Exercises on column space and nullspace

Problem 6.1: (3.1 #30. *Introduction to Linear Algebra:* Strang) Suppose $\mathbf{S}$ and $\mathbf{T}$ are two subspaces of a vector space $\mathbf{V}$.

a) **Definition:** The sum $\mathbf{S} + \mathbf{T}$ contains all sums $\mathbf{s} + \mathbf{t}$ of a vector $\mathbf{s}$ in $\mathbf{S}$ and a vector $\mathbf{t}$ in $\mathbf{T}$. Show that $\mathbf{S} + \mathbf{T}$ satisfies the requirements (addition and scalar multiplication) for a vector space.

b) If $\mathbf{S}$ and $\mathbf{T}$ are lines in $\mathbf{R}^m$, what is the difference between $\mathbf{S} + \mathbf{T}$ and $\mathbf{S} \cup \mathbf{T}$? That union contains all vectors from $\mathbf{S}$ and $\mathbf{T}$ or both. Explain this statement: *The span of* $\mathbf{S} \cup \mathbf{T}$ *is* $\mathbf{S} + \mathbf{T}$.

Solution:

a) Let $\mathbf{s}, \mathbf{s}'$ be vectors in $\mathbf{S}$, let $\mathbf{t}, \mathbf{t}'$ be vectors in $\mathbf{T}$, and let c be a scalar. Then

$$(\mathbf{s} + \mathbf{t}) + (\mathbf{s}' + \mathbf{t}') = (\mathbf{s} + \mathbf{s}') + (\mathbf{t} + \mathbf{t}') \text{ and } c(\mathbf{s} + \mathbf{t}) = c\mathbf{s} + c\mathbf{t}.$$

Thus $\mathbf{S} + \mathbf{T}$ is closed under addition and scalar multiplication; in other words, it satisfies the two requirements for a vector space.

b) If $\mathbf{S}$ and $\mathbf{T}$ are distinct lines, then $\mathbf{S} + \mathbf{T}$ is a plane, whereas $\mathbf{S} \cup \mathbf{T}$ is only the two lines. The span of $\mathbf{S} \cup \mathbf{T}$ is the set of all combinations of vectors in this union of two lines. In particular, it contains all sums $\mathbf{s} + \mathbf{t}$ of a vector $\mathbf{s}$ in $\mathbf{S}$ and a vector $\mathbf{t}$ in $\mathbf{T}$, and these sums form $\mathbf{S} + \mathbf{T}$.

Since $\mathbf{S} + \mathbf{T}$ contains both $\mathbf{S}$ and $\mathbf{T}$, it contains $\mathbf{S} \cup \mathbf{T}$. Further, $\mathbf{S} + \mathbf{T}$ is a vector space. So it contains all combinations of vectors in itself; in particular, it contains the span of $\mathbf{S} \cup \mathbf{T}$. Thus the span of $\mathbf{S} \cup \mathbf{T}$ is $\mathbf{S} + \mathbf{T}$.

Problem 6.2: (3.2 #18.) The plane $x - 3y - z = 12$ is parallel to the plane $x - 3y - x = 0$. One particular point on this plane is $(12, 0, 0)$. All points on the plane have the form (fill in the first components)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} \\ 0 \\ 1 \end{bmatrix}.$$

Solution: The equation $x = 12 + 3y + z$ says it all:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \left(= \begin{bmatrix} 12 + 3y + z \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} \boxed{12} \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} \boxed{3} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} \boxed{1} \\ 0 \\ 1 \end{bmatrix}.$$

Problem 6.3: (3.2 #36.) How is the nullspace $\mathbf{N}(C)$ related to the spaces $\mathbf{N}(A)$ and $\mathbf{N}(B)$, if $C = \begin{bmatrix} A \\ B \end{bmatrix}$?

Solution: $\mathbf{N}(C) = \mathbf{N}(A) \cap \mathbf{N}(B)$ contains all vectors that are in both nullspaces:

$$C\mathbf{x} = \begin{bmatrix} A\mathbf{x} \\ B\mathbf{x} \end{bmatrix} = 0$$

if and only if $A\mathbf{x} = 0$ and $B\mathbf{x} = 0$.