- The choice of the class of bounded continuous function.

depends on : ① what kind of $\{X_n\}$ we have.

② If we will consider the sum of r.v.s.

$\Rightarrow$ choose characteristic func.

Method of moment

$\underline{\{X^k\}_{k\in\mathbb{N}}}$ $\qquad \mathbb{E}X_n^k \to \mathbb{E}X^k \quad \forall k\in\mathcal{N}$ + Carlmen's condition.

① Not bounded actually

② requires the r.v. have all moment.

Pf of CLT is ok because. $\mathcal{N}(\mu,\sigma^2)$ have all moments.

Method of characteristic func. (Fourier transform)

$\{e^{itx}\}_{t\in\mathbb{R}} = \{\sin(tx), \cos(tx)\}_{t\in\mathbb{R}} \qquad \mathbb{E}e^{itX_n} \to \mathbb{E}e^{itX} \quad \forall t\in\mathbb{R}$

Method of Stieltjes transform

$\left\{\frac{1}{x-z}\right\}_{z\in\mathbb{C}^+} \qquad \mathbb{E}\frac{1}{X_n-z} \to \mathbb{E}\frac{1}{X-z} \quad \forall z\in\mathbb{C}^+$

① These classes are large enough.

② why choose ? → why fourier transform. ?

Because we are gonna consider S_n

$S_n = \sum_{i=1}^n \xi_i \qquad \xi_i$ are indept

$$\begin{tikzcd} & \xi_1 \arrow[d] & + & \xi_2 \arrow[d] \\ \text{density} & f_1 & & f_2 \end{tikzcd}$$

$$\underline{f_{\xi_1+\xi_2}}(z) = \int f_{\xi_1}(z-x) f_{\xi_2}(x)\,dx$$

$$= (f_{\xi_1} * f_{\xi_2})(z).$$

↓

Hard to compute.

other characteristics ?

$$\underset{\varphi_{\xi_1+\xi}}{\underset{\|}{\mathbb{E}\, e^{it(\xi_1+\xi_2)}}} = \underset{\varphi_{\xi_1}}{\underset{\|}{(\mathbb{E}\, e^{it\xi_1})}} \cdot \underset{\varphi_{\xi_2}}{\underset{\|}{(\mathbb{E}\, e^{it\xi_2})}}$$

$$\varphi_{\sum_{i=1}^{n}\xi_i} = \varphi_n \xrightarrow[?]{F_n = \psi^{-1}(\varphi_n) \quad F = \psi^{-1}(\varphi)} \varphi.$$

$$\Downarrow$$

$$F = \psi^{-1}(\varphi)$$

$\mathcal{D}$: space of all df

$\mathcal{C}$: space of all cf.

Def. c.f. of X

$\phi(t) = \mathbb{E}\, e^{itX} = \int e^{itx}\, dF(x).$

$\psi : \mathcal{D} \longrightarrow \mathcal{C}$

$F(x) \to \phi(t) = \int e^{itx}\, dF(x)$

Q: Is ψ invertible. $\qquad$ F is invertible.

Q: If $\phi_n \to \phi$ whether $\underset{\psi^{-1}(\phi_n)}{\underset{\|}{F_n}} \Rightarrow \underset{\psi^{-1}(\phi)}{\underset{\|}{F}}$ $\qquad$ ψ^{-1} is continuous.

Basic property

①. $\varphi = 1$ $\quad |\phi(t)| \le 1 \quad \forall t \in \mathbb{R}$

②. $\phi(t)$ is uniformly continuous.

whether it's differentiable or not.

$\forall \varepsilon > 0$ $\exists$ a uniform $\delta > 0$ so that $\forall$ $t \in \mathbb{R}$

$|\phi(t+\delta) - \phi(t)| \le \varepsilon.$

$$\Downarrow$$

$$\left|\mathbb{E}\, e^{i(t+\delta)x} - \mathbb{E}\, e^{itx}\right| \le \mathbb{E}\, \overset{=1}{\cancel{|e^{itx}|}}\, \underbrace{|e^{i\delta x} - 1|}_{}$$

$\to 0$ By BCT

Bounded by 2

when $\delta \to 0$ this part $\to 0$ a.s.

$|\mathbb{E}\, e^{itx}| \le |\mathbb{E} \cos tx + i\, \mathbb{E} \sin tx|$

triangle inequality.

$\le |\mathbb{E} \cos tx| + |i\, \mathbb{E} \sin tx|$

≤ 1

$\phi(t) = \int e^{itx}\, dF(x)$

$\phi'(0) = \int ix \cdot e^{itx}\, dF(x) \cdot \Big|_{t=0}$

$= i \int \underline{x\ dF(x)}.$ May not be Integrable.

$\phi''(0) = \int (ix)^2 e^{itx}\, dF(x)\, \Big|_{t=0} = -\int x^2\, dF(x).$

①. If $\phi^{(k)}(0)$ exists then $\begin{cases} \mathbb{E}|X|^k < \infty. & \text{If } k \text{ is even.} \\ \mathbb{E}|X|^{k-1} < \infty & \text{If } k \text{ is odd.} \end{cases}$

② If $\mathbb{E}|X|^k < \infty$ $\quad \varphi^{(k)}\underline{(0)}$ exist.

$$\phi(t) = \sum_{j=0}^{k} \frac{i^j \mathbb{E}X^j}{j!} t^j + o(t^k)$$

eg. X density $f(x)$

$f(x) \sim x^{-1-\varepsilon}$

$\mathbb{E}|X| < \infty$

$f(x) \sim x^{-2+\varepsilon}$

$\mathbb{E}|X| = \infty$

$\mathbb{E}|X| = \int |x| f(x)\, dx$,

Inversion thm.

Special case X: absolutely continuous with density $f(x)$

cf. $\varphi = \int e^{ity} f(y) \cdot dy$,

Thm.

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-itx} \varphi(t)\, dt .$$ (Not for all x)

Pf: $I(x) := \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-itx} \varphi(t)\, dt$. Aim: Show $I(x) = f(x)$.

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-itx} \left[\int e^{ity} f(y)\, dy \right] dt .$$

$$\frac{1}{2\pi} \int \left[\int_{-\infty}^{+\infty} \underbrace{e^{ity} e^{-itx}}_{\underline{e^{it(y-x)}}} dt \right] f(y)\, y$$

Not integrable

$$I_\varepsilon(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-itx}\left[\int e^{ity} f(y)\,dy\right]\cdot e^{-\frac{\varepsilon^2 t^2}{2}}\,dt \,..$$

$I_\varepsilon \to I$ if $\varepsilon \downarrow 0$.

$$I_\varepsilon(x) = \int f(y)\, f_\varepsilon(x-y)\,dy$$

↓

density function of $X + g_\varepsilon$.

$g_\varepsilon \sim N(0, \varepsilon^2)$ $X \perp\!\!\!\perp g_\varepsilon$.

$g_\varepsilon \rightsquigarrow 0$ when $\varepsilon \to 0$.

$\therefore I_\varepsilon \to I$

$$\bar F(b) - \bar F(a) = \lim_{T\to\infty} \int_{-T}^{T} \frac{e^{-iat} - e^{-ibt}}{2\pi i t}\,\phi(t)\,dt$$

Define

$$\bar F(x) = \frac{1}{2}\big(F(x) + \underbrace{F(x-)}_{\lim_{y\uparrow x} F(y)}\big)$$

Continuity Thm. given F_n with c.f. ϕ_n

If $\phi_n \to \phi$ (in certain sense)

then $F_n \Rightarrow F = \mathcal{F}^{-1}(\phi)$.

Lévy continuity Thm.

① If $F_n \Rightarrow F$ for some d.f. F

then $\phi_n \to \phi$ pointwise where ϕ is the c.f. of F

② If $\phi_n \to \phi$ pointwise for some function ϕ and ϕ is continuous at 0 then ϕ is the c.f. of some d.f. F

and $F_n \Rightarrow F$