SimOpt Model Descriptions

Benjamin Rees

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1 Newsvendor under Dynamic Consumer Substitution

1.1 Model Description

A retailer sells n substitutable products $j = 1, \ldots, n$ each with price p^j and cost c^j. The initial inventory levels are denoted by $\mathbf{x} = (x_1, \ldots, x_n)$.

Unlike in the classical newsvendor problem, the demand is not given by a predetermined distribution, but depends on the initial inventory levels x as well. The customers $t = 1, \ldots, T$ arrive in order and each can choose one product that is in-stock when he/she arrives, namely any element in

$$\mathrm{S}(x_t) = \{j : x_t^j > 0\} \cup \{0\},$$

where 0 denotes the no-purchase option.

Each customer t assigns a utility U_t^j to option $j = 0, \ldots, n$ and thus $\mathrm{U}_t = \left(\mathrm{U}_t^0, \mathrm{U}_t^1, \ldots, \mathrm{U}_t^n\right)$ are their vector of utilities. Note that U_t^j is the utility of product j net of the price p^j, and therefore could be negative. Since the no-purchase option 0 incurs neither utility nor cost, one can assume $\mathrm{U}_t^0 = 0$. Customer t makes their choice to maximise their utility by solving the following problem:

$$d\left(x_t, \mathrm{U}_t\right) = \underset{j \in \mathrm{S}(x_t)}{\operatorname{argmax}}\ \mathrm{U}_t^j.$$

The source of randomness comes from the Multinomial Logit model, such that $\mathrm{U}_t^0, \mathrm{U}_t^1, \ldots, \mathrm{U}_t^n$ are mutually independent random variables of the form:

$$\mathrm{U}_t^j = u^j + \epsilon_t^j$$

where u^j is some constant and ϵ_t^j, $j = 0, 1, \ldots, n$ are mutually independent Gumbel random variables with

$$\mathbb{P}\left(\epsilon_t^j \leq z\right) = \exp\left(-e^{-\left(\frac{z}{\mu+\gamma}\right)}\right),$$

with γ the Euler–Mascheroni constant.

1.2 Model Factors and Responses

The following are model Factors:

- Number of Products. This also defines the dimension of the model. The default is 10.
- Number of Customers. The default is 30.
- Constant of each products utility. This is the same size as the number of products. The default is $u^j = 5 + j$.
- μ for calculating Gumbel Random Variable. The default is 1.0.
- Initial inventory level. This is the same size as the number of products. The default is $(3, 3, \ldots, 3)$
- Sell price of products. This is the same size as the number of products. The default is $(9, 9, \ldots, 9)$
- Cost of each product. This is the same size as the number of products. The default is $(5, 5, \ldots, 5)$

There are four model responses. They are the following:

1. The profit gained by the vendor at the end of the period
2. The number of products which are out of stock at the end of the period.
3. The number of unmet customer orders.
4. The fraction of customer orders filled.

1.3 Optimisation Problem Description

Let $\omega = \{\mathrm{U}_t : t = 1, \ldots, T\}$ denote the sample path, and assume that ω follows the probability distribution $\mathbb{P}$. We consider the one-period inventory model and assume $\mathbb{P}(T < \infty) = 1$. The retailer knows the probability measure $\mathbb{P}$ and their objective is to choose the initial inventory level x that will maximise their profit in the model.

1.4 Optimisation Problem Factors

This is a maximisation problem with the decision variable being optimised is the **Initial Inventory Level**.

The problem is constrained by the fact that every component of the initial inventory level must be non-negative.

We begin at an initial solution of $(3, 3, \ldots, 3)$ and there is no known optimal solution.

The dimension of the problem is set to the model factor: number of products (this is default to be 10).

2 Network Queuing System Design

2.1 Model Description

The model represents a communication system where arriving messages are routed through a network based on chosen routing percentages. There are N random messages that arrive following a Poisson process with a rate of λ that need to go to a particular destination, and there are n networks available to process these messages. When a message arrives there is a p_i chance that it will be processed by network i. The per message processing cost is $c_1, c_2, \ldots, c_i$ depending on which network the message is routed through. It also takes time for a message to go through a network. This transit time is denoted by S_i for each network i and S_i follows a triangular distribution with lower limit a_i, upper limit b_i, and mode c_i. Each network behaves like a single-server queue with first-in-first-out service discipline. There is a cost for the length of time a message spends in network i measured by c_i per unit of time.

There are three sources of randomness present in the model. These being:

1. Interarrival time of a message.
2. The network a message is routed to.
3. The transit time of a message (depends on the network).

2.2 Model Factors and Responses

The following are model factors:

- The number of networks in the communications system. The default is 10.

- The probability that a message will go through a particular network i. This is the same dimension as the number of networks and the default is $(0.1, 0.1, \ldots, 0.1)$.
- The message processing cost of network i. This is the same dimension as the number of networks and the default value is defined as $c_i = 0.1/i$.
- The cost for the length of time a message spends in a network i per unit of time. This is the same dimension as the number of networks and the default is $(0.005, 0.005, \ldots, 0.005)$.
- The mode time of transit for network i following a triangular distribution. This is the same dimension as the number of networks and the default is $(1, 2, \ldots, \text{Num of Networks})$
- The lower limits for the triangular distribution for the transit time. This is the same dimension as the number of networks and the default is $a_i = i - 0.5$.
- The upper limits for the triangular distribution time. This is the same dimension as the number of networks and the default is $a_i = i + 0.5$.
- The arrival rate of messages following a Poisson process. The default is $\lambda = 1$.
- The number of messages that arrive and need to be routed. The default is 1000.

There is one model response. This is

1. The total cost spent to route and process all messages.

2.3 Optimisation Problem Description

The objective is to minimise total costs, the sum of time costs and network costs for all messages, by minimising the **probability that a message will go through a particular network** i.

$$\sum_{i=1}^{n}$$

2.4 Optimisation Problem Factors

The problem is constrained by the probability simplex properties on the routing probabilities $p_1, p_2, \ldots, p_n$:

$$0 \leq p_i \leq 1 \quad \forall i = 1, 2, \ldots, n$$
$$\sum_{i=1}^{n} p_i = 1$$

We have a starting solution of $(0.1, 0.1, \ldots, 0.1)$ and no known optimal solution.

3 Stochastic Activity Network

3.1 Model Description

Consider a stochastic activity network where each arc i is associated with a task with random duration X_i. Task durations are independent. SANs are also known as *PERT Networks* and are used in planning large-scale projects.

The source of randomness comes from the task durations being exponentially distributed with mean θ_i.

3.2 Model Factors and Responses

The following are the model factors:

- The number of nodes in the network. The default is 9.
- The list of arcs in the network. The default is 13 arcs
- The mean task durations for each arc. This is the same dimension as the list of arcs and the default is given as: $(1, 1, \ldots, 1)$

The following graph represents the default SAN network:

There is one Model response being the

1. length/duration of the longest path.

3.3 Optimisation Problem Description

Suppose that we can select $\theta_i > 0$ for each i, but there is an associated cost. In particular, we want to minimize the following:

$$\min_{\theta \in \mathbb{R}^n} \mathbb{E}\left[\mathrm{T}(\theta)\right] + f(\theta).$$

Where n is the number of arcs, $\mathrm{T}(\theta)$ is the random duration of the longest path from a to i and

$$f(\theta) = \sum_{i=1}^{n} \theta_i^{-1}.$$

The objective function is convex in θ. An IPA estimator of the gradient is also given in the code. The model factor being optimised is **the mean task durations for each arc**.

3.4 Optimisation Problem Factors

The only constraint is that for each $i = 1, \ldots, n$ $\theta_i > 0$. We begin with the starting solution for the mean task duration for each arc as $(8, 8, \ldots, 8)$ where the number of arcs is 13.

4 Amusement Park Queues

4.1 Model Description

This model simulates an amusement park with 7 attractions. Visitors arrive at each attraction according to a Poisson distribution with a rate $\gamma_i = 1$ for $i = 1, \ldots, 7$. Each attraction can only take one visitor at a time, while others wait in a queue with capacity c_i. If a visitor finds a queue full, they will immediately leave the park.

After visiting each attraction, a visitor leaves the park with probability 0.2. Otherwise, the visitor goes to another attraction according the following transition matrix

	1	**2**	**3**	**4**	**5**	**6**	**7**
1	0.1	0.1	0.1	0.1	0.2	0.2	0
2	0.1	0.1	0.1	0.1	0.2	0.2	0
3	0.1	0.1	0.1	0.1	0.2	0.2	0
4	0.1	0.1	0.1	0.1	0.2	0.2	0
5	0.1	0.1	0.1	0.1	0	0.1	0.3
6	0.1	0.1	0.1	0.1	0.1	0	0.3
7	0.1	0.1	0.1	0.1	0.1	0.1	0.2

The time that a visitor spends at an attraction follows an Erlang distribution with shape parameter $k = 2$ and rate $\lambda = 9$. The park opens at 9AM and closes at 5PM, and the time is measured in minutes. When the park closes, all visitors leave immediately.

There are three sources of randomness in this model:

1. The arrival rate of visitors as a Poisson distribution with rate of 1 for all $i = 1, \ldots, 7$.
2. The transition probability matrix that visitors follow after visiting each attraction.
3. The time spent at each attraction as an Erlang distribution with the shape parameter $k = 2$ and rate $\lambda = 9$.

The Erlang distribution is the distribution representing a sum of k independent exponential variables with mean λ^{-1} each. It is a special case of the gamma distribution wherein the shape of the distribution is discretized. The probability density function of the Erlang distribution is

$$f(x; k, \lambda) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!} \quad x, \beta \geq 0$$

4.2 Model Factors and Responses

The following are model factors of *AMUSEMENT*:

- The total number of visitors waiting for attractions that can be maintained through park facitilites, distributed across the attractions. The default is 350.
- The number of attractions in the park. The default is 7.
- The number of minutes per day the park is open. The default assumption is that the park is open 8AM-5PM (8 hours), leading to a default value of 480.
- The shape parameter of the Erlang distribution for each attraction duration. The default is $(2, 2, \ldots, 2)$ where the dimension of the vector is equal to the number of attractions.
- The scale parameter of the Erlang distribution for each attraction duration (this is the reciprocal of the rate value). The default is $(1/9, 1/9, \ldots, 1/9)$ and has a dimension equal to the number of attractions.
- The probability that a visitor will depart the park after visiting an attraction. The default value is $(0.2, 0.2, \ldots, 0.2)$ and has a dimension equal to the number of attractions.
- The capacity of the queues for the attractions based on the portion of facilities allocated. The default is $(50, 50, \ldots, 50)$ and has a dimension equal to the number of attractions.

- The gamma values for the Poisson distribution dictating the rate at which visitors entering the park arrive at each attraction. The default is $(1, 1, \ldots, 1)$ and has a dimension equal to the number of attractions.
- The transition matrix that describes the probability of a visitor visiting each attraction after their current attraction. The default is the transition matrix listed above.

The Amusement Park Queues has four model responses:

1. The total number of visitors to leave the park due to full queues.
2. The percentage of visitors to leave the park due to full queues.
3. The time average of the number of visitors in the system.
4. The percent utilizations for each attraction.

4.3 Optimisation Problem Description

The optimisation problem is to minimize the total number of departed visitors by minimising the **the capacity of the queues for the attractions**. The following constraints for the problem are:

$$\sum_{i=1}^{N} c_i = \text{park capacity}$$
$$c_i \geq 0 \quad \forall i = 1, \ldots, N$$

where N is the number of attractions.

4.4 Optimisation Problem Factors

The starting solution is the default queue capacity, $(50, 50, \ldots, 50)$, where the number of attractions is 7 and the park can hold 350 people.

There is no known optimal solution to the problem.

5 Chess Matchmaking Optimisation

5.1 Model Description

Chess players are rated using the ELO rating system, which assigns a (non-unique) number to each player based on their level of skill. The model simulates an online chess-matchmaking platform, which tries to match players of similar skill level.

The platform uses a search width x, which is the maximum allowable difference in ELO rating between two matched players. N players are drawn from a distribution of ELO ratings and arrive (independent of their rating) according to a stationary Poisson process with rate λ. When a player arrives, and there is an existing, unmatched player with ELO rating within x of the first player's ELO rating, they are matched. If not, then the player waits for an opponent with an appropriate ELO rating to arrive.

There are two sources of randomness present in the model:

1. A source of randomness in the model comes from creating the ELO distribution. First a normal distribution with mean 1200 and standard deviation $1200\left(\sqrt{2}\mathrm{cerf}^{-1}\left(0.02\right)\right)^{-1}$, where cerf^{-1} is the inverse complementary error function. This results in a distribution where the 1st percentile is at 0 and the 99th percentile is at 2400. The distribution is then truncated at 0 and 2400.
2. A stationary Poisson process with rate λ for arrivals.

5.2 Model Factors and Responses

The following are factors for the chess matchmaking model:

- The mean of the normal distribution for the ELO rating. The default is 1200.
- The standard deviation of the normal distribution for the ELO rating. The default is $1200\left(\sqrt{2}\mathrm{cerf}^{-1}\left(0.02\right)\right)^{-1}$.
- The rate of Poisson process for player arrivals. The default is $\lambda = 1$.
- The number of players. The default is $N = 1000$.
- The maximum allowable difference between ELO ratings. The default is $x = 150$.

The chess matchmaking model has two responses:

1. The average ELO difference between all pairs.
2. The average waiting time in the matchmaking queue.

5.3 Optimisation Problem Description

The optimisation problem attempts to minimise the average ELO difference between all pairs of matched players by finding an optimal solution for the model factor x.

The problem is constrained by the following factors:

$$0 \leq x \leq 2400$$
$$\frac{1}{\delta} \leq \lambda$$

5.4 Optimisation Problem Factors

There is no optimal solution known for the problem. The dimension of the problem is 1 and the starting solution is 150.

6 Continuous Newsvendor Problem

6.1 Model Description

A vendor orders a fixed quantity of product at the beginning of a day to be sold to customers throughout the day. The vendor pays a per-unit order cost c for the initial inventory and sells the product to customers at a per-unit price s. At the end of the day, any unsold product can be salvaged at a per-unit price w.

The randomness of the model comes from the demand of the product which follows a Burr Type XII distribution and is denoted by D. The parameters of the Burr Type XII distribution are α and β so that its cumulative distribution function is given by

$$\mathrm{F}(x) = 1 - (1 + x^{\alpha})^{-\beta},$$

where $x, \alpha, \beta > 0$.

6.2 Model Factors and Responses

The follwoing are model factors for the continuous newsvendor problem:

- The price at which the newsvendor purchases one unit of product. The default is $c = 5$.
- The price at which the newsvender sells on unit of product. The default is $s = 9$.

- The price at which any unsold product is sold for salvage. The default is $w = 1$.
- The parameter for the demand distribution. The default is $\alpha = 2$.
- The parameter for the demand distribution. The default is $\beta = 20$.
- The amount of product ordered at the beginning of the day. The default is $x = 0.5$.

There is one response of the model:

1. The daily profit: $st + w(x - t) - cx$ where t is the amount of product sold during the day. The daily profit can be negative if a loss is incurred.

6.3 Optimisation Problem Description

The optimisation problem is to maximise the vendors expected profit:

$$\underset{x \in \mathbb{R}}{\operatorname{argmax}} \{st + w(x - t) - cx\}$$

where t is the amount of product sold according to the demand distribution. This problem is constrained by the fact that the quantity of product must be non-negative:

$$x \geq 0$$

6.4 Optimisation Problem Factors

The dimension of the problem is 1. The optimal solution is given at:

$$x^* = (((1 - r)^{1/\beta})^{-1} - 1)^{1/\alpha}$$

For the default factors, this gives the optimal solution as:

$$x^* = 0.1878$$

Substituting this optimal solution into the model. The optimal objective function value under the default factors gives us a maximum expected profit of 0.4635.

7 Contamination Control Problem

7.1 Model Description

Consider a food supply chain consisting of n stages. Pathogenic microorganisms and other poisonous elements can contaminate some fraction of the food supply at each stage. Specifically, let the growth rate of contamination at stage i of the chain be denoted by some random variable Λ_i with $0 \leq \Lambda_i \leq 1$ for $i = 1, 2, \ldots, n$. If a prevention effort is made at stage i, the contamination decreases by the random rate Γ_i with $0 \leq \Gamma_i \leq 1$ with associated prevention cost c_i. Let the binary variable u_i represent whether a prevention measure is executed at stage i.

There are two sources of randomness in the model. These being:

1. The contamination rate, Λ_i, which is modelled by the distribution Beta $(\alpha_{\text{contam}}, \beta_{\text{contam}})$ for $i = 1, 2, \ldots, n$.
2. The restoration rate, Γ_i, which is modelled by the distribution Beta $(\alpha_{\text{restore}}, \beta_{\text{restore}})$ for $i = 1, 2, \ldots, n$.

7.2 Model Factors and Responses

The following are factors of the Contamination Control Model:

- The alpha parameter of the Beta distribution for growth rate of contamination at each stage. The default is $\alpha_{\text{contam}} = 1$.
- The beta parameter of the Beta distribution for growth rate of contamination at each stage. The default is $\beta_{\text{contam}} = 17/3$.
- The alpha parameter of the Beta distribution for rate that contamination decreases by after prevention effort. The default is $\alpha_{\text{restore}} = 1$.
- The beta parameter of the Beta distribution for rate that contamination decreases by after prevention effort. The default is $\beta_{\text{restore}} = 3/7$.
- The alpha parameter of the Beta distribution for initial contamination fraction. The default is 1
- The beta parameter of the Beta distribution for initial contamination fraction. The default is 30
- The stages of the food supply chain. The default is $n = 5$
- The prevention decision, this is a list where each element represents a stage of the food supply chain. The dimension is the same as the stages of the food supply chain. The default value is $(0, 0, \ldots, 0)$.

There is one response to the model, being:

1. A list of contamination levels over time.

7.3 Optimisation Problem Description

The Optimisation problem requires finding the optimal **prevention decision** to minimise the deterministic total cost of prevention efforts:

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{n} c_i u_i \\
s.t. \quad & \mathrm{X}_i = \Lambda_i(1-u_i)(1-\mathrm{X}_{i-1}) + (1-\Gamma_i u_i)\mathrm{X}_{i-1} \\
& \mathbb{P}\{\mathrm{X}_i \leq p_i\} > 1 - \epsilon_i \\
& u_i \in \{0,1\} \quad \forall i = 1,2,\ldots,n.
\end{aligned}
$$

The following constraints show that the contaminated fraction at stage i, X_i, should almost surely not exceed a pre-specified upper limit p_i with probability at least $1-\varepsilon_i$ and that the element of the previous decision (u_i) is binary.

The constraint on u_i can be relaxed to be continuous in the interval $[0,1]$. This relaxation leads to a continuous optimisation problem.

7.4 Optimisation Problem Factors

The optimisation problem has the following factors:

- The cost of prevention in each stage. The dimension is equal to the number of stages in the food supply chain. The default value is $(u_1, u_1, \ldots, u_n) = (1, 1, \ldots, 1)$.
- The allowable error probability in each stage. The dimension is equal to the number of stages in the food supply chain. The default value is $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) = (0.2, 0.2, \ldots, 0.2)$.
- The upper limit of amount of contamination in each stage. The dimension is equal to the number of stages in the food supply chain. The default value is $(p_1, p_2, \ldots, p_n) = (0.1, 0.1, \ldots, 0.1)$.

The dimension of the problem is equal to the number of stages in the food supply chain (n), and has no known optimal solution. The starting solution of the problem is $(1, 1, \ldots, 1)$.

8 Dual Sourcing System

8.1 Model Description

Consider a single-stage, incapacitated, manufacturing location facing stochastic demand. The manufacturer can buy the material from a 'regular' supplier at cost c_r per unit, or, if needed, they can get some or all of the material 'expedited' at some premium cost c_e per unit with $c_e > c_r$. Regular orders arrive after l_r periods while expedited orders arrive after l_e periods with $l_e < l_r$. Let the difference in lead times be $l = l_r - l_e \geq 1$.

If there is remaining on-hand inventory at the end of period n (after demand d_n is satisfied), these items are carried over to the next period ($I_n + 1 > 0$) at a holding cost per unit. However, if there is a stock-out ($I_n + 1 < 0$), there is a penalty cost per unit of unsatisfied demand.

We let the period n expediting order be based on the on-hand inventory plus the orders that will arrive within l_e periods (both regular and expedited). Regular orders that are due to arrive after l_e periods are not considered in expedited ordering decisions. The expedited order is placed to restore the expedited inventory position IP_n^e to some target parameter level z_e. The regular order X_n^r, on the other hand, is based on the regular inventory position (sum of on-hand inventory and all outstanding orders, including the expedited order placed in the current period). Similarly, it tries to restore the regular inventory position IP_n^r to the target parameter z_r. Thus, under this model, we carry two inventory positions, one for regular orders and another for expedited orders.

The only source of randomness in this model comes from the demand d_n which follows a normal distribution.

8.2 Model Factors and Responses

The following are model factors for the dual sourcing system:

- The number of days to simulate. The default is $n = 1000$.
- The initial inventory. The default is 40.
- The regular ordering cost per unit. The default is $c_r = 100$.
- The expedited ordering cost per unit. The default is $c_e = 110$.
- The lead time for regular orders in days. The default is $l_r = 110$.
- The lead time for expedited orders in days. The default is $l_e = 0$.
- The holding cost per unit per period. The default is 5.

- The penalty cost per unit per period for backlogging. The default is 495.
- The standard deviation of the demand distribution. The default is $\sigma = 10$.
- The mean of the demand distribution. The default is $\mu = 30$.
- The order-up-to level for regular orders. The default is 80
- The order-up-to level for expedited orders. The default is 50.

There are three responses for the model:

1. The average holding cost over the time period.
2. The average penalty cost over the time period.
3. The average ordering cost over the time period.

8.3 Optimisation Problem Description

The optimisation problem finds the optimal **order-up-to level for regular and expedited orders** by minimising the expected total cost. This is the summation of the three model responses:

- average holding cost
- average penalty cost
- average ordering cost.

The problem is constrained by the fact that the model factors, the order-up-to levels for regular and expedited orders are both non-negative.

8.4 Optimisation Problem Factors

The dimension of the problem is 2, where solutions in $\mathbb{R}^2$, have one component represent the order-up-to level for regular orders and the other component represents the order-up-to level for expedited orders.

There is no known optimal solution for the optimisation problem.

9 Facility Sizing

9.1 Model Description

The facility-sizing problem is formulated as follows:

m facilities are to be installed, each with capacity $x_i \geq 0$ for $i = 1, \dots, m$. The random demand ξ_i arrives at facility i, with a known joint distribution of the random factor $\xi = (\xi_1, \dots, \xi_m)$.

A realisation of the demand $\xi = (\xi_1, \dots, \xi_m)$ is said to be satisfied by the capacity x if $x_i \geq \xi_i$ for all $i = 1, \dots, m$.

The only source of randomness in the model comes from the random demand vector ξ which follows a multivariate normal distribution and correlation coefficients.

9.2 Model Factors and Responses

The following are model factors of the facility sizing model:

- The number of facilities that are to be installed. The default is $m = 3$.
- The inventory capacities of the facilities. The dimension is equal to the number of facilities. The default value is $(x_1, x_2, x_3) = (150, 300, 400)$.
- The mean vector of the multivariate normal distribution for demand. The dimension is equal to the number of facilities. The default is $\mu = (100, 100, \dots, 100)$.
- The covariance matrix of the multivariate distribution for the demand. This is a square $m \times m$ matrix, where m is the number of facilities to be installed. The default value is

$$\begin{bmatrix} 2000 & 1500 & 500 \\ 1500 & 2000 & 750 \\ 500 & 750 & 2000 \end{bmatrix}$$

The facility sizing model has three responses:

1. The stockout binary flag which is true when at least one of the facilities did not satisfy the demand and true when all facilities satisfied the demand.
2. number of facilities that cannot satisfy the demand.
3. The amount of total demand which cannot be satisfied.

9.3 Optimisation Problem Description

There are two optimisation problem associated with this model. Both require finding an optimal **inventory capacity of the facilities**:

The first optimisation problem attempts to minimise the total costs of installing capacity while keeping the probability of stocking out low.

The probability of failing to satisfy demand $\xi = (\xi_1, \ldots, \xi_m)$ is

$$p(x) = \mathbb{P}(\xi \leq x).$$

Let $\epsilon \in [0, 1]$ be a risk-level parameter, giving us the probabilistic constraint:

$$\mathbb{P}(\xi \leq x) \leq \epsilon.$$

Meanwhile, the unit cost of installing facility i is c_i and hence the total cost we wish to minimise is

$$\sum_{i=1}^{n} c_i x_i.$$

This problem is constrained by the following:

$$\begin{aligned} &\mathbb{P}(\text{stockout}) \leq \epsilon \\ &0 < x_i < \infty \quad \forall i \in \{1, \ldots, m\} \end{aligned}$$

The second optimisation problem attempts to maximise the probability of not stocking out subject to a budget constraint on the total cost of installing capacity. This problem follows the same box constraints on $x_i \in (0, \infty)$ for all $i = 1, \ldots, m$ and the deterministic constraint that the sum of facility capacity installation costs less than an installation budget.

9.4 Optimisation Problem Factors

Both optimisation problems have the factor of the cost to install a unit of capacity at each facility. This factor has a dimension equal to the number of facilities and the default value in both cases is $(1, 1, \ldots, 1)$.

The first optimisation problem has an additional factor for the maximum allowed probability of stocking out (ϵ), the default value being 0.05.

The second optimisation problem has a factor of the total budget for installation costs. The default value being 500.

For the first problem, the capacity starts with a value $(300, 300, \ldots, 300)$. For the second problem, the capacity starts with a value of $(100, 100, \ldots, 100)$.

Both problems do not have a known optimal solution.

10 Hotel Revenue Management

10.1 Model Description

Most of the revenue for a hotel comes from guests staying in its rooms. Assume a given hotel has only two rates:

- rack rate, which pays p_f per night
- discount rate, which pays p_r per night

Furthermore, let each different combination of length of stay, arrival date and rate paid be a 'product' so that the following 56 products are available to satisfy one week's worth of capacity (14 arriving Monday, 12 arriving Tuesday, ..., 2 arriving Sunday):

1. One night stay, rack rate arriving Monday
2. One night stay, discount rate arriving Monday
3. Two night stay, rack rate arriving Monday
4. Two night stay, discount rate arriving Monday

This goes on for all combination of 56 products.

For a given stay, the hotel collects revenue equal to the (rate paid multiplied by the length of stay). Lastly, let the arrival processes for each product be a stationary Poisson process with rate λ_i, noting that orders for a Monday night stay will stop arriving at 3AM Tuesday night; orders for a Tuesday night stay stop arriving at 3AM on a Wednesday night etc.

Booking limits $(b_1, \ldots, b_{56})$ are controls that limit the amount of capacity that can be sold to any particular product; i.e. they represent the maximum number of requests of product i we are willing to accept. The booking limits do not represent the number of rooms reserved for each product, rather, they represent the number of rooms available to this product and all products that use the same resources and have a higher booking limit. For example, if we have five products and all of them require the same resource (say capacity $C = 10$) and their corresponding booking limits are $b_1 = 10$, $b_2 = 8$, $b_3 = 4$, $b_4 = 2$, $b_5 = 1$, we know we can only take 1 request for product 5, 2 requests for product 4 and so on. However, this **does not mean that** 2 **rooms will be saved** until 2 requests for product 4 arrive; **rather**, it means that, **out of all requests accepted, at most** 2 **can be a product** 4. Note also that the maximum number of requests accepted in this case would be 10, as they all use the same resource, which has $C = 10$. Doing this ensures that those products with higher booking limits are always accepted if capacity is available while also accounting for the interconnectedness of the system.

Now, once the booking limits are set, a request for product i is accepted iff $b_i > 0$ and rejected otherwise. When a request for product i is accepted, all of the booking limits that require the resources used by product i must be updated to account for the decrease in available resources. For example, if a request for a 3-night stay arriving Monday is accepted, all products using a night on either Monday, Tuesday, or Wednesday must have their booking limits decreased by one.

We may see that $b_i \leq C$ (to avoid overbooking) and that the highest booking limit must equal capacity (we want to rent as many rooms as possible without going over capacity, thus, all rooms must be available to at least one product). Furthermore, since requests are only accepted when rooms are available ($b_i > 0$), we are guaranteed to never go over capacity.

In summary, a booking limit represents the maximum number of requests of product i that we are willing to accept given that we start with full availability. As soon as a request is accepted, available capacity changes and booking limits must be updated to account for this change.

There is one source of randomness in the model which is the stationary Poisson process with rate λ_i for guest arrivals for product i.

Although our interest is in modelling the full 56 products to find the optimal set of booking limits, to illustrate how booking limits are updated, one may look at the following, small-scale example:

10.1.1 Example of how Booking Limits are Updated:

Assume a hotel offers only the following 5 products:

1. Two night stay arriving Monday.
2. Two night stay arriving Tuesday.
3. Two night stay arriving Wednesday.
4. Three night stay arriving Wednesday.
5. Two night stay arriving Thursday.

If the booking limits for each product are $b_1 = 10$, $b_2 = 8$, $b_3 = 4$, $b_4 = 7$, $b_5 = 1$ and the following requests are received, the booking limits would be updated in the following way as decisions to accept or reject a given order are made: Note that, in this case product 1 has a final booking limit of 9 as only one room

	$b_1 = 10$	$b_2 = 8$	$b_3 = 4$	$b_4 = 7$	$b_5 = 1$	Request	Action
Initial Booking Limit	10	8	4	7	1	Product 5	Accept
P5 Sold, BL $3, 4, 5$ affected	10	8	3	6	0	Product 3	Accept
P3 Sold, BL $2, 3, 4$ affected	10	7	2	5	0	Product 4	Accept
P4 Sold, BL $2, 3, 4$ affected	10	6	1	4	0	Product 1	Accept
P1 Sold, BL $1, 2$ affected	9	5	1	4	0	Product 5	Reject
Rejected $b_5 = 0$	9	5	1	4	0		

has been sold on either Monday or Tuesday, which means that 9 rooms are still available on Monday night.

To simplify this, one may create a binary matrix A showing which products use which resources. Thus, we will let each row be a resource available and each column a product, having a 1 in entry (i, j) if product j uses resource i, and 0 otherwise. Then, if we accept a request for product i, we must update the booking limits of all products j such that $\mathrm{A}_j^T \mathrm{A}_i \geq 1$ (they share at least one of the resources). For this small example, we have:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

10.2 Model Factors and Responses

The following are model factors for the hotel revenue management model:

- The number of products (combining rate and length of stay). The default is 56 for 7 lengths of stays and 8 different rates.
- The arrival rates for each product. The number of arrival rates is equal to the number of products. The default is $\lambda_i = \frac{1}{168}, \frac{2}{168}, \frac{3}{168}, \frac{2}{168}, \frac{1}{168}, \frac{0.5}{168}, \frac{0.25}{168}$ for $i = 1, \ldots, 8$.
- The hotel capacity. The default is 100.
- The discount rate. The default is 100.
- The rack rate (full price for a product). The default is 200.
- The product incidence matrix. The default is seen in the example.
- The time after which orders of each product no longer arrive. The default is a list of 14 27's, 12 51's, 10 75's, 8 99's, 6 123's, 4 144's, 2 168's. The length of the list is equal to the number of products.
- The hours before $t = 0$ to start running. (e.g. 168 means start at time -168). The default value is 168.
- The booking limits. The tuple length is equal to the number of products. The default value is a tuple of 56 100's.

There is one model response being the expected revenue.

10.3 Optimisation Problem Description

The optimisation problem is to maximise the expected revenue by finding the optimal **booking limits**.

The problem is box constrained in the interval $(0, N)^n$, where n is the number of products and N is the total number of rooms.

10.4 Optimisation Problem Factors

The problem is n-dimensional, where n is the number of products on offer in the model. The initial solution is a tuple of n zeroes.

There is no known optimal solution to the problem.

11 Iron Ore Production with Exogenous Stochastic Price

11.1 Model Description

Iron ore is traded on the spot market, facing an exogenous, stochastic price. There is enormous demand for iron, and so for the purposes of a small or medium-sized iron ore mine, we assume that any quantity of ore can be instantaneously sold at current market rates.

Let there be T time periods (days), holding cost of h per unit, production cost of c per unit, maximum production per day of m units, and maximum holding capacity of K units. Let the iron ore market price for the day be P_t.

Let x_1 be the price at which to begin production, x_2 be the inventory level at which to cease production, x_3 be the price at which to cease production, and x_4 be the price at which to sell all current stock.

The daily order of operations in the simulations is as follows:

1. Sample the market price, P_t. Let the current stock price be s_t.
2. If production is already underway,
   (a) If $\mathrm{P}_t \leq x_3$ or $s_t \geq x_2$, cease production.
   (b) Else, produce $\min(m, K - s_t)$ at cost c per unit.
3. If production is not currently underway, and if $\mathrm{P}_t \geq x_1$ and $s_t < x_2$, begin production.
4. If $\mathrm{P}_t \geq x_4$, sell all stock (after production) at price P_t.

5. Charge a holding cost of h per unit (after production and sales).

There is one source of randomness in the model. That being P_t is a mean-reverting random walk, such that $\mathrm{P}_t = \mathrm{trunc}(\mathrm{P}_t - 1 + \mathrm{N}_t(\mu, \sigma))$ where N_t is a normal random variable with standard deviation σ and mean $\mu_t = \mathrm{sgn}(\mu_0 - \mathrm{P}_t - 1)(|\mu_0 - \mathrm{P}_t - 1|)^{\frac{1}{4}}$. Here $\mathrm{trunc}(x)$ truncates the price to lie between a specified minimum and maximum price.

11.2 Model Factors and Responses

The following are model factors to the iron ore production model:

- The mean iron ore price per unit. The default is 100.
- The maximum iron ore price per unit. The default is 200.
- The minimum iron ore price per unit. The default is 0
- The maximum holding capacity. The default is $K = 10000$.
- The standard deviation of random walk steps for price. The default is $\sigma = 7.5$.
- The holding cost per unit per period. The default is $h = 1$.
- The production cost per unit. The default is $c = 100$.
- The maximum units produced per day. The default is $m = 100$.
- The price level to start production. The default is $x_1 = 80$.
- The inventory level to cease production. The default is $x_2 = 7000$.
- The price level to stop production. The default is $x_3 = 40$.
- The price level to sell all stock. The default is $x_4 = 100$.
- The number of days to simulate. The default is $T = 365$.

There are three responses to the model. These are:

1. The total profit over the time period.
2. The fraction of days spent producing iron ore.
3. The average stocks over the time period.

11.3 Optimisation Problem Description

The optimisation problem aims to maximise the total profit over the time period T by changing the values x_1, x_2, x_3, x_4.

The constraints for the optimisation problem are the following:

$$x_3 \leq x_1 \leq x_4$$
$$x_1, x_2, x_3, x_4 \geq 0$$

11.4 Optimisation Problem Factors

The solution lies in $\mathbb{R}^4$ where

$$x^* = (x_1, x_2, x_3, x_4).$$

The initial solution is the following:

$$x_1 = 80 \qquad x_2 = 7000x_3 = 40 \qquad x_4 = 100$$

There is no known optimal solution to this problem.

12 M/M/1 Queue

12.1 Model Description

This is a model that simulates an $M/M/1$ queue with an exponential interarrival time distribution and an exponential service time distribution.

There are two sources of randomness in the model:

1. The exponential interarrival times.
2. The exponential service times.

12.2 Model Factors and Responses

The following are factors for the $M/M/1$ queue:

- The rate parameter for the distribution of interarrival times. The default is $\lambda = 1.5$.
- The rate parameter for the distribution of service times. The default is $\mu = 3$.

- The number of people as warmup before collecting statistics (burn-in). The default is 50.
- The number of people from which to calculate the average soujourn time. The default is 200.

The queue model has three responses:

1. The average of soujourn time of customers (time customers spend in the system).
2. The average waiting time of customers.
3. The fraction of customers who wait.

12.3 Optimisation Problem Description

The optimisation problem associated with this model attempts to find the optimal service rate parameter μ that minmimises the expected soujourn time plus a penalty for increasing the rate $c\mu^2$.

12.4 Optimisation Problem Factors

There is one problem factor associated with the cost for increasing μ. The default is $c = 0.1$.

The solution is 1-dimensional and the starting solution for μ is 3. There is no known optimal solution to the problem.

13 Parameter Estimation

13.1 Model Description

A model that simulates the maximum likelihood estimations (MLE) for the parameters of a two-dimensional Gamma distribution.

Say a simulation generates output data $\mathrm{Y}_i, \mathrm{Y}_j \in [0, \infty] \times [0, \infty]$, what are i.i.d. and are known to come from a distribution with the two-dimensional density function:

$$f(y1, y2; x^*) = \frac{e^{-y1} y_1^{x_1^* y_2 - 1}}{\Gamma(x_1^* y_2)} \frac{e^{-y2} y_2^{x_2^* - 1}}{\Gamma(x_2^*)},$$

where $y1, y2 > 0$ and $x^* = (x_1^*, x_2^*)$ is the unknown vector of parameters.

Noting that x^* maximises the function

$$g(x) = \mathbb{E}[\log(f(\mathrm{Y}; x))] = \int_0^\infty \log(f(y; x)) f(y; x^*)\, dy,$$

and that

$$\mathrm{G}_m(x) = \frac{1}{m} \sum_{j=1}^{m} \log(f(\mathrm{Y}_j; x))$$

is a consistent estimator of $g(x)$. Observations generated from the distribution specified by a given x^*.

There is one source of randomness in the model coming from the vector $y = (y_1, y_2)$. Both elements of y are gamma random variables.

13.2 Model Factors and Responses

The following are model factors of the MLE model:

- The unknown 2-D parameter that maximises the expected log likelihood function. The default is $x^* = (2, 5)$.
- A 2-D variable in the probability density function (pdf). The default is $x = (1, 1)$.

There is one model response which is the log likelihood of the pdf.

13.3 Optimisation Problem Description

The optimisation problem involves finding a **2-D variable x in the pdf** that maximises the log likelihood of the 2-D Gamma random variable.

The value x is constrained in the square $(0, 10) \times (0, 10)$.

13.4 Optimisation Problem Factors

The starting solution is $x = (1, 1)$ and the optimal solution to the optimisation problem is $x^* = (2, 5)$. It is easy to substitute this into the log-likelihood function to obtain the optimal objective function value.

14 Multi-Stage Revenue Management with Inter-Temporal Dependence

14.1 Model Description

Consider the following inventory system: A businessperson initially purchases b units of a product to sell to customers over a fixed time horizon. In the first period, a businessperson receives a certain demand for units at a low price p_{low} . The businessperson decision can choose how many units to sell right now. The businessperson may wish to reserve some units for customers arriving later who are willing to pay more for the product. The number of units reserved for the start of period t is denoted by r_t. This process of observing demand and choosing how many units to sell is repeated over T time periods. In order to make wise decisions about how much inventory to reserve in each time period, the businessperson must consider future demand for the product.

The demand in period t is denoted by $\mathrm{D}_t = \mu_t \mathrm{X} \mathrm{Y}_t$ where X has a gamma distribution with parameters $k > 0$ and $\theta > 0$ such that it has mean $k\theta = 1$ and standard deviation $\sqrt{k}\theta = (\sqrt{k})^{-1}$. $\mathrm{Y}_1, \ldots, \mathrm{Y}_T$ are i.i.d. exponential with mean 1 and μ_t are positive constants for all t.

There are two sources of randomness in the model used to generate the X's and Y_t's that form the demand.

14.2 Model Factors and Responses

The following are factors for the model:

- The period of time that is considered. The default is $T = 3$.
- The prices for each period. The dimension is equal to the time horizon T. The default is $p = (100, 300, 400)$.
- The mean demand for each period. The dimension is equal to the time horizon T. The default is $\mu = (50, 20, 30)$.
- The cost per unit of capacity at $t = 0$. The default is $c = 80$.
- The shape parameter of the Gamma distribution. The default is $k = 1$.
- The scale parameter of the Gamma distribution. The default is $\theta = 1$.
- The initial inventory of the product at $t = 1$. The default is $b = 100$.
- The inventory to reserve going into periods $2, 3, \ldots, T$. The dimension of the reservation quantity is equal to one less than the time horizon. The default value is $(r_2, r_3) = (50, 30)$.

There is one response from the model which is the total revenue of the given model.

14.3 Optimisation Problem Description

The objective of the optimisation problem is to find an **initial inventory** (b) and **reservation quantities** ($r_2, r_3, \ldots, r_T$) such that the total revenue of the model is maximised.

The only constraint on the problem is such that the reserve quantities are decreasing and less than the initial capacity:

$$b \geq r_2 \geq r_3 \geq \cdots \geq r_T$$

14.4 Optimisation Problem Factors

The starting solution is the following selection on the decision variables:

$$b = 100$$
$$r_2 = 50$$
$$r_3 = 30$$

There is no known optimal solution to the model.

15 (s,S) Inventory System

15.1 Model Description

Consider a (s, S) inventory model with full backlogging. Demand during each period, D_t is distributed exponential with mean μ. At the end of each period, the inventory position (IP_t = Stock on hand − Backorders + Outstanding Orders) is calculated and, if it is below s, an order to get back up to S is placed ($\mathrm{O}_t = \max(\mathbb{I}(IP_t < s)(S - IP_t), 0)$). Lead times have a Poisson distribution with mean θ days and all replenishment orders are received at the beginning of the period. Note that, since orders are placed at the end of the day, an order with lead time l placed in period n will arrive at the beginning of period $n + l + 1$.

A per unit holding cost h is charged for inventory on-hand; furthermore, there is a fixed order cost f and a variable, per unit, cost c.

15.2 Model Factors and Responses

The following are model factors for the inventory model:

- The mean of exponentially distributed demand in each period. The default is $\mu = 100$.
- The mean of the Poisson distributed over lead time. The default is $\theta = 6$.
- The cost per unit of demand not met with instock inventory. The default is 4.
- The holding cost per unit per period. The default is $h = 1$.
- The order fixed cost. The default is $f = 36$.
- The order variable cost per unit. The default is $c = 2$.
- The inventory threshold for placing the order. The default is $s = 1000$.
- The maximum inventory capacity. The default is $S = 2000$.
- The number of periods to simulate. The default is $N = 100$.
- The number of periods as warmup before collecting statistics. The default is 20

There are eight responses to the inventory model:

1. The average backorder costs per period.
2. The average order costs per period.
3. The average holding costs per period.
4. The fraction of demand met with stock on hand in store.
5. The fraction of periods an order was made.
6. The fraction of periods a stockout occurred.
7. The mean amount of product backordered given a stockout occurred.
8. The mean amount of product ordered given an order occurred.

15.3 Optimisation Problem Description

The optimisation problem aims to find values for the **inventory threshold** s and the **maximum inventory capacity** S in order to minimize the following objective function:

$$\mathbb{E}(\text{Total cost per period}).$$

The constraint on the problem is that the stockout rate δ, the fraction of demand not supplied from stock on-hand, is at most 10%.

15.4 Optimisation Problem Factors

The solution exists in $\mathbb{R}^2$ with the solution being defined as $x^* = (s, S)$. The starting solution is $x_0 = (1000, 2000)$.

There is no known optimal solution to the problem.

16 Restaurant Table Management

16.1 Model Description

Floor space in a restaurant is allocated to N different sizes of tables, with capacities $c = (c_1, c_2, \ldots, c_N)$ with $c_i \in \mathbb{Z}_+^n$ for $i = 1, 2, \ldots, N$. A table of capacity c_i can seat c_i customers or fewer. The restaurant can seat a maximum of K customers at a time, regardless of table allocation, and is open for T consecutive hours. The decision variable is the vector x representing the number of tables allocated to each size.

Customers arrive in groups of size $j \in \{1, \ldots, \max_i(c_i)\}$ according to a homogeneous Poisson process with rate λ_j. A group of customers is seated at the smallest possible available table. If there are no available tables large enough, the group of customers leaves immediately. Service time per group is exponential and revenue per group is fixed.

The source of randomness in the model comes from the group of customers arriving according to a homogenous Poisson process. Group size is randomly generated with probability proportional to each group's average arrival rate. Service time per group is exponential.

16.2 Model Factors and Responses

The following are factors to the table allocation model:

- The number of hours to simulate. The default is $T = 3$.
- The maximum total capacity of the restaurant. The default is $K = 80$.
- The capacity of each type of table. The default is $c = (2, 4, 6, 8)$.
- The average number of arrivals per hour. The default value is $\lambda = (3, 6, 3, 3, 2, 4/3, 6/5, 1)$. It is clear the the dimension of the vector lambda is equal to the $\max(c) = 8$.
- The mean service time in minutes. The default value is $\theta = (20, 25, 30, 35, 40, 45, 50, 60)$. The dimension of θ is equal to the dimension of λ.

- The revenue earned for each group size. The default is $(15, 30, 45, 60, 75, 90, 105, 120)$. The dimension of the vector for revenue of each group size is equal to the dimension of λ.
- The number of tables of each capacity. The default is $x = (10, 5, 4, 2)$. It is seen that $K = x \cdot c$.

There are two responses to the table allocation model, these are:

1. The total revenue earned over the simulation period.
2. The fraction of customer arrivals that are seated.

16.3 Optimisation Problem Description

The optimisation problem finds a value for the **number of tables of each capacity** x such that it maximises the total expected revenue for a restaurant operation.

The only constraint on the problem is that the number of the seats in the restaurant remains below capacity:

$$x \cdot c \leq K.$$

16.4 Optimisation Problem Factors

The starting solution for the number of tables is

$$x = (10, 5, 4, 2),$$

this corresponds to 10 tables of size 2, 5 tables of size 4, 4 tables of size 6, and 2 tables of size 8.

There is no known optimal solution to this problem.