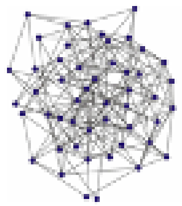
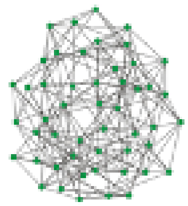
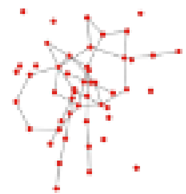
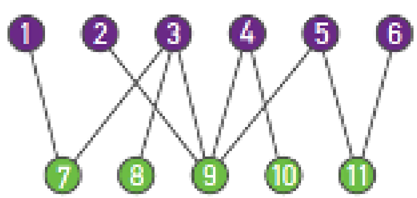




HOMEWORK ASSIGNMENT 1:
MATHEMATICS OF NETWORKS

PHYSICS 5300
NETWORKS THEORY SPRING 2021
PROFESSOR GABRIEL CWILICH

PROBLEM 1

Consider this bipartite network



a) Construct its adjacency matrix. Why is it a block-diagonal matrix?

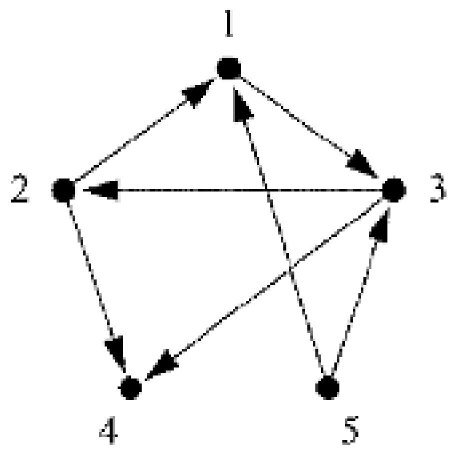
b) Construct the adjacency matrix of its two projections, on the purple and on the green nodes, respectively.

c) Calculate the average degree of the purple nodes and the average degree of the green nodes in the bipartite network.

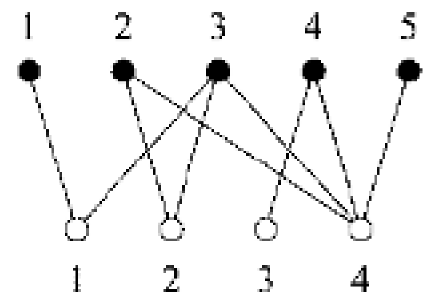
d) Calculate the average degree in each of the two network projections.
Is it surprising that the values are different from those obtained in point (c)?

PROBLEM 2

Consider the following two networks:



(a)



(b)

Network (a) is a directed network Network (b) is undirected but bipartite. Write down:

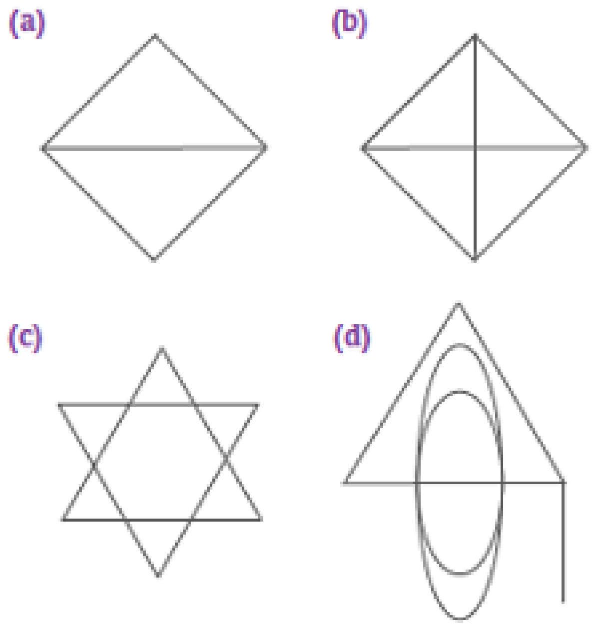
a) the adjacency matrix of network (a);

b) the cocitation matrix of network (a);

c) the incidence matrix of network (b);

d) the projection matrix (as discussed in class) for the projection of network (b) onto its black vertices.

PROBLEM 3



Which of the icons in this image can be drawn without raising your pencil from the paper, and without drawing any line more than once?

Why?

For each of the four, if it can be drawn as suggested, show how would you do it (show an actual path)

If it cannot, show which is the minimal number of edges that you can remove and obtain a figure than can be drawn in such a way.

PROBLEM 4

a) Consider a bipartite network, with its two types of vertices , and suppose that there are n_1 vertices of type 1 and n_2 vertices of type 2.

Show that the average degrees c_1 and c_2 of the two types are related by:

$$c_2 = \frac{n_1}{n_2} c_1$$

b) (quite easy) Show that a 3-regular graph (a graph in which **all** the nodes have degree exactly equal to 3) **must** have an even number of nodes.

PROBLEM 5

Let $\mathbf{A}$ be the adjacency matrix of an undirected network and $\mathbf{1}$ be the column vector whose elements are all ones.

In terms of these quantities write expressions for:

a) the vector $\mathbf{k}$ whose elements are the degrees k_i of the vertices;

b) the number m of edges in the network;

c) the matrix $\mathbf{N}$ whose element N_{ij} is equal to the number of common neighbors of vertices i and j;

d) the total number of triangles in the network (which we will see play a very big role), where a triangle means three vertices, each connected by edges to both of the others.

PROBLEM 6

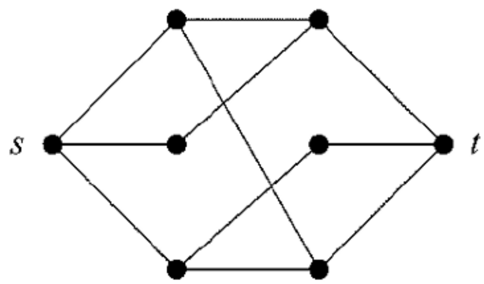
Consider any three nodes A, B and C in a network.

The edge connectivity of A and B is some number x. The edge connectivity of B and C is some other number y, with $y < x$.

What is the edge connectivity of A and C and **why**?

PROBLEM 7

What is the size k of the minimum vertex cut set between s and t in this network?



Prove your result by

a) finding one possible cut set of size k, and
b) one possible set of k independent paths between s and t.

Why do these two actions constitute a proof that the minimum cut set has size k? Isn't one of them enough?

PROBLEM 8

Consider the set of all paths from vertex a to vertex b on an undirected graph with adjacency matrix $\mathbf{A}$. Let us give each path a weight equal to α^r, where r is the length of the path, as we did in our alternative derivation of the Katz centrality in the lecture.

a) Show that the sum of the weights of all the paths from a to b with that weight is given by $\mathbf{M}_{ab}$, which is the ab element of the matrix

$$M = \frac{1}{\mathbf{I} - \alpha \mathbf{A}}$$

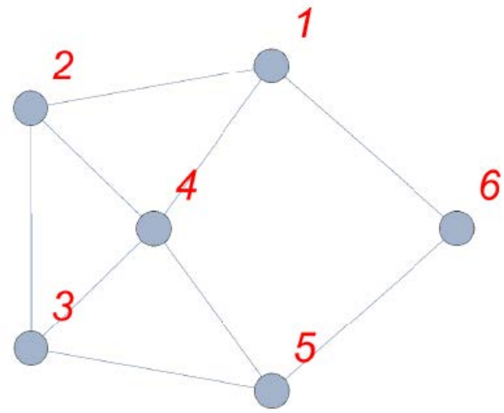
where $\mathbf{I}$ is the identity matrix.

b) What condition must a satisfy for the sum to converge?

c) Very interestingly, using the properties we learned of the connection between the elements of the matrix $\mathbf{A}^n$ and the number of walks among the corresponding nodes show that the length l_{ab} of the **geodesic** path from a to b, if it exists, is given by

$$M = \lim_{\alpha \to 0} \frac{\partial \log \mathbf{M}_{ab}}{\partial \log \alpha}$$

PROBLEM 9



Using the theory of the graph Laplacian analyze the following problem of diffusion in the network shown (If you prefer to use a different network it is ok but look at the note below)

Just a reminder that the method discussed in class involves

- finding the graph Laplacian.
- finding its eigenvalues and eigenvectors
- making sure they are normalized and orthogonal to each other. This last thing does not happen automatically if an eigenvalue is repeated.
- expressing the initial state you want to study as a combination of those eigenvectors.
- then writing the time evolution of that state.

The questions I want you to look at are the following:

(1) At which node should I inject one unit of some commodity so that in the minimum amount of time each node acquires at least 10% of it?
(2) How long will that time be?

(3) At which node should I inject one unit of that commodity so that it takes the longest amount of time until each node acquires at least 10% of it ?
(4) How long will that time be?
(5) Do your answers coincide with your intuition? Why or why not?

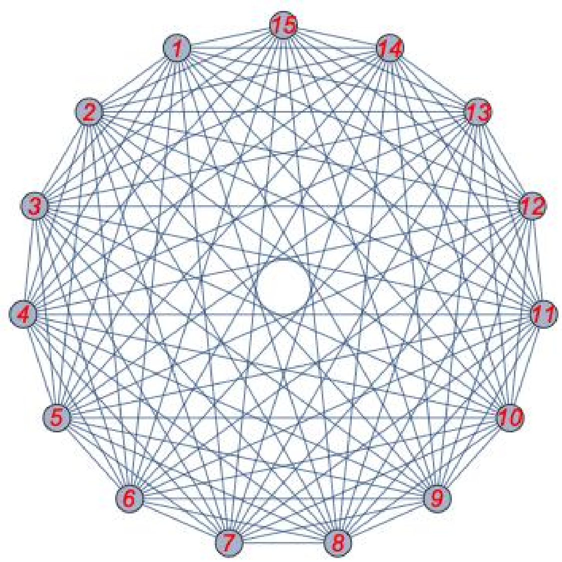
(6) Starting from all the commodity in a certain find how long will it take until the commodity is shared among all the nodes equally with a deviation of no more than 1% of the limiting value ?

Note:

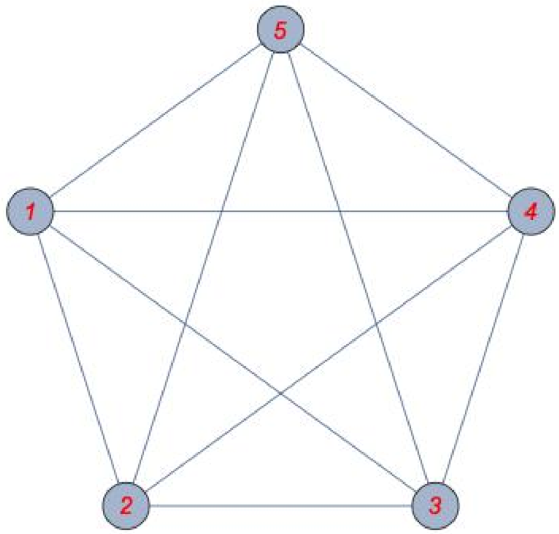
If you do not want to use this network you can create a different one of your choice but a) It should have not less than six nodes b) it should not be so simple and symmetric that all the nodes are equivalent. That would be too simple and anyways you will explore that possibility in the next problem.

PROBLEM 10

Consider the following graph:



Before you think I went mad, understand that this is just the complete graph for $n = 15$ and to see what I mean this is the complete graph for $n = 5$ for example



Assume that you inject a commodity of value 1 at a certain arbitrary node in the complete graph of $n = 15$ at time $t = 0$. Find out what will be the functions $f_i(t)$ that tell you you the amount of commodity at node i at time t.

You should realize that symmetry plays a big role in this problem. For example you do not need to find fifteen different function $f_i(t)$, but just two of them. Why?

Also if for example you solve the problem for a complete graph of , say, $n = 2, n = 3$ and perhaps $n = 4$ you will soon realize the way the Laplacian goes and what the eigenvalues for the case $n = 15$ are.

Finding all the 15 eigenvectors might be extremely tedious but actually you do not need that, since you know you are only looking for two different functions, and you know exactly their value at time $t = 0$ of course, and what they should be when $t \to \infty$, and that is enough to find them.

PROBLEM 11

For the same network you used in problem 9, study now the problem of a random walker in it.

(1) Start the walker at one given node (in other words P_o is the vector where all the entries are 0 except one of them which is the number 1). This would be for a walker starting at node 3 for example: $P_o = \{0, 0, 1, 0, 0, 0\}$.

(2) Find out by hand (not using any matrices) the distribution of probabilities P_1 after one step, the distribution of probabilities P_2 after two steps, and after three steps, P_3.

(3) Repeat this but starting from a different node. Check that the evolution of the system is different depending on where you start.

(4) Now, using the techniques discussed in class calculate the distribution after many many steps (say 50). You cannot do this by hand, obviously. You will need to get the matrix that gives the evolution of the system, find its eigenvalues and eigenvectors, express the initial state in terms of those eigenvectors, etc. Remember it is **not** the same matrix you used in problem 9, although it is the same network.

(5) Check that for the cases of $t = 1, 2, 3$ you get the same results that you obtained "by hand" . Check also that for t big enough the result is the same no matter where the walker starts, as discussed in class