Weak Convergence / Convergence in Distribution

Def. A seq of d.f. F_n converges weakly to a d.f. F.

$(F_n \Rightarrow F)$ if $F_n(x) \to F(x)$ $\forall$ continuity point of $F(x)$

Def. A seq of r.v. X_n converges weakly / in distribution to a r.v. X

to a r.v. X $(X_n \Rightarrow X$ or $X_n \xrightarrow{D} X)$ if $F_n \Rightarrow F$

we also write $X_n \Rightarrow F$

eg. (Glivenko - Cantelli thm)

Let $X_1\ X_2 \cdots$ i.i.d. with d.f. F for almost every ω.

$$F_n(x, \underset{\substack{\uparrow \\ \text{fixed}}}{\omega}) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}(X_i(\omega) \le x) \longrightarrow F(x) \quad \forall x \in \mathbb{R}.$$

eg. Converge of maxima

Let. $X_1\ X_2 \cdots$ i.i.d. $\sim F$ (d.f)

Let $M_n = \max_{1 \le i \le n} X_i$ $\quad [\ \mathbb{P}(M_n \le x) = \mathbb{P}(X_i \le y \text{ for all } i) = [F(x)]^n$

(i) If $F(x) = 1 - x^{-\alpha}$ $\quad x \ge 1 \quad \alpha > 0$

$$\mathbb{P}(M_n \le x) = (1 - x^{-\alpha})^n \longrightarrow 0$$

$$x^{-\alpha} \sim \frac{1}{n}$$

$$x = n^{\frac{1}{\alpha}} \cdot y$$

$$\mathbb{P}(n^{-\frac{1}{\alpha}} \cdot M_n \le y) = \mathbb{P}(M_n \le n^{\frac{1}{\alpha}} y) = (1 - \tfrac{1}{n} y^{-\alpha})^n \longrightarrow e^{-y^{-\alpha}} \quad y \ge 0.$$

(ii) If. $F(x) = 1-|x|^{\beta}$ $\quad -1 \le x < 0 \quad \beta > 0$

$$\mathbb{P}(M_n \le x) = (1-\underline{|x|^{\beta}})^n$$

$\downarrow$

$|x|^{\beta} \sim \frac{1}{n}$

$x \sim n^{-\frac{1}{\beta}} \cdot y$

$$\mathbb{P}(n^{\frac{1}{\beta}} M_n \le y) = \mathbb{P}(M_n \le n^{-\frac{1}{\beta}} \cdot y) = (1-\frac{1}{n}|y|^{\beta})^n \to e^{-|y|^{\beta}} \quad -\infty < y < 0$$

(iii) If $F(x) = 1-e^{-x}$ for $x \ge 0$

$$\mathbb{P}(M_n \le x) = (1-\underline{e^{-x}})^n$$

No const here.

$e^{-x} \sim \frac{1}{n} \Rightarrow x = \log n + y$

$$\mathbb{P}(M_n - \log n \le y) = \mathbb{P}(M_n \le \log n + y) = (1-\frac{1}{n}e^{-y})^n \to e^{-e^{-y}} \quad y \in \mathbb{R}$$

Gumbel distribution.

What's the general method to pf weak convergence.

Convergence in Lebesgue Integral $\longrightarrow$ Convergence in Expectation.

Thm. If. $F_n \Rightarrow F$ $(X_n \xrightarrow{D} X)$ then there exist some ~~probability~~ measurable space $(\Omega, \mathcal{F})$ s.t. there are $Y_n \cdots Y : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ s.t. $Y_n \overset{D}{=} X_n$ $\quad Y \overset{D}{=} X$ and $Y_n \xrightarrow{a.s.} Y$

If. $X_n \xrightarrow{D} X$ $\quad |X_n| \le k$. $\mathbb{E} X_n \to \mathbb{E} X$???

$|X_n| \le \underline{k}$

$\because X_n \xrightarrow{D} X \quad \exists Y_n \xrightarrow{a.s.} Y \quad Y_n \overset{D}{=} X_n \quad Y \overset{\Delta}{=} X$

$|Y_n| \le k$

$\mathbb{E} Y_n = \mathbb{E} X_n \quad \mathbb{E} Y = \mathbb{E} X$

$\therefore\ Y_n \xrightarrow{a.s.} Y \overset{BCT}{\Longrightarrow} \mathbb{E}Y_n \to \mathbb{E}Y \qquad \therefore\ \mathbb{E}X_n \to \mathbb{E}X$

$|Y_n| < k \qquad \mathbb{E}Y_n = \mathbb{E}X_n,\quad \mathbb{E}Y = \mathbb{E}X$

☆ Δ With this thm we can upgrade MCT, DCT, DCT to the $\xrightarrow{D}$ sense.

Given Distribution. Generate distribution.

· If. F_n F strictly ↑ (invertible).

Generate $U \sim \text{unif}[0,1]$

$F_n^{-1}(u) \overset{d}{=} F_n \qquad F^{-1}(u) \overset{d}{=} F$

Thm. If. $F_n \Rightarrow F$ ($X_n \xrightarrow{D} X$) then there exist some probability space $(\Omega, \mathcal{F})$ s.t. there are $Y_n \cdots Y : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ s.t. $Y_n \overset{D}{=} X_n$, $Y \overset{D}{=} X$ and $Y_n \xrightarrow{a.s.} Y$

Pf. in case F_n F strictly ↑ F continuous.)

$(\Omega, \mathcal{F}, \mathbb{P}) \qquad \Omega = [0,1] \quad \mathcal{F} = \mathcal{B}([0,1]) \quad \mathbb{P} \sim \text{unif}$

$Y_n(\omega) = F_n^{-1}(\omega) \overset{D}{=} F_n. \qquad F_n(t) \to F(t)$ for all t.

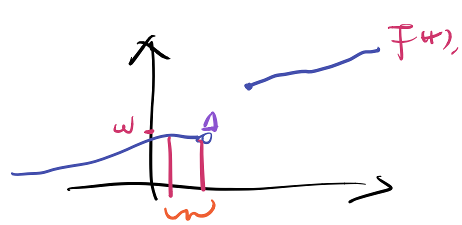
$Y(\omega) = F^{-1}(\omega) \overset{D}{=} F \qquad \Rightarrow F_n^{-1}(\omega) \to F^{-1}(\omega)$ for all ω. (*)

Stronger than a.s.

In general Let

$Y_n(\omega) = \sup\{x : F_n(x) < \omega\}$

$Y(\omega) = \sup\{x : F(x) < \omega\}$



(*) 会有 fail 的 ω

but countable many for F

都是 $Y(\omega)$ 但我们只 take sup Δ

Thm. $X_n \xrightarrow{D} X$ $(F_n \to F)$ iff for every bounded continuous function g we have $\mathbb{E}\, g(X_n) \to \mathbb{E}\, g(X)$

$$\int g(x)\, dF_n(x) \to \int g(x)\, dF(x)$$

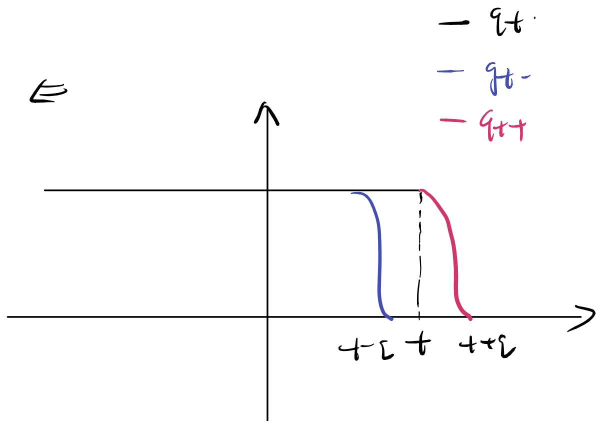
Pf: $(\Rightarrow)$ $\forall$ $X_n \xrightarrow{D} X$ $\quad \exists\, Y_n \xrightarrow{a.s.} Y \quad Y_n \overset{D}{=} X_n \quad Y \overset{D}{=} X$

Since g is continuous $\qquad g(Y_n) \xrightarrow{a.s.} g(Y)$

Since g is bounded $\qquad \mathbb{E}\, g(Y_n) \to \mathbb{E}\, g(Y)$

$$\underset{\mathbb{E}\, g(X_n)}{\overset{\shortparallel}{}} \qquad \underset{\mathbb{E}\, g(X)}{\overset{\shortparallel}{}}$$

$\Leftarrow$



F and $\mathbb{E}\, g(X)$ is the same fnel of they

$F_n(t) = \mathbb{P}(X_n \le t) = \mathbb{E}\, \mathbb{I}\{X_n \le t\}$

$F(t) = \quad - \; - \qquad = \mathbb{E}\, \mathbb{I}\{X \le t\}$

$g_t(x) \triangleq \mathbb{I}(x \le t)$

$F_n(t) = \mathbb{E}\, g_t(X_n)$

$F(t) = \mathbb{E}\, g_t(X)$

Consider $\{g_t\}$ $t \in$ continuous point.

$$g_{t-} \le g_t \le g_{t+}$$

$$g_{t-}(X_n) \le g_t(X_n) \le g_{t+}(X_n)$$

$$\mathbb{E}\, g_{t-}(X_n) \le \mathbb{E}\, g_t(X_n) \le \mathbb{E}\, g_{t+}(X_n)$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\mathbb{E}\, g_{t-}(X) \le \liminf_{n\to\infty} \mathbb{E}\, g_t(X_n) \le \limsup_{n\to\infty} \mathbb{E}\, g_{t+}(X_n) \le \mathbb{E}\, g_{t+}(X)$$

$$\left| g_{t\pm\varepsilon}(X) - g_t(X) \right| \le \mathbb{I}\left(X \in [t-\varepsilon, t+\varepsilon]\right)$$

$$\left| \mathbb{E}\, g_{t\pm}(X) - \mathbb{E}\, g_t(X) \right| \le \mathbb{E} \left| g_{t\pm}(X) - g_t(X) \right| \le \mathbb{E}\, \mathbb{I}\left(X \in [t-\varepsilon, t+\varepsilon]\right)$$

$$= \mathbb{P}\left(X \in [t-\varepsilon, t+\varepsilon]\right)$$

$$\mathbb{E}\, g_t(X) - \mathbb{P}(\cdots) \le \liminf \mathbb{E}\, g_t(X_n) \le \limsup \mathbb{E}\, g_t(X_n) \le \mathbb{E}\, g_t(X) + \mathbb{P}(\cdots)$$

send $\varepsilon \to 0$

$\therefore \quad \lim \mathbb{E}\, g_t(X_n) = \mathbb{E}\, g_t(X)$

⚠ if you want to pf $\xrightarrow{D}$ you can proof the $\mathbb{E}$ for a class of test functions.