Stochastic Differential Equations, Spring 2021

Homework 7

Due May 14, 2021

Name:_______________

1. For any point $\xi \in [t_i, t_{i+1}]$, we know that $\xi_i = t_i + \lambda(t_{i+1} - t_i)$, for some $\lambda \in [0,1]$. Similar as Riemann integral we can define the so–called λ–integral of $f(t, W_t)$ w.r.t Brownian motion W_t, or general Stratonovich integral, written in the following form

$$\lambda - \int_0^T f(s, W_s) dW_s = m.s. - \lim_{n\to\infty} \sum_{i=0}^{n-1} f(\xi_i, W_{\xi_i})(W_{t_{i+1}} - W_{t_i}),$$

with . Show that

$$\lambda - \int_0^T W_s dW_s = \frac{1}{2} W_T^2 + (\lambda - \frac{1}{2})T.$$

Hint: You may have other methods to deal with this, but one of the tricks (as we have used in class) is to construct $\sum (W_{t_{i+1}} - W_{t_i})^2$ which converges to T in mean square as is well known.

2. In order to prove the following connection between Itô and Stratonovich integrals

$$\int_0^t f(s, W_s) \circ dW_s = \int_0^t f(s, W_s) dW_s + \frac{1}{2} \int_0^t \frac{\partial f(s, W_s)}{\partial W_s} ds,$$

it is necessary to show that the all the Higher Order Terms converges to zero. Denote

$$\Delta W_i^{i+1} = W_{t_{i+1}} - W_{t_i}, \Delta W_i^{i+\frac{1}{2}} = W_{t_{i+\frac{1}{2}}} - W_{t_i}.$$

Prove that

(a) $m.s. - \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{\partial^2 f(t_i, W_{t_i})}{\partial t^2} (\frac{\Delta t}{2})^2 \Delta W_i^{i+1} = 0$;

(b) $m.s. - \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{\partial^2 f(t_i, W_{t_i})}{\partial t \partial W_t} \frac{\Delta t}{2} (\Delta W_i^{i+\frac{1}{2}}) \Delta W_i^{i+1} = 0$;

(c) $m.s. - \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{\partial^2 f(t_i, W_{t_i})}{\partial W_t^2} \big(\Delta W_i^{i+\frac{1}{2}}\big)^2 \Delta W_i^{i+1} = 0$;

(d) $m.s. - \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{\partial^{k+l} f(t_i, W_{t_i})}{\partial^k t \partial^l W_t} (\frac{\Delta t}{2})^k \big(\Delta W_i^{i+\frac{1}{2}}\big)^l \Delta W_i^{i+1} = 0$, $\forall k, l \geq 2$—this is optional and only for motivated students.

Remark: one then concludes that on the term $\frac{\partial f}{\partial W_t}$ is left after passing the limit(s).

3. Prove that the Itô integral $\int_0^t f(s, W_s) dW_s$ is $\mathcal{F}_t$–predictable, i.e.,

$$E\Big(\int_0^t f(u, W_u) dW_u | \mathcal{F}_s\Big) = \int_0^s f(u, W_u) dW_u.$$

Hint: use definition of the integral.