

# Stochastic Dimension-reduced Second-order Methods for Policy Optimization\*

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## Abstract

In this paper, we propose several new stochastic second-order algorithms for policy optimization that only require gradient and Hessian-vector product in each iteration, making them computationally efficient and comparable to policy gradient methods. Specifically, we propose a dimension-reduced second-order method (DR-SOPO) which repeatedly solves a projected two-dimensional trust region subproblem. We show that DR-SOPO obtains an  $\mathcal{O}(\epsilon^{-3.5})$  complexity for reaching approximate first-order stationary condition and certain subspace second-order stationary condition. In addition, we present an enhanced algorithm (DVR-SOPO) which further improves the complexity to  $\mathcal{O}(\epsilon^{-3})$  based on the variance reduction technique. Preliminary experiments show that our proposed algorithms perform favorably compared with stochastic and variance-reduced policy gradient methods.

## 1 Introduction

The policy gradient (PG) method, pioneered by Williams (1992), is a widely used approach for finding the optimal policy in reinforcement learning (RL). The main idea behind PG is to directly maximize the total reward by using the (stochastic) gradient of the cumulative rewards. PG is particularly useful for high dimensional continuous state and action spaces due to its ease in implementation. In recent years, the PG method has gained much attention due to its significant empirical successes in a variety of challenging RL applications (Lillicrap et al., 2015; Silver et al., 2014).

Despite its wide application and popularity, there is a growing concern about the high variance of classical PG method Williams (1992) and its variants Sutton et al. (2000); Baxter and Bartlett (2001). To address this issue, various approaches such as trust region method (Schulman et al., 2015a), proximal policy optimization (Schulman et al., 2017) and natural PG (Kakade, 2001) have been proposed to further improve the algorithm robustness and efficiency. Motivated by the variance reduction technique developed in stochastic optimization, (e.g. Johnson and Zhang (2013); Fang et al. (2018); Zhou et al. (2020)), recent work (Papini et al., 2018; Xu et al., 2020; Shen et al., 2019) have developed new variance-reduced policy gradient

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methods for model-free RL. In particular, Xu et al. (2020); Shen et al. (2019) obtain  $\mathcal{O}(\epsilon^{-3})$  sample complexity<sup>1</sup> for finding an  $\epsilon$ -first order stationary point, which improves the  $\mathcal{O}(\epsilon^{-4})$  complexity of vanilla PG method (Williams, 1992) by a factor of  $\mathcal{O}(\epsilon^{-1})$ . We address that all those variance-reduced PG methods only have guaranteed convergence to an  $\epsilon$  first-order stationary point ( $\epsilon$ -FOSP), which, due to the non-convexity of RL objective, is insufficient to ensure global optimality. By exploiting the objective landscape in policy optimization, much recent efforts have been directed towards finding global solutions beyond first-order stationary convergence. For example, see Agarwal et al. (2021); Bhandari and Russo (2019); Cen et al. (2022); Lan (2022) on global convergence of various PG methods and Zhang et al. (2021); Liu et al. (2020) for the variance-reduced PG methods. Despite much progress, the global optimality can be achieved only for special parameterized settings such as softmax tabular policy and Fisher-non-degenerate policy, or when function approximation error is controllable (Yuan et al., 2022).

For the more general setting, Zhang et al. (2020); Yang et al. (2021) have established the convergence of PG methods to second-order stationary point (SOSP), which successfully exclude saddle point solutions that have indefinite Hessian. This research line is driven by the recent progress on developing efficient (stochastic) gradient methods to escape saddle-point solution (Daneshmand et al., 2018; Jin et al., 2017). In addition to the first order methods, second-order methods, such as cubic regularized Newton method (Nesterov and Polyak, 2006; Cartis et al., 2011; Tripuraneni et al., 2018; Kohler and Lucchi, 2017) and trust region method Yuan (2015); Curtis et al. (2017); Curtis and Shi (2020), are known to have provable finite time convergence to SOSP solutions in nonconvex (and stochastic) optimization. However, a challenge to the second-order methods is that they involve some nontrivial subproblems which require either matrix decomposition or external iterative solvers to obtain desired solutions. For example, see (Nesterov and Polyak, 2006; Agarwal et al., 2017; Nocedal and Wright, 1999). To alleviate this issue, Wang et al. (2022) proposed a new stochastic cubic-regularized policy gradient method (SCP-PG) which only involves the cubic regularized Newton subproblem when converging to FOSP. While SCP-PG can save the additional Hessian-vector product from time to time, it has an unsatisfactory sample complexity of  $\mathcal{O}(\epsilon^{-7/2})$ , which is worse than the optimal rate of  $\mathcal{O}(\epsilon^{-3})$  (Arjevani et al., 2020).

In this paper, we aim to address some remaining issues in designing efficient second-order method for model-free policy optimization. Our work is motivated by the recent work in trust region optimization, and particularly, the dimension reduced second-order method (DRSOM) for nonconvex optimization (Ye, 2022; Zhang et al., 2022). DRSOM has a significant advantage in that it solves a relatively simple 2-dimensional trust region subproblem without the need of external solvers, resulting a computation overheads on a par with gradient descent methods. Building on this research direction, we develop several new stochastic second-order methods for policy optimizations. Our main contribution can be summarized as the following aspects.

- First, we propose a stochastic second-order method for policy optimization (DR-SOPO), based on the recently proposed dimension-reduced trust region method. We show that the proposed method achieves a convergence rate of  $\mathcal{O}(\epsilon^{-3.5})$  for approximating a first order stationary point and a second-order stationary point in certain subspace. Our method only involves a cheap dimension reduced subproblem and hence bypasses the computation burden in solving a full dimensional quadratic program for trust region method. As a consequence, the computational overheads of DR-SOPO is comparable to that of policy gradient.
- Second, to further enhance the efficiency of DR-SOPO, we propose a dimension and variance-reduced second-order method for policy optimization (DVR-SOPO). DVR-SOPO incorporates a Hessian-aided variance reduction technique (Shen et al., 2019) to DR-SOPO. The variance technique significantly improves the complexity in terms of the stochastic gradient queries, which is the dominating term in the complexity bound. We show that DVR-SOPO further improves the convergence rate of DR-SOPO to  $\mathcal{O}(\epsilon^{-3})$ .
- Finally, we provide some preliminary experiments to demonstrate the empirical advantage of our proposed methods against policy gradient method and variance-reduced policy gradient method. Our

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<sup>1</sup>For sample complexity, we mean the total number of samples involved in the estimation of gradient and hessian-vector product

theoretical analysis and empirical study show the great potential of directly using second-order information in policy optimization and reinforcement learning.

**Comparison with TRPO** Prior to our work, Schulman et al. (2015a) proposed the trust region policy optimization (TRPO) method, which has received great popularity in RL. However, it is important to note that TRPO is a first-order method that uses KL divergence to enforce optimization stability. This approach fundamentally differs from our work, which draw inspiration from trust region method in numerical optimization. Typically, a classic trust region method involves  $\ell_p$ -ball constraints to improve the convergence of second-order method. Recently, Jha et al. (2020) replaced the linear objective in TRPO by quadratic approximation, and proposed to solve the modified TRPO by Quasi-Newton methods. However, their work did not provide any theoretical guarantee for the proposed method.

## 1.1 More on Related Work

We give a more detailed review over two research directions in policy optimization that are closely related to our work: the study on the convergence to SOSP and the variance reduction technique for the PG methods.

**Convergence to SOSP.** Zhang et al. (2020) appears to be the first study showing that PG can escape FOSP and reach high-order stationary solutions. Specifically, they proposed a random horizon rollout for unbiased estimation of policy gradient. Furthermore, equipped with a periodically enlarged stepsize and the correlated negative curvature technique, they show that the a modified PG method an  $\mathcal{O}(\epsilon^{-9})$  sample complexity for converging to an  $(\epsilon, \sqrt{\epsilon})$ -SOSP. In a follow-up work Yang et al. (2021), the authors improved the sample complexity to  $\mathcal{O}(\epsilon^{-9/2})$  and extended the SOSP analysis to extensive policy optimization algorithms, though under a restrictive objective structure assumption. A more recent work (Wang et al., 2022) proposed to use stochastic cubic Newton method for policy optimization and further improved the sample complexity to  $\mathcal{O}(\epsilon^{-7/2})$ .

**Variance-reduction.** Recent work in policy optimization has made a strong effort to apply variance reduction (VR) techniques, inspired by their success in the oblivious stochastic setting, such as SARAH (Nguyen et al., 2017) and SPIDER (Fang et al., 2018). Hessian-aided VR (Shen et al., 2019) is one of the examples that succeed in non-oblivious setting. Besides, Xu et al. (2020) proposes a SRVR-PG method based on SARAH and use important sampling techniques to deal with the distribution shift, while Pham et al. (2020) provides a single-loop version by a hybrid approach. Based on a novel gradient truncation technique, Zhang et al. (2021) successfully removes a strong assumption on the variance bound of important sampling in earlier work. All of the above algorithms match the sample complexity of  $\mathcal{O}(\epsilon^{-3})$ . Motivated by the success of stochastic recursive momentum (i.e. STORM, Cutkosky and Orabona (2019)), Huang et al. (2020); Yuan et al. (2020) propose new STORM-like PG methods that blend momentum in the updates. These methods have the same  $\mathcal{O}(\epsilon^{-3})$  complexity but do not require alternating between large and small batch sizes.

**Paper structure** Our paper proceeds as follows. Section 2 introduces notations and background in policy optimization. Section 3 presents the dimension reduced second-order method and its convergence analysis. Section 4 presents the dimension and variance-reduced method based on the Hessian-aid VR technique. Section 5 elucidates more details about the implementation of trust region procedure. Section 6 conducts empirical study to show the advantage of our proposed methods. We draw conclusion in Section 7 and leave technical proof in the appendix sections.

## 2 Preliminaries

**Notation** For a square matrix  $A \in \mathbb{R}^{n \times n}$ , we define norm for matrix as  $\|A\| = \sqrt{\lambda_M}$ , where  $\lambda_M$  is the eigenvalue of  $A^T A$  with biggest absolute value. For a vector  $v \in \mathbb{R}^n$ , we use  $\|v\|$  to express the standard Euclidean norm.  $\|v\|_Q := \sqrt{v^T Q v}$  where  $Q$  is a positive-definite matrix.

Consider the Markov decision process  $\mathfrak{M} = \{\mathcal{S}, \mathcal{A}, \mathcal{P}, r, \gamma\}$ . Here,  $\mathcal{S}$  is the state space,  $\mathcal{A}$  is the action space,  $\mathcal{P}(s_{h+1} | s_h, a_h)$  is the transition kernel,  $r : \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}$  is the regret/cost function<sup>2</sup> and  $\gamma \in [0, 1)$  is a discount function. The agent's behavior is modeled by policy function  $\pi$ , and  $\pi(a | s)$  is the density of action  $a \in \mathcal{A}$  given the state  $s \in \mathcal{S}$ . We describe the policy  $\pi_\theta$  to reflect the fact the policy function is parameterized by a vector  $\theta \in \mathbb{R}^d$ . Let  $s_0$  follow from the initial distribution  $\rho(\cdot)$ . Let  $H$  be the length of truncated trajectory, and  $p(\tau; \pi_\theta)$  be the density of trajectory  $\tau = (s_0, a_0, \dots, s_{H-1}, a_{H-1})$  following from the MDP:  $p(\tau; \pi_\theta) := \rho(s_0) \prod_{h=0}^{H-1} \mathcal{P}(s_{h+1} | s_h, a_h) \pi_\theta(a_h | s_h)$ . For brevity, we use the notation  $p(\tau; \pi_\theta)$  and  $p(\tau; \theta)$  interchangeably. We consider the accumulated discounted regret:

$$\mathcal{R}(\tau) := \sum_{h=0}^{H-1} \gamma^h r(s_h, a_h).$$

We remark that for more general objective with infinite horizon, one can set  $H = \mathcal{O}(\log(\epsilon^{-1}))$  and view  $\mathcal{R}(\tau)$  as a truncated estimator. See more discussion in Appendix D. Our goal is to solve the following policy optimization problem which minimizes the expected discounted trajectory regret/cost

$$\min_{\theta \in \mathbb{R}^d} J(\theta) := \mathbb{E}_\tau[\mathcal{R}(\tau)] = \int \mathcal{R}(\tau) p(\tau; \pi_\theta) d\tau \quad (1)$$

Note that the above problem can be viewed as nonconvex stochastic optimization (e.g. Ghadimi and Lan (2013)). We say that a solution  $\theta$  is an  $(\varepsilon_1, \varepsilon_2)$ -second-order stationary point ( $(\varepsilon_1, \varepsilon_2)$ -SOSP) if i)  $\|\nabla J(\theta)\| \leq \varepsilon_1$  and ii)  $\lambda_{\min}(\nabla^2 J(\theta)) \geq -\varepsilon_2$ . Here  $\lambda_{\min}$  denotes the smallest eigenvalue. Moreover,  $\theta$  is an  $\varepsilon_1$ -first-order stationary point ( $\varepsilon_1$ -FOSP) if only condition i) holds. An unbiased estimator of the gradient  $\nabla J(\theta)$  is given by

$$g(\theta; \tau) := \sum_{h=0}^{H-1} \Psi_h(\tau) \nabla \log \pi_\theta(a_h | s_h) \quad (2)$$

where we denote  $\Psi_h(\tau) = \sum_{i=h}^{H-1} \gamma^i r(s_i, a_i)$ . Due to  $\nabla J(\theta) = \mathbb{E}_{\tau \sim p(\cdot; \theta)}[g(\theta; \tau)]$  we can derive an unbiased estimator of the Hessian  $\nabla^2 J(\theta)$  by

$$H(\theta; \tau) := \nabla g(\theta; \tau) + g(\theta; \tau) \nabla \log p(\tau; \theta)^T. \quad (3)$$

Let  $\mathcal{M}$  be a set of i.i.d. trajectories sampled from density  $p(\cdot; \pi_\theta)$ . We denote  $g(\theta; \mathcal{M}) = \frac{1}{|\mathcal{M}|} \sum_{\tau \in \mathcal{M}} g(\theta; \tau)$  and  $H(\theta; \mathcal{M}) = \frac{1}{|\mathcal{M}|} \sum_{\tau \in \mathcal{M}} H(\theta; \tau)$ .

### 3 DR-SOPO

#### 3.1 Algorithm

We first present the dimension-reduced second-order method for policy optimization (DR-SOPO), which extends dimension-reduced trust region technique in DRSOM to the non-oblivious stochastic setting. The novel ingredient in DRSOM is to determine the descent step by involving a two-dimensional trust-region subproblem, which is much simpler than the full dimensional quadratic program in standard trust region method. More specifically, let  $d_t = \theta_t - \theta_{t-1}$ ,  $g_t = g(\theta_t; \mathcal{M}_g)$ ,  $H_t = H(\theta_t; \mathcal{M}_H)$ , instead of updating  $\theta$  by the gradient descent step, we update

$$\theta_{t+1} = \theta_t - \alpha_t^1 g_t + \alpha_t^2 d_t,$$

where the step size  $\alpha_t = (\alpha_t^1, \alpha_t^2)^T$  is determined by solving the following dimension-reduced trust-region (DRTR) problem:

$$\begin{aligned} \min_{\alpha \in \mathbb{R}^2} \quad & m_t(\alpha) := J(\theta_t) + c_t^T \alpha + \frac{1}{2} \alpha^T Q_t \alpha \\ \text{s.t.} \quad & \|\alpha\|_{G_t} \leq \Delta, \end{aligned} \quad (4)$$

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<sup>2</sup>We use regret as opposed to the reward function in standard MDP literature. Our development is more aligned with optimization literature and results in minimization as opposed to maximization problem.

Table 1: Comparison of different policy optimization algorithms. We ignore the logarithmic terms, for simplicity.

Algorithms	Conditions	Guarantee	Sample complexity
REINFORCE (Williams, 1992)	Assumption 3.2,3.3	FOSP	$\mathcal{O}(\epsilon^{-4})$
SRVR-PG (Xu et al., 2020)	Assumption 3.2,3.3 $\text{Var}[\prod_{i \geq 0} \frac{\pi_{\theta_0}(a_i s_i)}{\pi_{\theta_t}(a_i s_i)}] < \infty$	FOSP	$\mathcal{O}(\epsilon^{-3})$
MBPG (Huang et al., 2020)	Assumption 3.2,3.3 Bounded variance of IM samplings	FOSP	$\mathcal{O}(\epsilon^{-3})$
TSIVR-PG (Zhang et al., 2021)	Assumption 3.2,3.3	FOSP	$\mathcal{O}(\epsilon^{-3})$
HAPG (Shen et al., 2019)	Assumption 3.2,3.3	FOSP	$\mathcal{O}(\epsilon^{-3})$
MRPG (Zhang et al., 2020)	Assumption 3.2,3.3, 3.4 Positive-definite Fisher information	SOSP	$\mathcal{O}(\epsilon^{-9})$
Yang et al. (2021)	Assumption 3.2,3.3, 3.4 All saddle points are strict	SOSP	$\mathcal{O}(\epsilon^{-\frac{9}{2}})$
SCR-PG Wang et al. (2022)	Assumption 3.2,3.3 3.4 Bounded variance of IM samplings	SOSP	$\mathcal{O}(\epsilon^{-\frac{7}{2}})$
DR-SOPO	Assumption 3.2,3.3 3.4 3.5	SOSPS	$\mathcal{O}(\epsilon^{-\frac{7}{2}})$
DVR-SOPO	Assumption 3.2,3.3 3.4 3.5	SOSPS	$\mathcal{O}(\epsilon^{-3})$

with

$$Q_t = \begin{bmatrix} g_t^T H_t g_t & -d_t^T H_t g_t \\ -d_t^T H_t g_t & d_t^T H_t d_t \end{bmatrix} \in \mathcal{S}^2,$$

$$c_t := \begin{pmatrix} -\|g_t\|^2 \\ g_t^T d_t \end{pmatrix}, \quad G_t = \begin{bmatrix} g_t^T g_t & -g_t^T d_t \\ -g_t^T d_t & d_t^T d_t \end{bmatrix},$$

and  $\|\alpha\|_{G_t} = \sqrt{\alpha^T G_t \alpha}$ .

The two-dimensional subproblem (4) has a closed-form solution, which can be found in Appendix A. Note that while DRTR conceptually utilizes the curvature information, it can be implemented without explicitly computing the Hessian. In fact, we only require two additional Hessian-vector products to formulate (4). Hence, DRTR only incurs a much lower computation cost than standard trust region method Nocedal and Wright (1999).

Moreover, the next lemma implies that while problem (4) is a two-dimensional trust region model, it can be equivalently transformed into a “full-scale” trust-region problem associated with projected Hessian  $\tilde{H}_t$ , which is pivotal to our theoretical analysis.

**Lemma 3.1.** *The subproblem (4) is equivalent to*

$$\begin{aligned} \min_{d \in \mathbb{R}^n} \tilde{m}_t(d) &:= J(\theta_t) + g_t^T d + \frac{1}{2} d^T \tilde{H}_t d \\ \text{s.t. } \|d\| &\leq \Delta_t, \end{aligned} \tag{5}$$

where  $\tilde{H}_t = V_t V_t^T H_t V_t V_t^T$  and  $V_t$  is the orthonormal bases for  $\mathcal{L}_t := \text{span}\{g_t, d_t\}$ .

For proof, see Zhang et al. (2022). Therefore,  $\tilde{H}_t$  can be viewed as an approximated Hessian matrix in the inexact Newton method, and DRTR may similarly be regarded as a cheap quasi-Newton method. For the ease of notation, let  $\tilde{\nabla} J(\theta_t) = V_t V_t^T \nabla J(\theta_t) V_t V_t^T$  be the projected Hessian onto the subspace  $\mathcal{L}_t$ .

With gradient estimator (2) and Hessian estimator (3), we present the basic DR-SOPO in Algorithm 1, in which we use a fixed trust region radius to make the complexity analysis more concise. A more practical implementation of DR-SOPO will be presented in Algorithm 3, which dynamically adjusts the trust region radius and gets better practical performance.

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**Algorithm 1** Basic DR-SOPO

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- 1: Given  $T, \Delta$
  - 2: **for**  $t = 1, \dots, T$  **do**
  - 3:   Collect sample trajectories  $\mathcal{M}_g$  and compute  $g_t$
  - 4:   Collect sample trajectories  $\mathcal{M}_H$  and compute  $H_t$
  - 5:   Compute stepsize  $\alpha_1, \alpha_2$  by solving the DRTR problem (4)
  - 6:   Update:  $\theta_{t+1} \leftarrow \theta_t - \alpha_1 g_t + \alpha_2 d_t$
  - 7: **end for**
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### 3.2 Complexity analysis

DR-SOPO can find a second-order stationary point in subspace  $((\epsilon, \sqrt{\epsilon})\text{-SOSPS})$ , which is defined as  $\theta$  such that

$$\|\nabla J(\theta)\| \leq c_1 \cdot \epsilon, \quad \lambda_{\min}(\tilde{\nabla}^2 J(\theta)) \geq -c_2 \cdot \sqrt{\epsilon},$$

where  $\tilde{\nabla}^2 J(\theta)$  is the projection of  $\nabla^2 J(\theta)$  in the particular searching space. Next, we will show that DR-SOPO converges to a  $(\epsilon, \sqrt{\epsilon})\text{-SOSPS}$  with a total sample complexity of  $\mathcal{O}(\epsilon^{-3.5})$ .

Before developing the convergence analysis, we need to make a few assumptions.

**Assumption 3.2.** There exists a constant  $R > 0$  such that  $|r(s, a)| \leq R$  for all  $a \in \mathcal{A}$  and  $s \in \mathcal{S}$ .

**Assumption 3.3** (Expected Lipschitzness and smoothness). There exist constants  $G > 0$  and  $L > 0$  such that

$$\mathbb{E}_\tau[\|\nabla \log \pi_\theta(a | s)\|^4] \leq G^4, \quad \mathbb{E}_\tau[\|\nabla^2 \log \pi_\theta(a | s)\|^2] \leq L^2,$$

for any choice of parameter  $\theta$  and state-action pair  $(s, a)$ .

**Assumption 3.4.** There exists  $M > 0$  such that for any  $\theta_1, \theta_2 \in \mathbb{R}^n$

$$\|\nabla^2 J(\theta_1) - \nabla^2 J(\theta_2)\| \leq M\|\theta_1 - \theta_2\|$$

**Assumption 3.5.** There exists a constant  $C > 0$  such that

$$\|(\nabla^2 J(\theta_t) - \tilde{\nabla}^2 J(\theta_t))d_{t+1}\| \leq MC\|d_{t+1}\|^2$$

*Remark 3.6.* Both Assumption 3.2 and 3.3 are fairly standard in the RL literature. Note that, however, 3.3 is weaker than the standard LS assumptions which are defined for  $\nabla \log(\pi_\theta(a | s))$  and  $\nabla^2 \log \pi_\theta(a | s)$  pointwisely. (e.g. Papini et al. (2018); Shen et al. (2019)). More detailed comparison can be found in Yuan et al. (2022). Assumption 3.4 on Lipschitz Hessian is standard for second-order methods, and for RL,  $M$  can be deduced by some other regularity conditions. For example, see Zhang et al. (2020). Assumption 3.5 is required for the convergence analysis of DRSOM Zhang et al. (2022), and is commonly used in the literature (Dennis and Moré, 1977; Cartis et al., 2011).

In view of Assumption 3.3, we characterize some properties of the stochastic gradient and stochastic Hessian, which paves the way for our convergence analysis of the stochastic algorithms.

**Lemma 3.7.** Under Assumptions 3.2 and 3.3, we have for all  $\theta \in \mathbb{R}^d$ ,

$$\begin{aligned} \mathbb{E}_\tau[\|g(\theta; \tau) - \nabla J(\theta)\|^2] &\leq \frac{G^2 R^2}{(1-\gamma)^3} := G_g^2, \\ \mathbb{E}_\tau[\|H(\theta; \tau) - \nabla^2 J(\theta)\|^2] &\leq \frac{H^4 G^4 R^2}{(1-\gamma)^2} + \frac{L^2 R^2}{(1-\gamma)^4} := G_H^2. \end{aligned}$$

Note that our bound is slightly better than the one in Shen et al. (2019), though under weaker Assumption 3.3. Moreover, based on the fact that future policy at time  $t'$  can't affect reward at time  $t$  if  $t < t'$ , we can construct a variance-reduced unbiased Hessian estimator, whose variance bound doesn't depend on  $H$ . More details can be found in B.2. Next, we derive variance bounds on the sampling gradient and Hessian in Algorithm 1.

**Lemma 3.8** (Variance bounds on stochastic estimators). *In Algorithm 1, by setting  $|\mathcal{M}_g| = \frac{144G_g^2}{\epsilon^2}$ , we have*

$$\mathbb{E}_t [\|g_t - \nabla J(\theta_t)\|^2] \leq \frac{\epsilon^2}{144M^2}.$$

Moreover, by setting  $|\mathcal{M}_H| = \frac{22 \times 24^2 G_H^2 \log(n)}{\epsilon}$ , we have

$$\mathbb{E}_t [\|H_t - \nabla^2 J(\theta_t)\|^2] \leq \frac{\epsilon}{24^2}.$$

where  $\mathbb{E}_t$  denotes the expectation conditioned on all the randomness before  $t$ -th iteration.

Based on Assumption 3.5, we obtain a regularity condition such that Hessian estimator  $H_t$  agrees with  $\tilde{H}_t$  along the directions  $d_{t+1}$  formed by past iterates  $\theta_t$ .

**Lemma 3.9.** *Under Assumption 3.5, then we have*

$$\mathbb{E}_t [|(H_t - \tilde{H}_t)d_{t+1}|] \leq M\tilde{C}\Delta^2,$$

where  $\tilde{C} = C + \frac{1}{24}$  and  $d_t \leq \Delta = \frac{2\sqrt{\epsilon}}{M}$ .

The following lemma implies a sufficient descent property in solving the trust region subproblem.

**Lemma 3.10** (Model reduction). *At the  $t$ -th iteration, let  $d_{t+1}$  and  $\lambda_t$  be the optimal primal and dual solution of (5). We have the following amount of decrease on  $\tilde{m}_t$ :*

$$\tilde{m}_t(d_{t+1}) - \tilde{m}_t(0) = -\frac{1}{2}\lambda_t\|d_{t+1}\|^2.$$

With all the tools at our hands, we derive the main convergence property of the DR-SOPO algorithm in the following theorem.

**Theorem 3.11** (Convergence rate of DR-SOPO). *Suppose Assumptions 3.2-3.5 hold. Let  $\Delta_t = \Delta = \frac{2\sqrt{\epsilon}}{M}$  and  $\Delta_J$  be a constant number that s.t.  $\Delta_J \geq J(\theta_0) - J^*$ , by setting  $M_g = \frac{144G_g^2}{\epsilon^2}$ ,  $|\mathcal{M}_H| = \frac{22 \times 24^2 G_H^2 \log(d)}{\epsilon}$ ,  $T = \frac{24M^2\Delta_J}{\epsilon^{3/2}}$  in Algorithm 1, we have*

$$\mathbb{E}[\|\nabla J(\theta_{\bar{t}})\|] \leq \frac{(3 + 4\tilde{C})}{M}\epsilon, \mathbb{E}[\lambda_{\min}(\tilde{\nabla}^2 J(\theta_{\bar{t}}))] \geq -3\sqrt{\epsilon},$$

where  $\bar{t}$  is sampled from  $\{1, \dots, T\}$  uniformly at random. Moreover, with probability at least  $\frac{7}{8}$ , we have

$$\|\nabla J(\theta_{\bar{t}})\| \leq \frac{(12 + 16\tilde{C})}{M}\epsilon.$$

Using the above result, we develop the sample complexity of Algorithm 1 to approximate second-order stationary condition.

**Corollary 3.12** (Sample complexity of DR-SOPO). *Under all the assumptions and parameter settings in Theorem 3.11, Algorithm 1 returns a point  $\hat{x}$  such that  $\mathbb{E}[\|\nabla J(\hat{x})\|] \leq \frac{(3 + 4\tilde{C})}{M}\epsilon$ ,  $\mathbb{E}[\lambda_{\min}(\tilde{\nabla}^2 J(\hat{x}))] \geq -3\sqrt{\epsilon}$  after performing at most*

$$\mathcal{O}\left(\frac{\Delta_J M^2 G_g^2}{\epsilon^{3.5}} + \frac{\Delta_J M^2 G_H^2}{\epsilon^{2.5}}\right) \tag{6}$$

gradient and Hessian-vector product queries.

## 4 DVR-SOPO

### 4.1 Algorithm

Note that the sample complexity (6) of Algorithm 1 is dominated by the component contributed by the stochastic gradient queries. In this section, we further improve the complexity of DR-SOPO by variance reduction. By incorporating the Hessian-aided variance reduction technique (HAVR, Shen et al. (2019)), we develop a dimension and variance-reduced second-order method for policy optimization, dubbed DVR-SOPO, which obtains the optimal  $\mathcal{O}(\epsilon^{-3})$  worst-case complexity bound.

**Variance reduction** We first introduce the variance reduction technique, which helps to construct a more sample-efficient gradient estimator  $g_t$ . Let  $\{\theta_s\}_{s=0}^t$  be the solution sequence. The gradient  $\nabla J(\theta_t)$  can be written in a path-integral form:  $\nabla J(\theta_t) = \nabla J(\theta_0) + \sum_{s=1}^t [\nabla J(\theta_s) - \nabla J(\theta_{s-1})]$ . Let  $\xi_s$  be an unbiased estimator for the gradient difference  $\nabla J(\theta_s) - \nabla J(\theta_{s-1})$ . We can recursively construct the following estimator for  $\nabla J(\theta_t)$ :

$$g_t = \begin{cases} g(\theta_t; \mathcal{M}_0) & \text{mod}(t, q) = 0, \\ g_{t-1} + \xi_t & \text{mod}(t, q) \neq 0 \end{cases}$$

where  $\mathcal{M}_0$  is a mini-batch of trajectories sampled from  $p(\cdot | \pi_\theta)$  and  $q$  is a given epoch length. In other words, we directly estimate the gradient using a mini-batch  $\mathcal{M}_0$  every  $p$  iterations, and we maintain an unbiased estimate by recursively adding the correction term  $\xi_t$  to the current estimate  $g_{t-1}$  in the other iterations.

**Construction of  $\xi_t$ .** We next describe the key idea of HAVR. Due to the non-oblivious stochastic setting of RL (which means that the distribution of random variable  $\tau$  is also influenced by independent variable  $\theta$ ), we can't have access to unbiased samples at  $\theta_t$  and  $\theta_{t-1}$  simultaneously as classical SVRG does. Therefore, we solve it by utilizing Hessian information: from the Taylor's expansion, the gradient difference can be written as

$$\begin{aligned} \nabla J(\theta_t) - \nabla J(\theta_{t-1}) &= \int_0^1 [\nabla^2 J(\theta(a)) \cdot v] da \\ &= \left[ \int_0^1 \nabla^2 J(\theta(a)) da \right] \cdot v, \end{aligned} \tag{7}$$

where we denote  $\theta(a) := a\theta_t + (1-a)\theta_{t-1}$  and  $v := \theta_t - \theta_{t-1}$ . Note that the integral in (7) can be understood as the expectation  $\mathbb{E}_a [\nabla^2 J(\theta(a))]$ , where  $a$  is a random variable uniformly sampled in between  $[0, 1]$ . Hence, with our unbiased Hessian estimator (3), we can rewrite the gradient difference as

$$\begin{aligned} &\nabla J(\theta_t) - \nabla J(\theta_{t-1}) \\ &= \mathbb{E}_{a \sim U[0,1], \tau(a) \sim p(\cdot; \pi_{\theta(a)})} [H(\theta(a); \tau(a))] \cdot v \end{aligned} \tag{8}$$

Let  $\widehat{\mathcal{M}}_g = \{(a, \tau(a))\}$  be set of samples, we can construct the estimator  $\xi_t$  by

$$\xi_t = \frac{1}{|\widehat{\mathcal{M}}_g|} \sum_{(a, \tau(a)) \in \widehat{\mathcal{M}}_g} H(\theta(a); \tau(a)) \cdot v \tag{9}$$

**Comparison with other VR techniques** We emphasize that other variance-reduced techniques, such as SRVR-PG, can also be applied in our methods to improve the estimation of  $\nabla J(\theta)$ . However, HAVR appears to have some advantages. In order to construct an unbiased estimator of (9), one relies on the assumption of expected LS, which is weaker than the point-wise LS often imposed on standard VR methods. Moreover, HAVR does not need to assume the bounded variance of importance sampling, which is also a strong assumption imposed by the standard VR techniques.

Based on the variance reduction technique, we develop the basic DVR-SOPO algorithm in Algorithm 2. A more detailed comparison of the sample complexity with other state-of-the-art policy gradient methods in

Table 1. Similar to the development of DR-SOPO, we will present a more practical version of DVR-SOPO in Algorithm 4.

---

**Algorithm 2** Basic DVR-SOPO

---

```

1: Given  $T, \Delta, q$ 
2: for  $t = 1, \dots, T$  do
3:   if  $\text{mod}(t, q) = 0$  then
4:     Compute  $g_t$  based on sampled trajectories  $\mathcal{M}_0$ 
5:   else
6:     Compute  $\xi_t$  based on sampled trajectories  $\widehat{\mathcal{M}}_g$  and update:

$$g_t = g_{t-1} + \xi_t$$

7:   end if
8:   Collect samples trajectories  $\mathcal{M}_H$  and construct  $H_t$ 
9:   Compute stepsize  $\alpha_1, \alpha_2$  by solving the DRTR problem (4)
10:  Update:  $\theta_{t+1} \leftarrow \theta_t - \alpha_1 g_t + \alpha_2 d_t$ 
11: end for

```

---

## 4.2 Complexity Analysis

DVR-SOPO relies on the same assumptions of DR-SOPO. With the help of HAVR, we can bound the variance of gradient estimator more sample-efficiently.

**Lemma 4.1** (Gradient variance bound for DVR-SOPO). *Recall the definition of  $g_t$  in the Algorithm 2. By setting  $\epsilon \leq \frac{G_H^2}{4}$ ,  $q = \frac{1}{8\epsilon^{1/2}}$ ,  $|\widehat{\mathcal{M}}_g| = \frac{288G_H^2}{M^2\epsilon^{3/2}}$ , and  $|\mathcal{M}_0| = \frac{288G_g^2}{\epsilon^2}$ , we have*

$$\mathbb{E} [\|g_t - \nabla J(\theta_t)\|^2] \leq \frac{\epsilon^2}{144M^2}.$$

**Theorem 4.2** (Convergence rate of DVR-SOPO). *Suppose Assumptions 3.2-3.5 hold. Let  $\Delta_t = \Delta = \frac{2\sqrt{\epsilon}}{M}$  and  $\Delta_J$  be a constant number that s.t.  $\Delta_J \geq J(\theta_0) - J^*$ . If we set  $\epsilon \leq \frac{G_H^2}{4}$ ,  $q = \frac{1}{8\epsilon^{1/2}}$ ,  $|\widehat{\mathcal{M}}_g| = \frac{288G_H^2}{M^2\epsilon^{3/2}}$ ,  $|\mathcal{M}_0| = \frac{288G_g^2}{\epsilon^2}$ ,  $|\mathcal{M}_H| = \frac{22 \times 24^2 G_H^2 \log(d)}{\epsilon}$ , and  $T = \frac{24M^2\Delta_J}{\epsilon^2}$  in Algorithm 2, then we have*

$$\mathbb{E}[\|\nabla J(\theta_{\bar{t}})\|] \leq \frac{(3 + 4\tilde{C})}{M}\epsilon, \quad \mathbb{E}[\lambda_{\min}(\tilde{\nabla}^2 J(\theta_{\bar{t}}))] \geq -3\sqrt{\epsilon},$$

where  $\bar{t}$  is uniformly sampled from  $\{1, \dots, T\}$ . Moreover, with probability at least  $\frac{7}{8}$ , we have

$$\|\nabla J(\theta_{\bar{t}})\| \leq \frac{(12 + 16\tilde{C})}{M}\epsilon.$$

**Corollary 4.3** (Sample complexity of DVR-SOPO). *Under all the assumptions and parameter settings in Theorem 4.2, Algorithm 2 returns a point  $\hat{x}$  such that  $\mathbb{E}[\|\nabla J(\hat{x})\|] \leq \frac{(3 + 4\tilde{C})}{M}\epsilon$ ,  $\mathbb{E}[\lambda_{\min}(\tilde{\nabla}^2 J(\hat{x}))] \geq -3\sqrt{\epsilon}$  after performing at most*

$$\mathcal{O} \left( \frac{\Delta_J(8M^2G_g^2 + G_H^2)}{\epsilon^3} + \frac{\Delta_J M^2 G_H^2}{\epsilon^{2.5}} \right)$$

gradient and Hessian-vector product queries.

*Proof.* In view of Lemma 4.1 and 3.8, for each iteration of DVR-SOPO, we need  $N_1 = \frac{288(8G_g^2 + \frac{G_H^2}{M^2})}{\epsilon^{3/2}}$  samples for gradient estimator  $g_t$  and  $N_2 = \frac{22 \cdot 576 G_H^2 \log(d)}{\epsilon}$  samples for Hessian estimator  $H_t$ . Following Theorem 4.2, the total number of samples required by DVR-SOPO is

$$(N_1 + N_2)T = \mathcal{O}\left(\frac{\Delta_J(8M^2G_g^2 + G_H^2)}{\epsilon^3} + \frac{\Delta_J M^2 G_H^2 \log(d)}{\epsilon^{2.5}}\right).$$

□

## 5 Practical Implementation

To achieve better empirical performance, we describe more practical versions of DR-SOPO and DVR-SOPO that dynamically adjust the trust region radius  $\Delta_t$ . Consider the reduction ratio for  $m_t^\alpha$  (4) at iterate  $\theta_t$

$$\rho_t := \frac{J(\theta_t) - J(\theta_t + d_{t+1})}{m_t(0) - m_t(\alpha_t)}. \quad (10)$$

If  $\rho_t$  is too small, our quadratic model is somehow inaccurate, which prompts us to reduce  $\Delta_t$ .

In practice, it is difficult to adjust properly  $\Delta_t$  in (4). To resolve this issue, we consider the following radius-free problem:

$$\beta_\alpha(\lambda_t) = \min_{\alpha \in \mathbb{R}^2} J(\theta_t) + c_t^\top \alpha + \frac{1}{2} \alpha^\top Q_t \alpha + \lambda_t \|\alpha\|_{G_t}^2 \quad (11)$$

as an alternative to (4). Due to the KKT condition,  $\Delta_t$  is implicitly defined by  $\lambda_t$ , and we can properly adjust  $\lambda_t$  to solve  $\beta_\alpha(\lambda_t)$  and get sensible amount of decrease. This strategy has been proven effective in Zhang et al. (2022) and hence is used in our implementation. The details of practical DR-SOPO and DVR-SOPO are presented in Appendix E.

A notable portion of the literature has focused on deriving a better choice of  $\Psi_h(\tau)$  in order to reduce the variance in estimating the policy gradient  $\nabla J(\theta)$ . Conventional approaches include actor-critic algorithms (Bhatnagar et al., 2007; Konda and Tsitsiklis, 1999), and adding baselines (Wu et al., 2018). The Generalized Advantage Estimation (GAE) proposed by Schulman et al. (2015b) is one of the best ways to approximate the advantage function. We emphasize that all these refined advantage estimators can be directly incorporated to our method by placing  $\Psi_h(\tau)$  correspondingly. Our experiments use GAE( $\gamma, 1$ ), then  $\Psi_h(\tau)$  becomes  $\Psi_h(\tau) = \sum_{i=h}^{H-1} \gamma^i r(s_i, a_i) - b(s_h)$ , where  $b$  is the linear baseline.

## 6 Numerical Results

In this section, we evaluate the performance of DR-SOPO and DVR-SOPO and compare them with REINFORCE (Sutton et al., 1999) and HAPG (Shen et al., 2019). We choose four Mujoco environments (Todorov et al., 2012): Walker2d-v2, Swimmer-v2, HalfCheetah-v2 and Ant-v2. We implement the four algorithms based on the Garage library garage contributors (2019) and PyTorch Paszke et al. (2019).

For all the environments, we use deep Gaussian policy, where the mean and variance are parameterized by a fully-connected neural network. We normalize the gradient estimator of all four algorithms for fair comparison. We also initialize all algorithms with the same random policy parameters. We repeat the experiment 10 times to reduce the impact of randomness and plot the mean and variance, using the same batch size for all algorithms. Note that the two variance-reduced algorithms DVR-SOPO and HAPG are double-loop procedures. We set the period and batch size of inner loop to be the same. For the two second-order algorithms DR-SOPO and DVR-SOPO, we set the additional hyper-parameters to be the same. More details of model architecture and hyper-parameter settings are shown in Appendix F.

We use the number of system probes (the number of state transitions) instead of the number of trajectories to measure the sample complexity. System probes is a better criterion since different trajectories might have different number of system probes when a failure flag returned from the environment. To be consistent with

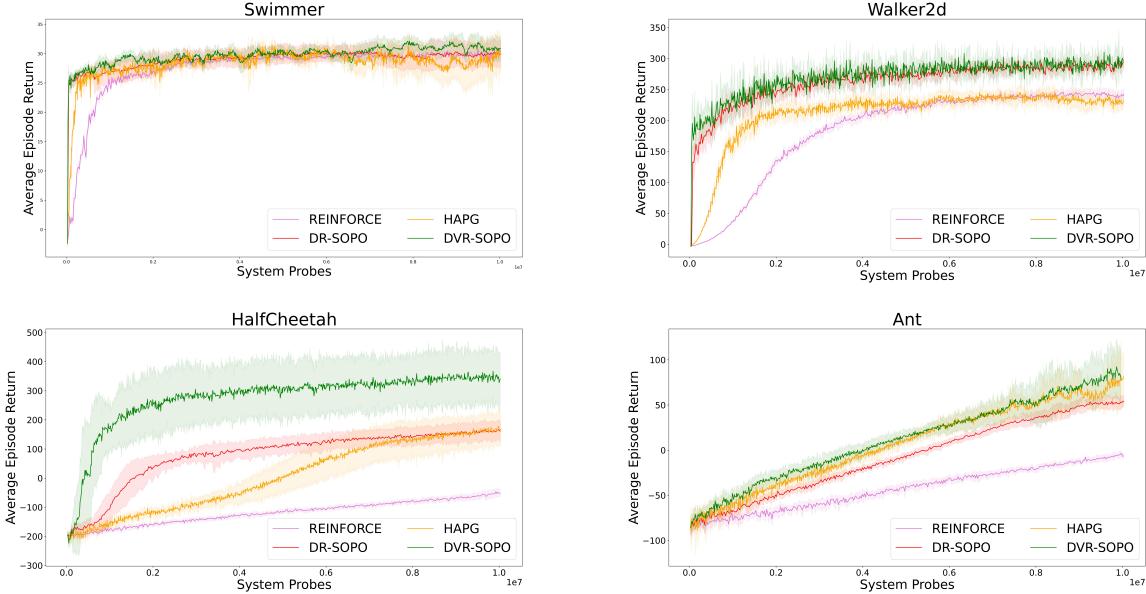


Figure 1: Performance of DR-SOPO, DVR-SOPO and REINFORCE, HAPG on four environments

previous empirical study, we measure the algorithm performance using the average episode return, which is the negation of the cost/regret. Hence the higher return indicates better result.

The experiment results are shown in Fig. 1. First, we observe that DR-SOPO consistently outperform REINFORCE in all four environments, and achieving significantly better final average return in Walker2d, HalfCheetah and Ant. Moreover, we observe similar comparison results between DVR-SOPO and HAPG, as DVR-SOPO converges faster in all four environments and achieves significantly better final average return on Walker2d and HalfCheetah. These experiment results indeed confirm the advantage of using second-order information. Next, we compare the performance of two second-order algorithms DR-SOPO and DVR-SOPO. In Swimmer, their convergence rates are similar, but DVR-SOPO yields better final average reward. In Walker2d, they yield similar final average reward but DVR-SOPO converge faster than DR-SOPO at the beginning. In HalfCheetah and Ant, DVR-SOPO not only converge faster but yield better final average reward than DR-SOPO. The experiment results suggest that variance reduction does improve the performance of second-order algorithms, which align with our theoretical analysis.

## 7 Discussion

In this paper, we propose several efficient second-order methods, namely DR-SOPO and DVR-SOPO, for policy optimization. We show that DR-SOPO obtains an  $\mathcal{O}(\epsilon^{-3.5})$  complexity for reaching first order stationary condition and a subspace second-order stationary condition. DVR-SOPO further improves the rate to  $\mathcal{O}(\epsilon^{-3})$  based on the recently proposed variance reduction technique. Our methods have the distinct advantage of only requiring the computation of gradient and Hessian-vector product in each iteration, without the need to solve much complicated quadratic subproblems. As a result, they can be implemented with efficiency comparable to standard PG methods.

It is worth noting that while we mainly focus on the dimension-reduced model, it is easy to extend our analysis to develop a "full-dimension" stochastic trust region algorithm that solves the standard trust region subproblem instead of the reduced problem (4). Due to the page limits, we give more detail analysis in Appendix section C.

One interesting direction is to further exploit the connection between our methods and TRPO. It would be interesting to see how to apply our algorithm to TRPO if the objective is replaced by a more accurate

quadratic approximation. Additionally, it would be valuable to investigate the performance of our methods for certain parameterized policy when global optimality is guaranteed.

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## A Optimal condition and solution for trust region problems

We introduce the widely known optimal conditions for trust region methods. The vector  $\alpha_t$  is the global solution to DRTR problem (4) if it is feasible and there exists a Lagrange multiplier  $\lambda_t \geq 0$  such that  $(\alpha_t, \lambda_t)$  is the solution to the following equations:

$$(Q_t + \lambda G_t) \alpha + c_t = 0, Q_t + \lambda G_t \succeq 0, \lambda (\Delta - \|\alpha\|_{G_t}) = 0. \quad (12)$$

Then by the optimal condition (12), we have the closed form solution of  $\alpha_t$ :

$$\alpha_t = -(Q_t + \lambda_t G_t)^{-1} c_t.$$

Since  $\alpha$  only has two dimensions, it can be easily solved numerically. Furthermore, recalling Lemma 3.1, by constructing  $d_{t+1} = V_t \alpha_t$ , we can prove that  $d_{t+1}$  is the solution to the full-scale problem (5) such that

$$(\tilde{H}_t + \lambda_t I) d_{t+1} + G_t = 0, \tilde{H}_t + \lambda_t I \succeq 0, \lambda_t (\|d_{t+1}\| - \Delta_t) = 0, \quad (13)$$

where  $\tilde{H}_t = V_t V_t^T H_t V_t V_t^T$ .

## B Detailed proofs

### B.1 Proof of Lemma 3.7

The proof can be found in Yuan et al. (2022), which is the best result though under a weaker expected-LS regularity. We know that our gradient estimator (PGT estimator, (Sutton et al., 2000)) is equivalent to GPOMDP estimator Baxter and Bartlett (2001):

$$g(\theta; \tau) = \sum_{h=0}^{H-1} \left( \sum_{t=0}^h \nabla_\theta \log \pi_\theta(a_t | s_t) \right) (\gamma^h r(s_h, a_h))$$

So we will also use the GPOMDP estimator definition if needed.

*Proof.*

$$\begin{aligned} \mathbb{E}_\tau [\|g(\theta; \tau)\|^2] &= \mathbb{E}_\tau \left[ \left\| \sum_{t=0}^{H-1} \gamma^{t/2} r(s_t, a_t) \gamma^{t/2} \left( \sum_{k=0}^t \nabla_\theta \log \pi_\theta(a_k | s_k) \right) \right\|^2 \right] \\ &\leq \mathbb{E}_\tau \left[ \left( \sum_{t=0}^{H-1} \gamma^t r(s_t, a_t)^2 \right) \left( \sum_{k=0}^{H-1} \gamma^k \left\| \sum_{k'=0}^k \nabla_\theta \log \pi_\theta(a_{k'} | s_{k'}) \right\|^2 \right) \right] \\ &\stackrel{\clubsuit}{\leq} \frac{R^2}{1-\gamma} \cdot \sum_{k=0}^{H-1} \gamma^k \mathbb{E}_\tau \left[ \left\| \sum_{k'=0}^k \nabla_\theta \log \pi_\theta(a_{k'} | s_{k'}) \right\|^2 \right] \\ &\stackrel{\diamondsuit}{=} \frac{R^2}{1-\gamma} \cdot \sum_{k=0}^{H-1} \gamma^k \sum_{k'=0}^k \mathbb{E}_\tau [\|\nabla_\theta \log \pi_\theta(a_{k'} | s_{k'})\|^2] \\ &\leq \frac{G^2 R^2}{1-\gamma} \cdot \sum_{k=0}^{H-1} \gamma^k (k+1) \\ &\leq \frac{G^2 R^2}{(1-\gamma)^3} := G_g^2, \end{aligned}$$

where ( $\clubsuit$ ) uses the Assumption 3.2, and ( $\diamondsuit$ ) is due to the fact that for any  $h \neq h'$ :

$$\mathbb{E}_\tau [\nabla_\theta \log \pi_\theta(a_h | s_h))^T (\nabla_\theta \log \pi_\theta(a_{h'} | s_{h'}))] = 0. \quad (14)$$

To derive the variance-reduced Hessian estimator and bound its variance, note that

$$\mathbb{E}_\tau[\|H(\theta; \tau)\|^2] \leq 2\mathbb{E}[\|\nabla_\theta g(\theta; \tau)\|^2] + 2\mathbb{E}_\tau[\|g(\theta; \tau)\nabla_\theta \log p(\tau | \theta)^T\|^2].$$

Plugging the GPOMDP definition of  $g(\theta; \tau)$  in this term yields

$$\begin{aligned} \mathbb{E}_\tau[\|\nabla g(\theta; \tau)\|^2] &\leq \mathbb{E}_\tau \left[ \left( \sum_{h=0}^{H-1} \gamma^h R^2 \right) \left( \sum_{h=0}^{H-1} \gamma^h \left\| \sum_{t'=0}^h \nabla_\theta^2 \log \pi_\theta(a_{t'} | s_{t'}) \right\|^2 \right) \right] \\ &\leq \frac{R^2}{1-\gamma} \cdot \sum_{h=0}^{H-1} \gamma^h \mathbb{E}_\tau \left[ \left\| \sum_{t'=0}^h \nabla_\theta^2 \log \pi_\theta(a_{t'} | s_{t'}) \right\|^2 \right] \\ &\leq \frac{R^2}{1-\gamma} \cdot \sum_{h=0}^{H-1} \gamma^h (h+1) \sum_{t'=0}^h \mathbb{E}_\tau [\|\nabla_\theta^2 \log \pi_\theta(a_{t'} | s_{t'})\|^2] \\ &\leq \frac{L^2 R^2}{1-\gamma} \cdot \sum_{h=0}^{H-1} \gamma^h (h+1)^2 \\ &\leq \frac{2L^2 R^2}{(1-\gamma)^4}. \end{aligned}$$

For the ease in notation, let us denote  $z_t = \nabla_\theta \log \pi_\theta(a_t | s_t)$ . It follows that

$$\begin{aligned} &\mathbb{E}_\tau [\|g(\theta; \tau)(\nabla_\theta \log p(\tau | \theta))^T\|^2] \\ &= \mathbb{E}_\tau \left[ \left\| \sum_{h=0}^{H-1} \gamma^h r(s_h, a_h) \left( \sum_{t=0}^h \nabla_\theta \log \pi_\theta(a_t | s_t) \right) \left( \sum_{t'=0}^{H-1} (\nabla_\theta \log \pi_\theta(a_{t'} | s_{t'}))^T \right) \right\|^2 \right] \\ &\stackrel{\diamond}{\leq} \mathbb{E}_\tau \left[ \left( \sum_{h=0}^{H-1} \gamma^h R^2 \right) \left( \sum_{h=0}^{H-1} \gamma^h \left\| \sum_{t=0}^h z_t \sum_{t'=0}^{H-1} z_{t'}^T \right\|^2 \right) \right] \\ &\leq \mathbb{E}_\tau \left[ \left( \sum_{h=0}^{H-1} \gamma^h R^2 \right) \left( \sum_{h=0}^{H-1} \gamma^h \left( \sum_{t=0}^h \|z_t\| \sum_{t'=0}^{H-1} \|z_{t'}\|^T \right)^2 \right) \right] \\ &\leq \frac{R^2}{1-\gamma} \mathbb{E}_\tau \left[ \sum_{h=0}^{H-1} \gamma^h \left( \sum_{t=0}^h \|z_t\| \sum_{t'=0}^{H-1} \|z_{t'}\|^T \right)^2 \right] \\ &\stackrel{\sharp}{\leq} \frac{R^2}{1-\gamma} \sum_{h=0}^{H-1} \gamma^h H^4 \mathbb{E}_\tau[\|z_t\|^4] \\ &\leq \frac{G^4 R^2}{1-\gamma} \sum_{h=0}^{H-1} \gamma^h H^4 \leq \frac{G^4 H^4 R^2}{(1-\gamma)^2} \end{aligned}$$

where  $(\diamond)$  is due to Cauchy-Schwarz inequality, and  $(\sharp)$  is due to the fact that for any matrix (or scalar)

$A_0, A_1, \dots, A_n$  with dimension  $d_1 \cdot d_2$ , we have

$$\begin{aligned}
\mathbb{E}[\|(\sum_{i=0}^n A_i \sum_{i=0}^n A_i^T)\|^2] &= \mathbb{E}[\|\sum_{i=0}^n A_i A_i^T + 2 \sum_{i \neq j} A_i A_j^T\|^2] \\
&\leq \mathbb{E}[(\sum_{i=0}^n \|A_i\|^2 + 2 \sum_{i \neq j} \|A_i\| \|A_j\|)^2] \\
&\leq \mathbb{E}[(n+1) \sum_{i=0}^n \|A_i\|^2]^2 \\
&= (n+1)^2 \cdot \mathbb{E}[(\sum_{i=0}^n \|A_i\|^2)^2] = (n+1)^3 \cdot \mathbb{E}[\sum_{i=0}^n \|A_i\|^4].
\end{aligned} \tag{15}$$

Hence, we have

$$\mathbb{E}_\tau \|H(\theta; \tau)\|^2 \leq 2 \mathbb{E}_\tau [\|g(\theta; \tau)(\nabla_\theta \log p(\tau | \theta))^T\|^2] + 2 \mathbb{E}_\tau [\|\nabla g(\theta; \tau)\|^2] \leq \frac{2H^4 G^4 R^2 (1-\gamma)^2 + 4L^2 R^2}{(1-\gamma)^4} := G_H^2.$$

Finally, using  $\mathbb{E}_\tau[(X - \mathbb{E}_\tau[X])^2] \leq \mathbb{E}_\tau[X^2]$  for all random variable X, we have

$$\begin{aligned}
\mathbb{E}_\tau \|g(\theta; \tau) - \nabla J(\theta)\|^2 &\leq \mathbb{E}_\tau \|g(\theta; \tau)\|^2 \leq G_g^2, \\
\mathbb{E}_\tau \|H(\theta; \tau) - \nabla^2 J(\theta)\|^2 &\leq \mathbb{E}_\tau \|H(\theta; \tau)\|^2 \leq G_H^2.
\end{aligned}$$

□

## B.2 More discussion about Hessian estimator

**1. A Variance-reduced unbiased estimator:** We claim here that by constructing a variance reduced Hessian estimator, the variance of  $H(\theta; \tau)$  can actually be bounded more tightly, which is without parameter H. To do so, note that

$$\begin{aligned}
&\mathbb{E}_\tau [g(\theta; \tau)(\nabla_\theta \log p(\tau | \theta))^T] \\
&= \mathbb{E}_\tau \left[ \sum_{h=0}^{H-1} \left( \sum_{t=0}^h \nabla_\theta \log \pi_\theta(a_t | s_t) \right) \gamma^h r(s_h, a_h) (\nabla_\theta \log p(\tau | \theta))^T \right] \\
&\stackrel{\diamond}{=} \mathbb{E}_\tau \left[ \sum_{h=0}^{H-1} \left( \sum_{t=0}^h \nabla_\theta \log \pi_\theta(a_t | s_t) \right) \gamma^h r(s_h, a_h) \left( \sum_{t'=0}^{H-1} (\nabla_\theta \log \pi_\theta(a_{t'} | s_{t'}))^T \right) \right] \\
&\stackrel{\heartsuit}{=} \mathbb{E}_\tau \left[ \sum_{h=0}^{H-1} u_h \cdot u_h^T \gamma^h r(s_h, a_h) \right],
\end{aligned}$$

where  $u_h = \sum_{t=0}^h \nabla_\theta \log \pi_\theta(a_t | s_t)$ ,  $(\diamond)$  is due to  $\nabla_\theta \mathcal{P}(s_{t'+1} | s_{t'}, a_{t'}) = 0$  and  $(\heartsuit)$  uses property (14).

Let  $H'(\theta; \tau) = \sum_{h=0}^{H-1} \gamma^h r(s_h, a_h) u_h \cdot u_h^T + \nabla g(\theta; \tau)$ , it is also an unbiased estimator of  $\nabla^2 J(\theta)$ . Moreover,

we have

$$\begin{aligned}
& \mathbb{E}_\tau \left[ \left\| \sum_{h=0}^{H-1} \gamma^h r(s_h, a_h) (u_h \cdot u_h^\top) \right\|^2 \right] \\
& \stackrel{\diamond}{\leq} \mathbb{E}_\tau \left[ \left( \sum_{h=0}^{H-1} \gamma^h R^2 \right) \left( \sum_{h=0}^{H-1} \gamma^h \|u_h \cdot u_h^\top\|^2 \right) \right] \\
& \leq \frac{R^2}{1-\gamma} \cdot \sum_{h=0}^{H-1} \gamma^h \mathbb{E}_\tau \left[ \|u_h \cdot u_h^\top\|^2 \right] \\
& \stackrel{\sharp}{\leq} \frac{R^2}{1-\gamma} \cdot \sum_{h=0}^{H-1} \gamma^h (h+1)^3 \sum_{t=0}^h \mathbb{E}_\tau \left[ \|\nabla_\theta \log \pi_\theta(a_t | s_t)\|^2 \right] \\
& \leq \frac{G^4 R^2}{1-\gamma} \cdot \sum_{h=0}^{H-1} \gamma^h (h+1)^4 \\
& \leq \frac{24 G^4 R^2}{(1-\gamma)^6},
\end{aligned}$$

where  $(\diamond)$  is due to Cauchy-Schwarz inequality, and  $(\sharp)$  is due to (15).

Hence, we have

$$\mathbb{E}_\tau \|H'(\theta; \tau)\|^2 \leq 2 \mathbb{E}_\tau \left[ \left\| \sum_{h=0}^{H-1} \gamma^h r(s_h, a_h) u_t u_t^\top \right\|^2 \right] + 2 \mathbb{E}_\tau [\|\nabla g(\theta; \tau)\|^2] \leq \frac{48 G^4 R^2 + 4 L^2 R^2 (1-\gamma)^2}{(1-\gamma)^6} := {G'_H}^2$$

In practice, this method requires parallelism and a significant amount of computing resources as it needs to be backpropagated multiple times ( $H$  times).

## 2. Biased Hessian Estimation:

Consider the biased estimator of  $\nabla^2 J(\theta)$ :

$$H_\mu(\theta; \tau) = \nabla g(\theta; \tau) + \mu g(\theta; \tau) \nabla \log p(\tau; \theta)^\top \quad (16)$$

Note that when  $\mu = 1$ , this bound becomes the unbiased Hessian estimator. However, a biased Hessian estimator which has better practical variance property. We can further derive its MSE bound:

$$\begin{aligned}
& \mathbb{E}_\tau [\|H_\mu(\theta; \tau) - \nabla^2 J(\theta)\|^2] \\
& = \text{Var}(H_\mu(\theta; \tau)) + \|\mathbb{E}_\tau[H_\mu(\theta; \tau)] - \nabla^2 J(\theta)\|^2 \\
& = \text{Var}(\nabla g(\theta; \tau) + \mu g(\theta; \tau) \nabla \log p(\tau; \theta)^\top) + \|\mathbb{E}_\tau[(\mu-1)g(\theta; \tau) \nabla \log p(\tau; \theta)^\top]\|^2 \\
& = (\mu-1)^2 \|\mathbb{E}_\tau[g(\theta; \tau) \nabla \log p(\tau; \theta)^\top]\|^2 + \text{Var}(\nabla g(\theta; \tau)) + \mu^2 \text{Var}[g(\theta; \tau)(\nabla_\theta \log p(\tau | \theta))^\top] \\
& \quad + 2\mu \mathbb{E}_\tau [[\nabla g(\theta; \tau) - \mathbb{E}_\tau[\nabla g(\theta; \tau)]] [g(\theta; \tau) \nabla \log p(\tau; \theta)^\top - \mathbb{E}_\tau[g(\theta; \tau) \nabla \log p(\tau; \theta)^\top]]] \\
& = (1-2\mu) \|\mathbb{E}_\tau[g(\theta; \tau) \nabla \log p(\tau; \theta)^\top]\|^2 + \mu^2 \mathbb{E}_\tau[\|g(\theta; \tau) \nabla \log p(\tau; \theta)^\top\|^2] + \text{Var}(\nabla g(\theta; \tau)) \\
& \quad + 2\mu \mathbb{E}_\tau [[\nabla g(\theta; \tau) - \mathbb{E}_\tau[\nabla g(\theta; \tau)]] [g(\theta; \tau) \nabla \log p(\tau; \theta)^\top - \mathbb{E}_\tau[g(\theta; \tau) \nabla \log p(\tau; \theta)^\top]]] \\
& \leq 2\mu \sqrt{\mathbb{E}_\tau [\|\nabla g(\theta; \tau) - \mathbb{E}_\tau[\nabla g(\theta; \tau)]\|^2] \mathbb{E}_\tau [\|g(\theta; \tau) \nabla \log p(\tau; \theta)^\top - \mathbb{E}_\tau[g(\theta; \tau) \nabla \log p(\tau; \theta)^\top]\|^2]} \\
& \quad + (\mu-1)^2 \mathbb{E}_\tau [\|g(\theta; \tau) \nabla \log p(\tau; \theta)^\top\|^2] + \text{Var}(\nabla g(\theta; \tau)) \\
& \leq \mathbb{E}_\tau [\|\nabla g(\theta; \tau)\|^2] + (\mu-1)^2 \mathbb{E}_\tau [\|g(\theta; \tau)(\nabla_\theta \log p(\tau | \theta))^\top\|^2] \\
& \quad + 2\mu \sqrt{\mathbb{E}_\tau [\|\nabla g(\theta; \tau)\|^2] \mathbb{E}_\tau [\|g(\theta; \tau)(\nabla_\theta \log p(\tau | \theta))^\top\|^2]} \\
& \leq \frac{2L^2 R^2}{(1-\gamma)^4} + (\mu-1)^2 \frac{G^4 H^4 R^2}{(1-\gamma)^2} + 2\mu \sqrt{\frac{2L^2 R^2}{(1-\gamma)^4} \frac{G^4 H^4 R^2}{(1-\gamma)^2}}.
\end{aligned}$$

By setting the value  $\mu = 1 - \frac{\sqrt{2}L}{(1-\gamma)G^2H^2}$ , we obtain the tightest bound:

$$\frac{2\sqrt{2}LR^2G^2H^2}{(1-\gamma)^3}.$$

In experiment, we find that using a biased estimator often leads to better performance, and hence adjust  $\mu$  as a hyper-parameter.

### B.3 Proof of Lemma 3.8

*Proof.*

$$\begin{aligned}\mathbb{E}_t \left[ \|g_t - \nabla J(\theta_t)\|^2 \right] &= \mathbb{E}_t \left[ \left\| \frac{1}{|\mathcal{M}_g|} \sum_{\tau \in \mathcal{M}_g} g(\theta, \tau) - \nabla J(\theta_t) \right\|^2 \right] \\ &= \frac{\mathbb{E}_t \left[ \left\| \sum_{\tau \in \mathcal{M}_g} g(\theta, \tau) - |\mathcal{M}_g| \cdot \nabla^2 J(\theta_t) \right\| \right]}{|\mathcal{M}_g|^2} \\ &= \frac{\mathbb{E}_t \left[ \|g(\theta, \tau) - \nabla J(\theta_t)\|^2 \right]}{|\mathcal{M}_g|} \\ &\leq \frac{\epsilon^2}{144M^2},\end{aligned}$$

where the last equality is due to the fact that samples are independent and  $M_g = \frac{144G_g^2}{\epsilon^2}$ .

As for the variance bound of Hessian, it is a direct result of the following auxiliary lemma (whose proof can be found in Arjevani et al. (2020). We derive it here just for completeness) by setting  $A_i = H(\theta; \tau)$ ,  $B = \nabla^2 J(\theta)$ .

**Lemma B.1.** Let  $(A_i)_{i=1}^m$  be a collection of i.i.d. matrices in  $\mathbb{S}^n$ , with  $\mathbb{E}[A_i] = B$  and  $\mathbb{E}\|A_i - B\|^2 \leq \sigma^2$ . Then it holds that

$$\mathbb{E} \left\| \frac{1}{m} \sum_{i=1}^m A_i - B \right\|^2 \leq \frac{22\sigma^2 \log n}{m}.$$

**Proof:** We drop the normalization by  $m$  throughout this proof. We first symmetrize. Observe that by Jensen's inequality we have

$$\begin{aligned}\mathbb{E} \left\| \sum_{i=1}^m A_i - B \right\|^2 &\leq \mathbb{E}_A \mathbb{E}_{A'} \left\| \sum_{i=1}^m A_i - A'_i \right\|^2 \\ &= \mathbb{E}_A \mathbb{E}_{A'} \left\| \sum_{i=1}^m (A_i - B) - (A'_i - B) \right\|^2 \\ &= \mathbb{E}_A \mathbb{E}_A \mathbb{E}_\epsilon \left\| \sum_{i=1}^m \epsilon_i ((A_i - B) - (A'_i - B)) \right\|^2 \leq 4 \mathbb{E}_A \mathbb{E}_\epsilon \left\| \sum_{i=1}^m \epsilon_i (A_i - B) \right\|^2,\end{aligned}$$

where  $(A')_{i=1}^m$  is a sequence of independent copies of  $(A_i)_{i=1}^m$  and  $(\epsilon_i)_{i=1}^m$  are Rademacher random variables. Henceforth we condition on  $A$ . Let  $p = \log n$ , and let  $\|\cdot\|_{S_p}$  denote the Schatten  $p$ -norm. In what follows, we will use that for any matrix  $X$ ,  $\|X\| \leq \|X\|_{S_{2p}} \leq e^{1/2} \|X\|$ . To begin, we have

$$\mathbb{E}_\epsilon \left\| \sum_{i=1}^m \epsilon_i (A_i - B) \right\|^2 \leq \mathbb{E}_\epsilon \left\| \sum_{i=1}^m \epsilon_i (A_i - B) \right\|_{S_{2p}}^2 \leq \left( \mathbb{E}_\epsilon \left\| \sum_{i=1}^m \epsilon_i (A_i - B) \right\|_{S_{2p}}^{2p} \right)^{1/p},$$

where the second inequality follows by Jensen. We now apply the matrix Khintchine inequality (Mackey et al. (2014), Corollary 7.4), which implies that

$$\begin{aligned} \left( \mathbb{E}_\epsilon \left\| \sum_{i=1}^m \epsilon_i (A_i - B) \right\|_{S_{2p}}^{2p} \right)^{1/p} &\leq (2p-1) \left\| \sum_{i=1}^m (A_i - B)^2 \right\|_{S_{2p}} \leq (2p-1) \sum_{i=1}^m \|(A_i - B)\|_{S_{2p}}^2 \\ &\leq e(2p-1) \sum_{i=1}^m \|(A_i - B)\|^2. \end{aligned}$$

Putting all the developments so far together and taking expectation with respect to  $A$ , we have

$$\mathbb{E} \left\| \sum_{i=1}^m A_i - B \right\|^2 \leq 4e(2p-1) \sum_{i=1}^m \mathbb{E}_{A_i} \|(A_i - B)\|^2 \leq 4e(2p-1)m\sigma^2.$$

To obtain the final result we normalize by  $m^2$  □

#### B.4 Proof of Lemma 3.9

Recalling  $\tilde{\nabla}^2 J(\theta_t) = V_t V_t^\top \nabla^2 J(\theta_t) V_t V_t^\top$ , we have

*Proof.*

$$\begin{aligned} &\mathbb{E}_t \|(H_t - \tilde{H}_t)d_{t+1}\| \\ &\leq \mathbb{E}_t [\|(\nabla^2 J(\theta_t) - \tilde{\nabla}^2 J(\theta_t))d_{t+1}\| + \|(H_t - \nabla^2 J(\theta_t))d_{t+1}\| + \|V_t V_t^\top (\nabla^2 J(\theta_t) - H_t) V_t V_t^\top d_{t+1}\|] \\ &\stackrel{(\diamond)}{\leq} MC\Delta^2 + \mathbb{E}_t [\|(H_t - \nabla^2 J(\theta_t))d_{t+1}\|] + \mathbb{E}_t [\|V_t V_t^\top\| \cdot \|(\nabla^2 J(\theta_t) - H_t)d_{t+1}\|] \\ &\stackrel{(\sharp)}{\leq} MC\Delta^2 + 2\mathbb{E}_t [\|(\nabla^2 J(\theta_t) - H_t)d_{t+1}\|] \\ &\stackrel{(\natural)}{\leq} MC\Delta^2 + 2 \cdot \frac{\sqrt{\epsilon}}{24} \Delta = M\tilde{C}\Delta^2, \end{aligned}$$

where  $(\diamond)$  is due to Assumption 3.5 and the fact that  $V_t V_t^\top d_{t+1} = d_{t+1}$ ;  $(\sharp)$  is due to the fact that  $V_t^\top V_t = I$  and  $V_t V_t^\top$  has the same non-zero eigenvalue with  $V_t^\top V_t$ ;  $(\natural)$  is due to Lemma 3.8 and the fact that  $d_{t+1} \leq \Delta_t = \Delta = \frac{2\sqrt{\epsilon}}{M}$ . Also,  $\tilde{C} = C + \frac{1}{24}$ . □

#### B.5 Proof of Theorem 3.11

*Proof.*

$$\begin{aligned} &J(\theta_{t+1}) - J(\theta_t) \\ &\leq \nabla J(\theta_t)^\top d_{t+1} + \frac{1}{2} d_{t+1}^\top \nabla^2 J(\theta_t) d_{t+1} + \frac{M}{6} \|d_{t+1}\|^3 \\ &= g_t^\top d_{t+1} + \frac{1}{2} (d_{t+1})^\top H_t d_{t+1} + (\nabla J(\theta_t) - g_t)^\top d_{t+1} + \frac{1}{2} (d_{t+1})^\top (\nabla^2 J(\theta_t) - H_t) d_{t+1} + \frac{M}{6} \|d_{t+1}\|^3 \quad (17) \\ &\stackrel{(\diamond)}{\leq} -\frac{1}{2} \lambda_t \|d_{t+1}\|^2 + \|\nabla J(\theta_t) - g_t\| \Delta_t + \frac{1}{2} \|(\nabla^2 J(\theta_t) - H_t)\| \Delta_t^2 + \frac{M}{6} \|d_{t+1}\|^3, \end{aligned}$$

where  $(\diamond)$  is due to Cauchy-Schwarz inequality and Lemma 3.10. At  $t$ -th iteration, if  $\|d_{t+1}\| = \Delta_t = \frac{2\sqrt{\epsilon}}{M}$ , then we can bound  $\|\lambda_t d_{t+1}\|$  by

$$\|\lambda_t d_{t+1}\| \leq \frac{M}{\sqrt{\epsilon}} (J(\theta_t) - J(\theta_{t+1})) + \|\nabla J(\theta_t) - g_t\| + \frac{1}{2} \|\nabla^2 J(\theta_t) - H_t\| \Delta_t + \frac{M}{6} \|d_{t+1}\|^2, \quad (18)$$

Otherwise, if  $d_{t+1} < \Delta_t$ , then by the optimal condition of (4),  $\lambda_t = 0$ , so the upper bound (18) still holds. Therefore, we have

$$\begin{aligned} \|\nabla J(\theta_{t+1})\| &\stackrel{(\natural)}{\leq} \|\nabla J(\theta_{t+1}) - \nabla J(\theta_t) - \nabla^2 J(\theta_t)d_{t+1}\| + \|\nabla J(\theta_t) - g_t\| \\ &\quad + \|(\nabla^2 J(\theta_t) - H_t)d_{t+1}\| + \|g_t + \tilde{H}_k d_{t+1}\| + \|(\tilde{H}_k - H_t)d_{t+1}\| \\ &\stackrel{(\diamond)}{\leq} \frac{M\Delta^2}{2} + \|\nabla J(\theta_t) - g_t\| + \Delta\|\nabla^2 J(\theta_t) - H_t\| + \|g_t + \tilde{H}_k d_{t+1}\| + M\tilde{C}\Delta^2 \\ &\stackrel{(\sharp)}{\leq} \frac{M\Delta^2}{2} + \|\nabla J(\theta_t) - g_t\| + \Delta\|\nabla^2 J(\theta_t) - H_t\| + \|\lambda_t d_{t+1}\| + M\tilde{C}\Delta^2, \end{aligned}$$

where  $(\natural)$  is due to the triangle inequality,  $(\diamond)$  is due to Taylor approximation, Cauchy-Schwarz inequality and Lemma 3.9,  $(\sharp)$  is due to the optimal condition of DRTR (13). Plugging the upper bound of  $\|\lambda_t d_{t+1}\|$  into the above equation, summing from  $t = 0$  to  $T - 1$  and then taking the expectation on both sides, we obtain

$$\begin{aligned} &\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla J(\theta_{t+1})\|] \\ &\leq \frac{M}{\sqrt{\epsilon}T} \mathbb{E} \left[ \sum_{t=0}^{T-1} (J(\theta_t) - J(\theta_{t+1})) \right] + \frac{2}{T} \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla J(\theta_t) - g_t\|] + \frac{3}{2T} \sum_{t=0}^{T-1} \Delta \mathbb{E}[\|\nabla^2 J(\theta_t) - H_t\|] + \frac{2+3\tilde{C}}{3} M\Delta^2 \\ &\leq \frac{M}{\sqrt{\epsilon}T} (J(\theta_0) - J^*) + \frac{(71+96\tilde{C})}{24M} \epsilon, \end{aligned}$$

where the last inequality is due to Lemma 3.8 and  $\Delta = \frac{2\sqrt{\epsilon}}{M}$ . Then we have the desired result by taking  $T = \frac{24M^2\Delta_J}{\epsilon^{\frac{3}{2}}}$  and  $\bar{t}$  be uniformly sampled from  $0, \dots, T - 1$ .

Moreover, by Markov inequality  $(\Pr(X \geq a) \leq \frac{\mathbb{E}[X]}{a})$  for any non-negative random variable  $X$ , with probability at least  $\frac{7}{8}$ , we have

$$\|\nabla J(\theta_{\bar{t}+1})\| \leq \frac{(12+16\tilde{C})}{M} \epsilon.$$

To derive the second-order condition, from (17) and the optimal condition (12) of DRTR, we have

$$\begin{aligned} &\mathbb{E}[J(\theta_{t+1}) - J(\theta_t)] \\ &\leq \mathbb{E} \left[ -\frac{1}{2} \lambda_t \|d_{t+1}\|^2 \right] + \mathbb{E} \left[ \|\nabla J(\theta_t) - g_t\| \Delta_t + \frac{1}{2} \|(\nabla^2 J(\theta_t) - H_t)\| \Delta_t^2 + \frac{M}{6} \|d_{t+1}\|^3 \right] \\ &\leq \mathbb{E}[\lambda_t | \lambda_t > 0] \cdot \Pr(\|\lambda_t\| > 0) \cdot \left( -\frac{2\epsilon}{M^2} \right) + \mathbb{E}[\lambda_t | \lambda_t = 0] \cdot \Pr(\|\lambda_t\| = 0) \cdot \left( -\frac{1}{2} d_k^2 \right) + \frac{5\epsilon^{3/2}}{3M^2} \\ &= \mathbb{E}[\lambda_t | \lambda_t > 0] \cdot \Pr(\|\lambda_t\| > 0) \cdot \left( -\frac{2\epsilon}{M^2} \right) + \frac{5\epsilon^{3/2}}{3M^2} \\ &= \mathbb{E}[\lambda_t] \cdot \left( -\frac{2\epsilon}{M^2} \right) + \frac{5\epsilon^{3/2}}{3M^2}. \end{aligned} \tag{19}$$

Summing from  $t = 0$  to  $T - 1$ , and plugging T into the equation, we have

$$\frac{1}{T} \mathbb{E} \left[ \sum_{t=1}^T (J(\theta_t) - J(\theta_{t-1})) \right] \geq \frac{1}{T} \cdot (J^* - J(\theta_0)) \geq -\frac{\epsilon^{3/2}}{24M^2}. \tag{20}$$

Combining (19) and (20), we have

$$\mathbb{E}[\lambda_{\bar{t}}] \leq \frac{41}{48} \sqrt{\epsilon}, \tag{21}$$

where  $\bar{t}$  is uniformly sampled from  $0, \dots, T - 1$ . By Assumption 3.4 and Lemma 3.8, we have

$$\begin{aligned}
-\frac{41}{48}\sqrt{\epsilon}I &\preceq -\mathbb{E}[\lambda_{\bar{t}}]I \preceq \mathbb{E}[\tilde{H}_{\bar{t}}] = \mathbb{E}[V_{\bar{t}}V_{\bar{t}}^T H_{\bar{t}+1} V_{\bar{t}}V_{\bar{t}}^T + \tilde{H}_{\bar{t}} - V_{\bar{t}}V_{\bar{t}}^T H_{\bar{t}+1} V_{\bar{t}}V_{\bar{t}}^T] \\
&= \mathbb{E}[V_{\bar{t}}V_{\bar{t}}^T H_{\bar{t}+1} V_{\bar{t}}V_{\bar{t}}^T + V_{\bar{t}}V_{\bar{t}}^T (H_{\bar{t}} - H_{\bar{t}+1}) V_{\bar{t}}V_{\bar{t}}^T] \\
&\preceq \mathbb{E}[V_{\bar{t}}V_{\bar{t}}^T H_{\bar{t}+1} V_{\bar{t}}V_{\bar{t}}^T + \|V_{\bar{t}}V_{\bar{t}}^T (H_{\bar{t}} - H_{\bar{t}+1}) V_{\bar{t}}V_{\bar{t}}^T\| I] \\
&\preceq \mathbb{E}[V_{\bar{t}}V_{\bar{t}}^T H_{\bar{t}+1} V_{\bar{t}}V_{\bar{t}}^T] + \mathbb{E}[\|V_{\bar{t}}V_{\bar{t}}^T\| \|H_{\bar{t}+1} - H_{\bar{t}}\| \|V_{\bar{t}}V_{\bar{t}}^T\| I] \\
&= \mathbb{E}[\tilde{H}_{\bar{t}+1}] + \mathbb{E}[\|H_{\bar{t}+1} - H_{\bar{t}}\| I] \\
&\preceq \mathbb{E}[\tilde{H}_{\bar{t}+1}] + \mathbb{E}[(\|H_{\bar{t}+1} - \nabla^2 J(\theta_{\bar{t}+1})\| + M \|d_{\bar{t}+1}\| + \|\nabla^2 J(\theta_{\bar{t}}) - H_{\bar{t}}\|) I] \\
&\preceq \mathbb{E}[\tilde{H}_{\bar{t}+1}] + \frac{25}{12}\sqrt{\epsilon}I,
\end{aligned}$$

which indicates that  $\mathbb{E}[\lambda_{\bar{t}+1}] \leq \frac{141}{48}$ . By Lemma 3.8, we have

$$\begin{aligned}
\mathbb{E}[\|\tilde{\nabla}^2 J(\theta_{\bar{t}+1}) - \tilde{H}_{\bar{t}+1}\|] &\leq \mathbb{E}[\|V_{\bar{t}+1}V_{\bar{t}+1}^T\| \cdot \|\nabla^2 J(\theta_{\bar{t}+1}) - H_{\bar{t}+1}\| \cdot \|V_{\bar{t}}V_{\bar{t}+1}^T\|] = \mathbb{E}[\|\nabla^2 J(\theta_{\bar{t}+1}) - H_{\bar{t}+1}\|] \leq \frac{1}{24}\sqrt{\epsilon} \\
\iff \mathbb{E}[\tilde{\nabla}^2 J(\theta_{\bar{t}+1})] &\succeq \mathbb{E}[\tilde{H}_{\bar{t}}] - \frac{1}{24}\sqrt{\epsilon}I \\
\iff \mathbb{E}[\lambda_{\min}(\tilde{\nabla}^2 J(\theta_{\bar{t}+1}))] &\geq \mathbb{E}[\lambda_{\min}(\tilde{H}_{\bar{t}+1})] - \frac{1}{24}\sqrt{\epsilon} = -\mathbb{E}[\lambda_{\bar{t}+1}] - \frac{1}{24}\sqrt{\epsilon} \geq -\frac{143}{48}\sqrt{\epsilon} \geq -3\sqrt{\epsilon}.
\end{aligned}$$

□

## B.6 Proof of Lemma 4.1

*Proof.* For ease of presentation, we consider  $t < q$ . The general case is a straightforward extension. Recall  $\theta(a) := a\theta_t + (1 - a)\theta_{t-1}$ .

$$\begin{aligned}
\mathbb{E}_t \|g_t - \nabla J(\theta_t)\|^2 &= \mathbb{E}_t \|g_{t-1} + \xi_t - \nabla J(\theta_t)\|^2 \\
&= \mathbb{E}_t \|\xi_t - (\nabla J(\theta_t) - \nabla J(\theta_{t-1})) + g_{t-1} - \nabla J(\theta_{t-1})\|^2 \\
&\stackrel{(\clubsuit)}{=} \mathbb{E}_t \|\xi_t - (\nabla J(\theta_t) - \nabla J(\theta_{t-1}))\|^2 + \mathbb{E}_t \|g_{t-1} - \nabla J(\theta_{t-1})\|^2
\end{aligned} \tag{22}$$

where  $(\clubsuit)$  is due to the fact that  $\xi_t$  is conditionally independent of  $g_{t-1}, \theta_{t-1}$ .

By our construction (9),  $\xi_t$  is an unbiased estimator of  $\nabla J(\theta_t) - \nabla J(\theta_{t-1})$ . Next we bound the variance

of estimator  $\xi_t$ . Let us denote  $m = |\widehat{\mathcal{M}}_g|$  and consider samples  $a_1, \tau_1, \dots, a_m, \tau_m$ . Then we have

$$\begin{aligned}
& \mathbb{E}_t \|\xi_t - (\nabla J(\theta_t) - \nabla J(\theta_{t-1}))\|^2 \\
&= \mathbb{E} \left[ \left\| \left( \frac{1}{m} \sum_{i=1}^m H(\theta(a_i), \tau(a_i)) - \int_0^1 \nabla^2 J(\theta(a)) da \right) \cdot v \right\|^2 \right] \\
&= \frac{1}{m^2} \sum_{i=1}^m \mathbb{E} \left[ \left\| \left( H(\theta(a_i), \tau(a_i)) - \int_0^1 \nabla^2 J(\theta(a)) da \right) \cdot v \right\|^2 \right] \\
&= \frac{1}{m^2} \sum_{i=1}^m \mathbb{E} \left[ \left\| \left( H(\theta(a_i), \tau(a_i)) - \nabla^2 J(\theta(a_i)) + \int_0^1 (\nabla^2 J(\theta(a_i)) - \nabla^2 J(\theta(a))) da \right) \cdot v \right\|^2 \right] \\
&= \frac{1}{m^2} \sum_{i=1}^m \mathbb{E} \left[ \| (H(\theta(a_i), \tau(a_i)) - \nabla^2 J(\theta(a_i))) \cdot v \|^2 \right] + \frac{1}{m^2} \sum_{i=1}^m \mathbb{E} \left[ \left\| \int_0^1 (\nabla^2 J(\theta(a_i)) - \nabla^2 J(\theta(a))) da \cdot v \right\|^2 \right] \\
&= \frac{1}{m^2} \sum_{i=1}^m \mathbb{E} \left[ \| H(\theta(a_i), \tau(a_i)) - \nabla^2 J(\theta(a_i)) \|^2 \right] \|v\|^2 + \frac{1}{m^2} \sum_{i=1}^m \mathbb{E} \left[ \int_0^1 \| \nabla^2 J(\theta(a_i)) - \nabla^2 J(\theta(a)) \|^2 da \right] \|v\|^2 \\
&\leq \frac{G_H^2}{m} \|v\|^2 + \frac{M^2}{m} \|v\|^4,
\end{aligned}$$

where the inequality uses the Hessian variance bound in Lemma 3.7 and the Hessian Lipschitzness (Assumption 3.4) to bound the two sums, respectively. Due to the trust region choice, we have  $\|v\| = \|\theta_{t+1} - \theta_t\| \leq \frac{2\sqrt{\epsilon}}{M}$ . It follows that

$$\mathbb{E}_t \|\xi_t - (\nabla J(\theta_t) - \nabla J(\theta_{t-1}))\|^2 \leq \frac{4G_H^2\epsilon}{mM^2} + \frac{16\epsilon^2}{mM^2}.$$

Since  $\epsilon \leq \frac{G_H^2}{4}$ , by telescoping (22) and taking the expectation over all the randomness, we obtain

$$\mathbb{E} \|g_t - \nabla J(\theta_t)\|^2 \leq \frac{8t \cdot G_H^2 \epsilon}{|\widehat{\mathcal{M}}_g| M^2} + \mathbb{E} \|g_0 - \nabla J(\theta_0)\|^2 \leq \frac{8q \cdot G_H^2 \epsilon}{|\widehat{\mathcal{M}}_g| M^2} + \frac{G_g^2}{|\mathcal{M}_0|}.$$

By setting  $q = \frac{1}{8\epsilon^{1/2}}$ ,  $|\widehat{\mathcal{M}}_g| = \frac{288G_H^2}{M^2\epsilon^{3/2}}$ , and  $|\mathcal{M}_0| = \frac{288G_g^2}{\epsilon^2}$ , the result of the lemma follows.  $\square$

## B.7 Proof of Theorem 4.2

The proof of Theorem 4.2 is nearly identical to that of Theorem 3.11. The only difference is that the bound for the variance of gradient estimator is based on Lemma 4.1.

## C Full dimension trust region method for policy optimization

Our proposed dimension-reduced method aims to alleviate the high computation cost associated with solving the trust region subproblem in the whole space. However, it should be noted that by invoking the full dimension trust region subproblem, we can achieve a second-order stationary point (SOSP) in the whole space with a sample complexity of  $\mathcal{O}(\epsilon^{-3})$ , although this method requires more Hessian-vector products in the subsolver.

Specifically, we need to solve the full-dimension trust region (FDTR) problem:

$$\begin{aligned} \min_{\alpha \in \mathbb{R}^n} \quad & m_t(d) := J(\theta_t) + g_t^T d + \frac{1}{2} d^T H_t d \\ \text{s.t.} \quad & \|d\| \leq \Delta, \end{aligned} \tag{23}$$

and the vector  $d_t$  is the global solution to FDTR problem (23) if it is feasible and there exists a Lagrange multiplier  $x_t \geq 0$  such that  $(d_t, x_t)$  is the solution to the following equations:

$$(H_t + \lambda) d + g_t = 0, Q_t + x I \succeq 0, x (\Delta - \|d\|) = 0. \tag{24}$$

To solve the full-dimension trust-region subproblem, the Steihaug-CG method (Steihaug, 1983) and dogleg method can be used. If we can solve the subproblem efficiently, by substituting  $\lambda_t$  of in the proof of Theorem 3.11 with  $x_t$  and following a similar procedure, we can prove the following theorem:

**Theorem C.1** (Convergence rate of full-dimension SOPO). *Suppose Assumptions 3.2-3.4 hold. Let  $\Delta_t = \Delta = \frac{2\sqrt{\epsilon}}{M}$  and  $\Delta_J$  be a constant number that s.t.  $\Delta_J \geq J(\theta_0) - J^*$ , by setting  $M_g = \frac{144G_g^2}{\epsilon^2}$ ,  $|\mathcal{M}_H| = \frac{22 \times 24^2 G_H^2 \log(d)}{\epsilon^2}$ ,  $T = \frac{24M^2 \Delta_J}{\epsilon^2}$  in Algorithm 1, we have*

$$\mathbb{E}[\|\nabla J(\theta_{\bar{t}})\|] \leq \frac{3}{M}\epsilon, \mathbb{E}[\lambda_{\min}(\nabla^2 J(\theta_{\bar{t}}))] \geq -2\sqrt{\epsilon},$$

where  $\bar{t}$  is sampled from  $\{1, \dots, T\}$  uniformly at random. Moreover, with probability at least  $\frac{7}{8}$ , we have

$$\|\nabla J(\theta_{\bar{t}})\| \leq \frac{12}{M}\epsilon.$$

**Theorem C.2** (Convergence rate of full-dimension VRSOPO). *Suppose Assumptions 3.2-3.4 hold. Let  $\Delta_t = \Delta = \frac{2\sqrt{\epsilon}}{M}$  and  $\Delta_J$  be a constant number that s.t.  $\Delta_J \geq J(\theta_0) - J^*$ . If we set  $\epsilon \leq \frac{G_H^2}{4}$ ,  $q = \frac{1}{8\epsilon^{1/2}}$ ,  $|\widehat{\mathcal{M}}_g| = \frac{288G_g^2}{M^2\epsilon^{3/2}}$ ,  $|\mathcal{M}_0| = \frac{288G_g^2}{\epsilon^2}$ ,  $|\mathcal{M}_H| = \frac{22 \times 24^2 G_H^2 \log(d)}{\epsilon^2}$ , and  $T = \frac{24M^2 \Delta_J}{\epsilon^2}$  in Algorithm 2, then we have*

$$\mathbb{E}[\|\nabla J(\theta_{\bar{t}})\|] \leq \frac{3}{M}\epsilon, \mathbb{E}[\lambda_{\min}(\tilde{\nabla}^2 J(\theta_{\bar{t}}))] \geq -2\sqrt{\epsilon},$$

where  $\bar{t}$  is uniformly sampled from  $\{1, \dots, T\}$ . Moreover, with probability at least  $\frac{7}{8}$ , we have

$$\|\nabla J(\theta_{\bar{t}})\| \leq \frac{12}{M}\epsilon.$$

## D Infinite setting

In the infinite setting, where the objective function  $J(\theta)$  is actually a truncated objective, and  $J_\infty(\theta)$  denotes the infinite setting objective. We claim that if we set  $H = \mathcal{O}(\log(\epsilon^{-1}))$ , then finding an SOSP for the truncated objective is sufficient for finding an SOSP for the infinite setting objective.

**Lemma D.1.** *There exists  $D, D'$  such that for all  $\theta \in \mathbb{R}^n$ , we have*

$$\|J_\infty(\theta) - J(\theta)\| \leq \frac{R}{1-\gamma} \gamma^H, \quad \|\nabla J_\infty(\theta) - \nabla J(\theta)\| \leq D \gamma^H, \quad \|\nabla^2 J_\infty(\theta) - \nabla^2 J(\theta)\| \leq D' \gamma^H.$$

*Proof.* The first two inequality can be found in Yuan et al. (2022), where  $D = \frac{GR}{1-\gamma} \sqrt{\frac{1}{1-\gamma} + H}$ . For the last inequality, we have

$$\begin{aligned} & \|\nabla^2 J_\infty(\theta) - \nabla^2 J(\theta)\|^2 \\ &= \left\| \mathbb{E}_\tau \left[ \sum_{h=H}^{\infty} \gamma^h r(s_h, a_h) u_h \cdot u_h^\top + \sum_{h=H}^{\infty} \gamma^h r(s_h, a_h) \sum_{t'=0}^h \nabla_\theta^2 \log \pi_\theta(a_{t'} | s_{t'}) \right] \right\|^2 \\ &\leq \mathbb{E}_\tau \left[ \left\| \sum_{h=H}^{\infty} \gamma^h r(s_h, a_h) u_h \cdot u_h^\top \right\|^2 \right] + \mathbb{E}_\tau \left[ \left\| \sum_{h=H}^{\infty} \gamma^h r(s_h, a_h) \sum_{t'=0}^h \nabla_\theta^2 \log \pi_\theta(a_{t'} | s_{t'}) \right\|^2 \right] \\ &\leq \mathbb{E}_\tau \left[ \left( \sum_{h=H}^{\infty} \gamma^h R^2 \right) \left( \sum_{h=H}^{\infty} \gamma^h \|u_h \cdot u_h^\top\|^2 \right) \right] + \mathbb{E}_\tau \left[ \left( \sum_{h=H}^{\infty} \gamma^h R^2 \right) \sum_{h=H}^{\infty} \gamma^h \left\| \sum_{t'=0}^h \nabla_\theta^2 \log \pi_\theta(a_{t'} | s_{t'}) \right\|^2 \right] \\ &\leq \frac{\gamma^H R^2}{1-\gamma} \sum_{h=H}^{\infty} \gamma^h (h+1)^3 \sum_{t'=0}^h \mathbb{E}_\tau [\|\nabla_\theta \log \pi_\theta(a_t | s_t)\|^4] + \frac{\gamma^H R^2}{1-\gamma} \sum_{h=H}^{\infty} \gamma^h (h+1) \sum_{t'=0}^h \mathbb{E}_\tau [\|\nabla_\theta^2 \log \pi_\theta(a_{t'} | s_{t'})\|^2] \\ &= \frac{\gamma^{2H} R^2 G^4}{1-\gamma} \sum_{h=0}^{\infty} \gamma^h (h+1+H)^4 + \frac{\gamma^{2H} R^2 L^2}{1-\gamma} \sum_{h=0}^{\infty} \gamma^h (h+1+H)^2 \\ &\leq \frac{\gamma^{2H} R^2 G^4}{(1-\gamma)^2} \left[ \frac{24}{(1-\gamma)^4} + \frac{24H}{(1-\gamma)^3} + \frac{12H^2}{(1-\gamma)^2} + \frac{4H^3}{(1-\gamma)} + H^4 \right] + \frac{\gamma^{2H} R^2 L^2}{(1-\gamma)^2} \left[ \frac{2}{(1-\gamma)^2} + \frac{2H}{(1-\gamma)} + H^2 \right], \end{aligned}$$

where the first inequality is due to Jensen's inequality, the second inequality is due to Cauchy-Schwarz inequality and the last inequality is due to Inequality of arithmetic and geometric means. It follows that

$$D' = \frac{RG^2}{1-\gamma} \sqrt{\frac{24}{(1-\gamma)^4} + \frac{24H}{(1-\gamma)^3} + \frac{12H^2}{(1-\gamma)^2} + \frac{4H^3}{(1-\gamma)} + H^4} + \frac{RL}{1-\gamma} \sqrt{\frac{2}{(1-\gamma)^2} + \frac{2H}{(1-\gamma)} + H^2}.$$

□

## E Practical versions of DR-SOPO and DVR-SOPO

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**Algorithm 3** Practical DR-SOPO algorithm

---

```

1: Given  $T, \Delta_1, \Delta_{max}, \eta$ 
2: for  $t = 1, \dots, T$  do
3:   Collect sample trajectories  $\mathcal{M}_g$  and compute  $g_t$ 
4:   Collect sample trajectories  $\mathcal{M}_H$  and compute  $H_t$ 
5:   Compute stepsize  $\alpha = (\alpha_1, \alpha_2)$  by solving the radius-free problem (11)
6:   if  $\|\alpha\| > \Delta_{max}$  then
7:      $\alpha = \alpha / \|\alpha\| * \Delta_{max}$ 
8:   end if
9:   Calculate  $\rho_t$  by 10
10:  if  $\rho_k > \eta$  then
11:    Update:  $\theta_{t+1} \leftarrow \theta_t - \alpha_1 g_t + \alpha_2 d_t$ 
12:  else
13:    Adjust the Lagrange multiplier  $\lambda_t$ 
14:  end if
15: end for
16: return  $\theta_{\bar{t}}$ , which is uniformly picked from  $\{\theta_t\}_{t=1, \dots, T}$ 

```

---

**Algorithm 4** Practical DVR-SOPO algorithm

---

```

1: Given  $T, \Delta_1, \Delta_{max}, \eta, q$ 
2: for  $t = 1, \dots, T$  do
3:   if  $\text{mod}(t, q) = 0$  then
4:     Collect sample trajectories  $\mathcal{M}_0$  and compute  $g_t$ 
5:   else
6:     Collect sample trajectories  $\widehat{\mathcal{M}}_g$  and compute  $\xi_t$ :
7:   end if
8:   sample  $|\mathcal{M}_H|$  trajectories to construct  $H_t$ 
9:   Compute stepsize  $\alpha = (\alpha_1, \alpha_2)$  by solving the radius-free problem (11)
10:  if  $\|\alpha\| > \Delta_{max}$  then
11:     $\alpha = \alpha / \|\alpha\| * \Delta_{max}$ 
12:  end if
13:  Calculate  $\rho_t$  by 10
14:  if  $\rho_k > \eta$  then
15:    Update:  $\theta_{t+1} \leftarrow \theta_t - \alpha_1 g_t + \alpha_2 d_t$ 
16:  else
17:    Adjust the lagrange multiplier  $\lambda_t$ 
18:  end if
19: end for
20: return  $\theta_{\bar{t}}$ , which is uniformly picked from  $\{\theta_t\}_{t=1, \dots, T}$ 

```

---

## F Hyper-parameter Settings

Table 2: Hyper-parameter Settings

Environment	Swimmer	Walker2d	HalfCheetah	Ant
Horizon	500	500	500	500
Baseline	Linear	Linear	Linear	Linear
Number of timesteps	$10^7$	$10^7$	$10^7$	$10^7$
NN size	$64 \times 64$	$64 \times 64$	$64 \times 64$	$64 \times 64$
NN activation function	Tanh	Tanh	Tanh	Tanh
HAPG learning rate	0.01	0.01	0.01	0.01
REINFORCE learning rate	0.01	0.01	0.01	0.01
REINFORCE $\mathcal{M}_g$	50	50	50	50
HAPG $\mathcal{M}_0$	50	50	50	50
HAPG $\tilde{\mathcal{M}}_g$	10	10	10	10
DR-SOPO $\mathcal{M}_g$	50	50	50	50
DR-SOPO $\mathcal{M}_H$	10	10	10	10
DVR-SOPO $\mathcal{M}_0$	50	50	50	50
DVR-SOPO $\tilde{\mathcal{M}}_g$	10	10	10	10
DVR-SOPO $\mathcal{M}_H$	10	10	10	10
HAPG $q$	10	5	5	5
DVR-SOPO $q$	10	5	5	5
DR-SOPO $\Delta_{max}$	2	0.2	0.02	0.05
DVR-SOPO $\Delta_{max}$	2	0.2	0.02	0.05
DR-SOPO $\eta$	0.001	0.001	0.001	0.001
DVR-SOPO $\eta$	0.001	0.001	0.001	0.001
$\mu^a$	$\frac{1}{500}$	$\frac{1}{500}$	$\frac{1}{500}$	$\frac{1}{500}$

---

<sup>a</sup>Parameter  $\mu$  is discussed in (16), and set as the same for all the four algorithms.