Mixed Clifford + non-Clifford execution via Clifford frames

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Consider a quantum circuit U composed of m gates, which may be Clifford or non-Clifford:

$$U = g_m g_{m-1} \cdots g_1,$$

Let the logical state at the kth computational step be

$$|\psi_k\rangle = g_k g_{k-1} \cdots g_1 |\psi_0\rangle. \tag{1}$$

We maintain:

- a classical *Clifford frame* $F_k \in \mathcal{C}_n$,

- a physical state $|\phi_k\rangle$,

such that the invariant

$$|\phi_k\rangle = F_k |\psi_k\rangle \tag{2}$$

holds for every k.

Base case: Initially,

$$F_0 = \mathsf{I}, \qquad |\phi_0\rangle = |\psi_0\rangle, \tag{3}$$

so (2) trivially holds at $k = 0$.

Clifford step: If $g_{k+1} = C \in \mathcal{C}_n$ is Clifford, we perform no physical operation and simply update

$$F_{k+1} := C\, F_k, \qquad |\phi_{k+1}\rangle := |\phi_k\rangle. \tag{4}$$

Since

$$|\psi_{k+1}\rangle = C|\psi_k\rangle, \tag{5}$$

we have by (2), (4), (5):

$$\begin{aligned}
F_{k+1}|\psi_{k+1}\rangle &= (CF_k)(C|\psi_k\rangle) \\
&= CF_k|\psi_k\rangle \\
&= C|\phi_k\rangle \\
&= |\phi_k\rangle = |\phi_{k+1}\rangle.
\end{aligned} \tag{6}$$

Thus the invariant (2) is preserved.

Non-Clifford step: Suppose $g_{k+1} = N$ is non-Clifford. Define the physical gate

$$N_{\text{phys}} := F_k\, N\, F_k^\dagger. \tag{7}$$

Apply N_{phys} to hardware and keep the frame unchanged:

$$|\phi_{k+1}\rangle := N_{\text{phys}}|\phi_k\rangle, \qquad F_{k+1} := F_k. \tag{8}$$

By (1),

$$|\psi_{k+1}\rangle = N|\psi_k\rangle. \tag{9}$$

Then, using (7),(8), (9),

$$\begin{aligned}
|\phi_{k+1}\rangle &= (F_k N F_k^\dagger)|\phi_k\rangle \\
&= (F_k N F_k^\dagger)(F_k|\psi_k\rangle) \\
&= F_k N|\psi_k\rangle \\
&= F_k|\psi_{k+1}\rangle.
\end{aligned} \tag{10}$$

Hence (2) also holds at $k+1$.

Final state. By induction using (1), (6), (10), the invariant (2) holds for all k. Thus

$$|\phi_m\rangle = F_m|\psi_m\rangle = F_m U|\psi_0\rangle. \tag{11}$$

I.e., the hardware state $|\phi_m\rangle$ is the logical output $U|\psi_0\rangle$ up to the final Clifford frame F_m. Thus the overall computation is equivalent to applying U directly.

If one requires the *exact* logical output on hardware, apply the known Clifford inverse at the end:

$$|\phi_m'\rangle := F_m^\dagger|\phi_m\rangle = U|\psi_0\rangle. \tag{12}$$

Conjugation of Pauli rotations. The only Non-Clifford gate necessary to reach universality is the single-qubit Z-rotation $R_Z(\theta) = e^{-i\theta/2Z}$. Since Cliffords conjugate Paulis to Paulis, the non-Cliffords remain on this form. I.e., if

$$N = e^{-i\theta/2P}, \qquad P \in \mathcal{P}_n,$$

the frame action satisfies

$$F_k N F_k^\dagger = e^{-i\theta/2(F_k P F_k^\dagger)} = R_{P'}(\theta), \tag{13}$$

That is, we just get another rotation by the same angle around a different Pauli axis $P' = F_k P F_k^\dagger$.

Unproblematic examples: Conjugating by single-qubit Cliffords and CX with control on the non-Clifford qubit don't increase complexity:

$$HR_Z(\theta)H^\dagger = He^{-i\theta/2Z}H = e^{-i\theta/2HZH} = e^{-i\theta/2X} = R_X(\theta). \tag{14}$$

$$\text{CX}\,(R_Z(\theta) \otimes I)\,\text{CX}^\dagger = \text{CX}\,e^{-i\theta/2(Z\otimes I)}\,\text{CX} \tag{15}$$

$$= e^{-i\theta/2\,\text{CX}\,(Z\otimes I)\,\text{CX}} = e^{-i\theta/2(Z\otimes I)} = R_Z(\theta) \otimes I. \tag{16}$$

Problematic example: Conjugating by CX with target on the non-Clifford qubit turns single-qubit rotation into two-qubit rotation:

$$\begin{aligned}
\text{CX}\,(I \otimes R_Z(\theta))\,\text{CX}^\dagger &= \text{CX}\,e^{-i\theta/2(I\otimes Z)}\,\text{CX} \\
&= e^{-i\theta/2\,\text{CX}\,(I\otimes Z)\,\text{CX}} = e^{-i\theta/2(Z\otimes Z)} = R_{ZZ}(\theta).
\end{aligned} \tag{17}$$

Happy example: Conjugating $R_{ZZ}(\theta)$ by CX with target on one of the qubits returns a single-qubit rotation:

$$\begin{aligned}
\text{CX}\,(R_{ZZ}(\theta))\,\text{CX}^\dagger &= \text{CX}\,e^{-i\theta/2(Z\otimes Z)}\,\text{CX} \\
&= e^{-i\theta/2\,\text{CX}\,(Z\otimes Z)\,\text{CX}} = e^{-i\theta/2(Z\otimes I)} = R_Z(\theta) \otimes I.
\end{aligned} \tag{18}$$

That is, CX can both increase and decrease the qubit count of the non-Clifford rotation.