Stochastic Differential Equations, Spring 2021

Homework 3

Due Apr 8, 2021

Name:______________

1. We now discuss the relationships between convergence almost surely and convergence in probability in the problem and the next. First of all, we want to show that convergence almost surely implies convergence in probability. The proof is a little bit tricky and we divide it into the following steps. Suppose that a.c–$X_n \to X$, then the set (σ–field) that $\lim X_n \neq X$ has a probability zero, i.e., for any $\epsilon > 0$
$$P(\omega; \lim_{n\to\infty} |X_n(\omega) - X(\omega)| \geq \epsilon) = 0$$
or
$$P(\omega; \lim_{n\to\infty} |X_n(\omega) - X(\omega)| \leq \epsilon) = 1,$$
and we want to show that
$$\lim_{n\to\infty} P(\omega; |X_n(\omega) - X(\omega)| \geq \epsilon) = 0.$$

We use the first definition.

Let $\mathcal{O} = \{\omega; \lim X_n(\omega) \neq X(\omega)\}$, then $P(\mathcal{O}) = 0$. Remark: $\mathcal{O}$ is not necessarily empty but its probability/measure is zero (think about the lengthy of a point but any singleton is not empty).

step 1 For each $\epsilon > 0$, let
$$A_n := \cup_{m\geq n}\{\omega; |X_m(\omega) - X(\omega)| \geq \epsilon\}.$$

Prove that $\{A_n\}$ is decreasing

step 2 According to probability theory (set theory), it is known that $\cap_{n\geq 1} A_n$ exists and we denote
$$A_\infty := \cap_{n\geq 1} A_n$$
hence $P(A_n) \to P(A_\infty)$. Let ω_0 be any point in $\mathcal{O}$. Show that ω_0 does not belong to A_n if n is sufficiently large.

step 3 Now we know that ω_0 does belong to A_∞. Use this to show that $P(\omega; |X_n(\omega) - X(\omega)| \geq \epsilon) \to 0$

2. The opposite of the statement above is not true. Let X_n be a random variable such that $P(X_n = 1) = \frac{1}{n}$ and $P(X_n = 0) = 1 - \frac{1}{n}$. Prove that $X_n \to X = 0$ in probability, but not almost surely.