Stochastic Differential Equations, Spring 2021

Homework 4

Due Apr 15, 2021

Name:______________

1. Prove that $X_t = e^{\sigma W_t + \mu t}$ is a submartingale with $\mu > 0$.
2. Use Doob's submartingale inequality to prove the followings for $\forall \lambda > 0$
   (a) $P(\max_{0 \le s \le t} W_t^2 \ge \lambda) \le \frac{t}{\lambda}$;
   (b) $P(\max_{0 \le s \le t} |W_t| \ge \lambda) \le \frac{\sqrt{2t/\lambda}}{\lambda}$.
3. Denote $X_t := \frac{\max_{0 \le s \le t} |W_s|}{t}$. Show that $X_t \to 0$ stochastically or in probability.
4. Show that Doob's submartingale inequality implies Markov's inequality.
5. Let $Y_t = tW_{\frac{1}{t}}$, $t > 0$, with $Y_0 = 0$. Find its distribution.
6. Write down the definition of a log–normal distribution random variable, with its pdf or cdf. Now show by definition that $X_t = e^{W_t}$ is log–normally distributed, i.e., find its distribution function.
7. Given a Brownian Motion, we are able to define or introduce its variants based on the phenomenon to model or describe. For example, the following process

$$X_t = W_t - tW_1, t \in [0, 1]$$

   is called a *Brownian Bridge*, since $W_0 = 0$ and $W_1 = 1$ and W_t is like a bridge connecting the points 0 and 1.
   (i) Show that X_t is normally distributed and find its pdf;
   (ii) Find $\mathrm{E}(X_t)$ and $\mathrm{Var}(X_t)$;
   (iii) Find $\mathrm{Cov}(X_s, X_t)$, $s, t \in [0, 1]$;
   (iv) Let $Y_t = X_t^2$. Find $\mathrm{E}(Y_t)$ and $\mathrm{Var}(Y_t)$.
8. I need you to review the definition and properties of the Riemann–Stieltjes integral.