# Stochastic Differential Equations, Spring 2021

## Homework 11, Due Jun 10

**Name**:______________

1. The following SDE serves as a counter–example of non-existence of solution after when $W_t$ hits 1

$$dX_t = X_t^3 dt + X_t^2 dW_t, X_0 = 1.$$

i) verify that $X_t = \frac{1}{1-W_t}$ is the solution for $t \in (0,1)$, i.e., it is a solution and it is unique. Remark: the solution does not exist after $T$, but it exists and is unique before this time;

ii) solve the SDE from scratch paper to get the solution above; hint: try $X_t = f(t, W_t)$;

iii) test that the solution blow-up by either solving the SDE numerically or plotting the explicit solution above; you might want to record the first time that $W_t$ hits 1, i.e., the blow-up time;

iv) note that the time $T$ above is also a random variable now that it depends on the path of $W_t$. It is called the stopping time as we shall see later in the class. Can you numerically give an estimate of the mean of $T$? One way you can do is to, for each trial, record the first time $X_t$ surpass a predetermined large value, say $X_t = 10^{16}$. Then find the average of all the trials;

2. Consider the SDE

$$dX_t = \Big(\sqrt{1+X_t^2} + \frac{1}{2}X_t\Big)dt + \sqrt{1+X_t^2}dW_t, X_0 = x_0.$$

(a). Show that there exists a unique solution to this problem;

(b). Solve this SDE.

3. Let us consider the following 1D classical heat equation

$$\begin{cases} u_t = Du_{xx}, & x \in \mathbb{R}, t > 0, \\ u(0,x) = \varphi(x), & x \in \mathbb{R}. \end{cases} \tag{0.1}$$

where $D$ is a positive constant, $\varphi$ is a function that decays exponentially as $|x| \to \infty$. Then the solution of (0.1) is given by

$$u(t,x) = \frac{1}{\sqrt{4\pi Dt}} \int_{\mathbb{R}} e^{\frac{(x-y)^2}{4Dt}} \varphi(y) dy. \tag{0.2}$$

i) use the probabilistic representation for the reverse heat equation to write $u(x,t)$ in terms of a conditional expectation. Explain this result in a physical model. Note $D$ is not necessarily $\frac{1}{2}$;

ii) evaluate the expectation in i) and show that it gives rise to (0.2);

iii) numerical studies through Monte Carlo simulations: set $D = 1$ and $\varphi$ be the characteristic function such that $\varphi \equiv 1$ for $|x| < 1$ and $\varphi \equiv 0$ for $|x| \geq 1$. Plot the solution $u(t,x)$ by evaluating the expectation in i) for $t = 0.01, 0.1, 0.5$ and 1. This should give us four curves of $x$ which describe the evolution of $u(t,x)$. Beautify your plots when necessary.

4. Consider the following parabolic equation

$$\begin{cases} u_t + u_x + \frac{1}{2}u_{xx} = -1, & x \in \mathbb{R}, t \in (0,T), \\ u(T,x) = 0, & x \in \mathbb{R}. \end{cases} \tag{0.3}$$

Represent the solution in terms of conditional expectation by the Feynman–Kac formula. Then find the solution by evaluating this conditional expectation. Hint: you should verify that your solution satisfies the PDE by substituting it into the equation.

5. Let me re-guide you along with the baby example of Feynman–Kac formula in class:

   Let $u(t,x)$ be the solution to the heat equation

$$\begin{cases} u_t + \frac{1}{2}u_{xx} = 0, & x \in \mathbb{R}, 0 < t < T, \\ u(T,x) = \psi(x), & x \in \mathbb{R}. \end{cases} \tag{0.4}$$

   According to the Feynman–Kac Theorem, we know that the solution can be written the following conditional expectation

$$u(t,x) = \mathbb{E}(\psi(W_T)|W_t = x).$$

   Here I used $\mathbb{E}$ instead of $E$ because the expectation is taken with respect to the risk–neutral measure.

   To evaluate the conditional expectation, we proceed as follows:

$$\mathbb{E}(\psi(W_T)|W_t = x) = \mathbb{E}(\psi(W_t + W_T - W_t)|W_t = x) = \mathbb{E}(\psi(x + \Delta W_t^T),$$

   where

$$\Delta W_t^T := W_T - W_t = \sqrt{\tau}Z,$$

   with $\tau = T - t$ and $Z \sim N(0,1)$. Note that we have applied the independency of $W_t$ and $\Delta W_t^T$. We also want to mention that sometimes the conditional expectation $\mathbb{E}(\cdot|W_t = x)$ is written as $\mathbb{E}_t^x(\cdot)$.

   Continue to find the expectation hence the solution of $u(t,x)$. Hint: Your solution should be an integral (a convolution).

6. We know that an option price, denoted by $V$, depends on time $t$ and also the stock price $S$, i.e., $V = V(t, S_t)$. According to the financial theories (assumptions) of Black and Scholes, $V(t,S)$ satisfies the following PDE

$$\begin{cases} V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0, \\ V(T,S) = \psi(S), \end{cases} \tag{0.5}$$

   where $V_t$, $V_S$ and $V_{SS}$ denote the partial derivatives. Use Feynman–Kac Theorem to find the solution to (0.5) in terms of conditional expectation, and then evaluate the integral.