Proposition 2 also shows that the optimal wealth and value function (if exist) are determined by the relative level of the reference point $\theta$ to the investor's current wealth scaled up by the riskless rate $w_0 r$. This suggests that when setting the reference point to be equal to or lower than current wealth scaled by the riskless rate in a dynamic context, as sometimes studied in the literature, the optimal portfolio would follow Case 1 of Proposition 2, and the investor behaves more like a standard expected utility maximizer. We will provide more discussions on this in Section 4.2.

## 3.3 Optimal Portfolio Allocation

Given the optimal wealth, we can further resolve the optimal portfolio allocation.

**Proposition 3.** *(Optimal portfolio allocation.) Suppose $A > \underline{A}$.*

1. *When $\theta \leq w_0 r$, the value of the optimal holdings of the risky asset is given by*

$$x^* = \frac{(k-1)(w_0 r - \theta)}{(u-r) + k(r-d)}, \tag{13}$$

*where $k = [\frac{p(u-r)}{(1-p)(r-d)}]^{\frac{1}{\gamma}}$.*

2. *When $\theta > w_0 r$, the optimal holdings of the risky asset is given by*

$$x^* = \begin{cases} x_+, & \textit{if} \quad \frac{p(u-r)^{1-\gamma}}{(1-p)(r-d)^{1-\gamma}} > 1; \\ x_-, & \textit{if} \quad \frac{p(u-r)^{1-\gamma}}{(1-p)(r-d)^{1-\gamma}} < 1; \\ x_+ \textit{ and } x_-, & \textit{if} \quad \frac{p(u-r)^{1-\gamma}}{(1-p)(r-d)^{1-\gamma}} = 1, \end{cases} \tag{14}$$

*where*

$$x_+ = \frac{(k_+ + 1)(\theta - w_0 r)}{(u-r) - k_+(r-d)} > 0, \qquad x_- = \frac{(k_- + 1)(\theta - w_0 r)}{(u-r) - k_-(r-d)} < 0,$$

*with $k_+ = [\frac{p(u-r)}{A(1-p)(r-d)}]^{\frac{1}{\gamma}}$ and $k_- = [\frac{Ap(u-r)}{(1-p)(r-d)}]^{\frac{1}{\gamma}}$.*

When $\theta \leq w_0 r$, (13) shows that the value of optimal holdings of stock $x^*$ is positive if and only if the risk premium is positive ($k > 1$). This result is typically found with standard expected utility. In this case, the optimal holdings are independent of loss aversion $A$.

When $\theta > w_0 r$, we have shown that the $EU$ has two local maximums. Proposition 3 shows that if $\frac{p(u-r)^{1-\gamma}}{(1-p)(r-d)^{1-\gamma}} > 1$, which is case 1 in Corollary 2, the global maximum occurs for $w_d < \theta < w_u$. In this case, $x_+ > \frac{\theta - w_0 r}{u-r} > 0$: the investor allocates more wealth in state $u$ than in $d$ by longing the risky asset. As a result, the optimal holdings $x^*$ are always positive, even if the risk premium is negative. Indeed, under the loss aversion utility, the sign of the optimal holdings are mainly determined by the relative levels of pricing kernel at states $u$ and $d$, which determine the tradeoff between losses in one state and gains in the other state. Because proportional scales in the expected returns at both states do not affect the condition $\frac{p(u-r)^{1-\gamma}}{(1-p)(r-d)^{1-\gamma}} > 1$, the investor can long an asset with an arbitrarily low risk premium.

However, if $\frac{p(u-r)^{1-\gamma}}{(1-p)(r-d)^{1-\gamma}} < 1$, the global maximum occurs for $w_u < \theta < w_d$. To allocate more wealth in state $d$, the investor must be short the risky asset: $x_- < -\frac{\theta - w_0 r}{r-d} < 0$. As a result, the optimal holdings are always negative, even with a positive risk premium. These results are different from that under standard expected utility. When the investment opportunity set contains a risky asset and a riskless asset, an investor with standard expected utility is always long the risky asset if it has a positive risky premium.

In the limiting case $\frac{p(u-r)^{1-\gamma}}{(1-p)(r-d)^{1-\gamma}} = 1$, the two local maximums have the same expected utility, and both $x_u$ and $x_d$ are optimal.

**Corollary 4.** *1. When $\theta = w_0 r$, the optimal holdings of stock are zero $x^* = 0$.*

*2. When $\theta < w_0 r$, $x^* > 0$ if and only if the risk premium is positive.*

*3. When $\theta > w_0 r$, $x^* > 0$ if and only if $\frac{p(u-r)^{1-\gamma}}{(1-p)(r-d)^{1-\gamma}} > 1$, independent of the sign of the risk premium.*

For $A > \underline{A}$, first-order risk aversion is applying at $w = \theta$ but second-order risk aversion at all $w > \theta$. As a result, an investor with loss aversion is reluctant to take on small risks when her wealth is close to $\theta$ but always accepts a gamble with a positive mean when her wealth is above $\theta$. Corollary 4 shows that when the reference point equals the investor's initial wealth level scaled up by the riskless rate, the investor is reluctant to invest in stocks. This is consistent with the literature that shows that loss aversion helps explain investors' reluctance to participate in the equity markets (e.g., Benartzi and Thaler, 1995; Gomes, 2005). Corollary 4 also shows that when the reference point is low ($\theta \leq w_0 r$), the investor always buys the risky asset with a positive risk premium, and when the reference point is high, the investor's demand is independent of the sign of risk premium.

Figure 5 illustrates the optimal portfolio weight $\phi^* \equiv x^*/w_0$ as a function of $u$ for the case $\theta > w_0 r$. When $u$ is small (i.e., $u < r + k_+(r-d)$), the pricing kernel $\xi_u$ is large. In this

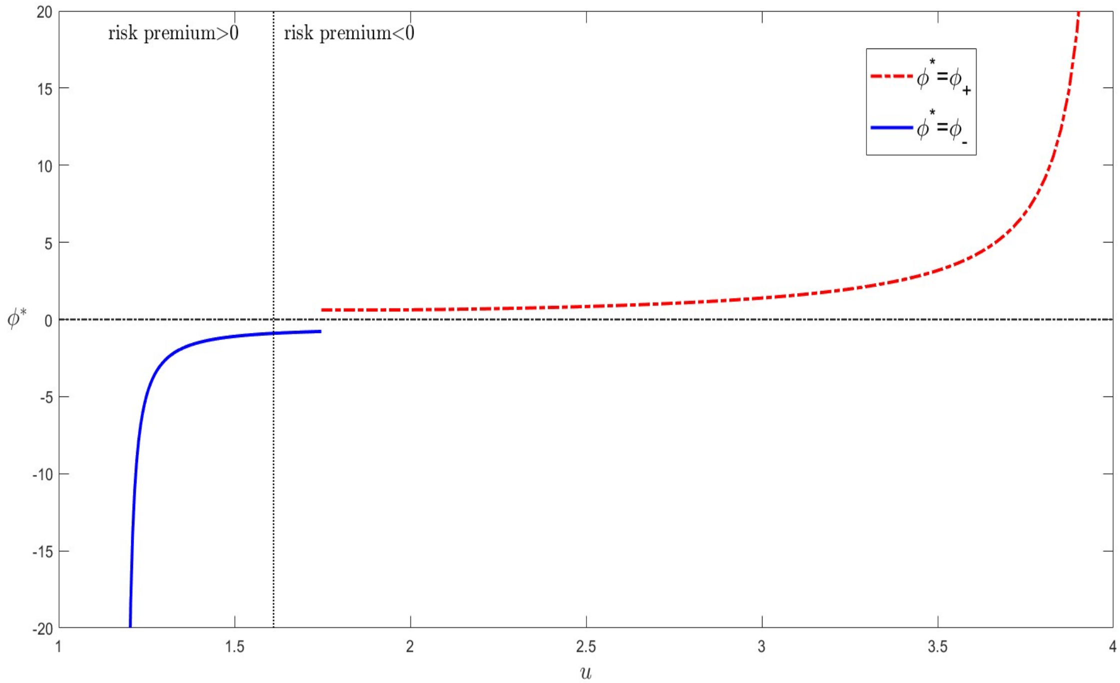


Figure 5: This figure illustrates the optimal portfolio weight $\phi^* \equiv x^*/w_0$ against $u$ for the case $\theta > w_0 r$. Here, $A = 2$, $\gamma = 0.5$, $d = 0.5$, $r = 1$, $w_0 = 0.8$, $\theta = 1$, and $p = 0.45$.

case, there is no optimal solution for the optimal choice problem as shown in Proposition 1. Similarly, there is also no optimal solution for sufficiently large $u$ (i.e., $u > r + k_-(r-d)$) that leads to low pricing kernel.

Now we consider the interval of $u$ in which there exist optimal solutions. When $\frac{p(u-r)^{1-\gamma}}{(1-p)(r-d)^{1-\gamma}} < 1$, that is, $u < \hat{u} \equiv r + \left[(1-p)/p\right]^{\frac{1}{1-\gamma}}(r-d)$, the optimal portfolio weight is illustrated by the blue solid line. In this case, the optimal portfolio weight is always negative, independent of the sign of risk premium. For example, the vertical dotted line illustrates the value of $u$ at which the risk premium is 0. The risk premium is positive when $u$ is greater than this value. Figure 5 shows that the optimal portfolio weight can be still negative even when the risk premium is positive. As $u$ decreases and approaches $r + k_-(r-d)$, the optimal portfolio approaches negative infinity. When $\frac{p(u-r)^{1-\gamma}}{(1-p)(r-d)^{1-\gamma}} > 1$, i.e., $u > \hat{u}$, the optimal portfolio weight is illustrated by the red dash-dot line. In this case, the optimal portfolio weight is always positive. As $u$ increases and approaches $r + k_+(r-d)$, the optimal portfolio approaches positive infinity.

In the extreme case of $\frac{p(u-r)^{1-\gamma}}{(1-p)(r-d)^{1-\gamma}} = 1$, i.e., $u = \hat{u}$, there are two different optimal portfolio weights $\phi^* = \phi_-(\equiv x_-/w_0)$ and $\phi^* = \phi_+(\equiv x_+/w_0)$, with one ($\phi_-$) being negative and one ($\phi_+$) positive. Both of them lead to the same $EU$s. In this case, the investor can achieve the highest expected utility by either buying the risky asset or short-selling it. As a result, the optimal portfolio weight can jump from positive to negative around this extreme point $u = \hat{u}$, for a small change in parameters (including both market parameters and utility parameters). We will show in Section 4.1 that with multiple states, there can be multiple local maximums of the $EU$ and hence multiple jumps in the optimal portfolio weight.

Due to this jump, the optimal portfolio weight is either $\phi^* \geq 0.61$ or $\phi^* \leq -0.78$ as illustrated in Figure 5. the portfolio weight cannot be optimal if it is small in magnitude ($-0.78 < \phi^* < 0.61$). Investors with a high reference point tend to take a large position, either long or short, in the risky asset, always participating into the stock market.

Proposition 3 directly leads to the following results on comparative statics.

**Corollary 5.** *(Comparative statics.)*

1. *The magnitude of optimal investment in stocks $|x^*|$ increases with $|\theta - w_0 r|$.*
2. *The optimal stock investment $x^*$ is independent of $A$ for $\theta \leq w_0 r$ and decreases with $A$ for $\theta > w_0 r$.*

Recall that when the reference point is the riskless rate $\theta = w_0 r$, the optimal stock investment is zero. Corollary 5 further shows that the larger the deviation of the reference point from the riskless rate $|\theta - w_0 r|$ is, the higher the magnitude of stock investment $|x^*|$.

The second part of Corollary 5 states the effect of the degree of loss aversion $A$ on the optimal holdings of stock. When the reference point is low $\theta \leq w_0 r$, the optimal demand is independent of $A$. When the reference point is high $\theta > w_0 r$, as loss aversion $A$ increases, the optimal holdings of stock always decreases for both the case $\frac{p(u-r)^{1-\gamma}}{(1-p)(r-d)^{1-\gamma}} > 1$ and $\frac{p(u-r)^{1-\gamma}}{(1-p)(r-d)^{1-\gamma}} \leq 1$, since the investor is averse to big losses. This further lowers expected utility.

## 3.4 The case $\gamma_+ \neq \gamma_-$

The above analyses focus on the case with the same curvature parameters, $\gamma_+ = \gamma_- \equiv \gamma$. This is considered in most studies. In this case, we have shown that the magnitude of loss aversion $A$ determines the existence of optimal solutions.

However, in the cumulative prospect theory developed in Tversky and Kahneman (1992), $\gamma_+$ and $\gamma_-$ can be different. The literature has different estimates of two coefficients. For example, Tversky and Kahneman (1992) estimates the same values of them: $\gamma_+ = \gamma_- = 0.12$; Wu and Gonzalez (1996) find $\gamma_-$ is higher ($\gamma_+ = 0.48$ and $\gamma_- = 0.63$), and Abdellaoui (2000) estimate a higher $\gamma_+$ ($\gamma_+ = 0.11$ and $\gamma_- = 0.08$). The following proposition discusses the existence of optimal solutions for problem (4) if $\gamma_+ \neq \gamma_-$.

**Proposition 4.** *(Existence of solutions when $\gamma_+ \neq \gamma_-$.)*

*When $\gamma_- < \gamma_+$, the EU always has global maximums.*

*When $\gamma_+ < \gamma_-$, the EU has no global maximum.*

Although loss aversion $A$ plays an important role in the existence of optimal solutions for the case $\gamma_- = \gamma_+$, Proposition 4 shows that it does not affect the existence when $\gamma_+ \neq \gamma_-$. If $\gamma_- < \gamma_+$, the penalty for losses is strong and the optimal portfolio weights always exist. However, if $\gamma_+ < \gamma_-$, there is no internal optimal portfolio weight, independent of $A$.

### 3.5 Positive Wealth Constraint

We have shown that with loss aversion utility, there exist optimal portfolios for half of the parameter space, but there is no optimal solution for the other half of the parameter space. To have solutions, the literature sometimes imposes wealth constraint. In this section, we examine its effects. Without loss of generality, we consider positive wealth constraint.[9] We show that imposing constraints often leads to corner solutions (extreme solutions).

We consider the nontrivial case $w_0 > 0$[10] and assume $\gamma_+ = \gamma_- \equiv \gamma$. The wealth constraint $w_u \geq 0$ for state $u$ together with the budget constraint $w_d = \frac{w_0 - p\xi_u w_u}{(1-p)\xi_d}$ leads to an upper bound of wealth in state $d$: $w_d \leq \frac{w_0}{(1-p)\xi_d}$. Similarly, there is also an upper bound for $w_u$ due to constraint $w_d \geq 0$. As a result, the wealth in the two states lies in the intervals:

$$w_u \in \left[0,\ \frac{w_0}{p\xi_u}\right], \qquad w_d \in \left[0,\ \frac{w_0}{(1-p)\xi_d}\right]. \tag{15}$$

With positive wealth constraint, the $EU$ should always have maximums, although many of them are corner solutions. Specifically, when $A < \underline{A}$, there is no optimal wealth for the unconstraint case. After imposing the constraint, there always exists optimal wealth and it is given by corner solutions:

$$\begin{cases} w_u^* = \frac{w_0}{p\xi_u} \\ w_d^* = 0 \end{cases} \quad \text{or} \quad \begin{cases} w_u^{**} = 0 \\ w_d^{**} = \frac{w_0}{(1-p)\xi_d} \end{cases}. \tag{16}$$

By implicitly imposing positive wealth constraint, Berkelaar et al. (2004, Proposition 3) find that the optimal wealth is either above the reference point or equal to 0. However, (16) shows that their findings depend on the initial level of wealth. With low $w_0$, the optimal wealth in both states can be lower than the reference point.

Now we consider $A > \underline{A}$. In this case, the unconstraint problem has optimal solutions. Especially, if $\theta > w_0 r$, the optimal wealth is negative at one of the two states. However,

[9] Imposing other levels of bound (either lower or upper bound) of wealth can be understood from our results by adjusting $\theta$ and $w_0$ by noting that our analyses generally hold for any values of them and that imposing a lower wealth bound is equivalent to certain upper wealth bound due to the budget constraint.

[10] The loss aversion preferences in Tversky and Kahneman (1992) allows negative wealth; however, there is no solution for $w_0 \leq 0$ under the positive wealth constraint.

imposing constraint in this state binds the wealth to be zero. These results show that wealth constraint can still have effects even if the choice has optimal solution in the unconstraint case. Imposing constraints can even qualitatively change the optimal strategy.

In addition, Proposition 2 implies that when $w_0$ is sufficiently low: $w_0 < \min\left\{p\xi_u\theta[A^{-\frac{1}{\gamma}}(\xi_u/\xi_d)^{-\frac{1}{\gamma}}+1],(1-p)\xi_d\theta[A^{-\frac{1}{\gamma}}(\xi_d/\xi_u)^{-\frac{1}{\gamma}}+1]\right\}$, the optimal solutions are always corner solutions, as one of the two bounds of $w_u$ in (15) must be reached.

When $A > \underline{A}$ and $\theta \le w_0 r$, the optimal wealth tends to be interior solutions ($w_{u,d} > 0$).[11] In this case, the wealth constraint takes no effect.

**Remark 1.** *With wealth constraints, the optimal portfolio weights always exist but are often given by corner solutions.*

It is worth noting that imposing wealth constraint is similar to changing utility function. For example, imposing positive wealth constraint is equivalent to setting utility as $-\infty$ for $w \in (-\infty, 0)$.[12] However, the original prospect theory utility is defined over all wealth levels in Tversky and Kahneman (1992).

# 4 Extensions

Section 3 focuses on a two-state static model with complete markets. In this section, we study extensions, e.g., with multiple states, in a dynamic setting, and in incomplete markets.

## 4.1 Multiple States

We extend the results in Section 3 to the general case with $S$ states in the world. Assume that there are $N = S$ assets and the markets are complete. We will examine incomplete markets in the next section.

### 4.1.1 Existence of Optimal Solutions

We divide all states into two sets, $\mathbb{S}_+$ and $\mathbb{S}_-$:

$$\mathbb{S}_+ = \{s : w_s \ge \theta\}, \qquad \mathbb{S}_- = \{s : w_s < \theta\}.$$

For notational simplicity, we use $1_+$ to denote $1_{s\in\mathbb{S}_+}$ and $1_-$ to denote $1_{s\in\mathbb{S}_-}$. Then we can extend the results on the existence condition for the binomial model in Proposition 1 to this general case with $S$ states.

[11] If $\theta < 0$, the optimal solutions can be given by corner solutions.

[12] This is similar to the utility studied in Dybvig, Rogers and Back (1999).

**Lemma 2.** *Given a non-trivial partition $\mathbb{S}_+$ and $\mathbb{S}_-$, if*

$$\mathbb{E}[\xi^{1-1/\gamma}(1_+ - A^{1/\gamma}1_-)] > 0,$$

*the $EU$ has no global maximum.*

Lemma 2 shows that if there exists a global maximum, we must have

$$\mathbb{E}\big[\xi^{1-1/\gamma}(1_+ - A^{1/\gamma}1_-)\big] \leq 0, \tag{17}$$

for any partition $\mathbb{S}_+$ and $\mathbb{S}_-$. For each partition, condition (17) leads to a lower bound for $A$:

$$A \geq \left(\frac{\mathbb{E}[\xi^{1-1/\gamma}1_+]}{\mathbb{E}[\xi^{1-1/\gamma}1_-]}\right)^\gamma = \left(\frac{\sum_{s\in\mathbb{S}_+} p_s\xi_s^{1-1/\gamma}}{\sum_{s\in\mathbb{S}_-} p_s\xi_s^{1-1/\gamma}}\right)^\gamma, \qquad \forall\mathbb{S}_+, \mathbb{S}_-, \tag{18}$$

where $p_s$ is the probability for state $s$, $s = 1, ..., S$. To find the maximum of these lower bounds, we only need to consider the cases where there is only 1 state with $w_s < \theta$ by noting that the state prices are positive. Thus, (18) is equivalent to

$$A \geq \left(\frac{\sum_{t\neq s} p_t\xi_t^{1-1/\gamma}}{p_s\xi_s^{1-1/\gamma}}\right)^\gamma, \qquad \forall s \in \{1, \cdots, S\}. \tag{19}$$

Since the $EU$ is a continuous function on a compact set, we have the following proposition on the conditions for the existence of global maximums.

**Proposition 5.** *Define*

$$\underline{A} = \max_s \left\{\left(\frac{\sum_{t\neq s} p_t\xi_t^{1-1/\gamma}}{p_s\xi_s^{1-1/\gamma}}\right)^\gamma\right\}. \tag{20}$$

1. *When $A > \underline{A}$, the $EU$ has global maximums.*
2. *When $A < \underline{A}$, the $EU$ has no global maximum.*
3. *When $A = \underline{A}$, the $EU$ has global maximums for $w_0 r \geq \theta$ and has no global maximum for $w_0 r < \theta$.*

Proposition 5 shows that there is a lower bound $\underline{A}$ for $A$ given by (20). The optimal portfolio choice problem has interior solutions only if $A$ is higher than this bound.

With $S$ states, it follows from (20) that the lower bound $\underline{A}$ must satisfy

$$\underline{A} \geq (S-1)^\gamma,$$

and the equality holds only if $p_s\xi_s^{1-1/\gamma} = p_t\xi_t^{1-1/\gamma}$, $\forall s, t$. When there are two states $S = 2$, the bound (20) reduces to (9) in Proposition 1, and it is greater than or equal to 1. As the number of states $S$ increases, the lower bound $\underline{A}$ increases. When $S$ is large, to have interior solutions, the agent must have an extremely high aversion to losses that imposes large penalty for losses. Because $\underline{A}$ increases with $S$ without bound, when $S$ is very large, e.g., a limiting case with an infinitely many states, the $EU$ is unlikely to have global maximum.

**Corollary 6.** *As the number of states $S$ increases, the conditions for the existence of interior solutions becomes stricter.*

Proposition 5 also implies that if the $EU$ has global optimums, we must have

$$A > \frac{p_t^\gamma \xi_t^{\gamma-1}}{p_s^\gamma \xi_s^{\gamma-1}}, \qquad \forall s \in \{1, \cdots, S\}. \tag{21}$$

Intuitively, we consider the asymptotic behavior with budget preserving changes in terminal wealth over any two states $s, t$, that is, $(\Delta w_s, \Delta w_t) = (\frac{\Delta}{p_s \xi_s}, -\frac{\Delta}{p_t \xi_t})$. if condition (21) is violated, the investor can always increase her expected utility without bound by allocating more wealth at state $t$ and less wealth at state $s$ accordingly. If there are two states $S = 2$, then (21) can be a necessary and sufficient condition for global maximums as shown in Proposition 1. When $S > 2$, it is not sufficient to guarantee the existence of global maximums and we need a higher bound $\underline{A}$.

### 4.1.2 Optimal Solutions

In this section, we study the optimal solutions. To this end, we assume that the $EU$ has global maximums. The optimal solutions are summarized in the following proposition.

**Proposition 6.** *(Optimal wealth and value function.) Suppose $A > \underline{A}$.*

1. *When $\theta \leq w_0 r$, the optimal wealth is given by*

$$w^* - \theta = \frac{w_0 - r^{-1}\theta}{\mathbb{E}[\xi^{1-1/\gamma}]} \xi^{-1/\gamma}, \tag{22}$$

*and the value function is given by*

$$J = \frac{\left(w_0 - r^{-1}\theta\right)^{1-\gamma}}{1-\gamma} \left(\mathbb{E}[\xi^{1-1/\gamma}]\right)^\gamma \geq 0. \tag{23}$$

2. *When $\theta > w_0 r$, for each non-trivial partition $\mathbb{S}_+$ and $\mathbb{S}_-$, the $EU$ has a local maximum, at which the wealth is given by*

$$\begin{aligned} (w - \theta)1_+ &= \frac{r^{-1}\theta - w_0}{\mathbb{E}[\xi^{1-1/\gamma}(-1_+ + A^{1/\gamma}1_-)]} \xi^{-1/\gamma} 1_+, \\ (\theta - w)1_- &= \frac{r^{-1}\theta - w_0}{\mathbb{E}[\xi^{1-1/\gamma}(-1_+ + A^{1/\gamma}1_-)]} A^{1/\gamma} \xi^{-1/\gamma} 1_-, \end{aligned} \tag{24}$$

*and the expected utility is given by*

$$EU = -\frac{(r^{-1}\theta - w_0)^{1-\gamma}}{1-\gamma} \Big(\mathbb{E}[\xi^{1-1/\gamma}(-1_+ + A^{1/\gamma}1_-)]\Big)^\gamma < 0. \tag{25}$$

*The global maximum is the greatest of these local maximums.*

Proposition 6 shows again that the optimal solutions exhibit starkly different properties for $\theta \le w_0 r$ and $\theta > w_0 r$. When $\theta \le w_0 r$, the $EU$ has a unique local maximum, which is the global maximum. At the global maximum, the terminal wealth in all states is higher than the reference point $w^* > \theta$ (if $w_0 r > \theta$) or equal to the reference point $w^* = \theta$ (if $w_0 r = \theta$). In this case, the expected utility is always positive (if $\theta < w_0 r$) or equals zero (if $\theta = w_0 r$).

When $\theta > w_0 r$, there are multiple local maximums, one for each non-trivial partition $\mathbb{S}_+$ and $\mathbb{S}_-$. The global maximum is achieved at the partition that leads to the highest $EU$, and Equation (25) shows that this partition leads to the smallest $\mathbb{E}[\xi^{1-1/\gamma}(-1_+ + A^{1/\gamma}1_-)]$. Note that $\mathbb{E}[\xi^{1-1/\gamma}(-1_+ + A^{1/\gamma}1_-)] = (A^{1/\gamma}+1)\mathbb{E}[\xi^{1-1/\gamma}1_-] - \mathbb{E}[\xi^{1-1/\gamma}]$. So the global maximum occurs for the partition with the lowest $\mathrm{E}[\xi^{1-1/\gamma}1_-]$. Furthermore, this lowest must be achieved for partitions with only 1 state in $\mathbb{S}_-$. That is, it is optimal to set wealth to be lower than the reference point in the state with the lowest $\xi_s^{1-1/\gamma}p_s$ and higher than the reference point in all the other states.

**Corollary 7.** *When $\theta \le w_0 r$, the optimal wealth is higher than the reference point $w^* > \theta$ in all states.*

*When $\theta > w_0 r$, the optimal wealth is lower than the reference point in the states with the lowest $\xi_s^{1-1/\gamma}p_s$ and higher than the reference point in all the other states.*

Corollary 7 shows that when the reference point is high ($\theta > w_0 r$), the optimal wealth is lower than the reference point in one state but higher in all the other states. In this case, the investor tends to pursuit negative skewness.[13] This is different from the literature that typically shows that investors with prospect theory preferences or disappointment aversion preferences dislike negative skewness (e.g., Ang et al., 2005; Barberis and Huang, 2008).

Figure 6 illustrates the impacts of return skewness on the optimal portfolio when the reference point is high ($\theta > w_0 r$). We consider a case with three states and three assets, a riskless asset and two risky assets (assets $A$ and $B$). To examine investor's demand for skewness, we let the gross returns of the two risky assets have the same mean 1.2 and same variance 0.2. We further fix asset $A$'s return skewness to be 0 and let asset $B$'s return skewness vary. Panel (a) illustrates asset $B$'s gross returns $R_b$ in the three states. Asset $B$'s return $R_{b,d}$ ($R_{b,u}$) in state $d$ ($u$) satisfies $R_{b,d} < r < R_{b,u}$, and as $R_{b,m}$ decreases, its return skewness changes from negative to positive. Panel (b) shows that for different levels of return

[13]In particular, with two states, Corollary 4 shows that for $\theta > w_0 r$, the investor is more likely to be short positively skewed asset but long negatively skewed asset than an investor who has CRRA utility: We consider two sets of parameters $u, d, p$ and $\hat{u}, \hat{d}, \hat{p}$, where $\hat{u} = d + 2p(u-d)$, $\hat{d} = u - 2(1-p)(u-d)$, and $\hat{p} = 1 - p$, under which the risky asset has the same expected return and return volatility. Without loss of generality, we assume $p < 0.5$. In this case, return skewness of the risky asset is positive under the first set of parameters and negative under the second set. If the investor has CRRA utility, her portfolio weight is the same under both sets of parameters. However, under loss aversion, the optimal portfolio weight tends to be negative under the first set of parameters but positive under the second set.

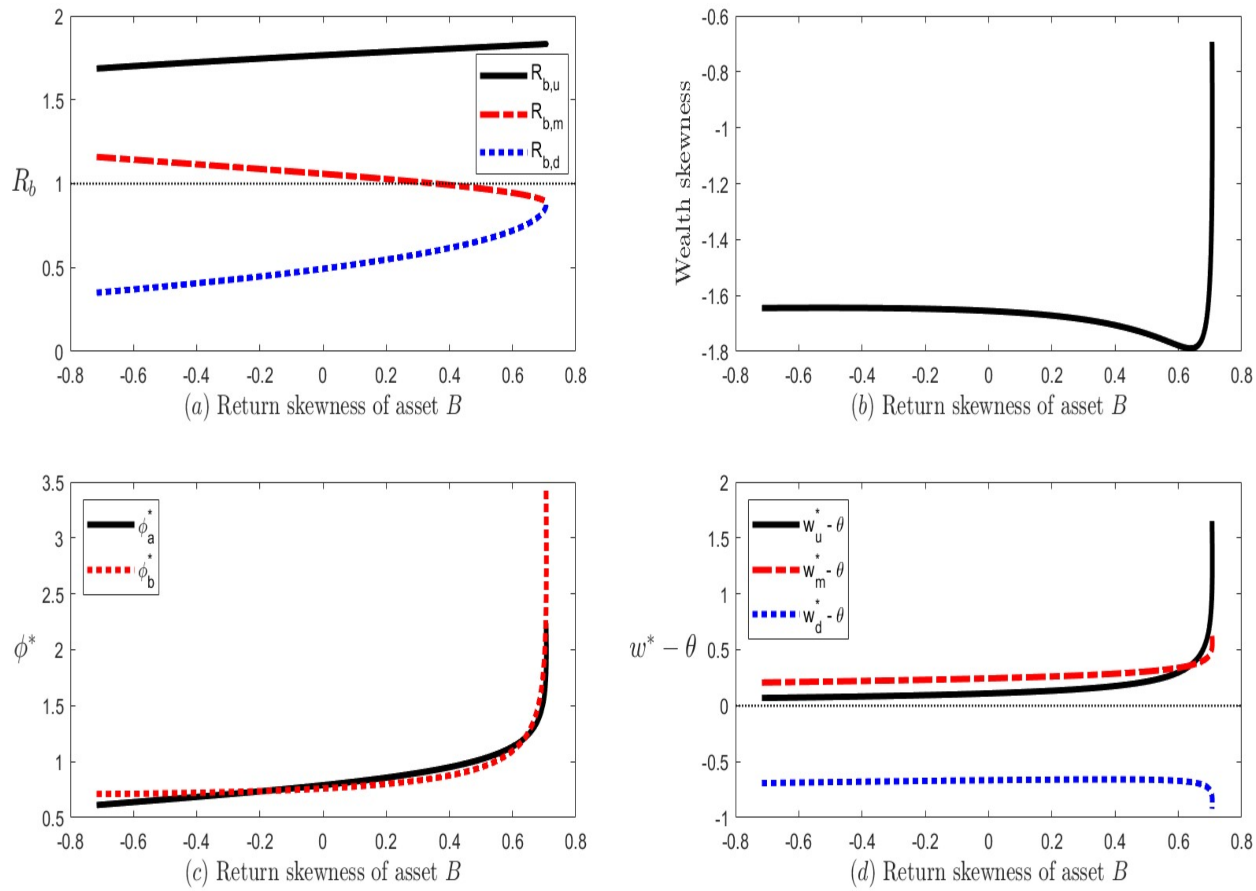


Figure 6: This figure illustrates the impacts of return skewness on the optimal portfolio for the case $\theta > w_0 r$. The investment opportunity set contains three assets: a riskless asset and two risky assets ($A$ and $B$). The gross returns of the two risky assets have the same mean 1.2 and same variance 0.2. We set asset $A$'s return skewness to be 0 and let asset $B$'s skewness vary. Panel ($a$) plots asset $B$'s returns in the three states ($u$, $m$, and $d$) against its return skewness. Panel ($b$) illustrates the return skewness of the optimal portfolio wealth. Panel ($c$) illustrates optimal portfolio weights of the two risky assets. Panel ($d$) illustrates the difference between the optimal portfolio wealth and the reference point ($w^* - \theta$) in the three states. Here, $A = 8$, $\gamma = 0.5$, $r = 1$, $w_0 = 0.8$, $\theta = 1$, and the probabilities of the three states $(p_1, p_2, p_3) = (1/2, 1/3, 1/6)$.

skewness of asset $B$, the optimal wealth return tends to be very negative. The skewness of the optimal wealth is much lower than the skewness of all individual assets. This is because the optimal wealth is higher than the reference point in two states ($u$ and $m$) but much lower than the reference point in the other (state $d$) as illustrated in panel (d). As a result, with low initial wealth, the agent actually seeks negative skewness.

Corollary 7 further shows that the "uninsured" state with wealth lower than the reference point is the one with the lowest $\xi_s^{1-1/\gamma} p_s$. It tends to be the worst state of the world with the highest state price against which it is most expensive to protect.

In particular, if multiple states have the same lowest $\xi_s^{1-1/\gamma} p_s$, then the $EU$ has multiple global maximums. In this case, there are multiple sets of optimal portfolio weights.

## 4.2 Dynamic Portfolio Choice

The prospect theory developed in Kahneman and Tversky (1979) and Tversky and Kahneman (1992) focuses on decision making in static settings. It is extended to a dynamic context to study asset pricing under prospect theory in, e.g., Barberis and Huang (2001) and Barberis et al. (2001). In this section, we study portfolio choice under loss aversion in a dynamic setting by following Barberis and Huang (2001) and Barberis et al. (2001).

### 4.2.1 Utility Function

For dynamic portfolio choice, the reference point can be time-varying. The utility function (1) becomes

$$u(w_{t+1}, \theta_t) = \begin{cases} -A\frac{1}{1-\gamma}(\theta_t - w_{t+1})^{1-\gamma} & \text{if } w_{t+1} \leq \theta_t; \\ \frac{1}{1-\gamma}(w_{t+1} - \theta_t)^{1-\gamma} & \text{if } w_{t+1} \geq \theta_t, \end{cases} \tag{26}$$

and the optimization problem of an investor who maximizes expected utility over terminal wealth $w_T$:

$$\max_{\boldsymbol{x}} \mathbb{E}\big[u(w_T, \theta_{T-1})\big] \tag{27}$$

where the utility function $u(\cdot)$ is given by (26).

Prior to examining time-varying reference point, we discuss a simplest example with a constant reference point.

**Example 1.** *(Binomial tree model with a constant reference point.) When $\theta \leq w_0 r$, it follows from Proposition 2 that at each note the optimal wealth $w_t^*$ (if exists) is higher than $\theta$, and the value function is always positive. In this case, the prospect theory investor is similar to a standard expected utility investor.*

*When $\theta > w_0 r$, the optimal wealth (if exists) at time 1 is higher than $\theta$ in one state and lower in the other state. If the former state happens, then the optimal wealth is higher than $\theta$*

*over all the following periods. If the latter state happens, then the optimal portfolio depends on the relative level of $w_1$ and $\theta/r$. However, once the optimal wealth is higher than $\theta/r$, it will stay higher than $\theta/r$ over the remaining periods, and the optimal portfolio is similar to that under standard expected utility.*

Example 1 shows that a dynamic portfolio problem with a constant reference point leads to similar properties of the optimal portfolio to a static problem. It would be more interesting to study time-varying, endogenous reference point. For example, Barberis and Huang (2001) and Barberis et al. (2001) use current wealth scaled up by the riskless rate as an endogenous reference point. Another interesting endogenous reference point can be investor's current consumption, as proposed, e.g., in the axiomatically motivated models of Dybvig and Rogers (2013), Dybvig et al. (2013), and Choi et al. (2022). However, an exogenous (time-varying) reference point is similar to a constant reference point in a static setting.

In addition, as suggested by Barberis and Huang (2001) and Barberis et al. (2001), in a dynamic setting, past gains/losses may also affect the preferences of investors. To characterize this effect, they assume that the utility function has an extra argument $z_t$ that measures past gains ($z_t < 1$) or losses ($z_t > 1$) from holding the risky asset relative to a "historical benchmark level".

To capture both the dynamic reference point and the effects of past gains/losses, we assume the following parsimonious model, which is in the spirit of and is a simpler version of Barberis and Huang (2001) and Barberis et al. (2001):[14]

$$\theta_t = \begin{cases} w_t r & \text{if } z_t \geq 1; \\ w_t z_t r & \text{if } z_t < 1. \end{cases} \tag{28}$$

When the investor has recently experienced losses ($z_t \geq 1$), she uses her current wealth scaled by the riskless rate as the reference point $\theta_t = w_t r$. In this case, the investor views stock return as a gain if it exceeds the riskless rate. With recent gains ($z_t < 1$), the investor has built up a cushion of prior gains that mitigates small subsequent losses and hence decreases the reference point $\theta_t = w_t z_t r$, as suggested by Barberis et al. (2001).[15]

[14]As pointed out by Barberis et al. (2001), the losses/gains, $w_{t+1} - \theta_t = w_{t+1} - w_t r$, calculated over total financial wealth as in this paper are consistent with those calculated over the risky asset alone: $w_t\phi_t R_t - w_t\phi_t r$ as in Barberis et al. (2001), since $w_t\phi_t R_t - w_t\phi_t r = [(1-\phi_t)r + \phi_t R_t]w_t - w_t r = w_{t+1} - w_t r$.

[15]In Barberis et al. (2001), loss aversion $A(z_t)$ is an increasing function of $z_t$ for $z_t \geq 1$, capturing the idea that future losses are penalized more heavily when the investor suffered losses recently. In their asset pricing model, $z_t$ is a function of stock returns and is endogenous. However, in our portfolio choice problem, it becomes exogenous since stock returns are exogenous. Our analysis is to provide a general analysis of the impact of loss aversion $A$ without imposing its dynamics in (26).

### 4.2.2 Optimal Portfolio Choice

We first look at the case without accounting for historical gains/losses, i.e., $z_t \equiv 1$. In this case, the reference point is the current wealth scaled by the riskless rate $\theta_t = w_t r$, which is usually used in static prospect theory models. At time $T-1$, the investor's reference point is $\theta_{T-1} = w_{T-1}r$, and her optimal terminal wealth is given by $w_T^* \equiv \theta_{T-1}$ as implied by Proposition 6. It is optimal to invest only in the riskless asset over the last time period $[T-1, T]$, and the value function is always 0, independent of the levels of wealth $w_t$, for $t \in [0, T]$. Furthermore, the holdings of the risky assets over $[0, T-1]$ are irrelevant and undeterminable.

**Corollary 8.** *In a dynamic portfolio choice problem* (26)–(27) *with a reference point* $\theta_t = w_t r$*, a prospect theory investor invests only in the riskless asset over the last time period* $[T-1, T]$*. Asset holdings over* $[0, T-1]$ *are undeterminable and irrelevant. The value function is always 0.*

Now we examine the effects of $z_t$ that accounts for historical gains/losses. By referring to historical gains/losses $z_t$, the optimal investment strategies (if exist) depend on the historical price path. This is because the reference point that is determined by $z_t$ depends on past prices. As a result, different historical price paths even with the same current price level and investment opportunity lead to different asset holdings.

Furthermore, the optimal portfolio (if exists) depends crucially on the historical benchmark $z_{T-1}$ that determines the reference point. For example, when $z_{T-1} \geq 1$, the optimal portfolio is described in Corollary 8. When $z_{T-1} < 1$, we have $\theta_{T-1} < w_{T-1}r$, and Proposition 6 implies that the optimal terminal wealth is given by $w_T^* - \theta_{T-1} = \frac{w_{T-1}(1-z_{T-1})}{\mathbb{E}[\xi^{1-1/\gamma}]}\xi^{-1/\gamma} > 0$. It is higher than the reference point in all states. As a result, the value function is positive. In addition, with the "cushion effect" ($z_t < 1$), a prospect theory investor tends to behave like a standard expected utility maximizer (characterized by Proposition 6 case 1).

**Example 2.** *(A two-period binomial-tree model) To further specify and examine the effects of* $z_t$*, we consider an IID 2-period binomial-tree model* ($T = 2$)*. Over each period, stock return can be either* $u$*, with probability* $p$*, or* $d$ ($< u$)*, with probability* $1-p$*. We assume that at the end of period 1,*

$$z_1 = \begin{cases} z_u < 1 & \textit{in state } u; \\ z_d \geq 1 & \textit{in state } d. \end{cases} \tag{29}$$

*That is, the investor experiences gains from holding the risky asset over the first period if the asset price goes up (in state* $u$*) and experiences losses if price goes down (in state* $d$*). This is consistent with Barberis et al. (2001) who consider stock price in the previous period as one benchmark to calculate historical gains/losses. In our portfolio choice problem, we only need to specify the relative level of* $z_t$ *to 1.*

*Proposition 2 implies that the (conditional) expected utility increases with wealth $w_u$ at node u but is always zero at node d. As a result, the investor can always achieve a higher expected utility without bound by increasing $w_u$ (i.e., longing more risky asset). This means that the dynamic portfolio choice problem does not have an internal solution. This nonexistence of optimal solutions is essentially because the penalty for losses (i.e., low wealth level) in state d is insufficient. In fact, the utility is the same for any level of wealth in state d, imposing no penalty for losses.*

*With the time-varying reference point, in good states the investor tends to decreases the reference point due to the "cushion effect", which directly amplifies utility gain in these states. In this case, the expected utility is more likely to approach positive infinity relative to the constant reference point problem (Example 1).*

## 4.3 Incomplete Markets

The above analyses focus on portfolio choice problems under complete markets. In this section, we consider incomplete markets. In this case, there are infinitely many strictly positive state prices, which lead to infinitely many constraints. We find transform the portfolio choice problem to one with finitely many constraints that can be solved with the Lagrangian approach. Numerical simulations show that the optimal solution exhibits similar patterns to that under complete markets.

With incomplete markets, there are infinitely many risk neutral measures. The maximization problem is:

$$\begin{aligned} \max \quad & \mathbb{E}[u(w)] \\ \text{subject to} \quad & \mathbb{E}^{\mathbb{Q}}[w/w_0] = r, \quad \forall \mathbb{Q} \in \mathcal{M}, \end{aligned} \tag{30}$$

where $\mathcal{M}$ is an infinite set of all risk neutral measures. Therefore, there are infinitely many constraints in problem (30). Indeed, these infinitely many restrictions can be reduced to finite ones. The intuitive reason is that there are finite states, so we can always find a finite base to represent the set $\mathcal{M}$. For simplicity, we use the following notations:

**Definition 1.** *Let $W = \big\{(q_1, \cdots, q_n) \in \mathbb{R}^n \big| q_1R_1 + \cdots + q_nR_n = r\big\}$, where $R_i$ is the gross return of an asset at the ith state, and $\mathcal{P}^+ = \{(q_1, \cdots, q_n)|q_1 + \cdots + q_n = 1, q_i \geq 0, \forall i = 1, 2, \cdots, n\}$. Then the set of risk-neutral measures is given by*

$$\mathcal{M} = W \cap \mathcal{P}^+. \tag{31}$$

We can always find a finite subset of $\mathcal{M}$, say $\mathcal{Q}$, such that every element in $\mathcal{M}$ can be represented as linear combination of $\mathcal{Q}$. We denote $\mathcal{Q} = \{\mathbb{Q}_1, \cdots, \mathbb{Q}_l\}$, where $l \leq n$ and $\mathbb{Q}_i$ satisfies $\mathbb{E}^{\mathbb{Q}_i}[R] = r$, $i = 1, \cdots, l$. Therefore, for any $\mathbb{Q} \in \mathcal{M}$, it can be written as a linear combination of $\mathbb{Q}_i$, that is,

$$\mathbb{Q} = \alpha_1\mathbb{Q}_1 + \cdots + \alpha_l\mathbb{Q}_l, \quad \alpha_1 + \cdots + \alpha_l = 1. \tag{32}$$

Hence $\mathbb{E}^{\mathbb{Q}}[R] = \alpha_1 \mathbb{E}^{\mathbb{Q}_1}[R] + \cdots + \alpha_l \mathbb{E}^{\mathbb{Q}_l}[R] = \alpha_1 r + \cdots + \alpha_l r = r$.

**Lemma 3.** *Maximization problem* (30) *with infinitely many constraints can be reduced to the following one with finitely many constraints:*

$$\begin{array}{ll} \max & \mathbb{E}[u(w)] \\ \textit{subject to} & \mathbb{E}^{\mathbb{Q}}[w/R_f] = w_0, \quad \forall \mathbb{Q} \in \mathcal{Q}. \end{array} \tag{33}$$

*The Lagrangian is given by* $\mathcal{L} = \mathbb{E}[u(w)] - \sum_{i=1}^{l} \lambda_i (\mathbb{E}^{\mathbb{Q}_i}[w/r] - w_0)$.

We reduce the portfolio choice problem under incomplete markets to one with a finite number of constraints. As a result, this problem can be solved with the Lagrangian approach.

### 4.3.1 Trinomial Model

We consider a trinomial model to illustrate our results. Suppose that there are only two assets, one risky and one riskless assets, and three different states, $\omega_1, \omega_2, \omega_3$. The return of the risky asset equals $u, m, d$ at the three states, respectively. Without loss of generality, we assume $d < m < u$.

For any risk neutral measure $\mathbb{Q} = (q_1, q_2, q_3)$, it must satisfies

$$\begin{cases} uq_1 + mq_2 + dq_3 = r, \\ q_1 + q_2 + q_3 = 1. \end{cases} \tag{34}$$

The fundamental sets of solution are given by

$$\mathbb{Q} = k\left(\frac{m-d}{u-m}, \frac{d-u}{u-m}, 1\right) + \left(\frac{r-m}{u-m}, \frac{u-r}{u-m}, 0\right), \tag{35}$$

where the range of $k$ is determined by the following inequalities:

$$\begin{cases} 0 \le k\dfrac{m-d}{u-m} + \dfrac{r-m}{u-m} \le 1, \\ 0 \le k\dfrac{d-u}{u-m} + \dfrac{u-r}{u-m} \le 1, \\ 0 \le k \le 1. \end{cases} \tag{36}$$

That is,

$$\max\left(0, \frac{m-r}{m-d}\right) \le k \le \frac{u-r}{u-d}. \tag{37}$$

Therefore, any risk neutral measure $\mathbb{Q}$ can be written as $\mathbb{Q} = \alpha\mathbb{Q}_1 + (1-\alpha)\mathbb{Q}_2$, where $\alpha \in [0,1]$, and the bases are given by

$$\begin{cases} \mathbb{Q}_1 = \left(0, \dfrac{r-d}{m-d}, \dfrac{m-r}{m-d}\right), \quad \mathbb{Q}_2 = \left(\dfrac{r-d}{u-d}, 0, \dfrac{u-r}{u-d}\right), & \text{if} \quad m \ge r, \\ \mathbb{Q}_1 = \left(\dfrac{r-m}{u-m}, \dfrac{u-r}{u-m}, 0\right), \quad \mathbb{Q}_2 = \left(\dfrac{r-d}{u-d}, 0, \dfrac{u-r}{u-d}\right), & \text{if} \quad m < r. \end{cases} \tag{38}$$

**Corollary 9.** *With three states, the optimal portfolio choice problem can be rewritten as*

$$\begin{aligned}
&\max && p_1 u(w_u) + p_2 u(w_m) + p_3 u(w_d) \\
&\text{subject to} && \mathbb{E}^{\mathbb{Q}_1}[w/r] = w_0, \\
& && \mathbb{E}^{\mathbb{Q}_2}[w/r] = w_0.
\end{aligned} \tag{39}$$

*The Lagrangian is given by* $\mathcal{L} = \mathbb{E}[u(w)] - \lambda_1(\mathbb{E}^{\mathbb{Q}_1}[w/r] - w_0) - \lambda_2(\mathbb{E}^{\mathbb{Q}_2}[w/r] - w_0)$.

In the case of three states, Corollary 9 shows that the optimization problem involves two constraints. In contrast, under complete markets, only one constraint is needed (see, e.g., (A.2)). To further illustrate the results, we consider an example.

**Example 3.** *Suppose that* $r = 1.03$, $u = 1.10, m = 1.00$, *and* $d = 0.95$. *Then the risk-neutral measure* $\mathbb{Q} = (q_1, q_2, q_3)$ *satisfies*

$$\begin{cases} 1.10q_1 + 1.00q_2 + 0.95q_3 = 1.03, \\ q_1 + q_2 + q_3 = 1. \end{cases}$$

*So the risk-neutral measure has the form* $\mathbb{Q} = (8/15+k, -3k, 7/15+2k)$, *where* $k \in [-7/30, 0]$. *Taking* $k = -7/30$ *and* $0$, *we can get the bases:*

$$\mathbb{Q}_1 = (3/10, 7/10, 0), \quad \mathbb{Q}_2 = (8/15, 0, 7/15).$$

*Therefore, our optimization problem is*

$$\begin{aligned}
&\max && p_1 u(w_u) + p_2 u(w_m) + p_3 u(w_d) \\
&\text{subject to} && 3/10 w_u + 7/10 w_m = w_0 r, \\
& && 8/15 w_u + 7/15 w_d = w_0 r,
\end{aligned} \tag{40}$$

*where* $\mathbb{P} = (p_1, p_2, p_3)$ *is the physical probability measure. Solving this optimization problem is sufficient to get optimal wealth.*

Under the parameters in Example 3, Figure 7 illustrates the expected utility function $EU$ as a function of the portfolio weight $\phi$. Here we set a high value of $A = 2$ to have global maximum.[16] In the left panel, $\theta \leq w_0 r$. The $EU$ has a unique local maximum (at $\phi^* = 3.14$), which is also the global maximum. In this case, the value function is positive. In the right panel of Figure 7, $\theta > w_0 r$. There are two local maximums, and the global maximum is the greater of them (at $\phi^* = 8.51$). In this case, the value function is negative. These results are similar to those under complete markets.

[16] If the portfolio choice problem under complete markets has optimal solutions, the corresponding problem under incomplete markets (i.e., with less assets available) should also have optimal solutions, since the $EU$ in the former case should be higher than or equal to that in the latter case.

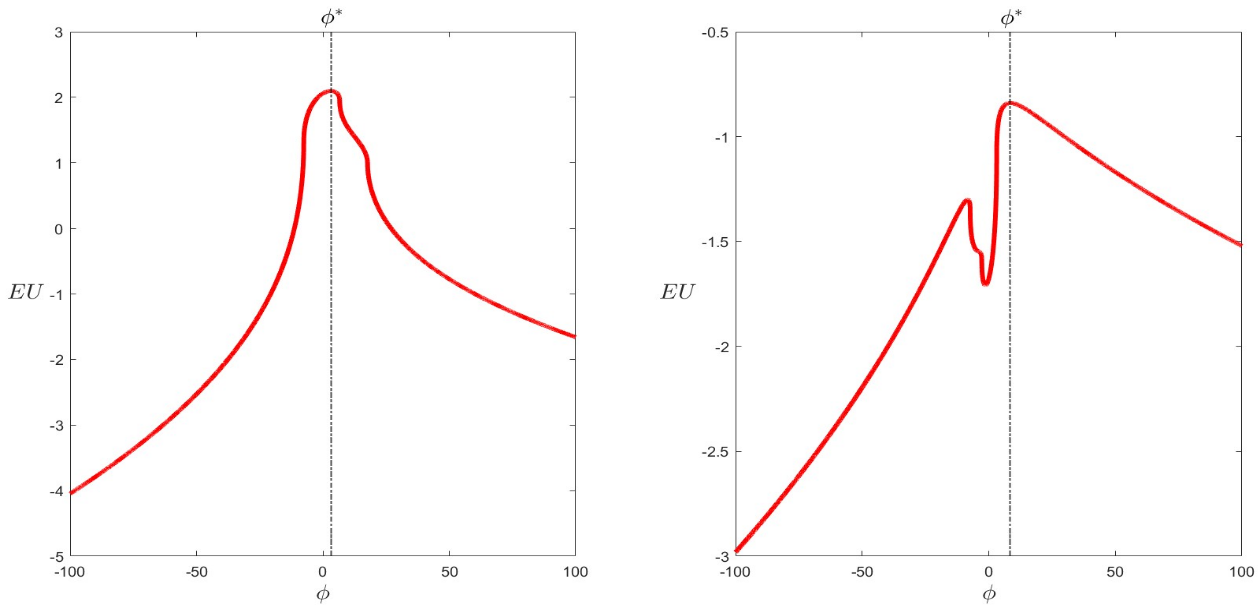


Figure 7: The figure plots the expected utility function $EU$ against the portfolio weight $\phi(\equiv x/w_0)$ for $\theta \leq w_0 r$ (the left panel) and $\theta > w_0 r$ (the right panel). Here, $A = 2$, $\gamma = 0.5$, $\theta = 1$, $r = 1.03$, $u = 1.1$, $m = 1$, $d = 0.95$, and $\mathbb{P} = (1/2, 1/3, 1/6)$. We also set $w_0 = 2$ $(\geq r^{-1}\theta)$ for the left panel and $w_0 = 0.8$ $(< r^{-1}\theta)$ for the right panel.

# 5 Portfolio Choice under Decision Weights

Sections 3–4 focus on the $S$-shaped loss aversion utility function proposed in prospect theory. In addition to loss aversion, probability weighting is another major component of prospect theory. In this section, we examine the effects of decision weights on portfolio choice.

## 5.1 Decision Weights

We start with describing the decision weights determined by cumulative prospect theory (Tversky and Kahneman, 1992). Suppose there are $S$ states in the world. Denote by $\tilde{w}_s$ the outcome in state $s$ and $p_s$ the associated physical probability satisfying $\sum_s p_s = 1$. Under cumulative prospect theory, the $S$ outcomes are first sorted in ascending order:

$$\tilde{w}_{-m} < \tilde{w}_{-m+1} < \cdots < \tilde{w}_{-1} < \tilde{w}_0 = 0 < \tilde{w}_1 < \cdots < \tilde{w}_n, \tag{41}$$

where $\tilde{w}_{-m}, \cdots, \tilde{w}_{-1}$ are losses, and $\tilde{w}_1, \cdots, \tilde{w}_n$ are gains relative to some reference point. In our setup with a reference point $\theta$, $\tilde{w}_s = w_s - \theta$. Then the decision weight $\pi_s$ for losses and gains $\tilde{w}_s$ is separately determined by probability weighting functions $\Omega^{\pm}$:

$$\begin{cases} \pi_{-i} = \Omega^-(p_{-m} + \cdots + p_{-i}) - \Omega^-(p_{-m} + \cdots + p_{-i-1}), & -i = -m, \cdots, -1, \\ \pi_j = \Omega^+(p_j + \cdots + p_n) - \Omega^+(p_{j+1} + \cdots + p_n), & j = 1, \cdots, n. \end{cases} \tag{42}$$

Tversky and Kahneman (1992) propose weighting functions of the form:

$$\Omega^{\pm}(P) = \frac{P^{\delta^{\pm}}}{[P^{\delta^{\pm}} + (1-P)^{\delta^{\pm}}]^{1/\delta^{\pm}}}, \tag{43}$$

with estimates of $\delta^{-} = 0.69$ and $\delta^{+} = 0.61$. They tend to inflate the probabilities for both tails.

Note that in (42)–(43), the decision weights may not sum up to 1. In the following analysis we impose that $\sum_s \pi_s = 1$ that allows us to apply the martingale approach. In addition, the decision weight for the status quo ($\tilde{w}_0 = 0$) is not directly assigned by (42). Our results show that this case occurs only when the investor holds only the riskless asset, and hence the outcome is deterministic, suggesting that $\pi_0 \equiv 1$ is endogenously determined in portfolio choice problems, and probably also in asset pricing problems.

## 5.2 Optimal Portfolio Choice

In the decision problems studied in Tversky and Kahneman (1992), the set of outcomes is predetermined. As a result, the decision weights can be also determined prior to making decision. However, in portfolio choice problems as we studied, the outcomes depend on investor's decisions, which depends on the decision weights. As a result, portfolio choice under prospect theory is a fixed-point problem. We solve this fixed-point problem in two steps. First, given a fixed ordering of outcomes (portfolio values across states), our results in Sections 3–4 can be used, leading to the optimal portfolios conditional on this ordering. In the second step, we can derive the optimal portfolios by maximizing the expected utility across all orderings.

### 5.2.1 Binomial Model

We consider portfolio choice under a binomial model. Section 3 shows that there can be five different orderings of outcomes:

$$\begin{cases} (a) \quad w_d < \theta < w_u; \\ (b) \quad w_u < \theta < w_d; \\ (c) \quad \theta < w_d < w_u; \\ (d) \quad \theta < w_u < w_d; \\ (e) \quad w_u = w_d = \theta. \end{cases} \tag{44}$$

Without loss of generality, we directly specify decision weights instead of deriving from weighting functions. Specifically, we denote by $\pi_u^{(k)}$ and $\pi_d^{(k)}$ the decision weights for states

$u$ and $d$, respectively, in ordering $(k)$. We further denote the pricing kernel in this ordering by $\xi_u^{(k)}$ and $\xi_d^{(k)}$, which satisfy $\xi_u^{(k)} = \frac{r-d}{\pi_u^{(k)}(u-d)r}$ and $\xi_d = \frac{u-r}{\pi_d^{(k)}(u-d)r}$.

The conditions for which the optimal solutions exist should be such that there exist optimal solutions in every ordering. Lemma 1 shows that to derive the asymptotic behavior, we only need to consider orderings $(a)$ and $(b)$. In this case, Proposition 1 leads to the following results on the solution existence conditions.

**Proposition 7.** *The optimal portfolio choice problem has no solution if $A < \underline{A}^{pw}$, where*

$$\underline{A}^{pw} = \max\left\{ \left(\frac{\pi_u^{(a)}}{\pi_d^{(a)}}\right)^{\gamma} \left(\frac{\xi_d^{(a)}}{\xi_u^{(a)}}\right)^{1-\gamma},\ \left(\frac{\pi_d^{(a)}}{\pi_u^{(a)}}\right)^{\gamma} \left(\frac{\xi_u^{(a)}}{\xi_d^{(a)}}\right)^{1-\gamma},\ \left(\frac{\pi_u^{(b)}}{\pi_d^{(b)}}\right)^{\gamma} \left(\frac{\xi_d^{(b)}}{\xi_u^{(b)}}\right)^{1-\gamma},\ \left(\frac{\pi_d^{(b)}}{\pi_u^{(b)}}\right)^{\gamma} \left(\frac{\xi_u^{(b)}}{\xi_d^{(b)}}\right)^{1-\gamma} \right\}. \tag{45}$$

Under the weighting functions (43) proposed in Tversky and Kahneman (1992), $\pi_u^{(b)} < p < \pi_u^{(a)}$. As a result, the probability weighting tends to lead to a stricter lower bound for $A$, relative to the case without probability weighting (9), that is, $\underline{A}^{pw} > \underline{A}$. Here the superscript "*pw*" stands for "probability weighting". This is because $\underline{A}^{pw}$ should be the maximum of $\underline{A}$ across all orderings.

Furthermore, under probability weighting, there may not exist optimal solution even if $A > \underline{A}^{pw}$ as shown shortly, unlike the case without probability weighting in Proposition 1.

**Corollary 10.** *Probability weighting leads to stricter conditions for the existence of optimal solutions.*

In the following analyses, we suppose $A > \underline{A}^{pw}$ and study the properties of the solutions.

When $w_0 r = \theta$, Proposition 2 shows that the investor holds only the riskless asset. In this case, there is only 1 outcome: $w^{(e)} = \theta$, i.e., ordering $(e)$. Then the decision weight is $\pi^{(e)} \equiv 1$, and the results on the optimal portfolios and value function are the same as the case without probability weighting.

**Proposition 8.** *Suppose that $A > \underline{A}^{pw}$ and $\theta = w_0 r$. Under prospect theory preferences, the optimal portfolio holdings are $x^* = 0$, the optimal wealth in the two states is the same given by $w = \theta$, and the value function is $J = 0$.*

When $\theta < w_0 r$, orderings $(c)$ and $(d)$ can occur. Within each of the two orderings, the optimal wealth is given by (see Proposition 2):

$$\begin{aligned} w_u^{(k)} - \theta &= \left(w_0 - r^{-1}\theta\right)\left[\pi_u^{(k)}(\xi_u^{(k)})^{1-\frac{1}{\gamma}} + \pi_d^{(k)}(\xi_d^{(k)})^{1-\frac{1}{\gamma}}\right]^{-1}(\xi_u^{(k)})^{-\frac{1}{\gamma}}, \\ w_d^{(k)} - \theta &= \left(w_0 - r^{-1}\theta\right)\left[\pi_u^{(k)}(\xi_u^{(k)})^{1-\frac{1}{\gamma}} + \pi_d^{(k)}(\xi_d^{(k)})^{1-\frac{1}{\gamma}}\right]^{-1}(\xi_d^{(k)})^{-\frac{1}{\gamma}}, \end{aligned} \quad \text{for} \quad k = c, d. \tag{46}$$

and the expected utility is given by

$$EU^{(k)} = \frac{(w_0 - r^{-1}\theta)^{1-\gamma}}{1-\gamma}[\pi_u^{(k)}(\xi_u^{(k)})^{1-\frac{1}{\gamma}} + \pi_d^{(k)}(\xi_d^{(k)})^{1-\frac{1}{\gamma}}]^{\gamma}. \tag{47}$$

The optimal wealth is summarized in the following proposition.

**Proposition 9.** *Suppose that $A > \underline{A}^{pw}$ and $\theta < w_0 r$.*

1. *When $w_u^{(c)} > w_d^{(c)}$ and $w_u^{(d)} < w_d^{(d)}$, the value function is given by $J = \max\{EU^{(c)}, EU^{(d)}\}$, and the optimal wealth is given by (46) with the ordering that has the higher EU.*

2. *When $w_u^{(c)} > w_d^{(c)}$ and $w_u^{(d)} > w_d^{(d)}$, ordering (d) cannot occur. The value function is $J = EU^{(c)}$, and the optimal wealth is given by $w_u^* = w_u^{(c)}$ and $w_d^* = w_d^{(c)}$.*

3. *When $w_u^{(c)} < w_d^{(c)}$ and $w_u^{(d)} < w_d^{(d)}$, ordering (c) cannot occur. The value function is $J = EU^{(d)}$, and the optimal wealth is given by $w_u^* = w_u^{(d)}$ and $w_d^* = w_d^{(d)}$.*

4. *When $w_u^{(c)} < w_d^{(c)}$ and $w_u^{(d)} > w_d^{(d)}$, both cannot occur, and there is no optimal solution.*

Situation 4 in Proposition 9 shows that even when $A > \underline{A}^{pw}$, the fixed point problem may not have a solution. In addition, it follows from Proposition 9 that when the optimal solution exists, the value function in this case tends to be higher than the case without probability weighting since probability weighting allows the investor to compare different orderings of portfolio values.

We can also derive the optimal wealth for the case $\theta > w_0 r$ based on the same analysis. In this case, there will be eight different situations, and the fixed point problem does not have solution in some situations.[17]

If there exist optimal solutions under the probability weighting, then the results in Corollaries 1–4 still hold true.

# 6 Conclusion

Prospect theory is one of the central topics in behavioural economics. When studying its implications for optimal choice and asset pricing, the literature often studies certain variations to the utility function proposed in Tversky and Kahneman (1992) to ensure that the optimal portfolio is bounded. However, the properties of the optimal policy under the original utility function are not well understood. In this paper, we study the optimal portfolio choice problem under prospect theory. We derive closed-form solutions of the optimal policy, as well as the existence conditions for it, that provide a rigorous understanding of the optimal policy.

We show that there is no optimal solution for half of the parameter space in which the loss aversion is low. This is essentially due to insufficient penalty for losses. However, there always exist optimal solutions for the other half of the parameter space. The lower bound of loss aversion above which optimal solutions exist increases without bound as asset returns are more skewed or the number of states of the world increases. Under wealth constraint, optimal solutions always exist but are often given by corner solutions. We also show that probability

[17] Results are available upon request.

weighting, which is another important component of prospect theory, further increases the lower bound of loss aversion above which optimal solutions exist.

The properties of the optimal solutions depend crucially on the relative level of the reference point to the initial wealth. When the investor has a sufficiently high reference point, the optimal portfolio distinctly differs from that under standard expected utility. In this case, the investor tends to seek negative skewness. The skewness of her wealth can be much lower than the skewness of all individual assets. The investor may be long (short) an asset with an arbitrarily low (high) risk premium. Furthermore, the investor tends to take a large position, either long or short, in the risky asset. Due to the nonconvex expected utility, the investor can even achieve the same highest expected utility by either longing or shorting the risky asset.

# A Proofs

## A.1 Proof of Lemma 1

Suppose $w_u$ is sufficiently large. Then $w_d$ is low by the budget constraint (5), and the expected utility becomes

$$EU = \frac{1}{1-\gamma}\Big[p(w_u-\theta)^{1-\gamma} - A(1-p)\Big(\theta - \frac{w_0 - p\xi_u w_u}{(1-p)\xi_d}\Big)^{1-\gamma}\Big].$$

When $w_u \to +\infty$,

$$EU \approx \Big[p - A(1-p)\Big(\frac{p\xi_u}{(1-p)\xi_d}\Big)^{1-\gamma}\Big]\frac{w_u^{1-\gamma}}{1-\gamma},$$

leading to the asymptotic behavior (7).

When $w_u$ is sufficiently low, the expected utility becomes

$$EU = \frac{1}{1-\gamma}\Big[-Ap(\theta-w_u)^{1-\gamma} + (1-p)\Big(\frac{w_0 - p\xi_u w_u}{(1-p)\xi_d} - \theta\Big)^{1-\gamma}\Big].$$

When $w_u \to -\infty$,

$$EU \approx \Big[-Ap + (1-p)\Big(\frac{p\xi_u}{(1-p)\xi_d}\Big)^{1-\gamma}\Big]\frac{(-w_u)^{1-\gamma}}{1-\gamma},$$

leading to the asymptotic behavior (8).

## A.2 Proof of Proposition 1

Lemma 1 shows that when $A > \underline{A}$, the $EU$ goes to $-\infty$ as $w_u \to \pm\infty$. Because the $EU$ is a continuous function on a compact set, the $EU$ has a global maximum.

When $A < \underline{A}$, the $EU$ goes to $+\infty$ as $w_u \to +\infty$ or as $w_u \to -\infty$. In this case, there is no optimal solution for problem (4).

When $A = \underline{A}$, $EU \geq 0$ for $w_0 r \geq \theta$ and $EU < 0$ for $w_0 r < \theta$ as shown in Appendix A.3. The $EU$ approaches zero as $w_u \to \pm\infty$. Thus, there is a global maximum for $w_0 r \geq \theta$ but no global maximum for $w_0 r < \theta$.

## A.3 Proof of Proposition 2

The $EU$ as a function of $w_u$ is given by

$$EU = \begin{cases} \frac{1}{1-\gamma}\big[p(w_u-\theta)^{1-\gamma} + (1-p)(\frac{w_0-p\xi_u w_u}{(1-p)\xi_d} - \theta)^{1-\gamma}\big], & \text{if} \quad w_u \geq \theta \text{ and } w_u \leq \hat{w}_u; \\ \frac{1}{1-\gamma}\big[p(w_u-\theta)^{1-\gamma} - A(1-p)(\theta - \frac{w_0-p\xi_u w_u}{(1-p)\xi_d})^{1-\gamma}\big], & \text{if} \quad w_u \geq \theta \text{ and } w_u \geq \hat{w}_u; \\ \frac{1}{1-\gamma}\big[-Ap(\theta-w_u)^{1-\gamma} + (1-p)(\frac{w_0-p\xi_u w_u}{(1-p)\xi_d} - \theta)^{1-\gamma}\big], & \text{if} \quad w_u \leq \theta \text{ and } w_u \leq \hat{w}_u; \\ -\frac{1}{1-\gamma}\big[A(p(\theta-w_u)^{1-\gamma} + (1-p)(\theta - \frac{w_0-p\xi_u w_u}{(1-p)\xi_d})^{1-\gamma}\big], & \text{if} \quad w_u \leq \theta \text{ and } w_u \geq \hat{w}_u, \end{cases}$$

where $\hat{w}_u \equiv \frac{w_0-(1-p)\xi_d\theta}{p\xi_u}$.

### A.3.1 $w_0 \geq r^{-1}\theta$

Note that $r^{-1} = \mathbb{E}[\xi] = p\xi_u + (1-p)\xi_d$. When $w_0 \geq r^{-1}\theta$, we have $\hat{w}_u \geq \theta$. Thus, the case $w_u < \theta$ and $w_u > \hat{w}_u$ cannot occur. The $EU$ becomes

$$EU = \begin{cases} \frac{1}{1-\gamma}\big[p(w_u-\theta)^{1-\gamma} + (1-p)(\frac{p\xi_u}{(1-p)\xi_d})^{1-\gamma}(\hat{w}_u - w_u)^{1-\gamma}\big], & \text{if} \quad w_u \in [\theta, \hat{w}_u]; \\ \frac{1}{1-\gamma}\big[p(w_u-\theta)^{1-\gamma} - A(1-p)(\frac{p\xi_u}{(1-p)\xi_d})^{1-\gamma}(w_u - \hat{w}_u)^{1-\gamma}\big], & \text{if} \quad w_u \in [\hat{w}_u, +\infty); \\ \frac{1}{1-\gamma}\big[-Ap(\theta - w_u)^{1-\gamma} + (1-p)(\frac{p\xi_u}{(1-p)\xi_d})^{1-\gamma}(\hat{w}_u - w_u)^{1-\gamma}\big], & \text{if} \quad w_u \in (-\infty, \theta]. \end{cases}$$

The $EU$ is concave when $w_u \in [\theta, \hat{w}_u]$; thus, it has a local maximum between $\theta$ and $\hat{w}_u$.[18] In addition,

$$EU = \begin{cases} \frac{w_u^{1-\gamma}}{1-\gamma}\big[p - A(1-p)(\frac{p\xi_u}{(1-p)\xi_d})^{1-\gamma}\big], & \text{when} \quad w_u \to +\infty; \\ \frac{(-w_u)^{1-\gamma}}{1-\gamma}\big[-Ap + (1-p)(\frac{p\xi_u}{(1-p)\xi_d})^{1-\gamma}\big], & \text{when} \quad w_u \to -\infty. \end{cases}$$

If $p - A(1-p)(\frac{p\xi_u}{(1-p)\xi_d})^{1-\gamma} > 0$, the $EU$ approaches $+\infty$ when $w_u \to +\infty$, the $EU$ has a local minimum in the interval $(\hat{w}_u, +\infty)$ and there is no global maximum.

If $-Ap + (1-p)(\frac{p\xi_u}{(1-p)\xi_d})^{1-\gamma} > 0$, the $EU$ approaches $+\infty$ when $w_u \to -\infty$, the $EU$ has a local minimum in the interval $[\theta, +\infty)$ and there is no global maximum.

Thus, to have a global maximum, we focus on the case $p - A(1-p)(\frac{p\xi_p u}{(1-p)\xi_d})^{1-\gamma} \leq 0$ and $-Ap + (1-p)(\frac{p\xi_u}{(1-p)\xi_d})^{1-\gamma} \leq 0$.

1. if $p - A(1-p)(\frac{p\xi_u}{(1-p)\xi_d})^{1-\gamma} < 0$, when $w_u \to +\infty$, the $EU$ approaches $-\infty$.
2. if $-Ap + (1-p)(\frac{p\xi_u}{(1-p)\xi_d})^{1-\gamma} < 0$, when $w_u \to -\infty$, the $EU$ approaches $-\infty$.
3. if $p - A(1-p)(\frac{p\xi_u}{(1-p)\xi_d})^{1-\gamma} = 0$, when $w_u \to +\infty$, the $EU$ approaches 0. In this case, we have $-Ap + (1-p)(\frac{p\xi_u}{(1-p)\xi_d})^{1-\gamma} < 0$.
4. if $-Ap + (1-p)(\frac{p\xi_u}{(1-p)\xi_d})^{1-\gamma} = 0$, when $w_u \to -\infty$, the $EU$ approaches 0. In this case, we have $p - A(1-p)(\frac{p\xi_u}{(1-p)\xi_d})^{1-\gamma} < 0$.

Finally, if $p - A(1-p)(\frac{p\xi_u}{(1-p)\xi_d})^{1-\gamma} \leq 0$, we have $\frac{\partial EU}{\partial w_u} < 0$ over $w_u \in [\hat{w}_u, +\infty)$, and if $-Ap + (1-p)(\frac{p\xi_u}{(1-p)\xi_d})^{1-\gamma} \leq 0$, we have $\frac{\partial EU}{\partial w_u} > 0$ over $w_u \in (-\infty, \theta]$. Therefore, the local maximum between $\theta$ and $\hat{w}_u$ is the global maximum.

Now, we derive the optimal wealth and value function. We use the Lagrangian approach to find local maximum. The Lagrangian is given by

$$\mathcal{L} = p(w_u - \theta)^{1-\gamma} + (1-p)(w_d - \theta)^{1-\gamma} - \lambda\big[p\xi_u w_u + (1-p)\xi_d w_d - w_0\big],$$

[18] Here we assume $\gamma > 0$. If $\gamma = 0$, the $EU$ is linear not concave in the interval $w_u \in [\theta, \hat{w}_u]$, and the optimum occurs at the boundary.

where $\lambda > 0$ is the Lagrange multiplier. The FOC for the local maximum is

$$(w_u - \theta)^{-\gamma} = \lambda \xi_u, \qquad (w_d - \theta)^{-\gamma} = \lambda \xi_d.$$

So

$$w_u - \theta = (\xi_d/\xi_u)^{\frac{1}{\gamma}} (w_d - \theta).$$

The budget constraint leads to

$$w_0 - r^{-1}\theta = p\xi_u(w_u - \theta) + (1-p)\xi_d(w_d - \theta) = \left[p\xi_u(\xi_d/\xi_u)^{\frac{1}{\gamma}} + (1-p)\xi_d\right](w_d - \theta).$$

The value function is

$$\begin{aligned} J &= \frac{1}{1-\gamma}\left[p(\xi_d/\xi_u)^{\frac{1-\gamma}{\gamma}} + (1-p)\right](w_d - \theta)^{1-\gamma} \\ &= \frac{(w_0 - r^{-1}\theta)^{1-\gamma}}{1-\gamma}\left[p(\xi_d/\xi_u)^{\frac{1-\gamma}{\gamma}} + (1-p)\right](w_d - \theta)^{1-\gamma}\left[p\xi_u(\xi_d/\xi_u)^{\frac{1}{\gamma}} + (1-p)\xi_d\right]^{\gamma-1} \\ &= \frac{(w_0 - r^{-1}\theta)^{1-\gamma}}{1-\gamma}\left[p\xi_u^{1-\frac{1}{\gamma}} + (1-p)\xi_d^{1-\frac{1}{\gamma}}\right]^{\gamma}. \end{aligned}$$

### A.3.2 $w_0 < r^{-1}\theta$

In this case, $\hat{w}_u < \theta$, we cannot have the case $w_u > \theta$ and $w_u < \hat{w}_u$. So the $EU$ is

$$EU = \begin{cases} \frac{1}{1-\gamma}\left[p(w_u - \theta)^{1-\gamma} - A(1-p)(\frac{p\xi_u}{(1-p)\xi_d})^{1-\gamma}(w_u - \hat{w}_u)^{1-\gamma}\right], & \text{if} \quad w_u \in [\theta, +\infty) \\ \frac{1}{1-\gamma}\left[-Ap(\theta - w_u)^{1-\gamma} + (1-p)(\frac{p\xi_u}{(1-p)\xi_d})^{1-\gamma}(\hat{w}_u - w_u)^{1-\gamma}\right], & \text{if} \quad w_u \in (-\infty, \hat{w}_u]; \\ -\frac{1}{1-\gamma}\left[A(p(\theta - w_u)^{1-\gamma} + (1-p)(\frac{p\xi_u}{(1-p)\xi_d})^{1-\gamma}(w_u - \hat{w}_u)^{1-\gamma}\right], & \text{if} \quad w_u \in [\hat{w}_u, \theta]. \end{cases}$$

The utility function is convex when $w_u \in [\hat{w}_u, \theta]$ and

$$EU = \begin{cases} \frac{w_u^{1-\gamma}}{1-\gamma}\left[p - A(1-p)(\frac{p\xi_u}{(1-p)\xi_d})^{1-\gamma}\right], & \text{when} \quad w_u \to +\infty; \\ \frac{(-w_u)^{1-\gamma}}{1-\gamma}\left[-Ap + (1-p)(\frac{p\xi_u}{(1-p)\xi_d})^{1-\gamma}\right], & \text{when} \quad w_u \to -\infty. \end{cases}$$

The $EU$ has a local minimum between $\theta$ and $\hat{w}_u$.

1. If $p - A(1-p)(\frac{p\xi_u}{(1-p)\xi_d})^{1-\gamma} > 0$, when $w_u \to +\infty$, the $EU$ approaches $+\infty$, and there is no global maximum.

2. If $p - A(1-p)(\frac{p\xi_u}{(1-p)\xi_d})^{1-\gamma} < 0$, when $w_u \to +\infty$, the $EU$ approaches $-\infty$, the $EU$ has a local maximum for $w_u \in (\theta, \infty)$ (and thus $w_d < \theta$).

   The FOC for the local maximum is

$$(w_u - \theta)^{-\gamma} = \lambda \xi_u, \qquad A(\theta - w_2)^{-\gamma} = \lambda \xi_2.$$

So

$$w_1 - \theta = \frac{(1/\xi_u)^{\frac{1}{\gamma}}}{(A/\xi_2)^{\frac{1}{\gamma}}}(\theta - w_d).$$

The budget constraint leads to

$$r^{-1}\theta - w_0 = -p\xi_1(w_u - \theta) + (1-p)\xi_d(\theta - w_d) = \Bigg(-p\xi_u \frac{(1/\xi_u)^{\frac{1}{\gamma}}}{(A/\xi_d)^{\frac{1}{\gamma}}} + (1-p)\xi_d\Bigg)(\theta - w_d).$$

The quasi-value function is

$$\begin{aligned} J &= \frac{1}{1-\gamma}\Bigg[p\Bigg(\frac{(1/\xi_u)^{\frac{1}{\gamma}}}{(A/\xi_d)^{\frac{1}{\gamma}}}\Bigg)^{1-\gamma} - A(1-p)\Bigg](\theta - w_d)^{1-\gamma} \\ &= \frac{(r^{-1}\theta - w_0)^{1-\gamma}}{1-\gamma}\Bigg[p\Bigg(\frac{(1/\xi_u)^{\frac{1}{\gamma}}}{(A/\xi_d)^{\frac{1}{\gamma}}}\Bigg)^{1-\gamma} - A(1-p)\Bigg]\Bigg[(1-p)\xi_d - p\xi_u\frac{(1/\xi_u)^{\frac{1}{\gamma}}}{(A/\xi_d)^{\frac{1}{\gamma}}}\Bigg]^{\gamma-1} \\ &= -\frac{(r^{-1}\theta - w_0)^{1-\gamma}}{1-\gamma}\Bigg((1-p)A^{\frac{1}{\gamma}}\xi_d^{1-\frac{1}{\gamma}} - p\xi_u^{1-\frac{1}{\gamma}}\Bigg)^{\gamma}. \end{aligned}$$

3. If $-Ap + (1-p)(\frac{p\xi_u}{(1-p)\xi_d})^{1-\gamma} > 0$, when $w_u \to -\infty$, the $EU$ approaches $+\infty$, and there is no global maximum.

4. If $-Ap + (1-p)(\frac{p\xi_u}{(1-p)\xi_d})^{1-\gamma} < 0$, when $w_u \to -\infty$, the $EU$ approaches $-\infty$, and there is a local maximum in the interval $[-\infty, \hat{w}_u]$. This case is symmetric to case (2).

5. If $p - A(1-p)(\frac{p\xi_u}{(1-p)\xi_d})^{1-\gamma} = 0$, when $w_u \to +\infty$, we have $EU \to 0$, and $\frac{\partial EU}{\partial w_1} > 0$ over $w_u \in [\theta, +\infty)$; thus, there is no local maximum over $w_u \in [\theta, +\infty)$.

6. If $-Ap + (1-p)(\frac{p\xi_u}{(1-p)\xi_d})^{1-\gamma} = 0$, when $w_u \to -\infty$, we have $EU \to 0$, and $\frac{\partial EU}{\partial w_1} < 0$ over $w_u \in (-\infty, \hat{w}_u]$; thus, there is no local maximum over $w_u \in (-\infty, \hat{w}_u]$.

Finally, if $p - A(1-p)(\frac{p\xi_u}{(1-p)\xi_d})^{1-\gamma} < 0$ and $-Ap + (1-p)(\frac{p\xi_u}{(1-p)\xi_d})^{1-\gamma} < 0$, there are two local maximum, one in $[\theta, +\infty)$ and one in $(-\infty, \hat{w}_u]$, and one of them is the global maximum.

## A.4 Proof of Corollary 2

When $w_0 r < \theta$ and $A > \underline{A}$, the $EU$ has two local maximum, and the global maximum is the greater of $EU_+$ and $EU_-$. Proposition 2 shows that $J_- < J_+$ if and only if $(1-p)\xi_d^{1-\frac{1}{\gamma}} < p\xi_u^{1-\frac{1}{\gamma}}$.

## A.5 Proof of Proposition 3

The expected utility (2) becomes

$$p\Big[(w_u-\theta)^{1-\gamma}1_{\{w_u \geq \theta\}} - A(\theta - w_u)^{1-\gamma}1_{\{w_u < \theta\}}\Big] + (1-p)\Big[(w_d-\theta)^{1-\gamma}1_{\{w_d \geq \theta\}} - A(\theta - w_d)^{1-\gamma}1_{\{w_d < \theta\}}\Big],$$

where $w_u$ and $w_d$ are the terminal wealth at states $u$ and $d$, respectively, given by

$$w_u = w_0\big[r + \phi(u - r)\big], \qquad w_d = w_0\big[r + \phi(d - r)\big]. \tag{A.1}$$

First consider the case when the optimal wealth is positive in both states. The FOC is

$$p[w_0 r - \theta + x(u - r)]^{-\gamma}(u - r) = -(1 - p)[w_0 r - \theta + x(d - r)]^{-\gamma}(d - r),$$

Let $k = [\frac{p(u-r)}{(1-p)(r-d)}]^{\frac{1}{\gamma}}$. The FOC becomes $w_0 r - \theta + x(u - r) = k[w_0 r - \theta + \phi(d - r)]$. The optimal holding $x$ is

$$x^* = \frac{(k - 1)(w_0 r - \theta)}{(u - r) + k(r - d)}.$$

Next we consider the case the optimal wealth is negative in one of the states. The FOC is

$$\begin{cases} p[w_0 r - \theta + x_+(u - r)]^{-\gamma}(u - r) = -(1 - p)A[\theta - w_0 r - x_+(d - r)]^{-\gamma}(d - r), & \text{if} \quad w_0 r + x_+(d - r) < \theta; \\ pA[\theta - w_0 r - x_-(u - r)]^{-\gamma}(u - r) = -(1 - p)[w_0 r - \theta + x_-(d - r)]^{-\gamma}(d - r), & \text{if} \quad w_0 r + x_-(u - r) < \theta. \end{cases}$$

Let $k_+ = [\frac{p(u-r)}{A(1-p)(r-d)}]^{\frac{1}{\gamma}}$ and $k_- = [\frac{Ap(u-r)}{(1-p)(r-d)}]^{\frac{1}{\gamma}}$. The FOC becomes

$$\begin{cases} w_0 r - \theta + x_+(u - r) = -k_d[w_0 r - \theta + x_+(d - r)], & \text{if} \quad w_0 r + x_+(d - r) < \theta; \\ -[w_0 r - \theta + x_-(u - r)] = k_u[w_0 r - \theta + x_-(d - r)], & \text{if} \quad w_0 r + x_-(u - r) < \theta. \end{cases}$$

The optimal $x$ is

$$x_\pm = \frac{(k_\pm + 1)(w_0 r - \theta)}{(u - r) - k_\pm(r - d)}.$$

## A.6 Proof of Proposition 4

When $w_u$ is sufficiently large, the expected utility becomes

$$EU = p\frac{(w_u - \theta)^{1-\gamma_+}}{1 - \gamma_+} - (1 - p)\frac{A}{1 - \gamma_-}\Big[\theta - \frac{w_0 - p\xi_u w_u}{(1 - p)\xi_d}\Big]^{1-\gamma_-}.$$

When $w_u \to +\infty$,

$$EU \approx p\frac{(w_u)^{1-\gamma_+}}{1 - \gamma_+} - (1 - p)A\frac{(w_u)^{1-\gamma_-}}{1 - \gamma_-}\Big[\frac{p\xi_u}{(1 - p)\xi_d}\Big]^{1-\gamma_-}.$$

It goes to $+\infty$ if $\gamma_+ < \gamma_-$ and goes to $-\infty$ if $\gamma_- < \gamma_+$.

When $w_u$ is sufficiently low, the expected utility becomes

$$EU = -pA\frac{(\theta - w_u)^{1-\gamma_-}}{1 - \gamma_-} + (1 - p)\frac{1}{1 - \gamma_+}\Big[\frac{w_0 - p\xi_u w_u}{(1 - p)\xi_d} - \theta\Big]^{1-\gamma_+}.$$

When $w_u \to -\infty$,

$$EU \approx -pA\frac{(-w_u)^{1-\gamma_-}}{1 - \gamma_-} + (1 - p)\frac{(-w_u)^{1-\gamma_+}}{1 - \gamma_+}\Big[\frac{p\xi_u}{(1 - p)\xi_d}\Big]^{1-\gamma_+}.$$

It also goes to $+\infty$ if $\gamma_+ < \gamma_-$ and goes to $-\infty$ if $\gamma_- < \gamma_+$.

## A.7 Proof of Lemma 2

The budget constraint is given by $\mathbb{E}[\xi\hat{w}] = w_0$, where $w$ is a wealth distribution across the $S$ states. Given a starting wealth distribution $\hat{w}$ that satisfies the budget constraint, any wealth $w$ under the budget constraint should satisfy the following budget-preserving change: $w - \hat{w} = \xi^{-1/\gamma}(1_+ - a1_-)\Delta$, with $a = \mathbb{E}[\xi^{1-1/\gamma}1_+]/\mathbb{E}[\xi^{1-1/\gamma}1_-]$. The $EU$ is given by

$$\begin{aligned}
EU =& \frac{1}{1-\gamma}\mathbb{E}\big[(w-\theta)^{1-\gamma}1_+ - A(\theta-w)^{1-\gamma}1_-\big] \\
=& \frac{1}{1-\gamma}\mathbb{E}\Big[\big(\hat{w}+\xi^{-1/\gamma}\Delta-\theta\big)^{1-\gamma}1_+ - A\big(\theta-\hat{w}+a\xi^{-1/\gamma}\Delta\big)^{1-\gamma}1_-\Big].
\end{aligned}$$

When $\Delta \to +\infty$,

$$\begin{aligned}
EU \approx& \frac{\Delta^{1-\gamma}}{1-\gamma}\big(\mathbb{E}[\xi^{1-1/\gamma}1_+] - Aa^{1-\gamma}\mathbb{E}[\xi^{1-1/\gamma}1_-]\big) \\
=& \frac{\Delta^{1-\gamma}}{1-\gamma}\big(\mathbb{E}[\xi^{1-1/\gamma}1_+]\big)^{1-\gamma}\big[(\mathbb{E}[\xi^{1-1/\gamma}1_+])^{\gamma} - A(\mathbb{E}[\xi^{1-1/\gamma}1_-])^{\gamma}\big].
\end{aligned}$$

This goes to $+\infty$ if $\mathbb{E}[\xi^{1-1/\gamma}(1_+ - A^{1/\gamma}1_-)] > 0$.

## A.8 Proof of Proposition 6

The objective function is given by

$$\max_{\{w_s\}} \mathbb{E}[u(w)] = \max_{\{w_s\}} \sum_s p_s u(w_s),$$

such that $w_0 = \sum_s p_s\xi_s w_s = \mathbb{E}[\xi w]$. We will consider interior solutions for the case. To this end, we use Lagrangian approach. The Lagrangian is given by

$$\mathbb{E}[u(w)] - \lambda\big(\mathbb{E}[\xi w] - w_0\big), \tag{A.2}$$

where $\lambda \geq 0$ is the Lagrange multiplier.

When $w_0 \geq \mathbb{E}[\xi]\theta$, the FOC is given by $(w-\theta)^{-\gamma} = \lambda\xi$. The budget constraint leads to $w_0 - \mathbb{E}[\xi]\theta = \mathbb{E}[\xi(w-\theta)] = \mathbb{E}\big[\xi(\lambda\xi)^{-1/\gamma}\big] = \lambda^{-1/\gamma}\mathbb{E}\big[\xi^{1-1/\gamma})\big]$. Therefore, the Lagrange multiplier is given by $\lambda = \left(\frac{w_0-\mathbb{E}[\xi]\theta}{\mathbb{E}[\xi^{1-1/\gamma}]}\right)^{-\gamma}$, leading to the optimal wealth and the value function given by (22) and (23).

When $w_0 < \mathbb{E}[\xi]\theta$, assume a local optimum occurs at one partition $S_+$ and $S_-$. The FOC is given by

$$(w-\theta)^{-\gamma}1_+ = \lambda\xi 1_+, \qquad A(\theta-w)^{-\gamma}1_- = \lambda\xi 1_-.$$

The budget constraint leads to

$$w_0 - \mathbb{E}[\xi]\theta = \mathbb{E}[\xi(w-\theta)] = \mathbb{E}\left[\xi\Big((\lambda\xi)^{-1/\gamma}1_+ - (\lambda\xi/A)^{-1/\gamma}1_-\Big)\right] = \lambda^{-1/\gamma}\mathbb{E}\big[\xi^{1-1/\gamma}(1_+ - A^{1/\gamma}1_-)\big].$$

Therefore, the Lagrange multiplier is given by

$$\lambda = (\mathbb{E}[\xi]\theta - w_0)^{-\gamma} \left(\mathbb{E}[\xi^{1-1/\gamma}(-1_+ + A^{1/\gamma}1_-)]\right)^{\gamma}. \tag{A.3}$$

Note that $\lambda > 0$, so $w_0 - \mathbb{E}[\xi]\theta > 0$ and $\mathbb{E}[\xi^{1-1/\gamma}(1_+ - A^{1/\gamma}1_-)] > 0$ has the same sign.

The expected utility function $EU$ is given by

$$\begin{aligned} EU &= \frac{1}{1-\gamma}\mathbb{E}\Big[\big((\lambda\xi)^{-1/\gamma}\big)^{1-\gamma}1_+ - A\big((\lambda\xi/A)^{-1/\gamma}\big)^{1-\gamma}1_-\Big] \\ &= \frac{\lambda^{1-1/\gamma}}{1-\gamma}\mathbb{E}[\xi^{1-1/\gamma}(1_+ - A^{1/\gamma}1_-)]. \end{aligned} \tag{A.4}$$

By substituting (A.3) into (A.4), the $EU$ becomes (25).

The global maximum is achieved only for partition with only 1 state in $\mathbb{S}_-$. Now consider partitions with only 1 state in $S_-$. Suppose $\mathbb{S}_- = \{t\}$, that is, state $t$ is the only state with wealth lower than $\theta$. In this case, $\mathbb{E}[\xi^{1-1/\gamma}1_-] = \xi_t^{1-1/\gamma}p_t$, and we can explicitly solve the budget constraint:

$$w_0 - \mathbb{E}[\xi]\theta = \sum_{s\neq t} p_s\xi_s(w_s - \theta) + p_t\xi_t(w_t - \theta).$$

Therefore,

$$\theta - w_t = \frac{\sum_{s\neq t} p_s\xi_s(w_s - \theta)}{p_t\xi_t} - (w_0 - \mathbb{E}[\xi]\theta). \tag{A.5}$$

The EU can be express in unconstrained variables

$$EU = \sum_{s\neq t} p_s\frac{(w_s-\theta)^{1-\gamma}}{1-\gamma} - Ap_t\frac{(\theta - w_t)^{1-\gamma}}{1-\gamma},$$

where $\theta - w_t$ is given by equation (A.5).

FOC for $w_s^*$, $s \neq t$, is

$$p_s(w_s^* - \theta)^{-\gamma} = Ap_t(\theta - w_t^*)^{-\gamma}\frac{p_s\xi_s}{p_t\xi_t}.$$

The Hessian matrix is

$$H_{s,s'} = -\gamma p_s(w_s^*-\theta)^{-\gamma-1}\delta_{s,s'} + Ap_t(\theta - w_t^*)^{-\gamma-1}\frac{p_s\xi_s p_{s'}\xi_{s'}}{(p_t\xi_t)^2}.$$

Using the FOC condition, the Hessian can be written as

$$\begin{aligned} H_{s,s'} &= -\gamma Ap_t(\theta - w_t^*)^{-\gamma}\frac{p_s\xi_s}{p_t\xi_t}(w_s^*-\theta)^{-1}\delta_{s,s'} + \gamma Ap_t(\theta-w_t^*)^{-\gamma-1}\frac{p_s\xi_s p_{s'}\xi_{s'}}{(p_t\xi_t)^2} \\ &= \gamma\frac{Ap_t}{\xi_t}(\theta - w_t^*)^{-\gamma}\frac{1}{p_t\xi_t}\left(-\frac{p_s\xi_s}{w_s^*-\theta}\delta_{s,s'} + \frac{p_s\xi_s p_{s'}\xi_{s'}}{p_t\xi_t(\theta - w_t^*)}\right). \end{aligned}$$

We should be able to show that $H$ is negative definite. Define $\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{\xi}{\mathbb{E}[\xi]}$, thus $q_s = p_s\xi_s / \sum_s(p_s\xi_s)$, which are the risk-neutral probabilities. The Hessian can be written as

$$\begin{aligned} H_{s,s'} &= \gamma \frac{Ap_t}{\xi_t q_t}(\theta - w_t^*)^{-\gamma}\left(-\frac{q_s}{w_s^* - \theta}\delta_{s,s'} + \frac{q_s q_{s'}}{q_t(\theta - w_t^*)}\right) \\ &= \gamma \frac{Ap_t}{\xi_t q_t^2}(\theta - w_t^*)^{-\gamma-1}\left(-\frac{\theta - w_t^*}{w_s^* - \theta} q_s q_t \delta_{s,s'} + q_s q_{s'}\right). \end{aligned}$$

Only terms in the bracket are relevant for determining whether $H$ is negative-definite,

$$h_{s,s'} = \left(-\frac{\theta - w_t^*}{w_s^* - \theta} q_s q_t \delta_{s,s'} + q_s q_{s'}\right) = \left(-a_s q_s q_t \delta_{s,s'} + q_s q_{s'}\right),$$

where $a_s = \frac{\theta - w_t^*}{w_s^* - \theta} q_s q_t \delta_{s,s'}$. Thus,

$$\begin{aligned} h &= \begin{pmatrix} -a_1 q_1 q_t + q_1 q_1 & q_1 q_2 & \dots & q_1 q_S \\ q_1 q_2 & -a_2 q_2 q_t + q_2 q_2 & \dots & q_2 q_S \\ \vdots & \vdots & \vdots & \vdots \\ q_1 q_S & q_2 q_S & \dots & -a_S q_S q_t + q_S q_S \end{pmatrix} \\ &= \begin{pmatrix} -a_1 q_t + q_1 & q_1 & \dots & q_1 \\ q_2 & -a_2 q_t + q_2 & \dots & q_2 \\ \vdots & \vdots & \vdots & \vdots \\ q_S & q_S & \dots & -a_S q_t + q_S \end{pmatrix} \Pi_s q_s. \end{aligned}$$

Subtracting column 1 from column $s$ for all $s \neq 1$, we get

$$h' = \begin{pmatrix} -a_1 q_t + q_1 & a_1 q_t & \dots & a_1 q_t \\ q_2 & -a_2 q_t & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ q_S & 0 & \dots & -a_S q_t \end{pmatrix}$$

The determinant of $h'$ is

$$(-a_1 q_t + q_1)\Pi_{s\neq 1}(-a_s q_t) - q_2 \Pi_{s \neq 2}(-a_s q_t) + \dots = \left(\Pi_s(-a_s q_t)\right)\left(1 - \sum_{s=1}^{S} \frac{q_s}{q_t a_s}\right).$$

If $w_0 - \mathbb{E}[\xi]\theta < 0$, then

$$\left(1 - \sum_{s=1}^{S} \frac{q_s}{q_t a_s}\right) = \frac{1}{q_t(\theta - w_t^*)}\left(q_t(\theta - w_t^*) - \sum_{s \neq t} q_s(w_s^* - \theta)\right) = \frac{-(\mathbb{E}[\xi]\theta - w_0)}{q_t(\theta - w_t^*)} > 0.$$

The above equation can be true for all principle minors of dimension $u$, $u = 1, \dots, S-1$, so the sign of their determinant is $(-1)^u$. This implies that the eigenvalues of $h$ are all negative; thus $h$ is negative definite. Therefore, the Hessian is negative definite, and the $EU$ has a local maximum at $w^*$.

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