

## Column space and nullspace

In this lecture we continue to study subspaces, particularly the column space and nullspace of a matrix.

### Review of subspaces

A vector space is a collection of vectors which is closed under linear combinations. In other words, for any two vectors  $\mathbf{v}$  and  $\mathbf{w}$  in the space and any two real numbers  $c$  and  $d$ , the vector  $c\mathbf{v} + d\mathbf{w}$  is also in the vector space. A subspace is a vector space contained inside a vector space.

A plane  $P$  containing  $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$  and a line  $L$  containing  $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$  are both sub-

spaces of  $\mathbb{R}^3$ . The union  $P \cup L$  of those two subspaces is generally not a subspace, because the sum of a vector in  $P$  and a vector in  $L$  is probably not contained in  $P \cup L$ . The intersection  $S \cap T$  of two subspaces  $S$  and  $T$  is a subspace. To prove this, use the fact that both  $S$  and  $T$  are closed under linear combinations to show that their intersection is closed under linear combinations.

### Column space of $A$

The *column space* of a matrix  $A$  is the vector space made up of all linear combinations of the columns of  $A$ .

#### Solving $A\mathbf{x} = \mathbf{b}$

Given a matrix  $A$ , for what vectors  $\mathbf{b}$  does  $A\mathbf{x} = \mathbf{b}$  have a solution  $\mathbf{x}$ ?

$$\text{Let } A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}.$$

Then  $A\mathbf{x} = \mathbf{b}$  does not have a solution for every choice of  $\mathbf{b}$  because solving  $A\mathbf{x} = \mathbf{b}$  is equivalent to solving four linear equations in three unknowns. If there is a solution  $\mathbf{x}$  to  $A\mathbf{x} = \mathbf{b}$ , then  $\mathbf{b}$  must be a linear combination of the columns of  $A$ . Only three columns cannot fill the entire four dimensional vector space – some vectors  $\mathbf{b}$  cannot be expressed as linear combinations of columns of  $A$ .

Big question: what  $\mathbf{b}$ 's allow  $A\mathbf{x} = \mathbf{b}$  to be solved?

A useful approach is to choose  $\mathbf{x}$  and find the vector  $\mathbf{b} = A\mathbf{x}$  corresponding to that solution. The components of  $\mathbf{x}$  are just the coefficients in a linear combination of columns of  $A$ .

The system of linear equations  $A\mathbf{x} = \mathbf{b}$  is *solvable* exactly when  $\mathbf{b}$  is a vector in the *column space* of  $A$ .

For our example matrix  $A$ , what can we say about the column space of  $A$ ? Are the columns of  $A$  *independent*? In other words, does each column contribute something new to the subspace?

The third column of  $A$  is the sum of the first two columns, so does not add anything to the subspace. The column space of our matrix  $A$  is a two dimensional subspace of  $\mathbb{R}^4$ .

## Nullspace of $A$

The *nullspace* of a matrix  $A$  is the collection of all solutions  $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$  to the equation  $A\mathbf{x} = \mathbf{0}$ .

The column space of the matrix in our example was a subspace of  $\mathbb{R}^4$ . The nullspace of  $A$  is a subspace of  $\mathbb{R}^3$ . To see that it's a vector space, check that any sum or multiple of solutions to  $A\mathbf{x} = \mathbf{0}$  is also a solution:  $A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{0} + \mathbf{0}$  and  $A(c\mathbf{x}) = cA\mathbf{x} = c(\mathbf{0})$ .

In the example:

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

the nullspace  $N(A)$  consists of all multiples of  $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ ; column 1 plus column 2 minus column 3 equals the zero vector. This nullspace is a line in  $\mathbb{R}^3$ .

## Other values of $\mathbf{b}$

The solutions to the equation:

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

do not form a subspace. The zero vector is not a solution to this equation. The set of solutions forms a line in  $\mathbb{R}^3$  that passes through the points  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  and

$$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \text{ but not } \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

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