## Solving Ax = 0: pivot variables, special solutions

We have a definition for the column space and the nullspace of a matrix, but how do we compute these subspaces?

## Computing the nullspace

The *nullspace* of a matrix A is made up of the vectors  $\mathbf{x}$  for which  $A\mathbf{x} = \mathbf{0}$ . Suppose:

$$A = \left[ \begin{array}{rrrr} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{array} \right].$$

(Note that the columns of this matrix A are not independent.) Our algorithm for computing the nullspace of this matrix uses the method of elimination, despite the fact that A is not invertible. We don't need to use an augmented matrix because the right side (the vector  $\mathbf{b}$ ) is  $\mathbf{0}$  in this computation.

The row operations used in the method of elimination don't change the solution to  $A\mathbf{x} = \mathbf{b}$  so they don't change the nullspace. (They do affect the column space.)

The first step of elimination gives us:

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix}.$$

We don't find a pivot in the second column, so our next pivot is the 2 in the third column of the second row:

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

The matrix *U* is in *echelon* (staircase) form. The third row is zero because row 3 was a linear combination of rows 1 and 2; it was eliminated.

The *rank* of a matrix A equals the number of pivots it has. In this example, the rank of A (and of U) is 2.

## **Special solutions**

Once we've found U we can use back-substitution to find the solutions x to the equation Ux = 0. In our example, columns 1 and 3 are *pivot columns* containing pivots, and columns 2 and 4 are *free columns*. We can assign any value to  $x_2$  and  $x_4$ ; we call these *free variables*. Suppose  $x_2 = 1$  and  $x_4 = 0$ . Then:

$$2x_3 + 4x_4 = 0 \implies x_3 = 0$$

and:

$$x_1 + 2x_2 + 2x_3 + 2x_4 = 0 \implies x_1 = -2.$$

So one solution is  $\mathbf{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$  (because the second column is just twice the

first column). Any multiple of this vector is in the nullspace.

Letting a different free variable equal 1 and setting the other free variables equal to zero gives us other vectors in the nullspace. For example:

$$\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

has  $x_4 = 1$  and  $x_2 = 0$ . The nullspace of A is the collection of all linear combinations of these "special solution" vectors.

The rank r of A equals the number of pivot columns, so the number of free columns is n-r: the number of columns (variables) minus the number of pivot columns. This equals the number of special solution vectors and the dimension of the nullspace.

## Reduced row echelon form

By continuing to use the method of elimination we can convert *U* to a matrix *R* in *reduced row echelon form* (rref form), with pivots equal to 1 and zeros above and below the pivots.

$$U = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R.$$

By exchanging some columns, R can be rewritten with a copy of the identity matrix in the upper left corner, possibly followed by some free columns on the right. If some rows of A are linearly dependent, the lower rows of the matrix R will be filled with zeros:

$$R = \left[ \begin{array}{cc} I & F \\ 0 & 0 \end{array} \right].$$

(Here I is an r by r square matrix.)

If *N* is the *nullspace matrix*  $N = \begin{bmatrix} -F \\ I \end{bmatrix}$  then RN = 0. (Here *I* is an n - r by n - r square matrix and 0 is an *m* by n - r matrix.) The columns of *N* are the special solutions.

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