Stochastic Differential Equations, Spring 2021

Homework 8

Due May 21, 2021

Name:________________

1. Let $\{X_n(t)\}$ be a sequence of square integrable and adapted/non-anticipating stochastic processes and it converges to $X(t)$ in the following sense

$$\lim_{n\to\infty} E\Big(\int_a^b |X_n - X|^2 dt\Big) = 0.$$

Prove that in mean square as $n \to \infty$

$$\int_a^b X_n(t)dW_t \to \int_a^b X(t)dW_t.$$

Hint: Use Isometry identity.

2. Let $f(t)$ be a function integrable in (a, b). Find

$$Cov\Big(W_t, \int_0^t f(s)dW_s\Big).$$

3. Let A_t be the average of the Brownian motion over $(0, t)$

$$A_t = \frac{1}{t}Z_t = \frac{1}{t}\int_0^t W_s ds.$$

Show that

$$dA_t = \frac{1}{t}\Big(W_t - \frac{1}{t}Z_t\Big)dt.$$

Hint: here and in the sequel you might want to use the fact that W_t is continuous in t.

4. Let G_t be the average of the geometric Brownian motion over $(0, t)$

$$G_t := \frac{1}{t}\int_0^t e^{W_s} ds.$$

Find dG_t.

5. Find the following increments

$$d(e^{W_t}); d((t + W_t)^k); d\Big(\frac{1}{t^\alpha}\int_0^t e^{W_s} ds\Big)$$

6. Let $R_t = \sqrt{X_t^2 + Y_t^2}$ be the Bessel process, where X_t and Y_t are two independent Brownian motions. Find dR_t and show that R_t satisfies the following stochastic differential equation
$$dR_t = dW_t + \frac{1}{2}\frac{dt}{R_t}.$$

7. Use Itô's lemma to show
$$\int_0^T W_s dW_s = \frac{1}{2}W_T^2 - \frac{1}{2}T.$$

8. Use Itô's lemma to verify that the Geometric Brownian Motion
$$S_t = S_0 e^{(\mu - \sigma^2/2)t + \sigma W_t}$$
satisfies stochastic differential equation
$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$

9. Let X_t and Y_t be two stochastic processes. Use
   (a) the definition
   (b) the Itô's lemma

   to show that
$$d\Big(\frac{X_t}{Y_t}\Big) = \frac{Y_t dX_t - X_t dY_t - dX_t dY_t}{Y_t^2} + \frac{X_t}{Y_t^3}(dY_t)^2.$$
   This is called the quotient rule. Now find $d(\frac{X_t}{f(t)})$ for f being a deterministic function of t.

10. Let $W_t^{(1)}$ and $W_t^{(2)}$ be two independent Brownian Motions and assume that there are two Itô diffusions $X_t^{(1)}$ and $X_t^{(2)}$ governed by
$$dX_t^{(1)} = \mu_1(t, W_t^{(1)})dt + \sigma_1(t, W_t^{(1)})dW_t^{(1)}$$
and
$$dX_t^{(2)} = \mu_2(t, W_t^{(2)})dt + \sigma_2(t, W_t^{(2)})dW_t^{(2)}.$$
   (a) Find the Itô's formula for $df(t, X^{(1)}, X^{(2)})$
   (b) Generalize this formula for $df(t, X^{(1)}, ..., X^{(n)})$ with
$$dX_t^{(i)} = \mu_i(t, W_t^{(i)})dt + \sigma_i(t, W_t^{(i)})dW_t^{(i)}$$
   for n mutually independent Brownian Motions $W_t^{(i)}$, $i = 1, 2..., n$.
   (c) Do the same when the correlation coefficient of $W_t^{(i)}$ and $W_t^{(j)}$ is δ_{ij}.

11. Find $d(tW_t^2)$

12. Let X_t be an Itô diffusion and $f(t)$ a deterministic function. Prove that $df(t)dX_t = 0$. Hint: this mean the co–variance is zero, so what is the definition?