Hybrid Quantum–Classical Computation via Clifford Frame Tracking

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Outline

Introduction

- Quantum hardware is noisy and expensive.
- Physical qubits are prone to decoherence.
- Many quantum gates are classically simulable.

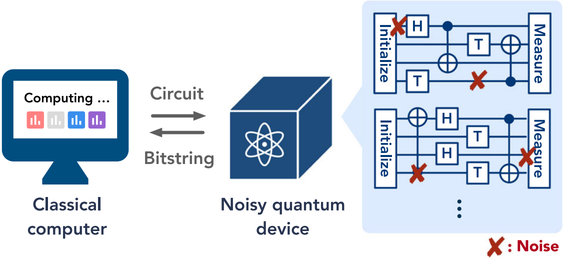

Figure: Schematic of a hybrid algorithm implementation (NISQ) [1]

General quantum gates

Any quantum circuit can be decomposed as

$$U = C_{T+1} N_T C_T \cdots N_1 C_1,$$

where $C_k \in \mathcal{C}_n$ are Clifford unitaries and N_k are non-Clifford unitaries.

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Example of universal quantum gate set

$$\{H, S, CNOT, T\}, \qquad T = e^{-i\frac{\pi}{8}Z}.$$

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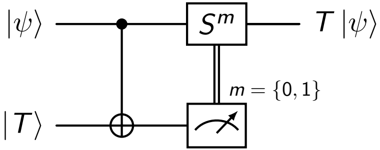
Physical implementation of T-state injection:

$$|T\rangle = \frac{|0\rangle + e^{i\pi/4}|1\rangle}{\sqrt{2}}$$



Clifford Simulation

From the stabilizer formalism,

$$\mathsf{Stab}(|\psi\rangle) = \langle P_1, \ldots, P_n \rangle \quad \Longleftrightarrow \quad \mathsf{Stab}(U|\psi\rangle) = \langle UP_1U^\dagger, \ldots, UP_nU^\dagger \rangle,$$

where $P_j \in \mathcal{P}_n$ and $U \in \mathcal{C}_n$.

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A stabilizer generator $P = P_j$ can be represented by bit-vectors:

$$P = i^{p(\mathrm{x},\mathrm{z})} X^{\mathrm{x}} Z^{\mathrm{z}}, \qquad (\mathrm{x}|\mathrm{z}) = \left(x_1\ x_2\ \cdots\ x_n \mid z_1\ z_2\ \cdots\ z_n\right).$$

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For each new generator $UPU^\dagger$,

$$(\mathrm{x}|\mathrm{z}) \;\mapsto\; (\mathrm{x}|\mathrm{z})\, M \;\in\; \mathbb{F}_2^{2n}$$

where $M \in \mathsf{Sp}(2n, \mathbb{F}_2)$ is the **symplectic matrix** of the conjugation.

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In the tableau format:

$$(\mathbf{D}, \mathbf{C})^\top = \mathrm{diag}(\mathbf{x}, \mathbf{z})M \quad \in \mathbb{F}_2^{2n\times 2n},$$

where $\mathbf{x} = (\mathrm{x}_1, \ldots, \mathrm{x}_n)^\top$ and $\mathbf{z} = (\mathrm{z}_1, \ldots, \mathrm{z}_n)^\top$.

Frame Tracking

Define

$$|\phi_k\rangle := F_k |\psi_k\rangle \, ,$$

where $F_k = C_k C_{k-1} \cdots C_1 \in \mathcal{C}_n$ is a sequence of unitaries interleaved by non-Clifford unitaries N.

Time evolution for non-Clifford gate

$$|\psi_{k+1}\rangle = N |\psi_k\rangle \, , \qquad |\phi_{k+1}\rangle = M |\phi_k\rangle \, ,$$

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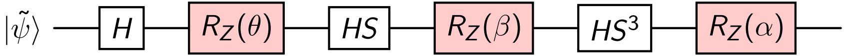
Pauli rotation as a non-Clifford gate:

$$N = e^{-i\theta/2P}, \quad P \in \mathcal{P}_n.$$

Frame Tracking: Single-qubit circuit

Exact circuit



Frame Tracking: Single-qubit circuit

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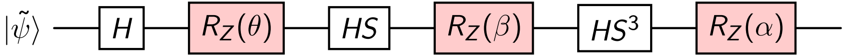
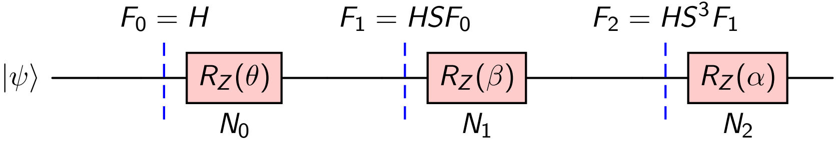


Logical circuit



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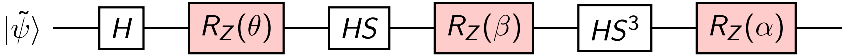

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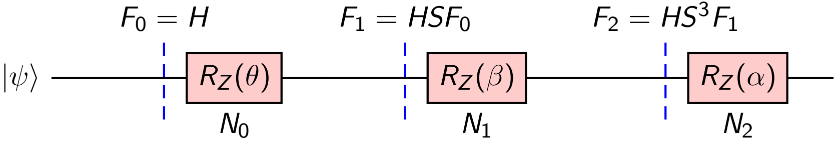
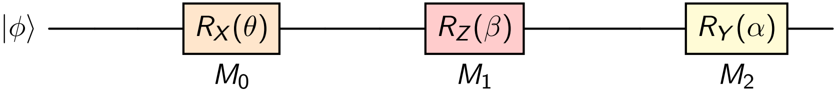


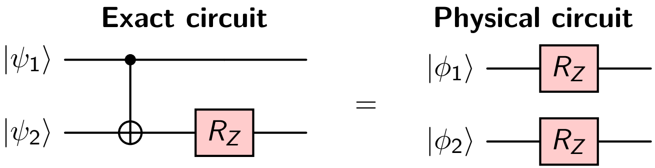
$$|\phi_{k+1}\rangle = F_k N F_k{}^\dagger |\phi_k\rangle$$

Physical circuit



Frame Tracking: Two-qubits circuits

Example 1

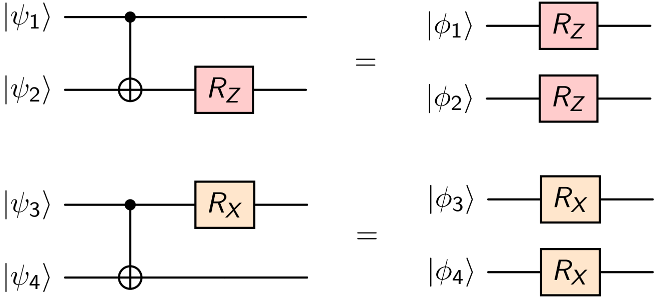


Frame Tracking: Two-qubits circuits

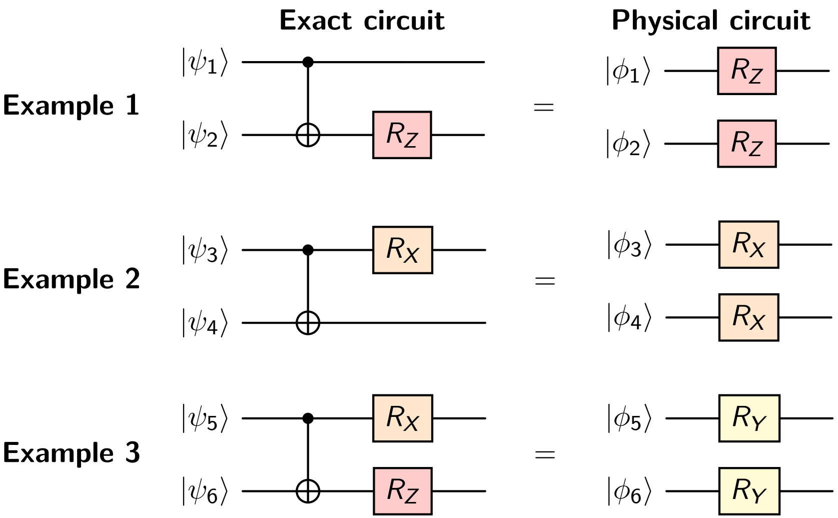
Exact circuit **Physical circuit**

Example 1

Example 2



Frame Tracking: Two-qubits circuits



CNOT gate

- Clifford propagation through CNOT gates [2].

$$CX\left(C_1 \otimes C_2\right) = \left(C_1' \otimes C_2'\right) CX.$$

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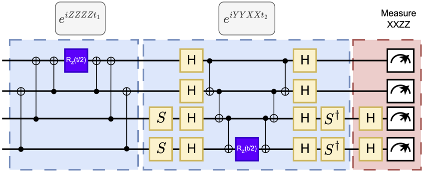

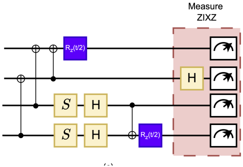

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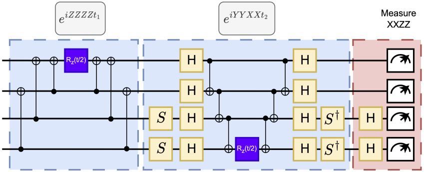

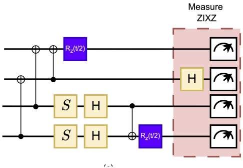

- Correction with ancilla qubit.

Logic of the implementation

$$U = g_m g_{m-1} \cdots g_1$$

Clifford-gate

$$F_{k+1} := C\,F_k,$$

$$|\phi_{k+1}\rangle := |\phi_k\rangle$$

Non-Clifford-gate

$$N_{\text{phys}} := F_k\,N\,F_k^\dagger$$

$$|\phi_{k+1}\rangle := N_{\text{phys}}|\phi_k\rangle, \quad F_{k+1} := F_k$$

Final state: $|\phi_m\rangle = F_m|\psi_m\rangle = F_m U|\psi_0\rangle$

References I

1. Chen, S., Cotler, J., Huang, H.-Y. & Li, J. The complexity of NISQ. en. *Nature Communications* **14,** 6001. ISSN: 2041-1723. https://www.nature.com/articles/s41467-023-41217-6 (2026) (Sept. 2023).
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References II

4. Liu, J., Gonzales, A., Huang, B., Saleem, Z. H. & Hovland, P. *QuCLEAR: Clifford Extraction and Absorption for Quantum Circuit Optimization.* en. arXiv:2408.13316 [quant-ph]. Feb. 2025. http://arxiv.org/abs/2408.13316 (2025).