$(\Omega, \mathcal{F}, \mu) \xrightarrow{f} (\mathbb{R}, \mathcal{B}(\mathbb{R}), *)$

$\int f \, d\mu.$ eq. if $(\Omega, \mathcal{F}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$

$\downarrow \quad \downarrow$

$\mu.$ d.f. F

$X \quad dF(x).$ $\int f(\omega) \, dF(\omega).$ $\int f(\omega) \, d\mu(\omega)$ $\int f(\omega) \cdot \mu(d\omega)$

$(\Omega, \mathcal{F}, \mu) \xrightarrow{f} (\mathbb{R}, \mathcal{B}(\mathbb{R}), \cdots)$

- (Jensen's inequality) suppose $\varphi : \mathbb{R} \to \mathbb{R}$ convex i.e.

$\lambda \varphi(x) + (1-\lambda) \cdot \varphi(y) \geq \varphi(\lambda \cdot x + (1-\lambda) y) \quad \forall \lambda \in [0,1] \quad x, y \in \mathbb{R}$

$$\varphi\left(\int f \, d\mu\right) \leq \int \varphi(f) \, d\mu.$$

$\varphi(\mathbb{E}X) \leq \mathbb{E}(\varphi(X))$

eq. $(\mathbb{E}X)^2 \leq \mathbb{E}X^2$

$\vee\!/$

0

eq. $f = x \cdot \mathbb{1}_A + y \cdot \mathbb{1}_{A^c}$

interchange the integral with the function φ

$$\varphi\left(\int f \, d\mu\right) \leq \int \varphi(f) \, d\mu.$$

$\downarrow$

we know the linearity of Lebesgue.

we know interchange linearity with $\int$

proof.

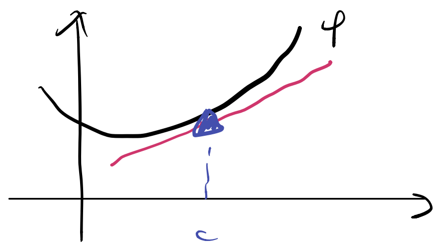
$l(x) = ax + b.$

$l\left(\int f \, d\mu\right) = a \int f \, d\mu + b.$

$= \int (af + b) \cdot d\mu. = \int l(f) \, d\mu.$

$$\varphi\left(\int f \, d\mu\right) \leq \int \varphi(f) \, d\mu.$$

$$\leq l\left(\int f \, d\mu\right) \leq \int l(f) \, d\mu.$$



$c = \int f \, d\mu.$

Hölder ineq If . $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$

then. $\int |fg| \, d\mu \le \underbrace{\left(\int |f|^p d\mu \right)^{\frac{1}{p}}}_{\|f\|_p} \underbrace{\left(\int |g|^q d\mu \right)^{\frac{1}{q}}}_{\|g\|_q}$

$\mathbb{E}|XY| \le (\mathbb{E}|X|^p)^{\frac{1}{p}} (\mathbb{E}|Y|^q)^{1/q}$

$p = q = 2$ Cauchy Schwarts.

f_n $n \ge 1$ $(\Omega, \mathcal{F}, \mu) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}), *)$

integrable.

If $f_n \to f$ in certain sense

Then $\lim_{n \to \infty} \int f_n \, d\mu \stackrel{?}{\Longrightarrow} \int f \, d\mu. = \int \lim_{n \to \infty} f_n \, d\mu.$

a.e. / a.s. convergence

$f_n \xrightarrow{a.s.} f$ $\rightsquigarrow$ basically pointwise convergence.

if. $\mu(\{\omega \in \Omega : f_n(\omega) \not\to f(\omega) \text{ as } n \to \infty\}) = 0.$

convergence in prob. / measure

$f_n \xrightarrow{m} f$

if $\forall \varepsilon > 0$ $\mu\{\omega \in \Omega : |f_n(\omega) - f(\omega)| > \varepsilon\} \to 0$

$$f_n(\omega) = \begin{cases} \sqrt{n} & \text{if } \omega \in [0, \frac{1}{n}) \\ 0 & \omega \in (\frac{1}{n}, 1) \end{cases}$$

$f_n \to f$ point wise.

$\omega \sim \text{Unif}[0,1]$ $\downarrow$ μ.

$f_n(\omega) \to f(\omega) \equiv 0 \quad \forall \omega \in (0,1)$

$\frac{1}{n} \times \sqrt{n} = \int f_n \, d\mu \not\to \int f \, d\mu = 0.$

Bounded Convergence thm.

$|f_n| \le M \quad f_n \xrightarrow{\mu} f \quad$ Then. $\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu.$

$\downarrow$

too strong

Pf. $\left| \int f_n \, d\mu - \int f \, d\mu \right| = \left| \int (f_n - f) \, d\mu \right|$

$\le \int |f_n - f| \, d\mu = \int |f_n - f| \mathbb{1}_{B_{n,\varepsilon}} \, d\mu + \int |f_n - f| \mathbb{1}_{B_{n,\varepsilon}^c} \, d\mu$

$\le \varepsilon + 2M \cdot \mu(B_{n,\varepsilon}) \to 0. \quad n \to \infty.$

$\varepsilon \to 0.$

$0 \le \limsup \left| \int f_n \, d\mu - \int f \, d\mu \right| \le \varepsilon.$

send $\varepsilon \downarrow 0 \Rightarrow \limsup_{n\to\infty} | \; | = 0$

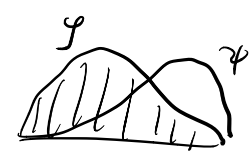
check $f_n(\omega) = \begin{cases} n & \omega \in (0, \frac{1}{n}) \\ 0 & \omega \in (\frac{1}{n}, 1) \end{cases}$

$\mu \sim \text{unif}[0,1]$

Fatou's Lemma

$f_n \ge 0$

$\liminf_{n\to\infty} \int f_n \, d\mu \ge \int \liminf_{n\to\infty} f_n \, d\mu.$



pf we have : Lebesgue Inte $\begin{cases} \text{monotonicity} \\ \text{linearility} \end{cases}$

BCT

need to construct a seq of func

$g_n = \inf_{m \ge n} f_m \uparrow$ take the inf of the tail part.

$g = \lim_{n\to\infty} g_n = \liminf_{n\to\infty} f_n$ Send n to ∞.

$f_n \geq g_n$,

$g_n \uparrow$ lim $\exists$

$$\liminf_{n\to\infty} \int f_n \, d\mu \geq \liminf_{n\to\infty} \int g_n \, d\mu \geq \liminf_{n\to\infty} \int g_n \wedge N \, d\mu$$

$$= \lim_{n\to\infty} \int g \wedge N \, d\mu$$

BCT $\leftarrow$

$$= \int g \wedge N \, d\mu$$

$$\liminf_{n\to\infty} \int f_n \, d\mu \geq \lim_{N\to\infty} \int g \wedge N \, d\mu = \int g \, d\mu$$

$$= \int \liminf_{n\to\infty} f_n \, d\mu$$

Monotone Convergence Theorem (MCT)

$f_n \geq 0$ $f_n \uparrow f$ a.s. as $n \to \infty$.

then. $\int f_n \, d\mu \uparrow \int f \, d\mu$.

Pf: By Fatou's Lemma.

$$\liminf_{n\to\infty} \int f_n \, d\mu \geq \int \underbrace{\liminf f_n}_{f} \, d\mu$$

$\geq$

$\int f_n \, d\mu \leq \int f \, d\mu$.

$\Downarrow$

$\limsup \int f_n \, d\mu \leq \int f \, d\mu$.

Dominated Convergence Theorem (DCT)

If $f_n \xrightarrow{a.s.} f$ $|f_n| \leq g$ some $g \geq 0$ uniform in n.

and $\int g \, d\mu < \infty$.

Then. $\int f_n \, d\mu \to \int f \, d\mu$.

Pf. $|f_n| \le g$ By Fatou's Lemma.

$f_n + g \ge 0$ $\liminf_{n\to\infty} \int f + g \, d\mu \ge$

$-f_n + g \ge 0$ $\int \liminf_{n\to\infty} (f_n + g)$.

①

$\therefore \liminf \int f_n \, d\mu + \cancel{\int g \, d\mu} \ge \int f \, d\mu + \cancel{\int g \, d\mu}$. Pf.

$\therefore \liminf_{n\to\infty} \int f_n \, d\mu \ge \int f \, d\mu$

②

$\liminf_{n\to\infty} \int -f_n + g \, d\mu \ge \int -f + g \, d\mu$.

$\Downarrow$

$\liminf_{n\to\infty} \left(-\int f_n \, d\mu\right) \ge -\int f \, d\mu$.

$\limsup_{n\to\infty} \int f_n \, d\mu \le \int f \, d\mu$.

$f_n(\omega) = \begin{cases} \sqrt{n} & \omega \in (0, \frac{1}{n}) \\ 0 & \omega \in [\frac{1}{n}, 1) \end{cases}$

$g(\omega) = \omega^{-\frac{2}{3}}$.

$\mu \sim \text{unif } [0,1]$

$\int g \, d\mu = \int_0^1 \omega^{-\frac{2}{3}} \, d\omega$.

$= \cdots < \infty$.

$\int f_n \, d\mu \to \int f \, d\mu = 0$. ✓

$f_n(\omega) = \begin{cases} n & \omega \in (0, \frac{1}{n}) \\ 0 & \omega \in [\frac{1}{n}, 1] \end{cases}$

$g(\omega) = \omega^{-1}$

$\int g(\omega) \, d\omega = \infty$. ✗

Thm.

Suppose $f_n \xrightarrow{a.s.} f$ Let g, h be two continuous functions satisfying

(i). $g \ge 0$ and $g(x) \to \infty$ as $|x| \to \infty$.

(ii) $\frac{h(x)}{g(x)} \to 0$ as $|x| \to \infty$

(iii) $\int g(f_n) \, d\mu \le K$ for all n.

$\Rightarrow \int h(f_n) \, d\mu \to \int h(f) \, d\mu$.