PhD Written Qualifying Exam on Probability

(2020, Fall)

Time: 2 hours

Q1: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $A_1, A_2, \cdots$ be a sequence of events in the sigma field $\mathcal{F}$. Define

$$B_n = \cup_{m=n}^{\infty} A_m, \qquad C_n = \cap_{m=n}^{\infty} A_m.$$

Clearly $C_n \subset A_n \subset B_n$. The sequences $\{B_n\}$ and $\{C_n\}$ are decreasing and increasing respectively with limits

$$\lim B_n = B = \cap_n B_n = \cap_n \cup_{m \geq n} A_m, \qquad \lim C_n = C = \cup_n C_n = \cup_n \cap_{m \geq n} A_m.$$

The events B and C are denoted $\limsup_{n\to\infty} A_n$ and $\liminf_{n\to\infty} A_n$, respectively. Show that
(a) $B = \{\omega \in \Omega : \omega \in A_n$ for infinitely many values of $n\}$,
(b) $C = \{\omega \in \Omega : \omega \in A_n$ for all but finitely many values of $n\}$,
We say that the sequence $\{A_n\}$ converges to a limit $A = \lim A_n$ if B and C are the same set A. Suppose that $A_n \to A$ and show that
(c)A is an event, i.e. $A \in \mathcal{F}$,
(d) $\mathbb{P}(A_n) \to \mathbb{P}(A)$.

Q2:

(1): Write down the definitions of four convergence modes of random variables: convergence in r-th mean, convergence in probability, almost surely convergence, convergence in distribution.

(2): State the implications among four convergence modes, i.e, which convergence mode implies which.

(3): Raise examples to show that the following statements are **NOT** true in general: (i) convergence in probability implies almost surely convergence; (ii) almost surely convergence implies convergence in r-th mean; (iii) convergence in r-th mean implies almost surely convergence.

Q3: Prove that X_n converges to X in probability, if and only if, for any subsequence X_{n_m}, there exists a further subsequence $X_{n_{m(k)}}$ that converges to X almost surely.

Q4: Let $X_1, X_2, \ldots$ be i.i.d. with $\mathbb{P}(X_i = (-1)^k k) = C/(k^2 \log k)$ for $k \geq 2$, where C is chosen to make the sum of the probabilities $= 1$. Show that $\mathbb{E}|X_i| = \infty$, but there is a finite constant μ so that $S_n/n \to \mu$ in probability.

Q5: Let X_i's be i.i.d. random variables. Consider the random power series

$$\sum_{n=0}^{\infty} X_n z^n$$

Is there any (almost surely) deterministic radius of convergence of the above series in the following two cases (a): $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{2}$, (b): $X_i \sim N(0,1)$? If so, find the radius.

Q6: Prove the following Lindeberg-Feller CLT: For each n, let $X_{nm}, 1 \leq m \leq n$ be independent random variables with $\mathbb{E}X_{nm} = 0$. Suppose (i): $\sum_{m=1}^{n} \mathbb{E}X_{nm}^2 \to \sigma^2 > 0$; (ii): For all $\varepsilon > 0$, $\lim_{n\to\infty} \sum_{m=1}^{n} \mathbb{E}X_{nm}^2 \mathbf{1}(|X_{nm}| \geq \varepsilon) = 0$. Then

$$S_n := \sum_{m=1}^{n} X_{nm} \xrightarrow{D} N(0, \sigma^2).$$