$$\frac{S_n - a_n}{a_n} \xrightarrow{\mathbb{P}} 0 \quad \text{WLLN}$$
$$\xrightarrow{D}$$

$$\frac{S_n - a_n}{b_n} \xrightarrow{a.s.} 0 \quad \text{SLLN} \quad \left(\begin{array}{c} \text{If } X_i\text{'s i.i.d.} \\ \frac{S_n}{n} \xrightarrow{a.s.} \mu \end{array} \right)$$

- why we need SLLN
- Truncation & Chebyshev Ieq useful but not enough
  $\Rightarrow$ Borel Cantelli Lemma.

Lemma. $\mathbb{P}(\{\omega \in \Omega : |X_n(\omega) - X(\omega)| > \varepsilon\})$

$A_n(\varepsilon) = \{|X_n - X| > \varepsilon\}$

$B_m(\varepsilon) = \bigcup_{n=m}^{\infty} A_n(\varepsilon)$.

① $X_n \xrightarrow{a.s.} X$ iff $\lim_{m\to\infty} \mathbb{P}(B_m(\varepsilon)) \to 0 \quad \forall \varepsilon > 0$

② $X_n \xrightarrow{a.s.} X$ if $\sum_{n=1}^{\infty} \mathbb{P}(A_n(\varepsilon)) < \infty \quad \forall \varepsilon > 0$

③ $X_n \xrightarrow{a.s.} X$ implies. $X_n \xrightarrow{\mathbb{P}} X$

If $\sum_{n=1}^{\infty} \mathbb{P}(A_n(\varepsilon)) < \infty$ then. $\boxed{\lim_{m\to\infty} \mathbb{P}(B_m(\varepsilon)) = 0}$

$\Downarrow$

$\mathbb{P}(\underbrace{A_n(\varepsilon) \text{ i.o.}}) = 0$

$\Downarrow$

$\{\omega : \omega \in A_n(\varepsilon) \text{ for infinite many } n\}$

$\mathbb{P}(B_m(\varepsilon)) \le \sum_{n=m}^{\infty} \mathbb{P}(A_n(\varepsilon))$

$$= \mathbb{P}\Big(\bigcap_{m=1}^{\infty} \underbrace{\bigcup_{n=m}^{\infty} A_n(\varepsilon)}_{B_m(\varepsilon)}\Big) = 0$$

Arbitrary events in $\mathcal{F}$ $(\Omega, \mathcal{F}, \mathbb{P})$ First B-C. Lemma.

Forget about r.v. now.

Borel – Cantelli Lemma. Let $A_1, A_2, \cdots \in \mathcal{F}$

① $\mathbb{P}(A_n \text{ i.o.}) = 0$ if. $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$

②. $\mathbb{P}(A_n \text{ i.o.}) = ?\ 1$ if. $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$.

and $A_1 \cdots$ are independent.

e.g.

Let.

$A_1 = A_2 = \cdots = A$

$\mathbb{P}(A) > 0$

$\mathbb{P}(A_n \text{ i.o})$

$= \mathbb{P}(A) > 0$

Δ Help us to deal with. liminf & limsup.

Δ Help us to pf $\xrightarrow{a.s.}$

Pf. ① $\mathbb{P}(A_n \text{ i.o.}) = \mathbb{P}(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n) = \lim_{m\to\infty} \mathbb{P}(\bigcup_{n=m}^{\infty} A_n)$

$\leq \lim_{m\to\infty} \sum_{n=m}^{\infty} \mathbb{P}(A_n) = 0$

② $\mathbb{P}(A_n \text{ i.o.}) = 1 \Leftrightarrow \mathbb{P}((A_n \text{ i.o.})^c) = 0$

$$\mathbb{P}(\bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} A_m^c) = \lim_{m\to\infty} \mathbb{P}(\bigcap_{n=m}^{\infty} A_m^c)$$

$$= \lim_{m\to\infty} \lim_{r\to\infty} \mathbb{P}(\bigcap_{n=m}^{r} A_n^c)$$ 有限交 无限并

$$= \lim_{m\to\infty} \lim_{r\to\infty} \prod_{n=m}^{r} (1 - \mathbb{P}(A_n))$$

$$= \lim_{m\to\infty} \prod_{n=m}^{\infty} (1 - \mathbb{P}(A_n))$$

$1+x \leq e^x$

$x \geq \ln(1+x)$

$$= \lim_{m\to\infty} \prod_{n=m}^{\infty} (1 - \mathbb{P}(A_n))$$

$1 - x \leq e^{-x}$

$1+x \leq e^x$

$$\leq \lim_{m\to\infty} \prod_{n=m}^{\infty} e^{-\mathbb{P}(A_n)}$$

$$= \underbrace{\lim_{m\to\infty} e^{-\boxed{\sum_{n=m}^{\infty} \mathbb{P}(A_n)}}}_{=0}, \quad = \infty.$$

Since $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$.

A Toy application of BC-lemma II.

(Infinite Monkey Problem)

Alphabet size N

String of letters. S with length m

name of the string is S.

Let E_k = event that the m-string starting at position k is S

Conclusion. $\mathbb{P}(E_k \text{ i.o.}) = 1$

eg $S = ab$. $E_1\ E_2$ cannot happen simultaneously.

$E_1\ E_2, \cdots$ are not independent

Sparsification.

Select a subsequence.

if i.o. for this subsequence then also for the original event.

E_{mj+1} $j = 0, 1, \cdots$

E_1 E_{m+1} E_{2m+1} $\cdots$ $\{E_{mj+1} \text{ i.o.}\} \subset \{E_k \text{ i.o.}\}$

$\mathbb{P}(E_{mj+1} \text{ i.o.}) = 1 \Rightarrow \mathbb{P}(E_k \text{ i.o.}) = 1$

in j

$\sim$ fixed number.

$\sum_{j=1}^{\infty} \mathbb{P}(E_{mj+1}) = \sum_{j=1}^{\infty} \left(\frac{1}{N}\right)^m = \infty.$

△ B-C. ① is useful in pf. a.s.

when A_n have big overlap.

maybe $\mathbb{P}(A_n \text{ i.o.}) = 0$ but still $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$.

$\Rightarrow$ Sparsification. Chose a subsequence.

Application of BC-lemma I

$a_1 \cdots a_n \cdots$

$a_n \to a \Leftrightarrow$ for any subseq $a_{n(m)}$. $\exists$ a further subseq $a_{n(m_k)}$ so that $a_{n(m_k)} \to a$

Thm. $X_n \xrightarrow{\mathbb{P}} X$

$\Leftrightarrow$ For $\forall$ subsequence $X_{n(m)}$. there is a further sub sequence $X_{n(m_k)}$ so that $X_{n(m_k)} \xrightarrow{a.s.} X$

pf. ($\Rightarrow$)

Fix a subsequence $n(m)$

Aim: Find a subseq $\to X$

Let ε_k be a seq of positive numbers that decrease to 0

For any k, there exists $n(m_k) > n(m_{k-1})$ so that

$$\mathbb{P}(\underbrace{|X_{n(m_k)} - X| > \varepsilon_k}_{\text{depends on } \varepsilon_k}) \le 2^{-k}$$

$\Downarrow$

$\sum_{k=1}^{\infty} \mathbb{P}(|X_{n(m_k)} - X| > \varepsilon_k) < \infty \Leftarrow$ BC Lemma I

$\therefore \mathbb{P}(|X_{n(m_k)} - X| > \varepsilon_k \text{ i.o.}) = 0$

original def. $\forall \varepsilon > 0$ $\mathbb{P}(|X_{n(m_k)} - X| > \varepsilon \text{ i.o.}) = 0$

Since $\varepsilon_k \downarrow 0$ $\forall$ fixed $\varepsilon > 0$.

$\varepsilon_k < \varepsilon$ for sufficiently large $k \geq k_\varepsilon$.

$$\{|X_{n(m_k)} - X| > \varepsilon\} \subset \{|X_{n(m_k)} - X| > \varepsilon_k\}$$

$k \geq k_\varepsilon$

$$\Rightarrow \{|X_{n(m_k)} - X| > \varepsilon \text{ i.o.}\} \subset \{|X_{n(m_k)} - X| \underline{> \varepsilon_k \text{ i.o.}}\}.$$

when k is large

this is easy to occur.

$\Leftarrow$

For any $X_{n(m)}$ $\exists$ $X_{n(m_k)} \xrightarrow{a.s.} X$

$\Rightarrow X_{n(m_k)} \xrightarrow{\mathbb{P}} X \Rightarrow \underline{\mathbb{P}(|X_{n(m_k)} - X| > \varepsilon)} \to 0$

let $a_n = \mathbb{P}(|X_n - X| > \varepsilon) \xrightarrow{?} 0$

For any $a_{n(m)}$ $\exists$ $\underline{a_{n(m_k)}} \to 0$

$X_n \xrightarrow{a.s.} X$

DCT : If $X_n \xrightarrow{\mathbb{P}} X$ and $|X_n| \leq Y$ $\mathbb{E}Y < \infty$

then, $\mathbb{E}X_n \to \mathbb{E}X$

pf. let. $a_n = \mathbb{E}X_n$.

$\left.\begin{array}{l}\text{for any } a_{n(m)} \ \exists \text{ a further sub seq} \\ a_{n(m_k)} = \mathbb{E}X_{n(m_k)} \to a = \mathbb{E}X\end{array}\right\}$ Aim.

If. $X_n \xrightarrow{\mathbb{P}} X$ $\forall$ $X_{n(m)}$ $\exists$ $X_{n(m_k)} \xrightarrow{a.s.} X$

$|X_{n(m_k)}| \leq Y$ apply DCT (a.s). to $X_{n(m_k)}$

$\mathbb{E}X_{n(m_k)} \xrightarrow{a.s.} \mathbb{E}X$