9.) $\underline{r}(t) = (1+\cos t,\ \sin t,\ 2\sin\frac{t}{2}) \qquad t \in [-2\pi, 2\pi]$

a.) $s = \int_{t_0}^{t_1} |\underline{r}'(t)|\,dt \ , \quad \underline{r}'(t) = (-\sin t,\ \cos t,\ \cos(\frac{t}{2}))$

$\Rightarrow |\underline{r}'(t)|^2 = 1 + \cos^2(t/2)$

$\therefore\ s = \int_{-2\pi}^{2\pi} \sqrt{1+\cos^2(\frac{t}{2})}\ dt \approx 15.28$ (Elliptic integral so evaluated numerically)

b.)

$$\kappa(t) = \frac{|\underline{r}'(t) \times \underline{r}''(t)|}{|\underline{r}'(t)|^3}$$

$\underline{r}(t) = (1+\cos t,\ \sin t,\ 2\sin\frac{t}{2})$

$\underline{r}'(t) = (-\sin t,\ \cos t,\ \cos\frac{t}{2})$

$\underline{r}''(t) = (-\cos t,\ -\sin t,\ -\frac{1}{2}\sin\frac{t}{2})$

$$|\underline{r}'(t)| = \sqrt{1+\cos^2\tfrac{t}{2}} = \sqrt{1+\tfrac{1}{2}+\tfrac{1}{2}\cos t} = \sqrt{3/2 + \tfrac{1}{2}\cos t} = \frac{\sqrt{3+\cos t}}{\sqrt{2}}$$

$$\therefore\ |\underline{r}'(t)|^3 = \frac{3+\cos t}{2}\sqrt{\frac{3+\cos t}{2}}$$

$$\underline{r}'(t) \times \underline{r}''(t) = \det\begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ -\sin t & \cos t & \cos\frac{t}{2} \\ -\cos t & -\sin t & -\frac{1}{2}\sin\frac{t}{2} \end{pmatrix}$$

$$= \left[-\tfrac{1}{2}\sin\tfrac{t}{2}\cos t + \sin t\cos\tfrac{t}{2},\ -\tfrac{1}{2}\sin\tfrac{t}{2}\sin t - \cos\tfrac{t}{2}\cos t,\ 1\right]$$

$$\therefore |\underline{r}'(t)\times\underline{r}''(t)|^2 = \tfrac{1}{4}\sin^2\tfrac{t}{2}\cos^2 t - \sin\tfrac{t}{2}\cos t\sin t\cos\tfrac{t}{2} + \sin^2 t\cos^2\tfrac{t}{2} + \tfrac{1}{4}\sin^2\tfrac{t}{2}\sin^2 t + \sin\tfrac{t}{2}\sin t\cos\tfrac{t}{2}\cos t + \cos^2\tfrac{t}{2}\cos^2 t + 1$$

$$= \tfrac{1}{4}\sin^2\tfrac{t}{2} + \sin\tfrac{t}{2}\sin t\cos\tfrac{t}{2}\cos t - \sin\tfrac{t}{2}\sin t\cos\tfrac{t}{2}\cos t + \cos^2\tfrac{t}{2} + 1$$

$$= \tfrac{1}{4}\sin^2\tfrac{t}{2} + \cos^2\tfrac{t}{2} + 1 = \tfrac{1}{4} - \tfrac{1}{4}\cos^2\tfrac{t}{2} + \cos^2\tfrac{t}{2} + 1$$

$$= \tfrac{3}{4}\cos^2\tfrac{t}{2} + \tfrac{5}{4} = \tfrac{3}{4}\left(\tfrac{1}{2}+\tfrac{1}{2}\cos t\right) + \tfrac{5}{4}$$

$$= \tfrac{3}{8} + \tfrac{3}{8}\cos t + \tfrac{5}{4} = \tfrac{13}{8} + \tfrac{3}{8}\cos t$$

$$\therefore\ |\underline{r}'(t)\times\underline{r}''(t)| = \frac{\sqrt{13+3\cos t}}{2\sqrt{2}}\ , \quad |\underline{r}'(t)|^3 = \frac{3+\cos t}{2}\sqrt{\frac{3+\cos t}{2}}$$

$$\therefore\ \kappa(t) = \frac{\sqrt{13+3\cos t}}{2\sqrt{2}} \cdot \frac{2\sqrt{2}}{(3+\cos t)^{3/2}} \Rightarrow \kappa(t) = \frac{\sqrt{13+3\cos t}}{(3+\cos t)^{3/2}}$$

$\underline{r}(t) = (1+\cos t,\ \sin t,\ 2\sin\frac{t}{2})$

$$\tau(t) = \frac{\underline{r}'\times\underline{r}''\cdot\underline{r}'''}{|\underline{r}'\times\underline{r}''|^2}\ , \quad \underline{r}'''(t) = \left(\sin t,\ -\cos t,\ -\tfrac{1}{4}\cos\tfrac{t}{2}\right)$$

$$= \frac{1}{\frac{13}{8}+\frac{3}{8}\cos t}\begin{pmatrix} -\frac{1}{2}\sin\frac{t}{2}\cos t + \sin t\cos\frac{t}{2} \\ -\frac{1}{2}\sin\frac{t}{2}\sin t - \cos\frac{t}{2}\cos t \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \sin t \\ -\cos t \\ -\frac{1}{4}\cos\frac{t}{2} \end{pmatrix}$$

$$= \frac{8}{13+3\cos t}\left[-\tfrac{1}{2}\sin\tfrac{t}{2}\sin t\cos t + \sin^2 t\cos\tfrac{t}{2} + \tfrac{1}{2}\sin\tfrac{t}{2}\sin t\cos t + \cos\tfrac{t}{2}\cos^2 t - \tfrac{1}{4}\cos\tfrac{t}{2}\right]$$

$$= \frac{8}{13+3\cos t}\cdot\frac{3}{4}\cos\frac{t}{2} = \frac{6\cos\frac{t}{2}}{13+3\cos t}$$

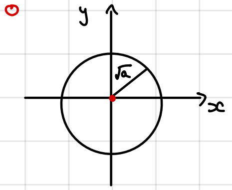
$$\therefore\ \tau(t) = \frac{6\cos(t/2)}{13+3\cos t}$$

11.) a.) $x^2 + y^2 - z^2 = a$, $a \in [-2, 2]$

In $x-y$:

$a > 0$

$a = 0$

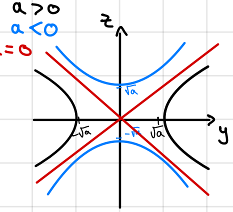


In $y-z$:

$a > 0$

$a < 0$

$a = 0$

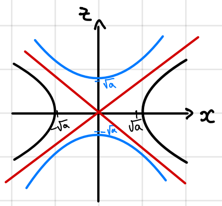


In $x-z$:

$a > 0$

$a < 0$

$a = 0$



12.) a.) $\underline{r}(u) = (f(u), 0, g(u))$, $f(u) \geq 0$

$$\begin{pmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ rotates by v about z axis.

Now we rotate $\underline{r}(u)$:

$$\begin{pmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f(u) \\ 0 \\ g(u) \end{pmatrix} = \begin{pmatrix} f(u)\cos v \\ f(u)\sin v \\ g(u) \end{pmatrix}$$

$\therefore$ After rotation by v:

$$\underline{r}(u, v) = (f(u)\cos v, f(u)\sin v, g(u))$$

12.c.) $\underline{r}(u,v) = (\cosh u \cos v, \cosh u \sin v, u)$

$\underline{r}_u = (\sinh(u)\cos v, \sinh(u)\sin v, 1)$

$\underline{r}_v = (-\cosh(u)\sin v, \cosh(u)\cos v, 0)$

$\underline{r}_u \times \underline{r}_v = (-\cosh(u)\cos v, -\cosh(u)\sin v, \sinh(u)\cosh(u))$

$$|\underline{r}_u \times \underline{r}_v|^2 = \cosh^2(u)\cos^2 v + \cosh^2(u)\sin^2 v + \sinh^2(u)\cosh^2(u)$$
$$= \cosh^2(u)(1 + \sinh^2(u))$$
$$= \cosh^4(u)$$

$\Rightarrow |\underline{r}_u \times \underline{r}_v| = \cosh^2(u)$

$$SA = \int_0^1 \int_0^{2\pi} \cosh^2(u) \, dv \, du$$
$$= 2\pi \int_0^1 \cosh^2(u) \, du = \pi \int_0^1 \cosh(2u) + 1 \, du$$
$$= \pi \left(\tfrac{1}{2}\sinh(2u) + u \Big|_0^1 \right)$$
$$= \frac{\pi}{2}\sinh(2) + \pi$$

$\therefore SA = \frac{\pi}{2}\sinh(2) + \pi$

13.) b.) $\underline{r}(u,v) = ((1-\cos u)\cos v, (1-\cos u)\sin v, u - \sin u)$

$\underline{r}_u = (\sin u \cos v, \sin u \sin v, 1-\cos u)$
$\underline{r}_v = (-(1-\cos u)\sin v, (1-\cos u)\cos v, 0)$

$\underline{r}_u \times \underline{r}_v = \left(-(1-\cos u)^2\cos v, -(1-\cos u)^2 \sin v, (1-\cos u)\sin u\right)$

$$\begin{aligned}|\underline{r}_u \times \underline{r}_v|^2 &= (1-\cos u)^4 + (1-\cos u)^2\sin^2 u \\ &= (1-\cos u)^2\left[(1-\cos u)^2 + \sin^2 u\right] \\ &= (1-\cos u)^2\left[2(1-\cos u)\right] \\ &= 2(1-\cos u)^3\end{aligned}$$

$\Rightarrow |\underline{r}_u \times \underline{r}_v| = \sqrt{2}(1-\cos u)^{3/2}$

$$SA = \int_0^{2\pi}\int_0^{2\pi} \sqrt{2}(1-\cos u)^{3/2}\,du\,dv$$

$$= 2\pi\sqrt{2}\int_0^{2\pi}(1-\cos u)^{3/2}\,du$$

$\cos(2x) = 1 - 2\sin^2 x$
$\Rightarrow 1 - \cos(x) = 2\sin^2(\frac{x}{2})$

$$= 2\pi\sqrt{2}\int_0^{2\pi}\left(2\sin^2(\tfrac{u}{2})\right)^{3/2}du$$

$$= 8\pi\int_0^{2\pi}\sin^3(\tfrac{u}{2})\,du$$

$t = u/2$
$2dt = du$

$$= 16\pi\int_0^{\pi}\sin^3 t\,dt = 16\pi\int_0^{\pi}\sin t\,(1-\cos^2 t)\,dt$$

$x = \cos t$
$-dx = \sin t\,dt$

$$= 16\pi\int_{-1}^{1} 1 - x^2\,dx = 16\pi\left[x - \frac{x^3}{3}\Big|_{-1}^{1}\right]$$

$$= 16\pi\left(1 - \frac{1}{3} + 1 - \frac{1}{3}\right)$$

$\Rightarrow$ Surface area $= \frac{64}{3}\pi$

14. b.) $\underline{S}(u,v) = \left((1-u\sin(\frac{v}{2}))\cos v, (1-u\sin(\frac{v}{2}))\sin v, u\cos(\frac{v}{2})\right)$

Centre of strip $\rightsquigarrow u = 0$

$\therefore \underline{S}(0,v) = (\cos v, \sin v, 0) = \underline{c}(v)$ the unit circle

c.) i.)
$\underline{S}(u,v) = \left((1-u\sin(\frac{v}{2}))\cos v, (1-u\sin(\frac{v}{2}))\sin v, u\cos(\frac{v}{2})\right)$

$\underline{S}_u = \left(-\sin(\frac{v}{2})\cos v, -\sin(\frac{v}{2})\sin v, \cos(\frac{v}{2})\right)$

$\underline{S}_v = \left(-\frac{u}{2}\cos(\frac{v}{2})\cos v - (1-u\sin(\frac{v}{2}))\sin v, -\frac{u}{2}\cos(\frac{v}{2})\sin v + (1-u\sin(\frac{v}{2}))\cos v, -\frac{u}{2}\sin(\frac{v}{2})\right)$
let $u=0$
$\therefore \underline{S}_u = \left(-\sin(\frac{v}{2})\cos v, -\sin(\frac{v}{2})\sin v, \cos(\frac{v}{2})\right)$
$\underline{S}_v = (-\sin v, \cos v, 0)$

$\therefore \underline{S}_u \times \underline{S}_v = \left(-\cos(\frac{v}{2})\cos v, -\cos(\frac{v}{2})\sin v, -\sin(\frac{v}{2})\right)$

$|\underline{S}_u \times \underline{S}_v|^2 = \cos^2(\frac{v}{2}) + \sin^2(\frac{v}{2}) = 1$

$\therefore$

$\underline{\hat{n}} = \left(\cos(\frac{v}{2})\cos v, \cos(\frac{v}{2})\sin v, \sin(\frac{v}{2})\right)$

15.b.) $x^3 + y^2 + z = 1$

let $f(x,y,z) = x^3 + y^2 + z$

$\nabla f|_{(1,0,0)}$ normal to $f = 1$

$\nabla f = (3x^2, 2y, 1) \Rightarrow \nabla f|_{(1,0,0)} = (3, 0, 1)$

$\Rightarrow \hat{\underline{n}} = \frac{1}{\sqrt{10}}\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$

c.) $(1,0,0)$ is on the plane.

$\frac{1}{\sqrt{10}}\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{3}{\sqrt{10}} \Rightarrow$ eq of plane is: $3x + z = 3$

16.) a.)

$\iint_S \nabla \cdot \underline{F} \, ds \quad , \quad \nabla \cdot \underline{F} = 2z - 2\sin y + 6z^2$

$\int_{-1}^{1}\int_{-1}^{1} 2z + 6z^2 - 2\sin y \, dx \, dy$

$= (2z + 6z^2)(4) - 2\int_{-1}^{1}\int_{-1}^{1} \sin y \, dx \, dy$

$= 8(3z^2 + z) - 4\int_{-1}^{1} \sin y \, dy$

$= 8z(3z + 1)$