$X_1 \cdots$

$S_n = \sum_{i=1}^{n} X_i$ $\quad \frac{S_n}{n} \xrightarrow{?} \mathbb{E}X_1$

$\frac{S_n}{n}$: stochastic $\quad \mathbb{E}X_1$: non-stochastic.

P / D WLLN $\quad \because X \xrightarrow{D} c \Rightarrow X \xrightarrow{\mathbb{P}} c$ ✓

a.s. SLLN.

$$\frac{S_n}{n} - \mathbb{E}X \sim n^{-\alpha}$$

$n^{\alpha}\left(\frac{S_n}{n} - \mathbb{E}X\right) \xrightarrow[D]{?}$ distribution. CLT

$\frac{S_n}{n} = \mathbb{E}X + n^{-\alpha}\xi_n$ ⇝ second order fluctuation.

$\mathbb{E}X$: first order limit.

$\mathbb{E}|X|^r \cdots$ is a tool in order to get us rid of the necessity to know $f_X(x)$ for $\xrightarrow{P}$'s proof.

Thm. (L^2-WLLN) Let $X_1\, X_2 \cdots$ be uncorrelated r.v.s with

$$\mathbb{E}X_i = \mu. \quad Var(X_i) \le c < \infty$$

set $S_n = \sum_{i=1}^{n} X_i$

then. $\frac{S_n}{n} \xrightarrow{2} \mu. \quad \frac{S_n}{n} \xrightarrow{\mathbb{P}} \mu.$

$$\mathbb{P}\left(\left|\frac{S_n}{n} - \mu\right| > \varepsilon\right) = \mathbb{P}\left(\left|\frac{S_n}{n} - \mu\right|^2 > \varepsilon^2\right) \le \frac{\mathbb{E}\left|\frac{S_n}{n} - \mu\right|^2}{\varepsilon^2} = \frac{nc}{n\varepsilon^2} \to 0$$

$\mathbb{E}\left|\frac{S_n}{n} - \mu\right|^2 = Var\left(\frac{S_n}{n}\right) = \frac{1}{n^2}Var(S_n) = \frac{1}{n}Var(X)$

$O\left(\frac{1}{n}\right)$

when $n \to \infty$

Application 1. Bernstein approx

f: continuous on $[0,1]$ (bounded)

闭区间上连续函数 bounded, uniform. continuous.

$$f_n(x) = \sum_{m=0}^{n} \binom{n}{m} x^m (1-x)^{n-m} f\left(\frac{m}{n}\right) \qquad x \in [0,1]$$

Then. $\sup_{x\in[0,1]} |f_n(x) - f(x)| \to 0 \qquad n \to \infty$

$\mathrm{Bin}(n,x) = X_1 + \cdots + X_n \qquad X_i \sim \mathrm{Bern}(x)$

$= S_{n,x}$

$$f_n(x) = \sum_{n=0}^{n} \mathbb{P}(S_{n,x} = m) \cdot f\left(\frac{m}{n}\right) = \mathbb{E} f\left(\frac{S_{n,x}}{n}\right)$$

Task

$$\sup_{x\in[0,1]} \left|\mathbb{E} f\left(\frac{S_{n,x}}{n}\right) - f(x)\right| \to 0$$

How to do it in a uniform way?

$? \quad \mathbb{E} f\left(\frac{S_{n,x}}{n}\right) - f(x) \to 0$

$\Uparrow \quad \because f\left(\frac{S_{n,x}}{n}\right)$ is bounded

$? \quad f\left(\frac{S_{n,x}}{n}\right) - f(x) \xrightarrow{\mathbb{P}} 0 \qquad$ so $\xrightarrow{\mathbb{P}} \Rightarrow \xrightarrow{\mathbb{E}}$

$\Uparrow$ continuous mapping thm.

By WLLN $\qquad \frac{S_{n,x}}{n} - x \xrightarrow{\mathbb{P}} 0$

Since f is continuous $\forall \varepsilon > 0 \quad \exists \delta = \delta(\varepsilon) > 0$

s.t. $\left|f\left(\frac{S_{n,x}}{n}\right) - f(x)\right| \le \varepsilon \quad$ if. $\left|\frac{S_{n,x}}{n} - x\right| \le \delta$

$\Rightarrow \quad$ if $\left|f\left(\frac{S_{n,x}}{n}\right) - f(x)\right| > \varepsilon \quad$ then. $\left|\frac{S_{n,x}}{n} - x\right| > \delta$

$\therefore \mathbb{P}\left(\left|f\left(\frac{S_{n,x}}{n}\right) - f(x)\right| > \varepsilon\right) \le \mathbb{P}\left(\left|\frac{S_{n,x}}{n} - x\right| > \delta\right) \to 0$

$\forall x$ we can choose the same δ for uniform continuity

$$\sup_{x\in[0,1]} \left|\mathbb{E} f\left(\frac{S_{n,x}}{n}\right) - f(x)\right| \le \sup_{x\in[0,1]} \mathbb{E}\left|f\left(\frac{S_{n,x}}{n}\right) - f(x)\right|$$

$$\le \sup_{x\in[0,1]} \mathbb{E}\left|f\left(\frac{S_{n,x}}{n}\right) - f(x)\right| \mathbb{1}\left(\left|\frac{S_{n,x}}{n} - x\right| \le \delta\right)$$

$$+ \sup_{x\in[0,1]} \mathbb{E}\left|f\left(\frac{S_{n,x}}{n}\right)-f(x)\right| \mathbb{1}\left(\left|\frac{S_{n,x}}{n}-x\right|>\delta\right)$$

$$\leq \varepsilon + 2M \sup_{x\in[0,1]} \mathbb{P}\left(\left|\frac{S_{n,x}}{n}-x\right|>\delta\right)$$

$$\mathbb{P}\left(\left|\frac{S_{n,x}}{n}-x\right|>\varepsilon\right) \leq \frac{\operatorname{Var}\left(\frac{S_{n,x}}{n}\right)}{\varepsilon^2} = \frac{\operatorname{Var}(S_{n,x})}{n^2\varepsilon^2} = \frac{\operatorname{Var}(X)}{n\varepsilon^2} = \frac{\overbrace{x(1-x)}^{\leq \frac{1}{4}}}{n\varepsilon^2}$$

$$\sup_{x\in[0,1]} \mathbb{P}\left(\left|\frac{S_{n,x}}{n}-x\right|>\varepsilon\right) \leq \frac{1}{4n\varepsilon^2}$$

Application. (Borel's geometric concentration.)

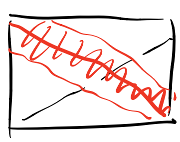
n-dim. cube $[-1,1]^n$

H: hyperplane $\perp$ $\underbrace{\text{principal diagonal}}_{\Downarrow\ (\underbrace{1,1\cdots 1}_{n})}$

$$H_r := \{x \in [-1,1]^n : \operatorname{dist}(x,H) \leq r\}$$

Conclusion. $\mu^n(H_{\varepsilon\sqrt{n}}) \to 1 \quad \forall \varepsilon>0.$

$n=2$



$H_{\varepsilon\sqrt{2}}$

Pf $\quad \mu_n(H_{\varepsilon\sqrt{n}})$

Let $X_1, X_2 \cdots$ be $\boxed{\text{i.i.d}} \sim \text{unif}[-1,1]$

$X=(X_1, X_2 \cdots X_n) \sim \mu_n.$

$$\mu_n(H_{\varepsilon\sqrt{n}}) = \mathbb{P}(X\in H_{\varepsilon\sqrt{n}})$$
$$= \mathbb{P}(\operatorname{dist}(X,H) \leq \varepsilon\sqrt{n})$$
$$= \mathbb{P}\left(\frac{|\langle X, (1\cdots 1)\rangle|}{\sqrt{n}} \leq \varepsilon\sqrt{n}\right)$$

$$= \mathbb{P}\left(\frac{|\Sigma x_i|}{n} \leq \varepsilon\right)$$

$$= 1 - \mathbb{P}\left(\frac{|\Sigma x_i|}{n} > \varepsilon\right)$$