Stochastic Differential Equations, Spring 2021

Homework 5

Due Apr 22, 2021

Name:_______________

1. As we mentioned in class, another important variant is the so-called Bessel process in 2D

$$R_t = \sqrt{W_{x,t}^2 + W_{y,t}^2}.$$

Think of a particle, now in 2D, moves by a combination of Brownian motions in both x and y direction, say $W_{x,t}$ and $W_{y,t}$, which are independent of each other. (Again I hope you know that two random variables have the same distribution does not mean they are dependent). We know that X_t, the distance of the particle from the origin at time t, follows the Wald's distribution. Find the distribution over the nD case for $n \geq 3$.

2. Let $X_t = \int_0^t W_s ds$. We have showed in class that $X_t \sim N(0, \frac{t^3}{3})$ (except the rigourous convergence of the random variable to X_t). Indeed, to find $E(W_t)$ alternatively, it is easy to see that

$$E(X_t) = E\Big(\int_0^t W_s ds\Big) = \int_\Omega \int_0^t W_s ds dP \overset{\text{Fubini}}{=} \int_0^t \int_\Omega W_s dP ds = \int_0^t E(W_s) ds = \int_0^t 0 ds = 0,$$

because here t is a constant and integration is a linear operation. Similarly, we can also use straightforward calculations to evaluate $\text{Var}(X_t)$ as follows

$$\begin{aligned}\text{Var}(X_t) =& E(X_t^2) - E^2(X_t)\\ =& E\Big\{\Big(\int_0^t W_s ds\Big)\Big(\int_0^t W_r dr\Big)\Big\}\\ \overset{\text{Fubini}}{=}& E\Big(\int_0^t \int_0^t W_s W_r ds dr\Big)...\end{aligned}$$

Now you can start from here to find $\text{Var}(X_t)$. We would like to remark that by straightforward calculations, one may find all kinds of statistics of X_t, such as mean, variance, kurtosis etc, however we can not tell if it is normally distributed as we showed in class.

3. Let us revisit the integrated Brownian motion by choosing a different sample point, say

$$\int_0^t W_s ds = m.s. - \lim_{N\to\infty} \sum_{i=1}^N W_{t_i} \Delta t,$$

under equidistant partition.

(i). What is the limit now?

(ii) What is the limit $\lim_{N\to\infty} \sum_{i=1}^N W_{t_i^*} \Delta t$ if we choose any sample point $t_i^* \in (t_i, t_{i+1})$, or $t_i^* = \lambda t_i + (1-\lambda) t_{i+1}$, $\lambda \in (0, 1)$. Does the limit depend on the choice of the sample point (compared to what I said in class)?

(iii). Do the same when the partition π is not equidistant, while the limit above with $t_i^* = \lambda t_i + (1 - \lambda)t_{i+1}$, $\lambda \in (0,1)$ is taken as $\max|\Delta t_i| \to 0^+$.

4. For any $a, b \in \mathbb{R}$, we consider equidistant of (a,b), denoted by $\pi : \{t_k; a = t_0 < t_1 < ... < t_{n-1} < t_n = b\}$ with $t_{k+1} - t_k = \Delta t_k$. Show that in mean square

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})^2 = b - a.$$

   Does the partition has to be equidistant? Provide a counter–example or prove why not.

5. Let $\pi_n = \{t_i\}_{i=0}^n$ be any partition of $(0,T)$, not necessarily be equidistant. Show that

$$m.s. - \lim_{\max|t_{i+1}-t_i|\to 0} \sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})^2 = T.$$

   Now similar as R–S integral, the Quadratic Variation does not depend on the choice of the partitions.

6. Consider the equidistant partition $0 = t_0 < t_1 < ... < t_{n-1} < t_n = T$. Show that

$$m.s. - \lim_{n\to\infty} \sum_{k=0}^{n-1} (W_{t_{k+1}} - W_{t_k})(t_{k+1} - t_k) = 0.$$

   Does the partition has to be equidistant? Provide a counter–example or prove why not.

7. Suppose that m.s.-$X_n \to X$ as $n \to \infty$ and X is square integrable. (i) Show that $\mathrm{Var}(X_n) \to \mathrm{Var}(X)$ as $n \to \infty$. (ii) Show that $\mathrm{Cov}(X_n, X) \to \mathrm{Cov}(X, X)$. Hint: You may need to first prove $X_n \to X$ in mean by Jensen's inequality.

8. Prove that

$$m.s \lim_{n\to\infty} (X_n + Y_n) = m.s \lim_{n\to\infty} X_n + m.s \lim_{n\to\infty} Y_n.$$

   Hint: First prove this for the case when $\lim_{n\to\infty} X_n = \lim_{n\to\infty} Y_n = 0$; here you may need to show that $E(X_n Y_n) \to 0$. Now for this general case, apply this results to $\tilde{X}_n = X_n - \lim X_n$ and $\tilde{Y}_n = Y_n - \lim Y_n$.

9. (i) Prove the squeeze theorem: Let X_n, Y_n and Z_n be sequences of random vairables on $(\Omega, \mathcal{F}, P)$ such that

$$X_n \le Y_n \le Z_n, \text{a.s}, \forall n \ge 1.$$

   If X_n and Y_n converge to L in mean square, then Z_n converges to L in mean square. Remark: the same conclusion holds for different convergence manner.

   (ii). Show that

$$m.s. - \lim_{t\to\infty} \frac{W_t \sin W_t}{t} = 0$$

10. Let us recall that, we say a function $f(t)$, $t \in (a,b)$ is of bounded variation if there exists a finite number C such that for any partition $\pi_n = \{t_i\}_{i=0}^n$ of (a,b)

$$\sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)| < C.$$

    Prove that a Wiener process (almost surely) does not have a bounded variation. Hint: argue by contradiction. Suppose that it has a bounded variation, then you can show that its quadratic variation converges to zero almost surely, which is impossible.

11. The Riemann–Stieltjes integral of a real function f with respect to g is denoted by

$$\int_a^b f(t)dg(t)$$

and it is evaluated as the limit of

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(\xi_i)(g(t_{i+1}) - g(t_i)),$$

where $\xi_i \in [t_i, t_{i+1}]$. When $g(t) = t$, it reduces to the baby Riemann integral. Here the limit is taken in the manner that $\max |t_{i+1} - t_i| \to 0$ as $n \to \infty$.

You are asked to find out and verify the followings:

(i). the integral is independent of the choices of the sample points ξ_i;

(ii). the integral is independent of the partition of the interval $[0, t]$, i.e., one does not have to choose $\Delta t_i = t_{i+1} - t_i$ to be equidistant. (Intuitively this is simple because

$$\lim_{n\to\infty} S_n = \lim_{2n\to\infty} S_{2n}.$$

Therefore for a partition with n intervals, you can find a finer portion such that it has $2n$ intervals with equidistant Δt_i (or if not, try kn until you have a equidistant partition). However you have to be rigourous to prove this.)

(iii). the integral does not exists for all functions f and g. However it exists if f is continuous and g is of bounded variation (BV) (Go to check the definition of BV above; *note that this condition is sufficient but not necessary*). You will see what it means by a function of BV and the trajectories (just the collection of the random variables W_{t_i}) of a Wiener process is not of BV.

I also want to remark that you need to be aware that here the integration limits a and b are taken for the integration variable t, but not $g(t)$, therefore a probably more rigourous and less confusing way to write it should be

$$\int_{g(a)}^{g(b)} f(t)dg(t),$$

yet again this can be misleading to some because now $f(t)$ is a function of t but not $g(t)$. Therefore if you want to write this correctly rigourously and also properly it should be

$$\int_{g(a)}^{g(b)} f(t^{-1}(g(t)))dg(t) = \int_{g(a)}^{g(b)} f(t^{-1}(X))dX,$$

however, this correct version is not reader–friendly as the conventional one on the top to most people. This applies when we write

$$\int_0^T f(t, W_t)dW_t$$

and the integration limits are taken for t but not W_t.

This problem is designed so you warm yourself up of integration in calculus and you do **NOT** need to turn in your answers.