# Introduction to SDEs, Spring 2021

**Test 1**

Apr 29th, 13:00-15:35

**Name**(Print):________________

**Student No**:________________

**Signature**:________________

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***There are 10 problems, 10 points each, 100 points in total.***

***Show* `details` *to get full credits. Make your justifications* `clear` *and* `direct`*.***

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**Leave the following table blank**

| Score Table | | |
|---|---|---|
| Problem | Points | Score |
| 1 | 10 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 10 | |
| 7 | 10 | |
| 8 | 10 | |
| 9 | 10 | |
| 10 | 10 | |
| | Total score | |

1. (1) Write down the definition that $X$ is a random variable;

   (2) Let $X$ be a random variable with mean $\mu$ and variance $\sigma^2$. Prove that for any positive number $\lambda$

$$P(\omega; X(\omega) \geq \lambda) \leq \frac{E(e^{tX})}{e^{\lambda t}}, \forall t > 0. \tag{1}$$

2. (1) Write down the definition that $X_t$ is a martingale or a sub-martingale ; (2) Let $X_t$ be a martingale and $\phi(\cdot)$ a smooth convex function. Prove that $\phi(X_t)$ is a sub-martingale.

3. (1) Write down the definition that $B_t$ is a Brownian Motion;

   (2) Prove that the Geometric Brownian Motion $X_t = e^{W_t}$ is not a martingale, but $e^{-\frac{t}{2}} X_t$ is a martingale.

4. (1) Write down the definition that that $X_n \to X$ in probability; (2) Write down the definition that that $X_n \to X$ almost surely; (3) Write down the definition that that $X_n \to X$ in distribution; (4) What are the connections between these three convergence manners? No proof needed!

5. Consider the following integrated Brownian Motion

$$Z_t := \int_0^t W_s ds.$$

(1) Write down the definition for this integral;

(2) Find the distribution of $Z_t$.

6. Revisit the integrated Brownian motion by choosing a different sample point, say

$$\int_0^t W_s ds = m.s. - \lim_{N\to\infty} \sum_{i=1}^{N} W_{t_i^*} \Delta t,$$

under equidistant partition.

What is the limit $\lim_{N\to\infty} \sum_{i=1}^{N} W_{t_i^*} \Delta t$ if we choose any sample point $t_i^* \in (t_i, t_{i+1})$, or $t_i^* = \lambda t_i + (1-\lambda) t_{i+1}$, $\lambda \in (0,1)$, as $\max |\Delta t_i| \to 0^+$.

7. (1) Let $X_t$ be a stochastic process. Write down the definition of the quadratic variation of $X_t$ over $[0, T]$;
   (2) Find the quadratic variation for $W_t$ over $[0, T]$. Proof of the claim is needed.

8. (1) Write down the definition of a stochastic integral with respect to Brownian Motion

$$I_T := \int_0^T F_t dW_t.$$

(2) Evaluate the integral $\int_0^T W_t dW_t$.

$\cdots$ continue here if necessary

9. Write down how to interpret the following differential identity

$$d(W_t^3) = 3W_t^2 dW_t + 3W_t dt.$$

No proof is needed at this stage.

10. (1) Write the definition of the Stratonovich integral

$$\int_0^T F_s \circ dW_t.$$

(2) Explain the differences between this integral and the corresponding Itô integral.