Lévy continuity Thm.

① If $F_n \Rightarrow F$ for some d.f. F

then. $\phi_n \to \phi$ pointwise. where. ϕ is the. c.f. of F

② If $\phi_n \to \phi$ pointwise for some function. ϕ. and ϕ is continuous at 0 then. ϕ is the c.f. of some d.f. F

and $F_n \Rightarrow F$

1. $F_n \to F \Rightarrow \phi_n \to \phi$ $\overset{\text{transform}}{\rightsquigarrow}$ $F_n^* \to F^*$

$\Uparrow$?

$\underline{\phi_n^*(t)} \to \underline{\phi^*(t)}$

① converge or not ?

② is it a c.f of some d.f.

Eg. $\phi_n^*(t)$ converges to $\phi^*(t)$ But it's not a c.f.

$X_n = nX \quad X \sim N(0,1) \Rightarrow X_n \sim N(0, n^2)$

$\phi_n(t) = \mathbb{E}\, e^{itX_n} = e^{-n^2t^2/2} \to \phi(t) = \begin{cases} 1 & t=0 \\ 0 & t \neq 0 \end{cases}$ not continuous.

uniformly continuous is the basic prop of c.f.

F_n, F. d.f we say $F_n \Rightarrow F$ if $F_n(x) \to F(x)$, all continuous point F

F may not be a d.f.

$$F_n(x) \begin{cases} 0 & \text{if } x<0 \\ \frac{1}{2} & \text{if } 0 \le x < n \\ 1 & \text{if } x \ge n \end{cases} \to F(x) = \begin{cases} 0 & \text{if } x<0 \\ \frac{1}{2} & \text{if } x \ge 0 \end{cases}$$

no longer a. d.f.

F_n. d.f. we say $F_n \to F$ vaguely

Lévy continuity Thm. ① $\Rightarrow$ vague convergence.

If ($F_n \to F$ at all continuous of F) and F is a d.f.

then. $\phi_n \to \phi$ pointwise. ϕ is the c.f. of F

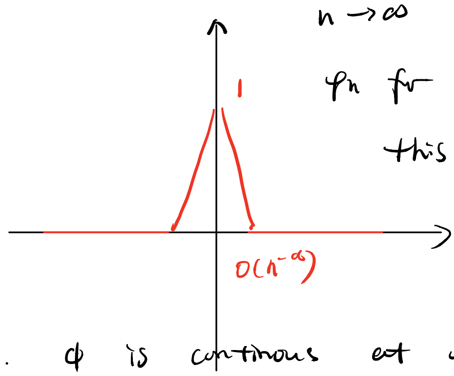
counterpart.

② If $\phi_n \to \phi$ pointwise for some function. ϕ. and ϕ is
(*) continuous at 0 then. ϕ is the c.f. of some d.f. F
and $F_n \Rightarrow F$

(*) is equivalent to
$\{F_n\}_{n=1}^{\infty}$ is tight.

Def. tightness : we say a seq of d.f. $\{F_n\}_{n=1}^{\infty}$ is tight
if $\forall \varepsilon > 0$ $\exists$ sufficient large M_ε
s.t. $\limsup_{n\to\infty} \mathbb{P}(|X_n| > M_\varepsilon) \le \varepsilon$.

$\limsup_{n\to\infty} [1 - F_n(M_\varepsilon) - F_n(-M_\varepsilon)] \le \varepsilon$



$n \to \infty$ the func become discontinuous at $x=0$.
ϕ_n for small n is continuous But
this is not so effective.

Pf. ϕ is continous at 0 $\to \{F_n\}$ is tight.

$\frac{1}{u}\int_{-u}^{u} (1-\phi_n(t))\, dt$ ①

$u \in N(0)$ $t \in N(0)$.

$\phi_n(t) \approx \phi_n(0)$.

the average of $(1-\phi_n(t))$ around 0

$\varepsilon > \frac{1}{u}\int_{-u}^{u} (1-\phi(t))\, dt$ ② ① $\to$ ② By BCT
choose $u = u_\varepsilon$ small enough.

$$① = \frac{1}{u}\int_{-u}^{u}\int (1-e^{itx})\,dF_n(x)\,dt.$$
$$= \int \left[\frac{1}{u}\int_{-u}^{u}(1-e^{itx})\,dt\right] dF_n(x)$$
$$= 2\int \Big(1-\underbrace{\boxed{\frac{\sin ux}{ux}}}_{|\cdot|\le 1}\Big)\,dF_n(x)$$
$$\geq 2\int_{|ux|>2}\Big(1-\frac{\sin ux}{ux}\Big)\,dF_n(x) \geq \cancel{2}\int_{|ux|>2}\cancel{\tfrac{1}{2}}\,dF_n(x)$$
$$= \int_{|x|>\frac{2}{u}} dF_n(x) \quad \text{tail prob.}$$
$$= 1-F_n(\underbrace{\tfrac{2}{u}}_{M_\varepsilon})+F(-\tfrac{2}{u}).$$

eg. WLLN $\quad X_1 \cdots$ i.i.d. $\mathbb{E}|X_1|<\infty$ $\mathbb{E}X_1=\mu$. Let $S_n=\sum_{i=1}^{n}X_i$

Then. $\frac{S_n}{n} \xrightarrow{D} \mu.$

$\mathbb{E}|X|^k<\infty$

$$\mathbb{E}e^{itX} = \mathbb{E}\Big[\sum_{j=0}^{k}\frac{(itX)^j}{j!}\Big] = \sum_{j=0}^{k}\frac{i^j t^j\,\mathbb{E}|X|^j}{j!} + o(|t|^k)$$

only when $t\to 0$ This expansion is right.

But in Lévy continuous Thm.

we have to check $t\in\mathbb{R}$

$\forall t\in\mathbb{R}$ fixed.

Pf: $$\mathbb{E}e^{it\frac{S_n}{n}} = \Big[\mathbb{E}e^{it\frac{X_1}{n}}\Big]^n = \Big[\mathbb{E}e^{i\frac{t}{n}\cdot X_1}\Big]^n$$
$$= \Big[1+\frac{it\mu}{n}+o(\tfrac{t}{n})\Big]^n \to e^{it\mu}. \quad \text{c.f. of } \mu.$$

eg. CLT $\quad$ Let $X_1 X_2 \cdots$ i.i.d. $\mathbb{E}|X|^2<\infty$

$\mathbb{E}X=\mu$. $\operatorname{Var}X=\sigma^2$ Let $S_n=\sum_{i=1}^{n}X_i$

Then. $\frac{S_n-n\mu}{\sqrt{n}} \xrightarrow{D} N(0,\sigma^2)$

Pf. $$\mathbb{E}e^{it\frac{S_n-n\mu}{\sqrt{n}}} = \mathbb{E}e^{it\frac{\sum(X_i-\mu)}{\sqrt{n}}} = \Big[\mathbb{E}e^{it\overbrace{\frac{X_i-\mu}{\sqrt{n}}}^{Y_i}}\Big]^n$$
$$= \Big(1+\cancel{\frac{it}{\sqrt{n}}\,\mathbb{E}(X_i-\mu)}^{\,0} - \frac{t^2}{n}\,\underbrace{\mathbb{E}(X_i-\mu)^2}_{\sigma^2} + o\big(\tfrac{t^2}{n}\big)\Big)^n.$$

$$= e^{-\frac{\sigma^2 t^2}{2}} \quad \text{c.f. of } N(0, \sigma^2)$$

Lindeberg - Feller CLT

Allow r.v. to have diff size.

$X_{1,1}$

$\vdots \quad \ddots$

$X_{n,1} \; --- \; X_{n,n}$.

$X_{n,m}$. $1 \le m \le n$ be independant

$\mathbb{E} X_{n,m} = 0$. Let. $S_n = \sum_{m=1}^{n} S_{n,m}$

(i) $\sum_{m=1}^{n} \mathbb{E} X_{n,m}^2 \to \sigma^2$

(ii) $\forall \varepsilon > 0 \quad \left(\sum_{m=1}^{n} \mathbb{E} X_{n,m}^2 \mathbb{1}(|X_{n,m}| > \varepsilon) \right) \to 0$ $\ast$

If $|X_{n,m}| \not\to 0$ then when ε is small

$\mathbb{1}(\cdot) = 1$ $(\ast)$ should $=$ (i)

So. (ii) actually tells us that. $|X_{n,m}| \to 0$.

Then we have.

$$S_n \xrightarrow{D} N(0, \sigma^2)$$

$\mathbb{E}|tX|^{k+1} = |t|^{k+1} \mathbb{E}|X|^{k+1}$

$\mathbb{E}|tX|^{k} = |t|^{k} \mathbb{E}|X|^{k}$

do not have $k+1$ moment.

An alternative Taylor expansion for c.f.

If. $\mathbb{E}|X|^k < \infty$

$$\left| \mathbb{E} e^{itX} - \sum_{j=0}^{k} \frac{\mathbb{E} X^j}{j!} (it)^j \right| \le \mathbb{E} \min\left(|tX|^{k+1}, 2|tX|^{k} \right)$$

t could be large.

$$\left| \mathbb{E} e^{itX} - \sum_{j=0}^{k} \frac{\mathbb{E} X^j}{j!} (it)^j \right| \le o(t^k) \quad \text{when. } t \downarrow 0$$

$$\mathbb{E} \min(\,,\,) \mathbb{1}\left(|X| \le t^{-\frac{1}{2(k+1)}}\right)$$

$$+ \mathbb{E} \min(\,,\,) \mathbb{1}\left(|X| > t^{-\frac{1}{2(k+1)}}\right)$$

$$\le \mathbb{E} |tX|^{k+1} \mathbb{1}\left(|X| \le t^{-\frac{1}{2(k+1)}}\right) + \mathbb{E} |tX|^{k} \mathbb{1}\left(|X| > t^{-\frac{1}{2(k+1)}}\right)$$

$$\le t^{k+1} t^{-\frac{1}{2}} + t^k \underbrace{\mathbb{E}|X|^k \mathbb{1}\left(|X| > t^{-\frac{1}{2(k+1)}}\right)}$$

$$\overset{\shortparallel}{o(t^k)}$$

$t \to 0$

$t^{-\frac{1}{2(k+1)}} \to \infty$.

$\mathbb{1}(\quad) \to 0$ a.s.

$|X|^k \mathbb{1}(\cdot) \to 0$ a.s.

$|X|^k \mathbb{1}(\cdot) \le |X|^k$.

$\mathbb{E}|X|^k \; \exists$

$\mathbb{E}|X|^k \mathbb{1}(\cdot) \to \mathbb{E}(0) = 0$.

[illegible] By DCT