① what if the sequences of r.v.s. depend on n.

$$\boxed{\frac{X_1}{n}} + \frac{X_2}{n} \cdots + \frac{X_n}{n}$$

$$\frac{X_1}{\sqrt{n}} + \frac{X_2}{n} \cdots + \frac{X_n}{n}$$

The normalization can fluctuate

The r.v. can change also.

$$\frac{Y_1}{n} + \frac{X_2}{n} + \cdots + \frac{X_n}{n}$$

$$\frac{X_1}{n+1} + \frac{X_2}{n+1} \cdots + \frac{X_{n+1}}{n}$$

Triangle Array

eg. $X_1 \quad X_2 \cdots$

$$X_{n,i} = \frac{X_i}{n}$$

$$\begin{array}{lllll} X_{1,1} & \cdots & X_{1,m(1)} & & \\ X_{2,1} & X_{2,2} & \cdots & X_{2,m(2)} & \\ \vdots & & & & \\ X_{n,1} & X_{n,2} & \cdots & X_{n,n} & \cdots\cdots X_{n,m(n)} \end{array}$$

$$S_n = \sum_{i=1}^{n} X_{n,i}$$

① triangle array
② fluctuate normalization.
③ impose bound of S_n directly rather than on that $X_1\ X_2 \cdots$ are uncorrelated.
④. Do not assum the independent or identical / uncorrelated of $X_1\ X_2 \cdots X_n$

$$\frac{S_n - n\mu}{n^{\frac{1}{2}+\delta}} \xrightarrow{P} 0 \qquad S_n = n\mu + o(n^{\frac{1}{2}+\delta})$$

$$\frac{S_n - n\mu}{n} \xrightarrow{P} 0 \qquad S_n = n\mu + o(n)$$

$$\frac{S_n - n\mu}{n^{1+\delta}} \xrightarrow{P} 0 \qquad S_n = n\mu \cancel{+} \underbrace{o(n^{1+\delta})}$$

may be the first order is still here.

Thus normalization no larger than the order of $\mathbb{E}S_n$.

Choose ---> second order term of S_n.

Thm. WLLN for triangular array

$\mu_n = \mathbb{E} S_n \quad \sigma_n^2 = Var(S_n)$ If. $\frac{\sigma_n^2}{b_n^2} \to 0$

for some $b_n > 0$ then.

$\frac{S_n - \mu_n}{b_n} \xrightarrow{\mathbb{P}} 0$ ✓

$\Downarrow$

$\frac{Var(S_n)}{b_n^2} \to 0$

$\mathbb{E}\left(\left(\frac{S_n - \mu_n}{b_n}\right)^2\right) \to 0$

- Compute $\mathbb{E} S_n$ $Var(S_n)$
- Have the freedom to choose b_n.

eg. Coupon. Collection. Problem.

$X_1 \; X_2 \cdots$ be i.i.d. uniform. on. $\{1, 2 \cdots n\}$

but it in. depend on n.

Let. $\tau_k^n := \inf\{m : |\{X_1 \cdots X_m\}| = k\}$

$T_n := \tau_n^n$

Q asymptotic behavior of T? if $n \to \infty$

Solution. $\tau_1^n = 1 \qquad \tau_0^n = 0$

Let. $X_{n,k} = \tau_k^n - \tau_{k-1}^n$

$T_n = \tau_n^n = \sum_{k=1}^{n} X_{n,k}$.

①. Compute $\mathbb{E} S_n$. $Var(S_n)$

② choose. a $b_n <$ order of $\mathbb{E} S_n$. $b_n >$ order of $\sqrt{Var}$.

Claim. $X_{n,k}$'s are $\perp\!\!\!\perp$

distribution of $X_{n,k} \sim \text{Geom}(1-\frac{k-1}{n})$

$$\mathbb{P}(X_{n,k}=l) = \left(\frac{k-1}{n}\right)^{l-1} \cdot \left(1-\frac{k-1}{n}\right)$$

Def of $Y \sim \text{Geo}(p)$, the waiting time of the first success first XX try

$$p(Y=l) = (1-p)^{l-1} p$$

$$\mathbb{E} Y = \frac{1}{p}$$

$$\text{Var} = \frac{1-p}{p^2} \leq \frac{1}{p^2}$$

$$\mathbb{E} T_n = \sum_{k=1}^{n} \mathbb{E} X_{n,k} = \sum_{k=1}^{n} \left(1-\frac{k-1}{n}\right)^{-1} = n \cdot \sum_{m=1}^{n} m^{-1} = n \cdot \log n \,(1+o(1))$$

$$\text{Var}(T_n) = \sum_{k=1}^{n} \text{var} X_{n,k} \leq \sum_{k=1}^{n} \left(1-\frac{k-1}{n}\right)^{-2} = n^2 \sum_{m=0}^{n} m^{-2} \leq n^2 C$$

Let $b_n = n \log n$.

$$\frac{T_n - \mathbb{E} T_n}{n \log n} \xrightarrow{\mathbb{P}} 0 \qquad \frac{T_n}{n \log n} \xrightarrow{\mathbb{P}} 1$$

Δ X_{nk} independent but not identical.

Δ no longer a story of average just sum.

eg 2. An occupancy problem.

r balls at random in n boxes, all n^r assignment are equally likely. Let

A_i := event that the i-th box in empty

N_n := # of empty boxes. $= \sum_{i=1}^{n} \mathbb{1}_{A_i}$ depend on n. $=: \sum_{i=1}^{n} X_{n,i}$

Q: If $r \equiv r(n)$ $\frac{r}{n} \to c$ if $n \to \infty$ asymptotic behavior of N_n.

$$\mathbb{E} N_n = \sum_{i=1}^{n} \mathbb{E} \mathbb{I}_{A_i} = \sum_{i=1}^{n} \mathbb{P}(A_i) = \sum_{i=1}^{n} (1-\frac{1}{n})^r = n(1-\frac{1}{n})^r$$

$$\frac{\mathbb{E} N_n}{n} = (1-\frac{1}{n})^{\frac{-n \cdot r}{-n}} \to e^{-\frac{r}{n}} = e^{-c}.$$

If $1 \cdots n-1$ all empty n has to have balls.

$\therefore A_1 \cdots A_n$ are not independent.

$$\mathrm{Var}(N_n) = \mathbb{E}(N_n^2) - (\mathbb{E} N_n)^2$$

$$= \mathbb{E}(\sum_{i=1}^{n} \mathbb{I}_{A_i})^2 - [\mathbb{E}(\sum_{i=1}^{n} \mathbb{I}_{A_i})]^2$$

$$= \mathbb{E}(\sum_{i,j} \mathbb{I}_{A_i} \mathbb{I}_{A_j}) - \sum_{i,j} \mathbb{E}\mathbb{I}_{A_i} \mathbb{E}\mathbb{I}_{A_j}$$

$$= \mathbb{E} \sum_{i,j} \mathbb{I}(A_i \cap A_j) - \sum_{i,j} \mathbb{P}(A_i)\mathbb{P}(A_j)$$

$$= \sum_{i,j} \mathbb{P}(A_i \cap A_j) - \sum_{i,j} \mathbb{P}(A_i)\mathbb{P}(A_j)$$ cancel each other

$$= \underbrace{\sum_{i \neq j}}_{n(n-1)} \underbrace{\mathbb{P}(A_i \cap A_j)}_{(1-\frac{2}{n})^r \to e^{-2c}} - \underbrace{\mathbb{P}(A_i)\mathbb{P}(A_j)}_{(1-\frac{1}{n})^{2r} \to e^{-2c}}$$

$$+ \underbrace{\sum_{i}}_{n} (\underbrace{\mathbb{P}(A_i)}_{(1-\frac{1}{n})^r \to e^{-c}} - \underbrace{[\mathbb{P}(A_i)]^2}_{(1-\frac{1}{n})^{2r} \to e^{-2c}}) = o(n^2)$$

Var is smaller than n

Let. $b_n = n$

$$\frac{N_n - \mathbb{E} N_n}{n} \xrightarrow{\mathbb{P}} 0 \qquad \frac{N_n}{n} \xrightarrow{\mathbb{P}} e^{-c}$$

Δ idendical but not independent

Maybe we still have WLLN if $\underbrace{\text{Var}(S_n)}$ does not exist.

Help us to pf the $\xrightarrow{P}$ by first prove $\xrightarrow{2\text{-th}}$ first.

Can we prove $\mathbb{E}\left|\frac{S_n - n\mu}{b_n}\right| \to 0$? X

small part → using Var

large part → only talk about $\xrightarrow{P}$