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Multi-agent team cooperation: A game theory approach☆

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ABSTRACT

The main goal of this work is to design a team of agents that can accomplish consensus over a common value for the agents' output using cooperative game theory approach. A semi-decentralized optimal control strategy that was recently introduced by the authors is utilized that is based on minimization of individual cost using local information. Cooperative game theory is then used to ensure team cooperation by considering a combination of individual cost as a team cost function. Minimization of this cost function results in a set of Pareto-efficient solutions. Among the Pareto-efficient solutions the Nash-bargaining solution is chosen. The Nash-bargaining solution is obtained by maximizing the product of the difference between the costs achieved through the optimal control strategy and the one obtained through the Pareto-efficient solution. The latter solution results in a lower cost for each agent at the expense of requiring full information set. To avoid this drawback some constraints are added to the structure of the controller that is suggested for the entire team using the linear matrix inequality (LMI) formulation of the minimization problem. Consequently, although the controller is designed to minimize a unique team cost function, it only uses the available information set for each agent. A comparison between the average cost that is obtained by using the above two methods is conducted to illustrate the performance capabilities of our proposed solutions.

1. Introduction

Sensor networks (SN), and in general, unmanned system networks (UMSN) are currently one of the strategic areas of research in different disciplines, such as communications, controls, and mechanics. These networks can potentially consist of a large number of agents, such as unmanned aerial vehicles (UAV), unmanned ground vehicles (UGV), unmanned underwater vehicles (UUV), and satellites. Wireless UMSN provide significant capabilities and numerous applications in various fields of research are being considered and developed. Some of these applications are in home and building automation, intelligent transportation systems, health monitoring and assisting, space explorations, and commercial applications (Sinopoli, Sharp, Schenato, Schafferthim, & Sastry, 2003). There are also military applications in intelligence, surveillance, and reconnaissance (ISR) missions in the presence of environmental disturbances, vehicle failures, and in battlefields subject to unanticipated uncertainties and adversarial actions (Bošković, Li, & Mehra, 2002).

One of the prerequisites for these networked agents that are intended to be deployed in challenging missions is team cooperation and coordination for accomplishing predefined goals and requirements. Cooperation in a network of unmanned systems, known as formation, network agreement, flocking, consensus, or swarming in different contexts, has received extensive attention in the past several years. Several approaches to this problem have been investigated within different frameworks and by considering different architectures (Arcak, 2006; Gazi, 2002; Lee & Spong, 2006; Olfati-Saber & Murray, 2002, 2003b,c, 2004; Paley, Leonard, & Sepulchre, 2004; Ren, 2007; Ren & Beard, 2004; Semsar & Khorasani, 2006; Stipanović, Inalhan, Teo, & Tomlin, 2004; Xiao & Wang, 2007).

An optimal approach to team cooperation problem is considered in Raffard, Tomlin, and Boyd (2004) and Inalhan, Stipanović, and Tomlin (2002) for formation keeping and in Bauso, Giarre, and Pesenti (2006) and Semsar-Kazerooni and Khorasani (2007a,b,c, 2008, 2009) for consensus seeking. The approach in Inalhan et al. (2002) is based on individual agent cost optimization for achieving team goals under the assumption that the states of the other team members are constant. The concept of Nash equilibrium is used for design of optimal controllers. In order to solve an optimal consensus problem, the authors in Bauso et al. (2006) have assumed an individual agent cost for each team member. In evaluating the minimum value of each individual cost, the states of the other agents

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are assumed to be constant. The work in Semsar-Kazerooni and Khorasani (2007a,b,c, 2008, 2009) have avoided and removed the above restricting assumptions by decomposing the control input of each team member into local and global components. The global component (referred to as the interaction term) is designed such that individual agent cost function is minimized in a distributed manner. In all the above referenced work the optimal problem is based on the individual cost definition for team members. However, to the best of the authors' knowledge, a single team cost function formulation has been proposed in only a few works (Fax, 2002; Raffard et al., 2004; Semsar & Khorasani, 2007). In Fax (2002), optimal control strategy is applied for formation keeping and a single team cost function is utilized. The authors in Raffard et al. (2004) assumed a distributed optimization technique for formation control in a leader–follower structure. The design is based on dual decomposition of the local and global constraints. However, in that approach, the velocity and position commands are assumed to be available to the *entire* team. In Semsar and Khorasani (2007), a *centralized* solution is obtained by using a game theoretic approach.

It is worth noting that a very few work in this domain use a design-based approach. In fact, many of the earlier work in the literature have focused on analysis only (Jadbabaie, Lin, & Morse, 2003; Olfati-Saber, 2006; Olfati-Saber & Murray, 2004; Ren & Beard, 2005; Tanner, Jadbabaie, & Pappas, 2007). However, the main contribution of this paper is to introduce a novel design-based approach to formally design a controller (the consensus algorithm) to address the output consensus over a common value using a single team cost function within a game theoretic framework. The advantage of minimizing a cost function that describes the total performance of the team is that it can provide a better insight into performance of the entire team when compared to individual agent performance indices. However, the potential main disadvantage of this formulation is clearly the requirement of availability of full information set for control design purpose. In the present work this problem is alleviated and the imposed information structure of the team is taken into account by using a linear matrix inequality (LMI) formulation. For this purpose, a decentralized optimal control strategy that was initially introduced in Semsar-Kazerooni and Khorasani (2007b, 2008) is used to design controllers based on minimization of individual costs. Since in this approach the solution is obtained through minimization of local cost functions we have at most a person-by-person optimality. However, if a cost function describing the total performance is minimized a lower team cost as well as lower individual costs may be achieved. Subsequently, the idea of cooperative game theory is used to minimize a team cost function which is a linear combination of the cost functions that are used in the optimal approach. This will guarantee that individual cost functions have the minimum possible values for the given team mission. To obtain a solution that is subject to a given information structure as well as to guarantee consensus achievement, a set of LMIs is used to constrain the controller to be designed for the entire team.

The organization of the paper is as follows: In Section 2, background information is presented. In Section 3, cooperative game theory is introduced. Application of the game theory to the multi-agent team problem, the design of a semi-decentralized optimal control, and solutions to the corresponding min–max problem are presented in Section 4. Finally, simulation results are conducted and conclusions are stated in Sections 5 and 6, respectively.

2. Background information

Multi-agent teams: Assume a set of agents $\Omega = \{i = 1, \ldots, N\}$, where N is the number of agents. Each member of the team which is denoted by i is placed at a vertex of the network information graph. The dynamical representation of each agent is governed by

$$\dot{X}^i = A^i X^i + B^i u^i, \quad X^i \in R^n, u^i \in R^m, i = 1, \ldots, N \tag{1}$$

$$Y^i = C^i X^i, \quad Y^i \in R^q \tag{2}$$

where X^i denotes the state vector, u^i denotes the input vector, and Y^i denotes the output vector of agent i and A^i, B^i and C^i are matrices of appropriate dimensions.

Information structure and neighboring sets: In order to ensure cooperation and coordination among team members, each member has to know the status of the other members, and therefore members have to communicate with each other. For a given agent i, the set of agents connected to it via communication links is called a neighboring set N^i. The existence of a link between two agents in general may refer to the availability of information of one agent to the other one, in other words $\forall\, i = 1, \ldots, N$, $N^i = \{j = 1, \ldots, N | (i, j) \in E\}$, where E is the edge set that corresponds to the underlying graph of the network. It is assumed that the graph describing the information structure is connected.

Laplacian matrix (Fax & Murray, 2004): This matrix is used to describe the graph associated with information exchanges in a network of agents, i.e. G, and is defined as $L = [L(i, j)]_{N\times N}$

$$L(i,j) = \begin{cases} d(i) & i = j \\ -1 & (i,j) \in E \text{ and } i \neq j \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

where $d(i)$ is equal to the cardinality of the set N^i (Olfati-Saber & Murray, 2003a), $|N^i|$, and is called the degree of vertex i. For an undirected graph, the degree of a vertex is the number of edges incident to that vertex (total number of links connected to that vertex). For directed graphs, instead of the degree either the in-degree or the out-degree might be used (the total number of the links entering or leaving a node). This matrix has vector $\mathbf{1}$ as its right eigenvector corresponding to its zero eigenvalue.

Leaderless (LL) structure: In this structure no external command is provided to the members of the team, and the goal is to make the agents' output, e.g. velocity, converge to a common value which is decided upon by the team members.

Model of interaction between the team members: Assume that the dynamical model of each agent is given by Eqs. (1) and (2). This model defines an isolated agent of the team, but in reality agents have some interactions through the information flow that exists among the neighboring agents. In Semsar-Kazerooni and Khorasani (2007b, 2008), it was shown that each member's dynamics can be described by the following model that incorporates the interaction terms, namely

$$\begin{cases} \dot{X}^i = A^i X^i + B^i u^i \\ u^i = u^i_l + u^i_g, \quad u^i_g = \sum\limits_{j\in N^i} F^{ij} Y^j \\ Y^i = C^i X^i \end{cases} \tag{4}$$

where u^i_l, u^i_g are the decomposition of the input signal into the "local" and the "global" control terms. As discussed in Semsar-Kazerooni and Khorasani (2007c, 2008), the local term for each agent is designed using the agents own output vector whereas the global control utilizes the information received from other agents in its neighboring set. The "global" control term $u^i_g(Y^j) = \sum_{j\in N^i} F^{ij} Y^j$ is also denoted as the interaction term, where F^{ij} is the interaction matrix to ensure compatibility in the agent's input and output channels dimensions. The proposed interaction terms are used in order to overcome the relative specifications and dependencies of individual agent goals on other agents' outputs or states as described in Semsar-Kazerooni and Khorasani (2007c, 2008). This reveals why the proposed method is called semi-decentralized. Whereas the local part is a function of only agents own output (decentralized design), the global part incorporates the effects of neighbors information (output) into the control law and so the control input is not fully decentralized in the conventional sense.

Remark 1. For an optimal control design strategy and given the fact that this method is used in this work for comparison purposes only, we assume that the dynamical equation of each agent is a simple double integrator. However, our approach based on the game theory is general for an arbitrary linear model of agents. For a large number of systems of interest, specifically in a team of mobile robots, the dynamical model may be converted (by invoking appropriate state-space coordinate transformations) into a standard form, which is a generalized form of integrators. Without loss of generality, assume that the agents are mobile robots for which the linear system (or the nonlinear equivalent linearized dynamical system) is transformed into a simple double integrator model (Semsar-Kazerooni & Khorasani, 2007c; Stipanović et al., 2004). The dynamical model of each agent consists of position and velocity states. However, since the main objective in this work is to have a common output, namely velocity, and the cost function that will be defined subsequently in (6) depends on the outputs, for analysis we will only consider the velocity dynamics to describe the behavior of the agents. However, for the purpose of numerical simulations subsequently in Section 5, the position dynamics is also included. In other words, the dynamics of each agent is governed by

$$\begin{cases} \dot{r}^i = v^i \\ \dot{v}^i = u_l^i + u_g^i \\ u_g^i = \sum_{j \in N^i} F^{ij} Y^j, \quad Y^i = v^i, i = 1, \ldots, N \end{cases} \tag{5}$$

in which $r^i, v^i \in R^m$ are the position and the velocity vectors, respectively.

Problem Definition: Our main goal in this paper is to ensure that agents' output, e.g. velocity, converge to the same value, i.e. $Y^i \to Y^j, \forall i, j$. In other words, we require that the team reaches a consensus. For this purpose, we apply the optimal control strategy that was introduced in Semsar-Kazerooni and Khorasani (2007b, 2008). Following this approach we utilize a game theoretic methodology to provide a "more" cooperative solution with lower cost values for consensus seeking. In order to accomplish consensus, in the next subsection we define individual cost functions that are to be used in the upcoming sections.

Definition of a cost function: Let us define the following cost function for each agent

$$J^i = \int_0^{\mathrm{T}} \left\{ \sum_{j \in N^i} [(Y^i - Y^j)^{\mathrm{T}} Q^{ij} (Y^i - Y^j)] + (u_l^i)^{\mathrm{T}} R^i u_l^i \right\} \mathrm{d}t, \quad T > 0 \tag{6}$$

where Q^{ij}, R^i are symmetric and positive definite (PD) matrices. By minimizing the above cost function for the given controllable and observable system (4), one may guarantee that all the agents in a neighboring set will have the same output vector in steady state, i.e. consensus is achieved (Semsar-Kazerooni & Khorasani, 2007b, 2008). In other words, the output vector $Y = [(Y^1)^{\mathrm{T}} \ \ldots \ (Y^N)^{\mathrm{T}}]^{\mathrm{T}}$ will converge to a vector in the subspace S spanned by the vector $\mathbf{1} = [1 \ 1 \ \ldots \ 1]^{\mathrm{T}}$. This vector is in fact an eigenvector of the Laplacian matrix corresponding to the underlying graph.

3. Cooperative game theory

In this section, we provide a general description of the "cooperative game theory" and in the next section we modify the formulation introduced here to make it compatible with our specific problem, i.e. consensus seeking problem. Assume a team of N players with the following dynamical model

$$\dot{x} = Ax + \sum_{i=1}^{N} B^i u^i \tag{7}$$

where the matrix A has an arbitrary structure. Each player wants to optimize its own cost

$$J^i = \int_0^T (x^{\mathrm{T}} Q^i x + (u^i)^{\mathrm{T}} R^i u^i) \mathrm{d}t \tag{8}$$

in which Q^i and R^i are symmetric matrices and R^i is a positive definite matrix.

If the players decide to minimize their cost in a non-cooperative manner, a strategy (control input u^i) chosen by the ith player can increase the cost of other players through dynamics of the system that couples different players together. However, if players decide to cooperate, individual costs may be minimized if each agent is aware of others' decisions and can reduce their team cost by selecting a suitable cooperative strategy. Hence, in a cooperative strategy depending on which agent requires more resource the resulting minima can be different. The cooperation ensures that the total cost of the team is less than any other non-cooperative optimal solution obtained.

In a cooperative approach it is intuitively assumed that if a set of strategies for a team results in a lower cost for all the members, all players will switch to that set. Hence, by excluding this situation the set of desired solutions consists of those strategies that if the team strategy changes to another one at least one of the players ends up with a higher cost. In other words, there is no alternative strategy that improves all the members' cost simultaneously. This property can be formally defined by the set of Pareto-efficient solutions as follows.

Pareto-efficient strategies (Engwerda, 2005): A set of strategies $U^* = [u^{1*}, \ldots, u^{N*}]$ is Pareto efficient if the set of inequalities $J^i(U) \leqslant J^i(U^*), i = 1, \ldots, N$, with at least one strict inequality does not have a solution for U. The point $J^* = [J^1(U^*), \ldots, J^N(U^*)]$ is called a Pareto solution. This solution is never fully dominated by any other.

Now consider the following optimization problem and assume that $Q^i \geq 0$, specifically

$$\min_{u^i \in \mathcal{U}^i} J^i = \int_0^T (x^{\mathrm{T}} Q^i x + (u^i)^{\mathrm{T}} R^i u^i) \mathrm{d}t$$

$$\text{s.t. } \dot{x} = Ax + \sum_{i=1}^{N} B^i u^i$$

where $\mathcal{U}^i$ is the set of all strategies for player i. The above is a convex optimization problem. It can be shown that the following set of strategies results in a set of Pareto-efficient solutions for this problem. In other words, the solution to the following minimization problem cannot be dominated by any other solution

$$U^*(\alpha) = \arg\min_{U \in \mathcal{U}} \sum_{i=1}^{N} \alpha^i J^i(U) \tag{9}$$

where $\alpha \in \mathcal{A}$, $\mathcal{A} = \{\alpha = (\alpha^1, \ldots, \alpha^N) | \alpha^i \geq 0 \text{ and } \sum_{i=1}^{N} \alpha^i = 1\}$, $\mathcal{U}$ is the set of all strategies for all players and the corresponding cost values will be $J^1(U^*(\alpha)), \ldots, J^N(U^*(\alpha))$. It is worth noting that although this minimization is over the set of strategies $\mathcal{U}$, the controller parameters (matrices) are in fact being optimized. In other words, the control strategies $\mathcal{U}$ are assumed to be in the form of state feedback and the coefficient matrices are obtained through the above optimization problem.

The strategies obtained from the above minimization as well as the optimal cost values, here referred to as the "solutions", are functions of the parameter α. Therefore, the Pareto-efficient solution is in general not unique and the set of these solutions, i.e. Pareto frontier, is denoted by $\mathcal{P}$ which is an edge in the space of possible solutions (cost values), i.e. Ξ. It can be shown that in both infinite horizon and finite horizon cases, the Pareto frontier will be a smooth function of α (Engwerda, 2005). Due to the non-uniqueness of Pareto solutions the next step is to decide how to choose one solution among the set of Pareto solutions (or to choose an α from the set of α's). This solution should be selected according to a certain criterion as our final strategy for the team cooperation problem. For this purpose, we need to solve the bargaining problem as defined below.

Bargaining problem (Engwerda, 2005): In this problem two or more players have to agree on the choice of some strategies from a set of solutions while they may have conflicting interests over this set. However, the players understand that better outcomes may be achieved through cooperation when compared to the non-cooperative outcome (called threat-point). One of the approaches to the bargaining problem is the axiomatic approach. Some of the well-known axiomatic approaches to this problem are Nash bargaining, Kalai-Smorodinsky, and Egalitarian.

Applying any of the above mentioned methods to the Pareto-efficient solutions will yield a unique cooperative solution. Due to the interesting properties of the Nash-bargaining solution such as symmetry and Pareto optimality (Engwerda, 2005), we invoke this method for obtaining a unique solution among the set of Pareto-efficient solutions obtained above.

Nash-bargaining solution (NBS) (Engwerda, 2005): In this method a point in Ξ, denoted by Ξ^N, is selected such that the product of the individual costs from d is maximal ($d = [d^i]^{\mathrm{T}}$ is the threat-point or the non-cooperative outcome of team agents), namely $\Xi^N(\Xi, d) = \arg\max_{J\in\Xi} \prod_{i=1}^{N}(d^i - J^i), J \in \Xi$ with $J \preceq d$, in which d^i's are the cost values calculated by using the non-cooperative solution that is obtained by minimizing the cost in (8) individually and constrained to (7) (the threat point). It can be shown that the Nash-bargaining solution is on the Pareto frontier and therefore the above maximization problem is equivalent to the problem $\alpha^N = \arg\max_{\alpha} \prod_{i=1}^{N}(d^i - J^i(\alpha, U^*)), J \in \mathcal{P}$ with $J \preceq d$, in which $J = [J^i]^{\mathrm{T}}$, and where J^i's are calculated by using the set of strategies given in (9). By solving the latter maximization problem, a unique value for the coefficient α can be found.

Remark 2. Theorem 6.10 in Engwerda (2005) can be used to determine the relationship that exists between the coefficients $\alpha^i, i = 1, \ldots, N$ and the achievable improvements in the individual costs due to cooperation in the team. According to this theorem the following relationship holds between the value of the cost functions at the NBS, $(J^{1*}(\alpha^*, U^*), \ldots, J^{N*}(\alpha^*, U^*))$, the threat-point d, and the optimal weight $\alpha^* = (\alpha^{1*}, \ldots, \alpha^{N*})$, that is, $\alpha^{1*}(d^1 - J^{1*}(\alpha^*, U^*)) = \cdots = \alpha^{N*}(d^N - J^{N*}(\alpha^*, U^*))$ or $\alpha^{j*} = \frac{\prod_{i\neq j}(d^i - J^{i*}(\alpha^*, U^*))}{\sum_{i=1}^{N}\prod_{k\neq i}(d^k - J^{k*}(\alpha^*, U^*))}$.

The above expressions describe the kind of cooperations that exist among the players. They show that if during the team cooperation, i.e. minimization of the team cost, a player has improved its cost more, it will receive a lower weight in the minimization scheme (Pareto solution) whereas the one who has not gained a great improvement as a result of participation in the team cooperation receives a greater weight. Therefore, all the players benefit from the cooperation in almost a similar manner, and hence have the incentive to participate in the team cooperation.

Remark 3. Selection of the NBS is motivated by the fact that this solution enjoys several appealing properties (axioms). As pointed out in Engwerda (2005), in this method each agent does not need to have information about the utility value or the threat point of other agents. In other words, no "interpersonal comparison" of utility functions is required. Moreover, this solution satisfies four axioms namely, Pareto optimality, symmetry, independence of irrelevant alternatives, and affine transformation invariance, which are all defined in Engwerda (2005).

4. Application of cooperative game theory to the consensus seeking in a multi-agent team

According to the discussions in the previous section, cooperation in a team of N players (agents), e.g. consensus seeking, can be solved in the framework of cooperative games. Our goal is to develop a cooperative solution that utilizes decentralized cost functions and combines them in a team cost function. This will ensure improvements in minimizing individual costs by utilizing the game-theoretic method. Towards this end, we try to find a set of Pareto optimal solutions for minimization of the team cost function through solving (9). An NBS solution can be selected among this set of Pareto-efficient solutions by solving the maximization problems for $\Xi^N(\Xi, d)$ and α^N.

The game theory techniques that were introduced in the previous section are now applied to a leaderless team of agents as described in Section 2. The dynamical model of each agent and the related cost functions are described in (4) and (6), respectively. To clearly describe the method, we combine the individual cost functions in (6) into a team cost function according to the following

$$
\begin{aligned}
J^c &= \sum_{i=1}^{N} \alpha^i J^i(U) = \int_0^{\mathrm{T}} [Y^{\mathrm{T}}\hat{Q}Y + U^{\mathrm{T}}RU]\mathrm{d}t \\
&= \int_0^{\mathrm{T}} [X^{\mathrm{T}}QX + U^{\mathrm{T}}RU]\mathrm{d}t
\end{aligned}
\tag{10}
$$

in which $\alpha = (\alpha^1, \ldots, \alpha^N) \in \mathcal{A}$, $J^i(U)$ is the cost function for the ith agent (player) that is defined in (6) and $U(\alpha) = [(u_l^1)^{\mathrm{T}} \ \ldots \ (u_l^N)^{\mathrm{T}}]^{\mathrm{T}}$ is the vector of all the agents' local input vectors and

$$
\begin{aligned}
&R = \mathrm{Diag}\{\alpha^1 R^1, \ldots, \alpha^N R^N\}, \qquad \hat{Q} = [\delta_{hk}]_{N\times N}, \\
&Q = C^{\mathrm{T}}\hat{Q}C, \\
&\delta_{hh} = \sum_{j\in\mathcal{N}^h} \alpha^j Q^{jh} + \alpha^h \sum_{k\in N^h} Q^{hk}, \\
&\delta_{hk} = \begin{cases} -\alpha^h Q^{hk} - \alpha^k Q^{kh} & \text{for } k \in N^h \\ 0 \quad \text{otherwise} \end{cases}
\end{aligned}
\tag{11}
$$

where $\mathcal{N}^h$ denotes the set of indices of the neighboring sets to which agent h belongs. Each agent belongs to only those clusters in which one of the agent's neighbors exists. Therefore, the total number of these clusters is the same as the number of neighbors of that agent, i.e. $\mathcal{N}^h = N^h$. The associated dynamical model constraint is given by

$$
\dot{X} = AX + BU, \qquad Y = CX \tag{12}
$$

in which X, U and Y are the state, input, and output vectors of the entire team that are obtained from the concatenation of all the agents' state, input, and output vectors and are given by

$$
\begin{aligned}
&X = [(X^1)^{\mathrm{T}} \ \ldots \ (X^N)^{\mathrm{T}}]^{\mathrm{T}}, \qquad U = [(u_l^1)^{\mathrm{T}} \ \ldots \ (u_l^N)^{\mathrm{T}}]^{\mathrm{T}}, \\
&Y = [(Y^1)^{\mathrm{T}} \ \ldots \ (Y^N)^{\mathrm{T}}]^{\mathrm{T}}.
\end{aligned}
\tag{13}
$$

Furthermore, the matrices A, B and C are defined as follows

$$A = \begin{bmatrix} A^1, 0, \ldots, B^1F^{1j}C^j, \ldots, 0 \\ \vdots \\ 0, \ldots, B^NF^{Nj}C^j, \ldots, 0, A^N \end{bmatrix}, \qquad B = \mathrm{Diag}\{B^1, \ldots, B^N\}, \qquad C = \mathrm{Diag}\{c^1, \ldots, c^N\} \tag{14}$$

where A^i, B^i, and C^i are defined in (4). The matrix $B^iF^{ij}C^j$ represents the interaction term that is incorporated in the dynamical model of each agent.

The Pareto-efficient solution for minimizing the team cost function (10) is achieved by invoking the following strategy

$$U^*(\alpha) = \arg\min_{U\in\mathcal{U}} \sum_{i=1}^{N} \alpha^i J^i(U) = \arg\min_{U\in\mathcal{U}} J^c(\alpha), \qquad \alpha = (\alpha^1, \ldots, \alpha^N) \in \mathcal{A} \tag{15}$$

in which $U^*(\alpha) = [(u_l^{1*})^{\mathrm{T}} \ \ldots \ (u_l^{N*})^{\mathrm{T}}]^{\mathrm{T}}$ is the vector of all the agents' local input vectors.

The set of solutions to the minimization problem (15) is a function of the parameter α which provides a set of Pareto-efficient solutions. Among these solutions, a unique solution can be obtained by using one of the methods that was mentioned earlier, e.g., the Nash-bargaining solution. Using this method the unique solution to the problem (a unique α) is given by α^N in which J^i's are defined in (6) and are calculated by applying the solution of the minimization problem (15) to the system that is given in (12), and hence are functions of the parameter α. The terms d^i's are the values of the cost defined in (6) which is obtained by applying a non-cooperative approach to the individual subsystems in (5). By solving the maximization problem α^N, the parameter α can be found and substituted in the set of control strategies that are obtained in (15). This solution guarantees that the product of the distances between d^i's (non-cooperative solution) and J^i's (cooperative solution) is maximized, implying that the individual costs in the latter case are minimized as much as possible.

Let us first solve the minimization problem (15) and then apply an algorithm given in Engwerda (2005) to solve the maximization problem α^N. In order to solve the minimization problem (15), the cost function (10) should be minimized subject to the dynamical constraint (12). This is a standard linear quadratic regulator (LQR) problem and its solution for an infinite horizon case (i.e. $T \to \infty$) will result in the following control law

$$U^*(\alpha, X) = -R^{-1}B^{\mathrm{T}}PX \qquad Q - PBR^{-1}B^{\mathrm{T}}P^{\mathrm{T}} + PA + A^{\mathrm{T}}P = 0. \tag{16}$$

The control U^* can be constructed if the above algebraic Riccati equation (ARE) has a solution for P. However, some issues arise when the above control law is applied to the dynamical system (12). In fact, given that the matrix P is not guaranteed to be block-diagonal, the control signal U^* yields a *centralized strategy* in the sense that its components, i.e. u_l^{i*}, are dependent on the information from the *entire* team. Moreover, the solution suggested by (16) does not guarantee that a non-zero consensus is achieved for an arbitrary parameter selection. Hence, to ensure that a desirable consensus solution is obtained that also satisfies the constraints on the availability of information, one needs to impose additional constraints on the original minimization problem. However, by adding additional constraints to the cost function (10), e.g. by considering a barrier function, the problem will no longer be a convex optimization problem which may not necessarily have a unique solution. To remedy this problem, the original cost function is kept unchanged, however the optimization problem is now formulated as an LMI (linear matrix inequality) problem so that the constraints due to the consensus and the controller structure are incorporated as convex constraints.

4.1. Solution of the minimization problem: An LMI formulation

In Willems (1971), Boyd, Ghaoui, Feron, and Balakrishnan (1994) and Ait-Rami and Zhou (2000), it is pointed out that the LQR problem can be formulated as a maximization or a minimization problem subject to a set of LMIs. In other words, instead of solving the ARE (16), an LMI can be solved. The following is one of the formulations that can be used for this purpose using a semi-definite programming framework (Ait-Rami & Zhou, 2000), namely

$$\max \mathrm{trace}(P) \text{ s.t. } \mathcal{R}(P) = PA + A^{\mathrm{T}}P - PBR^{-1}B^{\mathrm{T}}P + Q \geq 0, \qquad P \geq 0 \tag{17}$$

where the optimal control law is then selected as $U^* = -R^{-1}B^{\mathrm{T}}PX$. Another formulation of the LQR problem is the following

$$\min X(0)^{\mathrm{T}}PX(0) \text{ s.t. } PA + A^{\mathrm{T}}P - PBR^{-1}B^{\mathrm{T}}P + Q \leq 0, \quad P \geq 0 \tag{18}$$

which can be transformed into an LMI optimization problem by introducing new variables $K = -R^{-1}B^{\mathrm{T}}P$, $X = P^{-1}$, and $Y = KP^{-1}$ (Jadbabaie, 1997). However, for the purpose of this work the first formulation is more convenient. This formulation can be translated into an LMI maximization problem by using the Schur complement decomposition, and given that $R > 0$, can be stated as the following problem.

Problem A. The LQR problem can be formulated as a maximization problem subject to a set of LMIs, namely

$$\max \mathrm{trace}(P) \text{ s.t. } \begin{bmatrix} PA + A^{\mathrm{T}}P + Q & PB \\ B^{\mathrm{T}}P & R \end{bmatrix} \geq 0, \quad P \geq 0. \tag{19}$$

It can be shown that the above maximization problem has a solution if and only if the following ARE has a solution

$$Q - PBR^{-1}B^{\mathrm{T}}P^{\mathrm{T}} + PA + A^{\mathrm{T}}P = 0. \tag{20}$$

Moreover, if $R > 0$ and $Q \geq 0$, the unique optimal solution to the maximization Problem A is the maximal solution to the ARE in (20) (Ait-Rami & Zhou, 2000).

4.1.1. Discussion on the existence and uniqueness of solutions

Existence of a solution: To guarantee existence of a solution to Problem A, Eq. (20) should have a solution. On the other hand, in order to have an optimal stabilizing solution to the ARE (20), the pair (A, B) should be stabilizable and the pair (A, Q) (or (A, D), $Q = DD^{\mathrm{T}}$) should be detectable (Anderson & Moore, 1990). Each of these two conditions can be checked through an LMI set of conditions (Boyd et al., 1994), namely

1. (A, B) stabilizable $\Leftrightarrow$ $AP_1 + P_1A^{\mathrm{T}} < BB^{\mathrm{T}}$ has a positive definite solution for P_1.
2. (A, Q) detectable $\Leftrightarrow$ $A^{\mathrm{T}}P_2 + P_2A < Q$ has a positive definite solution for P_2.

In our case, matrix A is a function of the interaction gains, i.e. F^{ij}, and therefore can be considered as a design parameter. Note that in case matrices B, Q, and the matrix A having no interaction terms, i.e. $F^{ij} \equiv 0$, $\forall i, j$, satisfy the above two conditions, the existence of a solution is clearly guaranteed. However, if the above conditions are not satisfied we may then select the F^{ij} gains such that these conditions are satisfied. Moreover, if required an internal loop can be added to each individual subsystem so that the above conditions are satisfied (by adding diagonal elements to the matrix A). Finally, if the ARE (20) has a solution then the LMIs of Problem A also has a solution.

Uniqueness of the solution: In the above discussion we showed how to formulate the optimization problem as a set of LMIs. The solutions to this set of LMIs which also minimize the cost function

(10) guarantee the consensus seeking, i.e. $X \to \xi\mathbf{1}$, where ξ is a constant coefficient of the consensus value. However, among these solutions a possible solution is when $\xi = 0$, that is when the closed-loop system is asymptotically stable and converges to the origin. This solution is not ideally desirable since it is a trivial solution of the consensus seeking problem and should be excluded. Towards this end, we may add the consensus seeking condition in the subspace S to Problem A, i.e. condition $(A - BR^{-1}B^{\mathrm{T}}P)S = 0$ should be incorporated into Problem A, where S is the unity vector, i.e. $S = \mathbf{1}$. This constraint will guarantee that the closed-loop matrix has a zero eigenvalue and is not Hurwitz. Therefore, if the rest of the eigenvalues of this matrix are negative, i.e. stable, then the system trajectories will move toward a constant non-zero state which is in the consensus space S. On the other hand, by adding other constraints to the LMI problem as will be discussed later on in this section, stability of the closed-loop matrix would be guaranteed as well. We now have a new formulation to our problem as stated next.

Problem B. The LQR minimization problem for consensus seeking can be formulated as a maximization problem subject to a set of LMIs, namely

max trace(P) s.t.

$$\begin{cases} 1.\ \begin{bmatrix} PA + A^{\mathrm{T}}P + Q & PB \\ B^{\mathrm{T}}P & R \end{bmatrix} \geq 0, \quad P \geq 0 \\ 2.\ (A - BR^{-1}B^{\mathrm{T}}P)S = 0 \end{cases} \tag{21}$$

where the optimal control law is selected as $U^* = -R^{-1}B^{\mathrm{T}}PX$ and P is obtained by solving the above set of LMIs.

4.1.2. Consensus seeking subject to a predefined information structure

As discussed previously, the solution to Problem B as well as the one given in (16) requires *full* network information for each agent. However, each agent has only access to its neighboring set information. Therefore, one needs to impose a constraint on the controller structure in order to satisfy the corresponding availability of information. For sake of notational simplicity assume that each agent has a one-dimensional state-space representation, i.e. A^i in (4) is a scalar. The case of a non-scalar system matrix can be treated similarly. The controller coefficient, i.e. $R^{-1}B^{\mathrm{T}}P$ in Problem B should have the same structure as the Laplacian matrix so that the neighboring set constraint holds. However, due to their definitions both R and B are block diagonal matrices. Therefore, it suffices to restrict P to have the same structure as the Laplacian matrix, i.e. $P(i,j) = 0$ if $L(i,j) = 0$, where $L(i,j)$ designates the (i,j) entry of the Laplacian matrix L. We may now solve the following problem to minimize the cost function (10) while simultaneously satisfying all the problem constraints, namely we now have:

Problem C.

max trace(P) s.t.

$$\begin{cases} 1.\ \begin{bmatrix} PA + A^{\mathrm{T}}P + Q & PB \\ B^{\mathrm{T}}P & R \end{bmatrix} \geq 0, \quad P \geq 0 \\ 2.\ (A - BR^{-1}B^{\mathrm{T}}P)S = 0 \\ 3.\ P(i,j) = 0 \quad \text{if } L(i,j) = 0, \forall i,j = 1,\ldots,N. \end{cases} \tag{22}$$

This problem is an LMI maximization problem in terms of P.

Up to now, we have formulated the minimization problem (15) as a set of LMIs. Let us try to solve the maximization problem that is given by α^N. For this purpose, we need to calculate the individual agent costs J^i by utilizing a given method. In this work we intend to use the semi-decentralized optimal control strategy (developed previously by the authors in Semsar-Kazerooni and Khorasani (2007b, 2008)) that is introduced in the following subsection. These values are considered as the non-cooperative outcome of the team, referred to as d^i's in $\Xi^N(\Xi, d)$.

Remark 4. It is worth noting that any algorithm which guarantees consensus seeking can be considered as a cooperative algorithm. However, in the context of the present work, the approach based on the semi-decentralized optimal control is classified as non-cooperative. The reason for such designation follows from our previous discussions and definitions where the game theoretic framework yields more characteristics of a cooperative solution when compared to the solution that is obtained by the optimal control strategy.

A brief description of our semi-decentralized optimal control method is presented below. The reader is referred to Semsar-Kazerooni and Khorasani (2007b, 2008) for further detail.

4.2. Application of the semi-decentralized optimal control to a leaderless multi-agent team

Our proposed optimal control strategy that results from the "individual" minimization of the agent cost functions (6) subject to the dynamical model (5) is provided in the following lemma.

Lemma 5 (*Semsar-Kazerooni & Khorasani, 2007b, 2008*). *Assume a team of agents is given whose dynamics are governed by the double integrator equations in (5) and are embedded with interactions among the agents based on the neighboring sets. Assume that the control input of each agent is decomposed into local and global components as described in (4). The global and local control laws that are proposed below minimize the cost functions (6) so that a consensus on the velocity is guaranteed where* $u_g^i = \sum_{j\in N^i} F^{ij}v^j$, $F^{ij} = 2K^i(t)^{-1}Q^{ij}$, $\forall i,j$, $u_l^i = -\frac{1}{2}(R^i)^{-1}K^i(t)v^i$, $i = 1,\ldots,N$, *in which* K^i *satisfies the differential Riccati equation (DRE) that is given by,* $-\dot{K}^i = 2|N^i|Q^{ij} - \frac{1}{2}K^i(R^i)^{-1}K^i$, $K^i(T) = 0$. *Moreover, in the cost function (6) if the time horizon is assumed to be infinite, i.e.* $T \longrightarrow \infty$, *then the combined control law reduces to the* agreement protocol *that is given by* $u^i(X^i, X^j) = u_l^i(X^i) + u_g^i(Y^j) = \Gamma^i(v^i - \frac{\sum_{j\in N^i} v^j}{|N^i|})$, $\Gamma^i = -\frac{1}{2}(R^i)^{-1}K^i$, $i = 1,\ldots,N$, *where* $|N^i|$ *is the cardinality of the neighboring set that is defined in Section 2.* ■

The above lemma provides a control strategy for consensus seeking using the individual agent cost functions minimization. Earlier, the individual cost of each agent was combined into a team cost function and cooperative game theory approach was utilized to "increase" the team cooperation and minimize the associated individual costs. The cost values that are obtained in the present section are referred to as the "non-cooperative" outcomes (or threat points) in the context of the game theoretic approach. We are now in a position to develop our algorithm for determining a Nash-bargaining solution to the cooperative game theory problem.

4.3. An algorithm for finding a Nash-bargaining solution

Up to this point we have shown that for any given $\alpha > 0$ the maximization Problem C should first be solved. We now need an algorithm for solving the maximization problem for α^N over different values of α so that a suitable and unique α can be found. In Engwerda (2005), two numerical algorithms for solving this maximization problem are given. With minor modifications made to one of these algorithms, the following algorithm will be used for our numerical simulation purposes. Namely, we have

Algorithm I.

- **Step 1** Start with an initial selection for $\alpha_0 \in \mathcal{A}$ (e.g. $\alpha_0 = [1/N,\ldots,1/N]$ is a good choice).
- **Step 2** Compute $U^*(\alpha_0) = \arg\min_{U\in\mathcal{U}} \sum_{i=1}^N \alpha_0^i J^i(U)$ by solving the maximization Problem C.

- **Step 3** Verify if $J^i(U^*) \leq d^i, i = 1, \ldots, N$, where d^i is the optimal value of (6) when the control laws in Lemma 5 are applied to the dynamical system (5). If this condition is not satisfied, then there is at least one i_0 for which $J^{i_0}(U^*) > d^{i_0}$. In that case, update $\alpha_0^{i_0} = \alpha_0^{i_0} + 0.01,\ \alpha_0^i = \alpha_0^i - \frac{0.01}{N-1}$, for $i \neq i_0$ and return to Step 2 (similarly extend the update rule for more than one i_0).
- **Step 4** Calculate

$$\tilde{\alpha}^j = \frac{\prod_{i\neq j}(d^i - J^i(U^*(\alpha_0)))}{\sum_{i=1}^{N}\prod_{k\neq i}(d^k - J^k(U^*(\alpha_0)))}, \quad j = 1, \ldots, N.$$

- **Step 5** Apply the update rule $\alpha_0^i = 0.9\alpha_0^i + 0.1\tilde{\alpha}^i$. If $|\tilde{\alpha}^i - \alpha_0^i| < 0.01, i = 1, \ldots, N$, then terminate the algorithm and set $\alpha = \tilde{\alpha}$, else return to Step 2.

The above discussions are now summarized in the following theorem.

Theorem 6. *Consider a team of agents with individual dynamical representation (4) or the team dynamics (12), the individual cost function (6), and the team cost function (10) with the corresponding parameters (11) and (14). Furthermore, assume that the desirable value of the parameter α is found by using Algorithm I. Moreover, the control law U^* is designed as $U^* = -R^{-1}B^{\mathrm{T}}PX$, and P is the solution to the following optimization problem*

max trace(P) *s.t.*

$$\begin{cases} 1.\ \begin{bmatrix} PA + A^{\mathrm{T}}P + Q & PB \\ B^{\mathrm{T}}P & R \end{bmatrix} \geq 0, \quad P \geq 0 \\ 2.\ A_c = (A - BR^{-1}B^{\mathrm{T}}P), \qquad A_cS = 0 \\ 3.\ P(i,j) = 0 \quad \text{if } L(i,j) = 0, \forall i, j = 1, \ldots, N \end{cases} \tag{23}$$

where $S = \mathbf{1}$. It then follows that

(a) *In the infinite horizon scenario, i.e. $T \to \infty$, the above controller solves the min–max problem $U^* = \arg\min_{U\in\mathcal{U}} \sum_{i=1}^{N} \alpha^i J^i(U)$, $\alpha \in \mathcal{A}$, $\mathcal{A} = \{\alpha = (\alpha^1, \ldots, \alpha^N)|\alpha^i \geq 0$ and $\sum_{i=1}^{N}\alpha^i = 1\}$, with $\alpha^* = \arg\max_\alpha \prod_{i=1}^{N}(d^i - J^i(\alpha, U^*)), J\preceq d$. The solution to this min–max problem guarantees consensus achieving for the proposed team of agents, i.e. $X_{ss} \to \xi\mathbf{1}$, where ξ is a constant coefficient of the consensus value.*

(b) *In addition, the suggested control law guarantees a "stable" consensus of agents output to a common value subject to the dynamical and information structure constraints of the team in a cooperative manner, if for at least one connected subgraph of the original graph, we have $A_c(i,j) \neq 0$ if $L_{sub}(i,j) \neq 0, \forall i, j = 1, \ldots, N$, where L_{sub} denotes the Laplacian matrix of any such arbitrary connected subgraph.*

(c) *Moreover, the optimal value of the cost function (10) has a finite infimum of $X^{\mathrm{T}}(0)PX(0) - \xi^2 \sum_i \sum_j P(i,j)$, where P is obtained from (23) and $X(0)$ is the initial value of the entire team state vector.*

Proof. The proof is provided in Appendix. ■

We can now conclude that by using the above results, the team consensus goal can be achieved in a decentralized cooperative manner while simultaneously satisfying all the given information constraints of the team.

5. Simulation results

The simulation results that are presented in this section correspond to a team of four agents that are being controlled by using two control strategies, namely the semi-decentralized optimal control law that is given by Lemma 5 and the cooperative game theoretic-based control law that is given by Theorem 6. The first set of numerical simulations corresponds to the application of the control laws in Lemma 5 to the individual agent model (5). In the second set, the numerical simulation results are obtained by applying the control law $U = KX$ with $K = -R^{-1}B^{\mathrm{T}}P$ to the team dynamics that is described by (12). Matrix P is obtained through the LMI set in (23) and by utilizing the maximization Algorithm I. Results shown below are conducted through Monte Carlo simulation runs to capture the average behavior of our proposed control strategies. The average team responses are due to 15 different randomly selected initial conditions.

The simulation parameters for both control approaches are selected as follows: $A^i = 0_{2\times 2}, R^i = I_{2\times 2}, C^i = I_{2\times 2}, B^i = \begin{pmatrix} 4 & -3 \\ -2 & 3 \end{pmatrix}$, and $Q^{ij} = \begin{pmatrix} 10 & 3 \\ 3 & 4 \end{pmatrix}$. The random initial conditions of the velocity vector for the Monte Carlo simulations are considered as $v_0^1 = [r(6,8) \;\; r(1,3)]^{\mathrm{T}}$, $v_0^2 = [r(5,7) \;\; r(3,5)]^{\mathrm{T}}$, $v_0^3 = [r(2,4) \;\; r(1,3)]^{\mathrm{T}}$, $v_0^4 = [r(-5,-3) \;\; r(-4,-2)]^{\mathrm{T}}$, where $r(x,y)$ represents a random variable in the interval $[x, y]$.

In cooperative game theory strategy the initial value for the parameter α is selected as $\alpha_0 = [1/4, \ldots, 1/4]$ and its optimal average value $\alpha = [0.2276\ 0.2005\ 0.2486\ 0.3232]$ is obtained by using the procedure that is outlined in Algorithm I for 15 Monte Carlo simulation runs. The interaction gains are selected as $F^{ij} = 1.6I_{2\times 2}$. It can be shown that the above parameters satisfy the required existence conditions for solutions to the original optimization problem A, i.e. they guarantee the required detectability and stabilizability conditions (refer to Section 4.1.1).

Table 1 compares the average cost values that are obtained by running the Monte Carlo simulations for the four agents under the two proposed control approaches for a period of 2 s. As expected the average costs for the cooperative game theory approach are less than those that are obtained from the optimal control approach. However, it should be noted that this achievement is at the expense of an increased computational complexity. In fact, in the former method two optimization problems, i.e. a maximization and a minimization problem should be solved as compared to the semi-decentralized approach where only a single minimization problem needs to be solved. Therefore, there is a tradeoff between the control computational complexity and the achievable control performance.

Remark 7. It should be noted that the final values that are obtained for the semi-decentralized optimal control strategy (graphs are not shown due to space limitations) are the average of the states initial values. In fact the control law provided by $u^i(X^i, X^j)$ in Lemma 5 is a weighted average consensus protocol which results in consensus on the average value of the initial state vector. However, cooperative game theory protocol that is obtained by solving the set of LMIs given in (23) is not necessarily an average consensus protocol. Therefore, in this case the consensus value can be any arbitrary number.

6. Conclusions

A control strategy based on cooperative game theory is applied to the problem of cooperation in a team of unmanned systems. The main goal of the team is to reach at a common output, i.e. to have consensus. To achieve this goal a linear combination of the locally defined cost functions is considered as the team cost function. The Nash-bargaining solution is chosen as "the best solution" among the Pareto-efficient ones that are found by optimizing the team cost function. Unfortunately, to implement this "centralized" strategy one requires to have access to the

Table 1
A comparative evaluation of the average value of the performance index corresponding to the two control design strategies for the cost functions defined in (6) for $T = 2$ s.

Control scheme	Average performance index			
	Agent 1	Agent 2	Agent 3	Agent 4
Semi-decentralized optimal control	14,648	14,781	11,629	12,519
Cooperative game theory	8,545	7,730	6,138	8,370

full team information set. To remedy this major shortcoming the corresponding optimization problem is formulated as a set of LMIs where the available information ("decentralized") structure is enforced onto the control structure. Consequently, one can now guarantee consensus achievement with minimum individual cost by maximizing the difference between the cost obtained through cooperative and non-cooperative approaches. For comparison purpose, the results obtained by utilization of a semi-decentralized optimal control strategy that the authors have recently developed in Semsar-Kazerooni and Khorasani (2007b, 2008) are considered as the non-cooperative solution of the consensus problem. Furthermore, the consensus achievement condition is added as a constraint to the set of LMIs. By performing a comparative study between the cooperative game theory and the optimal control strategies, it is concluded that the former approach results in lower individual as well as team cost as predicted. Moreover, whereas the optimal control approach provides a local optimal solution (agent-by-agent), cooperative game theory approach results in a global optimal solution that is subject to the imposed communications constraints. In future work, a quantified index will be presented to measure the effects of decentralization of information on the increase of the team cost. In other words, we will quantify the effects of the connectedness of the information graph on the performance of the team. This quantization criteria may provide an insight into the tradeoffs that exist between the availability of information and the team cost based on the two proposed methods.

Appendix

Proof of Theorem 6. (a) Follows from the constructive results that are derived in this section.

(b) For stability analysis of the closed-loop system we should note that Condition 3 in (23) guarantees that matrix P has at least the same zeros as the Laplacian matrix of the information graph, L (i.e. it may also have more zeros). Also, since both B and R are block diagonal matrices, the term $BR^{-1}B^{\mathrm{T}}P$ has at least the same zero elements as L. On the other hand, based on the definition given in (14), matrix A has also this property too. Therefore, the closed-loop system matrix A_c has a structure similar to the Laplacian of a subgraph of the original graph but this subgraph may not be in general a connected one (due to the extra zero entries that may appear in A_c). However, the condition $A_c(i,j) \neq 0$ if $L_{sub}(i,j) \neq 0, \forall i,j = 1,\dots,N$, guarantees that A_c does not have any other zeros besides the ones that exist in the Laplacian matrix of at least one of the connected subgraphs of the original graph. Therefore, matrix A_c has the minimum required non-zero elements to describe a Laplacian matrix of a "strongly connected" graph. Moreover, since A_c has the "structure" of a Laplacian matrix and it also satisfies Condition 2 in (23), it is in fact the Laplacian matrix of a weighted and strongly connected graph. From the graph theory, and in particular as shown in Olfati-Saber and Murray (2004), it is known that the Laplacian matrix of any strongly connected and directed graph has one and only one zero eigenvalue and $N-1$ negative eigenvalues (Godsil & Royle, 2001). Therefore, it is a stable matrix with only one zero eigenvalue.

(c) Assume that P^* is obtained from the optimization problem (23). It follows clearly that P^* also satisfies (17). Furthermore, $\int_0^{\mathrm{T}} \frac{\mathrm{d}}{\mathrm{d}t}(X^{\mathrm{T}}P^*X)\mathrm{d}t = \int_0^{\mathrm{T}}[(AX+BU)^{\mathrm{T}}P^*X + X^{\mathrm{T}}P^*(AX+BU)]\mathrm{d}t = X^{\mathrm{T}}(T)P^*X(T) - X^{\mathrm{T}}(0)P^*X(0)$, so that we have $\int_0^{\mathrm{T}}[(AX+BU)^{\mathrm{T}}P^*X + X^{\mathrm{T}}P^*(AX+BU)]\mathrm{d}t + X^{\mathrm{T}}(0)P^*X(0) - X^{\mathrm{T}}(T)P^*X(T) = 0$. By adding the above expression to the cost function (10) we get

$$\begin{aligned} J^c &= \int_0^{\mathrm{T}}[X^{\mathrm{T}}QX + U^{\mathrm{T}}RU + (AX+BU)^{\mathrm{T}}P^*X + X^{\mathrm{T}}P^*(AX+BU)]\mathrm{d}t \\ &\quad + X^{\mathrm{T}}(0)P^*X(0) - X^{\mathrm{T}}(T)P^*X(T) \\ &= \int_0^{\mathrm{T}}[X^{\mathrm{T}}(Q + A^{\mathrm{T}}P^* + P^*A)X + U^{\mathrm{T}}RU + U^{\mathrm{T}}B^{\mathrm{T}}P^*X \\ &\quad + X^{\mathrm{T}}P^*BU]\mathrm{d}t + X^{\mathrm{T}}(0)P^*X(0) - X^{\mathrm{T}}(T)P^*X(T). \end{aligned} \tag{24}$$

Since P^* satisfies (17), one gets $\mathcal{R}(Q,R,P^*) = P^*A + A^{\mathrm{T}}P^* - P^*BR^{-1}B^{\mathrm{T}}P^* + Q \geq 0$. Hence,

$$\begin{aligned} J^c &\geq \int_0^{\mathrm{T}}[X^{\mathrm{T}}(P^*BR^{-1}B^{\mathrm{T}}P^*)X + U^{\mathrm{T}}RU + U^{\mathrm{T}}B^{\mathrm{T}}P^*X + X^{\mathrm{T}}P^*BU]\mathrm{d}t \\ &\quad + X^{\mathrm{T}}(0)P^*X(0) - X^{\mathrm{T}}(T)P^*X(T) \\ &= \int_0^{\mathrm{T}}[(U + R^{-1}B^{\mathrm{T}}P^*X)^{\mathrm{T}}R(U + R^{-1}B^{\mathrm{T}}P^*X)]\mathrm{d}t \\ &\quad + X^{\mathrm{T}}(0)P^*X(0) - X^{\mathrm{T}}(T)P^*X(T). \end{aligned} \tag{25}$$

In order to minimize the integral part of the above cost function one may select the control input as $U^* = -R^{-1}B^{\mathrm{T}}P^*X$, which is already satisfied due to the definition of the control law. Moreover, since P^* is obtained through the optimization problem (23), it is guaranteed that in steady state consensus will be achieved. In other words, if we assume that T is sufficiently large (i.e. $T \to \infty$) so that the system reaches a steady state, then $X(T) = \xi[1\ 1\ \dots\ 1]^{\mathrm{T}}$. Correspondingly, it can be shown that $X^{\mathrm{T}}(T)PX(T) = \xi^2 \sum_i \sum_j P^*(i,j)$, where $P^*(i,j)$ represents the ij'th entry of the matrix P^*. Therefore, when $T \to \infty$, the optimal cost has a lower bound that is given by $J^{c*} \geq X^{\mathrm{T}}(0)P^*X(0) - \xi^2 \sum_i \sum_j P^*(i,j)$, where P^* is obtained by solving the maximization problem (23). ■

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