# Stochastic Differential Equations, Spring 2021

## Homework 1

Due Mar 25, 2021

**Name**:______________

1. In the class, we define the $\sigma$–algebra $\mathcal{F}$ to be any subset of $2^\Omega$. However in practice, since all about $\Omega$ are known, it is reasonable to take $\mathcal{F} = 2^\Omega$. We shall assume this throughout this course unless otherwise stated. I wish there is no confusion here.

   Let us recall our example in class that $\Omega = \{H, T\}$, for which it is easy to find that $2^\Omega = \{\emptyset, \{H\}, \{T\}, \{H, T\}\}$, which has 4 elements. Let us see a slightly general example: suppose that there are $k$ possible outcomes in the sample space given by $\Omega = \{\omega_1, \omega_2, ..., \omega_k\}$. Prove that there are $2^k$ elements in its $\sigma$–algebra $2^\Omega$. Remark: this is why formally one puts $\Omega$ on the shoulder.

2. We say that $X(\omega)$ is a measurable function with respect to $\sigma$–algebra $\mathcal{F}$ if

   $$\{\omega \in \Omega | a < X(\omega) < b\} \in \mathcal{F}, \forall a < b \in \mathbb{R}.$$

   Prove by using this definition that $X(\omega)$ is a measurable if

   $$\{\omega \in \Omega | X(\omega) < k\} \in \mathcal{F}, \forall k \in \mathbb{R}.$$

   Remark: therefore only one inequality is needed in the definition. Hint: if $A \in \mathcal{F}$, then its complement $A^c \in \mathcal{F}$; if $A_i \in \mathcal{F}$, then $\cup_{i=1}^\infty A_i \in \mathcal{F}$. Choose different $k$'s and repeat using these facts.

3. Let $X(\omega) : \Omega \to \mathbb{R}$ be a function that takes only finite many values $x_1, x_2, ..., x_n \in \mathbb{R}$. Prove that $X$ is a random variable if and only if $X^{-1}(x_1) \in \mathcal{F}$. Remark: this is also true is *finite* is replaced by *countably*. Throughout this course, we shall always assume that a random variable $X$ is measurable on its probability space $(\Omega, \mathcal{F}, P)$.

4. Let $X$ be a random variable with mean $\mu$ and variance $\sigma^2$. Prove that for any positive number $\lambda$

   $$P(\omega; X(\omega) \geq \lambda) \leq \frac{E(e^{tX})}{e^{\lambda t}}, \forall t > 0 \tag{0.1}$$

   and

   $$P(\omega; X(\omega) \leq \lambda) \geq \frac{E(e^{tX})}{e^{\lambda t}}, \forall t > 0. \tag{0.2}$$

5. (0.1) and (0.2) are called Chernoff bounds and $m(t) := E(e^{tX})$ here is called the *moment-generating function*(mgf) of $X$. These estimates sometimes can be very useful in various applications if you use them properly. For example, let $X$ be a normally distributed random variable with expectation $\mu$ and variance $\sigma^2$, which can be written as $X \sim N(\mu, \sigma^2)$ for simplicity. Here I assume that you know the probability density function of $X$ is given by

   $$p_X(x) = \frac{1}{\sqrt{2\sigma^2\pi}} e^{-\frac{|x-\mu|^2}{2\sigma^2}}, x \in \mathbb{R}.$$

Prove that for any $\lambda > \mu$

$$P(X \geq \lambda) \leq e^{-\frac{(\lambda-\mu)^2}{2\sigma^2}}. \tag{0.3}$$

Hint: You can break your proof into the following steps:

Step 1: Prove that the mgf of $X$ is $m(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$;

Step 2: Use Chernoff bounds to obtain an upper bound for $P(X \geq \lambda)$;

Step 3: Find the minimum of the quadratic function to prove (0.3).