Stochastic Differential Equations, Spring 2021

Homework 9

Due May 27, 2021

Name:_______________

1. Show that
$$\int_0^T e^{\frac{t}{2}+iW_t}dW_t = i(1-e^{\frac{T}{2}+iW_T}).$$
Hint: Use Euler's formula.

2. Use Integration by parts to show that
$$\int_0^T \frac{1}{1+W_t^2}dW_t = \arctan W_T + \int_0^T \frac{W_t}{(1+W_t^2)^2}dt, \forall T > 0.$$
Now verify that
$$E(\arctan W_T) = -\int_0^T E\Big(\frac{W_t}{(1+W_t^2)^2}\Big)dt.$$

3. Show that
$$\int_0^T W_t e^{W_t} dW_t = W_T e^{W_T} + 1 - e^{W_T} - \frac{1}{2}\int_0^T e^{W_t}(1+W_t)dt.$$
Now find $E(W_t e^{W_t})$, $\mathrm{Cov}(W_t, e^{W_t})$ and $\mathrm{Corr}(W_t, e^{W_t})$. Find the limit of $\mathrm{Corr}(W_t, e^{W_t})$ as $t \to 0$ and $t \to \infty$.

4. Show that
$$\int_0^T \frac{2W_t}{1+W_t^2}dW_t = \ln(1+W_T^2) - \int_0^T \frac{1-W_t^2}{(1+W_t^2)^2}dt,$$
and
$$-\frac{T}{8} \le E(\ln(1+W_T^2)) \le T.$$
You can also use Jensen's inequality to show that
$$E(\ln(1+W_T^2)) \le \ln(1+T).$$
Is there a contradiction in the upper bounds in two estimates above?

5. Evaluate the stochastic integral
$$\int_a^b t^{-\frac{3}{2}} W_t e^{-\frac{W_t^2}{2t}} dW_t.$$

6. Evaluate the stochastic integral
$$\int_a^b \frac{\partial f(t, W_t)}{\partial W_t} dW_t$$
if $f(t,x)$ satisfies the heat equation with source
$$\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} = G(t).$$

7. Solve the following stochastic differential equations and your solution X_t should be in the form of an integrated Itô diffusion, (e.g. $\int_0^t W_s ds$ which is just a Riemann integral and easy to calculate), but not stochastic integral (e.g., $\int_0^t W_s dW_s$, which is an Itô integral and, in general, hard to calculate). Indeed, this is the main idea/motivation of the stochastic integration by parts.
   (a) $dX_t = dt + W_t dW_t, X_0 = 1$;
   (b) $dX_t = (W_t - 1)dt + W_t^2 dW_t$, $X_0 = 0$;
   (c) $dX_t = (W_t + 3t^2)dt + tdW_t$, $X_0 = 0$;

8. Use the heat equation method to find
   (a) $\int_0^T e^{-\frac{\lambda^2 t}{2} \pm \lambda W_t} dW_t$;
   (b) $\int_0^T e^{\frac{\lambda^2 t}{2}} \cos(\lambda W_t) dW_t$;
   (c) $\int_0^T e^{\frac{\lambda^2 t}{2}} \sin(\lambda W_t) dW_t$.

   Remark: well, the solutions have been given in lecture.

9. Let us revisit the example considered in class

$$\int_0^T W_t^2 dW_t,$$

   for which we tried to evaluate by the so–called heat equation method.

   To this end, here we observe that $\phi(x) = s^2$ and therefore we would like to find $f(t, x)$ that satisfies

$$\begin{cases} \frac{\partial f}{\partial x} = x^2, \\ \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} = 0. \end{cases} \tag{0.1}$$

   From the first equation we find that f takes the form

$$f(t, x) = \frac{x^3}{3} + c(t)$$

   where $c(t)$ is a function of ONLY variable t. Substituting this form into the heat equation gives us $c'(t) = -x$ hence

$$c(t) = -tx + c_0,$$

   c_0 being an arbitrary constant. However, this is impossible because $c(t)$ should only be a function of t as we have mentioned above.

   It is worth noting that the heat equation method indeed works after minor modifications. To illustrate how, let us write

$$\int_0^T W_t^2 dW_t = \int_0^T W_t^2 + g(t) dW_t - \int_0^T g(t) dW_t$$

   for some $g(t)$ to be determined. Since $\int_0^T g(t) dW_t$ is Wiener and its distraction is known, we only need to evaluate $\int_0^T W_t^2 + g(t) dW_t$. Now finish this by using heat equation method for this new integral.

10. Let us revisit the integrated process and for the simplicity of demonstration let us consider

$$I_t := \int_0^t W_s^2 ds, t > 0.$$

   Suppose that we do not know its distribution, and one way to evaluate its statistics is to conduct trials through realized paths for sufficiently many times and then approximate the desired statistics by its limit.

For example, let us try to find $E(I_1)$. Generate a sample path of W_s up to time $t = 1$, and then find the area enclosed by this path and the $x-$axis. Then repeat this for N times and find the mean value of this collected areas A_i (of the i-th trial)

$$E^N = \frac{1}{N}\sum_{i=1}^{N} A_i.$$

You need to use MATLAB or other programs to generate these paths and then find the areas.

Plot E_N against N, and find the limit. Compare the limit with your analytical result of $E(I_1)$.