$(\Omega, \mathcal{F}, \mathbb{P}) \xrightarrow{X} (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbb{P}_{\circ} X^{-1})$

$(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}) \xrightarrow{\bar{X}} (\mathbb{R}, \mathcal{B}(\mathbb{R}), \tilde{\mathbb{P}}_{\circ} \bar{X}^{-1})$

My be d.f.s. are the same.

eg. indicator function.

arbitary $(\Omega, \mathcal{F}, \mathbb{P})$ Let $A \in \mathcal{F}$

$\mathbb{1}_A : \Omega \to \mathbb{R}$

$$\mathbb{1}_A(\omega) := \begin{cases} 0 & \omega \in A^c \\ 1 & \omega \in A \end{cases}$$

Whether it's a r.v.

$$\mathbb{1}_A^{-1}((-\infty, b]) = \begin{cases} \phi & b < 0 \\ A^c & 0 \leq b < 1 \\ \Omega & 1 \leq b \end{cases}$$

$\because A \in \mathcal{F}$ $\therefore A^c \in \mathcal{F}$ ✓

$\therefore \mathbb{1}_A$ is a r.v. $\rightsquigarrow$ Bernoulli

$\mathbb{P}\, \mathbb{1}_A^{-1}(\{1\}) = \mathbb{P}(A)$

$\mathbb{P}\, \mathbb{1}_A^{-1}(\{0\}) = \mathbb{P}(A^c) = 1 - \mathbb{P}(A)$

on any $(\Omega, \mathcal{F}, \mathbb{P})$ we can define r.v. $\mathbb{1}_A$. $\mathbb{P}(A) = p$

the distr $\sim Ber(p)$

$\therefore$ diff space, diff r.v. may save distribution.

Ask sth. of X, Y as the func of ω. ⇒ have to ref to ω.

Ask sth of distribution of X Y ⇒ No need care about ω.

a.s., P, p-th. care ω. define on same $(\Omega, \mathcal{F}, \mathbb{P})$.

d., no need care about ω. no need same $(\Omega, \mathcal{F}, \mathbb{P})$

$$\begin{array}{ccc} \omega & & \\ \Psi & & \\ (\Omega, & \mathcal{F}, & \mathbb{P}) \\ (\mathbb{R}, & \mathcal{B}(\mathbb{R}), & \mathbb{P}\circ X^{-1}) \\ \Psi & & \\ x & & \end{array}$$

$$\begin{aligned} &\mathbb{P}(X^{-1}[-\infty, x]) \\ &\quad \| \\ &\mathbb{P}(\{\omega : X(\omega) \in [-\infty, t]\}) \\ &\quad \| \\ &= \mathbb{P}(X \in [-\infty, t]) \end{aligned}$$

c.d.f. $F(t) := \mathbb{P}\circ X^{-1}((-\infty, t]) = \mathbb{P}(X \le t)$

$F \circ X^{-1}((a, b]) = F(b) - F(a)$.

prop. (i) $F(t)$ non decreasing

(ii) $F(t)$ right-continuous.

(iii) $\lim_{t\to-\infty} F(t) = 0$ $\quad \lim_{t\to\infty} F(t) = 1$

(i) discrete distribution.

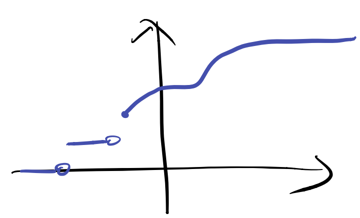
(ii) continuous distribution.

$\exists f: \mathbb{R} \to [0, \infty)$ s.t. density func.

$F(t) = \int_{-\infty}^{t} f(x)\,dx$

(iii) singularity continuous distribution.

Cantor function.



Lebesgue decomposition Thm.

$\mu = $ p.p. a.c. + s.c.

measurable map. $(\mathbb{R}\ \mathcal{B}(\mathbb{R})) \rightsquigarrow$ random. variable

Def. Let $(\Omega, \mathcal{F})$, $(S, \mathcal{S})$ be measurable spaces.

$X: \Omega \to S$

If $X^{-1}(B) \in \mathcal{F}$ $\forall B \in \mathcal{S}$

then. we call X a measurable map.

(i). $(S, \mathcal{S}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ r.v.

(ii). $(S, \mathcal{S}) = (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ r.vector.

(iii) $(S, \mathcal{S}) = (C[0,1], \mathcal{B}(C[0,1]))$

continuous random. process.

$C[0,1]$ = space of all continuous function. $[0,1]$

$d(x,y) = \sup_{t \in [0,1]} |x(t) - y(t)|$ $x, y \in C[0,1]$

$\Downarrow$

Define topo $\longrightarrow$ define $\mathcal{B}(C[0,1])$

X is a random vector $\Leftrightarrow$ $X_1\ X_2 \cdots X_d$ are r.v.s.

defined as (ii)

$\Leftarrow$ check $X^{-1}(B) \in \mathcal{F}$ $\forall B \in \mathcal{B}(\mathbb{R}^d)$

$\Uparrow$

$\underline{X^{-1}(A_1 \times A_2 \times \cdots \times A_d)} \in \mathcal{F}$ $\forall A_i = (a_i, b_i]$

$\parallel$

$\{w : X(w) = (X_1(w) \cdots X_d(w)) \in A_1 \times \cdots \times A_d\}$

$= \{w : X_i(w) \in A_i \ \forall i = 1, \cdots d\} = \bigcap_{i=1}^{d} \{w : X_i(w) \in A_i\}$

$= \bigcap_{i=1}^{d} X_i^{-1}(A_i)$

$$\Rightarrow \quad X^{-1}(A_1 \times \mathbb{R} \times \cdots \mathbb{R}) \in \mathcal{F}$$

$$\parallel$$

$$X_1^{-1}(A_1) \cap \underbrace{X_2^{-1}(\mathbb{R}) \cap \cdots \cap X_d^{-1}(\mathbb{R})}_{\Omega}$$

$$= X_1^{-1}(A_1)$$

$\Rightarrow$ X_i's are random variable

Thm. $X: (\Omega, \mathcal{F}) \to (S, \mathcal{S})$ — measurable.

$f: (S, \mathcal{S}) \to (T, \mathcal{T})$ — measurable.

Then. $f \circ X: (\Omega, \mathcal{F}) \to (T, \mathcal{T})$ is also measurable.

$\forall B \in \mathcal{T}$

$$(f \circ X)^{-1}(B) = X^{-1} \circ f^{-1}(B).$$

$$\frac{\overset{\in}{\mathcal{S}}}{\overset{\in}{\mathcal{F}}}$$

Choose Both $(S, \mathcal{S})$ and $(T, \mathcal{T}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$

$$(\Omega, \mathcal{F}) \xrightarrow{X} (\mathbb{R}, \mathcal{B}(\mathbb{R})) \xrightarrow{f} (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$

If f is continuous func.

the $f(X)$ is a r.v. measurable function.

continuity :

$f(x)$ is continuous

$\forall\, O \in \mathcal{B}(\mathbb{R})$ open $\quad f^{-1}(O)$ is again open. $\in \mathcal{B}(\mathbb{R})$

$\Downarrow$

Def in topo $\therefore$ continuity $\gg$ measurable.

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$ $\rightarrow$ measurable but not continuous.

If X is a r.v.

cx^2 $\sin x$ e^x are r.v.s.

App. $(S, \mathcal{S}) = (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$

$(T, \mathcal{T}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

$f: (S, \mathcal{S}) \to (T, \mathcal{T})$ is measurable.

then $f(X)$ is also a r.v.s.

$X_1\ X_2 \cdots X_d$ r.v.s. $\Leftrightarrow$ $X = (X_1 \cdots X_d)$ is a random vector

$f(x) = \sum_{i=1}^{d} x_i$

use the original topo of

ω is much more ideal.

X is discrete.

$p(x) = \mathbb{P} \circ X^{-1}(\{x\}) = \mathbb{P}(X = x)$ p.m.f.

$\mathbb{P} \circ X^{-1}(\{x_i\})$

$\mathbb{E}X = \sum x_i\, p(x_i)$ $\quad \sum_j X(\omega_j)\, \mathbb{P}(\{\omega_j\})$ (suppose Ω is discrete)

$\mathbb{E}\, \underline{g(x)} = \sum_i g(x_i)\, p(x_i) = \sum_i g(x_i)\, p \circ x^{-1}(\{x_i\})$

Y $\quad P_Y$ $\quad \sum y \cdot P_Y(y_i)$

X is continuous $\quad f(x)$ pdf

$$\mathbb{E}X = \int x \cdot f(x)\,dx$$

$$\mathbb{E}g(X) = \int g(x) \cdot f(x)\,dx$$