A Topological Approach to Magnetic Nulls

Ben Israeli[1,†], Christopher Berg Smiet[1,2], Amitava Bhattacharjee[1]

[1]Princeton Plasma Physics Laboratory, Princeton, 08540 NJ, USA,

[2]Leiden University, Leiden, 2300 RA, Netherlands

[†] bisraeli@pppl.gov

Introduction

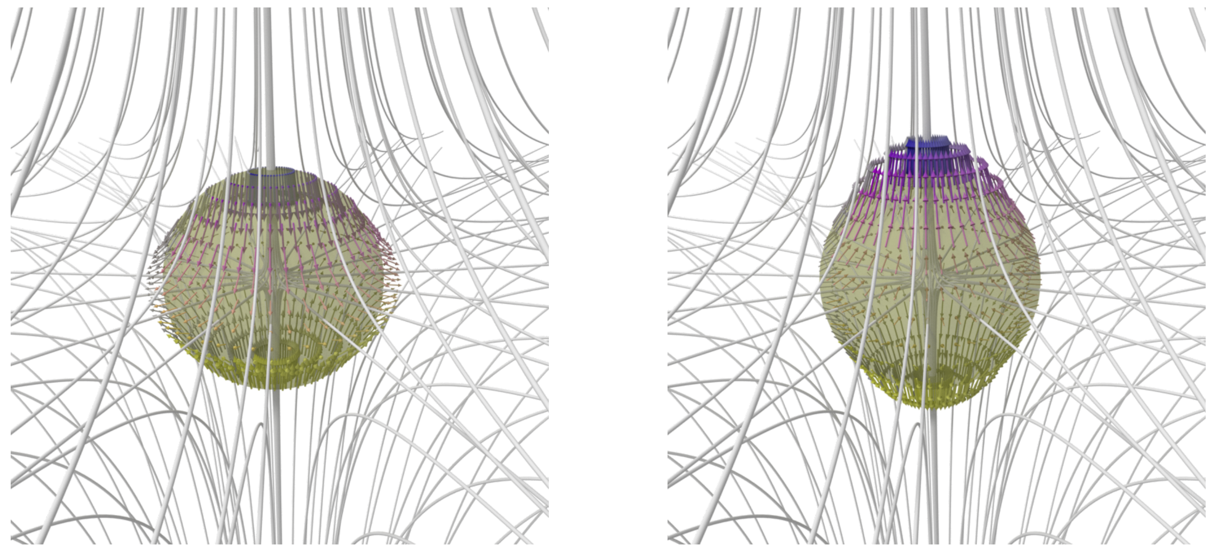
Three-dimensional magnetic nulls play an important role in astrophysical plasma physics and earth-based fusion concepts such as the FRC. We present a novel method of conceptualizing the movement of these nulls by treating them as the convergence point of isotropes (iso=same, tropos = direction), lines in space along which the magnetic field points in the same direction. It is shown that the isotropes can be recovered as the stream lines of the isotrope field, which is defined via a geometric formula from the magnetic field, and for which nulls act as charges analogous to point charges generating an electric field. In this way, the index theorem for magnetic nulls can be reframed as a Gauss's Law on the isotrope field.

Background

Type A and Type B Nulls

Because magnetic fields are divergence free, nulls of 3D magnetic fields are constrained to be one of two types [1, 2, 3]:

Type A (Index -1) **Type B (Index +1)**



This can be understood via the eigenvalues of the Jacobian of the magnetic field at the null:

$$J_{ij} = \partial_j B_i. \tag{1}$$

$\nabla \cdot \mathbf{B} = 0$ implies $\mathrm{Tr}(J) = 0$. As a result, J must have either all real eigenvalues or a pair of conjugate eigenvalues and one real eigenvalue. Generically, two eigenvalues will share a sign of their real part, and one will necessarily have opposite sign. The paired eigenvalues have eigenvectors that span a "fan plane" extending from the null, while the remaining eigenvector follows a "spine" passing through this plane at the null.

Type A nulls are those with one positive real eigenvalue, and **type B** nulls are those with one negative real eigenvalue, with flows along the spine and fan correspondingly directed.

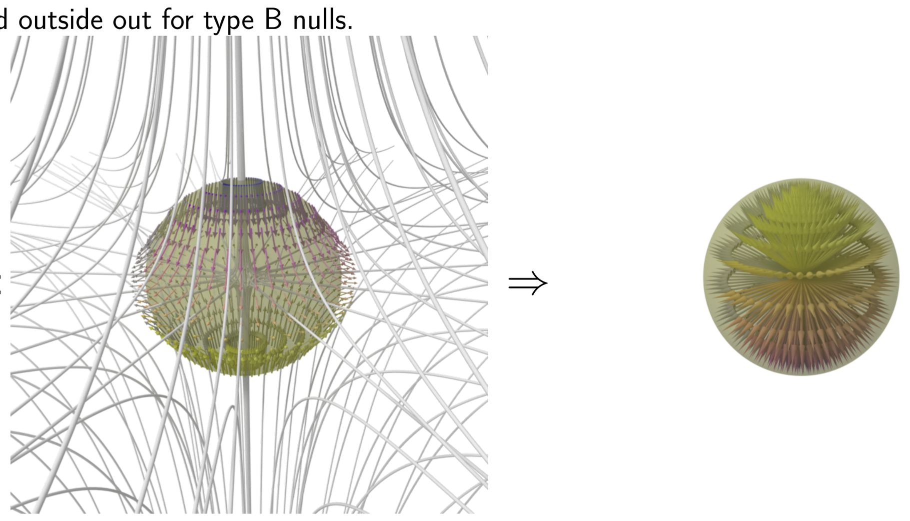
Topological Index

Define the map $g : \mathbb{R}^3 \to S^2$ that sends points in space to the unit vectors in the direction of the field $\mathbf{B}$ (i.e. points on the unit sphere):

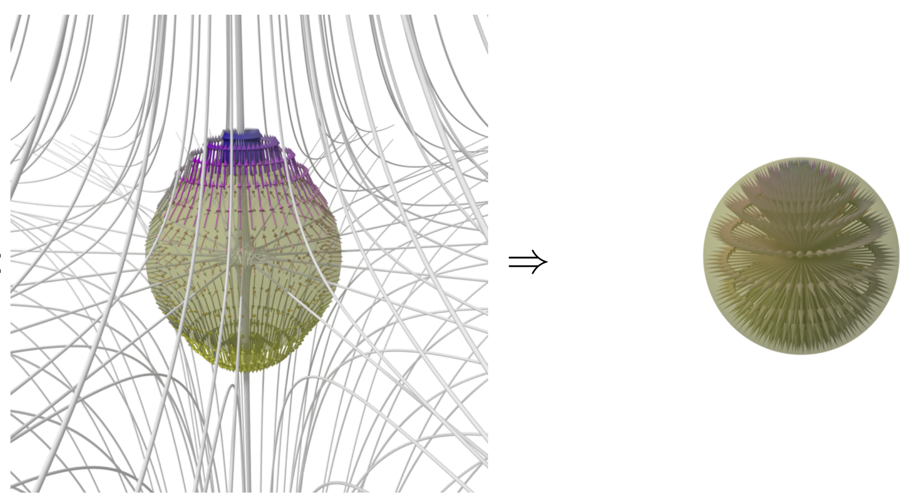
$$g = \frac{\mathbf{B}(\mathbf{x})}{|\mathbf{B}(\mathbf{x})|} \tag{2}$$

Note that on a sufficiently small sphere around a null, this map is 1-to-1, inside out for type A nulls and outside out for type B nulls.

A:



B:



The degree of a map is the number of times the domain wraps around the range. For the surface of a solid sphere containing a single null, the degree of the mapping $g|_{\partial D^3_{x_0}}$ is -1 for a type A null and $+1$ for a type B null. This is the **topological index** of an isolated null.

$$\mathrm{Ind}(x_0) = \mathrm{Deg}(g|_{\partial D^3_{x_0}}) \tag{3}$$

The degree of any map between spheres is always in $\mathbb{Z}$ [4], so $\mathrm{Deg}(g|_{\partial D^3}) \in \mathbb{Z}$. If D^3 contains more than one isolated null, then the following theorem holds:

$$\boxed{\mathrm{Deg}(g|_{\partial D^3}) = \sum_{x_0 \in D^3} \mathrm{Ind}(x_0)} \tag{4}$$

The degree of the map counts the topological index of the nulls, and is robust to perturbation.

The Isotrope Field

Isotrope Lines

The magnetic field direction defines a map from space to the sphere $g : \mathbb{R}^3 \to S^2$, mapping a 3 dimensional manifold to a 2 dimensional manifold. Since this map loses a degree of freedom, at any point in $\mathbb{R}^3$ (with the exception of null points and singularities), there is a direction in which the direction of the magnetic field remains constant. We call the resulting paths of constant direction the isotropes of the magnetic field. These lines cannot start or end, except at nulls. As described by the topological index of nulls, the magnetic field direction on a closed surface surrounding a single null takes on all possible values (mapping over all of S^2). Therefore any null is the termination of isotropes for every magnetic field direction.

Motivating the Isotrope Field

We see that nulls:

- can be counted by integration over an enclosing surface.
- function as sources and sinks for isotrope lines.

In analogy to electrostatic point charges, this suggests constructing a field tangent to the isotrope lines (E-field lines) that has a Gauss's Law for topological index (electric charge).

Deriving the Isotrope Field

We want to construct some vector field v such that integrating it over a closed surface gives the total topological index of the enclosed nulls.

$$\boxed{\int_{\partial U} v \cdot da = 4\pi \sum_{x_i \in U} \mathrm{Ind}(x_i)} \tag{5}$$

Calculating total index by counting covers of S^2 by the surface of a region $U \in \mathbb{R}^3$ can be written:

$$4\pi \sum_{x_i \in U} \mathrm{Ind}(x_i) = \int_{g_* \partial U} \omega \tag{6}$$

where we have defined some properly normalized area two-form ω on S^2. We need to transfer our notion of area on S^2 (defined by ω) to an object that can be integrated over a surface in R^3. This is just the pull-back through our map $g : \mathbb{R}^3 \to S^2$ of the area two-form ω:

$$4\pi \sum_{x_i \in U} \mathrm{Ind}(x_i) = \int_{\partial U} g^* \omega \tag{7}$$

But we wanted a vector field, not a two-form, so we then take the Hodge dual:

$$v = \star g^* \omega \tag{8}$$

which yields, as desired:

$$4\pi \sum_{x_i \in U} \mathrm{Ind}(x_i) = \int_{\partial U} v \cdot da \tag{9}$$

Working with some arbitrary coordinates (α, β) on S^2 with $\omega = d\alpha \wedge d\beta D(\alpha, \beta)$, we can derive the general form of v and verify it indeed lies tangent to the isotrope lines.

$$4\pi \sum_{x_i \in U} \mathrm{Ind}(x_i) = \int_{g_* \partial U} \omega = \int_{g_* \partial U} d\alpha d\beta D(\alpha, \beta) = \int_{\partial U} dxdy \begin{vmatrix} \partial_x \alpha & \partial_x \beta \\ \partial_y \alpha & \partial_y \beta \end{vmatrix} D(\alpha, \beta) = \int_{\partial U} v \cdot da \tag{10}$$

For any two vectors a, b:

$$v \cdot (a \times b) = g^* \omega(a, b) = \begin{vmatrix} \partial_a \alpha(\vec{x}) & \partial_a \beta(\vec{x}) \\ \partial_b \alpha(\vec{x}) & \partial_b \beta(\vec{x}) \end{vmatrix} D(\alpha(\vec{x}), \beta \vec{x})) = (\nabla \alpha \times \nabla \beta) \cdot (a \times b) D(\alpha, \beta) \tag{11}$$

So v can be written as:

$$\boxed{v = (\nabla \alpha \times \nabla \beta) D(\alpha, \beta)} \tag{12}$$

We can immediately see that $g : \vec{x} \mapsto (\alpha, \beta)$ is constant on streamlines of v, so the streamlines of v are by definition the previously defined isotropes of $\mathbf{B}$.

$$v \perp \nabla \alpha, \nabla \beta \Rightarrow \partial_v \vec{g} = 0 \tag{13}$$

If we choose to parameterize magnetic field direction with standard spherical coordinates, and use the usual solid angle definition of area on S^2, this becomes:

$$\boxed{v = (\nabla \theta \times \nabla \phi) \sin \phi} \tag{14}$$

The isotrope field is directed away from nulls of index $+1$ and towards nulls of index -1. In this sense the isotrope field is analogous to the electric field of point charges, and it's integral over a surface calculates the topological charge enclosed.

Type A (Index -1) **Type B (Index +1)**

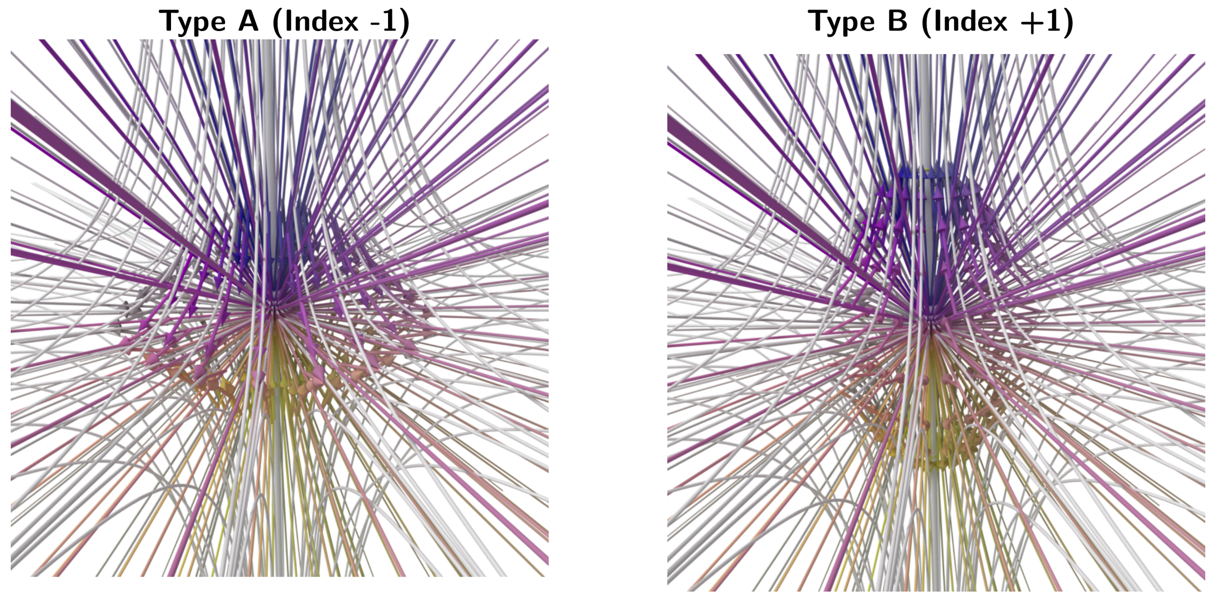


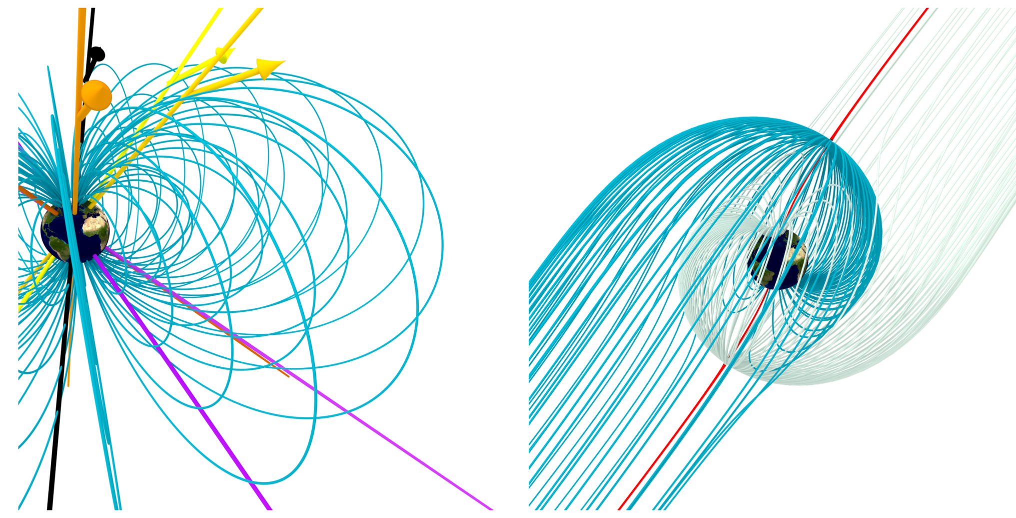
Figure: The fieldlines (white) and isotrope lines (colored) of nulls with diagonal Jacobian $\pm(1, 1, -2)$, whose isotrope fields then correspond precisely to the fields of electrostatic point charges of unit charge. Color and arrows indicate the magnetic field direction of each isotrope (compare to left).

Local Magnetic Fields in Guide Fields

Isotropes of a Localized Field Embedded in a Guide Field

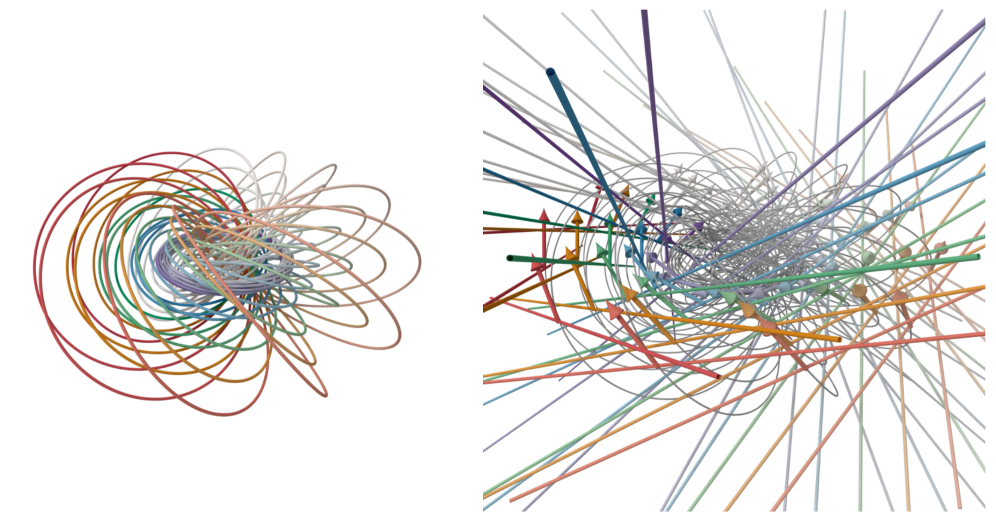
The problem of locating the nulls in a localized field embedded in a homogeneous guide field becomes immediately tractable using isotropes.

- Nulls appear on the isotrope (of the local field) corresponding with the direction opposite to the direction of the guide field, at its intersection with surfaces of constant field strength equalling that of the guide field.
- Sum of indices of all nulls must be 0.
- Isotrope pierces each surface of constant field strength twice ($+1$ on entering, -1 on exiting).
- Applicable to systems such as the magnetospheres of celestial objects and the field configuration in an FRC.



(a) (b)

Figure: a) The magnetic field of a dipole. The magnetic field lines are given by the blue curves. Seven isotropes are shown in different colors, with vectors indicating the direction of the isotropes. b) The magnetic field of a dipole embedded in a guide field of direction $(-1, 0, -1)$. The spines are colored red, the fan corresponding to the index -1 null is dark blue and the fan corresponding to the index $+1$ null is light blue.



(a) (b)

Figure: a) The structure of the Hopf field, a toroidal magnetic field constructed from the Hopf map $S^3 \to S^2$ through stereographic projection. [5] b) Several isotropes of the Hopf field with some of their directions labelled. Their color corresponds to the fieldline from (a) which they intersect. Note that the isotropes are straight lines.

Movement of Nulls

- As the guide field is varied smoothly, the location of nulls changes smoothly.
- As guide field strength is varied, nulls move along the isotropes for the guide field direction.
- Nulls of opposite type move in opposite directions along isotropes, and can be annihilated/created where they meet.

Singular Surfaces of the Jacobian

- The Jacobian J of the magnetic field is independent of the (homogeneous) guide field, and therefore so are its eigenvalues.
- As a result, the system is split into regions (invariant w.r.t. the guide field) of $|J| \lessgtr 0$, where nulls of only one type or the other can appear.
- These are separated by surfaces (or regions) where $|J| = 0$ and therefore $\nabla \times B = 0$.
- As a guide field is varied, if a null drifts onto a singular surface, it necessarily annihilates against a null of opposite type from the other side.
- Conversely, nulls can only be created (by varying a guide field) in pairs at singular surfaces.
- $|B|$ reaches a local extremum along the path of an isotrope as it crosses a singular surface.

Nulls Near Singular Surfaces

- Using the identities $J \cdot v = Bv \cdot \nabla \ln |B|$ (globally) and $v \cdot \nabla |B| = 0$ (at the surface), $J \cdot v = 0$.
- The isotrope field is a zero eigenvector of J at the singular surface.
- Eigenvectors and eigenvalues vary smoothly in R^3, so near the surface there is an eigenvector $\approx v$ for the eigenvalue that vanishes at the singular surface.
- On a path connecting two nulls across a singular surface, the eigenvalue that changes sign along the path corresponds to a fan plane eigenvector for each null.
- As nulls merge due to a varying guide field, **they move towards each other along an isotrope on their fan planes.**

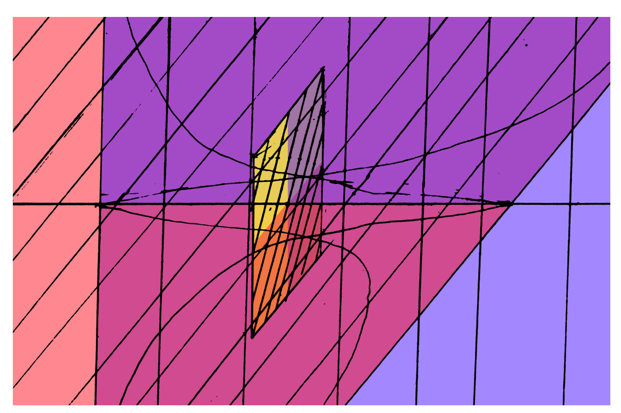


Figure: Two nulls of opposite type merging. The fan plane of the null on the left is in red, and the fan plane of the null on the right is in blue. Each null's fan plane terminates at the other's spine. The singular surface is shown in yellow. The isotrope the nulls are following is a horizontal straight line.

Further Directions

Isotrope Lines as Constraints

- Application: Null search algorithm via integration of isotropic field, as opposed to previously known method via volume bisection and calculation of index. [6])
- Classification of null bifurcation by study of intersection of isotrope lines with isosurfaces of $|B|$.

Charts on S^2 and Straightening Fields

- The derivation of the isotrope field did not assume any particular coordinates or area form on S^2.
- These can be chosen arbitrarily, resulting in **different isotrope fields**.
- In particular, consider a change of basis that diagonalizes J at a null. Using the standard area form on S^2 in these coordinates results in the fan plane coinciding with the equator of S^2 and the spine coinciding with the poles, regardless of the tilt of the fan plane relative to the spine in R^3.
- A proper change of coordinates might also be used to straighten an inhomogeneous externally applied fields, making the results for homogeneous guide fiels more broadly applicable.

Isolines of Eigenvectors of J

- Where J has real eigenvalues, $(\nabla \times B = 0)$, eigenvectors of J are in $\mathbb{R}P^2$, which is a 2 dimensional manifold, so there are lines along which an eigenvector is constant.

Topological Defects and Connections to Other Fields

- Framing nulls as topological defects (in this case characterized by the second homotopy group of the sphere) raises the possibility of connecting to or using results from other fields with similar topological phenomena (defects in nematic liquid crystals[7] etc).
- Topologically invariant structures have previously been found in other electromagnetic phenomena, lending the possibility of relating these results to broader theories.[8, 9]

References

[1] Yun-Tung Lau and John M Finn.
Three-dimensional kinematic reconnection in the presence of field nulls and closed field lines.
The Astrophysical Journal, 350:672–691, 1990.

[2] CE Parnell, JM Smith, T Neukirch, and ER Priest.
The structure of three-dimensional magnetic neutral points.
Physics of Plasmas (1994-present), 3(3):759–770, 1996.

[3] John M Greene.
Geometrical properties of three-dimensional reconnecting magnetic fields with nulls.
Journal of Geophysical Research: Space Physics (1978–2012), 93(A8):8583–8590, 1988.

[4] Luitzen Egbertus Jan Brouwer.
Über abbildung von mannigfaltigkeiten.
Mathematische Annalen, 71(1):97–115, 1911.

[5] Christopher Berg Smiet, Simon Candelaresi, and Dirk Bouwmeester.
Ideal relaxation of the hopf fibration.
Physics of Plasmas, 24(7):072110, 2017.

[6] John M Greene.
Locating three-dimensional roots by a bisection method.
Journal of Computational Physics, 98(2):194–198, 1992.

[7] Gareth P. Alexander, Bryan Gin-ge Chen, Elisabetta A. Matsumoto, and Randall D. Kamien.
Colloquium: Disclination loops, point defects, and all that in nematic liquid crystals.
Rev. Mod. Phys., 84:497–514, Apr 2012.

[8] Amy Thompson, Alexander Wickes, Joe Swearngin, and Dirk Bouwmeester.
Classification of electromagnetic and gravitational hopfions by algebraic type.
Journal of Physics A: Mathematical and Theoretical, 48, 2014.

[9] A F Ranada.
Topological electromagnetism.
Journal of Physics A: Mathematical and General, 25(6):1621–1641, 1992.

Source

This code uses the BlenDaViz library: `https://github.com/SimonCan/BlenDaViz` to generate the images.

To make the entire poster (including the images):

```
$ > git clone --recurse-submodules
https://github.com/BenYI/Topological_Nulls/
$ > cd Topological_Nulls
$ > make
```