A finite difference approach to solving the Navier-Stokes equations for the 2-D Lid Driven Cavity problem

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Introductory remarks

The Navier-Stokes equations are a set of non linear, partial differential equations that govern the viscous motion of fluids. If used correctly, these equations can be used as a powerful tool to model the flow of fluids in many different scenarios, including weather systems, viscous fluids moving around barriers and even in computer game simulations. Due to the non linear nature of these equations there are very few analytically solvable problems that have been found. Due to this, these equations must be approximately solved by numerical methods and their solutions calculated with the help of a computer. I will be using MATLAB to implement and execute my numerical method for solving these equations for my cavity problem.

The Problem

The Lid Driven Cavity problem has been thoroughly studied and results from different numerical methods approaches agree and are well documented. The problem involves a cavity (usually square or rectangular) where there are non slip (zero velocity) boundary conditions on three edges and a constant velocity imposed on the fourth. Figure 1 shows a diagrammatic view of this. This problem is commonly used to test out the accuracy of newly written fluid mechanics software due to its simple boundary and initial conditions. I will be modelling the time evolution of the velocity field for 10 seconds in intervals of 0.01 seconds and superimposing it onto a 15x15 cavity by use of a quiver plot. I will also plot the vorticity of the flow using a contour plot and will produce a separate surface plot of the pressure field for the cavity. The equations that govern the system are as follows, the conservation of x momentum is

$$\frac{\partial}{\partial t}\int_{\mathrm{V}} udV = -\oint_S u\boldsymbol{u}\cdot\boldsymbol{n}dS - \frac{1}{\rho}\oint_S pn_x dS + v\oint_S \nabla u\cdot\boldsymbol{n}dS$$

where V is the size of the control volume (spoken about in 'My approach'), S is the surface area of the control volume, $\boldsymbol{n}$ is a unit normal to the surface S with x component n_x, ρ is the density of the fluid, p is the pressure, $\boldsymbol{u}, \boldsymbol{v}$ are the x and y velocity vectors with magnitudes u, v. The equation for conservation of y momentum is such

$$\frac{\partial}{\partial t}\int_{\mathrm{V}} vdV = -\oint_S v\boldsymbol{u}\cdot\boldsymbol{n}dS - \frac{1}{\rho}\oint_S pn_y dS + v\oint_S \nabla v\cdot\boldsymbol{n}dS$$

where n_y is the y component of the normal vector to the surface S. The final equation governing the system relates to the conservation of mass in the system

$$\oint_S \boldsymbol{u}\cdot\boldsymbol{n}dS = 0$$

There is no explicit equation for pressure here so I have to use a special method, discussed later, to calculate the pressure.

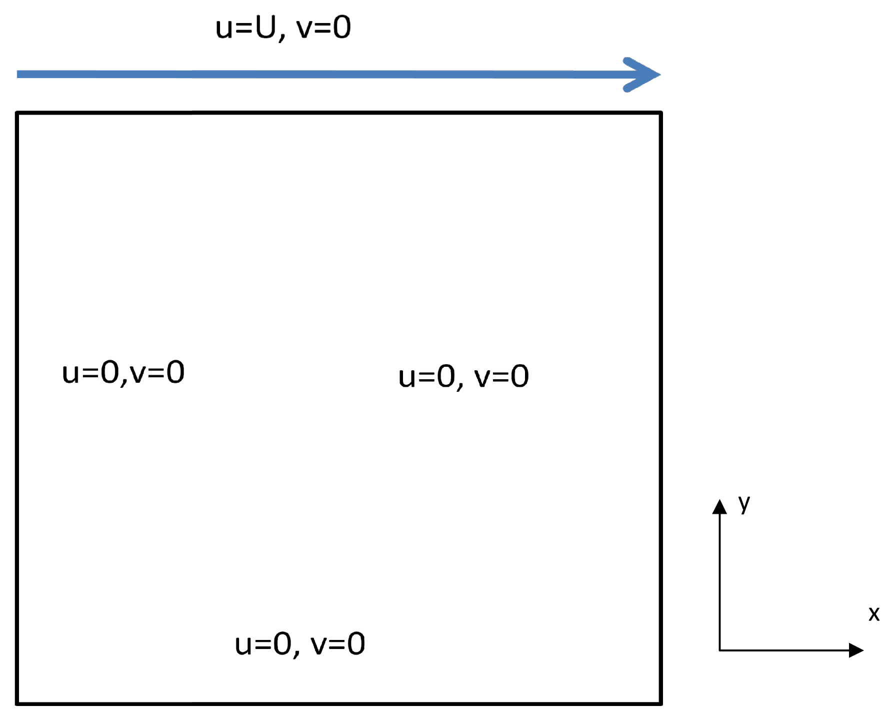

Figure 1: I will be modelling a square cavity where the top face has a constant velocity in the x direction denoted by U. u and v are velocity vectors in the x and y directions respectively. The other three faces have a non slip boundary condition whereby u and v are zero.

My approach

Starting off I knew that I had to solve the Navier Stokes equations for three variables: u and v velocities and the pressure p. By defining a standard 'control' volume I can place the u velocities on the vertical columns, v velocities on the next set of vertical columns, offset by half a step. The pressure can be held in the centre of the control volume, this whole set up can be seen in Figure 2. This method is known as a Method and Cell, MAC, scheme.

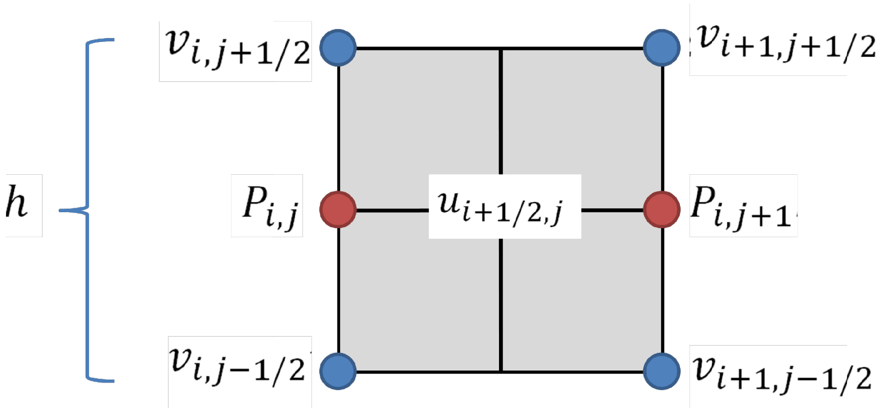

Figure 2: Here i and j label the x and y positions on the grid. u and v relate to the x and y velocities in the fluid and P labels the pressure. h is the spatial separation between i and i+1 or j and j+1.

To implement these equations into MATLAB code I need to discretise them. This is done by approximating the integrals with values of u,v and P on the grid. The conservation of mass integral $\oint_S \boldsymbol{u} \cdot \boldsymbol{n} dS = 0$ can be approximated as such:

$$hu_{i+1/2,j}^{n+1} - hu_{i-1/2,j}^{n+1} + hv_{i,j+1/2}^{n+1} - hv_{i,j-1/2}^{n+1} = 0$$

dividing by h gives

$$u_{i+1/2,j}^{n+1} - u_{i-1/2,j}^{n+1} + v_{i,j+1/2}^{n+1} - v_{i,j-1/2}^{n+1} = 0$$

where n labels the time step.

To approximate the unsteady term $\frac{\partial}{\partial t}\int_{\mathrm{V}} udV$ for the x momentum we can use the u velocity at the n'th time period and at the n+1st time period. The integral can be discretised to give:

$$\frac{\partial}{\partial t}\int_{\mathrm{V}} udV \cong \frac{u_{i+1/2,j}^{n+1} - u_{i+1/2,j}^{n}}{\Delta t} h^2$$

similarly for the y momentum

$$\frac{\partial}{\partial t}\int_{V} v dV \cong \frac{v_{i+1/2,j}^{n+1} - v_{i+1/2,j}^{n}}{\Delta t} h^2$$

Next, the x component of the advection term $\oint_S u\boldsymbol{u}\cdot\boldsymbol{n}dS$ can be approximated in a similar fashion

$$\oint_S u\boldsymbol{u}\cdot\boldsymbol{n}dS \cong \left[\left(\frac{1}{2}\left(u^n{}_{i+\frac{3}{2},j} + u^n_{i+\frac{1}{2},j}\right)\right)^2 - \left(\frac{1}{2}\left(u^n{}_{i+\frac{1}{2},j} + u^n_{i-\frac{1}{2},j}\right)\right)^2\right.$$
$$+\left(\frac{1}{2}\left(u^n{}_{i+\frac{1}{2},j} + u^n_{i-\frac{1}{2},j+1}\right)\right)\left(\frac{1}{2}\left(v^n{}_{i,j+\frac{1}{2}} + v^n_{i+1,j+\frac{1}{2}}\right)\right)$$
$$\left.-\left(\frac{1}{2}\left(u^n{}_{i+\frac{1}{2},j} + u^n_{i-\frac{1}{2},j-1}\right)\right)\left(\frac{1}{2}\left(v^n{}_{i,j-\frac{1}{2}} + v^n_{i+1,j-\frac{1}{2}}\right)\right)\right] h$$

In my code the squared terms are referred to as *'uu'* and the non squared terms are referred to as *'uv'*. A similar discretisation can be found for the y component. The pressure term can be approximated to give

$$\frac{1}{\rho}\oint_S p n_x dS \cong \frac{1}{\rho}\left(P_{i+1,j} - P_{i,j}\right)h$$

Moving on to the diffusion term $v\oint_S \nabla u\cdot\boldsymbol{n}dS$, to approach this we need to rewrite $\nabla u\cdot\boldsymbol{n}$ explicitly in terms of derivatives then approximate those derivatives.

$$v\oint_S \nabla u\cdot\boldsymbol{n}dS = v\oint_S \left(\frac{\partial u}{\partial x}n_x + \frac{\partial u}{\partial y}n_y\right)dS$$
$$\cong v\left[\left(\frac{\partial u}{\partial x}\right)^n_{i+1,j} - \left(\frac{\partial u}{\partial x}\right)^n_{i,j} + \left(\frac{\partial u}{\partial y}\right)^n_{i+\frac{1}{2},j+\frac{1}{2}} - \left(\frac{\partial u}{\partial y}\right)^n_{i+\frac{1}{2},j+\frac{1}{2}}\right] h$$

These derivatives can be approximated by looking at sequential terms in the grid, for example

$$\left(\frac{\partial u}{\partial x}\right)_{i,j}^{n} \cong \frac{u_{i+\frac{1}{2},j}^{n+1} - u_{i-\frac{1}{2},j}^{n}}{h}$$

Approximating each derivative this way then substituting into the previous equation gives

$$v\oint_{S} \nabla u \cdot \boldsymbol{n} dS \cong v\left(u^{n}{}_{i+\frac{3}{2},j} + u^{n}{}_{i-\frac{1}{2},j} + u^{n}{}_{i+\frac{1}{2},j-1} + u^{n}{}_{i+\frac{1}{2},j+1} - 4u^{n}{}_{i+\frac{1}{2},j}\right)$$

Putting all of these equations together and dividing by a factor of h^2 gives two fully discretised equations for conservation of momentum.

$$\frac{u^{n+1}{}_{i+\frac{1}{2},j} - u^{n}{}_{i+\frac{1}{2},j}}{\Delta t}$$

$$= -\frac{1}{h}\left[\left(\frac{1}{2}\left(u^{n}{}_{i+\frac{3}{2},j} + u_{i+\frac{1}{2},j}^{n}\right)\right)^{2} - \left(\frac{1}{2}\left(u^{n}{}_{i+\frac{1}{2},j} + u_{i-\frac{1}{2},j}^{n}\right)\right)^{2}\right.$$

$$+\left(\frac{1}{2}\left(u^{n}{}_{i+\frac{1}{2},j} + u_{i-\frac{1}{2},j+1}^{n}\right)\right)\left(\frac{1}{2}\left(v^{n}{}_{i,j+\frac{1}{2}} + v_{i+1,j+\frac{1}{2}}^{n}\right)\right)$$

$$\left.-\left(\frac{1}{2}\left(u^{n}{}_{i+\frac{1}{2},j} + u_{i+\frac{1}{2},j-1}^{n}\right)\right)\left(\frac{1}{2}\left(v^{n}{}_{i,j-\frac{1}{2}} + v_{i+1,j-\frac{1}{2}}^{n}\right)\right)\right]$$

$$-\frac{1}{h}\left(P_{i+1,j} - P_{i,j}\right) + \frac{v}{h^2}\left(u^{n}{}_{i+\frac{3}{2},j} + u^{n}{}_{i-\frac{1}{2},j} + u^{n}{}_{i+\frac{1}{2},j+1} + u^{n}{}_{i+\frac{1}{2},j-1} - 4u^{n}{}_{i+\frac{1}{2},j}\right)$$

For x momentum conservation and for y momentum conservation (top of next page)

$$\frac{{v^{n+1}}_{i,j+\frac{1}{2}} - {v^{n}}_{i,j+\frac{1}{2}}}{\Delta t}$$

$$= -\frac{1}{h}\Bigg[\left(\frac{1}{2}\left({v^n}_{i,j+\frac{3}{2}} + v^n_{i,j+\frac{1}{2}}\right)\right)^2 - \left(\frac{1}{2}\left({v^n}_{i,j+\frac{1}{2}} + v^n_{i,j-\frac{1}{2}}\right)\right)^2$$

$$+ \left(\frac{1}{2}\left({u^n}_{i+\frac{1}{2},j} + u^n_{i-\frac{1}{2},j+1}\right)\right)\left(\frac{1}{2}\left({v^n}_{i,j+\frac{1}{2}} + v^n_{i+1,j+\frac{1}{2}}\right)\right)$$

$$- \left(\frac{1}{2}\left({u^n}_{i+\frac{1}{2},j} + u^n_{i+\frac{1}{2},j-1}\right)\right)\left(\frac{1}{2}\left({v^n}_{i,j-\frac{1}{2}} + v^n_{i+1,j-\frac{1}{2}}\right)\right)\Bigg]$$

$$-\frac{1}{h}\left(P_{i,j+1} - P_{i,j}\right) + \frac{v}{h^2}\left({v^n}_{i,j+\frac{3}{2}} + {v^n}_{i-1,j+\frac{1}{2}} + {v^n}_{i+1,j+\frac{1}{2}} + {v^n}_{i+\frac{1}{2},j-\frac{1}{2}} - 4{v^n}_{i,j+\frac{1}{2}}\right)$$

The conservation of mass equation

$$u^{n+1}_{i+1/2,j} - u^{n+1}_{i-1/2,j} + v^{n+1}_{i,j+1/2} - v^{n+1}_{i,j-1/2} = 0$$

still holds.

The two conservation of momentum equations now need to be written in vector form so that we can find an equation to find the pressure. The vector equation is

$$\frac{\boldsymbol{u}^{n+1}_{i,j} - \boldsymbol{u}^{n+1}_{i,j}}{\Delta t} = -\boldsymbol{J}^n_{i,j} - \nabla P_{i,j} + \boldsymbol{B}^n_{i,j}$$

where $\boldsymbol{J}$ represents the advection terms and $\boldsymbol{B}$ represents the diffusion terms. I used a method I found called the projection method to find equations for the pressure. This method involves using a temporary velocity using only the advection and diffusion equations.

$$\frac{\boldsymbol{u}^{t}_{i,j} - \boldsymbol{u}^{n}_{i,j}}{\Delta t} = -\boldsymbol{J}^n_{i,j} + \boldsymbol{B}^n_{i,j}$$

and

$$\frac{\boldsymbol{u}_{i,j}^{n+1} - \boldsymbol{u}_{i,j}^{t}}{\Delta t} = -\nabla P_{i,j}$$

Using these two equations the pressure needed to make the flow incompressible i.e. $\nabla^2 P_{i,j} = \frac{1}{\Delta t}\nabla \cdot \boldsymbol{u}_{i,j}^{t}$. After this, the velocity is corrected by adding in the pressure term that was calculated. This lead to three equations for the pressure, one for point in on the grid that did not touch any boundaries, one for the points that touch one boundary and a third for the corner points that touch two boundaries. The pressure equations are as follows, for i=3,4,…,nx-1 and j=3,4,…,ny-1for a grid of size (nx,ny)

$$P_{i,j} = \frac{1}{4}\beta\left(\left(P_{i+1,j} + P_{i-1,j} + P_{i,j+1} + P_{i,j-1}\right) - \frac{h}{\Delta t}\left(u_{i,j}^{t} - u_{i-1,j}^{t} + v_{i,j}^{t} - v_{i,j-1}^{t}\right)\right) + (1-\beta)P_{i,j}$$

where β is a constant, 1.2 for water, and $u_{i,j}^{t}$ and $v_{i,j}^{t}$ are the temporary velocities. For boundary pressures i=2; i=nx j=2;j=ny

$$P_{i,j} = \frac{1}{3}\beta\left(\left(P_{i+1,j} + P_{i-1,j} + P_{i,j+1} + P_{i,j-1}\right) - \frac{h}{\Delta t}\left(u_{i,j}^{t} - u_{i-1,j}^{t} + v_{i,j}^{t} - v_{i,j-1}^{t}\right)\right) + (1-\beta)P_{i,j}$$

finally for corner pressures where i=2,j=2…i=2, j=ny…i=nx, j=2…and i=nx,j=ny

$$P_{i,j} = \frac{1}{2}\beta\left(\left(P_{i+1,j} + P_{i-1,j} + P_{i,j+1} + P_{i,j-1}\right) - \frac{h}{\Delta t}\left(u_{i,j}^{t} - u_{i-1,j}^{t} + v_{i,j}^{t} - v_{i,j-1}^{t}\right)\right) + (1-\beta)P_{i,j}$$

Now we have all of the equations needed to calculate and plot what we want! I will also calculate and plot the vorticity of the fluid as contour lines by calculating the 2-D curl of the velocity vector

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u}.$$

Results/What I've learnt

Plotting my results was slightly more difficult than I anticipated since the points on my grid did not line up with real space velocity positions. To combat this I averaged sequential velocity terms to give me the correct velocities in real space. An animation of the time evolution of the velocity and vorticity fields can be viewed on Imgur via this link https://imgur.com/l8YbS9J. The vorticity field is more intense when the contour lines are closer together and the colour is more red and yellow than blue. The pressure field surface plot at time t=10 seconds can be seen in figure 3 below. Pressure

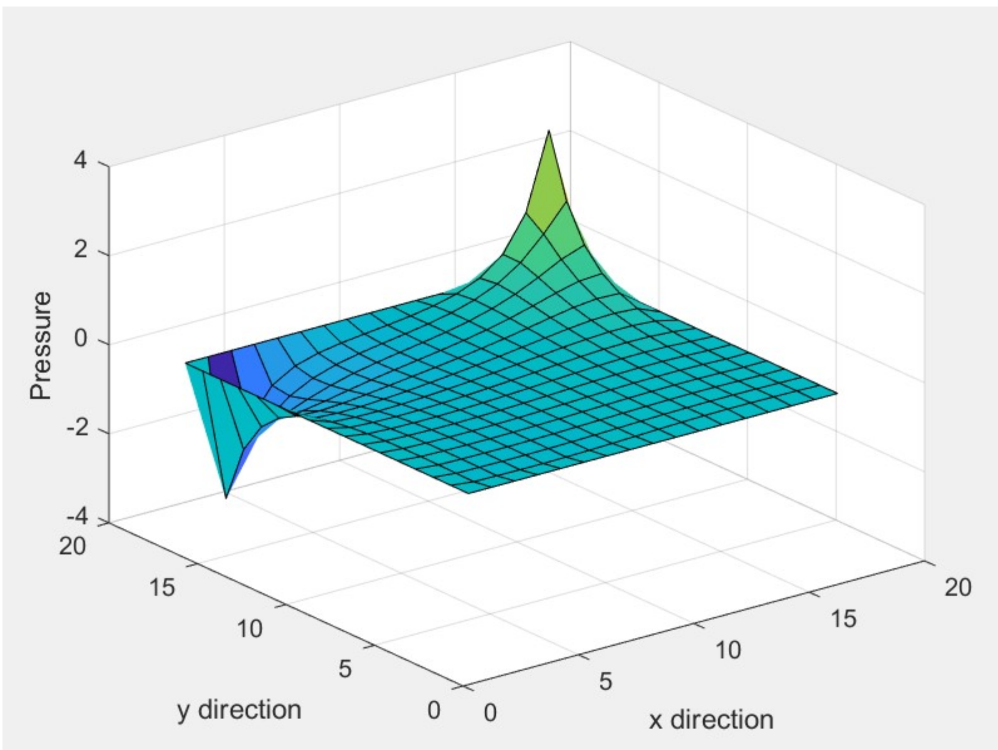

Figure 3: Pressure is highest in the top right where the flow bunches up and is lowest where the flow moves away from the top left corner.