MATH 5411 - Advanced Probability I
Final exam

Due date: December 17, 2020

Q1 (15'): Suppose one independently flips an infinite sequence of weighted coins. Let E_n be the event that the n-th coin is Heads. Let $\{E_n \text{ i.o.}\} = \cap_{m\geq 1} \cup_{n\geq m} E_n$ be the event that Heads show up infinitely often.
(a) If $\mathbb{P}(E_n) = \frac{1}{n}, n = 1, 2, \cdots$, find $\mathbb{P}(E_n \text{ i.o.})$.
(b) If $\mathbb{P}(E_n) = (\frac{99}{100})^n, n = 1, 2, \cdots$, find $\mathbb{P}(E_n \text{ i.o.})$.
(c) Now, suppose that the coins are all fair, i.e., $\mathbb{P}(E_n) = \frac{1}{2}, n = 0, 1, 2\cdots$. Let A_n be the event that starting from the n-th flip one gets k consecutive Heads for some fixed number k, i.e., $A_n = \cap_{m=n}^{n+k-1} E_m$. Find $\mathbb{P}(A_n \text{ i.o.})$, where $\{A_n \text{ i.o.}\} = \cap_{m\geq 1} \cup_{n\geq m} A_n$.

Q2 (10'): The interval $[0, 1]$ is partitioned into k disjoint sub-intervals with lengths $p_1, \ldots, p_k$, and the *entropy* of this partition is defined to be

$$h = -\sum_{i=1}^{k} p_i \log p_i.$$

Let $X_1, X_2, \ldots$ be independent random variables having the uniform distribution on $[0, 1]$, and let $Z_n(i)$ be the number of the $X_1, X_2, \ldots, X_n$ which lie in the ith interval of the partition above. Show that

$$R_n = \prod_{i=1}^{k} p_i^{Z_n(i)}$$

satisfies $n^{-1} \log R_n \to -h$ almost surely as $n \to \infty$.

Q3 (20'): Let $S_n = X_1 + \cdots X_n$ in the following problems.

(a): Suppose that X_i's are independent and $\mathbb{P}(X_i = i) = \mathbb{P}(X_i = -i) = \frac{i^{-\alpha}}{4}$ and $\mathbb{P}(X_i = 0) = 1 - \frac{i^{-\alpha}}{2}$ for some nonnegative parameter α. Find $a_n(\alpha), b_n(\alpha)$ such that $(S_n - a_n(\alpha))/b_n(\alpha) \Rightarrow N(0, 1)$ when $\alpha \in (0, 1)$ and prove this CLT.

(b):Suppose that X_i's are independent and $\mathbb{P}(X_i = 1) = \frac{1}{i} = 1 - \mathbb{P}(X_i = 0)$. Find a_n and b_n such that $(S_n - a_n)/b_n \Rightarrow N(0, 1)$ and prove this CLT.

Q4 (20'): Suppose $X_1, X_2, \ldots$ are independent and $P(X_j = j) = P(X_j = -j) = 1/(2j^\beta) = (1 - P(X_j = 0))/2$ with $\beta > 0$. Prove that (i) if $\beta > 1$, $\sum_n X_n < \infty$ a.s.; (ii) if $\beta < 1$, then $S_n/n^{(3-\beta)/2} \to N(0, \sigma^2)$ for some $\sigma^2 > 0$; (iii) if $\beta = 1$, S_n/n converges in distribution to a distribution with characteristic function $\exp(-\int_0^1 (1 - \cos xt)/x \mathrm{d}x)$.

Q5 (20'): (a) Suppose that $X_n \xrightarrow{\mathbb{P}} X$. Show that $\{X_n\}$ is *Cauchy convergent in probability*, i.e. , for all $\varepsilon > 0$, $\mathbb{P}(|X_n - X_m| > \varepsilon) \to 0$ as $n, m \to \infty$.

(b) Is the following converse statement of (a) also true: If $\{X_n\}$ is *Cauchy convergent in probability*, then there exists a random variable X such that $X_n \xrightarrow{\mathbb{P}} X$? If so, prove it.

(c) Let $\{X_n\}$ and $\{Y_n\}$ be sequences of random variables such that the pairs (X_i, X_j) and (Y_i, Y_j) have the same distributions for all i, j. If $X_n \xrightarrow{\mathbb{P}} X$, show that Y_n converges in probability to some limit Y having the same distribution as X.

Q6 (15'): Find an application of the advanced probability theory in your own research field. In case it is hard to find any in your field, you can also choose an application in other fields. The application can be related to any concepts or results you have learnt in this course. (*Your score of this problem will depend on how well you explain your application and the connection with the advanced probability theory*)