# Stochastic Differential Equations, Spring 2021

## Homework 12, Due Jun 24

**Name**:______________

1. Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded domain with smooth boundary $\partial\Omega$. According to PDE theory, there exists a smooth solution $u = u(x)$ of the following equation

$$\begin{cases} -\frac{1}{2}\Delta u = 1, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{0.1}$$

Prove that for each point $x \in \Omega$, $u(x) = E(\tau_x)$, where $\tau_x$ is the first time that a Brownian motion starting at $x$ hits $\partial\Omega$.

2. Consider the following equation

$$\begin{cases} -\frac{1}{2}\Delta u + cu = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{0.2}$$

where $c = c(x)$ and $f = f(x)$ are smooth functions, and $c \geq 0$ in $\Omega$. Prove that, for each $x \in \Omega$,

$$u(x) = E\Big(\int_0^{\tau_x} f(X_t) e^{-\int_0^t ic(X_s)ds} dt\Big),$$

where $X_t = W_t + x$, i.e., the n-D Brownian motion statuting at $x$ and $\tau_x$ is the first time that $X_t$ hits $\partial\Omega$. Hint: find $d\big(u(X_t)e^{-\int_0^t c(X_s)ds}\big)$ first and then mimic the proof of Feynman–Kac Theorem. This is an extension of the previous problem.