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Topical Review

Stochastic resetting and applications

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Abstract

In this topical review we consider stochastic processes under resetting, which have attracted a lot of attention in recent years. We begin with the simple example of a diffusive particle whose position is reset randomly in time with a constant rate r, which corresponds to Poissonian resetting, to some fixed point (e.g. its initial position). This simple system already exhibits the main features of interest induced by resetting: (i) the system reaches a nontrivial nonequilibrium stationary state (ii) the mean time for the particle to reach a target is finite and has a minimum, optimal, value as a function of the resetting rate r. We then generalise to an arbitrary stochastic process (e.g. Lévy flights or fractional Brownian motion) and non-Poissonian resetting (e.g. power-law waiting time distribution for intervals between resetting events). We go on to discuss multiparticle systems as well as extended systems, such as fluctuating interfaces, under resetting. We also consider resetting with memory which implies resetting the process to some randomly selected previous time. Finally we give an overview of recent developments and applications in the field.

Keywords: stochastic resetting, non equilibrium stationary state, search optimisation

(Some figures may appear in colour only in the online journal)

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1. Introduction

1.1. Search processes

Sometimes it is best just to give up and start all over again! Imagine some mundane task such as locating one's keys in the morning. After a fruitless, haphazard search, which has taken one to areas far away from where the keys should normally be, it is perhaps best to go back to the starting point of the search and try again. Similarly, in visual search [1], where one tries to locate a face in a crowd, the eye typically flicks back to some chosen starting point after darting around in the vicinity of this point. In both cases one has a search process that entails a local and to a greater or lesser extent random search procedure interspersed with resetting or restart events.

Generally search processes are ubiquitous in nature and human behaviour [2, 3]: from the search for the holy grail and the Higgs boson all the way to animals searching for food [4, 5] and biomolecules searching for a binding site such as proteins on DNA [6–9]. Depending on the specific search problem there are different protocols, but what is common to these problems is to find an optimal search strategy. Different classes of search strategies have been identified, see e.g. [10–14] and prominent among them is *intermittent search* strategy wherein there is a mixture of local steps and long-range moves [15–17]. During the local step actual searching takes place whereas during the long relocation move the searcher moves but is not actively searching. Such strategies have been shown to be advantageous in a variety of contexts such as animal foraging and the target search of proteins on DNA molecules [18, 19].

1.2. From stochastic algorithms to chemical reactions

Another example of such an intermittent search strategy is realised in computer simulations of dynamics on complex (free) energy landscapes, such as in simulated annealing. Here one starts from some initial configuration and tries to locate the global minimum of the landscape. However at low temperature the system may get trapped in a metastable, local minimum for a long time. To speed up the search it has been observed empirically that it helps to halt the process and restart from the initial configuration, the rationale being that this allows the exploration of new pathways on the landscape. More generally the advantage of restarting has been exploited in various stochastic algorithms. The idea is that a stochastic algorithm may get stuck before completing the intended task and therefore it is advantageous to simply restart the algorithm [20–24]. Some variants of these problems have been studied in the probability [25, 26] and combinatorics [27] literature.

We also mention chemical reactions where it has been pointed out [28] that a complex chemical process to produce some product is much like a complex stochastic process which may benefit from restarting. In this case the restarting can be effected by the unbinding of an enzyme which forms the initial catalyst for the process.

1.3. Nonequilibrium states

Resetting a stochastic process is also of interest as a paradigm which stops a system attaining an equilibrium state (as it is continually returned to its initial condition). However the system will still attain a stationary state which will be off-equilibrium in nature, i.e. there will be probability currents in the system which would vanish if the system were allowed to relax to thermal equilibrium [29]. The resetting moves dynamically generate an effective potential which drives the system out of equilibrium.

1.4. Catastrophes in population dynamics

Perhaps the first instances of stochastic processes with resetting appeared in literature on birth–death processes in which there is an absorbing state (for example, when the population size reaches zero). In the case where the population is always absorbed, a restart process to a finite population size generates interesting stationary properties [30–32]. On the other hand if the population tends to increase exponentially in time, resetting to a finite initial population, from which there is a finite probability of absorption, renders the mean time to absorption finite [33, 34]. More generally one can consider the effect of catastrophes with a distribution of sizes on growing populations and study various stationary properties [35–38]. The same applies to queueing systems where catastrophes reset the length of the queue to zero [39–41].

1.5. Purpose of this review

In recent years there has been a surge in the study of stochastic processes subject to resetting (for general formulations see for example [42–69]—we refer the reader to [70] for an historical perspective). This is a very general problem as resetting to the initial condition can be applied to any stochastic process. The purpose of this review is to describe these developments in a pedagogical manner focussing on simple models and the derivation of quantitative analytical results.

We will begin by considering a single diffusing particle subject to reset in one or higher dimensions [43, 44, 46]. In this example we first show how a nontrivial nonequilibrium stationary state emerges. Then by introducing a target for the diffusing particle to search for, we show how the mean time for the particle to locate the target (the mean first passage time or mean time to absorption of the target) is minimised for an optimal choice of the resetting rate. These features turn out to be very general and hold for various other stochastic processes, which may correspond to extended, many-particle systems. We shall also explore various reset protocols, beginning with the simplest one which is Poissonian resetting (with a constant rate) to a fixed initial configuration. We then generalise to non-Poissonian resetting and resetting which uses memory of the past history. We also give an overview of recent extensions of the subject in various directions.

2. Single particle process

2.1. Diffusion with Poissonian resetting

First let us define diffusion with Poissonian resetting in one space dimension. We consider a single particle on the real line with initial position x_0 at $t = 0$ and resetting with rate r to position X_r. We stress here that the initial position x_0 and resetting position X_r are in general distinct, although at the end of some calculations it is convenient to set them to be equal.

The position $x(t)$ of the particle at time t is updated by the following stochastic rule [43]: in a small time interval dt the position $x(t)$ is updated to

$$x(t + dt) = X_r \quad \text{with probability } r\, dt$$

$$= x(t) + \xi(t)(dt)^{1/2} \quad \text{with probability } (1 - r\, dt) \tag{2.1}$$
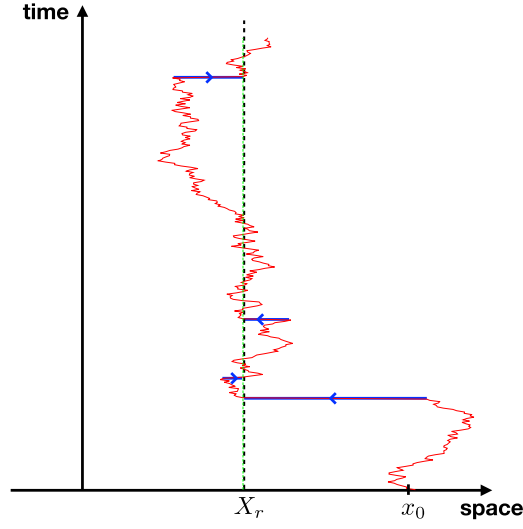
Figure 1. Illustration in $d = 1$ of the diffusion with resetting process: the particle starts at initial position x_0 and resets to position X_r with rate r.

where $\xi(t)$ is a Gaussian random variable, uncorrelated in time, with mean zero and variance given by

$$\langle \xi(t) \rangle = 0 \tag{2.2}$$

$$\langle \xi(t)^2 \rangle = 2D. \tag{2.3}$$

The dynamics thus consists of a stochastic mixture of resetting to the initial position with rate r (long range move) and ordinary diffusion (local move) with diffusion constant D (see figure 1).

The probability density for the particle to be at position x at time t, having started from position x_0 at time $t = 0$ with resetting to position X_r, should, in principle, be written as $p(x, t | x_0; X_r)$. However in the following, when the context is sufficiently clear, we shall frequently use $p(x, t | x_0)$ (omitting the dependence on X_r) or simply $p(x, t)$ (omitting the dependence on both x_0 and X_r).

The forward master equation for the probability density for diffusion with resetting rate r to point X_r is easily obtained from the update (2.1): averaging over events in time t to $t + dt$ we obtain

$$p(x, t + dt) = rdt\, \delta(x - X_r) + (1 - rdt) \int_{-\infty}^{\infty} \mathcal{D}\xi\, p(x - \xi(dt)^{1/2}, t), \tag{2.4}$$

where $\int_{-\infty}^{\infty} \mathcal{D}\xi$ denotes an integral over random variables ξ with a Gaussian distribution. Expanding in dt yields

$$p(x, t + dt) = rdt\, \delta(x - X_r) + (1 - rdt) \int_{-\infty}^{\infty} \mathcal{D}\xi$$

$$\times \left[p(x, t) - (dt)^{1/2} \xi \frac{\partial p(x, t)}{\partial x} + dt \frac{\xi^2}{2} \frac{\partial^2 p(x, t)}{\partial x^2} + \cdots \right].$$

Performing the integrals using (2.2) and (2.3) and taking the limit $\mathrm{d}t \to 0$ we obtain

$$\frac{\partial p(x,t)}{\partial t} = D\frac{\partial^2 p(x,t)}{\partial x^2} - rp(x,t) + r\delta(x - X_r), \tag{2.5}$$

with initial condition $p(x,0) = \delta(x - x_0)$. The first term on the right hand side (rhs) of (2.5) expresses the diffusive spread of probability; the second term expresses the loss of probability from x due to resetting to X_r; the final term corresponds to the gain of probability at X_r due to resetting from all other positions. We shall refer to (2.5) as the forward master equation.

In an analogous way one can derive the backward master equation in which the initial position x_0 is the variable, i.e. averaging over events in the interval $[0, \mathrm{d}t]$ yields

$$p(x, t + \mathrm{d}t|x_0) = r\mathrm{d}t\, p(x, t|X_r) + (1 - r\mathrm{d}t)\int_{-\infty}^{\infty} \mathcal{D}\xi\, p(x, t|x_0 + \xi(\mathrm{d}t)^{1/2}), \tag{2.6}$$

from which one obtains

$$\frac{\partial p(x, t|x_0)}{\partial t} = D\frac{\partial^2 p(x, t|x_0)}{\partial x_0^2} - rp(x, t|x_0) + rp(x, t|X_r). \tag{2.7}$$

Note that the gain term from resetting (i.e. the final term on rhs) now involves the probability density of reaching x at time t having started from the resetting position X_r.

2.2. Renewal equation approaches

Instead of beginning from these master equations one can write down renewal equations (which indeed give the solution to (2.5) and (2.7)) in a simple and intuitive way as follows.

We first note that in the absence of resetting ($r = 0$), the diffusive Green function (also known as the propagator for the diffusion equation) which we denote $G_0(x, t|x_0)$, satisfies

$$\frac{\partial G_0(x, t|x_0)}{\partial t} = D\frac{\partial^2 G_0(x, t|x_0)}{\partial x^2}, \tag{2.8}$$

with initial condition $G_0(x, t = 0|x_0) = \delta(x - x_0)$, and is given by the familiar Gaussian expression

$$G_0(x, t|x_0) = \frac{1}{(4\pi Dt)^{1/2}} \exp\left[-\frac{|x - x_0|^2}{4Dt}\right]. \tag{2.9}$$

The probability density in the presence of resetting, $p(x, t|x_0)$, is a sum over two contributions: one which comes from trajectories where no resetting events have occurred in time t and a second contribution which comes from summing over trajectories where the *last* resetting event occurred at time $\tau_1 = t - \tau$ (see figure 2). For Poissonian resetting (with constant rate r), the probability of no resetting events having occurred up to time t is e^{-rt} and the probability density of the last resetting event having occurred at $\tau_1 = t - \tau$ (and no resetting events since) is $r\mathrm{e}^{-r\tau}$. Thus the full time-dependent solution to (2.5) can be written down as

$$p(x, t|x_0) = \mathrm{e}^{-rt}G_0(x, t|x_0) + r\int_0^t \mathrm{d}\tau\, \mathrm{e}^{-r\tau}G_0(x, \tau|X_r). \tag{2.10}$$

We refer to this equation as a *last renewal* equation as it involves the time of the last reset $\tau_1 = t - \tau$. Note that this renewal equation holds for more general stochastic processes, with
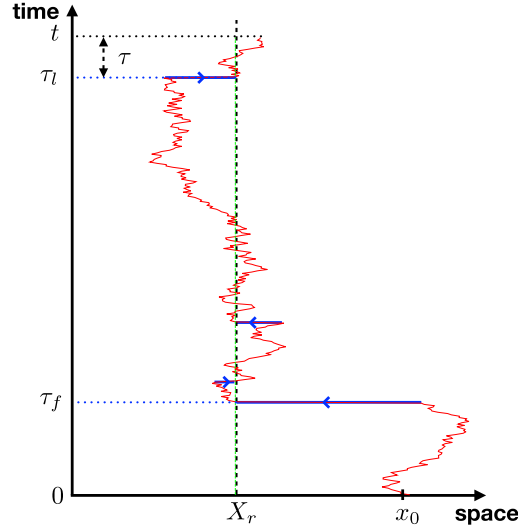
Figure 2. Same trajectory as in figure 1 of 1D-diffusion with resetting: here $\tau_1 = t - \tau$ and τ_f denote respectively the time at which the last and first resetting events happen.

propagator denoted by $G_0(x, \tau | x_0)$, which can be different from the diffusive case we have considered so far.

We will also consider *first renewal* equations where instead of the last resetting, we consider the *first resetting* at time τ_f having started from $t = 0$ (see figure 2). Subsequently, the particle diffuses from τ_f until time t, under resetting. It is again straightforward to write down an equation for the probability:

$$p(x, t | x_0) = e^{-rt} G_0(x, t | x_0) + r \int_0^t d\tau_f\, e^{-r\tau_f} p(x, t - \tau_f | X_r), \tag{2.11}$$

where the second term now integrates over trajectories in which there has been a first reset to X_r between time τ_f and $\tau_f + d\tau_f$ and then there can be multiple resets in the remaining time $t - \tau_f$ which is why $p(x, t - \tau_f | X_r)$ now appears inside the integral.

The equivalence between (2.10) and (2.11) may be shown by taking Laplace transforms of both equations (see appendix A). For the time being we note that the Laplace transform of the solution to (2.10) is given by

$$\tilde{p}(x, s | x_0) = \tilde{G}_0(x, r + s | x_0) + \frac{r}{s} \tilde{G}_0(x, r + s | X_r), \tag{2.12}$$

where

$$\tilde{p}(x, s | x_0) = \int_0^\infty dt\, e^{-st}\, p(x, t | x_0) \tag{2.13}$$

is the Laplace transform of $p(x, t | x_0)$ and similarly $\tilde{G}_0(x, s | x_0)$ is the Laplace transform of $G_0(x, t | x_0)$.

2.3. Nonequilibrium stationary state

The stationary state is attained as $t \to \infty$ where (2.10) tends to the stationary distribution

$$p^*(x) = r \int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{-r\tau} G_0(x, \tau | X_r), \tag{2.14}$$

thus the stationary distribution under resetting is related to the Laplace transform (with Laplace variable r) of the propagator in the absence of resetting. This is actually a generic property, valid for more general processes with a propagator $G_0(x, \tau | x_0)$, when resetting is Poissonian.

In order to evaluate the integral (2.14) in the case of the diffusive propagator (2.9) we use the identity (equation 3.471.9 of [71])

$$\int_0^\infty \mathrm{d}t\, t^{\nu-1}\, \mathrm{e}^{-\frac{\beta}{t} - \gamma t} = 2\left(\frac{\beta}{\gamma}\right)^{\nu/2} K_\nu(2\sqrt{\beta\gamma}), \tag{2.15}$$

where K_ν is the modified Bessel function of the second kind of order ν. The relevant case of this identity here is $\nu = 1/2$ and using the definition

$$K_{1/2}(y) = \left(\frac{\pi}{2y}\right)^{1/2} \mathrm{e}^{-y}, \tag{2.16}$$

equation (2.15) becomes

$$\int_0^\infty \mathrm{d}t\, t^{-1/2}\, \mathrm{e}^{-\frac{\beta}{t} - \gamma t} = \left(\frac{\pi}{\gamma}\right)^{1/2} \mathrm{e}^{-2(\beta\gamma)^{1/2}}. \tag{2.17}$$

Then one obtains from (2.9) (with $x_0 = X_r$) and (2.14),

$$p^*(x) = \frac{\alpha_0}{2}\, \mathrm{e}^{-\alpha_0 |x - X_r|} \tag{2.18}$$

where

$$\alpha_0 = \left(\frac{r}{D}\right)^{1/2}. \tag{2.19}$$

Of course, we can check directly that (2.18) satisfies (2.5) with the left hand side (lhs) set to zero by using the identity

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\, \mathrm{e}^{-\alpha_0 |x - X_r|} = \alpha_0^2\, \mathrm{e}^{-\alpha_0 |x - X_r|} - 2\alpha_0 \delta(x - X_r). \tag{2.20}$$

The first thing to note is that the stationary distribution in the presence of resetting (2.18) exhibits exponential decay away from the resetting position X_r in both the $x - X_r > 0$ and $x - X_r < 0$ directions. The double exponential decay is known as a Laplace distribution. Thus the distribution is localised around X_r over a length $1/\alpha_0$ and there is a cusp singularity at $x = X_r$ (see figure 3).

Also note that (2.18) is a nonequilibrium stationary state (NESS) by which it is meant that there is circulation of probability, in contrast to an equilibrium state where detailed balance holds and probability currents vanish. This is because resetting implies a source of probability at X_r while probability is lost through resetting from all other values of $x \neq X_r$.
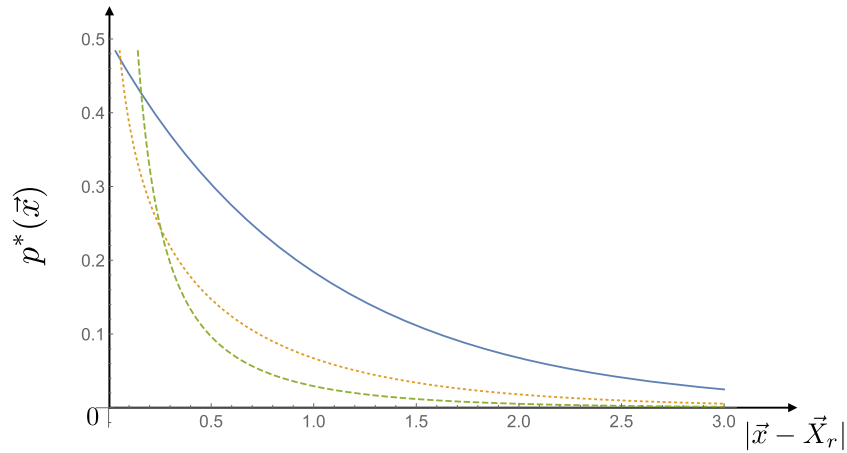
Figure 3. The stationary probability densities $p^*(\vec{x})$ given by (2.33) for the case $\alpha_0 = 1$ and $\vec{X}_r = 0$: $d = 1$ full lines; $d = 2$ dotted lines; $d = 3$ dashed lines.

2.4. Diffusion with resetting in potentials

As an illustration of the utility of (2.14) one can consider a diffusive particle with a constant drift μ in the positive x direction under Poissonian resetting with rate r. This corresponds to an unbounded linear potential. For this case the Green function in the absence of resetting is

$$G_0(x, t|x_0) = \frac{1}{(4\pi Dt)^{1/2}} \, e^{-\frac{|x-x_0-\mu t|^2}{4Dt}}, \qquad (2.21)$$

and one finds the stationary state under resetting, using (2.14) and (2.17), to be

$$p^*(x) = \frac{r}{(4Dr + \mu^2)^{1/2}} \, e^{\frac{(x-X_r)\mu}{2D} - \frac{(4Dr+\mu^2)^{1/2}}{D}|x-X_r|}. \qquad (2.22)$$

Here the stationary distribution is asymmetric about the resetting position X_r with different exponential decays in the downstream ($x > X_r$) and upstream ($x < X_r$) directions. This case has been studied in detail in [72] and the Péclet number $\text{Pe} = X_r\mu/(2D)$ identified as a key governing dimensionless variable.

We also note that (2.14) allows one to write down the stationary state under Poissonian resetting for various confining potentials (linear or quadratic) or unstable potentials [73]. One just requires the knowledge of G_0, the propagator in each case.

2.5. Relaxation to stationary state for diffusion with resetting

In addition to knowing the stationary state, it is also important to understand how the system relaxes to this state. In order to investigate this relaxation, we start with the exact solution in equation (2.10), valid at all time t, and analyse it for large but finite t [50]. For simplicity, we will set $X_r = x_0$, i.e. we reset the particle to its initial position. It is further convenient to rescale the time $\tau = wt$ and rewrite (2.10) as

$$p(x, t) = \frac{e^{-t\Phi(1, (x-X_r)/t)}}{\sqrt{4\pi Dt}} + \frac{rt^{1/2}}{\sqrt{4\pi D}} \int_0^1 \frac{dw}{w^{1/2}} \, e^{-t\Phi(w, (x-X_r)/t)} \qquad (2.23)$$
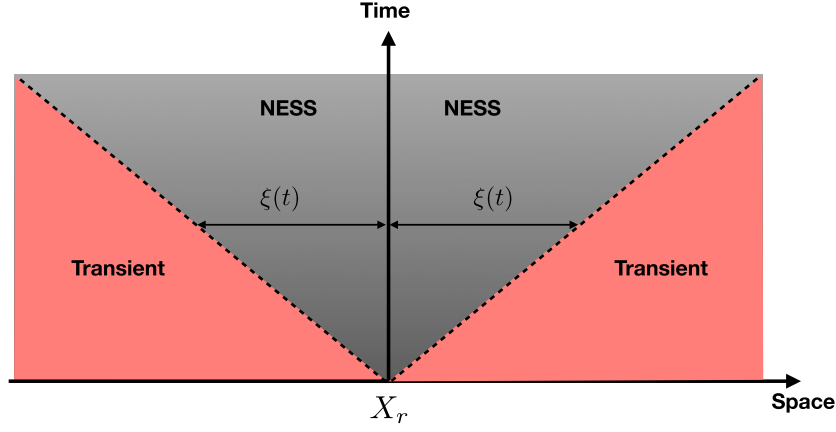
Figure 4. A NESS gets established in a core region around the resetting centre X_r whose frontiers $\xi(t)$ grow with time as $\xi(t) \propto t$. Outside the core region, the system is transient.

where we have defined

$$\Phi(w, y) = rw + \frac{y^2}{4Dw}. \tag{2.24}$$

For large t the integral in the second term in (2.23) can be analysed by the saddle-point method. We keep $y = (x - X_r)/t$ fixed and take the $t \to \infty$ limit. The saddle point of this integral, if it exists, occurs at

$$w^* = \frac{|y|}{\sqrt{4Dr}}, \tag{2.25}$$

which minimises the function $\Phi(w, y)$, for fixed y. If $w^* < 1$, the saddle point occurs within the integration limits $w \in [0, 1]$ and one gets, from (2.23) $p(x, t) \sim e^{-t\,\Phi(w^*, (x-X_r)/t)}$ for large t, where $\Phi(w^*, y) = \alpha_0|y|$ where α_0 is given by (2.19). In contrast, for $w^* > 1$, the function $\Phi(w, y)$ has its lowest value in $w \in [0, 1]$ at $w = 1$. Hence the integrand in the second term is dominated by the regime at $w = 1$ (and is of the same order as the first term). Physically, this corresponds to trajectories which have undergone zero (or almost zero) resettings up to time t. One then gets $p(x, t) \sim e^{-t\,\Phi(1, (x-X_r)/t)}$, with $\Phi(1, y) = r + y^2/(4D)$. Summarising, we obtain

$$p(x, t) \sim e^{-tI\left((x-X_r)/t\right)}, \tag{2.26a}$$

where the function $I(y)$ is called the rate function or the large deviation function (LDF). In this case, it is given by

$$I(y) = \begin{cases} \alpha\,|y| & \text{for } |y| < y^*, \\ r + \dfrac{y^2}{4D} & \text{for } |y| > y^*, \end{cases} \tag{2.26b}$$

with $y^* = \sqrt{4Dr}$.

The appearance of the factor $(x - X_r)/t$ as the argument of the rate function I in (2.23) indicates that there is a growing length scale $\xi(t) \sim t$, much larger than the typical diffusion length scale $\sim \sqrt{t}$. The linearity of the LDF for $|y| < y^*$ implies that, for any large but finite t,

there is an interior spatial region $-y^*t < x - X_r < y^*t$, where the NESS has been achieved, since $p^*(x, t) \sim \exp(-\alpha|x - X_r|)$ becomes independent of t, in agreement with (2.23). However, there is still an exterior region $|x - X_r| > y^*t$ that has not yet relaxed to the NESS (see figure 4). The boundaries between the two regions move at a constant speed y^*. From (2.23), it is easy to check that while $I(y)$ and its first derivative are both continuous at $y = \pm y^*$, its second derivative has a discontinuity at $y = \pm y^*$. This signifies a second order dynamical phase transition.

What is the physical significance of this phase transition? The probability density $p(x, t)$ can also be interpreted as the density at time t of a swarm of independent Brownian motions, each subjected to stochastic resetting with rate r, all starting from the origin at $t = 0$. Our calculation shows that at time t the density for $|x| < y^*t$ becomes stationary, while is still time dependent for $|x| > y^*t$. From the analysis above, it is clear that, for $|x| > y^*t$, the density is typically of the form $\sim e^{-rt}G_0(x, t)$ in (2.23), i.e., it corresponds to particles that have undergone almost no resetting up to time t. This is of course a very rare event and these particles in the outer region thus have very atypical trajectories. In contrast, the particles in the inner core region correspond to typical trajectories that have undergone a large number of resettings—leading to a stationary behaviour in this regime. The LDF $I(y)$ in (2.26a) probes precisely the separation between these two regions, i.e., between the typical and the atypical trajectories; the singularity in the LDF signifies a sharp separation between these two types of particles.

In any typical application of resetting, for instance in the optimisation of search algorithms, we would ideally like to keep, at any given finite time t, only the typical trajectories and not the atypical ones—since the latter ones do not feel the resetting at all. The LDF $I(y)$ and its associated singularity, that sharply separates the two types of trajectories, thus provides a very useful and practical way to select the typical ones at any given time t. Even though we discuss it here in the context of a single particle diffusion, it turns out that this physical picture associated with the second order dynamical phase transition is quite generic [50] and holds for arbitrary stochastic processes undergoing resetting and even for spatially extended systems, such as fluctuating interfaces [74] that we discuss later.

2.6. Diffusion with resetting in arbitrary spatial dimension

It is straightforward to generalise the formalism of sections 2.1–2.5 to diffusion with resetting in arbitrary spatial dimension [46]. The particle now moves in $\mathbb{R}^d$ with initial position $\vec{x}_0$ at $t = 0$ and resetting to position $\vec{X}_r$. In a small time interval dt each component x_i of the position vector $\vec{x}(t)$ becomes

$$x_i(t + dt) = (X_r)_i \quad \text{with probability } r\, dt$$

$$= x_i(t) + \xi_i(t)(dt)^{1/2} \quad \text{with probability } (1 - r\, dt) \tag{2.27}$$

where $\xi_i(t)$ is a Gaussian random variable, uncorrelated in time, with mean $\langle \xi_i(t) \rangle = 0$ and covariance $\langle \xi_i(t)\xi_j(t) \rangle = 2D\delta_{ij}$. The forward master equation for the probability density for diffusion with resetting rate r to point $\vec{X}_r$ now reads

$$\frac{\partial p(\vec{x}, t)}{\partial t} = D\nabla^2 p(\vec{x}, t) - rp(\vec{x}, t) + r\delta^d(\vec{x} - \vec{X}_r), \tag{2.28}$$

with initial condition $p(\vec{x}, 0) = \delta^d(\vec{x} - \vec{x}_0)$, where $\delta^d(\vec{x} - \vec{x}_0)$ is the d-dimensional Dirac delta function centred on $\vec{x}_0$.

As before, we can write down a last renewal equation, which is the solution to (2.28), as

$$p(\vec{x}, t) = e^{-rt} G_0(\vec{x}, t | \vec{x}_0) + r \int_0^t d\tau \, e^{-r\tau} G_0(\vec{x}, \tau | \vec{X}_r), \tag{2.29}$$

where the d-dimensional diffusive propagator is now

$$G_0(\vec{x}, t | \vec{x}_0) = \frac{1}{(4\pi Dt)^{d/2}} \exp\left[-\frac{|\vec{x} - \vec{x}_0|^2}{4Dt} \right]. \tag{2.30}$$

The stationary distribution for the resetting problem is again related to the Laplace transform of the propagator in the absence of resetting

$$p^*(\vec{x}) = r \int_0^\infty d\tau \, e^{-r\tau} G_0(\vec{x}, \tau | \vec{X}_r). \tag{2.31}$$

The integral in (2.31) may be evaluated using (2.15) where the relevant case is now

$$\nu = 1 - d/2, \tag{2.32}$$

and one obtains from (2.30) and (2.31)

$$p^*(\vec{x}) = \left(\frac{\alpha_0^2}{2\pi} \right)^{1-\nu} (\alpha_0 |\vec{x} - \vec{X}_r|)^\nu K_\nu(\alpha_0 |\vec{x} - \vec{X}_r|), \tag{2.33}$$

where α_0 is, as before, given by (2.19).

Expression (2.33) holds for arbitrary d and one can continue it to noninteger d. Of course the cases of integer d are of special interest (see figure 3 for a plot). For $d = 1$, one recovers the result given before in (2.18), which has a cusp singularity at the resetting point $x = X_r$. We note that for $d = 2$ the singularity at X_r becomes logarithmic. For $d = 3$ one can use the identity $K_{-1/2}(y) = K_{1/2}(y) = \left(\frac{\pi}{2y} \right)^{1/2} e^{-y}$ to find a simple form

$$p^*(\vec{x}) = \frac{\alpha_0^2}{4\pi |\vec{x} - \vec{X}_r|} \exp(-\alpha_0 |\vec{x} - \vec{X}_r|). \tag{2.34}$$

In general, using the asymptotic behaviour $K_\nu(r) \sim r^{-|\nu|}$ as $r \to 0$, one finds that near the resetting position $\vec{X}_r$, the stationary PDF behaves as

$$p^*(\vec{x}) \sim \begin{cases} O(1), & d < 2 \\ -\ln(|\vec{x} - \vec{X}_r|), & d = 2 \\ |\vec{x} - \vec{X}_r|^{-(d-2)}, & d > 2. \end{cases} \tag{2.35}$$

Thus, in $d \geqslant 2$ the stationary PDF $p^*(\vec{x})$ diverges at the resetting position $\vec{X}_r$ and the divergence gets stronger as the dimension increases. Note that, despite the singularity at the resetting point X_r, $p^*(\vec{x})$ remains integrable (and normalisable to unity) because in the integral $\int d\vec{x} \, p^*(\vec{x})$, after making the change of variable $\vec{x}' = \vec{x} - \vec{X}_r$, the divergence of $p^*(\vec{x} - \vec{X}_r)$ at the origin gets compensated by the volume factor $\propto |\vec{x}'|^{d-1}$.

2.7. Resetting distribution and spatially dependent resetting

We now consider some simple generalisations of the resetting dynamics. First let us consider resetting to a distribution of sites rather than to a single preordained site. We define a *resetting distribution* $\mathcal{P}_r(X_r)$ for the resetting process such that the process is reset to $X_r + \mathrm{d}X_r$ with probability $\mathcal{P}(X_r)\mathrm{d}X_r$. Then the renewal equation for the probability distribution of the process (2.5) becomes

$$p(x, t|x_0) = \mathrm{e}^{-rt}G_0(x, t|x_0) + r\int_0^t \mathrm{d}\tau\, \mathrm{e}^{-r\tau}\int \mathrm{d}X_r\mathcal{P}_r(X_r)G_0(x, \tau|X_r). \qquad (2.36)$$

In the long time limit we find that the stationary distribution is given by

$$p^*(x) = \int \mathrm{d}X_r\, \mathcal{P}_r(X_r)\, p^*(x|X_r), \qquad (2.37)$$

where here $p^*(x|X_r)$ is the stationary distribution with reset to fixed position X_r. This equation is intuitively obvious: the stationary state is just that of resetting to a fixed position, averaged over the resetting position distribution. For the case of a finite number N of resetting positions $X_{r_i}\, i = 1, \ldots, N$, each chosen at a resetting event with probability $\mathcal{P}_i$, one has

$$p^*(x) = \sum_{i=1}^n \mathcal{P}_i\, p^*(x|X_{r_i}). \qquad (2.38)$$

We now turn to a space-dependent resetting rate $r(x)$: the particle at position x at time t is reset in time t to $t + \mathrm{d}t$ with probability $r(x)\mathrm{d}t$. In this case, the simplest is to use the forward master equation, i.e. a generalization of (2.7), which for the case of a one-dimensional diffusive process reads

$$\frac{\partial p(x, t)}{\partial t} = D\frac{\partial^2 p(x, t)}{\partial x^2} - r(x)p(x, t) + \int \mathrm{d}x'r(x')p(x', t)\delta(x - X_r). \qquad (2.39)$$

Although it appears difficult to solve this equation generally, a specific case of resetting outside of a window ($r(x) = 0$ for $|x| < a$ and $r(x) = r$ for $|x| > a$) has been studied in [44]. Also in [75] the case of $r(x)$ decaying with x has been considered and the conditions for which a stationary state exists have been derived. A general path integral approach to the space-dependent resetting problem has been developed in [76].

2.8. Non-Poissonian resetting

So far we have considered resetting to occur at constant rate r which we refer to as Poissonian resetting. More generally one can define the resetting process through the waiting time distribution $\psi(t)$ between resetting events [54–56, 77], i.e. after a reset the next reset occurs in time interval $(t, t + \mathrm{d}t]$ with probability $\psi(t)\mathrm{d}t$. The probability, $\Psi(t)$, of no resets up to time t is given by

$$\Psi(t) = \int_t^\infty \mathrm{d}t'\psi(t') = 1 - \int_0^t \mathrm{d}t'\psi(t'). \qquad (2.40)$$

For Poissonian resetting (constant r) one obtains as before $\psi(t) = r\mathrm{e}^{-rt}$ and $\Psi(t) = \mathrm{e}^{-rt}$. One realisation of non-Poissonian resetting is to have a time-dependent resetting rate, $r(t)$, then $\psi(t) = r(t)\mathrm{e}^{-R(t)}$ where $R(t) = \int_0^t r(t')\mathrm{d}t'$ and $\Psi(t) = \mathrm{e}^{-R(t)}$ [55]. A time-dependent rate is

often referred to as a *time-inhomogeneous* Poisson process. However we stress that here the resetting rate is itself reset, so that $r(t)$ depends on the time t since the last reset rather than absolute time from the initial condition. The latter scenario would be strongly non-Markovian in nature as discussed in [78]. However, here we consider the scenario where the whole history of the process is reset. This means that when a reset happens, the system no longer remembers what happened before resetting. Thus the process is still Markovian. Non-Poissonian simply means, in this context, that the waiting time distribution $\psi(t)$ is different from a pure exponential as in Poissonian resetting.

For non-Poissonian resetting it is more difficult to write down a forward master equation analogous to (2.5) as one must in addition keep track of the time since the last reset. This results in a generalised master equation [54]. Here, we use the renewal approach (see e.g. [32, 55, 59–61, 68, 69, 74]) which we now review.

In the case of time-dependent resetting, one can again exploit the renewal structure of the process in a simple and straightforward way. We consider a time interval $[0, t]$ and the particle starts initially at x_0. We want to compute the probability distribution $p(x, t|x_0)$ in the presence of resetting. In this time interval $[0, t]$ there can be no resetting events, one resetting, two resettings, etc. Consider for example the case of no resetting. The probability for this event is simply $\Psi(t)$ and hence the contribution to the probability distribution representing no resetting in $[0, t]$ is therefore $\Psi(t)G_0(x, t|x_0)$ where $G_0(x, t|x_0)$ is the bare propagator. If there is one resetting event, say at time $t_1 \in [0, t]$, the contribution to the probability is given by

$$\int_0^t dt_1 \psi(t_1)\Psi(t - t_1)G_0(x, t - t_1|X_r), \tag{2.41}$$

where $\psi(t_1)dt_1$ is the probability that a reset event happens in $(t_1, t_1 + dt_1]$, followed by no resetting in the interval $(t_1, t]$ during which the particle propagates freely. Similarly, if there are two resetting events, the contribution to the probability is

$$\int_0^t dt_1 \int_0^{t-t_1} dt_2 \, \psi(t_1)\psi(t_2)\Psi(t - t_1 - t_2)G_0(x, t - t_1 - t_2|X_r). \tag{2.42}$$

The same pattern holds for n resetting events and we need to sum over all $n \geqslant 1$. The convolution structure of these terms suggests that it is simpler to work in the Laplace space. Taking the Laplace transform and summing over all resetting events, using the geometric series, one immediately obtains

$$\tilde{p}(x, s|x_0) = \int_0^\infty dt \, e^{-st} \Psi(t)G_0(x, t|x_0) + \frac{\tilde{\psi}(s)}{1 - \tilde{\psi}(s)}\int_0^\infty dt \, e^{-st} \Psi(t)G_0(x, t|X_r). \tag{2.43}$$

If we now set $X_r = x_0$, a simplification occurs and one gets

$$\tilde{p}(x, s|x_0) = \frac{1}{s\tilde{\Psi}(s)}\int_0^\infty dt \, e^{-st} \Psi(t)G_0(x, t|x_0), \tag{2.44}$$

where we used the relation $\tilde{\Psi}(s) = (1 - \tilde{\psi}(s))/s$. Here we denote by $\tilde{\Psi}(s)$ and $\tilde{\psi}(s)$ the Laplace transform of $\Psi(t)$ and $\psi(t)$ respectively. Note that one can also obtain the result in (2.44) just by renewing the process after the first resetting (we referred to this as the first renewal equation

in section 2.2)

$$p(x, t|x_0) = \Psi(t)G_0(x, t|x_0) + \int_0^t d\tau \psi(\tau) p(x, t - \tau|X_r). \qquad (2.45)$$

Taking the Laplace transform of this equation, upon setting $X_r = x_0$, and using the relation $\tilde{\Psi}(s) = [1 - \tilde{\psi}(s)]/s$, one recovers (2.44).

Also one can work from the last renewal equation

$$p(x, t|x_0) = \Psi(t)G_0(x, t|x_0) + \int_0^t d\tau_1 \Upsilon(\tau_1)\Psi(t - \tau_1)G_0(x, t - \tau_1|X_r), \qquad (2.46)$$

where $\Upsilon(\tau_1)$ is the probability density for a reset to occur in $(\tau_1, \tau_1 + d\tau_1]$ (without specifying when a previous reset occurred) and $\Psi(t - \tau_1)$ is the probability that there are no further resets after this. The distribution $\Upsilon(\tau)$ is implied by $\psi(\tau)$ but is difficult to write down in closed form. However in the Laplace domain it is simply given by $\tilde{\Upsilon}(s) = \tilde{\psi}(s)/(1 - \tilde{\psi}(s))$ and the Laplace transform of (2.46) recovers (2.43).

A stationary state will only exist as $t \to \infty$ in the case when $\psi(\tau)$ decays to zero quickly enough. The stationary state is given by the coefficient of $1/s$ in (2.44) in the limit $s \to 0$ thus

$$p(x, t \to \infty|x_0, X_r = x_0) \to p^*(x|x_0, X_r = x_0) = \frac{\int_0^\infty dt \Psi(t)G_0(x, t|x_0)}{\int_0^\infty dt \Psi(t)} \qquad (2.47)$$

provided the limit exists. A sufficient condition for this is

$$\int_0^\infty dt \, \Psi(t) < \infty. \qquad (2.48)$$

This condition implies that the waiting time distribution $\psi(t)$ should decay to zero more quickly than $1/t^2$. In the case where $\psi(t)$ decays slower than $1/t^2$ then the system does not reach any stationary state [56].

We end up this section by mentioning that the case of a resetting rate that depends on absolute time t elapsed from the initial condition, rather than time since last reset, was considered in [78].

2.9. Discrete time random walks and Lévy flights with resetting

Up to now we have considered continuous time stochastic processes with resetting. However, in some cases, it is relevant to consider discrete time processes. This might be the case, for instance, when studying animal movements which typically consist of discrete jumps. The simplest example of such processes is the discrete time random walk (RW) subject to resetting.

We thus consider a random walker on a line, starting from x_0 and evolving according to the following rules [47]

$$x_n = \begin{cases} X_r & \text{with probability } r \\ x_{n-1} + \eta_n & \text{with probability } 1 - r, \end{cases} \qquad (2.49)$$

where r denotes here the probability (and not a probability rate) of a resetting event, and hence $0 < r < 1$. In (2.49) the jumps η_n's are independent and identically distributed (i.i.d.) random variables each drawn from a probability distribution function (PDF) $f(\eta)$. Here we will restrict our attention to the case where $f(\eta)$ is continuous and symmetric. We may consider ordinary

random walks, corresponding to jump distributions $f(\eta)$ with a well defined second moment $\sigma^2 = \int_{-\infty}^{+\infty} \eta^2 f(\eta) \, d\eta$ (and in that case the RW converges for large n to Brownian motion), as well as Lévy flights, corresponding to heavy-tailed jump distribution $f(\eta) \sim |\eta|^{-1-\mu}$ with $0 < \mu < 2$. The tail behaviour of $f(\eta)$ is encoded in the small k behaviour of the Fourier transform $\hat{f}(k) = \int_{-\infty}^{+\infty} f(\eta) \, e^{-ik\eta} \, d\eta$ of the jump distribution,

$$\hat{f}(k) = 1 - |a\,k|^\mu + o(|k|^\mu), \quad 0 < \mu \leqslant 2, \tag{2.50}$$

where a sets the characteristic scale of the jumps and μ is called the Lévy index. The case $\mu = 2$ thus corresponds to ordinary random walk while $0 < \mu < 2$ corresponds to Lévy flights.

We denote by $p(x, n|x_0, X_r)$ the probability density to find the particle at x at step n, starting from x_0 with resetting to position X_r. As before, we will use the shorthand notations $p(x, n|x_0)$ or even simply $p(x, n)$ where there is no ambiguity. From the evolution (2.49), it is straightforward to derive a forward master equation for $p(x, n|x_0, X_r)$. It reads

$$p(x, n) = (1 - r) \int_{-\infty}^{+\infty} p(x - \eta, n - 1) f(\eta) \, d\eta + r\delta(x - X_r), \tag{2.51}$$

starting from the initial condition $p(x, 0) = \delta(x - x_0)$. This equation (2.51) is the discrete time counterpart of the continuous time forward equation derived in (2.5).

This forward equation (2.51) can be solved via the use of Fourier transform. If one denotes by $\hat{p}(q, n) = \int_{-\infty}^{+\infty} e^{iqx} p(x, n) \, dx$ the Fourier transform of $p(x, n)$ with respect to x, one obtains from (2.51) that it satisfies the equation

$$\hat{p}(q, n) = (1 - r)\hat{f}(q)\,\hat{p}(q, n - 1) + r\,e^{iqX_r}, \tag{2.52}$$

starting from $\hat{p}(q, 0) = e^{iqx_0}$. This recurrence equation (2.52) can be easily solved with the result

$$\hat{p}(q, n) = (1 - r)^n \left[\hat{f}(q)\right]^n \left(e^{iqx_0} - \frac{r\,e^{iqX_r}}{1 - (1 - r)\hat{f}(q)}\right) + \frac{r\,e^{iqX_r}}{1 - (1 - r)\hat{f}(q)}. \tag{2.53}$$

Let us focus here on the limit $n \to \infty$ of expression (2.53). Clearly, since $|\hat{f}(q)| \leqslant \int_{-\infty}^{+\infty} f(\eta) \, d\eta = 1$, one has $|(1 - r)\hat{f}(q)| < 1$ and therefore in the limit $n \to \infty$ the only term that remains in (2.53) is the last one. Hence, one finds that $p(x, n)$ reaches a stationary distribution which is given by the inverse Fourier transform of the last term in (2.53), independently of x_0,

$$p(x, n) \underset{n \to \infty}{\to} p^*(x) = \int_{-\infty}^{+\infty} \frac{dq}{2\pi} \, e^{-iq(x - X_r)} \hat{p}^*(q), \tag{2.54}$$

$$\text{where } \hat{p}^*(q) = \frac{r}{1 - (1 - r)\hat{f}(q)}. \tag{2.55}$$

For an arbitrary jump distribution $f(\eta)$, it is very hard to compute explicitly $p^*(x)$ from this formula (2.54) for all x (one exception being the double exponential jump distribution, see below). However, from (2.54) one can rather easily extract the large x behaviour of the stationary distribution $p^*(x)$, which turns out to be very different for the two cases $\mu = 2$ and $0 < \mu < 2$.

The case $\mu = 2$. It is instructive to study the case of a double exponential jump distribution $f(\eta) = 1/(2a)e^{-|x|/a}$, for which the integral over q in (2.54) can be performed explicitly and one finds,

$$p^*(x) = r\delta(x - X_r) + (1 - r)\frac{\sqrt{r}}{2a}\,e^{-\sqrt{r}\,\frac{|x - X_r|}{a}}. \tag{2.56}$$

In this case $\hat{f}(q) = 1/(1 + (aq)^2) \approx 1 - (aq)^2$ for small q (and hence indeed $\mu = 2$ from (2.50)) which is very similar to the stationary state found for continuous time diffusion (2.18), apart from the term $r\delta(x - X_r)$ which exists only in the case of discrete time RW. (We note that the delta-peak at the resetting position is a generic feature in discrete time resetting problems.) In particular, in the limit of large $|x|$ the stationary distribution has an exponential tail

$$p^*(x) \approx e^{-|x|/\xi(r)}, \quad |x| \to \infty, \tag{2.57}$$

with $\xi(r) = a/\sqrt{r}$. In fact, such an exponential tail is quite generic for $\mu = 2$. The reason is that for $\mu = 2$, $\hat{p}^*(q)$ in (2.53) is an analytic function in the complex q-plane and the large x behaviour of the integral over q in (2.53) will be dominated, say for $x \to +\infty$, by the pole of smallest modulus of the integrand (in the lower half complex q-plane, i.e. for $\text{Im}(q) < 0$). Therefore $p^*(x)$ decays exponentially as in (2.57) where $\xi(r)$ is the largest solution of $1 - (1 - r)\hat{f}(i/z) = 0$ for $z > 0$.

The case $0 < \mu < 2$. In this case, the situation is quite different since $\hat{p}^*(q)$ is non analytic near $q = 0$, where it behaves as $\hat{p}^*(q) \approx (1 - |aq|^\mu(1 - r)/r)$. Hence, for large $|x|$ the integral over q in (2.53) is dominated by this non-analyticity which implies that $p^*(x)$ decays as a power law for $|x| \to \infty$

$$p^*(x) \sim A_\mu(r)\,|x|^{-1-\mu}, \quad A_\mu(r) = a^\mu \,\frac{1-r}{r}\,\sin\left(\frac{\pi\mu}{2}\right)\frac{\Gamma(\mu + 1)}{\pi}, \tag{2.58}$$

which is markedly different from the exponential decay (2.57) for $\mu = 2$.

3. Survival in the presence of an absorbing target

We now consider stochastic processes under resetting with a target to be achieved by the process. In the case of diffusive processes we consider a spatial target which absorbs the diffusive particle (the searcher for the target) and arrests the process.

We begin by considering the one-dimensional diffusive case of section 2.1. The particle (or searcher) starts at the initial position x_0 and undergoes diffusion with diffusion constant D and stochastic resetting to X_r with a constant rate r. When it reaches the target, the particle is absorbed (see figure 5). We wish to compute the survival probability, $Q_r(x_0, t|X_r)$, of a diffusive particle at time t, having *started* at x_0 at $t = 0$ with resetting to X_r. The subscript r emphasises that this quantity pertains to the process with resetting. As we have already seen, the results are simplified when the initial position coincides with the resetting position $X_r = x_0$. In the following we will have recourse to $Q_0(x_0, t|X_r)$ which denotes the survival probability in the absence of resetting.

There are several approaches to compute the survival probability: for example, one can use the forward master equation, the backward master equation or a renewal equation approach. Here we will present the renewal equation approach. We refer the reader to the literature [43, 44, 46] for the backward master equation approach.

3.1. Renewal equation approach for Poissonian resetting

For Poissonian resetting, for a generic process, it is possible to relate in a simple way the survival probability with resetting, Q_r, to that without resetting, Q_0. A convenient way to establish
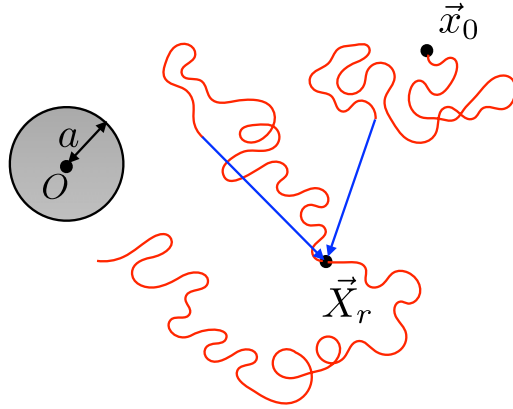
Figure 5. Illustration in $d = 2$ of the diffusion of a particle with initial position $\vec{x}_0$ and resetting to $\vec{X}_r$, in the presence of an absorbing trap of radius a with centre at the origin O.

this relation is to use a last renewal equation which reads

$$Q_r(x_0, t) = e^{-rt} Q_0(x_0, t) + r \int_0^t d\tau \, e^{-r\tau} Q_0(X_r, \tau) Q_r(x_0, t - \tau), \tag{3.1}$$

where, to lighten the notation, we have used the shorthand $Q_r(x_0, t|X_r) = Q_r(x_0, t)$ and similarly for Q_0. The first term in (3.1) represents trajectories in which there has been no resetting. The second term represents trajectories in which resetting has occurred. The integral is over τ, the time elapsed since the last reset and we have a convolution of survival probabilities: survival starting from x_0 with resetting up to time $t - \tau$ (the time of the last reset) and survival starting from X_r in the absence of resetting for duration τ (see figure 2).

We now define the Laplace transform

$$\tilde{Q}_r(x_0, s) = \int_0^\infty dt \, e^{-rt} Q_r(x_0, t). \tag{3.2}$$

Then Laplace transforming (3.1) yields

$$\tilde{Q}_r(x_0, s) = \tilde{Q}_0(x_0, r + s) + r\tilde{Q}_0(X_r, r + s)\tilde{Q}_r(x_0, s) \tag{3.3}$$

from which we readily obtain

$$\tilde{Q}_r(x_0, s) = \frac{\tilde{Q}_0(x_0, r + s)}{1 - r\tilde{Q}_0(X_r, r + s)}. \tag{3.4}$$

This is a very general result for Poissonian resetting, relating the Laplace transform of the survival probability in the presence of resetting to that in the absence of resetting. We shall use it repeatedly in this section.

In the specific case where the initial position and resetting position coincide, i.e. $x_0 = X_r$, (3.4) simplifies to

$$\tilde{Q}_r(X_r, s) = \frac{\tilde{Q}_0(X_r, r + s)}{1 - r\tilde{Q}_0(X_r, r + s)}. \tag{3.5}$$

From these expressions (3.4) and (3.5) we obtain the survival probability with Poissonian reset-
ting from that without resetting, as claimed above. Various first-passage observables in the
presence of resetting can then be computed.

For example, the mean time to absorption (MTA), with coincident initial and resetting posi-
tions $X_r = x_0$, can be computed from the survival probability $Q_r(X_r, t)$. First note that the
first-passage time density is given by $-\frac{\partial Q_r(X_r,t)}{\partial t}$. Averaging the first-passage time over this
density, then integrating by parts yields

$$
\begin{aligned}
\langle T(X_r) \rangle &= -\int_0^\infty \mathrm{d}t\, t \frac{\partial Q_r(X_r, t)}{\partial t} \\
&= \tilde{Q}_r(X_r, s = 0) = \frac{\tilde{Q}_0(X_r, r)}{1 - r\tilde{Q}_0(X_r, r)}.
\end{aligned}
\tag{3.6}
$$

The last equality follows from (3.5) and it relates the MTA with resetting to the Laplace trans-
form of the survival probability without resetting, for any arbitrary stochastic process with
Poissonian resetting. Let us now consider an application of these results to our prototypical
case of one-dimensional diffusion.

3.2. Survival probability for diffusion with resetting

The expression for $Q_0(x_0, t)$, the survival probability of a diffusive particle starting from
x_0 and its Laplace transform $\tilde{Q}_0(x_0, s)$ are standard results in the literature (see e.g. [79]).
For completeness we derive $\tilde{Q}_0(x_0, s)$ here, first using a general renewal equation approach
for a first-passage process and then using the backward master equation for the diffusive
case.

We can write a general equation for the propagator from x_0 to x as an integral over the *first*
time to reach x

$$
G_0(x, t|x_0) = \int_0^t \mathrm{d}\tau\, \phi_0(x, \tau|x_0) G_0(x, t - \tau|x)
\tag{3.7}
$$

where $\phi_0(x, \tau|x_0)$ is the probability density of reaching x for the first time at t. Taking the
Laplace transform yields

$$
\tilde{\phi}_0(x, s|x_0) = \frac{\tilde{G}_0(x, s|x_0)}{\tilde{G}_0(x, s|x)}.
\tag{3.8}
$$

This is a general result for the first-passage distribution for Markovian processes, which
expresses its Laplace transform in terms of the Laplace transform of the propagator G_0 for
the process.

Now $\phi_0(x, s|x_0)$ is equivalent to the rate of absorption at an absorbing target at x, thus for
our case of an absorbing target at the origin

$$
\phi_0(0, t|x_0) = -\frac{\partial}{\partial t} Q_0(x_0, t).
\tag{3.9}
$$

Taking the Laplace transform yields

$$
\tilde{\phi}_0(0, s|x_0) = 1 - s\tilde{Q}_0(x_0, s).
\tag{3.10}
$$

Thus we obtain from (3.8)

$$\tilde{Q}_0(x_0, s) = \frac{1}{s} - \frac{1}{s} \frac{\tilde{G}_0(0, s|x_0)}{\tilde{G}_0(0, s|x)}. \tag{3.11}$$

This is a general result relating survival probability and hence first passage distribution to the propagator of the process.

Finally using the form of the Laplace transform of the diffusive propagator $\tilde{G}_0(x, s|x_0)$

$$\tilde{G}_0(x, s|x_0) = \frac{1}{2(Ds)^{1/2}} \, e^{-(s/D)^{1/2}|x-x_0|} \tag{3.12}$$

we obtain

$$\tilde{Q}_0(x_0, s) = \frac{1 - e^{-(s/D)^{1/2}x_0}}{s}. \tag{3.13}$$

We note for future reference that the Laplace transform in (3.13) can be simply inverted (see e.g. [79]), yielding

$$Q_0(x_0, t) = \mathrm{erf}\left(\frac{x_0}{2(Dt)^{1/2}}\right). \tag{3.14}$$

Using (3.4) we deduce

$$\tilde{Q}_r(x_0, s) = \frac{1 - \exp(-\alpha x_0)}{s + r \, \exp(-\alpha X_r)} \tag{3.15}$$

where

$$\alpha(s) = \left(\frac{r + s}{D}\right)^{1/2}. \tag{3.16}$$

We note that $\alpha(0) = \alpha_0$ given by (2.19). In the case where the resetting position X_r coincides with the initial position x_0 we have

$$\tilde{Q}_r(X_r, s) = \frac{1 - \exp(-\alpha X_r)}{s + r \, \exp(-\alpha X_r)}. \tag{3.17}$$

Having obtained this expression for the survival probability in the presence of resetting one would ideally wish to invert the Laplace transform in order to obtain $Q_r(X_r, t)$. However, it is a difficult task to invert the Laplace transform (3.15) explicitly for all parameters. We will discuss the late time asymptotics in section 3.5.

For completeness let us also derive (3.13) from the backward Fokker–Planck equation for the survival probability which reads

$$\frac{\partial Q_0(x_0, t)}{\partial t} = D \frac{\partial^2 Q_0(x_0, t)}{\partial x_0^2} \tag{3.18}$$

with boundary condition $Q_0(0, t) = 0$ and initial condition $Q_0(x_0, t = 0) = 1$ for $x_0 \neq 0$. The Laplace transform obeys

$$D \frac{\partial^2 \tilde{Q}_0(x_0, s)}{\partial x_0^2} - s \tilde{Q}_0(x_0, s) = -1, \tag{3.19}$$

whose general solution, satisfying the additional boundary condition that $\lim_{x_0 \to \pm \infty} \tilde{Q}_0(x_0, s) < \infty$, is given by

$$\tilde{Q}_0(x_0, s) = A \, e^{-(s/D)^{1/2}|x_0|} + \frac{1}{s}. \tag{3.20}$$

The constant A is then fixed by the boundary condition of (3.18) at $x_0 = 0$ which translates to $\tilde{Q}_0(0, s) = 0$ and we obtain

$$\tilde{Q}_0(x_0, s) = \frac{1}{s} \left[1 - e^{-(s/D)^{1/2}|x_0|} \right] \tag{3.21}$$

which recovers (3.13).

3.3. Mean time to absorption for diffusion with resetting

The mean time to absorption in the case of one-dimensional diffusion with resetting is obtained from (3.6) by setting $s = 0$ in (3.15)

$$\langle T(X_r) \rangle = \frac{1}{r} \left(\exp\left(\alpha_0 X_r \right) - 1 \right), \tag{3.22}$$

where we recall that

$$\alpha_0 = \left(\frac{r}{D} \right)^{1/2}. \tag{3.23}$$

We note that $\langle T(X_r) \rangle$ diverges as $r \to 0$ as $\langle T(X_r) \rangle \sim r^{-1/2}$, which recovers the well-known result that the mean time for a diffusive particle to reach the origin (in the absence of resetting) is infinite. Also $\langle T(X_r) \rangle$ diverges as $r \to \infty$, the explanation being that as the reset rate increases the diffusing particle has less time between resets to reach the origin. In between these two divergences there is a single minimum of $\langle T(X_r) \rangle$ (see figure 6) which we now study.

For convenience we introduce the dimensionless quantity

$$\gamma = \alpha_0 X_r \tag{3.24}$$

which is the ratio of two length scales: X_r is the distance from the resetting position to the target and $1/\alpha_0$ is the typical diffusion length between resets.

We now seek to minimise the MTA with respect to r. The equation

$$\frac{d\langle T(X_r) \rangle}{dr} = 0 \tag{3.25}$$

reduces in terms of γ to the transcendental equation

$$\frac{\gamma}{2} = 1 - e^{-\gamma} \tag{3.26}$$

which has a unique non-zero solution $\gamma^* = 1.5936\ldots$. Thus the minimal mean time to locate the target is achieved when the ratio of the distance X_r to the target to the typical distance diffused between resets is γ^*.
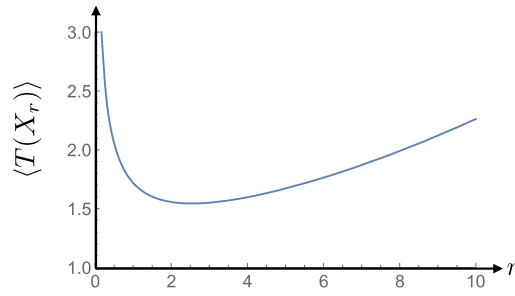
Figure 6. Mean time to absorption as a function of r (for $D = 1$ and $X_r = x_0 = 1$).

3.4. Optimal resetting: general considerations for diffusive problems

As we have seen the resetting rate r to position X_r may be chosen to minimise the mean time to absorption for a target at the origin. This is our first instance of optimal resetting i.e. choosing the resetting rate or the distribution of resetting sites so as to optimise some measure of efficiency such as the mean time to absorption by a target. Of course in more realistic search problems we may have only partial information about the target. For example, we may merely know that the target position x_T is drawn from some distribution $\mathcal{P}_T(x_T)$. In [44] various optimisation problems concerning optimal resetting with a target distribution were considered. Optimal resetting implies choosing (most generally) the space-dependent resetting rate $r(x)$, or the resetting distribution $\mathcal{P}_r(X_r)$ (see section 2.7), to minimise the mean time to absorption for a given target distribution $\mathcal{P}_T(x_T)$.

In the case of a precisely located target (where $\mathcal{P}_T$ is a delta function distribution) it was shown in [44] how a non-resetting window around the coincident initial and resetting positions $x_0 = X_r$ can reduce the mean time to absorption, provided that x_0 is sufficiently far from the target. The non-resetting window is given by a space-dependent resetting rate

$$r(x) = \begin{cases} 0 & \text{for } |x - x_0| < a \\ r & \text{for } |x - x_0| \geqslant a. \end{cases} \tag{3.27}$$

For an exponentially decaying target distribution centred at the origin $\mathcal{P}_T(x_T) = (\beta/2)e^{-\beta|x_T|}$, it was shown in [44] that a transition in the optimal resetting distribution occurs as β decreases i.e. the target distribution broadens. The critical value of β is $\beta_c = 2\alpha_0$ where $\alpha_0 = \sqrt{r/D}$. For a narrow target distribution $\beta > \beta_c$ the optimal resetting distribution is simply a delta function at the origin. However, for a broader target distribution $\beta < \beta_c$ the optimal resetting distribution becomes a delta function at the origin plus an exponentially decaying piece:

$$\mathcal{P}_r^*(X_r) = \begin{cases} \delta(X_r) & \text{for } \beta > \beta_c \\ \dfrac{\beta}{4}\left[1 - \dfrac{\beta^2}{4\alpha_0^2}\right] e^{-\beta|X_r|/2} + \dfrac{\beta^2}{4\alpha_0^2}\delta(X_r) & \text{for } \beta \leqslant \beta_c. \end{cases} \tag{3.28}$$

A related optimisation question is: when does diffusion with resetting perform better than diffusion in a confining potential? In the former scenario the resetting process confines the particle and creates a nonequilibrium stationary state, whereas in the latter scenario the confining potential creates an equilibrium stationary state. The question is: which class of dynamics gives

the lower mean time to absorption? In [80] it was shown that the optimal mean time to absorption under resetting with optimised constant rate r^*, is less than that for an effective equilibrium Langevin process with a potential that generates the same stationary distribution. In [81] the optimal potential for a Langevin process (without resetting) for a given target distribution was computed exactly. The mean time to absorption was then compared to that of resetting with a constant rate r^* optimised for the target distribution. Whether the Langevin dynamics in a potential or the resetting dynamics performs better depends on the particular choice of target distribution.

As mentioned in the introduction other search strategies may be considered e.g. Gelenbe [11] considered searchers that have some probabilistic lifetime after which another searcher will be sent out, and computed mean times to absorption. In [82], several searchers under resetting to a single home base were considered and the optimisation of the search time and associated search cost (i.e., the number of searchers times the search time) was studied. Also, in the mathematical literature the mean first-passage time for random walkers that have the option of restarting at the initial position has been considered [25].

Returning to our original problem of optimising the MTA by tuning the resetting rate, we have seen that for diffusive processes with Poissonian resetting in one dimension, there exists an optimal resetting rate r^* that minimises the MTA to the target. However, it turns out that this optimisation paradigm holds for a wide class of stochastic processes with both Poissonian and non-Poissonian resetting. These generalisations to arbitrary stochastic processes will be discussed in section 5.

3.5. Late time asymptotics of the survival probability and connection with extreme value statistics

We now consider the inversion of the Laplace transform of the survival probability (3.17)

$$Q_r(X_r|t) = \int_{-i\infty+c}^{+i\infty+c} \frac{ds\,e^{st}}{2\pi i} \frac{1 - \exp(-\alpha X_r)}{s + r\,\exp(-\alpha X_r)} \tag{3.29}$$

where c is a real number chosen so that the integration contour is to the right of any singularities in the complex s plane.

The singularity structure of the integrand is as follows. There is a simple pole at s_0 given by the solution of

$$s_0 + r\,\exp\left[-\left(\frac{r + s_0}{D}\right)^{1/2} X_r\right] = 0. \tag{3.30}$$

There is also a branch point singularity at $s = -r$. One can check that $0 > s_0 > -r$, which implies that for large t the dominant contribution to the inversion will come from the pole and therefore

$$Q_r(X_r|t) \simeq A\,e^{s_0 t} \tag{3.31}$$

where the constant A is determined by the residue of the pole as

$$A = \frac{1 + s_0/r}{1 + s_0 X_r/(2(r + s_0)^{1/2} D^{1/2})}. \tag{3.32}$$

Now let us consider the limit $\gamma = \sqrt{r/D}\,X_r \gg 1$ in which case

$$s_0 \simeq -r\,e^{-\gamma} \tag{3.33}$$

is very small and we find from (3.31)

$$Q_r(X_r|t) \simeq \exp(-rt\,e^{-\gamma}). \tag{3.34}$$

Interestingly expression (3.34) has the form of a Gumbel distribution which occurs in the theory of extreme value statistics of i.i.d. random variables [83].

To understand better the reason for this we can make an heuristic derivation of the survival probability. After a long time t we expect $N = rt$ resets (with corrections of order $t^{1/2}$) to have occurred. After each reset the diffusive particle will perform an excursion from the reset position X_r, which is independent of the previous excursions. For survival until t, each excursion must not reach the origin. We have already seen that the survival probability for a diffusive particle (in the absence of resetting) is given by (3.14). The duration of each excursion is distributed exponentially, thus the survival probability for an excursion $Q_0(X_r, \tau)$, averaged over the duration of the excursion τ is given by

$$\overline{Q_0(X_r)} = \int_0^\infty d\tau\, r\, e^{-r\tau} \mathrm{erf}\left(\frac{X_r}{2(D\tau)^{1/2}}\right) = 1 - e^{-(r/D)^{1/2}X_r}, \tag{3.35}$$

where we have used the result for the Laplace transform of an error function (3.13). Thus we deduce

$$Q_r(X_r, t) \approx \left[1 - e^{-\gamma}\right]^{rt} \tag{3.36}$$

and if γ is large this recovers (3.34). We note that the only approximation in this argument is that we fix the number of resets to be $N = rt$ and allow fluctuations in the times between resets, rather than fixing the total duration of the resets to be t.

The connection with extreme value statistics is now clear. The renewal picture implies that we have a large number $N \simeq rt$ of resets and we require the probability that amongst these the largest excursion to the left is less than X_r. This coincides with the Gumbel distribution which is a cumulative probability that the largest of N i.i.d. random variables is less than some value. The Gumbel distribution indeed applies when the distribution of each of the random variables has a tail which decays exponentially or faster, which is the case here (see (3.35)). When the distribution of the random variables has a power-law tail, the corresponding distribution of the maximum belongs to the so-called Fréchet class (for a recent pedagogical review on extreme value statistics see [84]). In the context of resetting, if the time between resets is drawn from a power law distribution (as e.g. in non-Poissonian resetting discussed in section 2.8), one can show that the distribution of the maximum of the reset process is given by the Fréchet law, appropriately centred and scaled. This was in fact demonstrated for a ballistic process with reset in one dimensoin [62]. Another classical extreme value distribution is the Weibull distribution, which occurs when the i.i.d. random variables are each drawn from a bounded distribution [84]. Hence, one would expect the Weibull distribution to appear in the resetting problem by appropriately choosing the time interval between resets (see e.g. [62]).

3.6. Quasi-stationary state for diffusion with resetting and absorption

It is of interest to consider the distribution of the particle when we condition on survival—at long times this converges to a quasi-stationary state [85, 86]:

$$p(x, t|x_0) \to Q_r(x_0, t)p_{qs}(x) \tag{3.37}$$

where $Q_r(x_0, t)$ is the survival probability and $p_{qs}(x)$ is the quasi-stationary state.

For one-dimensional diffusion with resetting, the forward master equation reads

$$\frac{\partial p(x,t)}{\partial t} = D\frac{\partial^2 p(x,t)}{\partial x^2} - rp(x,t) + rQ_r(x_0,t)\delta(x - X_r), \tag{3.38}$$

with initial condition $p(x,0) = \delta(x - x_0)$ and boundary condition $p(0,t) = 0$ due to the absorbing target at the origin. As before, $Q_r(x_0,t)$ is the survival probability at time t having started from x_0.

Substituting (3.37) into the forward master equation and dividing by $Q_r(x_0,t)$ yields

$$D\frac{\partial^2 p_{\text{qs}}(x)}{\partial x^2} - \left(r + \frac{1}{Q_r(x_0,t)}\frac{\partial Q_r(x_0,t)}{\partial t}\right)p_{\text{qs}}(x) = -r\delta(x - X_r). \tag{3.39}$$

As we have seen [see equation (3.31)], for $t \gg 1$, $Q_r(x_0,t) \sim e^{s_0 t}$ and using this we obtain

$$D\frac{\partial^2 p_{\text{qs}}(x)}{\partial x^2} - (r + s_0)p_{\text{qs}}(x) = -r\delta(x - X_r), \tag{3.40}$$

with boundary condition $p_{\text{qs}}(0) = 0$.

The solution of this equation is obtained by standard means as

$$p_{\text{qs}}(x) = \frac{\alpha(s_0)}{e^{\alpha(s_0)X_r} - 1}\sinh(\alpha(s_0)x) \quad \text{for } x < X_r \tag{3.41}$$

$$= \frac{\alpha(s_0)}{e^{\alpha(s_0)X_r} - 1}\sinh(\alpha(s_0)X_r)e^{-\alpha(s_0)(x - X_r)} \quad \text{for } x > X_r, \tag{3.42}$$

where

$$\alpha(s_0) = \left(\frac{r + s_0}{D}\right)^{1/2}. \tag{3.43}$$

The distribution is shown in figure 7 which illustrates the asymmetric distribution decaying more steeply to zero at $x = 0$ and with a cusp at the resetting site X_r.

3.7. Arbitrary spatial dimension

As for the stationary state we can easily generalise the calculation of the survival probability to arbitrary spatial dimension d. However we have to generalise the point target at the origin in the one-dimensional case to an absorbing d-dimensional sphere of radius a (see figure 5 for a two-dimensional illustration) centred at $\vec{x} = 0$. The particle starts at the initial position $\vec{x}_0$ (with $|\vec{x}_0| > a$) and undergoes diffusion with diffusion constant D and stochastic resetting to $\vec{X}_r$ with a constant rate r. When it reaches the surface of the target sphere, the particle is absorbed.

There is now an additional length scale in the system, a the radius of the trap. We generalise the dimensionless variable γ (3.24)

$$\gamma = \alpha_0 R_r \tag{3.44}$$

where $R_r = |\vec{X}_r|$ is the distance from the resetting position to the target and we define an additional dimensionless reduced variable

$$\epsilon = \frac{a}{R_r} \tag{3.45}$$

which is simply the ratio of the radius of the absorbing sphere to the distance R_r of the reset point to the target at the origin.
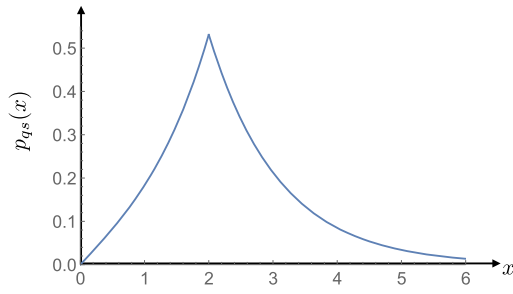
Figure 7. Plot of the quasi-stationary state distribution given by equations (3.41) and (3.42). Here $X_r = 2$ and $r = D = 1$, such that $s_0 \simeq -0.16$ from (3.30).

As before, we write down a last renewal equation satisfied by the survival probability

$$Q_r(\vec{x}_0, t) = e^{-rt} Q_0(\vec{x}_0, t) + r \int_0^t d\tau \, e^{-r\tau} Q_0(\vec{X}_r, \tau) Q_r(\vec{x}_0, t - \tau) \tag{3.46}$$

where the first term represents trajectories in which there has been no resetting and the integral in the second term is over τ, the time elapsed since the last reset (see figure 2).

We define the Laplace transform

$$\tilde{Q}_r(\vec{x}_0, s) = \int_0^\infty dt \, e^{-rt} Q_r(\vec{x}_0, t) \tag{3.47}$$

and we obtain from (3.46) on setting $\vec{x}_0 = \vec{X}_r$,

$$\tilde{Q}_r(\vec{X}_r, s) = \frac{\tilde{Q}_0(\vec{X}_r, r + s)}{1 - r\tilde{Q}_0(\vec{X}_r, r + s)}. \tag{3.48}$$

The expression for $\tilde{Q}_0(\vec{x}_0, s)$, the Laplace transform of the survival probability of a diffusive particle starting from $\vec{x}_0$ with an absorbing sphere at the origin, is given by (see e.g. [79])

$$\tilde{Q}_0(\vec{x}_0, s) = \frac{1}{s} - \frac{1}{s} \frac{R_r^\nu}{a^\nu} \frac{K_\nu(\alpha_0 R_r)}{K_\nu(\alpha_0 a)}, \tag{3.49}$$

where $K_\nu(z)$ is the modified Bessel function of the second kind, with index $\nu = 1 - d/2$, and we obtain

$$\tilde{Q}_r(\vec{X}_r, s) = \frac{a^\nu K_\nu(\alpha a) - R_r^\nu K_\nu(\alpha R_r)}{r R_r^\nu K_\nu(\alpha R_r) + s a^\nu K_\nu(\alpha a)}, \tag{3.50}$$

where $R_r = |\vec{X}_r|$. Expression (3.50) is the exact expression for the Laplace transform of the survival probability with resetting in arbitrary dimension d.

3.8. Partial absorption by a target

A natural generalisation of the absorbing target, is a target which has some reduced probability of absorbing the process, i.e. a partially absorbing target. The formula (3.5) may still be applied for Poissonian resetting. Thus the problem reduces to finding the survival probability for diffusion with a partially absorbing target at the origin.

This problem may be formulated by introducing an 'absorption velocity' b (note that we use b here rather than a as in [87] to avoid a clash of notation with other sections). The limit $b \to \infty$ corresponds to the absorbing target and the limit $b \to 0$ corresponds to a reflecting boundary at the target (no absorption). In [87] it is shown that b may be implemented by either applying a boundary condition (sometimes referred to as a radiation boundary condition [88, 89]) to the survival probability (without resetting)

$$\left. \frac{\partial Q_0(x_0, t)}{\partial x_0} \right|_{x_0=0} = \frac{b}{D} Q_0(0, t) \tag{3.51}$$

or adding a sink term at the origin to a master equation for the survival probability. For example the backward master equation becomes

$$\frac{\partial Q_0(x_0, t)}{\partial t} = D \frac{\partial^2 Q_0(x_0, t)}{\partial x_0^2} - b Q_0(0, t)\delta(x_0). \tag{3.52}$$

We begin from (3.52), the Laplace transform of which becomes

$$D \frac{\partial^2 \tilde{Q}_0(x_0, s)}{\partial x_0^2} - s\tilde{Q}_0(x_0, s) + 1 = b\tilde{Q}_0(0, s)\delta(x_0). \tag{3.53}$$

The general solution of the homogeneous equation (where the right hand side has been set to zero), satisfying the additional boundary condition that $\lim_{x_0 \to \pm\infty} \tilde{Q}_0(x_0, s) < \infty$, and the condition that $\tilde{Q}_0(x_0, s)$ is continuous at $x_0 = 0$, is

$$\tilde{Q}_0(x_0, s) = A\, e^{-(s/D)^{1/2}|x_0|} + \frac{1}{s}. \tag{3.54}$$

The constant A is then fixed by the discontinuity condition on the first derivative at $x_0 = 0$ (which comes from integrating (3.53) over $x_0 = 0$)

$$\lim_{\epsilon \to 0} \left[\frac{\partial \tilde{Q}_0(x_0, s)}{\partial x_0} \right]_{x_0=-\epsilon}^{x_0=+\epsilon} = \frac{b}{D} \tilde{Q}_0(0, s), \tag{3.55}$$

which yields

$$\tilde{Q}_0(x_0, s) = -\frac{b}{2\sqrt{sD}} \tilde{Q}_0(0, s) e^{-(s/D)^{1/2}|x_0|} + \frac{1}{s}. \tag{3.56}$$

Setting $x_0 = 0$ fixes $\tilde{Q}_0(0, s)$ self-consistently as

$$\tilde{Q}_0(0, s) = \frac{1}{s} \frac{2\phi}{1 + 2\phi} \tag{3.57}$$

where $\phi = \sqrt{sD}/b$. Finally we obtain

$$\tilde{Q}_0(x_0, s) = \frac{1}{s} \left[1 - \frac{1}{1 + 2\phi} e^{-(s/D)^{1/2}|x_0|} \right]. \tag{3.58}$$

This expression may then be inserted into (3.5) to obtain the Laplace transform of the survival probability in the presence of resetting [87]. Note that the limit $b \to \infty$ of (3.58) recovers our previous result for a fully absorbing target (3.13). Also note that in the limit $b \to 0$, $\tilde{Q}_0(x_0, s) \to 1/s$, which implies $Q_0(x_0, t) = 1$, consistent with no absorption at the target.

3.9. Survival probability with resetting on a finite domain

So far we have seen that the introduction of resetting can render finite an expected time to complete a task that would otherwise diverge. Our archetypal example is diffusion with resetting rate r. In the absence of resetting the mean for a diffusive process to locate a target, or equivalently the mean time to absorption (MTA), diverges in any dimension. Introducing resetting results in a finite MTA. However we have so far assumed that the domain of the diffusive process is infinite. If, instead, the diffusion is on a finite domain, for example a finite interval in one dimension with reflecting boundaries, then the mean time to locate a target is always finite. In this case resetting to an initial condition may or may not reduce the mean time to absorption.

Several recent works have studied diffusion with resetting on a finite domain [49, 90]. Christou and Schadschneider [49] considered the problem just alluded to, that of diffusion with resetting in the interval $[0, L]$ with reflecting boundaries. They considered an arbitrary number N of possible resetting positions (see section 2.7) X_{r_i} with $i = 1, \ldots, N$, each chosen at a resetting event with probability $\mathcal{P}_{X_i}$. The equation for the stationary distribution $p^*(x)$ reads

$$D\frac{\partial^2 p^*(x)}{\partial x^2} = rp^*(x) - \frac{r}{N}\sum \mathcal{P}_{X_{r_i}}\delta(x - X_{r_i}) \tag{3.59}$$

with boundary conditions

$$\left.\frac{\partial p^*}{\partial x}\right|_{x=0,L} = 0. \tag{3.60}$$

The problem may be solved by using a decomposition in terms of eigenfunctions $\phi_n(x)$ with eigenvalue $\epsilon_n = -n^2\pi^2/L^2$ of the Laplacian with boundary conditions (3.60)

$$\phi_n(x) = \sqrt{2/L}\,\cos(n\pi x/L) \tag{3.61}$$

for n integer. This results in the solution of (3.59)

$$p^*(x) = p_0 + \sum_{n=1}^{\infty}\frac{2r}{NL(r - D\epsilon_n)}\,\cos(n\pi x/L)\sum_{i=1}^{N}\cos(n\pi X_{r_i}/L) \tag{3.62}$$

where p_0 is chosen to normalise the probability.

In [49] the survival probability $Q_r(x_0, t)$ for a single resetting site X_r, on a finite domain $0 \leqslant x \leqslant L$ with reflecting boundaries and a partially absorbing site with absorption velocity b was considered. The solution was worked out from the forward master equation for the survival probability $Q_r(x_0, t)$ and an eigenfunction expansion. We note that it is also straightforward to obtain a closed form solution using the general result (3.4) for Poissonian resetting. Then one just needs to obtain $Q_0(x_0, t)$ the survival probability for a diffusive particle on a finite domain with a partial absorption site.

As the expression for MTA in [49] involves an infinite sum, rather than minimising the MTA the authors optimised the resetting rate by minimising s_0 which appears in the survival probability as $Q_r \sim e^{s_0 t}$ and, as discussed in section 3.5, is the dominant pole in the Laplace transform. Then the value optimal value of the resetting rate, r^*, is that which minimises the survival probability at late times. It was found that the $r^* > 0$ only in a region where the resetting site X_r is sufficiently close to the target site X_B. Typically only if $|X_B - X_r| \leqslant L/2$ is resetting advantageous.

In [91] the case of a searcher diffusing on a disc with a target on the perimeter was considered. When the searcher reaches the perimeter of the disc the diffusion becomes one-dimensional i.e. the searcher then sticks to the perimeter. Resetting occurs whereby the searcher is returned to the initial condition, somewhere in the disc away from the perimeter. Thus the search process comprises two steps: two-dimensional diffusion from the initial position to the perimeter then one-dimensional diffusion around the perimeter (with periodic boundary conditions) and an absorbing site. The diffusion around the circular perimeter with an absorbing site is equivalent to a finite domain with absorbing boundaries at both ends.

The latter problem has also been studied in the case where absorption at the different ends is distinguished [60, 92, 93] so that absorption at one end is interpreted as successful completion of a task but absorption at the other end is a failure of the task. Splitting probabilities and conditional MTAs are computed and conditions for the resetting to expedite successful completion of the task are analysed in [60, 92, 93].

3.10. Survival probability and mean time to absorption for non-Poissonian resetting

We now derive expressions for the survival probability for non-Poissonian resetting, as defined in section 2.8, in terms of transforms of the survival probability in the absence of resetting.

In the non-Poissonian case it is convenient to use a first renewal equation for the survival probability which reads

$$Q_r(x_0, t) = \Psi(t)Q_0(x_0, t) + \int_0^t d\tau_f \, \psi(\tau_f)Q_0(x_0, \tau_f)Q_r(X_r, t - \tau_f), \tag{3.63}$$

where $\psi(t)$ is the distribution of the time period between two successive resets and $\Psi(t) = \int_t^\infty \psi(t') \, dt'$. The first term in (3.63) represents trajectories in which there has been no resetting. The second term represents trajectories in which resetting has occurred. The integral is over τ_f, the time of the first reset and we have a convolution of survival probabilities: survival starting from x_0 without resetting up to time τ_f and survival starting from X_r in the presence of resetting for duration $t - \tau_f$ (see figure 2).

We now take the Laplace transform of (3.63) and get

$$\tilde{Q}_r(x_0, s) = \int_0^\infty dt \, e^{-st}\Psi(t)Q_0(x_0, t) \tag{3.64}$$

$$+ \tilde{Q}_r(X_r, s)\int_0^\infty d\tau_f \, e^{-st\tau_f}\psi(\tau_f)Q_0(x_0, \tau_f). \tag{3.65}$$

Setting $x_0 = X_r$, we obtain

$$\tilde{Q}_r(X_r, s) = \frac{\int_0^\infty dt \, e^{-st}\Psi(t)Q_0(X_r, t)}{1 - \int_0^\infty dt \, e^{-st}\psi(t)Q_0(X_r, t)}. \tag{3.66}$$

The formula for Poissonian resetting (3.5) is recovered when we take $\Psi(t) = e^{-rt}$ and $\psi(t) = re^{-rt}$, in which case the integrals on the rhs reduce to Laplace transforms.

Integration by parts in the denominator allows formula (3.66) to be written as

$$\tilde{Q}_r(X_r, s) = \frac{\int_0^\infty dt \, e^{-st}\Psi(t)Q_0(X_r, t)}{s\tilde{\Psi}(s) - \int_0^\infty dt \, e^{-st}\Psi(t)\frac{\partial Q_0(X_r, t)}{\partial t}}. \tag{3.67}$$

28

The mean time to absorption becomes

$$\langle T(X_r) \rangle = - \frac{\int_0^\infty dt \, \Psi(t) Q_0(X_r, t)}{\int_0^\infty dt \, \Psi(t) \frac{\partial Q_0(X_r, t)}{\partial t}}. \tag{3.68}$$

In the case of a diffusive particle, it was shown in [55] that the mean time to absorption is optimised for deterministic resetting with suitably chosen period t^* i.e.

$$\psi(t) = \delta(t - t^*). \tag{3.69}$$

We shall see later that this is a general feature which is valid beyond the case of simple diffusion—see section 5.

3.11. Mean time to absorption for discrete time random walks and Lévy flights with resetting

In this section, we study the survival probability for the discrete random walk model with resetting discussed in section 2.9. We have in mind a searcher that moves in discrete time, on a line, according to the resetting dynamics specified in (2.49), starting from $x_0 > 0$. Here we consider a broad class of continuous jump distributions characterized by a Lévy index μ [see (2.50)] with $0 < \mu \leqslant 2$. We recall that the case $\mu = 2$ corresponds to ordinary random walks, while $\mu < 2$ describes Lévy flights, where the jumps are broadly distributed.

The target is located at the origin and, for simplicity, we restrict our discussion to the case where the resetting position X_r coincides with the initial position $X_r = x_0$. To characterise the efficiency of the search process, it is useful to compute the MTA $\langle T(X_r) \rangle$ which, for a fixed resetting position X_r, depends here on the probability of a resetting event r as well as on the Lévy index μ. To compute the MTA we introduce the cumulative distribution function

$$Q_r(X_r, n) = \text{Prob}\left[T(X_r) \geqslant n \right], \tag{3.70}$$

which is precisely the survival probability, i.e. the probability that the walker, starting at X_r, does not cross the origin up to step n. Of course, the MTA can then be computed from the relation (analogous to (3.6) in the continuous time case)

$$\langle T(X_r) \rangle = \sum_{n \geqslant 0} Q_r(X_r, n). \tag{3.71}$$

As done in the case of continuous time processes (3.1), one can write a last renewal equation for $Q_r(X_r, n)$ [47]

$$Q_r(X_r, n) = \sum_{m=0}^{n-1} r(1-r)^m Q_r(X_r, n - m - 1) Q_0(X_r, m) + (1-r)^n Q_0(X_r, n), \tag{3.72}$$

where, as before, $Q_0(X_r, n)$ is the survival probability in the absence of resetting (i.e. $r = 0$). The first term on the right hand side of (3.72) accounts for the event where the last resetting before step n takes place at step $n - m$ (see figure 8) with $0 \leqslant m \leqslant n - 1$. The evolution from step $n - m$ to step n occurs without resetting and the survival probability during this period is $Q_0(X_r, m)$, while $Q_r(X_r, n - m - 1)$ accounts for the survival probability from step 1 to step $n - m - 1$ in the presence of resetting. The last term in (3.72) corresponds to the case where there is no resetting event at all up to step n, which occurs with probability $(1 - r)^n$.

Figure 8. Illustration of a random walk in one dimension with resetting to the initial position x_0 and first passage to the target at the origin. The integers n and m here illustrate the notation in the renewal equation (3.72).

Equation (3.72) can be solved by introducing the generating function $\tilde{Q}_r(X_r, z)$ $= \sum_{n \geqslant 0} Q_r(X_r, n) z^n$. Multiplying both sides of (3.72) by z^n and summing over n, we arrive at the result

$$\tilde{Q}_r(X_r, z) = \frac{\tilde{Q}_0(X_r, (1-r)z)}{1 - rz\tilde{Q}_0(X_r, (1-r)z)}. \tag{3.73}$$

This formula (3.73) relates the survival probability in the presence of resetting ($r \geqslant 0$) to that without resetting ($r = 0$). It is reminiscent of the relation (3.5) obtained for continuous time processes. Interestingly the Laplace transform of the survival probability in the absence of resetting $\tilde{Q}_0(X_r, z)$, with respect to X_r, can be computed using the so-called Pollaczek–Spitzer formula [94–97]

$$\int_0^\infty \tilde{Q}_0(X_r, z) e^{-\lambda X_r} dX_r = \frac{1}{\lambda\sqrt{1-z}} \varphi(z, \lambda), \tag{3.74a}$$

$$\varphi(z, \lambda) = \exp\left[-\frac{\lambda}{\pi} \int_0^\infty \frac{dq}{\lambda^2 + q^2} \ln\left(1 - z\hat{f}(q)\right)\right], \tag{3.74b}$$

which is valid for any continuous and symmetric jump distribution $f(\eta)$, including Lévy flights (we recall that $\hat{f}(q) = \int_{-\infty}^{+\infty} e^{iq\eta} f(\eta) d\eta$). Therefore (3.73) together with (3.74) allow one to compute the cumulative distribution of $T(X_r)$ (3.70). In fact by noting the identity $\langle T(X_r) \rangle = \tilde{Q}_r(X_r, 1)$, which follows directly from the definition (3.71), one obtains from (3.73)

$$\langle T(X_r) \rangle = \tilde{Q}_r(X_r, 1) = \frac{\tilde{Q}_0(X_r, 1-r)}{1 - r\tilde{Q}_0(X_r, 1-r)}, \tag{3.75}$$

where $\tilde{Q}_0(X_r, 1-r)$ can, in principle, be computed from (3.74).

In reference [47] a detailed analysis of this formula for the MTA (3.75) was performed for the class of Lévy stable jump distributions, characterized by $\hat{f}(k) = e^{-|ak|^\mu}$, with Lévy index $0 < \mu \leqslant 2$ (and we set $a = 1$ in what follows). We summarise here the main results obtained there and refer the reader to [47] for more details. For a fixed X_r, it is natural to minimise the

MTA $\langle T(X_r)\rangle$ with respect to the two parameters μ and r and find the optimal parameters $\mu^*(X_r)$ and $r^*(X_r)$ as a function of X_r.

It turns out that these optimal values $\mu^*(X_r)$ and $r^*(X_r)$ exhibit a rather rich and surprizing behaviour, as functions of X_r. Indeed there exists a critical value $X_r^* \simeq 0.58$ (determined numerically) such that the optimal strategy depends crucially on whether $X_r > X_r^*$ or $X_r < X_r^*$. When $X_r > X_r^*$, the optimal parameters are independent of X_r, and are given by

$$\mu^*(X_r > X_r^*) = 0, \quad r^*(X_r > X_r^*) = r_>^*, \tag{3.76a}$$

$$\text{where } r_>^* = \frac{\sqrt{e-1}}{2}(\sqrt{e} - \sqrt{e-1}) = 0.22145\ldots. \tag{3.76b}$$

In (3.76a), $\mu^* = 0$ actually means the limit $\mu^* \to 0$. On the other hand, for $X_r < X_r^*$, the optimal values $\mu^*(X_r)$ and $r^*(X_r)$ depend continuously on X_r, both of them being monotonously decreasing functions of X_r. Interestingly, it was found in [47] that the optimal parameters $\mu^*(X_r)$ and $r^*(X_r)$ exhibit a discontinuity as x_0 crosses the value X_r^*. This behaviour is a typical characteristic of a first order transition at X_r^*. Note that related phase transitions have also been reported for the case of random walks with exponentially distributed flights under resetting [48].

Here, we have studied Lévy flights by considering discrete time random walks with heavy-tailed jump distributions $f(\eta) \propto |\eta|^{-1-\mu}$ as $|\eta| \to \infty$, in the limit of a large number of steps $n \gg 1$. It is also possible to study Lévy flights in the context of continuous time random walks and we refer the reader to [98–100] for a study of the MTA for continuous time Lévy flights with resetting.

4. Multiparticle diffusive systems

So far we have considered the problem of single particle dynamics under resetting. We now turn to the problem of multiple non-interacting particles and show how the time to find a target is affected [43].

We consider N independent particles labelled by $j = 1, \ldots, N$, each of which is reset to its own resetting position X_j. We will consider here the Poissonian case where each particle is reset with the same rate r. One may think of a team of searchers seeking a target and the whole process stops when any one of the searchers finds the target (see e.g. [82, 101]). We are interested in the survival probability $P_s(t)$ of the target, i.e. the probability that none of the particles have reached the target up to time t. To simplify matters, we also take the initial position of each particle to be identical to its resetting position.

4.1. Average and typical behaviour: annealed and quenched averages

As the N particles move independently, the survival probability of the target is given by

$$P_s(t) = \prod_{j=1}^{N} Q_r(X_j, t) \tag{4.1}$$

where $Q_r(X_j, t)$ is the survival probability, in the presence of resetting, in the single particle problem considered in section 3.

We consider the N resetting positions to be random variables distributed uniformly with density ρ and consequently, $P_s(t)$ will itself have a distribution. Its average is simply $P_s^{\text{av}}(t) = \langle P_s(t)\rangle_X$ where $\langle\cdot\rangle_X$ denotes averages over X_j's. However, as we shall see, the *typical*

value of the survival probability $P_s(t)$ is not captured by the average. This is because the average may be dominated by rare samples of the resetting positions of the searchers for which the survival probability is much larger than the typical value of the survival probability. We will discuss this effect in more detail in section 4.4.

To compute the average behaviour of (4.1) we write

$$P_s^{\text{av}}(t) = \left[\langle Q_r(X, t)\rangle_X\right]^N \tag{4.2}$$

$$= \exp\left\{N \ln\left[1 - \langle 1 - Q_r(X, t)\rangle_X\right]\right\}. \tag{4.3}$$

We begin by considering each X_j to be distributed uniformly over a finite interval $[-L/2, L/2]$ (and later take the limit $L \to \infty$). We obtain

$$\langle 1 - Q_r(X, t)\rangle_X = \frac{1}{L}\int_{-L/2}^{L/2} dX\,(1 - Q_r(X, t)). \tag{4.4}$$

Letting $N, L \to \infty$ but keeping the density of walkers $\rho = N/L$ fixed, we obtain

$$P_s^{\text{av}}(t) \to \exp\left[-2\rho I_1(t)\right], \tag{4.5}$$

where we define

$$I_1(t) = \int_0^\infty dX(1 - Q_r(X, t)), \tag{4.6}$$

and we have assumed that $Q_r(X, t)$ is an even function of X, i.e. $Q_r(X, t) = Q_r(-X, t)$.

On the other hand the typical behaviour of $P_s(t)$ can be found by first averaging the logarithm of $P_s(t)$ followed by exponentiating

$$P_s^{\text{typ}}(t) = \exp\left[\langle \ln P_s(t)\rangle_X\right]. \tag{4.7}$$

One can draw an analogy to a disordered system with $P_s(t)$ playing the role of a partition function Z and X_μ's corresponding to disorder variables. Thus the average and typical behaviour correspond respectively to the *annealed* average (where one averages the partition function Z) and the *quenched* average (where one averages the free energy $\ln Z$) in disordered systems. In the limit $N, L \to \infty$, with density of walkers $\rho = N/L$ fixed, we can express $P_s^{\text{typ}}(t)$ as

$$P_s^{\text{typ}}(t) = \exp\left\{N\langle \ln[Q_r(X, t)]\rangle_X\right\} \to \exp\left[-2\rho I_2(t)\right], \tag{4.8}$$

where

$$I_2(t) = -\int_0^\infty dX\,\ln Q_r(X, t), \tag{4.9}$$

assuming again that $Q_r(X, t)$ is an even function of X.

Thus the determination of the average and typical behaviour reduces to the evaluation of two integrals $I_1(t)$ in (4.6), $I_2(t)$ in (4.9).

4.2. Average behaviour for one-dimensional diffusion with resetting

The Laplace transform of (4.6) $\tilde{I}_1(s) = \int_0^\infty I_1(t)e^{-st}dt$ can be determined as follows:

$$\tilde{I}_1(s) = \int_0^\infty dX \left[\frac{1}{s} - \tilde{Q}_r(X,s) \right]$$

$$= \frac{(r+s)}{s} \int_0^\infty dX \left[\frac{e^{-\alpha X}}{s + r\,e^{-\alpha X}} \right] \tag{4.10}$$

$$= \frac{(r+s)}{sr\alpha} \ln\left(\frac{s+r}{s} \right), \tag{4.11}$$

where, in the second line, we have used the expression of $\tilde{Q}_r(X,s)$ given in (3.15). The Laplace transform can be inverted to yield [43]

$$I_1(t) = \left(\frac{D}{r} \right)^{1/2} \int_0^{rt} dv \frac{1 - e^{-v}}{v} \left[\mathrm{erf}[(rt-v)^{1/2}] + \frac{e^{-(rt-v)}}{\sqrt{\pi(rt-v)}} \right]. \tag{4.12}$$

The asymptotic behaviours of $I_1(t)$ are given by

$$I_1(t) \simeq \begin{cases} \dfrac{2}{\sqrt{\pi}}(Dt)^{1/2} & \text{for } rt \ll 1, \\[2mm] \left(\dfrac{D}{r} \right)^{1/2} [\ln(rt) + \gamma_e] & \text{for } rt \gg 1, \end{cases} \tag{4.13}$$

where γ_e is Euler's constant which may be defined as

$$\gamma_e = \int_0^\infty du\,e^{-u} \ln u = 0.577216\ldots. \tag{4.14}$$

The first limit $rt \ll 1$ can be obtained from the small v expansion of the integral (4.12). The $rt \gg 1$ behaviour can be most easily obtained from the $s \to 0$ of the Laplace transform (4.11)

$$\tilde{I}_1(s) = \frac{2}{s\alpha_0} \ln s + O(1/s), \tag{4.15}$$

together with the identity

$$\int_0^\infty dt\,e^{-st} \ln t = \frac{1}{s} \int_0^\infty du\,e^{-u} \ln \frac{u}{s} = -\frac{\gamma_e}{s} - \frac{\ln s}{s}. \tag{4.16}$$

By substituting the result for $I_1(t)$ given in (4.13) in (4.5), one finds that the long-time behaviour ($rt \gg 1$) of the average survival probability is a power law with an exponent that varies continuously with the density

$$P_s^{\mathrm{av}}(t) \sim t^{-2\rho(D/r)^{1/2}}. \tag{4.17}$$

Such behaviour is somewhat unexpected, since in the absence of resetting the average survival probability decays like $P_s(t) \sim e^{-b\sqrt{t}}$ [102–104].

4.3. Typical behaviour for one-dimensional diffusion with resetting

For the typical behaviour (the quenched case) we recall that in the long time limit the inversion of the Laplace transform $\tilde{Q}_r(X,s)$ is dominated by a pole at s_0 (3.30) in the complex s plane

endowing $Q_r(X, t)$ with exponential dependence on time. Therefore at large times, $I_2(t)$ in (4.9) becomes

$$I_2(t) \simeq -t \int_0^\infty dX \, s_0(X). \tag{4.18}$$

The integral may be computed exactly as follows. We first define u_0 through

$$s_0 = r(u_0 - 1). \tag{4.19}$$

Then we may express X as a function of u_0 from (3.30)

$$u_0 = 1 - e^{-\gamma u_0^{1/2}}, \tag{4.20}$$

where γ is defined in (3.24). In particular we have

$$\alpha_0 X = -\frac{\ln(1 - u_0)}{u_0^{1/2}}, \tag{4.21}$$

allowing one to transform the integration from X to u with range $0 < u < 1$. One finds

$$-\int_0^\infty dX \, s_0(X) = \frac{r}{\alpha_0} \int_0^1 du \left[\frac{(1-u)}{u^{3/2}} \ln(1-u) + \frac{1}{u^{1/2}} \right] \tag{4.22}$$

$$= (Dr)^{1/2} 4(1 - \ln 2). \tag{4.23}$$

Thus, the asymptotic decay of the typical total survival probability is exponential with explicit decay constant [43]

$$P_s^{\text{typ}}(t) \sim \exp\left[-t\rho(Dr)^{1/2} 8(1 - \ln 2) \right], \tag{4.24}$$

which behaves quite differently from the average survival probability (4.17).

4.4. Explanation in terms of extreme value statistics

It is important to note that the average and typical survival probabilities have distinct asymptotic behaviours—the average behaviour (4.17) decays far more slowly than the typical behaviour (4.24). In fact the average behaviour has a different functional form, power law rather than exponential decay with time, which is a surprising result.

In order to understand the asymptotic form of the average survival probability (4.17), we consider the following simple picture. At long times we assume that the absorption probability of the target will be dominated by the searcher which started nearest to the target (taken to be at the origin). Denoting the position of this searcher by y, the average survival probability for the many searcher problem should then be recovered by averaging the single searcher's survival probability (3.31) over the distribution of the position y. What we will now show is that this average is dominated by rare configurations of the searcher initial positions where y is large.

In order to obtain the distribution of the distance y of the nearest searcher to the origin, we consider first the probability that a single searcher, distributed uniformly in a box of size L, starts at distance $X > y$ from the origin

$$\text{Prob}(X > y) = \frac{2}{L} \int_y^{L/2} dX = \left[1 - \frac{2y}{L} \right]. \tag{4.25}$$

Then it follows that the probability that all N searchers start at distance $X > y$ from the origin is given by, in the limit of large N, L with ρ fixed,

$$[\text{Prob}(X > y)]^N \to \exp(-2\rho y) \tag{4.26}$$

and we obtain the distribution of the distance y of the nearest searcher from the target as

$$P(y) = -\frac{\text{d}}{\text{d}y} \exp[-2\rho y] = 2\rho \exp(-2\rho y). \tag{4.27}$$

As we have seen for a single searcher starting at y the long time behaviour of the survival probability, in the presence of resetting, is given by (3.31)

$$Q_r \simeq \exp\left[-|s_0(y)|t\right], \tag{4.28}$$

where the function $s_0(y)$ is given by (3.30). Within the approximation that the survival probability of the target is dominated by the searcher initially nearest to the target, at distance y, we obtain $P_s^{\text{av}}(t)$ as the average of (4.28) with respect to (4.27)

$$P_s^{\text{av}}(t) \simeq \int_0^\infty \text{d}y \, \exp[-2\rho y + ts_0(y)]. \tag{4.29}$$

For large t, we expect the integral to be dominated by the value y^* that maximises the integrand with respect to y. Thus

$$-2\rho + ts_0'(y^*) = 0. \tag{4.30}$$

For large t we expect y^* to be large and $s_0(y^*) \simeq -r\exp(-\alpha_0 y^*)$ small. The maximum y^* is then given by

$$-2\rho + r\alpha_0 t \, \text{e}^{-\alpha_0 y^*} = 0 \tag{4.31}$$

which implies that asymptotically

$$y^* \sim \frac{\ln t}{\alpha_0}. \tag{4.32}$$

The dominant behaviour of the integral (4.29) is then

$$P_s^{\text{av}}(t) \sim \exp\left[-2\rho y^*\right] \tag{4.33}$$

which recovers the asymptotic result (4.17) of subsection 4.2.

Thus we have deduced that at long times t the average survival probability is dominated by initial arrangements of searchers in which the nearest searcher is at distance $y^* \sim \ln t/\alpha_0$. These initial configurations of searchers are atypical as may be seen by comparing with the distribution of nearest searcher distance (4.27). As time progresses rarer and rarer initial configurations of the searchers, with nearest searcher at distance $y \sim \ln t$ from the target, dominate the average. This reflects a strong dependence on the initial conditions whose memory is retained through resetting.

5. General resetting and first-passage processes

5.1. Arbitrary stochastic process with Poissonian resetting

In the previous sections, we mainly discussed diffusive processes with Poissonian and non-Poissonian resetting (see figure 1). The resetting can be generalized to arbitrary stochastic processes, going beyond simple diffusion as follows:

- Consider any process $x(t)$ evolving freely under its own dynamics (it can be deterministic or stochastic) during a certain interval of time.
- At the end of this random period, the process is reset to a new starting point X_r (which can in particular be the initial position $X_r = x_0$) and then its dynamics restarts afresh.
- The interval of free evolution between resets is drawn independently from a distribution $\psi(\tau)$ (hence naturally it is a renewal process). For Poissonian resetting, $\psi(\tau) = r\,e^{-r\tau}$.

In this subsection, we first focus on Poissonian resetting. Exploiting the renewal structure, we can then relate observables in the presence of resetting to the same observables in the absence of resetting, for arbitrary stochastic processes, as was done for diffusive processes before (see e.g. (2.10)). For example, $p_r(x, t|x_0)$, defined as the probability density to reach x at time t in the presence of resetting is related to the propagator without resetting, $G_0(x, t|x_0)$,

$$p_r(x, t|x_0) = e^{-rt}G_0(x, t|x_0) + r\int_0^t d\tau\, e^{-r\tau}G_0(x, \tau|X_r). \tag{5.1}$$

This is the analogue of (2.10) for the diffusive process. The derivation of this relation (5.1) is straightforward, as in the diffusive case. The first term refers to no resetting in $[0, t]$. In the second term, τ denotes the time between t and the last resetting before t. In this term, $rd\tau e^{-r\tau}$ denotes the probability that there is no resetting during τ followed by a resetting during τ and $\tau + d\tau$. During this interval τ, the particle evolves freely with the propagator p_0, since there is no resetting event in the interval $(t - \tau, t]$ (as in figure 2). One then takes the product of these two terms and integrate over all τ in $[0, t]$. Therefore, if we know the free propagator G_0, in principle one can compute p_r in the presence of resetting. Finally, the stationary state, if it exists, can be obtained by taking the limit $t \to \infty$ in (5.1). This gives

$$p_r^*(x) = r\int_0^\infty d\tau\, e^{-r\tau}G_0(x, \tau|X_r). \tag{5.2}$$

The stationary state is thus given by the Laplace transform of the free propagator (up to the constant factor r), provided the integral in (5.2) is finite. This is then a very general relation for any stochastic process with resetting.

One can also relate other observables between processes with and without resetting, going beyond the one-point function discussed above. For instance, let us define the two-point correlation function for any process as

$$C(t_1, t_2) = \langle x(t_1)x(t_2)\rangle - \langle x(t_1)\rangle\langle x(t_2)\rangle. \tag{5.3}$$

For Poissonian resetting, one can relate the correlator $C_r(t_1, t_2)$ for the process with reset to $C_0(t_1, t_2)$ referring to the correlator in the absence of resetting. For resetting to the initial condition $X_r = x_0$, this exact relation has been derived recently in [105] and it reads (for $t_1 \leqslant t_2$)

$$C_r(t_1, t_2) = e^{-r(t_2 - t_1)}\left[e^{-rt_1}C_0(t_1, t_2) + r\int_0^{t_1} d\tau\, e^{-r\tau}C_0(\tau, t_2 - t_1 + \tau)\right]. \tag{5.4}$$

The derivation of this relation exploits the renewal structure of the reset process. This relation was then used to obtain the power spectrum of various stochastic processes with reset, such as the fractional Brownian motion (fBm) [105].

Similarly, one can relate the survival probability for arbitrary stochastic processes with Poissonian resetting to that in the absence of resetting via the renewal equation

$$Q_r(x_0, t) = e^{-rt}Q_0(x_0, t) + r\int_0^t d\tau \, e^{-r\tau}Q_0(X_r, \tau)Q_r(x_0, t - \tau), \tag{5.5}$$

which was already presented in section 3.1 and was exploited to obtain explicit results for diffusive processes. Consequently, the MTA for the process with resetting can be related to the Laplace transform of the survival probability without resetting, for any arbitrary stochastic process with Poissonian resetting [see (3.6)].

As an application of these general results valid for arbitrary processes with Poissonian resetting, going beyond the simple diffusion, we just give the example of a run and tumble particle (RTP) subject to stochastic resetting [61, 106]. The position of an RTP in one dimension, in the absence of resetting, evolves via the stochastic equation of motion

$$\frac{dx}{dt} = v_0\,\sigma(t) \tag{5.6}$$

where $\sigma(t)$ is a dichotomous noise that switches between two states $\sigma(t) = \pm 1$ with rate γ. The correlation function of the noise then decays as $\langle\sigma(t_1)\sigma(t_2)\rangle = e^{-2\gamma|t_1-t_2|}$. For finite γ, the noise has thus a memory. This motion is sometimes referred to as a persistent random walk and has been the subject of renewed recent interest in the context of active particles. Consider this RTP being subjected to Poissonian resetting, and using the general results above, one can compute various observables in the presence of resetting in terms of those without resetting. For example, the propagator $G_0(x, t|x_0 = 0)$ for an RTP starting at the origin with equal probability for the initial velocity to be ± 1 can be computed exactly. Its Laplace transform reads simply

$$\tilde{p}_0(x, s|0) = \int_0^\infty dt \, e^{-st}\, G_0(x, t|x_0 = 0) = \frac{\lambda(s)}{2s}\, e^{-\lambda(s)|x|}, \quad \text{where } \lambda(s) = \sqrt{\frac{s(s + 2\gamma)}{v_0}}. \tag{5.7}$$

Under Poissonian resetting of the position to the initial position $x_0 = 0$ with rate r, and randomisation of the velocity after each resetting, the stationary state $p_r^*(x)$ is then given by (5.2) and one gets [61]

$$p_r^*(x) = r\,\tilde{p}_0(x, s = r|0) = \frac{\lambda(r)}{2}\, e^{-\lambda(r)\,|x|}, \quad \text{where } \lambda(r) = \sqrt{\frac{r(r + 2\gamma)}{v_0}}. \tag{5.8}$$

It turns out that this stationary state is robust, i.e., it does not depend on the precise velocity randomisation protocol following each reset of the position [61]. In the limit $v_0 \to \infty$, $\gamma \to \infty$ keeping the ratio $v_0^2/(2\gamma) = D$ fixed, the RTP is known to reduce to ordinary diffusion. By taking this limit in (5.8), one indeed recovers the diffusive stationary state given in equations (2.18) and (2.19). Similarly, one can also derive the survival probability and the MTA of the RTP with reset [61], from (5.5) using the known result for the survival probability of the RTP without reset [107, 108]. We also mention that the telegrapher's equation, which naturally arises in the context of RTP, has been studied under resetting [109].
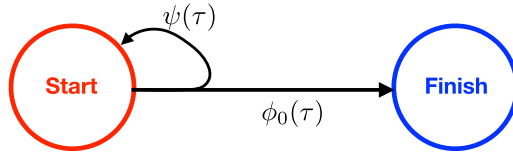
Figure 9. Schematic illustration of task to be completed with bare completion distribution $\phi_0(\tau)$ and a restart process waiting time $\psi(\tau)$ [58].

5.2. General resetting and completion times

So far we have discussed Poissonian resetting for arbitrary stochastic processes. One can easily generalise these ideas to non-Poissonian resettings, as we have already seen for the diffusive process. The stationary state for non-Poissonian resetting and arbitrary stochastic process can be read off from (2.47), with $G_0(x, t|x_0)$ the bare propagator of the stochastic process without reset. For non-Poissonian resetting, the first-passage probability with reset can also be generalised to arbitrary processes, as we discuss below.

As already noted in section 2.8 one can also choose non-Poissonian resetting wherein the distribution of the waiting time τ to the next reset, $\psi(\tau)$, is specified. The Poissonian case corresponds to $\psi(\tau) = r \exp(-r\tau)$. For this, one can of course study the standard first-passage probability. This first-passage probability to find a target can be thought of, in a more general context, as the distribution of the time to complete a task; let us call this distribution $\phi_0(\tau)$ (see figure 9). The diffusive first-passage problem then is an example of a distribution $\phi_0(\tau)$ which decays asymptotically as $\phi_0(\tau) \sim \tau^{-3/2}$ i.e. it has infinite mean and variance. As we have seen, in this case resetting dramatically improves the mean time to completion.

More broadly one can consider a general completion time distribution $\phi_0(\tau)$ for the stochastic process, with for example finite mean, and ask whether resetting will improve the completion rate of the task [58].

The first renewal equation of section 3.10 is exactly applicable in this case and one obtains the result (3.66) which now reads

$$\tilde{Q}_r(s) = \frac{\int_0^\infty dt\, e^{-st} \Psi(t) Q_0(t)}{1 - \int_0^\infty dt\, e^{-st} \psi(t) Q_0(t)}, \tag{5.9}$$

where the survival probabilities, $Q_*(t)$ with $* = 0, r$, are now the probabilities that the task is not completed in time t ($Q_r(t)$ is the survival probability with resetting while $Q_0(t)$ is the survival probability without resetting).

If we define the completion time distribution of the reset process as $\phi_r(\tau)$, we then have

$$\phi_*(\tau) = -\frac{\partial Q_*(\tau)}{\partial \tau}, \tag{5.10}$$

so that the Laplace transforms are related through

$$\tilde{\phi}_*(s) = 1 - s\tilde{Q}_*(s), \tag{5.11}$$

again with $* = 0, r$. Substituting (5.11) in (5.9) and using integration by parts one finds

$$s \int_0^\infty dt\, e^{-st} \Psi(t) Q_0(t) = 1 - \int_0^\infty dt\, e^{-st} \left[\psi(t) Q_0(t) + \Psi(t) \phi_0(t) \right], \tag{5.12}$$

which yields

$$\tilde{\phi}_r(s) = \frac{\int_0^\infty dt\, e^{-st} \Psi(t) \phi_0(t)}{1 - \int_0^\infty dt\, e^{-st} \psi(t) Q_0(t)}. \tag{5.13}$$

Equations (5.9) and (5.13) are the general results for resetting time distribution $\psi(\tau)$ and completion time distribution $\phi_0(\tau)$, for the underlying stochastic process. As noted they are derived in [55] for the diffusive process and generalised to an arbitrary stochastic process in [60].

5.3. Poissonian reset and general completion time distribution

In the case of Poissonian reset the formulas (5.9) and (5.13) simplify to

$$\tilde{Q}_r(s) = \frac{\tilde{Q}_0(r+s)}{1 - r\tilde{Q}_0(r+s)}, \tag{5.14}$$

and

$$\tilde{\phi}_r(s) = \frac{(r+s)\tilde{\phi}_0(r+s)}{s + r\tilde{\phi}_0(r+s)}, \tag{5.15}$$

where, as usual, $\tilde{f}(s)$ is the Laplace transform with Laplace variable s of the function $f(t)$ in the time domain. Formula (5.15) was derived by Reuveni [58] using an alternative recursion relation for the completion time under reset T_r:

$$T_R = \begin{cases} T & \text{for } T < R \\ R + T_R' & \text{for } T \geqslant R \end{cases} \tag{5.16}$$

where T_R and T_R' are two completion times drawn from $\phi_r(t)$, R is a reset time drawn from $\psi(t)$ and T is a completion time drawn from $\phi_0(t)$. This recursion involving the i.i.d.random variables T_R, T_R' contains essentially the same information as the first renewal equation for the probability distribution of T_R.

From (5.15) all moments of the completion time T_R can easily be computed, in particular

$$\langle T_r \rangle = \frac{1}{r} \frac{1 - \tilde{\phi}_0(r)}{\tilde{\phi}_0(r)} \tag{5.17}$$

$$\langle T_r^2 \rangle = \frac{2}{r^2} \frac{r\frac{d\tilde{\phi}_0(r)}{dr} - \tilde{\phi}_0(r) + 1}{\tilde{\phi}_0(r)^2}. \tag{5.18}$$

Then the optimal resetting rate is given by extremising $\langle T_r \rangle$ which yields

$$\frac{d\tilde{\phi}_0(r)}{dr} = \frac{\tilde{\phi}_0(r)(1 - \tilde{\phi}_0(r))}{r} \tag{5.19}$$

at $r = r^*$.

Reuveni made the interesting observation that at $r = r^*$ the coefficient of variation (defined as standard deviation over mean) is unity,

$$\frac{\sigma(T_{r^*})}{\langle T_{r^*} \rangle} = 1, \tag{5.20}$$

implying universality as the result does not depend on $\phi_0(\tau)$. Furthermore, (5.20) trivially implies that

$$\langle T_{r^*} \rangle = \frac{1}{2} \frac{\langle T_{r^*}^2 \rangle}{\langle T_{r^*} \rangle} \tag{5.21}$$

which may be interpreted as follows. At the optimal resetting rate the mean time to completion (from the initial condition) is equal to the *mean residual life time* [110] which is the mean time to completion of the process without resetting from a randomly chosen time during an (incomplete) run.

5.4. General resetting time distribution and general completion time distribution

Pal and Reuveni [59] considered general resetting time and general completion time distributions and sought to answer the question posed in [55, 82] of whether a sharp restart distribution $\psi(\tau) = \delta(\tau - \tau^*)$, with τ^* suitably chosen is always optimal, i.e. minimises $\langle T_r \rangle$. They showed through a probabilistic argument that this is indeed the case and moreover at the optimal resetting the coefficient of variation obeys

$$\frac{\sigma(T_r)}{\langle T_r \rangle} \leqslant 1. \tag{5.22}$$

Here we present a subsequent alternative proof, given by Chechkin and Sokolov [60], that a sharp reset distribution is optimal, which follows easily from (5.11).

The mean time to completion $\langle T_r \rangle$ is given, as usual, by setting $s = 0$ in the expression of the Laplace transform of the survival probability (5.9)

$$\langle T_r \rangle = \frac{\int_0^\infty dt \, \Psi(t) Q_0(t)}{1 - \int_0^\infty dt \, \psi(t) Q_0(t)}. \tag{5.23}$$

Defining F_0 as the probability of completion before time t

$$F_0(t) = 1 - Q_0(t), \tag{5.24}$$

the denominator of (5.23) becomes $\int_0^\infty F_0(t)\psi(t)$ (assuming $\int_0^\infty dt \, \psi(t) = 1$). The numerator of (5.23) may be rewritten using the definition (2.40) as

$$\int_0^\infty dt \, \Psi(t) Q_0(t) = \int_0^\infty dt' \psi(t') \int_0^{t'} dt \, Q_0(t) = \int_0^\infty dt' \psi(t') \left[t' - \int_0^{t'} dt F_0(t) \right]. \tag{5.25}$$

Thus, after a trivial relabelling of integration variables, (5.23) becomes

$$\langle T_r \rangle = \frac{\int_0^\infty dt \, \psi(t) F_0(t) H(t)}{\int_0^\infty dt \, \psi(t) F_0(t)}, \tag{5.26}$$

where

$$H(t) = \frac{t - \int_0^t dt' F_0(t')}{F_0(t)}. \tag{5.27}$$

(We note that we have used the notation $H(t)$ rather than $G(t)$ of [60] to avoid a clash of notation with Green function.) Since $\psi(t)F_0(t)$ is a positive quantity we immediately get a lower bound from (5.26)

$$\langle T_r \rangle \geqslant \min_{0 \leqslant t < \infty} H(t). \tag{5.28}$$

The lower bound is achieved by choosing $\psi(t) = \delta(t - t_r)$ where t_r gives the global minimum of $H(t)$, as can be easily checked from (5.26). Thus the minimum mean time to completion is realised by a sharp resetting distribution.

The optimal value of the resetting time, t_r, can be obtained by extremising $H(t)$ (assuming the minimum is an extremum rather than a boundary value) to obtain

$$\frac{1 - F_0(t)}{F_0(t)} - \frac{t - \int_0^t dt' F_0(t')}{F_0^2(t)} F_0'(t) = 0. \tag{5.29}$$

Note that this equation can be written in a slightly more compact form in terms of $Q_0(t) = 1 - F_0(t)$ as [62]

$$Q_0(t) - Q_0^2(t) + Q_0'(t) \int_0^t dt' Q_0(t') = 0. \tag{5.30}$$

If this equation has no solution the minimum will be at $t_r \to \infty$, i.e. in the limit of no resetting. In addition, if $Q_0(t) = e^{-\alpha t}$ with $\alpha > 0$, one finds that this equation (5.30) is automatically satisfied for all time t, which means that there is no optimal t_r in this case.

5.5. Michaelis–Menten reaction scheme (MMRS)

An interesting context in which to frame general questions of restarting a complex stochastic process has been proposed by Reuveni, Urbakh and Klafter and entails a (generalised) Michaelis–Menten reaction scheme (MMRS) [28, 111]. These authors envisage a molecular interaction involving an enzyme molecule which, when bound to a substrate, triggers the start of a complex process leading to the production of a product. This process has a bare (in the absence of resetting) completion time distribution $\phi_0(\tau)$. The enzyme unbinds and binds reversibly, thus stopping and restarting the process, and the unbinding rate corresponds to the resetting rate. Also note that in this scenario the unbound enzyme is in a quiescent state where the stochastic process has been switched off. Schematically the reaction scheme is

$$E + S \rightleftharpoons ES \to E + P.$$

Most generally there are *three* waiting times with associated distributions here: the duration of the quiescent period when the enzyme is unbound, the duration of the period during which the enzyme is bound and the production process is active, and the time to completion of the production process (given that the enzyme is bound for sufficiently long time). Reuveni *et al* initially considered an exponential distribution for the unbinding times which corresponds to Poissonian resetting. In this case they obtained using a recursion similar to (5.16), namely [28]

$$\langle T_r \rangle = \frac{\langle \tau \rangle}{\tilde{\phi}_0(r)} + \frac{1}{r} \frac{1 - \tilde{\phi}_0(r)}{\tilde{\phi}_0(r)} \tag{5.31}$$

where $\langle \tau \rangle$ is the mean duration of the quiescent period when the enzyme is unbound, which we shall refer to as the refractory period. Note that (5.31) is equation (4) of [28] transcribed into our current notation. The equation illustrates that the refractory period gives an additive

constant to the mean time to completion. The coefficient of variation of the completion time at the optimal restart rate is now [58]

$$\frac{\sigma(T_{r^*})}{\langle T_{r^*}\rangle} = \sqrt{1 + \frac{\langle \tau^2\rangle - 2\langle \tau\rangle^2}{\tilde{\phi}_0(r)\langle T_{r^*}\rangle^2}}, \tag{5.32}$$

which is now no longer universal as it depends on $\tilde{\phi}_0(r)$.

However, one can usefully define a critical value of the coefficient of variation

$$C_v^* = \sqrt{1 + \frac{2\langle \tau\rangle}{\langle T_r\rangle}} \tag{5.33}$$

which allows one to categorise the behaviour of $\langle T_r\rangle$ at small r, i.e. how the introduction of resetting changes the mean completion time: if $C_v < C_v^*$ there is an inhibitory effect, i.e. the mean time to completion increases; if $C_v > C_v^*$ there is an excitatory effect, i.e. the mean time to completion decreases (linearly with r); if $C_v \to \infty$ there is a superexcitatory effect i.e. the mean time to completion decreases nonanalytically with r; if $\langle T_0\rangle \to \infty$ there is a restorative effect i.e. an infinite mean time to completion is rendered finite. The example of diffusion with resetting falls into the restorative category, which simply means that $\langle T_r\rangle$ decreases as r increases for small r. However, there can be situations where the opposite happens, i.e., $\langle T_r\rangle$ increases with r for small r. In fact, by changing system parameters, such as reaction rates, it is possible to induce a transition between the two scenarios—such 'restart transitions' in a generic setting have been discussed in several recent papers [72, 111–113].

It was also shown in [111] how the equation for the optimal resetting rate (5.19) generalises in the case of a refractory period with mean $\langle \tau\rangle$ to

$$\frac{\tilde{\phi}_0(r)(1 - \tilde{\phi}_0(r))}{r^2\tilde{\phi}_0'(r)} - \frac{1}{r} = \langle \tau\rangle \tag{5.34}$$

at $r = r^*$.

5.6. Effects of refractory period

The effects of a refractory period have been further studied in [114, 115] where a spatial stochastic process with propagator $G_0(x, t)$ was considered in the presence of resetting with a refractory period. (Note that in that work the convention is taken that the refractory period occurs after a reset, so that the initial condition is slightly different from [28, 111].) A first renewal equation was used to derive the probability distribution in the absence of an absorbing target and the Laplace transform of the survival probability in the presence of an absorbing target.

For the case of Poissonian resetting with rate r to the origin ($X_r = 0$) the nonequilibrium stationary state has the interesting feature of a delta peak at the resetting position, due to the refractory period [115]

$$p^*(x) = \frac{r}{1 + r\langle \tau\rangle} \left[\tilde{G}_0(x, r) + \langle \tau\rangle\delta(x) \right]. \tag{5.35}$$

The relative weight of the peak is equal to the ratio of the mean refractory period to the mean resetting period. If the mean refractory period diverges then $p^*(x) \to \delta(x)$. The emergence of the peak has been analysed and it was shown how slow relaxation can emerge when the refractory period distribution $W(\tau)$ has a power law tail.

In addition, the case of a *correlated* resetting time and refractory period was considered, a simple example being Poissonian resetting with rate r but now with a correlated refractory period

$$H(t, \tau) = r\,\mathrm{e}^{-rt}W(\tau|t) \tag{5.36}$$

where $W(\tau|t)$ is the refractory period distribution conditioned on a preceding resetting time t. Finally, the joint active time and first-passage time distribution was calculated.

An extension of the idea of a refractory period following a reset is to have a different dynamics, which commences on resetting and returns the process to the origin over some finite time. Such two-phase, reset-return processes have been studied in [116–118].

6. Extended systems with resetting

Even though we have so far considered resetting of a single particle stochastic process (with the exception of section 4 which is still a system of non-interacting particles with random initial conditions). Resetting dynamics can be easily generalised to any extended system with interacting degrees of freedom as we discuss in this section.

6.1. Resetting dynamics for general extended systems

Consider any extended system, e.g. a fluctuating interface where heights of the interface at different space points are the relevant degrees of freedom that fluctuate in time according to some prescribed stochastic dynamics. Similarly, one can consider for instance an Ising model where the spins are the degrees of freedom that evolve, under say the Glauber dynamics at some temperature T. One can also think of a polymer chain consisting of N monomers (the degrees of freedom) that evolve via say the Rouse dynamics. Let $p_0(\mathcal{C}, t|\mathcal{C}_\mathrm{in})$ denote the probability that the system is in a given configuration $\mathcal{C}$ at time t, starting from the initial configuration $\mathcal{C}_\mathrm{in}$. For example, in the case of the Ising model, a configuration $\mathcal{C}$ corresponds to a spin configuration. For a fluctuating interface, $\mathcal{C}$ is specified by a height profile $h(x, t)$ in $1 + 1$ dimensions. All these systems have their own microscopic dynamics by which the configuration $\mathcal{C}$ evolves in time, but at this point, we do not need to specify the dynamics.

Now imagine that we introduce the resetting process whereby the configuration $\mathcal{C}$ at time t is reset to a specific reset *configuration* $\mathcal{C}_r$ with a constant rate r. This is a generalisation of the single particle case discussed earlier where the configuration $\mathcal{C}$ is specified by the position of the particle x and X_r denotes the resetting position. Here we will discuss only the Poissonian resetting for simplicity, though it can be easily generalised to non-Poissonian resetting as well.

More precisely, in time $\mathrm{d}t$, the configuration $\mathcal{C}$ is reset to $\mathcal{C}_r$ with probability $r\mathrm{d}t$ and, with the complementary probability $1 - r\mathrm{d}t$, the system continues to evolve by its own dynamics. Let $p_r(\mathcal{C}, t|\mathcal{C}_\mathrm{in})$ denote the probability that the system is in configuration $\mathcal{C}$ in the presence of resetting with rate r, starting from the initial configuration $\mathcal{C}_\mathrm{in}$. Then, as in the single particle case (see section 2.2), one can express $p_r(\mathcal{C}, t|\mathcal{C}_\mathrm{in})$ in terms of $p_0(\mathcal{C}, t|\mathcal{C}_\mathrm{in})$ using a renewal approach, which takes into account the event of last resetting before time t. This reads

$$p_r(\mathcal{C}, t|\mathcal{C}_\mathrm{in}) = \int_0^t r\,\mathrm{e}^{-r\tau}p_0(\mathcal{C}, \tau|\mathcal{C}_r)\,\mathrm{d}\tau + \mathrm{e}^{-rt}\,p_0(\mathcal{C}, t|\mathcal{C}_\mathrm{in}). \tag{6.1}$$

The second term represents the case where there is no resetting in the interval $[0, t]$, which happens with probability e^{-rt}—in this case the system evolves by its own dynamics (without

reset) from time 0 till time t, explaining the occurrence of $p_0(\mathcal{C}, t|\mathcal{C}_{\text{in}})$ in the second term. The first term can also be explained easily. Let the last resetting event before time t occur at time $\tau_1 = t - \tau$. Looking backwards in time from the instant t, this means that there is no resetting in the interval $[0, \tau]$ followed by a resetting event between τ and $\tau + d\tau$—the probability for this event is $r\,e^{-r\tau}d\tau$. During this time interval τ followed by the last resetting, the system evolves freely (without resetting) from configuration $\mathcal{C}_r$ to $\mathcal{C}$ by its own dynamics, which happens with probability $p_0(\mathcal{C}, \tau|\mathcal{C}_r)$.

Even though the system's own dynamics may not lead to a stationary state, the resetting drives the systems into a non-equilibrium stationary state (as in the single particle case). The corresponding stationary state is obtained by taking the $t \to \infty$ limit in (6.1), leading to

$$p_r^{\text{stat}}(\mathcal{C}) = \int_0^\infty r\,e^{-r\tau} p_0(\mathcal{C}, \tau|\mathcal{C}_r)\,d\tau. \tag{6.2}$$

Note that even though the stationary state is independent of the initial configuration $\mathcal{C}_{\text{in}}$, it does depend on the resetting configuration $\mathcal{C}_r$.

Various models of extended systems subject to resetting have been studied recently. This includes fluctuating interfaces [74, 119], reaction diffusion systems [120], exclusion processes [121], Ising model [122] etc. In the following, we discuss in detail a specific example of an extended system under resetting, namely a fluctuating $1 + 1$-dimensional interface.

6.2. Fluctuating interfaces: non-equilibrium steady states

We consider a $1 + 1$ dimensional interface characterized by a height field $H(x, t)$ at position x and time t. Starting from an initially flat profile: $H(x, 0) = 0\ \forall x$, the heights evolve according to the Kardar–Parisi–Zhang (KPZ) equation [123–125]:

$$\frac{\partial H}{\partial t} = \nu \frac{\partial^2 H}{\partial x^2} + \frac{\lambda}{2}\left(\frac{\partial H}{\partial x}\right)^2 + \eta(x, t), \tag{6.3}$$

where ν is the diffusivity, λ accounts for the nonlinear term, while $\eta(x, t)$ is a Gaussian noise of zero mean and correlations $\langle \eta(x, t)\eta(x', t') \rangle = 2D\delta(x - x')\delta(t - t')$. In the case where $\lambda = 0$, the non-linear term disappears and the height field $H(x, t)$ becomes Gaussian—in this case the KPZ equation (6.3) reduces to the Edwards–Wilkinson (EW) equation [126].

For an interface of length L evolving according to (6.3), the spatially averaged height $\overline{H(x, t)} = \int_0^L dx\, H(x, t)/L$ grows with time with velocity $v_\infty = (\lambda/2)\int_0^L dx \langle (\partial H/\partial x)^2 \rangle$. Let us define the height fluctuation as

$$h(x, t) = H(x, t) - \overline{H(x, t)}. \tag{6.4}$$

For a given sample of the interface, we define the (empirical) variance of the height fluctuation as

$$\sigma^2(L, t) = \frac{1}{L}\int_0^L h^2(x, t)\,dx. \tag{6.5}$$

Note that $\sigma^2(L, t)$ is still a random variable, fluctuating from sample to sample. The interface width $W \equiv W(L, t)$ is then defined as

$$W(L, t) = \sqrt{\langle \sigma^2(L, t) \rangle}. \tag{6.6}$$

where the average $\langle \cdot \rangle$ is an ensemble average over different realisations of the noise $\eta(x, t)$. In the thermodynamic limit $L \to \infty$, we expect that $\sigma^2(L, t)$ approaches its expectation value. As time grows beyond a non-universal microscopic time scale $T_{\text{micro}} \sim O(1)$, the width initially grows as a power law $W(L, t) \sim t^\beta$ as long as $T_{\text{micro}} \ll t \ll T^* \sim L^z$, where β and z are known as the growth and the dynamical exponents respectively. For times $t \gg T^*$, the width saturates to an L-dependent value $\sim L^\alpha$. In other words

$$W(L, t) \sim \begin{cases} t^\beta, & T_{\text{micro}} \ll t \ll T^* \sim L^z \\ L^\alpha, & t \gg T^*. \end{cases} \tag{6.7}$$

The former is called the 'growing' regime while the latter is called the 'stationary' regime. The width in these two regimes is connected via the Family–Vicsek scaling form [127]: $W(L, t) \sim L^\alpha \mathcal{W}(t/T^*)$ where the scaling function $\mathcal{W}(s)$ behaves as a constant as $s \to \infty$, and as s^β as $s \to 0$ where $\beta = \alpha/z$. For the one-dimensional KPZ equation (6.3) with $\lambda \neq 0$, the dynamical exponent is $z = 3/2$ while the width exponent $\alpha = 1/2$, and hence the growth exponent $\beta = 1/3$. In contrast, for $\lambda = 0$ (i.e. the EW equation), the exponents are $z = 2$, $\alpha = 1/2$ and $\beta = 1/4$. Indeed, generically at long times $t \gg T^*$, the full probability distribution of the height fluctuations (6.4) reaches a stationary state in a *finite* system at long times $t \gg T^*$. In fact, both for the KPZ and the EW cases in $(1 + 1)$ dimensions, the stationary height distribution $p_0^{\text{stat}}(h, L)$ turns out to be a simple Gaussian [128]. If, however, we take the $L \to \infty$ limit first, such that T^* diverges, the system never reaches a stationary state and it is always in a 'growing' regime where the height fluctuations typically grow as a power law in time and where the distribution of the height fluctuations is time-dependent.

In the following, we will restrict ourselves to this growing regime where $t \ll T^* \sim L^z$ and switch on the resetting that interrupts the growth and restarts the system from its initial flat configuration. This resetting move drives the system to a non-equilibrium stationary state (NESS) as discussed in (6.2) in the general context. Below we characterise precisely the height distribution in this reset driven NESS. We characterise a configuration by $\mathcal{C} = \{h(x, t)\}_{0 \leqslant x \leqslant L}$ that specifies the height fluctuation at each space point. Furthermore, we integrate out the heights at all points except one, say at the origin at $x = 0$ and denote by $p_r(h, t)$ as the height distribution at $x = 0$ at time t, in the presence of the resetting at constant rate r. Focussing thus on this marginal distribution $p_r(h, t)$ at $x = 0$, the general (6.2) then reads

$$p_r(h, t) = \int_0^t r\, e^{-r\tau} p_0(h, \tau)\, d\tau + e^{-rt} p_0(h, t), \tag{6.8}$$

where $p_0(h, \tau)$ is the height distribution at time τ in the growing regime, starting from a flat configuration in the absence of resetting. This equation is valid for arbitrary time t. Taking the $t \to \infty$ limit, the stationary height distribution at $x = 0$ is given by

$$p_r^{\text{stat}}(h) = \int_0^\infty d\tau\, r\, e^{-r\tau} p_0(h, \tau). \tag{6.9}$$

We start with the simpler EW case where $\lambda = 0$ in (6.3). The resulting linear equation can be trivially solved using Fourier transform and it gives a Gaussian distribution for $p_0(h, \tau)$

$$p_0(h, \tau) = \frac{1}{\sqrt{2\pi W^2}}\, e^{-\frac{h^2}{2W^2}}, \tag{6.10}$$

where $W \equiv W(L \to \infty, \tau)$ is the time-dependent width of the interface in the thermodynamic limit $L \to \infty$ and is given by

$$W(\tau) = D\sqrt{\frac{2\tau}{\nu \pi}}. \tag{6.11}$$

Plugging this result (6.10) in (6.9), the reset induced stationary height distribution can be expressed in the scaling form

$$p_r^{\text{stat}}(h) \sim \sqrt{\gamma}\, r^{1/4}\, G^{\text{EW}}(h\sqrt{\gamma}r^{1/4}), \tag{6.12}$$

where $\gamma = \sqrt{\pi\nu}/(D2^{3/2})$ and $G^{\text{EW}}(x)$ is given by

$$G^{\text{EW}}(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{dy}{y^{1/4}}\, \exp\left(-y - \frac{x^2}{\sqrt{y}}\right), \tag{6.13}$$

which is symmetric in x, $G^{\text{EW}}(-x) = G^{\text{EW}}(x)$, yielding zero mean, and variance $\int_{-\infty}^{+\infty} x^2 G^{\text{EW}}(x)\,dx = \sqrt{\pi/4}$. From the scaling form in (6.12), one obtains the scaling of the stationary width with r as $W_r^{\text{EW}} \sim r^{-1/4}$. One can show that $G^{\text{EW}}(x)$ behaves asymptotically as [74]

$$G^{\text{EW}}(x) \sim \begin{cases} \dfrac{1}{\sqrt{\pi}}\left[\Gamma\left(\dfrac{3}{4}\right) - x^2\Gamma\left(\dfrac{1}{4}\right) + \dfrac{8}{3}\sqrt{\pi}|x|^3\right], & x \to 0, \\[2mm] c|x|\exp[-3/2^{2/3}\,|x|^{4/3}], & x \to \pm\infty, \end{cases} \tag{6.14}$$

where $\Gamma(x)$ is the Gamma function and c is a computable constant. Interestingly, due to the $|x|^3$ term in (6.14), $G^{\text{EW}}(x)$ is non-analytic close to $x = 0$. In the limit $x \to \pm\infty$, the stretched exponential behaviour (6.14) is significantly different from a Gaussian tail. These analytical results have been also verified numerically in [74].

We now turn to the KPZ case. Here, it is known that for times $T_{\text{micro}} \ll t \ll T^*$, and for a flat initial profile, the interface height $H(x,t)$ has a deterministic linear growth with stochastic $t^{1/3}$ fluctuations [129–137]:

$$H(x,t) = v_\infty t + (\Gamma t)^{1/3}\chi(x). \tag{6.15}$$

Here, $\Gamma \equiv \Gamma(\nu, \lambda, D)$ is a constant, while χ is a time-independent random variable distributed according to the celebrated Tracy–Widom distribution corresponding to Gaussian orthogonal ensemble (GOE), $f_1(\chi) = F_1'(\chi)$, which can be written explicitly in terms of the Hastings–McLeod solution of the Painlevé II equation [136]. In particular, $f_1(\chi)$ has asymmetric non-Gaussian tails [136, 137]:

$$f_1(\chi) \sim \begin{cases} \exp(-|\chi|^3/24), & \chi \to -\infty \\ \exp(-2\chi^{3/2}/3), & \chi \to +\infty. \end{cases} \tag{6.16}$$

Equation (6.15) gives

$$h = (\Gamma t)^{1/3}\left[\chi - (1/L)\int_0^L dx\, \chi(x)\right]. \tag{6.17}$$

Knowing that $f_1(\chi)$ has a finite mean $\langle\chi\rangle < 0$, it follows from the law of large numbers that in the limit $L \to \infty$, the second term on the rhs converges to $\langle\chi\rangle$, so that $\langle h\rangle = 0$. In this case, in the limit $\tau \to \infty$, $h \to \infty$, keeping $h/\tau^{1/3}$ fixed, $p_0(h,\tau)$ takes the scaling form

$$p_0(h,\tau) \sim \frac{1}{(\Gamma\tau)^{1/3}} \widehat{f}_1\left(\frac{h}{(\Gamma\tau)^{1/3}}\right), \tag{6.18}$$

where $\widehat{f}_1(x) \equiv f_1(x + \langle\chi\rangle)$. Note that this scaling form is valid only in the large τ limit. In contrast, in (6.9), the integral is over all τ. Therefore, unfortunately, we can not replace $p_0(h,\tau)$ by its scaling form (6.18) which is only valid in the large τ limit. However, this is possible in the $r \to 0$ limit. To see this, we make a change of variable $\tau' = r\tau$ in (6.9) and get

$$p_r^{\text{stat}}(h) = \int_0^\infty d\tau' \, e^{-\tau'} p_0(h,\tau'/r). \tag{6.19}$$

One now sees that, in the limit $r \to 0$, the effective time τ'/r inside $p_0(h,\tau'/r)$ becomes large and, hence, we can replace it by its scaling form (6.18). Hence, for $r \to 0$, $h \to \infty$, with $h\,r^{1/3}$ fixed, we get

$$p_r^{\text{stat}}(h) \sim (r\Gamma^{-1})^{1/3} G^{\text{KPZ}}\left[(r\Gamma^{-1})^{1/3}h\right], \tag{6.20}$$

where the scaling function $G^{\text{KPZ}}(x)$ is given by

$$G^{\text{KPZ}}(x) = \int_0^\infty dy \frac{e^{-y}}{y^{1/3}} \widehat{f}_1\left(\frac{x}{y^{1/3}}\right). \tag{6.21}$$

In contrast to $G^{\text{EW}}(x)$, $G^{\text{KPZ}}(x)$ is not symmetric in x. Since $\hat{f}_1$ has zero mean, it follows that G^{KPZ} has also vanishing mean, but is still asymmetric, with a variance $\int_{-\infty}^{+\infty} x^2 G^{\text{KPZ}}(x)dx \approx 1.44$. From (6.20), the stationary width scales as $W_r^{\text{KPZ}} \sim r^{-1/3}$. Its asymptotic behaviours for $x \to \pm\infty$, obtained from the corresponding behaviours of $\hat{f}_1(x)$ combined with a saddle point analysis, are

$$G^{\text{KPZ}}(x) \approx \begin{cases} \exp(-|x|^{3/2}/\sqrt{6}), & x \to -\infty \\ \exp(-3^{1/3}x), & x \to +\infty. \end{cases} \tag{6.22}$$

Equation (6.21) implies that $G^{\text{KPZ}}(x)$ has a non-analytic behaviour as $x \to 0$: $G^{\text{KPZ}}(x) \sim A + Bx + Cx^2\ln x$, with A, B, C being constants. Non-analyticity at the resetting value was also observed for the EW interfaces, (6.14), and, hence, is a generic feature of stochastic resetting. A quick comparison between equations (6.14) and (6.22) shows that the resetting induced steady state height distribution is rather different in the two cases. This is in contrast to the stationary state in a finite system of size L without resetting where both have the identical Gaussian distribution. Thus resetting is able to distinguish between the two cases.

Non-Poissonian resetting (see section 2.8) of the interface with a power-law waiting time distribution between resets $\psi(\tau) \sim \tau^{-(1+\alpha)}$ with $\alpha > 0$ has been considered in [119]. For $\alpha > 1$ the height attains a stationary distribution, which exhibits heavy tails, whereas for $\alpha < 1$ the distribution remains time dependent for large times.

6.3. Fluctuating interfaces: relaxation to the stationary state

To study the relaxation to the reset induced non-equilibrium stationary state in an extended system such as fluctuating interfaces, our starting point is the finite t renewal equation (6.8) for the height distribution at fixed point in space, say at $x = 0$. We now want to analyse this equation for finite but large time t. For this purpose, we first note from equations (6.10) and (6.18) that there is a scaling regime for large t, large h, keeping the ratio h/t^β fixed such that the single-site height distribution in the absence of resetting $p_0(h, t)$ exhibits a scaling form, both for the EW and the KPZ equation,

$$p_0(h, t) \approx \frac{1}{(\Gamma t)^\beta} \, g\left(\frac{h}{(\Gamma t)^\beta} \right). \tag{6.23}$$

In the above equation, Γ is a microscopic constant, the growth exponent $\beta = 1/4$ (for EW) and $\beta = 1/3$ (for KPZ) and the scaling function $g(x)$ is also different in the two cases. For EW, $g(x)$ is a simple Gaussian while, for the KPZ, $g(x)$ is the shifted Tracy–Widom GOE as discussed below (6.18). Unlike the EW case where $g(x) \sim e^{-x^2}$ for large x both on the positive and the negative side, for the KPZ the scaling function $g(x)$ has asymmetric tails [see (6.16)] with $g(x) \sim e^{-|x|^3/24}$ for $x \to -\infty$ while $g(x) \sim e^{-(2/3)x^{3/2}}$ for $x \to +\infty$. Hence to investigate the approach to the NESS, we consider the generic case when $g(x) \sim \exp(-a_\pm |x|^{\gamma_\pm})$ as $x \to \pm\infty$. For example, for the KPZ with flat initial condition, $\gamma_+ = 3/2$, $a_+ = 2/3$ and $\gamma_- = 3$, $a_- = 1/24$.

In order to know $p_r(h, t)$ from (6.8) we need to know $p_0(h, \tau)$ for all $\tau \in [0, t]$. However, except for the EW interface, where $p_0(h, \tau)$ is an exact Gaussian at all times τ, we typically have information on $p_0(h, \tau)$ only in the scaling limit when τ and h both are large, while the ratio h/τ^β is held fixed, as discussed in (6.23). Thus, to use this scaling form in the equation for $p_r(h, t)$ in (6.8) we focus in the regime where h is large (i.e., in the scaling regime). We substitute this scaling form (6.23) in (6.8) and rescale, as before, the time $\tau = wt$. This gives

$$p_r(h, t) \approx (\Gamma t)^{-\beta} \, e^{-rt} g\left((\Gamma t)^{-\beta} h \right)$$

$$+ rt(\Gamma t)^{-\beta} \int_0^1 dw \, w^{-\beta} \, e^{-rtw} g\left((\Gamma t)^{-\beta} h w^{-\beta} \right). \tag{6.24}$$

This solution has been analysed in detail in reference [50]. The main result is the following. We consider a scaling regime where h and t large but the ratio $h/t^\nu_\pm$ is fixed (where $\pm$ refers to positive or negative h) and the exponent

$$\nu_\pm = \frac{\gamma_\pm}{1 + \beta\gamma_\pm}, \tag{6.25}$$

where we recall that β is the growth exponent and $\gamma_\pm$ specify the behaviour at the tails of the scaling function $g(x)$ on the positive and negative side respectively (see the previous paragraph). In this scaling regime, it has been shown that $p_r(h, t)$ admits a large deviation form

$$p_r(h, t) \sim e^{-t \, I(h \, t^{-1/\nu_\pm})}, \tag{6.26a}$$

where the rate function is given by

$$I(y) = \begin{cases} \dfrac{r \, |y|^{\nu_\pm}}{\beta\nu_\pm (y_\pm^*)^{\nu_\pm}} & \text{for } |y| < y_\pm^*, \\ r + b_\pm |y|^{\gamma_\pm} & \text{for } |y| > y_\pm^*, \end{cases} \tag{6.27a}$$
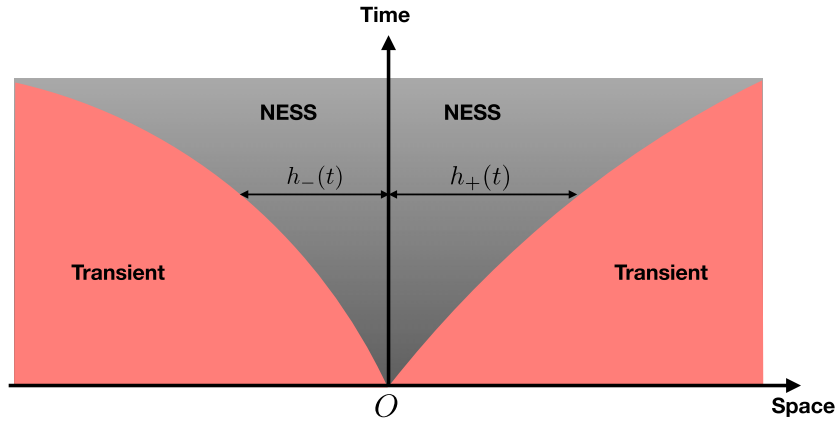
Figure 10. A NESS gets established in a core region around the resetting centre O (corresponding here to a flat profile $h = 0$) whose right and left frontiers $h_\pm(t)$ grow with time with (*a priori*) different exponents $h_\pm(t) \sim t^{1/\nu_\pm}$. Outside the core region, the system is transient.

where the singular points of the rate function $y_\pm^*$ on both sides have been computed explicitly [50]. The second derivative of $I(y)$ is discontinuous at both $y_\pm^*$, indicating a second order dynamical phase transition. Essentially, in the height space, there are two growing length scales $h_\pm(t) \sim t^{1/\nu_\pm}$ growing in the opposite direction. For $h_-(t) < h < h_+(t)$, the distribution $p_r(h, t)$ becomes independent of time and reaches a NESS, while for h outside this range, the height distribution still depends on t and is transient (see figure 10).

7. Resetting with memory of history

So far we have mainly considered resetting to a fixed reset point which may be chosen to coincide with the initial condition. We saw in section 2.7 that this may be easily generalised to a resetting distribution from which the reset point is sampled at each reset event.

In this section we review works where the process is reset to its value at some selected time in its history. This is most naturally illustrated in the case of a discrete time random walk in which with some reset probability the walker is returned to its previous position at a randomly selected time from the past. These dynamics are examples of a more general class of models referred to as reinforced random walks [138, 139].

7.1. Preferential visit model

Boyer and Solis-Salas [140] considered a minimal model for animal mobility proposed in the ecological literature [141, 142], which they called the *preferential visit model* (PVM). The idea is that an animal can either explore territory locally (by random walk dynamics) or relocate to places visited in the past (via a stochastic resetting move). For simplicity, we start by presenting the model on a lattice, the generalization to the continuum space is discussed later. At each discrete time step, $t \to t + 1$, the walker moves with probability $1 - r$ to a randomly chosen nearest neighbour site on the lattice and, with probability r, the walker relocates to a site it has visited in the past. In the simplest case this relocation is implemented by selecting a previously visited site with a probability proportional to the number of past visits to that site. One of the key observations that leads to the solvability of some aspects of this model is the fact that this

relocation protocol is exactly equivalent to choosing a past time at random [140]. Note that this model is different from the so-called 'elephant random walk' [143] which is also a non-Markovian process but there the stochastic rules are different from the PVM discussed above. In the 'elephant random walk', one again chooses a past time uniformly at random but one actually resets the increment of the jump rather than the position of the walker.

For simplicity we consider a one-dimensional lattice and let $P(n, t)$ denote the probability that the walker is at site n at time t. One can write down the master equation for the evolution of $P(n, t)$ and it reads

$$P(n, t+1) = \frac{1-r}{2}P(n-1, t) + \frac{1-r}{2}P(n+1, t) + \frac{r}{t+1}\sum_{n'=-\infty}^{\infty}\sum_{t'=0}^{t}P(n', t; n, t'), \quad (7.1)$$

where $P(n', t; n, t')$ denotes the joint probability that the walker is at site n' at time t and at site n at time $t' \leqslant t$. The first two terms on the rhs of (7.3) represent the standard random walk dynamics. The last term can be explained as follows. Suppose that the particle is at n' at time t and makes a transition to site n at time $t+1$ via the relocation move. For this transition to occur, the walker must have been at site n at some previous time t'. The probability for this event of being at n' at time t and at n at time t' is simply the joint probability $P(n', t; n, t')$. The prefactor $r/(t+1)$ in the third term in (7.1) is just the probability of the relocation via this event. Finally, the transition can occur from any site n' at time t to site n at time $t+1$—hence one has to sum over all possible n'. In addition, one has to sum over all possible t'. This explains the third term in (7.1). Fortunately, when one sums the two-point probability distribution over all n', one gets back a one-point distribution

$$\sum_{n'=-\infty}^{\infty} P(n', t; n, t') = P(n, t'). \quad (7.2)$$

Consequently, (7.1) becomes a closed equation for $P(n, t)$ [140, 144]

$$P(n, t+1) = \frac{1-r}{2}P(n-1, t) + \frac{1-r}{2}P(n+1, t) + \frac{r}{t+1}\sum_{t'=0}^{t}P(n, t'). \quad (7.3)$$

Even though this equation is linear, it is nonlocal in time and hence the solution is nontrivial as we will see below.

As a first step, consider the mean squared displacement

$$M_2(t) = \sum_{n=-\infty}^{\infty} n^2 P(n, t). \quad (7.4)$$

It obeys an equation obtained by summing (7.3)

$$M_2(t+1) = (1-r) + (1-r)M_2(t) + \frac{r}{t+1}\sum_{t'=0}^{t}M_2(t'). \quad (7.5)$$

The solution to this equation, with initial condition $M_2(0) = 0$ is given by

$$M_2(t) = \frac{1-r}{r}\sum_{k=1}^{t}\frac{1-(1-r)^k}{k} \quad (7.6)$$

as may be checked by substitution into (7.5). For large t

$$M_2(t) \simeq \frac{1-r}{r} \left[\ln(rt) + \gamma_e\right] \tag{7.7}$$

where γ_e is Euler's constant [see (4.14)] and we have used the large t asymptotics

$$\sum_{k=1}^{t} \frac{1}{k} \simeq \ln t + \gamma_e \tag{7.8}$$

and

$$\sum_{k=1}^{t} \frac{(1-r)^k}{k} \simeq -\ln r. \tag{7.9}$$

Thus the width of the distribution grows as $(\ln t)^{1/2}$. Similarly, equations for higher moments of the displacement may be written down and eventually solved for large t. Indeed, it can be shown that the full distribution converges at late times to a Gaussian form [140]

$$P(n,t) \simeq \frac{1}{\sqrt{2\pi M_2(t)}} \, e^{-\frac{n^2}{2M_2(t)}}, \tag{7.10}$$

where the variance $M_2(t)$ grows extremely slowly, i.e., logarithmically at late times, as given in (7.7). Thus the PVM provides a simple mechanism for anomalously slow sub-diffusive growth. Similar slow subdiffusion is known to arise in diffusion in disordered medium, such as the Sinai model. There the slowdown occurs due to the disorder that blocks the particle motion. In contrast, in the PVM, the slow subdiffusion arises even in the absence of disorder, simply by the dynamics of memory-driven resetting.

In [144] the PVM model was generalised to include biased sampling of the past history during relocations. Specifically at a relocation event at time t the past time t' is selected with a probability

$$F(t-t') = B(t-t'+1)^{-\beta} \tag{7.11}$$

where $\beta \geqslant 0$ and B is a normalisation constant. The case $\beta = 0$ recovers the previous PVM model. The master equation (7.3) is modified to

$$P(n,t+1) = \frac{1-r}{2}P(n-1,t) + \frac{1-r}{2}P(n+1,t) + r\sum_{t'=0}^{t} F(t-t')P(n,t'). \tag{7.12}$$

In this case the large time behaviour of the mean squared displacement depends on the value of β [144]

$$\beta > 2 \qquad M_2(t) \simeq \left(\frac{1-r}{1+r\langle\tau\rangle}\right) t, \tag{7.13}$$

$$1 < \beta < 2 \quad M_2(t) \propto t^{\beta-1}, \tag{7.14}$$

$$\beta < 1 \qquad M_2(t) \propto \ln t. \tag{7.15}$$

Moreover, in the large time limit, the probability distribution of the position takes the scaling form

$$P(n, t) \simeq \frac{1}{\sqrt{M_2(t)}} g\left(\frac{n}{\sqrt{M_2(t)}}\right) \tag{7.16}$$

where the scaling function $g(y)$ was found to be Gaussian in the cases $\beta > 2$ and $\beta < 1$ but to have a nontrivial form for $1 < \beta < 2$.

Thus for $\beta > 2$ (fast decaying memory) the probability distribution is that of the simple random walk but with a modified diffusion constant. For $\beta < 1$ (slowly decaying memory) one obtains a narrow Gaussian distribution with width $\sim (\ln t)^{1/2}$ and for $1 < \beta < 2$ one obtains an anomalous width $\sim t^{(\beta-1)/2}$ and nontrivial scaling distribution [144, 145].

7.2. Interpolation between the preferential visit model and resetting to the initial condition

A continuous time and space version of resetting with memory was considered in [146]. The master equation now reads

$$\frac{\partial p(x, t)}{\partial t} = D \frac{\partial^2 p(x, t)}{\partial x^2} - rp(x, t) + r \int d\tau K(\tau, t) p(x, \tau). \tag{7.17}$$

The third term represents the gain of probability into x by choosing a time τ in the past with probability density $K(\tau, t)$ and relocating to x with probability density $p(x, \tau)$. The memory kernel $K(\tau, t)$ is normalised so that

$$\int_0^t d\tau K(\tau, t) = 1. \tag{7.18}$$

In [146] a memory kernel was chosen that allows one to recover the case of resetting to the initial condition and to interpolate to the PVM:

$$K(\tau, t) = \frac{\phi(\tau)}{\int_0^t d\tau \phi(\tau)}. \tag{7.19}$$

Thus $K(\tau, t)$ only depends on the present time t through the normalisation denominator in (7.19). By choosing $\phi(\tau) = \delta(\tau)$ one recovers resetting to the initial condition, which has a stationary state, and by choosing $\phi = 1$ one recovers a continuous time and space version of PVM, which does not have a stationary state but exhibits a time-dependent distribution whose width grows as $(\ln t)^{1/2}$. For the case $\phi = 1$ the exact time-dependent distribution can be obtained and for the exponential kernel $\phi(\tau) = \lambda e^{-\lambda \tau}$ with $\lambda > 0$ the exact stationary state can be obtained.

Interestingly, it was shown in [146] (see also [147] for some rigorous results) that a whole range of behaviour is made possible by choosing different $\phi(\tau)$. These behaviours are classified as follows:

- When $\phi(\tau)$ decays faster than $1/\tau$ for large τ, i.e. $\tau\phi(\tau) \to 0$ as $\tau \to \infty$ there is a stationary distribution $p^*(x)$.
- When $\phi(\tau)$ increases, or decays as, or more slowly than, $1/\tau$ for large τ, i.e., $\tau\phi(\tau) > 0$ as $\tau \to \infty$ there is no stationary distribution. Instead there is a late time behaviour in which the time-dependent distribution takes a Gaussian form with variance $\sigma^2(t)$. The time dependence of the variance exhibits various distinct behaviours depending on $\phi(\tau)$:

    * For $\phi(\tau) \sim 1/\tau$, $\sigma^2(t) \sim \ln \ln t$;

* For $\phi(\tau) \sim \tau^\alpha$ with $\alpha > -1$, $\sigma^2(t) \sim \ln t$;
* For $\phi(\tau) \sim \exp(a\tau^\beta)$ where $0 < \beta < 1$ and a is a positive constant, $\sigma^2(t) \sim t^\beta$;
* For $\phi(\tau) \sim \exp(a\tau)$ where a is a positive constant, $\sigma^2(t) \simeq \left(\frac{2a}{a+r}\right) Dt$
* For $\phi(\tau) \sim \exp(a\tau^\beta)$ where $\beta > 1$ and a is a positive constant, $\sigma^2(t) = 2Dt$.

Thus in addition to the logarithmic growth of the variance, which we have seen in the previous subsection, an ultraslow $\ln \ln t$ growth of the variance occurs for $\phi(\tau) \sim 1/\tau$. Such a double logarithmic growth with time has been reported in data on human mobility [148].

7.3. Localization–delocalization transition induced by preferential resetting

Recently, the PVM described above was studied in the presence of a single defect site where the random walk has a finite probability to stay [149, 150]. We recall that, in the absence of the defect site, the walker is always delocalized, i.e, $P(n,t)$ always depends on time t and the variance increases like $\ln t$ at late times. Remarkably, the presence of one single defect site is able to induce a transition from a delocalised to localised phase (where $P(n,t)$ becomes independent of time at late times). More precisely, the model is defined as follows. Again, we consider a single random walker on a d-dimensional lattice. At any generic site other than the origin, the walker performs the same dynamics as in the PVM, namely with probability $1 - r$ (with $0 \leqslant r \leqslant 1$), it diffuses to any neighbouring site chosen randomly and, with probability r, it relocates to any previously visited site by choosing a past time at random (which is equivalent to choosing a previously visited site with a probability proportional to the number of past visits to that site). The origin is a special site where, with probability $\gamma \in [0, 1]$ the walker stays there and with the complementary probability $1 - \gamma$, it either diffuses (with probability $(1 - \gamma)(1 - r)$) or relocates preferentially [with probability $(1 - \gamma)r$]. The two parameters in this model are thus γ and r. For $\gamma = 0$, it reduces to the PVM described earlier, where the walker is always delocalized.

The analysis of this model with a finite γ exhibits an interesting phase transition for $d > 2$ in the (γ, r) plane, across a critical line $r_c(\gamma)$ (see figure 11). For $r < r_c(\gamma)$, the walker is delocalized and the variance increases with time. In contrast, for $r > r_c(\gamma)$, the walker gets localised, i.e. $P(n,t)$ becomes stationary at late times and the stationary distribution has an exponential tail with a characteristic localization length scale $\xi(r)$ that diverges as one approaches the critical line as $\xi \simeq (r - r_c)^{-\nu}$ where ν takes the same value as in the self-consistent theory of Anderson localization of waves in random media [149, 150]. The critical value $r_c(\gamma)$ was shown to be related to the probability $P_{\text{no−return}}$ of no-return to the origin for the free random walker, i.e, without resetting, via the simple relation

$$r_c(\gamma) = \frac{(1 - \gamma)P_{\text{no-return}}}{\gamma + (1 - \gamma)P_{\text{no-return}}}. \tag{7.20}$$

It turns out that this relation is quite general and holds for random walks with arbitrary jump length on the lattice, e.g., for Lévy flights. For a general random walk with a jump distribution $p(\ell)$, with its Fourier transform given by $\tilde{p}(k)$, the probability of no-return has a simple general expression

$$P_{\text{no-return}} = \left(\frac{1}{(2\pi)^d} \int_{\mathcal{B}_d} \frac{d\vec{k}}{1 - \hat{p}(\vec{k})} \right)^{-1}, \tag{7.21}$$
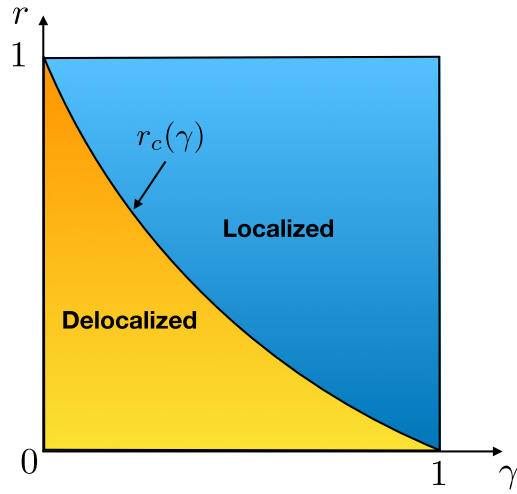
Figure 11. Phase diagram for the PVM model (for $d > \mu$) with relocation probability r to a previous site, in the presence of a defect site with a remain probability γ.

where $\mathcal{B}_d$ is the d-dimensional first Brillouin zone. For example, for nearest-neighbour random walk on a d-dimensional hyper-cubic lattice,

$$\tilde{p}(k) = \frac{1}{d}\sum_{i=1}^{d} \cos k_i. \tag{7.22}$$

In this case, $P_{\text{no-return}} > 0$ only for $d > 2$, i.e. the walk is recurrent for $d \leqslant 2$ and is transient for $d > 2$. For Lévy flights with Lévy index μ, such that $p(\ell) \propto \ell^{-1-\mu}$ for large ℓ with $0 < \mu < 2$, $\tilde{p}(k)$ behaves as $\tilde{p}(k) \simeq 1 - |a\,k|^{\mu}$ as $k \to 0$, where a is a characteristic jump length. In this case, from (7.21), one finds that $P_{\text{no-return}} = 0$ for $d \leqslant \mu$, i.e., the walk is recurrent. In contrast, for $d > \mu$, $P_{\text{no-return}} > 0$ and the Lévy walk is transient. Therefore, this delocalisation–localisation transition in this resetting model with defect will occur, even for Lévy flights, as long as $d > \mu$. For example, if $\mu = 1/2$, this transition will be there even in $d = 1$, as was seen in numerical simulations [149, 150]. Note that, for a recurrent walk with $d < \mu$, $P_{\text{no-return}} = 0$ and hence $r_c(\gamma) = 0$: this indicates that for any finite r the walker is always in the localised phase. Finally, we mention that several variants of this simple model with one defect site were recently studied in reference [150].

To conclude this subsection, we point out that, in the absence of resetting, i.e. $r = 0$, for any finite γ, the walker is always delocalized (see figure 11). This is consistent with the well known fact that a single defect is not enough to localise a diffusing particle in dimension $d > \mu$. However, introducing a finite resetting rate $r > r_c(\gamma)$ can localise the walker in $d > \mu$.

7.4. Resetting to the past maximum

In this section we consider another example of resetting protocol using the memory of the full history of the process. In this case we consider a random walk for which the resetting move returns the walker to the previous maximum [51].

To be specific we consider a one-dimensional random walk on the lattice. At any given time step n if the position $x(n)$ of the walker is less than the maximum position

$$m(n) = \max\left[x(0) = 0, x(1), x(2), \ldots, x(n)\right] \tag{7.23}$$

then in the next time step the position is reset to $m(n)$ with probability r and moves to either the right or left nearest neighbour site with equal probability $(1 - r)/2$. If the position is $x(n) = m(n)$ then the walker moves to the right or left nearest neighbour site with equal probability $1/2$.

It is useful to consider the distance y of the walker at time n from the maximum position upto time n: $y(n) = m(n) - x(n)$. The master equation of the joint probability $P(y, m, n)$ for the maximum to take value m and the distance from the maximum to take value y obeys, for $y > 1$ [51]

$$P(y, m, n) = \left[\frac{1-r}{2} + \frac{r}{2} \delta_{y,1} \right] P(y - 1, m, n - 1) + \frac{1-r}{2} P(y + 1, m, n - 1)$$

(7.24)

and for $y = 0$

$$P(0, m, n) = \frac{1}{2} P(0, m - 1, n - 1) + \frac{1-r}{2} P(1, m, n - 1) + r \sum_{y=1}^{\infty} P(y, m, n - 1)$$

(7.25)

with initial condition $P(y, m, 0) = \delta_{y,0} \delta_{m,0}$.

These equations may be solved by generating function techniques [51]. The main results of interest for our purposes concern how resetting to the maximum affects the statistics of the maximum. It turns out that for non zero r the average value of the maximum for large n increases ballistically in time as

$$\langle m(n) \rangle \simeq v(r) n$$

(7.26)

where the speed $v(r)$ is given by

$$v(r) = \frac{r(1 - r)}{r - 2r^2 + \sqrt{r(2 - r)}}.$$

(7.27)

Note that the speed vanishes as $v(r) \simeq \sqrt{r/2}$ signalling a crossover from ballistic behaviour when $r > 0$ to diffusive behaviour $m(n) \sim n^{1/2}$ for $r = 0$.

Moreover, the distribution of the distance from the maximum reaches a stationary state for y held fixed as $n \to \infty$

$$P_y(y, n) \sim \left(\frac{1 - r}{1 + \sqrt{r(2 - r)}} \right)^y.$$

(7.28)

Now if one looks at the relaxation to this stationary state one finds a dynamical transition reminiscent of the relaxation front described in section 2.5. Indeed, for large n the probability distribution of y obeys a large deviation principle

$$P_y(y = wn, n) \sim \exp(-nH(w))$$

(7.29)

where the rate function takes the form

$$H(w) = \begin{cases} w \ln \left[\dfrac{1 + \sqrt{r(2 - r)}}{1 - r} \right] & \text{for } w < w^* \\[4mm] \dfrac{w}{2} \ln \left[\dfrac{1 + w}{1 - w} \right] + \ln \left[\dfrac{\sqrt{1 - w^2}}{1 - r} \right] & \text{for } w > w^* \end{cases}$$

(7.30)

with $w^* = \sqrt{r(2-r)}$. Thus for $y < y^* = w^* n$ the distribution of y has reached the stationary state but for $y > y^*$ the distribution still depends on time n.

8. Other topics associated to resetting

In this section we briefly review some exciting new developments that extend the resetting paradigm we have described in previous sections.

8.1. Thermodynamics of resetting and integral theorems

The concept of resetting raises several issues in stochastic thermodynamics, which have been identified and addressed by Fuchs *et al* [151]. First, resetting to some fixed configuration or finite region of phase space implies a change in information content since information about the preceding state is lost, which in turn implies a thermodynamic cost. Second, an important question in biological systems is the efficiency of computation [152–154]. For a biomolecular search process involving resetting, this boils down to tensioning the informational efficiency of the search against the thermodynamic cost of resetting. Finally the nonequilibrium stationary states generated by resetting exhibit currents which imply entropy production and it is of importance to characterise this.

In [151] resetting of colloidal particle in a potential was considered and used to obtain a first law of thermodynamics and to identify the thermodynamic work done by resetting. The resetting entropy production rate was derived for this system with space-dependent resetting rate $r(x)$ to resetting position X_r

$$\dot{S}^{\text{reset}} = \int \mathrm{d}x\, r(x) p(x) \ln \frac{p(x)}{p_{X_r}} \tag{8.1}$$

and from this a second law of thermodynamics including resetting was proposed (see also [155]).

Building on the identification of entropy change due to resetting, Pal and Rahav [156] considered how integral fluctuation theorems apply to resetting problems. The integral theorems may be thought of as generalising the second law to equalities involving averages over fluctuations [157]. They showed how the Hatano–Sasa integral fluctuation theorem [158] which pertains to nonequilibrium steady states is also valid for systems with resetting. Further integral theorems have been considered in [156, 159].

In [160] the authors considered a probe in contact with a bath held out of equilibrium by a resetting process. The bath entails particles in a harmonic potential which are reset to a fixed position with a Poissonian rate. The probe is coupled to the bath-particles and for large bath is governed by an effective Langevin equation which generates non-Gaussian fluctuations.

8.2. Large deviations

As we have already seen in sections 2.5 and 7.4, many probabilities associated with stochastic processes obey a large deviation principle. In particular, additive observables, which are the time integral of a fluctuating quantity such as time integrated current or the area A_T under the space time trajectory,

$$A_T = \frac{1}{T} \int_0^T X_t \mathrm{d}t, \tag{8.2}$$

are expected to have a probability distribution which can be written for large T as

$$P(A_T = a) = e^{-TI(a) + o(T)}. \tag{8.3}$$

This is the large deviation form (often referred to as large deviation principle) and the function $I(a)$ is the rate function or large deviation function. The rate function is usually determined by considering the generating function

$$\mathcal{G}(k, t) = \langle e^{tkA_t} \rangle \tag{8.4}$$

and finding the asymptotic behaviour for large t

$$\mathcal{G}(k, t) \sim e^{\lambda(k)t}. \tag{8.5}$$

Then $I(a)$ can be found by the Legendre transform

$$I(a) = \sup_k [ka - \lambda(k)], \tag{8.6}$$

see [161] as well as [162] for reviews.

Using the renewal approach Meylahn *et al* [163] showed how the generating function, $\mathcal{G}_r$, i.e. the equivalent of (8.4) for an additive observable in a stochastic process subject to Poissonian resetting, can be written simply in terms of the generating function in the absence of resetting, $\mathcal{G}_0$, through the relation in Laplace domain

$$\tilde{\mathcal{G}}_r(k, s) = \frac{\tilde{\mathcal{G}}_0(k, s + r)}{1 - r\tilde{\mathcal{G}}_0(k, s + r)}. \tag{8.7}$$

Then to extract the large time behaviour (8.5) just requires the knowledge of the poles of the rhs of (8.7). Calculations were carried out explicitly for an Ornstein Uhlenbeck process under reset. It was also pointed out that stochastic observables that do not ordinarily obey a large deviation principle may acquire one under resetting.

Further work [164] has shown how a variational formula involving large deviation rate functions without resetting may be used to obtain the rate function with resetting. Three examples of additive observables for diffusion with resetting, positive occupation time, area and absolute area, were worked out.

In [165] observables J_n (generalised currents) that are not reset to zero but retain their value on resetting were considered under discrete time dynamics. It was shown how phase transitions in the large deviation function may occur between regimes where the current fluctuations are optimally realised by a finite frequency of resets to a regime where the current fluctuation is optimally realised by not resetting. Very recently, another functional of Brownian motion with resetting has been studied—this is the local time spent by the particle at at given position in space [166].

8.3. Coupling of stochastic process and resetting

Most of this review has been based on independent processes for the underlying stochastic dynamics and resetting which has allowed simple renewal equations to be written. A natural extension to explore is when the stochastic dynamics becomes coupled to resetting, e.g. through the dynamics depending on the time to the last reset. Preliminary examples which have been studied are Brownian particles which are attracted towards each other but reset on contact to an initial separation [167], and random walk dynamics where the mean or variance of the microscopic steps of the walk depend on the time since resetting [165].

8.4. Quantum dynamics with reset

The idea of introducing a resetting to an arbitrary classical stochastic process, such as diffusion, can be easily generalized to the dynamics of a quantum system. Consider, for simplicity, a generic quantum system with a time independent Hamiltonian H, prepared in an initial pure state $|\psi_0\rangle$. In the absence of resetting, this state evolves under the unitary dynamics

$$|\psi(t)\rangle = e^{-iH\,t}|\psi(0)\rangle. \tag{8.8}$$

The density matrix

$$\hat{\rho}(t) = |\psi(t)\rangle\langle\psi(t)| \tag{8.9}$$

evolves via

$$\hat{\rho}(t) = e^{iHt}\hat{\rho}(0)e^{-iHt} \quad \hat{\rho}(0) = |\psi(0)\rangle\langle\psi(0)|. \tag{8.10}$$

One can now introduce resetting via the following protocol [168]: the state $|\psi(t)\rangle$ evolves from time t to $t + dt$ as follows

$$|\psi(t+dt)\rangle = \begin{cases} |\psi(0)\rangle, & \text{with proba. } r\,dt \\ [1 - iHdt]|\psi(t)\rangle, & \text{with proba. } 1 - r\,dt, \end{cases} \tag{8.11}$$

where we have set $\hbar = 1$ for convenience. Here $r \geqslant 0$ denotes the resetting rate with which the system is projected back to the initial state. Thus, in a small time interval dt, the system either goes back to its initial state with probability rdt, or, with the complementary probability $(1 - rdt)$, it evolves unitarily with its Hamiltonian H. Now the density matrix at time t is denoted by

$$\hat{\rho}_r(t) = |\psi(t)\rangle\langle\psi(t)|. \tag{8.12}$$

Note that for any $r > 0$, the dynamics is a mixture of stochastic and deterministic evolution and the density matrix in (8.12) is stochastic in the sense that it varies from one realisation of the reset process to another. Hence, the observed density matrix at time t is obtained by averaging over all possible reset histories

$$\rho_r(t) = \mathbb{E}\left[\hat{\rho}_r(t)\right] \tag{8.13}$$

where $\mathbb{E}[\cdot]$ denotes the classical expectation value over all stochastic evolutions. Our goal is to investigate how a nonzero r modifies the time evolution of the quantum state, or equivalently the associated density matrix in (8.13). Following the same renewal structure that was described for the classical systems, one can then write the last renewal equation for the evolution of the density matrix as

$$\rho_r(t) = e^{-rt}\,e^{-iHt}\hat{\rho}_0\,e^{iHt} + \int_0^t r\,e^{-r\tau}\,e^{-iH\tau}\hat{\rho}_0\,e^{iH\tau}d\tau, \tag{8.14}$$

where $\hat{\rho}_0 = |\psi(0)\rangle\langle\psi(0)|$. Here the first term corresponds to no resetting and the second term counts the events following the last resetting before time t, that occurs at time $t - \tau$. Now, at long times t, the first term vanishes exponentially and the density matrix $\rho(t)$ approaches a stationary value (as $t \to \infty$)

$$\rho^* = \int_0^\infty r\,e^{-r\tau}\,e^{-iH\tau}\hat{\rho}_0\,e^{iH\tau}d\tau, \tag{8.15}$$

where the superscript * denotes the stationary density matrix. This stationary density matrix was analysed in [168] and it was pointed out that it has non-zero off-diagonal elements in the eigenbasis of H. This is at variance with the pure unitary evolution, i.e. $r = 0$, where the density matrix, in the eigenbasis of H, becomes diagonal. These general results were then applied to various quantum models [168]. For example, the evolution of non-interacting fermions on a one-dimensional lattice, starting from a step initial condition for the density of fermions, and subjected to resetting was shown to lead to a non-trivial steady state density profile. Other models include Dirac fermions and the Bose–Hubbard model in an optical lattice. We refer the readers to reference [168] for further details.

There are of course other interesting questions associated to resetting in quantum systems. For example, the spectral properties of quantum systems subjected to resetting have been studied in [169]. Another quantum setup where resetting can play an important role is the quantum random walk subjected to perturbations caused by repeated measurements [170–173]. Another situation akin to resetting corresponds to making a stroboscopic series of projective measurements on the initial state.

8.5. Further topics

There have been some very recent developments that we have not been able to cover in this review, unfortunately. These include, e.g. the connection between home range search and resetting [174], the interplay between population dynamics and resetting [175, 176], branching processes and resetting [177, 178], resetting of the scaled Brownian motion with a time-dependent diffusion coefficient, $D(t) \propto t^{\alpha-1}$ with $\alpha > 0$ [179, 180].

9. Conclusion

To summarise, in this review we have attempted to provide a survey of recent developments in the theory of stochastic processes subjected to random resettings. The basic idea is very simple and general: any stochastic process evolving under its own natural dynamics is interrupted at random times and brought back (reset) to a fixed state, say its initial state. The intervals between successive reset events are statistically independent and are drawn from some specified distribution $\psi(t)$. A particularly simple and illustrative case is the Poissonian resetting where $\psi(t) = re^{-rt}$ with r denoting the constant reset rate. One can ask how this stochastically interrupted 'reset process' evolves with time and what are its statistical properties?

There are two principal effects of resetting that we emphasized in this review. First, such 'reset' interruptions drive the system to a non-trivial nonequilibrium stationary state with a nonzero current in the configuration space—indeed detailed balance is manifestly violated by the reset moves. Thus resetting provides a very simple and natural way to generate a nonequilibrium stationary state. The second effect of resetting concerns the mean time to search or capture a target. We have shown that in several models, resetting not only reduces the mean search time, but there is typically an optimal resetting rate r^* (for Poissonian resetting) at which the mean capture time becomes minimum. Thus resetting typically makes the search process efficient.

While the theory of stochastic resetting has seen rather rapid progress in recent years, there have been little progress on the experimental side so far. The recent preliminary results from the experiments in an optimal trap set-up from the group of Ciliberto seem promising [181]. The point about experiments is not just to reproduce the theoretical results (which can be easily done by simulations), but often real experiments require new

resetting protocols that have not been theoretically studied. For example, in a theoretical model one often assumes instantaneous resetting which is impossible to achieve experimentally. Thus experimentalists need to devise different types of resetting protocols, which in turn pose interesting theoretical challenges. We hope that this synergy between theory and experiments will advance the field of stochastic resetting even further in the coming years.

Finally, the idea of resetting is so simple and natural that it can be used and adapted to ask interesting questions in many different fields, going beyond classical stochastic processes. For example, we have seen how resetting in a quantum system leads to a nontrivial steady state density matrix with non-zero off-diagonal elements, giving rise to new non-diagonal ensembles. The idea of resetting has led to interesting new observations in stochastic thermodynamics, population dynamics, chemical reactions, just to name a few. We hope that this review will stimulate further new ideas in this rapidly developing field of research.

Acknowledgments

Appendix A. Equivalence of forward and backward renewal equations

In this appendix we show that the last renewal equation and the first renewal equation are equivalent by showing that they share the same solution.

The last renewal equation (2.10) reads

$$p(x,t|x_0) = \mathrm{e}^{-rt}G_0(x,t|x_0) + r\int_0^t \mathrm{d}\tau\,\mathrm{e}^{-r\tau}G_0(x,\tau|X_r). \tag{A.1}$$

Taking the Laplace transform yields

$$\tilde{p}(x,s|x_0) = \tilde{G}_0(x,r+s|x_0) + r\int_0^\infty \mathrm{d}t\,\mathrm{e}^{-st}\int_0^t \mathrm{d}\tau\,\mathrm{e}^{-r\tau}G_0(x,\tau|X_r). \tag{A.2}$$

The final term on the rhs becomes

$$r\int_0^\infty \mathrm{d}\tau\int_0^\infty \mathrm{d}t'\,\mathrm{e}^{-st'-(r+s)\tau}G_0(x,\tau|X_r) = \frac{r}{s}\tilde{G}_0(x,r+s|X_r).$$

Thus the Laplace transform of the solution to (2.10) is given by

$$\tilde{p}(x,s|x_0) = \tilde{G}_0(x,r+s|x_0) + \frac{r}{s}\tilde{G}_0(x,r+s|X_r). \tag{A.3}$$

Now consider the first renewal equation (2.11)

$$p(x,t|x_0) = \mathrm{e}^{-rt}G_0(x,t|x_0) + r\int_0^t \mathrm{d}\tau_\mathrm{f}\,\mathrm{e}^{-r\tau_\mathrm{f}}p(x,t-\tau_\mathrm{f}|X_r). \tag{A.4}$$

Taking the Laplace transform yields

$$\tilde{p}(x,s|x_0) = \tilde{G}_0(x, r+s|x_0) + r \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-st} \int_0^t \mathrm{d}\tau\, \mathrm{e}^{-r\tau} p(x, t-\tau|X_r). \qquad (A.5)$$

The final term on the rhs becomes

$$r \int_0^\infty \mathrm{d}\tau \int_0^\infty \mathrm{d}t'\, \mathrm{e}^{-st'-(r+s)\tau} p(x, t'|X_r) = \frac{r}{r+s} \tilde{p}(x,s|X_r)$$

so that the Laplace transform of the solution to the first renewal equation obeys

$$\tilde{p}(x,s|x_0) = \tilde{G}_0(x, r+s|x_0) + \frac{r}{r+s} \tilde{p}(x,s|X_r). \qquad (A.6)$$

Assuming that the solutions of the first renewal and last renewal equations are the same we may subtract (A.3) from (A.6) to obtain

$$\frac{r}{s} \tilde{G}_0(x, r+s|X_r) = \frac{r}{r+s} \tilde{p}(x,s|X_r) \qquad (A.7)$$

which is indeed consistent with the solution (A.3).

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