eg. flip two coins.

$\Omega = \{HH, HT, TH, TT\}$

$X(\omega) = \#$ of H's in ω.

UG: r.v. X : physical outcome $\downarrow$ $\mathbb{R}$

$\Omega \xrightarrow{X} \mathbb{R}$

eg. rope of length 1. $[0,1]$

cut the rope at ω.

$\Omega = [0,1]$

$X(\omega) = \min\{\omega, 1-\omega\}$.

uncertainty of ω

$\Downarrow$

uncertainty of $X(\omega)$.

Q: probability of $X \in B$ $\quad B \in \mathbb{R}$ $\forall$ interval.

- what kind of B we want to consider

$\mathbb{P}(X \in B) = \mathbb{P}\{\omega \in \Omega : X(\omega) \in B\} = \mathbb{P}(X^{-1}(B))$

$(\Omega, \mathcal{F}, \mathbb{P})$ $\quad$ $\{\omega \in \Omega : X(\omega) \in B\}$ $=$ a subset of ω.

$\underbrace{X^{-1}(B)\text{: preimage of } B.}$

$\downarrow$

You have to guarantee $X^{-1}(B)$ $\forall B \in \mathcal{F}$

Def. Given a measurable space $(\Omega, \mathcal{F})$

We call $X: \Omega \to \mathbb{R}$ a random variable,

if $X^{-1}(B) \in \mathcal{F}$ $\forall$ $\boxed{\text{Borel set}}$ B.

$\downarrow$

replaced by $(a,b]$, (a,b), $(-\infty, b]$

Claim. X^{-1} can be interchange with any set operation.

eg. $X^{-1}(A \cup B) = X^{-1}(A) \cup X^{-1}(B)$.

$X^{-1}(A^c) = (X^{-1}(A))^c$.

In this case we say X is $\mathcal{F}$-measurable.

$\boxed{\text{written as } X \in \mathcal{F}}$

$$(\Omega, \mathcal{F}) \xrightarrow{X} (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$

$(\Omega, \mathcal{F}, \mathbb{P})$ $\quad \Omega = \{HH \ HT \ TH \ TT\}$

eg. $X(HH) = 2 \quad X(HT) = X(TH) = 1 \quad X(TT) = 0$

$$X^{-1}((-\infty, b]) = \begin{cases} \phi & b < 0 \\ TT & 0 \le b < 1 \\ HT \cup TH \cup TT & 1 \le b < 2 \\ \Omega & b \ge 2 \end{cases}$$

$\{\omega : X(\omega) \in (-\infty, b]\}$

eg. Assume $\mathcal{F} = 2^{\Omega}$. $\to$ Yes X is a r.v.

eg. Assume $\mathcal{F} = \{\phi, \Omega\} \to X$ is not a r.v.

Modern viewpoint.

$$(\Omega, \mathcal{F}, \mathbb{P}) \xrightarrow{X} (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbb{Q})$$

$\omega \quad\quad X(\omega)$

$\mathbb{Q} : \mathcal{B}(\mathbb{R}) \to [0, 1] \quad \mathbb{Q}(B) = \mathbb{Q}(X \in B) \quad B \in \mathcal{B}(\mathbb{R})$

$\mathbb{P} : \mathcal{F} \to [0, 1] \quad \mathbb{P}(E) = \mathbb{P}(\omega \in E) \quad E \in \mathcal{F}$

set fn on $\mathcal{B}(\mathbb{R})$

$\mathbb{Q}(B) := \mathbb{P}(X^{-1}(B)) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \in B\})$

set fn on $\mathcal{F}$

check

$\mathbb{Q}(\emptyset) = 0$ $\mathbb{Q}(\mathbb{R}) = 1$ countable additivity

Denote $\mathbb{Q}$ by $\boxed{\mathbb{P} \circ X^{-1}}$

probability measure

push forward. measure.

induced measure.

distribution of X