σ-field generated by r.v. X

$\sigma(X) = \{ X^{-1}(C) : C \in \mathcal{B}(\mathbb{R}) \} \subset \mathcal{F}$

- smallest σ-field makes X a r.v.
- info carried by r.v. X

Claim. If $\mathcal{H} \perp\!\!\!\perp \mathcal{G}$ $X \in \mathcal{H}$ $Y \in \mathcal{G}$

$\Updownarrow$

$\sigma(X) \subset \mathcal{H}$

Extension to more than 2 object

① σ-field $\mathcal{F}_1, \mathcal{F}_2 \cdots \mathcal{F}_n \subset \mathcal{F}$ are independent

if for any $A_i \in \mathcal{F}_i$ $i=1,\cdots n$

$\mathbb{P}(\bigcap_{i=1}^{n} A_i) = \prod_{i=1}^{n} \mathbb{P}(A_i)$

choose $B_i = \mathbb{R}$ 则和③ 叙述一样 用

② r.v.s. $X_1 \cdots X_n$ are independent if $\forall B_i \in \mathcal{B}(\mathbb{R})$ $i=1,\cdots n$ 了即可

$\mathbb{P}(\bigcap_{i=1}^{n} X_i^{-1}(B_i)) = \prod_{i=1}^{n} \mathbb{P}(X_i^{-1}(B_i))$

③. events $A_1 \cdots A_n \in \mathcal{F}$ are independent if

$\forall$ any $I \subset \{1,\cdots n\}$

因为不一定所有 $A_i = \Omega$

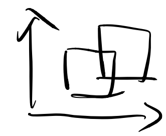
$P(\bigcap_{i\in I} A_i) = \prod_{i\in I} P(A_i)$

Sufficient condition for independence.

π-system. We say a collect. $\mathcal{A}$ is a π-system if it's nonempty if it's closed under intersection

that is $A, B \in \mathcal{A}$ then $A \cap B \in \mathcal{A}$

eg. collection of all rectangles in $\mathbb{R}^d$.



eg. $\{X^{-1}((-\infty, x]) : x \in \mathbb{R}\} = \mathcal{A}$

$X^{-1}((-\infty, a]) \cap X^{-1}((-\infty, b])$ $\quad a < b$

$= X^{-1}((-\infty, a]) \in \mathcal{A}$

Δ The set operation can interchange with the X^{-1}

Def Collection of sets $\mathcal{A}_1 \cdots \mathcal{A}_n \subset \mathcal{F}$

are said to be independent if for all $A_i \in \mathcal{A}_i$

$\forall I \subset \{1, \cdots n\}$

$P(\bigcap_{i \in I} A_i) = \prod_{i \in I} P(A_i)$

Thm Suppose $\mathcal{A}_1 \cdots \mathcal{A}_n$ are independent.

and all $\mathcal{A}_i$'s are π-system

$\Rightarrow \sigma(\mathcal{A}_1) \cdots \sigma(\mathcal{A}_n)$ are independent.

Thm. In order to show $X_1 \cdots X_n$ are independent

it's sufficient to check for all $x_1 \cdots x_n \in \mathbb{R}$ s.t.

$\mathbb{P}(\bigcap_{i=1}^{n} X_i^{-1}(-\infty, x_i]) = \prod_{i=1}^{n} \mathbb{P}(X_i^{-1}(-\infty, x_i])$

$\mathbb{P}(X_1 \le x_1 \cdots X_n \le x_n) = \prod_{i=1}^{n} \mathbb{P}(X_i \le x_i)$

- pre image of half line is a π-system.
- pre image of half line can generate the $\sigma(X)$

Thm.

$$X_{1,1} \cdots X_{1,m(1)} \to f(X_{1,1} \cdots X_{1,m(2)})$$

$$X_{2,1} \cdots X_{2,m(2)} \to f(X_{2,1} \cdots X_{2,m(2)})$$

$$\vdots \qquad\qquad \vdots$$

$$X_{n,1} \cdots X_{n,m(n)}$$

$f_1 \cdots f_n$ are independent