Stochastic Differential Equations, Spring 2021

Homework 2

Due Apr 1, 2021

Name:______________

1. Let X be an integrable random variable on $(\Omega, \mathcal{F}, P)$ and $\mathcal{F}_t$ be a filtration. Prove that $X_t = E[X|\mathcal{F}_t]$ is a martingale. Verify the definition of a martingale for the proof, here and in the following problems.

2. Let X_n, $n \geq 0$ be a sequence of integrable independent random variables and $\mathcal{F}_n$ be a filtration. Let $S_n = \sum_{k=0}^{n} X_k$, $n = 0, 1, 2, \ldots.$ Prove that

   (a) $S_n - E[S_n]$ is an $\mathcal{F}_n$–martingale;

   (b) if $E[X_n] = 0$ and X_n is square integrable for each $n = 0, 1, \ldots$, then $S_n^2 - \mathrm{Var}[S_n]$ is an $\mathcal{F}_n$–martingale.

3. Let X_n, $n \geq 0$ be a sequence of integrable independent random variables.

   (a) Suppose that their product $P_n = \prod_{k=0}^{n} X_k = X_0 \cdot X_1 \cdot X_2, \ldots \cdot X_n$ is an $\mathcal{F}_n$–martingale, what condition(s) on X_n do you need?

   (b) Suppose that for each $n \geq 0$ X_n is normally distributed $N(\mu, \sigma^2)$ with $\mu \neq 0$. Prove that there exists a unique nonzero θ such that $E[e^{\theta X_n}] = 1$.

   (c) Suppose that $X_n \sim N(\mu, \sigma^2)$ with $\mu \neq 0$ and let $S_n = \sum_{k=0}^{n} X_k$, $n = 0, 1, 2, \ldots.$ Prove that $Z_n = e^{\theta S_n}$ is a martingale, where θ is given in part (b).

4. Let W_t be a Wiener process. Show that for any $s, t > 0$ their covariance is

   $$\mathrm{Cov}(W_s, W_t) = \min\{s, t\}$$

   and their correlation is

   $$\mathrm{Cov}(W_s, W_t) = \frac{\min\{s, t\}}{\max\{s, t\}}.$$

5. Here and in the sequel and always denote W_t as a Brownian motion (as we mentioned in class that a Brownian Motion and a Wiener process can be applied interchangeably). For any $s, t > 0$, find their covariance $\mathrm{Cov}(W_s, W_t)$ and correlation $\mathrm{Corr}(W_s, W_t)$.

6. The process $X_t = |B_t|$ is called Brownian motion reflected at the origin. Find $E(X_t)$ and $\mathrm{Var}(X_t)$.

7. (1) Find $E((W_t^2 - t)(W_s^2 - s))$;

   (2) Find $E(W_t^2 W_s^2)$;

   (3) Find $\mathrm{Cov}(W_t^2, W_s^2)$.

8. Let $X_t = e^{W_t}$. Find $\mathrm{Cov}(X_s, X_t)$ by direction calculation.

9. Prove that B_t and $B_t^2 - t$ are martingales by definition. Now use Martingale's properties to find the followings for any $s, t \geq 0$

   (i) Find $E((B_t^2 - t)(B_s^2 - s))$;

   (ii) Find $E(B_t^2 B_s^2)$;

   (iii) Find $\mathrm{Cov}(B_t^2, B_s^2)$.

10. Consider $X_t = e^{W_t}$. This is called the Geometric Brownian Motion or GBM for short. Show that X_t is not a martingale, however $e^{-\frac{t}{2}} X_t$ is a martingale. For the latter case, you can actually show in general that $Y_t = e^{aW_t - \frac{a^2 t}{2}}$ is a martingale for any constant $a \in \mathbb{R}$. Now use the martingale property to find $\mathrm{Cov}(X_s, X_t)$. Remark: I want to emphasize that in almost all textbooks or lecture notes, you may find a statement like *a geometric Brownian motion is a martingale* with respect to... blabla...However there is no contradiction to the fact that X_t is not a martingale because it is $X_t = \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right\}$ how they define a GBM. And then they state that a GBM is a martingale when the drift parameter is 0, which is exactly our general case above with $a = \sigma$.

11. Let $X_t = e^{W_t}$. Find $\mathrm{Cov}(X_s, X_t)$ by direction calculation and verify it with your answer to the problem above.