

## Solving $Ax = b$ : row reduced form $R$

When does  $Ax = b$  have solutions  $x$ , and how can we describe those solutions?

### Solvability conditions on $b$

We again use the example:

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}.$$

The third row of  $A$  is the sum of its first and second rows, so we know that if  $Ax = b$  the third component of  $b$  equals the sum of its first and second components. If  $b$  does not satisfy  $b_3 = b_1 + b_2$  the system has no solution. If a combination of the rows of  $A$  gives the zero row, then the same combination of the entries of  $b$  must equal zero.

One way to find out whether  $Ax = b$  is solvable is to use elimination on the augmented matrix. If a row of  $A$  is completely eliminated, so is the corresponding entry in  $b$ . In our example, row 3 of  $A$  is completely eliminated:

$$\begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{bmatrix}.$$

If  $Ax = b$  has a solution, then  $b_3 - b_2 - b_1 = 0$ . For example, we could choose

$$b = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}.$$

From an earlier lecture, we know that  $Ax = b$  is solvable exactly when  $b$  is in the column space  $C(A)$ . We have these two conditions on  $b$ ; in fact they are equivalent.

### Complete solution

In order to find all solutions to  $Ax = b$  we first check that the equation is solvable, then find a particular solution. We get the complete solution of the equation by adding the particular solution to all the vectors in the nullspace.

#### A particular solution

One way to find a particular solution to the equation  $Ax = b$  is to set all free variables to zero, then solve for the pivot variables.

For our example matrix  $A$ , we let  $x_2 = x_4 = 0$  to get the system of equations:

$$\begin{aligned} x_1 + 2x_3 &= 1 \\ 2x_3 &= 3 \end{aligned}$$

which has the solution  $x_3 = 3/2$ ,  $x_1 = -2$ . Our particular solution is:

$$\mathbf{x}_p = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}.$$

### Combined with the nullspace

The general solution to  $A\mathbf{x} = \mathbf{b}$  is given by  $\mathbf{x}_{\text{complete}} = \mathbf{x}_p + \mathbf{x}_n$ , where  $\mathbf{x}_n$  is a generic vector in the nullspace. To see this, we add  $A\mathbf{x}_p = \mathbf{b}$  to  $A\mathbf{x}_n = \mathbf{0}$  and get  $A(\mathbf{x}_p + \mathbf{x}_n) = \mathbf{b}$  for every vector  $\mathbf{x}_n$  in the nullspace.

Last lecture we learned that the nullspace of  $A$  is the collection of all combinations of the special solutions  $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$ . So the complete solution to the equation  $A\mathbf{x} = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$  is:

$$\mathbf{x}_{\text{complete}} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix},$$

where  $c_1$  and  $c_2$  are real numbers.

The nullspace of  $A$  is a two dimensional subspace of  $\mathbb{R}^4$ , and the solutions to the equation  $A\mathbf{x} = \mathbf{b}$  form a plane parallel to that through  $\mathbf{x}_p = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$ .

## Rank

The rank of a matrix equals the number of pivots of that matrix. If  $A$  is an  $m$  by  $n$  matrix of rank  $r$ , we know  $r \leq m$  and  $r \leq n$ .

### Full column rank

If  $r = n$ , then from the previous lecture we know that the nullspace has dimension  $n - r = 0$  and contains only the zero vector. There are no free variables or special solutions.

If  $A\mathbf{x} = \mathbf{b}$  has a solution, it is unique; there is either 0 or 1 solution. Examples like this, in which the columns are independent, are common in applications.

We know  $r \leq m$ , so if  $r = n$  the number of columns of the matrix is less than or equal to the number of rows. The row reduced echelon form of the

matrix will look like  $R = \begin{bmatrix} I \\ 0 \end{bmatrix}$ . For any vector  $\mathbf{b}$  in  $\mathbb{R}^m$  that's not a linear combination of the columns of  $A$ , there is no solution to  $A\mathbf{x} = \mathbf{b}$ .

### Full row rank

If  $r = m$ , then the reduced matrix  $R = \begin{bmatrix} I & F \end{bmatrix}$  has no rows of zeros and so there are no requirements for the entries of  $\mathbf{b}$  to satisfy. The equation  $A\mathbf{x} = \mathbf{b}$  is solvable for every  $\mathbf{b}$ . There are  $n - r = n - m$  free variables, so there are  $n - m$  special solutions to  $A\mathbf{x} = \mathbf{0}$ .

### Full row and column rank

If  $r = m = n$  is the number of pivots of  $A$ , then  $A$  is an invertible square matrix and  $R$  is the identity matrix. The nullspace has dimension zero, and  $A\mathbf{x} = \mathbf{b}$  has a unique solution for every  $\mathbf{b}$  in  $\mathbb{R}^m$ .

### Summary

If  $R$  is in row reduced form with pivot columns first (rref), the table below summarizes our results.

	$r = m = n$	$r = n < m$	$r = m < n$	$r < m, r < n$
$R$	$I$	$\begin{bmatrix} I \\ 0 \end{bmatrix}$	$\begin{bmatrix} I & F \end{bmatrix}$	$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$
# solutions to $A\mathbf{x} = \mathbf{b}$	1	0 or 1	infinitely many	0 or infinitely many

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