Stochastic Differential Equations, Spring 2021

Homework 10

Due Jun 3, 2021

Name:_______________

1. If $dX_t = (2X_t + e^{2t})dt + b(t, W_t, X_t)dW_t$. Find $E(X_t)$. This problem, similar as the example problem in class, indicates that in order to evaluate X_t, sometimes it is first to obtain the SDE of dX_t and then using it to establish the ODE of $E(X_t)$. Then solving the ODE gives rise to the desired expectation.

   Using this approach to work the following problems

2. Find $E(W_t^2 e^{W_t})$ and $E(W_t e^{kW_t})$, $\forall k \in \mathbb{R}$.

3. Find $E(\sin(t + \sigma W_t))$ and $E(\cos(t + \sigma W_t))$ with $\sigma \in \mathbb{R}$.

4. Use the SDE of Brownian motion to show that for $a \neq 0$

$$\int_{-\infty}^{\infty} x e^{-ax^2+bx} dx = \sqrt{\frac{\pi}{a}} \left(\frac{b}{2a}\right) e^{\frac{b^2}{4a}}$$

   and

$$\int_{-\infty}^{\infty} x^2 e^{-ax^2+bx} dx = \sqrt{\frac{\pi}{a}} \frac{1}{2a} \left(1 + \frac{b^2}{2a}\right) e^{\frac{b^2}{4a}}.$$

5. Consider the following linear stochastic differential equation

$$dX_t = (2 - X_t)dt + e^{-t} W_t dW_t, X_0 = 0.$$

   (a) show that $E(X_t) \to 2$ as $t \to \infty$ without solving the SDE; is the convergence in mean square?

   (b) solve the SDE and do the same as in (a).

6. The following equation is called the Ornstein–Uhlenbeck equation, where r and σ are constants

$$X_t = (r - X_t)dt + \sigma dW_t, X_0 = x_0 \in \mathbb{R}.$$

   (a) Solve the SDE;

   (b) Find its mean and variance with/without solving the SDE;

   (c) Find $\mathrm{Cov}(X_s, X_t)$ for $0 < s < t$.

7. Verify that the following SDE is exact

$$dX_t = (2tW_t^3 + 3t^2(1 + W_t))dt + (3t^2 W_t^2 + 1)dW_t, X_0 = 0;$$

   then find its solution.

8. I failed to cover another (but not the last) method to solve SDE due to time constraints. Then I feel it is better that we cover it in the HW for consistency, rather than postponing it to the next week now that we start something new.

   This method is called the variation of parameter method, and I believe you have seen its analogy in ODE. Let me proceed as follows: suppose that we want to solve the baby SDE

   $$dX_t = X_t dW_t, X_0 = 1.$$

   We formally solve it without knowing the Ito lemma and then have that

   $$X_t = e^{W_t + C}, C \in \mathbb{R}.$$

   However, we know that this is not the solution and the parameter C can not be constant. Instead, we vary the parameter C by making it a function of both t and W_t such that

   $$X_t = e^{W_t + C(t, W_t)}, \tag{0.1}$$

   and this is referred to as the variation of parameter C. Now we substitute (0.1) into the SDE and find that $C(t, W_t) = -\frac{t}{2}$. I skip the details.

   Use the method of variation of parameters to solve the following SDE

   $$dX_t = X_t W_t dW_t.$$

9. Use the method of variation of parameters to solve the SDE with μ and σ being constants

   $$dX_t = \mu X_t dt + \sigma X_t dW_t.$$

   Hint: you want to writing the total differential form for X_t.

10. Let X_t satisfy the SDE

    $$dX_t = f(t, X_t)dt + g(t)X_t dW_t,$$

    f and g being continuous deterministic functions. Verify that the integrating factor

    $$I(t) = e^{-\int_0^t g(s)dW_s + \frac{1}{2}\int_0^t g^2(s)ds},$$

    satisfies

    $$d(I_t X_t) = I_t f(t, X_t)dt.$$

    Remark: In order to understand the integrating factor of SDE, it is better that you understand that of the ODE first (and also how to derive this integrating factor as we have illustrated in class). Then in light of this identity, $Y_t = I_t X_t$ satisfies

    $$dY_t = I_t f(t, Y_t/I_t)dt$$

    which (might) can be solved by other methods. It seems necessary to mention that, sometimes there are several methods that one can solve a SDE and sometimes not, therefore it is necessary for you to know how to tackle one problem from different approaches in order to better understandings.

11. In the lecture and the previous HW, you are given the integrating factor and asked to verify that it is one. However, for one to apply it without the expression at hand, it is necessary to solve for this formula. Find the SDE for I_t such that $dX_t = f(t, X_t)dt + g(t)X_t dW_t$. Then solve for I_t. Your solution should be in the form given above.

12. Solve the following SDEs by the method of integrating factors

    (a) $dX_t = \alpha X_t dW_t$;

(b) $dX_t = rdt + \alpha X_t dW_t$;

(c) $dX_t = X_t dt + \alpha X_t dW_t$;

(d) $dX_t = \frac{1}{X_t} dt + \alpha X_t dW_t$, $X_0 > 0$.

13. Let X_t be the solution of the stochastic equation $dX_t = \sigma X_t dW_t$, σ being a constant. Let A_t be the *stochastic average* of X_t

$$A_t := \frac{1}{t} \int_0^t X_s dW_s.$$

(a) Find the SDE of A_t;

(b) Find the mean and variance of A_t;

(c) Do the same when X_t is a geometric Brownian Motion, i.e., $dX_t = \mu X_t dt + \sigma X_t dW_t$. What is(are) your observation(s) on the effect of μ?

14. Use any method you like to solve *two* of the following SDEs with initial condition X_0 for each problem, unless otherwise stated

(a) $dX_t = (4X_t - 1)dt + 2dW_t$;

(b) $dX_t = (3X_t - 2)dt + e^{3t} dW_t$;

(c) $dX_t = (X_t + 1)dt + e^t W_t dW_t$;

(d) $dX_t = (4X_t + t)dt + e^{4t} dW_t$;

(e) $t^3 dX_t = (3t^2 X_t + t)dt + t^6 dW_t$, $X_1 = 0$;

(f) $dX_t = (\frac{1}{2} X_t + t)dt + e^t \sin W_t dW_t$;

(g) $dX_t = -X_t dt + e^{-t} dW_t$;

(h) $dX_t = X_t^3 dt + X_t^2 dW_t$, $X_0 = 1$;

(i) $dX_t = \left(\sqrt{X_t^2 + 1} + \frac{1}{2} X_t\right) dt + \sqrt{X_t^2 + 1} dW_t$;

(j) $d(\ln r_t) = (\theta(t) - \alpha(t) \ln r_t)dt + \sigma(t) dW_t$; this is called Black–Karasinki Model

Remark: It is strongly recommended that you apply all the methods learned to work out each problem if applicable, though not required for the HW; moreover, it is also a good practice for you to evaluate the mean and variance with or without solving the SDEs; find the covariance and distribution etc.

15. Solve the SDEs

$$dX_t = rX_t dt + \alpha dW_t, t > 0$$

and

$$dY_t = rY_t dt + e^{\sqrt{2r} W_t} dW_t, t > 0.$$

Find the expectation and variacne of X_t and Y_t. Note that X_t is Gaussian but Y_t is not.

16. For any $a, b \in \mathbb{R}$ being fixed.

(1) Solve the following SDE of Brownian Bridge

$$dX_t = \frac{b - X_t}{1 - t} dt + dW_t, 0 \le t < 1, X_0 = a.$$

(2) Find $\mathrm{Cov}(X_s, X_t)$ for $0 < s < t$.

(3) Show that a.c. $X_t \to b$ as $t \to \infty$. Hint: It is equivalent as showing

$$U_t = (1 - t) \int_0^t \frac{1}{1 - s} dW_s.$$

Argue by contradiction by using Markov's inequality.

17. Let the constants r and $\sigma > 0$ be fixed.

(1) Solve the following Ornstein–Uhlenbeck process

$$dX_t = (r - X_t)dt + \sigma dW_t$$

(2) Find $E(X_t)$, $\mathrm{Var}(X_t)$ and $\mathrm{Cov}(X_s, X_t)$ for $0 < s < t$.

(3) Find the limit of $E(X_t)$, $\mathrm{Var}(X_t)$ as $t \to \infty$.