Stochastic Differential Equations, Spring 2021

Homework 6

Due Apr 29, 2021

Name:________________

1. Recall that we have verified that the integrated Brownian motion, which is approximated by the mean square limit of a finite sum, does not depend on the choice of sample points. Indeed, this is true for a general integrated stochastic process such as

$$I := \int_0^t f(s, W_s) ds,$$

where f is assumed to be continuous and integrable (hence the integral makes sense). Now let us define

$$I_n^* = \sum_{i=1}^n f(t_i^*, W_{t_i^*}) \Delta t,$$

with $\{t_i\}_{i=0}^n$ being an equi–distant partition of $(0, t)$, $t_i^* \in [t_{i-1}, t_i]$ an sample point. Prove that I does not depend on the choice of the sample points. Hint: choose another finite sum

$$I_n^{**} = \sum_{i=1}^n f(t_i^{**}, W_{t_i^{**}}) \Delta t.$$

Then prove that $I_n^* - I_n^{**} \to 0$ in mean square. Hence we can show that the integral does not depend on the sample points by triangle's inequality. Remark: of course the equi–distant partition is not required as long as $\max |t_i| \to 0$.

2. Let $0 \leq a < b < \infty$. Find the Itô integral

$$\int_a^b W_s dW_s.$$

3. We know that $(dW_t)^2 = dt$ is (and also should be) understood as the differential form of the quadratic variation $[W_t, W_t]|_T = T$, or precisely the mean square limit

$$m.s. - \lim_{n\to\infty} \sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})^2 = T.$$

Similarly one should understand $dtdW_t = 0$ in the previous HW as

$$m.s. - \lim_{n\to\infty} \sum_{i=0}^{n-1} (t_{i+1} - t_i)(W_{t_{i+1}} - W_{t_i}) = 0.$$

(1). How do you understand the following identity

$$d(W_t^3) = 3W_t^2 dW_t + 3W_t dt.$$

Though you do not that why this holds at this stage.

(2). Now, prove your claim in (1).

Remark 1. *The identity above is a very simple application of Itô's lemma which states that*

$$df(W_t) = f'(W_t)dW_t + \frac{1}{2}f''(W_t)dt, \tag{0.1}$$

the verification of which involves quite lengthy and tedious calculations, hence I do not require it at this time; actually, I do not even need you to know this formula at all. However, we shall see in our coming lectures that (0.1) follows easily from the quadratic variation of W_t (at least intuitively/formally). Moreover, this gives (or will) you a better understanding that the (stochastic) differential equation indeed is the differential form of the mean square convergence of the stochastic process.

4. In some textbooks, you see that the Stratonovich integral is defined as

$$\int_0^t f(s, W_s) \circ dW_s = m.s. - \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{f(t_i, W_{t_i}) + f(t_{i+1}, W_{t_{i+1}})}{2}(W_{t_{i+1}} - W_{t_i}),$$

i.e., choosing the mean value of $f(s, W_s)$ at the end points, instead of the value at the middle point. For $f(s, W_s) = W_s$, in this setting, we can easily see that

$$\int_0^t W_s \circ dW_s = m.s. - \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{W_{t_{i+1}} + W_{t_i}}{2}(W_{t_{i+1}} - W_{t_i}) = m.s. - \lim_{n\to\infty} \frac{1}{2}\sum_{i=0}^{n-1}(W_{t_{i+1}}^2 - W_{t_i}^2) = \frac{W_t^2}{2}.$$

What if you choose the value of $\lambda f(t_i, W_{t_i}) + (1-\lambda)f(t_{i+1}, W_{t_{i+1}}), \lambda \in [0,1]$. Find the Stratonovich integral

$$\int_0^t W_s \circ d_\lambda W_s$$

and compare it with the answer above.

5. Denote

$$X_t := \int_0^t W_s dW_s, \forall t > 0,$$

and then we already know that

$$X_t = \frac{W_t^2}{2} - \frac{t}{2}$$

hence can find its expectation and variance.

Now, find $E(X_t)$ and $\mathrm{Var}(X_t)$ by the its mean square limit property, but not using the explicit formula above.