Problem 6.7: Using helicity amplitudes, calculate the differential cross section for the $e^-\mu^- \to e^-\mu^-$ scattering in the following steps

a) From the Feynman rules for QED, show that the lowest order QED matrix element for the scattering is

$$\mathcal{M}_{fi} = -\frac{e^2}{(p_1 - p_3)^2} g_{\mu\nu} \left[\bar{u}(p_3)\gamma^\mu u(p_1)\right] \left[\bar{u}(p_4)\gamma^\nu u(p_2)\right], \tag{1}$$

where p_1 and p_3 are the four-momenta of the initial and final state e^-, and p_2 and p_4 are the four momenta of the initial and final state μ.

b) Working in the COM frame, writing the four momenta of the initial and final state e's as

$$p_1^\mu = \begin{pmatrix} E_1 & 0 & 0 & p \end{pmatrix} \qquad \text{and} \qquad p_3^\mu = \begin{pmatrix} E_1 & p\sin\theta & 0 & p\cos\theta \end{pmatrix} \tag{2}$$

respectively, show that the electron currents for the four possible helicity combinations are

$$\bar{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) = 2\begin{pmatrix} E_1 c & ps & -ips & pc \end{pmatrix} \tag{3}$$

$$\bar{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) = 2\begin{pmatrix} ms & 0 & 0 & 0 \end{pmatrix} \tag{4}$$

$$\bar{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) = 2\begin{pmatrix} E_1 c & ps & ips & pc \end{pmatrix} \tag{5}$$

$$\bar{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1) = -2\begin{pmatrix} ms & 0 & 0 & 0 \end{pmatrix} \tag{6}$$

where $s = \sin(\theta/2)$ and $c = \cos(\theta/2)$.

a) This will be a t-channel interaction. The 1-3-gamma vertex gives us a

$$\bar{u}(p_3)(ie\gamma^\mu)u(p_1), \tag{7}$$

the 2-4-gamma vertex gives us a

$$\bar{u}_4(ie\gamma^\nu)u(p_2), \tag{8}$$

and interaction is mediated by a photon,

$$\frac{-ig_{\mu\nu}}{q^2}, \tag{9}$$

where q is the 4-momentum transfer $p_1 - p_3$. Putting these all together,

$$-iM_{fi} = \left[\bar{u}(p_3)(ie\gamma^\mu)u(p_1)\right] \frac{-ig_{\mu\nu}}{(p_1 - p_3)^2} \left[\bar{u}_4(ie\gamma^\nu)u(p_2)\right]. \tag{10}$$

We then combine all the i's and being the metric tensor and 4-momentum term out front.

$$M_{fi} = -\frac{g_{\mu\nu}}{(p_1 - p_3)^2} \left[\bar{u}(p_3)(e\gamma^\mu)u(p_1)\right] \left[\bar{u}_4(e\gamma^\nu)u(p_2)\right] \tag{11}$$

b) Recalling, from Thompson, that the two matter helicity spinors are

$$u_\uparrow(p) = \sqrt{E+m}\begin{pmatrix} c \\ se^{i\phi} \\ \frac{p}{E+m}c \\ \frac{p}{E+m}se^{i\phi} \end{pmatrix} \qquad u_\downarrow(p) = \sqrt{E+m}\begin{pmatrix} -s \\ ce^{i\phi} \\ \frac{p}{E+m}s \\ -\frac{p}{E+m}ce^{i\phi} \end{pmatrix} \tag{12}$$

So, we write, explicitly

$$u_\uparrow(p_3) = N\begin{pmatrix} c \\ s \\ \frac{p}{E+m}c \\ \frac{p}{E+m}s \end{pmatrix} \quad u_\downarrow(p_3) = N\begin{pmatrix} -s \\ c \\ \frac{p}{E+m}s \\ -\frac{p}{E+m}c \end{pmatrix} \quad u_\uparrow(p_1) = N\begin{pmatrix} 1 \\ 0 \\ \frac{p}{E+m} \\ 0 \end{pmatrix} \quad u_\downarrow(p_1) = N\begin{pmatrix} 0 \\ 1 \\ 0 \\ -\frac{p}{E+m} \end{pmatrix} \tag{13}$$

where $N = \sqrt{E+m}$. The rest is just turning the crank on these calculations (using Eqs (Th 6.12-15)).

- $\bar{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1)$

$$\bar{u}_\downarrow(p_3)\gamma^0 u_\downarrow(p_1) = (E+m)(0 + c - 0 + \tfrac{p^2}{(E+m)^2}c) = \tfrac{(E+m)^2+E_1^2-m^2}{E+m}c = 2E_1c \tag{14}$$
$$\bar{u}_\downarrow(p_3)\gamma^1 u_\downarrow(p_1) = N^2[\tfrac{p}{E+m}s + 0 + \tfrac{p}{E+m}s + 0] = 2ps \tag{15}$$
$$\bar{u}_\downarrow(p_3)\gamma^2 u_\downarrow(p_1) = -iN^2[\tfrac{p}{E+m}s + \tfrac{p}{E+m}s] = -2ips \tag{16}$$
$$\bar{u}_\downarrow(p_3)\gamma^3 u_\downarrow(p_1) = N^2[0 + \tfrac{p}{E+m}c + 0 + \tfrac{p}{E+m}c] = 2pc \tag{17}$$

Therefore $\bar{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) = 2\begin{pmatrix} E_1c & ps & -ips & pc \end{pmatrix}$

- $\bar{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1)$

$$\bar{u}_\uparrow(p_3)\gamma^0 u_\downarrow(p_1) = (E+m)[0 + s + 0 - \tfrac{E_1^2-m^2}{(E+m)^2}s] = \tfrac{E^2+m^2+2mE-E^2+m^2}{E+m}s = 2ms \tag{18}$$
$$\bar{u}_\uparrow(p_3)\gamma^1 u_\downarrow(p_1) = N^2[-\tfrac{p}{E+m}c + 0 + \tfrac{p}{E+m}c + 0] = 0 \tag{19}$$
$$\bar{u}_\uparrow(p_3)\gamma^2 u_\downarrow(p_1) = -iN^2[-\tfrac{p}{E+m}c - 0 + \tfrac{p}{E+m}c - 0] = 0 \tag{20}$$
$$\bar{u}_\uparrow(p_3)\gamma^3 u_\downarrow(p_1) = N^2[0 + \tfrac{p}{E+m}s0 - \tfrac{p}{E+m}s] = 0 \tag{21}$$

Therefore $\bar{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) = 2\begin{pmatrix} ms & 0 & 0 & 0 \end{pmatrix}$.

- $\bar{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1)$

$$\bar{u}_\uparrow(p_3)\gamma^0 u_\uparrow(p_1) = (E+m)\left[c + 0 + \tfrac{p^2}{(E+m)^2}c + 0\right] = 2E_1c \tag{22}$$
$$\bar{u}_\uparrow(p_3)\gamma^1 u_\uparrow(p_1) = (E+m)\left[0 + \tfrac{p}{E+m}s + 0 + \tfrac{p}{E+m}s\right] = 2ps \tag{23}$$
$$\bar{u}_\uparrow(p_3)\gamma^2 u_\uparrow(p_1) = -i(E+m)\left[0 - s\tfrac{p}{E+m} + 0 - s\tfrac{p}{E+m}\right] = 2ips \tag{24}$$
$$\bar{u}_\uparrow(p_3)\gamma^3 u_\uparrow(p_1) = (E+m)\left[c\tfrac{p}{E+m} - 0 + c\tfrac{p}{E+m} - 0\right] = 2pc \tag{25}$$

Therefore $\bar{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) = 2\begin{pmatrix} E_1c & ps & ips & pc \end{pmatrix}$

- $\bar{u}_{\downarrow}(p_3)\gamma^{\mu}u_{\uparrow}(p_1)$

$$\bar{u}_{\downarrow}(p_3)\gamma^0 u_{\uparrow}(p_1) = (E+m)\left[-s + 0 + s\tfrac{p^2}{(E+m)^2} + 0\right] = -2ms \tag{26}$$

$$\bar{u}_{\downarrow}(p_3)\gamma^1 u_{\uparrow}(p_1) = (E+m)\left[0 + c\tfrac{p}{E+m} + 0 - c\tfrac{p}{E+m}\right] = 0 \tag{27}$$

$$\bar{u}_{\downarrow}(p_3)\gamma^2 u_{\uparrow}(p_1) = -i(E+m)\left[0 - c\tfrac{p}{E+m} + 0 + c\tfrac{p}{E+m}\right] = 0 \tag{28}$$

$$\bar{u}_{\downarrow}(p_3)\gamma^3 u_{\uparrow}(p_1) = (E+m)\left[-s\tfrac{p}{E+m} - 0 + s\tfrac{p}{E+m} - 0\right] = 0 \tag{29}$$

Therefore $\bar{u}_{\downarrow}(p_3)\gamma^{\mu}u_{\uparrow}(p_1) = -2\begin{pmatrix} ms & 0 & 0 & 0 \end{pmatrix}$