$(\Omega, \mathcal{F}, \mu) \xrightarrow{g} (\mathbb{R}, \mathcal{B}(\mathbb{R}), \cdots)$

μ: probability measure

eg. $(\Omega, \mathcal{F}, \mu) \xrightarrow{g} (\mathbb{R}, \mathcal{B}(\mathbb{R}), \cdots)$

eg. $(\Omega, \mathcal{F}, \mu) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$

Notation $\mathbb{E}g(\omega) = \int g(\omega)\cdot dF(\omega) = \int g(\omega)\cdot d\mu(\omega)$.

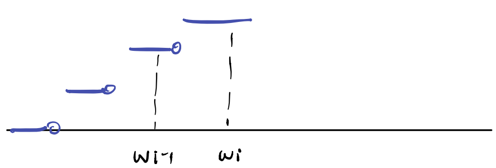
First try: Riemann – Stieltjes integral.

$$\int g(\omega)\cdot d\omega \triangleq \lim_{\max\{\omega_i - \omega_{i-1}\} \to 0} \sum_i g(\omega_i^*)\cdot(\omega_i - \omega_{i-1}) \qquad \forall \omega^* \in [\omega_{i-1}, \omega_i]$$

Nature number has no real order

$$\int g(\omega)\, dF(\omega)) = \lim_{\max|\omega_i - \omega_{i-1}| \to 0} \sum_i g(\omega_i^*).(F(\omega_i) - F(\omega_{i-1}))$$

$\sim f(\omega_i^*)(\omega_i - \omega_{i-1})$



eg. Dirichlet function.

$\omega \sim U[0,1]$

$g(\omega) = \begin{cases} 1 & \omega \text{ is rational} \\ 0 & \omega \text{ is irrational} \end{cases}$

Bernoulli

$$\mathbb{E}g(\omega) = \lim_{\max|\omega_i - \omega_{i-1}| \to 0} \sum_i g(\omega_i^*)(\omega_i - \omega_{i-1})$$

also always an irrational num.

always a rational num.

$\mathbb{E}\, g(\omega) = 1\cdot \mu\{\omega\in\Omega : g(\omega)=1\}$

$+ 0\cdot \mu\{\omega\in\Omega : g(\omega)=0\}$

$= 1\cdot\mu\{\mathbb{Q}\cap[0,1]\} + 0\cdot\mu\{[0,1]\setminus\mathbb{Q}\} = 0$

The diff is how you decompose the interval.

Lebesgue integral.

Step 1. (simple function)

$A_i \in \mathcal{F}$

$A_i \in \mathcal{F}$ A_i's disjoint.

A_i's disjoint

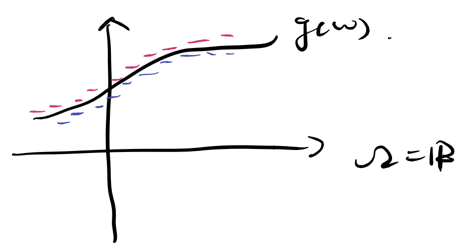
$g(\omega) = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i}(\omega)$

($\bigcup_{i=1}^{n} A_i$ may not Ω)

(a_i's could be the same).

Def. $\int g \, d\mu = \sum_{i=1}^{n} a_i \mu(A_i)$

Step. 2. (bounded function)



property If φ ψ are simple.

$\varphi > \psi$ a.s. $\mu\{\omega\in\Omega : \psi(\omega) > \varphi(\omega)\} = 0$

(i) If $\varphi \geq 0$ a.e. then $\int \varphi \, d\mu \geq 0$

(ii) If $c \in \mathbb{R}$ then $\int c\varphi \, d\mu = c \int \varphi \, d\mu$.

(iii) $\int \varphi + \psi \, d\mu = \int \varphi \, d\mu + \int \psi \, d\mu$.

(iv) If $\psi \geq \varphi$ a.s. then $\int \psi \, d\mu \geq \int \varphi \, d\mu$.

(v) $=$ $=$

(vi) $\left|\int \varphi \, d\mu\right| \leq \int |\varphi| \, d\mu$.

$\sup_{\omega} |g(\omega)| \le M$

$E_k := \{\omega \in \Omega : \frac{(k-1)}{N} \cdot M \le g(\omega) \le \frac{k}{N} M\}$

N is any positive integer $\quad -N \le k \le N$

$\varphi_N := \sum_{k=-N}^{N} \frac{(k-1)M}{N} \mathbb{I}_{E_k} \qquad \varphi_N \le g \le \psi_N$

$\psi_N := \sum_{k=-N}^{N} \frac{kM}{N} \mathbb{I}_{E_k}$.

$\sup_{\substack{\varphi \le g \\ \varphi \text{ is simple}}} \int \varphi \, d\mu \ge \int \varphi_N \, d\mu = \int \psi_N \, d\mu - \frac{M}{N}$

$\ge \inf_{\substack{\psi \ge g \\ \psi \text{ is simple.}}} \int \psi \, d\mu - \frac{M}{N}$

nothing to do with N.

send N to ∞.

$\le$

$\Downarrow$

$=$

Generalize the property to bounded function.

Step 3. (Nonnegative function)

from prop

$\int g \wedge N_1 \, d\mu \le \int g \wedge N_2 \, d\mu$.

if $N_1 \le N_2$

$\int g \, d\mu := \lim_{N \uparrow \infty} \boxed{\int \underbrace{g \wedge N}_{\min\{g, N\}} \, d\mu}$

always exists $[0, \infty]$

If $< \infty$ we say

g is Lebesgue integrable.

Step. 4 (General Integrable function)

We say g is Lebesgue integrable if $\int |g|\, d\mu < \infty$. We define

$$\int g\, d\mu := \int g_+\, d\mu - \int g\, d\mu.$$

$g_+ = g \vee 0 \qquad g_- = (-g) \vee 0$

$g = g_+ - g_- \qquad |g| = g_+ + g_-$

absolutely convergent .

$$\int |g|\, d\mu = \sum_{i=1}^{\infty} |a_i|\, \mu(A_i) < \infty$$

$$\int g\, d\mu = \sum_{i=1}^{\infty} a_i\, \mu(A_i)$$

$$g = \sum_{i=1}^{\infty} a_i \mathbb{1}_{A_i}$$