Stochastic Optimal Control Pair Trading

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Contents

HJB equation

Theorem

Let $\{X_t^\pi\}$ be a controlled Markov diffusion given by

$$\mathrm{d}X_t^\pi = \mu(t, X_t^\pi, \pi(t, X_t^\pi))\mathrm{d}t + \sigma(t, X_t^\pi, \pi(t, X_t^\pi))\mathrm{d}W_t$$

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and let u be the dynamic value function given by

$$u(t, X_t) = \sup_{\pi \in \mathcal{A}_{t,T}} \mathbb{E}[\psi(X_T^\pi)|\mathcal{F}_t]$$

for a bounded continuous function ψ.

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for a bounded continuous function ψ. Then u satisfies the PDE

$$\sup_{\pi \in \mathbb{R}} \left\{ \frac{\partial u}{\partial t} + \mu(t, x, \pi)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2(t, x, \pi)\frac{\partial^2 u}{\partial x^2} \right\} = 0$$

$$u(T, x) = \psi(x)$$

Underlying Assets

- We following the steps of[1]
- Risk-free asset $\mathrm{d}S_t^{(0)} = rS_t^{(0)}\mathrm{d}t$

[1] Thomas Nanfeng Li and Agnès Tourin. "Optimal pairs trading with time-varying volatility". In: *International Journal of Financial Engineering* 3.03 (2016), p. 1650023, p. 1.

[2] Robert F Engle and Clive WJ Granger. "Co-integration and error correction: representation, estimation, and testing". In: *Econometrica: journal of the Econometric Society* (1987), pp. 251–276, p. 2.

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- $W_t^{(1)}$ and $W_t^{(2)}$ has correlation $\rho \in (-1, 1)$

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- $W_t^{(1)}$ and $W_t^{(2)}$ has correlation $\rho \in (-1, 1)$
- Cointegrated vector

$$z_t = a + bt + \log S_t^{(1)} + \beta \log S_t^{(2)}$$

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The dynamics of z_t

$$\mathrm{d}\log S_t^{(1)} = \left(\mu_1 - \frac{\sigma_1^2}{2} + \delta_1 z_t\right)\mathrm{d}t + \sigma_1 \mathrm{d}W_t^{(1)}$$

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$$\begin{aligned}
\mathrm{d}z_t &= b + \mathrm{d}\log S_t^{(1)} + \beta \mathrm{d}\log S_t^{(2)} \\
&= \left(b + \mu_1 - \frac{\sigma_1^2}{2} + \beta\mu_2 - \beta\frac{\sigma_2^2}{2} + \delta_1 z_t + \beta\delta_2 z_t\right)\mathrm{d}t + \sigma_1 \mathrm{d}W_t^{(1)} \\
&+ \beta\sigma_2 \mathrm{d}W_t^{(2)} \\
&:= \alpha(\eta - z_t)\mathrm{d}t + \sigma_\beta \mathrm{d}W_t
\end{aligned}$$

Optimization problem

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\mathrm{d}V_t &= \pi_t^{(1)} V_t \frac{\mathrm{d}S_t^{(1)}}{S_t^{(1)}} + \pi_t^{(2)} V_t \frac{\mathrm{d}S_t^{(2)}}{S_t^{(2)}} + (1 - \pi_t^{(1)} - \pi_t^{(2)}) V_t \frac{\mathrm{d}S_t^{(0)}}{S_t^{(0)}} \\
&= [\pi_t^{(1)}(\mu_1 + \delta_1 z_t) + \pi_t^{(2)}(\mu_2 + \delta_2 z_t) + r(1 - \pi_t^{(1)} - \pi_t^{(2)})] V_t \mathrm{d}t \\
&+ \pi_t^{(1)} \sigma_1 V_t \mathrm{d}W_t^{(1)} + \pi_t^{(2)} \sigma_2 V_t \mathrm{d}W_t^{(2)}
\end{aligned}
$$

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&= [\pi_t^{(1)}(\mu_1+\delta_1 z_t) + \pi_t^{(2)}(\mu_2+\delta_2 z_t) + r(1-\pi_t^{(1)}-\pi_t^{(2)})] V_t \mathrm{d}t \\
&+ \pi_t^{(1)} \sigma_1 V_t \mathrm{d}W_t^{(1)} + \pi_t^{(2)} \sigma_2 V_t \mathrm{d}W_t^{(2)}
\end{aligned}
$$

- Denote

$$
x_t = \log S_t^{(1)}, \quad y_t = \log S_t^{(2)}
$$

- Maximize the value function

$$
u(t,v,x,y) = \sup_{\pi^{(1)},\pi^{(2)}} \mathbb{E}[U(V_T^{t,v,x,y,\pi^{(1)},\pi^{(2)}})]
$$

The HJB equation

The HJB equation of this problem can be written as

$$
\begin{aligned}
u_t + \sup_{\pi_1,\pi_2} \Big\{ & [\pi_1 (\mu_1 + \delta_1 z) + \pi_2 (\mu_2 + \delta_2 z) \\
& + r (1 - \pi_1 - \pi_2)] v u_v + \left(\mu_1 - \frac{1}{2}\sigma_1^2 + \delta_1 z \right) u_x \\
& + \left(\mu_2 - \frac{1}{2}\sigma_2^2 + \delta_2 z \right) u_y + (\pi_1 \sigma_1^2 v + \pi_2 \rho \sigma_1 \sigma_2) u_{vx} \\
& + (\pi_2 \sigma_2^2 v + \pi_1 \rho \sigma_1 \sigma_2) u_{vy} + \frac{1}{2} (\pi_1^2 \sigma_1^2 + \pi_2^2 \sigma_2^2 \\
& + 2\pi_1 \pi_2 \rho \sigma_1 \sigma_2) v^2 u_{vv} + \frac{1}{2}\sigma_1^2 u_{xx} + \frac{1}{2}\sigma_2^2 u_{yy} \\
& + \rho \sigma_1 \sigma_2 u_{xy} \Big\} = 0
\end{aligned}
$$

Solving the PDE analytically

- Separation of variable and reduce the dimension

$$u(t,v,x,y) = \frac{1}{\gamma} v^{\gamma} h(t,z)$$

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- Taking derivative with respect to (π_1, π_2)

$$\begin{cases} \pi_1 \sigma_1^2 + \pi_2 \rho \sigma_1 \sigma_2 = \dfrac{\mu_1 - r + \delta_1 z}{1-\gamma} + \dfrac{\sigma_1^2 + \beta \rho \sigma_1 \sigma_2}{1-\gamma} \dfrac{h_z}{h} \\ \pi_1 \rho \sigma_1 \sigma_2 + \pi_2 \sigma_2^2 = \dfrac{\mu_2 - r + \delta_2 z}{1-\gamma} + \dfrac{\beta \sigma_2^2 + \rho \sigma_1 \sigma_2}{1-\gamma} \dfrac{h_z}{h} \end{cases}$$

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- $\rho \neq \pm 1$!

Optimal Controls(initial)

$$\pi_1^* = \frac{\mu_1 - r + \delta_1 z}{\sigma_1^2(1-\gamma)\left(1-\rho^2\right)} - \rho\frac{\mu_2 - r + \delta_2 z}{\sigma_1\sigma_2(1-\gamma)\left(1-\rho^2\right)} + \frac{h_z}{(1-\gamma)h}$$
$$\pi_2^* = \frac{\mu_2 - r + \delta_2 z}{\sigma_2^2(1-\gamma)\left(1-\rho^2\right)} - \rho\frac{\mu_1 - r + \delta_1 z}{\sigma_1\sigma_2(1-\gamma)\left(1-\rho^2\right)} + \frac{\beta h_z}{(1-\gamma)h}$$

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- Set $h = \frac{1}{1-\gamma}\phi^{1-\gamma}$ for an unknown function ϕ
- Try the transformation

$$\phi(t, z) = \exp\{f_0(t) + f_1(t)z + f_2(t)z^2\}$$

Denote

$$c_1 = \sigma_1^2 + \beta^2\sigma_2^2 + 2\beta\rho\sigma_1\sigma_2 > 0, c_2 = \frac{\alpha}{2(1-\gamma)c_1} > 0,$$

$$c_0 = \frac{\alpha^2}{2(1-\gamma)^2 c_1} - \frac{\gamma}{2(1-\gamma)^2\left(1-\rho^2\right)}\left(\frac{\delta_1^2}{\sigma_1^2} + \frac{\delta_2^2}{\sigma_2^2} - 2\rho\frac{\delta_1\delta_2}{\sigma_1\sigma_2}\right)$$

$$c_3 = 2b + \frac{2r\gamma(1+\beta)}{\gamma-1} - \frac{1}{\gamma-1}\left[2\left(\mu_1 + \beta\mu_2\right)\right] - \left(\sigma_1^2 + \beta\sigma_2^2\right)$$

$$c_4 = \frac{\gamma}{2(\gamma-1)^2\left(1-\rho^2\right)}\left[\frac{\left(\mu_1 - r\right)\delta_1}{\sigma_1^2} + \frac{\left(\mu_2 - r\right)\delta_2}{\sigma_2^2}\right.$$
$$\left. -2\rho\left(\frac{\delta_1\left(\mu_2 - r\right)}{\sigma_1\sigma_2} + \frac{\delta_2\left(\mu_1 - r\right)}{\sigma_1\sigma_2}\right)\right]$$

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$$\left.-2\rho\left(\frac{\delta_1\left(\mu_2-r\right)}{\sigma_1\sigma_2} + \frac{\delta_2\left(\mu_1-r\right)}{\sigma_1\sigma_2}\right)\right]$$

- The linear ODE system

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$$\left.-2\rho\left(\frac{\delta_1\left(\mu_2-r\right)}{\sigma_1\sigma_2} + \frac{\delta_2\left(\mu_1-r\right)}{\sigma_1\sigma_2}\right)\right]$$

- The linear ODE system

$$\begin{cases} f_2'(t) + 2c_1(f_2(t)-c_2)^2 - c_0 = 0 \\ f_1'(t) + [-2c_1c_2 + 2c_1f_2(t)]f_1(t) + c_3f_2(t) + c_4 = 0 \end{cases}$$

when $c_0 > 0$, the solution of f_2 can be expressed analytically

$$f_2(t) =$$

$$c_2\left(1-\frac{c_0}{2c_1c_2^2}\right)\frac{\sinh\left(\sqrt{2c_1c_0}(T-t)\right)}{\sinh\left(\sqrt{2c_1c_0}(T-t)\right)+\frac{1}{c_2}\sqrt{\frac{c_0}{2c_1}}\cosh\left(\sqrt{2c_1c_0}(T-t)\right)}$$

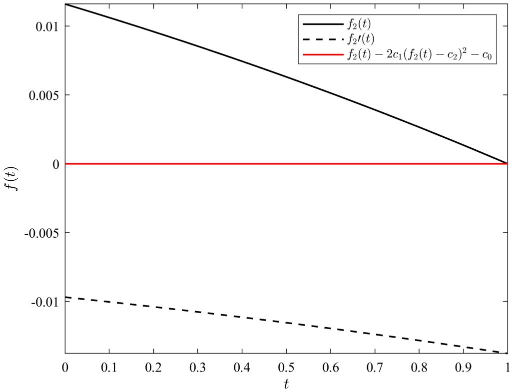

Figure 1: Varify the solution of f_2

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$$f_1(t) = \int_t^T [c_3 f_2(s) + c_4] \exp\left\{ 2c_1 \int_t^s (f_2(u) - c_2)\mathrm{d}u \right\} \mathrm{d}s$$

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Optimal Controls

$$\pi_1^* = \frac{\mu_1 - r + \delta_1 z}{\sigma_1^2(1-\gamma)\left(1-\rho^2\right)} - \rho\frac{\mu_2 - r + \delta_2 z}{\sigma_1\sigma_2(1-\gamma)\left(1-\rho^2\right)} + 2f_2(t)z + f_1(t)$$

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where $\theta_1, \theta_2 \in (-1, 0)$

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- Leverage effect

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- Solving initial π^*

$$\begin{aligned}
\pi_1^* =& \frac{\mu_1 - r + \delta_1 z}{\sigma_1^2(t,x)(1-\gamma)\left(1-\rho^2\right)} - \rho \frac{\mu_2 - r + \delta_2 z}{\sigma_1(t,x)\sigma_2(t,y)(1-\gamma)\left(1-\rho^2\right)} \\
&+ \frac{g_x}{(1-\gamma) g}, \\
\pi_2^* =& \frac{(\mu_2 - r + \delta_2 z)}{\sigma_2^2(t,y)(1-\gamma)\left(1-\rho^2\right)} - \rho \frac{\mu_1 - r + \delta_1 z}{\sigma_1(t,x)\sigma_2(t,y)(1-\gamma)\left(1-\rho^2\right)} \\
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\end{aligned}$$

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- The optimal controls involving ϕ

$$\pi_1^* = \frac{\mu_1 - r + \delta_1 z}{\sigma_1^2(t,x)(1-\gamma)\left(1-\rho^2\right)} - \rho\frac{\mu_2 - r + \delta_2 z}{\sigma_1(t,x)\sigma_2(t,y)(1-\gamma)\left(1-\rho^2\right)} + \frac{\phi_x}{\phi}$$
$$\pi_2^* = \frac{\mu_2 - r + \delta_2 z}{\sigma_2^2(t,y)(1-\gamma)\left(1-\rho^2\right)} - \rho\frac{\mu_1 - r + \delta_1 z}{\sigma_1(t,x)\sigma_2(t,y)(1-\gamma)\left(1-\rho^2\right)} + \frac{\phi_y}{\phi}$$

Contents

Constant volatility case

In this part, we set parameters as

$$
\begin{aligned}
& r = 0.01, \mu_1 = 0.2, \mu_2 = 0.08, \sigma_1 = 0.4, \sigma_2 = 0.45 \\
& \beta = -0.6, a = -0.01, b = -0.01, \delta_1 = -0.1, \delta_2 = 0.1 \\
& \theta_1 = -0.2, \theta_2 = -0.15, \gamma = 0.1, \rho = 0.5, S_0^{(1)} = 12.18 \\
& S_0^{(2)} = 20.09
\end{aligned}
$$

The Dynamics of Stock Prices

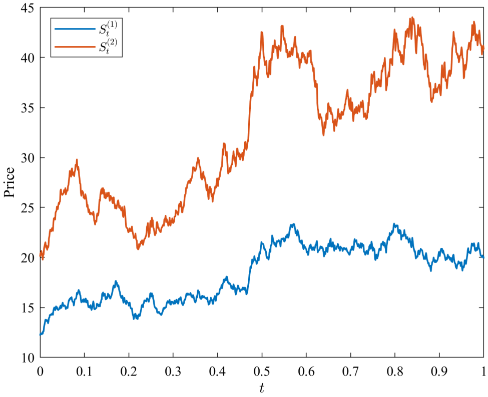

Figure 2: The dynamics of two stock prices in constant volatility case

The Dynamics of z_t

Ornstein-Uhlenbeck Process–Mean reverting!

$$\mathrm{d}z_t = \alpha(\eta - z_t)\mathrm{d}t + \sigma_\beta \mathrm{d}W_t$$

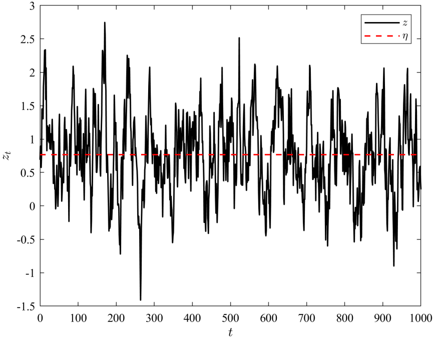

Figure 3: The dynamics of z_t in constant volatility case

The Dynamics of π_t

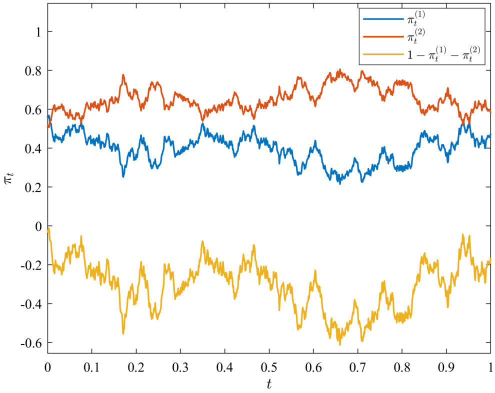

Figure 4: The dynamics of π_t in constant volatility case

Comparison

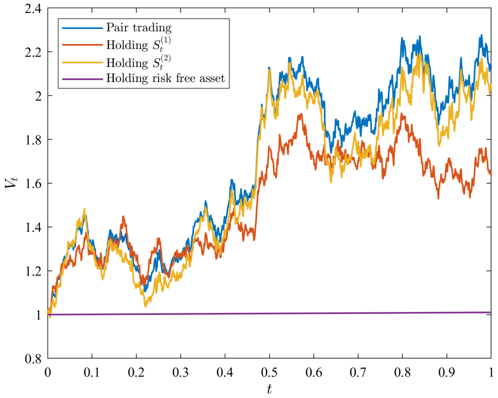

Figure 5: Comparison different strategies in constant volatility case

Time-varying Volatility

Main steps of are shown as following

1. Calculate the dynamics of x, y, z.
2. Solving the PDE by finite difference method[3].
3. Calculate the dynamics of π_1 and π_2

The parameters are the same as the constant volatility case, except for

$$\theta_1 = -0.2, \quad \theta_2 = -0.15$$

[3] K Ma and PA Forsyth. "An unconditionally monotone numerical scheme for the two-factor uncertain volatility model". In: *IMA Journal of Numerical Analysis* 37.2 (2017), pp. 905–944.

Solving the PDE by numerical method

- Discretise x, y, z as

$$x_i = x_{min} + i\Delta x, \quad y_j = y_{min} + j\Delta y$$
$$z_{i,j}^k = a + b(T - k\Delta t) + x_i + \beta y_j$$

where $(i,j) \in \{0, 1, \cdots, I\} \times \{0, 1, \cdots, J\}$, with $I\Delta x = x_{max} - x_{min}$ and $J\Delta y = y_{max} - y_{min}$. Also, $k \in \{0, 1, \cdots, K\}$ with $K\Delta t = T$.

- Denote $\phi_{i,j}^{k} := \phi(t_k, x_i, y_j)$

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$$\phi_x \approx \frac{\phi_{i+1,j}^k - \phi_{i,j}^k}{\Delta x}$$

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- $$\phi_{xx} \approx \frac{\phi_{i-1,j}^{k} - 2\phi_{i,j}^{k} + \phi_{i+1,j}^{k}}{\Delta x^2}$$

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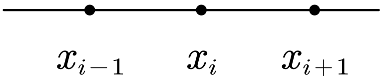
- $$\phi_{xx} \approx \frac{\phi^k_{i-1,j} - 2\phi^k_{i,j} + \phi^k_{i+1,j}}{\Delta x^2}$$
- $$\phi_t \approx \frac{\phi^{k+1}_{i,j} - \phi^k_{i,j}}{\Delta t}$$



Solving the PDE by numerical method

- When $\rho \geq 0$

$$\phi_{xy} \approx \frac{2\phi^k_{i,j} + \phi^k_{i+1,j+1} + \phi^k_{i-1,j-1} - \phi^k_{i+1,j} - \phi^k_{i-1,j} - \phi^k_{i,j+1} - \phi^k_{i,j-1}}{2\Delta x \Delta y}$$

Solving the PDE by numerical method

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- When $\rho < 0$

$$\phi_{xy} \approx \frac{-2\phi^k_{i,j} - \phi^k_{i+1,j-1} - \phi^k_{i-1,j+1} + \phi^k_{i+1,j} + \phi^k_{i-1,j} + \phi^k_{i,j+1} + \phi^k_{i,j-1}}{2\Delta x \Delta y}$$

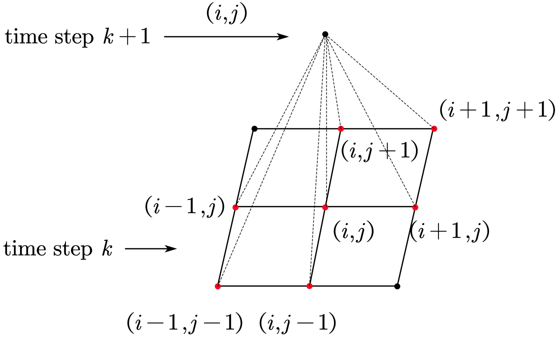

Figure 6: Illustration of finite difference method scheme when $\rho \geq 0$

The Dynamics of Stock Prices

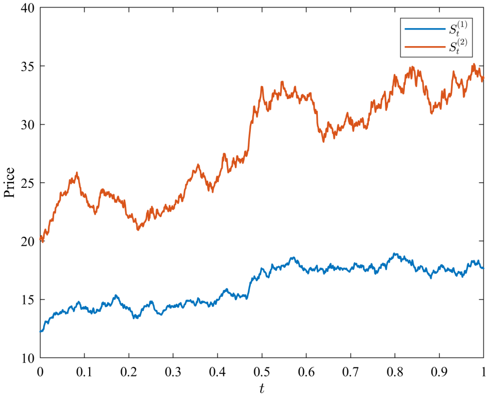

Figure 7: The dynamics of two stock prices in time-varying volatility case

The Dynamics of z_t

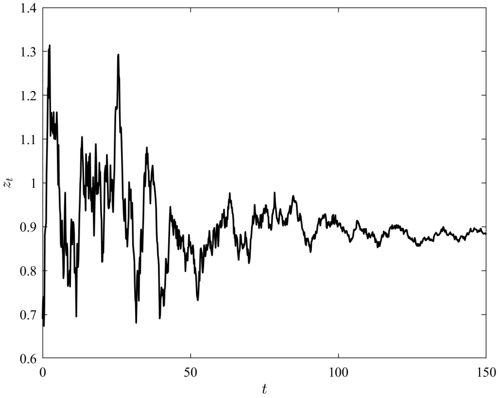

Figure 8: The dynamics of z_t in time-varying volatility case. It is not OU process but has a long run mean $\frac{1}{\alpha}(b + \mu_1 + \beta\mu_2)$

The dynamics of π_t

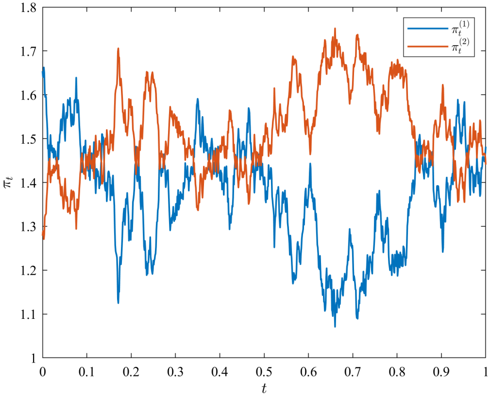

Figure 9: The dynamics of $\pi_t^{(1)}$ and $\pi_t^{(2)}$

Comparison

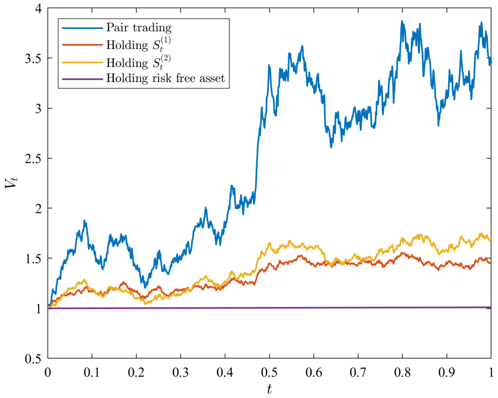

Figure 10: Dynamics of total wealth

Contents

Generalized Method of Moments

- Suppose the data $\{Y_t\}$ is generated by a weakly stationary ergodic stochastic process

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- GMM estimator

$$\hat{\theta} = \arg\min_{\theta \in \Theta} g(\theta)^T W g(\theta)$$

where W is positive-definite weighting matrix.

Generalized Method of Moments

- Discretize

$$\ln S_{t+1}^i - \ln S_t^i = \Big[\mu_i - \frac{1}{2}\sigma_i^2 e^{2\theta_i \ln S_t^i} + \delta_i \left(a + bt + \ln S_t^1 + \beta \ln S_t^2\right)\Big] \Delta t + \epsilon_{t+1}^i$$

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$$\mathbb{E}\left[\epsilon_{t+1}^1 \epsilon_{t+1}^2\right] = \rho\sigma_1 e^{\theta_1 \ln S_t^1} \sigma_2 e^{\theta_2 \ln S_t^2} \Delta t$$

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- Define λ the parameter vector with elements

$$\mu_1, \sigma_1, \delta_1, \theta_1, \mu_2, \sigma_2, \delta_2, \theta_2, \rho, a, b, \beta$$

- moment functions

$$f_t(\lambda) = \begin{bmatrix} \epsilon_{t+1}^1 \\ \epsilon_{t+1}^2 \\ \left(\epsilon_{t+1}^1\right)^2 - \sigma_1^2 e^{2\theta_1 \ln S_t^1} \Delta t \\ \left(\epsilon_{t+1}^2\right)^2 - \sigma_2^2 e^{2\theta_2 \ln S_t^2} \Delta t \\ \epsilon_{t+1}^1 \epsilon_{t+1}^2 - \sigma_1 e^{\theta_1 \ln S_t^1} \sigma_2 e^{\theta_2 \ln S_t^2} \rho \Delta t \end{bmatrix} \otimes \begin{bmatrix} 1 \\ t \\ \ln S_t^1 \\ \ln S_t^2 \end{bmatrix}$$

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- where $\otimes$ denotes Kronecker product

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- Sample average

$$g(\lambda) = \frac{1}{T} \sum_{t=1}^{T} f_t(\lambda)$$

- moment functions

$$f_t(\lambda) = \begin{bmatrix} \epsilon_{t+1}^1 \\ \epsilon_{t+1}^2 \\ \left(\epsilon_{t+1}^1\right)^2 - \sigma_1^2 e^{2\theta_1 \ln S_t^1} \Delta t \\ \left(\epsilon_{t+1}^2\right)^2 - \sigma_2^2 e^{2\theta_2 \ln S_t^2} \Delta t \\ \epsilon_{t+1}^1 \epsilon_{t+1}^2 - \sigma_1 e^{\theta_1 \ln S_t^1} \sigma_2 e^{\theta_2 \ln S_t^2} \rho \Delta t \end{bmatrix} \otimes \begin{bmatrix} 1 \\ t \\ \ln S_t^1 \\ \ln S_t^2 \end{bmatrix}$$

- where $\otimes$ denotes Kronecker product
- Moment condition $\mathbb{E}[f_t(\lambda)] = 0$
- Sample average

$$g(\lambda) = \frac{1}{T} \sum_{t=1}^{T} f_t(\lambda)$$

- Estimator

$$\hat{\lambda} = \arg\min_{\lambda \in \Theta} g(\lambda)^T W g(\lambda)$$

- For simplicity, choose$W = I$

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- To this end, select 12 out of 20 moment functions. Just-identified.

We choose Apple and Microsoft stock price from 2018-01-01 to 2020-01-01

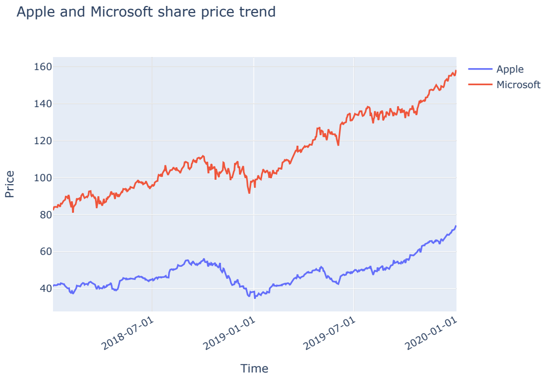

Figure 11: Apple and Microsoft stock price

Estimators

$$\mu_1 = 0.6324, \sigma_1 = 0.2438, \delta_1 = -0.00096, \theta_1 = -0.00553$$
$$\mu_2 = 0.8360, \sigma_2 = 0.3401, \delta_2 = -0.0013, \theta_2 = -0.0549$$
$$\rho = -0.0439, a = -113.6311, b = -174.3466, \beta = 171.8160$$

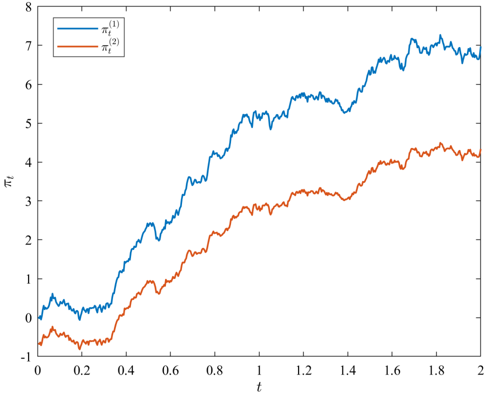

Figure 12: Propotion of total wealth invested to Apple and Microsoft

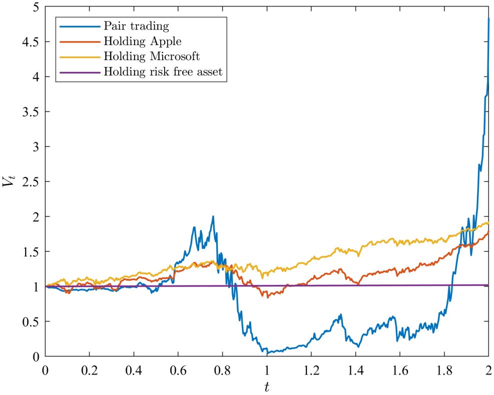

Figure 13: Comparing holding strategy with pair trading strategy

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Thanks for listening!