$(X_n)_{n\in\mathbb{N}}$ independent. $\sum_{n=1}^{\infty} X_n$ converges almost surely in $\mathbb{R}$ for $A>0$

iff: (i) (ii) (iii) hold for $A>0$

(i) $\sum_{n=1}^{\infty} \mathbb{P}(|X_n|>A)$ converges.

(ii) $\sum_{n=1}^{\infty} \mathbb{E}[X_n \mathbb{I}\{|X_n|\le A\}]$ converges.

(iii) $\sum_{n=1}^{\infty} \mathrm{Var}(X_n \mathbb{I}(|X_n|\le A))$ converges.

Thm. Kolmogorov's maximal inequality

Suppose $X_1 \cdots X_n$. $\perp\!\!\!\perp$ with. $\mathbb{E}X_i=0$ $\mathrm{Var}(X_i)<\infty$ $S_n = X_1+X_2+\cdots+X_n$.

$$\mathbb{P}\left(\max_{1\le k\le n}|S_k|\ge x\right) \le x^{-2}\,\mathrm{Var}(S_n)$$

Thm. When $\sum_{n=1}^{\infty} X_n(\omega)$ converges?

Suppose $X_1\, X_2 \cdots$ $\perp\!\!\!\perp$ $\mathbb{E}X_i=0$ $\sum_{n=1}^{\infty}\mathrm{Var}(X_n)<\infty$ then.

with. probability one $\sum_{n=1}^{\infty} X_n(\omega)$ converges.

pseudo pf:

Aim. $S_n(\omega)$ is a Cauchy sequence.

$\Rightarrow \lim_{n\to\infty} S_n(\omega)$ exists.

$$w_M = \sup_{n,m\ge M} |S_m - S_n|$$

$$\mathbb{P}(w_M > 2\varepsilon) \le \mathbb{P}\left(\sup_{m\ge M}|S_m - S_M| > \varepsilon\right) \to 0$$

Thm. Kolmogorov's three-series. Thm.

$X_1, X_2 \cdots$ $\perp\!\!\!\perp$ $Y_i \triangleq X_i \mathbb{I}(|X_i|\le A)$

$\sum_{n=1}^{\infty} X_n$ converges. a.s. $\Leftrightarrow$ (i). $\sum_{i=1}^{n}\mathrm{Var}(Y_i)<\infty$.

(ii) $\sum_{i=1}^{n}\mathbb{E}(Y_i)$ converges.

(iii) $\sum_{n=1}^{\infty}\mathbb{P}(|X_n|>A)<\infty$

Thm. The strong Law of Large Number.

$X_1\ X_2 \cdots$ i.i.d. $\mathbb{E}|X_i| < \infty$ $\mathbb{E}X_i = \mu$. $S_n = X_1 + \cdots X_n$.

Then. $\frac{S_n}{n} \xrightarrow{a.s.} \mu$. $n \to \infty$

Rate of Convergence.

Thm. elementary Rate.

$X_1\ X_2 \cdots$ iid. r.v. with. $\mathbb{E}X_i = 0$ $\mathbb{E}X_i^2 = \sigma^2 < \infty$ $S_n = X_1 + \cdots + X_n$.

then. $S_n / n^{1/2} (\log n)^{1/2+\delta} \to 0$ a.s.

Thm. second Rate

$X_1\ X_2 \cdots$ iid. r.v. with. $\mathbb{E}X_i = 0$ $\mathbb{E}|X_1|^p < \infty$ $1 < p < 2$

$S_n = X_1 + \cdots + X_n$. then. $S_n / n^{1/p} \to 0$ a.s.

P 165 1490 **Rates of Convergence (Berry-Esseen.)**

Thm. $X_1 \cdots$ iid. with. $\mathbb{E}X_i = 0$ $\mathbb{E}X_i^2 = \sigma^2$ $\mathbb{E}|X_i|^3 = \rho < \infty$

If $F_n(x)$ is + . d.f of. $(X_1 + \cdots + X_n) / \sigma\sqrt{n}$

and. $\mathcal{N}(x)$ is std then.

$$|F_n(x) - \mathcal{N}(x)| \le 3\rho / \sigma^3 \sqrt{n}.$$