THE QUILLEN-SULLIVAN EQUIVALENCE IN RATIONAL HOMOTOPY THEORY

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Abstract. This survey was written as part of a "Project with Colloquium" at the Technical University of Munich in the fall of 2025. The goal is to give an introduction to *Rational Homotopy Theory* and build up the theory to state the *Quillen-Sullivan Equivalence*. We begin by explaining the main ideas and motivation behind rational homotopy theory and what role the Quillen-Sullivan equivalence plays. We then give overviews of some concepts regarding commutative differential graded algebras, simplicial sets and model categories. Using this we define the two functors required to state our main theorem, which establishes a correspondence between certain rational homotopy types and certain algebraic objects.

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1. Introduction

The goal of this survey is to build up the theory to present Theorem 6.6, which describes a correspondence between certain *rational homotopy types* and certain *commutative differential graded algebras*. Following [Hol21], we

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call this the *Quillen-Sullivan Equivalence*, in honor of the two main architects of rational homotopy theory.

We will give some basic overviews of rational homotopy theory, simplicial sets and model categories, though we will rarely give proofs. We further expect that the reader has some basic knowledge of algebraic topology and category theory.

There are several versions of the main theorem. In 1969 Daniel Quillen [Qui69] first proved the equivalence of the homotopy category of simply connected rational spaces and the homotopy categories of certain algebraic objects[1]. Though the proof relied on long chains of equivalences which made it inconvenient in practice. A bit later Dennis Sullivan [Sul77] recognized the usefulness of the polynomial de Rham algebra, which we will see in Definition 3.5, and using this, defined a direct one-to-one correspondence between rational homotopy types of nilpotent spaces and certain commutative graded algebras. We will see this how this is done in Definition 6.1. There have, of course, been many more developments in this corner of topology, but for a more thorough historical treatment of this topic we recommend [Hes99].

Many sources on this topic (e.g., [FHT12], [Hes06] and [Qui69]) restrict to the case of simply connected spaces, as this makes the theory slightly easier. However, we will mostly follow Julian Holstein's lecture notes on rational homotopy theory [Hol21], which are based on [BG76] and which only restrict to nilpotent spaces. These are more general than simply connected spaces, but are still nice enough to work with.

There are, however, also other generalizations to non-simply connected spaces such as [Mor10] and [GTHT00]. Further, [Iva22] gives an overview of four rationalization theories in a more general setting than simply connected spaces.

A word on notation and conventions. For categories $\mathcal{C}, \mathcal{D}$ and objects $X, Y \in \mathcal{C}$, we write $\mathcal{C}(X, Y)$ for the hom-set, or the set of morphisms from X to Y, and $[\mathcal{C}, \mathcal{D}]$ for the category of functors from $\mathcal{C}$ to $\mathcal{D}$. As is common in category theory, we will often write fg instead of $f \circ g$ for the composition of two morphisms. Further, as we will see objects with *cohomological grading*, we note that a superscript n is always an index, unless otherwise stated. For example, A^n may denote the n-th piece of a chain complex (A, d).

We will always assume that a topological space comes equipped with a base point and will therefore usually refrain from writing them explicitly. And as we did above, when considering chain complexes (A, d), we assume they are *cohomologically graded*, that is, the differential increases $d : A^n \to A^{n+1}$.

[1]Namely, reduced graded Lie algebras and 2-reduced graded co-algebras respectively.

2. Rational Homotopy Theory

One of the fundamental problems in algebraic topology is to classify topological spaces up to homotopy equivalence. In order to show that two spaces are *not* homotopy equivalent one has many algebraic invariants to choose from.

One of these invariants consists of the *homotopy groups* $\pi_n(X)$ of some pointed space X. Two topological spaces X and Y then have the same *weak homotopy type* or are *weakly equivalent* if there exists a *weak homotopy equivalence*[2] between them, that is to say, a map $f : X \to Y$ that induces isomorphisms on all homotopy groups. The following theorem by Whitehead [Whi49] shows why this is such an important concept. Nowadays, there exist many generalizations of this theorem, including a generalization to model categories which we will see later.

Theorem 2.1 (Whitehead)**.** *If X and Y are CW complexes, then a continuous map $f : X \to Y$ is a weak equivalence if and only if it is a homotopy equivalence.*

The problem with homotopy groups is that they are usually very difficult to compute. The idea of *rational* homotopy theory is to compute homotopy groups only "modulo torsion", which loses information, but which, thanks to the Quillen-Sullivan equivalence, is much easier compute. To do this, one computes the ordinary homotopy groups of the *rationalization* of a space. We will now demonstrate what that is and why it makes sense.

Definition 2.2. A path connected topological space X is called *rational* if for all $k \geq 1$ the homology group $H_k(X, \mathbb{Z})$ is a $\mathbb{Q}$-vector space.

At the end of this section, we will see why defining rational spaces using homology tells us something about its homotopy groups. Note also that we only use homology groups of degrees 1 and higher, and that for a path connected space X we have $H_0(X) \cong \mathbb{Z}$. Since the ordinary homology groups can be described by the *reduced* homology groups $\tilde{H}_k(X)$ as

$$H_n(X) = \begin{cases} \tilde{H}_k(X) \oplus \mathbb{Z} & \text{if } \ k = 0, \\ \tilde{H}_k(X) & \text{else,} \end{cases}$$

it follows that condition on homology groups is equivalent to saying that *all* reduced homology groups $\tilde{H}_k(X)$ are $\mathbb{Q}$-vector spaces.

A trivial example of a rational space is any contractible space, for example the n-disk D^n. This holds since all of its (reduced) homology groups are trivial and can therefore be seen as the trivial vector space over $\mathbb{Q}$. But the close

[2]Or more generally, a zig-zag of homotopy equivalences.

relative of D^n, the n-sphere S^n is already not rational, since $H_n(S^n, \mathbb{Z}) \cong \mathbb{Z}$ is not a $\mathbb{Q}$ vector space. However, we can "make" it rational.

Example 2.3. Let $(p_i)_{i=1}^{\infty}$ be the sequence of primes $2, 3, 2, 3, 5, 2, 3, 5, 7, 2 \ldots$ and so on. It is a well known fact that $\operatorname{colim}(\mathbb{Z} \xrightarrow{\cdot p_1} \mathbb{Z} \xrightarrow{\cdot p_2} \mathbb{Z} \xrightarrow{\cdot p_3} \cdots) = \mathbb{Q}$, where $\cdot p_i$ denotes the multiplication by p_i map. Using this and the fact that $\pi_n(S^n) \cong \mathbb{Z}$, we can define the *rational n-sphere* $S^n_{\mathbb{Q}}$ as the *homotopy colimit* of the sequence $S^n \xrightarrow{q_1} S^n \xrightarrow{q_2} S^n \xrightarrow{q_3} \cdots$, where the q_i are maps that induce $\cdot p_i$ on the n-th homotopy group. It can then be shown that the reduced homology groups of $S^n_{\mathbb{Q}}$ are

$$\tilde{H}_k(S^n_{\mathbb{Q}}, \mathbb{Z}) \cong \begin{cases} \mathbb{Q} & \text{if } \ k = n, \\ 0 & \text{else.} \end{cases}$$

Hence, $S^n_{\mathbb{Q}}$ is a rational space.

For more details on this particular construction of $S^n_{\mathbb{Q}}$ using homotopy colimits, see Example 1.7 in [Moe17]. We will not (explicitly) need homotopy (co)limits for the rest of this survey, but the interested reader can find out more about them in e.g. Part I of [Rie14].

One can also construct the rational n-sphere more explicitly, namely as

$$S^n_{\mathbb{Q}} = \left(\bigvee_{i=0}^{\infty} S_i \right) \cup_h \left(\coprod_{j=1}^{\infty} D_j^{n+1} \right)$$

where the S^n_i's and D^n_j's are n-spheres and n-disks respectively. Here we glue D^{n+1}_j to $S^n_{j-1} \vee S^n_j$ using a map $S^n \to S^n_{j-1} \vee S^n_j$ that represents the homotopy class $[\iota_{j-1}] - p_i[\iota_j]$, where p_i is as in the example above and ι_j is the inclusion map $S^n \hookrightarrow \bigvee_{i=0}^{\infty} S_i$ into the j-th summand. For this construction see the beginning of Chapter 9 in [FHT12] or Example 1.2 in [Hes06].

Now recall that D^{n+1} can be identified with the cone of the n-sphere, i.e. the space $(S^n \times I)/(S^n \times \{0\})$. Mimicking this, we can define the *rational n-disk* as the cone $D^{n+1}_{\mathbb{Q}} := (S^n_{\mathbb{Q}} \times I)/(S^n_{\mathbb{Q}} \times \{0\})$.

This may seem like a pointless thing to define, since we just claimed that the ordinary n-disk D^n is already a rational space. But using this, we can rationalize simply connected CW complexes, giving us an endless supply of rational spaces. This rationalization is done in an obvious way: Given a CW complex X we just replace its n-cells D^n and their boundary S^n by $D^n_{\mathbb{Q}}$ and $S^n_{\mathbb{Q}}$. This results in a rational space $X_{\mathbb{Q}}$ that is "rationally equivalent" to X.

Of course, it is not quite so simple, as one also has to change the attaching maps. A more thorough treatment of this topic can be found in Chapter 2 of [Moe17].

Let's now turn to the n-sphere again. We know that $H_n(S^n, \mathbb{Z}) \cong \mathbb{Z}$ has no torsion and therefore the Universal Coefficient Theorem tells us that

$$H_n(S^n, \mathbb{Q}) \cong H_n(S^n, \mathbb{Z}) \otimes \mathbb{Q} \cong \mathbb{Z} \otimes \mathbb{Q} \cong \mathbb{Q},$$

and by the same argument we have

$$H_n(S^n_{\mathbb{Q}}, \mathbb{Q}) \cong H_n(S^n_{\mathbb{Q}}, \mathbb{Z}) \otimes \mathbb{Q} \cong \mathbb{Q} \otimes \mathbb{Q} \cong \mathbb{Q}.$$

So, since both spaces have non trivial homology only in degrees 0 and n, it follows that $H_k(S^n, \mathbb{Q})$ and $H_k(S^n_{\mathbb{Q}}, \mathbb{Q})$ are isomorphic for all $k \geq 1$. But we can say even more, namely, that the natural inclusion $i : S^n \hookrightarrow S^n_{\mathbb{Q}}$ induces such isomorphisms $i_* : H_k(S^n, \mathbb{Q}) \xrightarrow{\cong} H_k(S^n_{\mathbb{Q}}, \mathbb{Q})$ for all $k \geq 1$. This motivates the following definition.

Definition 2.4. Let X be a path connected topological space. A *rationalization* of X is a rational space $X_{\mathbb{Q}}$ with a continuous map $f : X \to X_{\mathbb{Q}}$ that induces isomorphisms $f_* : H_k(X, \mathbb{Q}) \to H_k(X_{\mathbb{Q}}, \mathbb{Q})$ for all $k \geq 1$.

So in the words of this definition, the rational n-sphere $S^n_{\mathbb{Q}}$ is a *rationalization* of the regular n-sphere S^n. As with our previous examples, we usually write $X_{\mathbb{Q}}$ for the rationalization of some space X, though in some books one may find the notation X_0 instead of $X_{\mathbb{Q}}$.

In general, if a continuous map $f : X \to Y$ between path connected spaces induces isomorphisms $f_* : H_k(X, \mathbb{Q}) \to H_k(Y, \mathbb{Q})$ for all $k \geq 1$, we call f a *rational equivalence*. Naturally, if such an f exists X and Y are called *rationally equivalent* and it can be easily verified that this defines an equivalence relation. We call the equivalence class of a space under this relation its *rational homotopy type*.

Since again we only consider path connected spaces, these definitions can be equivalently made using reduced homology in all degrees.

It may seem strange that we call it rational *homotopy* type, when we defined it using homology. But when restricting our class of spaces somewhat, it starts to make more sense. As we've mentioned, in this survey we consider *nilpotent* spaces.

In order to define what nilpotent spaces are recall that the *lower central series* of a group G is the sequence of normal subgroups $\cdots \trianglelefteq G_1 \trianglelefteq G_0 = G$ where G_{i+1} is defined as the commutator $[G_i, G]$.

Now let G act on another group H. Following Dror [Dro06], we call the sequence $\cdots \subseteq H_1 \subseteq H_0 = H$, where the abelian group H_{i+1} is generated by $\{g(h) \cdot h^{-1} \mid g \in G,\ h \in H\}$, the *lower central series* associated to the group action of G on H.

We can now see that the lower central series of a group G is just the lower central series associated to the group action of G on itself by conjugation. Using this we can make the following definition.

Definition 2.5. Let G be a group that acts on an abelian group A. Then we say that

(i) G *acts nilpotently* on H if the lower central series associated to the group action terminates after a finite number of steps[3],
(ii) G is *nilpotent* if it acts nilpotently on itself by conjugation.

Given a path $\alpha : I \to X$ from x_0 to x_1, we can now define a base change isomorphism $\pi_n(X, x_0) \to \pi_n(X, x_1)$ by deforming S^n into $S^n \vee I$ so that the base point is the far end of I and then applying $g \vee \alpha$ with $g \in \pi_n(X, x_0)$. If α is now a loop, this is an action of $\pi_1(X)$ an $\pi_n(X)$. For $n = 1$, this is just the standard group action of a group on itself by conjugation. This action is also discussed in Section 9.5 of [May99] and a slightly different way of defining it can be found in Chapter 4 of [Hat02]. With this action in mind, we can now define nilpotent spaces.

Definition 2.6. A path connected space X is *nilpotent* if its fundamental group acts nilpotently on all homotopy groups.

Note that this implies that the fundamental group is nilpotent. It is also clear that every simply connected space X is nilpotent, as simply connected means the fundamental group $\pi_1(X)$ is trivial and we therefore get $\{g(a) - a \mid g \in \pi_1(X),\ a \in \pi_n(X)\} = \{0\}$.

A prime example of a nilpotent space that is not simply connected is S^1! We know that $\pi_1(S^1) = \mathbb{Z}$ and $\pi_n(S^1) = 0$ for $n > 1$. So $\pi_1(S^1)$ being abelian implies that it is nilpotent and since the higher homotopy groups are trivial, the lower central series associated to the group action of $\pi_1(S^1)$ on $\pi_n(S^1)$ is trivial.

The class of nilpotent spaces also includes *simple* spaces, which are path connected spaces whose fundamental group acts trivially on all homotopy groups. All this is to say that considering nilpotent spaces instead of simply connected spaces gives us a lot more flexibility.

Having defined nilpotent spaces, we can formulate the following theorem, which is one of the key ingredients to showing why the Quillen-Sullivan equivalence is useful for computing homotopy groups modulo torsion.

Theorem 2.7. *Every nilpotent space X has a rationalization $f : X \to X_{\mathbb{Q}}$ such that $\pi_n(f) \otimes \mathbb{Q} : \pi_n(X) \otimes \mathbb{Q} \to \pi_n(X_{\mathbb{Q}})$ is an isomorphism for all $n \geq 1$.*

Note that $\pi_1(X)$ is only assumed to be nilpotent and might not be abelian. We therefore denote by $\pi_1(X) \otimes \mathbb{Q}$ the *Mal'cev completion* of $\pi_1(X)$. It is not important to know precisely what this means, but it is essentially a group

[3]By this we mean that for the lower central series $\cdots \subseteq H_1 \subseteq H_0 = H$, there exists an $n \in \mathbb{N}$ such that $H_k = \{0\}$ for all $k \geq n$.

one obtains from $\pi_1(X)$ that has no torsion elements and that, if $\pi_1(x)$ is abelian, coincides with the tensor product with $\mathbb{Q}$.

The existence of a rationalization is Theorem 5.3.2 of [MP12] and the isomorphism on homotopy groups then follows from Chapter V Proposition 3.2 of [BK72].

So why is this useful? Given a nilpotent space X we now know that there is a rationalization $X_{\mathbb{Q}}$ and by the second part of the previous theorem we can compute the homotopy groups of X modulo torsion, if we can compute the homotopy groups of $X_{\mathbb{Q}}$.

As it turns out, using the Quillen-Sullivan equivalence we can obtain an algebraic object corresponding to $X_{\mathbb{Q}}$ with which we can compute the homotopy groups of $X_{\mathbb{Q}}$ in a much easier way.

3. Commutative Differential Graded Algebras

We mentioned that Theorem 6.6 establishes a correspondence between certain rational homotopy types and certain *commutative differential graded algebras*, or *cdga*'s for short. We briefly recall what these are.

Definition 3.1. Let R be a commutative ring. Then a *commutative differential graded algebra* is a *commutative monoid object* in the symmetric monoidal category $(\mathbf{Ch}_R, \otimes, R[0])$ of chain complexes over R.

This means that a cdga $((A,d), e, \mu)$ consists of a chain complex $A = (A,d)$, a *unit map* $e : R[0] \to A$ and a *product map* $\mu : A \otimes A \to A$. These satisfy certain associativity, unitality and commutativity diagrams. Let's unpack more concretely what this boils down to.

Our underlying structure is of course the chain complex (A,d). Now, given two elements $a \in A^n$ and $b \in A^m$, we can multiply them by $a \cdot b := \mu(a \otimes b)$ and since by the definition of the tensor product of chain complexes $a \otimes b$ lies in $(A \otimes A)^{n+m}$, the product $a \cdot b$ lies in A^{n+m}.

It is then a technical detail of the symmetric braiding of $(\mathbf{Ch}_R, \otimes, R[0])$ that the commutativity condition on μ is actually a *graded* commutativity condition, which means we have $a \cdot b = (-1)^{|a||b|} b \cdot a$, where $|a| = n$ denotes the degree of $a \in A^n$.

Hence, A with the just defined multiplication becomes a commutative graded ring. And since all components A^n are in particular R-modules, this assembles into a *commutative graded algebra* over R.

We also have the differential d, which we now write as d_A to avoid confusing it with the induced differential $d_{A\otimes A}$ on the tensor product $A \otimes A$. Since μ is a chain map we get $d_A(a \cdot b) = d_A(\mu(a \otimes b)) = \mu(d_{A\otimes A}(a \otimes b))$. Recall that the differential of tensor products of chain complexes is defined to be

$d_{A\otimes A}(a\otimes b) = d_A(a)\otimes b + (-1)^{|a|} a\otimes d_A(b)$. Combining this, we find that the differential d on A satisfies the usual *Leibniz rule*

$$d(a\cdot b) = d(a)\cdot b + (-1)^{|a|} a\cdot d(b).$$

Of course, instead of $((A,d),e,\mu)$ we will usually just write (A,d), or A if the differential is obvious or not of importance.

The graded commutativity of cdga's is the reason why they are sometimes also called *graded-commutative differential algebras* to reflect that they are not strictly commutative.

From now on, we will only consider the case where $R=\mathbb{Q}$, hence we write **CDGA** for the category that has as objects *non-negatively graded* cdga's[4] over $\mathbb{Q}$ and where morphisms are algebra homomorphisms that preserve the grading and differential.

Furthermore, as a cdga (A,d) is just a chain complex with extra data, we can ignore that extra data and take it's cohomology groups $H^*(A)$. As usual, we call a morphism between two cdga's a *quasi-isomorphism* if it induces an isomorphism on cohomology.

Example 3.2. Some trivial examples of cdga's are the following:

- (i) Every algebra with graded commutative structure can be viewed as a cdga with the differential being the constant zero map.
- (ii) Similarly, every commutative algebra can be viewed as a cdga where every element is homogeneous of degree 0.
- (iii) In particular, $\mathbb{Q}$ and the trivial algebra are cdga's. These are the initial and terminal object of the category **CDGA**.

Example 3.3 (Non-Example)**.** From algebraic topology one may remember that we can turn the *singular chain complex* $C^\bullet(X;R)$ of a topological space X with coefficients in a commutative k-algebra R into a *differential graded algebra* with the cup product. But this is only graded commutative up to chain homotopy and only becomes graded commutative on the nose after taking cohomology $H^\bullet(X;R)$.

There is another class of important examples, namely *free* cdga's. These are defined as follows.

Given a graded $\mathbb{Q}$-vector space $V=\bigoplus_{n\in Z} V^n$, we define the *odd* and *even* part of V as

$$V_{\text{odd}} = \bigoplus_{n\in\mathbb{Z}} V_{2n+1} \quad \text{and} \quad V_{\text{even}} = \bigoplus_{n\in\mathbb{Z}} V_{2n}$$

respectively. The *free commutative graded algebra* of V is then defined as $\wedge V := \mathrm{S}(V_{\text{even}})\otimes \mathrm{E}(V_{\text{odd}})$, where $\mathrm{S}(V_{\text{even}})$ is the *symmetric algebra* of V_{even}

[4]That is, cdga's A satisfying $A^k=\{0\}$ for $k<0$.

and $\mathrm{E}(V_{\mathrm{odd}})$ is the *exterior algebra* of V_{odd}. Alternatively, [FHT12] defines it as the *tensor algebra* $\mathrm{T}(V)$ of V modulo the relation $a \otimes b = (-1)^{|a||b|} b \otimes a$, where a and b are homogeneous elements. In any case we write $a \otimes b$ as $a \cdot b$ or ab.

If one has never heard of the symmetric, exterior or tensor algebra, this definition may seem quite intimidating. However, as with most "free" constructions, it is really just the simplest thing one can think of, we just need to define it in a complicated way because of the graded commutative structure.

Elements of $\wedge V$ can simply be thought of as being free products[5] of elements of V such that their degree is additive and the free products commute up to grading, so $ab = (-1)^{|a||b|} ba$ for $a, b \in V$, extending to arbitrary products.

If V furthermore has a differential d, or in other words, if V is a chain complex, then we can extend d to a differential $d_\wedge$ on $\wedge V$ by setting $d_\wedge(v) = d(v)$ for $v \in V$ and forcing it to satisfy the Leibniz rule, that is by setting $d_\wedge(vw) := d_\wedge(v)w + (-1)^{|v|} v d_\wedge(w)$ for any $v, w \in \wedge V$. This then extends uniquely to a differential on $\wedge V$.

A cdga $(\wedge V, d)$ where $\wedge V$ is a free commutative graded algebra, but where d may be any differential on $\wedge V$ is called *semi-free*.

One does not necessarily need to know what the de Rham complex of a smooth manifold is, but it leads to a very important example that will play an important role in the Quillen-Sullivan equivalence.

Example 3.4. Let $M = \mathrm{int}(|\Delta^n|)$ be the interior of the geometric n-simplex, i.e. $M = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n t_i = 1,\ 0 < t_i < 1\}$. One can then show that the cotangent bundle is $T^*M \cong \mathcal{C}^\infty(\mathbb{R}^{n+1}) \otimes \mathbb{R}[dt_0, \dots, dt_n]/J$, where $J = (\sum_{i=0}^n dt_i = 0)$. Restricting the coefficient functions to only be polynomials gives us a cdga

$$\mathbb{R}[t_0, \dots, t_n]/I \xrightarrow{d} \mathbb{R}[t_0, \dots, t_n]/I \otimes \mathbb{R}[dt_0, \dots, dt_n]/J \xrightarrow{d} \cdots$$

where d is the exterior derivative, $I = (\sum_{i=0}^n t_i = 1)$ and the higher degree terms are generated from the ones in degrees 0 and 1.

Specializing our coefficients to $\mathbb{Q}$ we get the following definition.

Definition 3.5. The *polynomial de Rham algebra of the n-simplex* $\Omega(n)$ is the free cdga generated in degree 0 by elements $t_0, \dots, t_n$ satisfying the condition $\sum_{i=0}^n t_i = 1$ and in degree 1 by elements $dt_0, \dots, dt_n$ with $dt_i = d(t_i)$.

From the condition on the t_i's and because of $dt_i = d(t_i)$ it follows that $\sum_{i=0}^n dt_i = 0$. From these two conditions on the generators it follows that

[5] That is, formal symbols $v_1 v_2 \cdots v_n$ with $v_i \in V$, where the product is concatenation of the symbols and we have obvious distributive laws.

$\Omega(n)$ is isomorphic to a free cdga with n generators in degrees 0 and 1. However, viewing it as having $n+1$ generators will be useful later.

From the definition above, it is easy to read off that $\Omega(0) = \mathbb{Q}$. Furthermore, as dt_i sits in degree 1 we get $dt_i \cdot dt_j = (-1)^{1 \cdot 1} dt_j \cdot dt_i = -dt_j \cdot dt_i$. Therefore the case $i = j$ shows that $dt_i \cdot dt_i = 0$. We can now deduce that the homogeneous elements of degree d in $\Omega(n)$ are sums of products $f dt_{i_1} \cdots dt_{i_d}$, where f is a polynomial with rational coefficients and $i_j \neq i_k$ for $i \neq k$. In particular it follows that $\Omega(n)^d = \{0\}$ for $d > n$.

Now recall, that for any $A \in \mathbf{CDGA}$ we had the *unit map* $e : \mathbb{Q}[0] \to A$, where $\mathbb{Q}[0]$ is the chain complex with $\mathbb{Q}$ concentrated in degree 0, so we may consider it as a homomorphism $e : \mathbb{Q} \to A^0$. An *augmentation* of A is a retraction or left-inverse of e, i.e. a homomorphism $\epsilon : A \to \mathbb{Q}$ such that $\epsilon \circ e = \mathrm{id}_{\mathbb{Q}}$. A cdga is called *augmented* if it has an augmentation. Of course, it isn't *a priori* clear that a cdga has an augmentation and therefore we write $\mathbf{CDGA}_{/\mathbb{Q}} \subset \mathbf{CDGA}$ for the subcategory of augmented cdga's and cdga homomorphisms that preserve augmentations. If one is familiar with *slice categories* this notation may seem familiar and indeed $\mathbf{CDGA}_{/\mathbb{Q}}$ can be viewed as a slice category. It is not important to know what this means, but it is a technical detail that is useful when talking about model structures on $\mathbf{CDGA}_{/\mathbb{Q}}$.

We say that a cdga A is *connected* if it is non-negatively graded and if the unit map $e : \mathbb{Q} \to A^0$ is an isomorphism. From the second condition it immediately follows that every connected cdga has an augmentation and in particular that this augmentation is unique.

A similar-looking concept to connectedness is *homological connectedness*. A cdga B is called *homologically connected* if it is non-negatively graded and there is an isomorphism $H^0(B) \cong \mathbb{Q}$. By the non-negative gradedness of B the second condition is the same as saying that there is an isomorphism $\ker(d^0) \cong \mathbb{Q}$. So being homologically connected is really a condition on the differential, whereas being connected has nothing to do with the differential.

We will later see that augmented and homologically connected cdga's are in some sense analogous to pointed and path connected spaces.

3.1. **Minimal models.** We will now look at *minimal models*, sometimes called *minimal Sullivan algebras*. These are certain cdga's with a relatively simple structure, as they are connected, semi-free and satisfy further conditions on the differential. It will later turn out that in many cases we can replace a cdga by a suitable minimal model. In order to give the definition we first need to introduce some auxiliary concepts.

Let $(A, d) \in \mathbf{CDGA}$. Then we define $A(n)$ to be the subalgebra of A generated by A^i for $0 \leq i \leq n$ and dA^n, and further we let $A(-1) = \mathbb{Q} \subseteq A^0$.

Now let $A(n, 0) = A(n-1)$ and inductively define $A(n, p)$ to be the subalgebra generated by $A(n, p-1)$ and those elements $x \in A^n$ such that $d(x) \in A(n, p-1)$.

Definition 3.6. A *minimal model* is a connected semi-free cdga M such that $M(n) = \bigcup_{p\in\mathbb{N}} M(n, p)$ for all $n \in \mathbb{N}$. A *minimal model for* a cdga A is a minimal model M with a quasi-isomorphism $f : M \to A$.

Of course the interesting condition here is $M(n) = \bigcup_{p\in\mathbb{N}} M(n, p)$. Clearly we have $M = \bigcup_{n\in\mathbb{N}} M(n)$, so for every generator $a \in M$ there is some $n \in \mathbb{N}$ such that $d(a) \in M(n)$ but $d(a) \notin M(n-1)$. The condition from above then implies that there is also some $p \geq 1$ such that $d(a) \in M(n, p)$ but $d(a) \notin M(n, p-1)$. In particular, this means that the differential of any generator of M is not a generator itself, but a product of other generators.

Further note that by our previous remarks, M being connected implies it has a unique augmentation, namely the two-sided inverse of the unit map $e_M : \mathbb{Q} \to M^0$.

We said that we later intend to replace a cdga by its minimal model, but in general we don't know whether a given cdga even has a minimal model. However, part of the following theorem gives a sufficient condition for the existence.

Theorem 3.7 (Existence and Uniqueness of Minimal Models)**.** *Every homologically connected cdga A has a minimal model M that is unique up to isomorphism.*

The idea for proving the existence is to construct a minimal model step by step, that is to say, to define cdga's $M(n, p)$, with properties as above and morphisms $f_{n,p} : M(n, p) \to A$ that induce isomorphisms in cohomology up to degree $n-1$ and injections in degree n.

This is slightly differently stated in Theorem 14.12 of [FHT12]. Alternatively, in [Hol21] Holstein gives a full proof of the existence in Theorem 2.21 and a more general statement for uniqueness in Theorem 4.11, which actually holds for minimal models of any cdga.

We will later define *minimal models of topological spaces*, which will just be minimal models of certain cdga's. Using this minimal model of a topological space X we can then compute the rational homotopy groups of X using the theory of Sullivan.

For now, this will suffice as our overview of the algebraic side of things and we will turn to the topological side via simplicial sets.

4. Simplicial Sets

Simplicial sets are an abstract categorical tool that is often used to combinatorially describe topological spaces. Similarly to simplicial complexes, the idea is to use them to triangulate a space, though simplicial sets have nicer categorical properties and give a lot more freedom than simplicial complexes. It even turns out that the categories of simplicial sets and topological spaces are in some sense equivalent, we will see the precise statement in Theorem 5.15.

Unfortunately, we will only be able to give a short overview, but there are several good introductions to simplicial sets, such as [Rie11], freely available online. A more visual approach can be found in Chapter 2 of [Ste21].

Let $\mathbf{\Delta}$ denote the *simplex category.* $\mathbf{\Delta}$ has as objects finite well ordered sets $[n] = \{0, 1, 2, \ldots, n\}$ and as morphisms order preserving maps. As partially ordered sets can be viewed as categories, this can be viewed as a subcategory of the category of (small) categories **Cat**.

We have two special kinds of maps in $\mathbf{\Delta}$. The unique injective (order preserving) map $\delta_i : [n-1] \to [n]$[6] such that $i \notin \mathrm{im}(\delta_i)$ is the *i-th co-face map* and the unique surjective map $\sigma_i : [n+1] \to [n]$ such that $|\sigma_i^{-1}(\{i\})| = 2$ is the *i-th co-degeneracy map*[7]. Concretely, these maps are defined as

$$\delta_i : [n-1] \to [n],\ j \mapsto \begin{cases} j & j < i \\ j+1 & j \geq i \end{cases}$$

$$\sigma_i : [n+1] \to [n],\ j \mapsto \begin{cases} j & j \leq i \\ j-1 & j > i \end{cases}$$

Using either definition it is not difficult to show that these maps satisfy the following so called *co-simplicial identities*:

$$\delta_j \delta_i = \delta_i \delta_{j-1} \quad \text{if} \ \ i < j$$

$$\sigma_j \sigma_i = \sigma_i \sigma_{j+1} \quad \text{if} \ \ i \leq j$$

$$\sigma_j \delta_i = \begin{cases} \delta_i \sigma_{j-1} & \text{if} \ \ i < j \\ \mathrm{id}_{[n]} & \text{if} \ \ i = j, j+1 \\ \delta_{i-1} \sigma_j & \text{if} \ \ i > j+1 \end{cases}$$

The following proposition, a proof of which can be found in Chapter VII Section 5 of [ML98], then shows that the co-face and co-degeneracy maps

[6] We disregard the dependence on n in our notation, since usually it is clear from context.

[7] We will later define face and degeneracy maps. Sometimes these terms are used for both this and the dual version of these maps.

with the co-simplicial identities fully determine the morphisms of the simplex category.

Proposition 4.1. *Every map $\alpha : [n] \to [m]$ in $\mathbf{\Delta}$ has a unique factorization $\alpha = \delta_{i_1} \cdots \delta_{i_r} \sigma_{j_1} \cdots \sigma_{j_s}$ where the i_k are non-increasing and the j_k are non-decreasing.*

Having thus fully described the simplex category, we can now come to the definition of a simplicial set.

Definition 4.2. A *simplicial set* is a functor $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$. We write $\mathbf{Set}_{\mathbf{\Delta}} := [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$ for the category of simplicial sets and write $X_n = X([n])$ for $X \in \mathbf{Set}_{\mathbf{\Delta}}$.

Since a simplicial set is a *contravariant* functor, a map $\alpha : [n] \to [m]$ in $\mathbf{\Delta}$ induces a map $\alpha^* = X(\alpha) : X_m \to X_n$. Hence, the co-face and co-degeneracy maps induce maps $d_i = X(\delta_i) : X_n \to X_{n-1}$ and $s_i = X(\sigma_i) : X_n \to X_{n+1}$, which we will call the *face* and *degeneracy maps.* These satisfy dual identities to their counterparts, naturally called *simplicial identities*:

$$d_i d_j = d_{j-1} d_i \quad \text{if} \quad i < j$$
$$s_i s_j = s_{j+1} s_i \quad \text{if} \quad i \leq j$$
$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{if} \quad i < j \\ \mathrm{id}_{X_n} & \text{if} \quad i = j, j+1 \\ s_j d_{i-1} & \text{if} \quad i > j+1 \end{cases}$$

Furthermore, by functoriality and Proposition 4.1, any map $a : X_m \to X_n$ of X[8] has a unique factorization $a = s_{i_1} \cdots s_{i_s} d_{j_1} \cdots d_{j_r}$ with the i_k being non-increasing and the j_k being non-decreasing.

These facts then imply that all the data one has to give in order to define a simplicial set are sets X_n for $n \in \mathbb{N}$ and maps $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$ satisfying the simplicial identities.

Since we defined the category of simplicial sets as the functor category $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$, a morphism $f : X \to Y$ between simplicial sets is a natural transformation between the functors $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ and $Y : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$. That is, for every $n \in \mathbb{N}$ we have a map of sets $f_n : X_n \to Y_n$ such that for any morphism $\alpha : [m] \to [n]$ in $\mathbf{\Delta}$ the following square commutes.

$$\begin{tikzcd} X_n \arrow[r, "f_n"] \arrow[d, "X(\alpha)"'] & Y_n \arrow[d, "Y(\alpha)"] \\ X_m \arrow[r, "f_m"'] & Y_m \end{tikzcd}$$

[8]That is, any map in **Set** that is the image of a map in $\mathbf{\Delta}$ under X.

By our previous remarks however, it is already enough if all those squares commute when the vertical maps are face or degeneracy maps.

We call an element $\sigma \in X_n$ an *n-simplex* of X. Often 0-simplices are called *vertices* and 1-simplices *edges*. Further, an n-simplex is *degenerate* if it is in the image of a degeneracy map, else we naturally call it *non-degenerate*. When trying to picture a simplicial set, the non-degenerate simplices are really all that matters and degenerate simplices, as the name implies, don't contribute anything to our geometric picture.

We define a *pointed simplicial set* (X, x) to be a simplicial set $X \in \mathbf{Set}_{\mathbf{\Delta}}$ together with a distinguished vertex $x \in X_0$. In category theory, pointed objects are usually defined as objects A with a morphism $* \to A$ from the terminal object to A. Since the terminal object in $\mathbf{Set}_{\mathbf{\Delta}}$ is just a simplicial set with precisely one n-simplex for every $n \geq 0$, these two notions coincide. We further write $\mathbf{Set}_{\mathbf{\Delta}*}$ for the category of pointed simplicial sets.

We note that a simplicial set is by definition a *presheaf* on the simplex category $\mathbf{\Delta}$. Let $\mathcal{C}$ be a locally small category and $A \in \mathcal{C}$ an object. We denote by $h_A : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$, $B \mapsto \mathcal{C}(B, A)$ the contravariant hom-functor and recall that the *Yoneda Lemma* says that for any presheaf $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ and object $A \in \mathcal{C}$, there is a natural isomorphism between elements of $F(A)$ and natural transformations $\mathrm{Nat}(h_A, F)$.

If now $\mathcal{C} = \mathbf{\Delta}$ and $A = [n]$, we will call the contravariant hom-functor $\Delta^n := h_{[n]} = \mathbf{\Delta}(-, [n])$ the *standard n-simplex*. The importance of this example can hardly be overstated, since by what we just remarked about the Yoneda Lemma, we find that the n-simplices X_n of some simplicial set X are in natural bijection with the maps of simplicial sets $\Delta^n \to X$.

Applying this remark to the standard n-simplex itself shows us that a k-simplex of the standard n-simplex, i.e. a map of sets $\sigma : [k] \to [n]$, can be thought of as a map of simplicial sets $\sigma : \Delta^k \to \Delta^n$. It turns out that a k-simplex $\sigma \in (\Delta^n)_k$ is non-degenerate if it is injective as a map $[k] \to [n]$, and degenerate else. Considering the non-degnerate k-simplices $\sigma : [k] \to [n]$ as k-simplices in the geometric sense, we then get the usual geometric picture of an n-simplex.

Example 4.3. Let $\Lambda^n_i \subset \Delta^n$ denote the *i-th horn* of Δ^n, that is to say, the simplicial subset of the standard simplex Δ^n where the identity $[n] \to [n]$ and the i-th face map $\delta_i : [n-1] \to [n]$ is removed. This is an important example, as we will later see. The notation and name of Λ^n_i comes from the fact that if we interpret Δ^n as a solid n-simplex, then the i-th horn is the subset where we remove the i-th *face* and the n-dimensional filling of Δ^n, making it somewhat look like a horn for $n = 3$ and 2.

Example 4.4 (Key Example)**.** Another example of a simplicial set is the polynomial de Rham algebra $\Omega(*)$ from Definition 3.5! Indeed, we can view $\Omega(*)$ and every graded piece $\Omega(*)^d$ as a simplicial set by saying $\Omega(n)$ or $\Omega(n)^d$ is its set of n-simplices and by defining suitable face and degeneracy maps. To avoid confusing the face map with the differential we write ∂_i for the face map, instead of the usual d_i. Then $\partial_i : \Omega(n) \to \Omega(n-1)$ and $s_i : \Omega(n) \to \Omega(n+1)$ are defined to be cdga homomorphisms such that on the generators $t_0, \dots, t_n$ of $\Omega(n)^0$ they are

$$\partial_i(t_k) = \begin{cases} t_{k-1} & \text{if } i < j \\ 0 & \text{if } i = k, \\ t_k & \text{if } i > k \end{cases} \quad s_i(t_k) = \begin{cases} t_{k+1} & \text{if } i < j \\ t_k + t_{k+1} & \text{if } i = k. \\ t_k & \text{if } i > k \end{cases}$$

On $\Omega(n)^1$ and above the rest follows by setting

$$\partial_i(dt_j) = d(\partial_i(t_j)), \quad s_i(dt_j) = d(s_i(t_j)).$$

Showing that these maps indeed satisfy the simplicial identities is then simply an exercise in writing out what the composition of two of these maps amounts to.

4.1. **Geometric Realization and Singular Simplicial Sets.** As we said, our goal is to assign some geometric or topological meaning to simplicial sets. We have seen some geometric interpretation of the standard n-simplex Δ^n, but what topological space should we assign to Δ^n, or in general to some simplicial set?

We suggestively write $|\Delta^n| = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0,\ \sum_{k=0}^n x_k = 1\}$ for the *geometric n-simplex*. Further, given some $\alpha : [m] \to [n]$ we define $\alpha_* : |\Delta^m| \to |\Delta^n|$, $(x_0, \dots, x_m) \mapsto (y_0, \dots, y_n)$ where $y_i = \sum_{k \in \alpha^{-1}(i)} x_k$. This then assembles into a functor $|\Delta^{(-)}| : \mathbf{\Delta} \to \mathbf{Top}$, $[n] \mapsto |\Delta^n|$.

The notation of $|\Delta^n|$ is suggestive because of the following definition.

Definition 4.5. The *geometric realization functor* $|-| : \mathbf{Set}_{\mathbf{\Delta}} \to \mathbf{Top}$ is the *left Kan extension* of $|\Delta^{(-)}|$ along the Yoneda embedding $h : \mathbf{\Delta} \to \mathbf{Set}_{\mathbf{\Delta}}$

$$\begin{tikzcd} \mathbf{\Delta} \arrow[r, "|\Delta^{(-)}|"] \arrow[d, "h"'] & \mathbf{Top} \\ \mathbf{Set}_{\mathbf{\Delta}} \arrow[ur, "|-|"'] & \end{tikzcd}$$

This is all well and good, but what does $|X|$ even look like? When considering the sets X_n as discrete spaces, the geometric realization of X can be concretely given as the quotient space $|X| = \left(\coprod_n X_n \times |\Delta^n|\right)/_\sim$ where $\sim$ is the equivalence relation generated by setting

$$X_m \times |\Delta^m| \ni (\alpha^*(x), y) \sim (x, \alpha_*(y)) \in X_k \times |\Delta^k|$$

for every map $\alpha : [m] \to [k]$ in $\mathbf{\Delta}$, where $\alpha_* : |\Delta^{(\alpha)}|$ and $\alpha^* = X(\alpha)$.

Intuitively the geometric realization of a simplicial set X consists of geometric n-simplices for every element $\sigma \in X_n$, which are further glued according to the face and degeneracy maps of X.

It then also follows that the geometric realization of the standard n-simplex $|\Delta^n|$ is (homeomorphic to) the geometric n-simplex $|\Delta^n|$.

Along with the geometric realization, we get a right adjoint of $|-|$, which we call the *singular simplicial set functor* and which is defined as

$$\mathrm{Sing} : \mathbf{Top} \to \mathbf{Set}_{\mathbf{\Delta}},\ X \mapsto \mathbf{Top}(|\Delta^{(-)}|, X).$$

Naturally, we then call $\mathrm{Sing}(X)$ the *singular simplicial set* of X. This should already be familiar from singular homology as $\mathrm{Sing}(X)_n$ is precisely the set of singular n-simplices. Unraveling the definition of the map induced by the face maps in $\mathbf{\Delta}$, one can also see that these are precisely the differentials used to define the homology groups.

In Theorem 5.15, we will see that this adjunction is in some sense an equivalence. But to state this theorem, we first need to introduce model categories.

5. Model Categories

Model categories were first introduced in the 1960s by Daniel Quillen [Qui06]. Their purpose was to unify different homotopy theoretical notions and provide a general framework to do homotopy theory.

A model category is bicomplete, i.e. it has all small limits and colimits, and it comes equipped with three distinct classes of morphisms satisfying certain axioms. These morphisms are supposed to mimic certain properties of fibrations, cofibrations and weak homotopy equivalences in topology, allowing for the definition of some concepts from homotopy theory in the model category. In order to describe the axioms of model categories, it is convenient to introduce some terminology.

A diagram of commuting solid arrows

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dotted, "q"] & Y \end{tikzcd}$$

is called a *lifting problem.* If there exists a morphism $q : B \to X$ as indicated, making the diagram commute, we say that q *solves the lifting problem* or that q is a *lift* of the diagram.

Moreover, if for fixed i and p every such lifting problem admits a solution, we say that p has the *right lifting property* against i, or equivalently that i has the *left lifting property* against p.

It is sometimes convenient to talk about lifting properties on a larger scale. Therefore, given a class of morphisms S, we will write ${}_\perp S$ and $S_\perp$ for the collections of morphisms with the left and right lifting property against all morphisms in S.

Recall that a retraction of a map $i : X \to Y$, is a left-inverse $r : Y \to X$, i.e. a map such that $r \circ s = id_X$, and that in the case that a retraction exists X is a *retract* of Y. This can also be expressed by saying that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[rr, bend right, "\mathrm{id}_X"'] & Y \arrow[r, "r"] & X \end{tikzcd}$$

Similarly, if $f : Y \to Y'$ is some morphism, then a *retract* of f is a morphism $g : X \to X'$ such that there is a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "g"] \arrow[rr, bend left, "\mathrm{id}_X"] & Y \arrow[r] \arrow[d, "f"] & X \arrow[d, "g"] \\ X' \arrow[r] \arrow[rr, bend right, "\mathrm{id}_{X'}"'] & Y' \arrow[r] & X' \end{tikzcd}$$

With this, we can come to the main definition of this section.

Definition 5.1. A *model category* is a bicomplete category $\mathcal{C}$ with three classes of morphisms: *fibrations* $\mathrm{Fib}_\mathcal{C}$, *cofibrations* $\mathrm{Cof}_\mathcal{C}$ and *weak equivalences* $\mathrm{W}_\mathcal{C}$. Fibrations/cofibrations which are also weak equivalences are called *trivial* (or sometimes *acyclic*) fibrations/cofibrations. These classes of morphisms have to satisfy the following axioms:

M1 Every lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

where i is a cofibration, p is a fibration and either i or p is also a weak equivalence, admits a solution.

M2 Every morphism f in $\mathcal{C}$ can be functorially[9] factored as $f = pi$ in two ways:
  (i) p is a trivial fibration and i is a cofibration,
  (ii) p is a fibration and i and trivial cofibration.

M3 The three classes $\mathrm{Fib}_{\mathcal{C}}, \mathrm{Cof}_{\mathcal{C}}, \mathrm{W}_{\mathcal{C}}$ are closed under retracts. More concretely, if g is a retract of f and f is one of these classes, then so is g.

M4 Weak equivalences satisfy the *2-out-of-3 property*, that is, if we have a commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "h"] & \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

in $\mathcal{C}$ and two out of the three maps are weak equivalences, then so is the third.

The three collections of maps $\mathrm{Cof}_{\mathcal{C}}$, $\mathrm{Fib}_{\mathcal{C}}$ and $\mathrm{W}_{\mathcal{C}}$ satisfying axioms M1-M4, are often called a *model structure* on $\mathcal{C}$, though some authors consider the bicompleteness of $\mathcal{C}$ to be a prerequisite to have a model structure. Usually categories admit multiple different model structures, some of which may be "equivalent" and some of which may not.

If one looks at Quillen's original definition in [Qui06], one may also realize that there are some significant differences. In fact, what Quillen originally called model categories is actually strictly weaker than our definition. Instead, what we call model category is much closer to what he called *closed model category*, though Quillen only assumed the existence of *finite* limits and colimits. For most purposes finite (co)limits suffice, but since in many important examples all small (co)limits exist, most authors nowadays require them.

One defining feature that separated (closed) model categories from Quillens original weaker notion was that any two of the three classes of morphisms determine the third. The following proposition describes this correspondence.

Proposition 5.2. *Let $\mathcal{C}$ be a model category, then the following equalities hold:*
  (i) $\mathrm{Fib}_{\mathcal{C}} = (\mathrm{Cof}_{\mathcal{C}} \cap \mathrm{W}_{\mathcal{C}})_{\perp}$
  (ii) $\mathrm{Cof}_{\mathcal{C}} = {}_{\perp}(\mathrm{Fib}_{\mathcal{C}} \cap \mathrm{W}_{\mathcal{C}})$
  (iii) $f \in \mathrm{W}_{\mathcal{C}}$ *if and only if* $f = pi$, *where* $p \in (\mathrm{Cof}_{\mathcal{C}})_{\perp}$ *and* $i \in {}_{\perp}(\mathrm{Fib}_{\mathcal{C}})$

[9] Here "functorially" basically means that f is factored without making any choices or breaking any commutative diagrams.

We now give some examples of model structures on categories that are of interest to us. As we have mentioned, some categories may admit multiple different model structures. However, when we use the following examples as model categories later, we will not mention the model structure we mean explicitly, but simply mean the ones given here.

In general, showing that a category is a model category is lengthy and difficult, so we will give references to proofs of the given model structure. Because of the previous proposition, it further suffices to give a description of only two of the three classes of morphisms, as the third is then determined by the other two. In fact, it is often the case that this is the most convenient description of this third class of maps.

As we have hinted at, we will use model categories to show that topological spaces and simplicial sets are in some sense equivalent. For this we use the model structures of the following two propositions. Proofs that these are in fact model structures can be found in most books on model categories, in particular, in Quillen's original paper [Qui06]. Nowadays, a standard reference for model categories is [Hov07], where the model structure on **Top** is discussed in Theorem 2.4.19 and that on $\mathbf{Set}_{\boldsymbol{\Delta}}$ in the beginning of Section 3.2.

Proposition 5.3. *The category* **Top** *of topological spaces becomes a model category with*

(i) weak equivalences being weak homotopy equivalences,
(ii) fibrations being Serre-fibrations,
(iii) cofibrations being the morphisms with the left lifting property against all trivial fibrations.

Proposition 5.4. *The category* $\mathbf{Set}_{\boldsymbol{\Delta}}$ *of simplicial sets, becomes a model category with*

(i) weak equivalences being maps whose realization is a weak homotopy equivalence,
(ii) cofibrations being inclusions of simplicial sets,
(iii) fibrations being the morphisms with the right lifting property against all trivial cofibrations.

This model structure is often called the *Kan-Quillen* model structure on simplicial sets.[10] As in the other model structures, the fibrations are defined to be the maps that have the right lifting property against all trivial cofibrations. It turns out however that a morphism in this model structure is a fibration if and only if it has the right lifting property against all *horn*

[10]There are other interesting model structures on $\mathbf{Set}_{\boldsymbol{\Delta}}$, which model different homotopy theories, but not the homotopy theory of spaces.

inclusions, that is, inclusion maps $\Lambda_i^n \hookrightarrow \Delta^n$, where Λ_i^n is the i-th horn as in Example 4.3. These fibrations are often also called *Kan fibrations*. We will also meet *Kan complexes* later, which are closely related.

One other example of a model structure that Quillen gave was on the category of chain complexes. This is not very important to us, but it turns out that there is a similar model structure on **CDGA**, meaning we can talk about homotopies of cdga's!

Proposition 5.5. *The category* **CDGA** *of non-negatively graded cdga's over* $\mathbb{Q}$ *becomes a model category with*

(i) weak equivalences being quasi-isomorphisms,
(ii) fibrations being cdga homomorphisms f such that f_n is surjective for all $n \in \mathbb{N}$,
(iii) cofibrations being the morphisms with the left lifting property against all trivial fibrations.

A proof that this indeed defines a model structure can be found in Section 4.2 of [Hol21] or Chapter 4 of [BG76].

One can prove that a *slice category* $\mathcal{C}_{/X}$ of a model category $\mathcal{C}$ "inherits" a model structure, making it a model category as well. The precise statement and why it holds is shown in Theorem 15.3.6 of [MP12]. As we have mentioned, the category $\mathbf{CDGA}_{/\mathbb{Q}}$ is a slice category of **CDGA** and we therefore get the following corollary.

Corollary 5.6. *The category* $\mathbf{CDGA}_{/\mathbb{Q}}$ *of augmented non-negatively graded cdga's over* $\mathbb{Q}$ *is a model category.*

In a similar way, $\mathbf{Set}_{\mathbf{\Delta}*}$ can be seen as the slice category $(\mathbf{Set}_{\mathbf{\Delta}})_{*/}$ and hence it also inherits a model structure.

It is also useful to note that if $(\mathcal{C}, \mathrm{Fib}_{\mathcal{C}}, \mathrm{Cof}_{\mathcal{C}}, \mathrm{W}_{\mathcal{C}})$ is a model category, then its opposite category is also a model category $(\mathcal{C}^{\mathrm{op}}, \mathrm{Cof}_{\mathcal{C}}, \mathrm{Fib}_{\mathcal{C}}, \mathrm{W}_{\mathcal{C}})$ with fibrations becoming cofibrations and vice versa. We therefore also get model structures on the opposites categories of all the previous examples.

5.1. **Homotopy theory in model categories.** As we have mentioned, we want to do homotopy theory in model categories. For this we will construct the *homotopy category* $\mathrm{Ho}(\mathcal{C})$ of a model category $\mathcal{C}$. For this we will first have to define *homotopies* in model categories.

Since a model category has all limits and colimits, it in particular has binary products and coproducts. Now let $\mathcal{C}$ be a model category, $A, B \in \mathcal{C}$ two objects and recall that a product of B with itself consists of an object $B \times B$ and two projection maps $\mathrm{pr}_1, \mathrm{pr}_2 : B \times B \to B$. Given two maps $f, g : A \to B$, we can now, using the universal property of the product, define a map $(f,g) : A \to B \times B$ as the unique morphism making the following diagram commute:

$$\begin{tikzcd}
A \arrow[dr, dashed, "(f,g)"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& B \times B \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] & B \\
& B &
\end{tikzcd}$$

In the case where $A = B$ and $f = g = \mathrm{id}_A$, we write Δ_A for (id_A, id_A) and call it the *diagonal map*.

Dually, we define the map $f \amalg g : A \amalg A \to B$ as the unique morphism that satisfies dual properties, namely $g = (f \amalg g) \circ \mathrm{in}_2$ and $f = (f \amalg g) \circ \mathrm{in}_1$, where $\mathrm{in}_i : A \to A \amalg A$ are the inclusions of the coproduct. If $A = B$ and $f = g = \mathrm{id}_A$ we write ∇_A for $\mathrm{id}_A \amalg \mathrm{id}_A$ and call it the *fold map*.

Now recall that we can factor any morphism in a model category in the two ways described in Definition 5.1 M2. We then define a *path object* for A to be an object $\mathrm{Path}(A)$ such that the diagonal map Δ_A factors as $A \xrightarrow{j} \mathrm{Path}(A) \xrightarrow{(q_1,q_2)} A \times A$, where j is a trivial cofibration and q is a fibration.

On the other hand, a *cylinder object* for A is an object $\mathrm{Cyl}(A)$ such that the fold map ∇_A factors as $A \amalg A \xrightarrow{i_1 \amalg i_2} \mathrm{Cyl}(A) \xrightarrow{p} A$, where p is a trivial fibration and i a cofibration.

Note that both cylinder and path objects for A are weakly equivalent to A, which suggests that the maps $i_1 \amalg i_2$ and (q_1, q_2) from above may carry some interesting bits of information.

Using these maps, we can now define homotopies in model categories.

Definition 5.7. Let $\mathcal{C}$ be a model category and let $f, g : A \to B$ be two morphisms in $\mathcal{C}$, then

(i) a *right homotopy* between f and g is a map $H : A \to \mathrm{Path}(B)$ such that $(q_1, q_2) \circ H = (f, g)$,
(ii) a *left homotopy* between f and g is a map $H : \mathrm{Cyl}(A) \to B$ such that $H \circ (i_1 \amalg i_2) = f \amalg g$,

where the maps $A \amalg A \xrightarrow{i_1 \amalg i_2} \mathrm{Cyl}(A)$ and $\mathrm{Path}(A) \xrightarrow{(q_1,q_2)} A \times A$ are from the factorization that defined $\mathrm{Cyl}(A)$ and $\mathrm{Path}(A)$.

Naturally, we call two maps *left homotopic* if there exists a left homotopy between them and *right homotopic* if there exists a right homotopy. Of course, it is *a priori* not clear that two maps are left homotopic if and only if they are right homotopic and indeed this is not always the case. Note moreover that the relation of being left or right homotopic in a model category is not necessarily transitive and therefore not an equivalence relation!

Since homotopies in model categories are obviously motviated by homotopies in topology, we want left and right homotopies to at least coincide there. To see this, let's take a closer look at the model category **Top**.

Example 5.8. Let $X \in \mathbf{Top}$ be a topological space and I the unit interval, then $X \times I$ is a cylinder object and the mapping-space $\mathrm{Map}(I, X)$[11] a path object for X. Hence, the usual homotopies from topology are left-homotopies in the model category **Top**.

It is a basic result in topology that for any locally compact Hausdorff space Z the functor $- \times Z$ taking a product with Z is left-adjoint to the functor $\mathrm{Map}(Z, -)$ taking the mapping-space *out of* Z, meaning we have a natural bijection $\mathbf{Top}(X \times Z, Y) \cong \mathbf{Top}(X, \mathrm{Map}(Z, Y))$.

Since the interval I is compact and Hausdorff it follows that there exists a left homotopy between continuous maps $f, g : X \to Y$ if and only if there exists a right homotopy.

As we said, this nice example is not the general case, so to somewhat remedy the shortcomings of left and right homotopies in general model categories, we now introduce special kinds of objects. Recall that a model category has all small limits and colimits and so, in particular, an initial and a terminal object.

Definition 5.9. Let $\mathcal{C}$ be a model category with initial object 0 and terminal object 1. Then an object $X \in \mathcal{C}$ is called

(i) *fibrant* if the unique map $X \to 1$ is a fibration,
(ii) *cofibrant* if the unique map $0 \to X$ is a cofibration.

Of course, a category may have multiple terminal (or initial) objects, but all are uniquely isomorphic. It can be shown that if $X \to 1$ is a fibration, then for any other terminal object $1'$ the unique map $X \to 1'$ is also fibration. An analogous statement holds for cofibrant objects and hence both of these are well-defined. If X is both fibrant and cofibrant we call it *fibrant-cofibrant*.

Using fibrant and cofibrant objects one can then deduce some nice results about left and right homotopies, such as the following lemma. This statement can be found as Corollary 1.2.6 in [Hov07].

Lemma 5.10. *Let $f, g : B \to X$ be two morphisms in a model category $\mathcal{C}$, with X being fibrant and B cofibrant, then*

[11] $\mathrm{Map}(A, B)$ is just the set of continuous maps from A to B equipped with the *compact-open topology* making it into a topological space. The exact definition of this topology isn't important now.

(i) f and g are left homotopic if and only if they are right homotopic,
(ii) if f and g are left or right homotopic, then they are so via any cylinder or path object.

In any of the equivalent cases of the first point of the previous lemma we say that f and g are *homotopic* and write $f \simeq g$. We further write $[B, X]$ for the set of homotopy classes of maps from B to X. With this, we can also define homotopy equivalences in model categories: If X and Y are fibrant-cofibrant objects in a model category, then $f : X \to Y$ is called a *homotopy equivalence* if there exists a map $g : Y \to X$ such that $fg \simeq \mathrm{id}_Y$ and $gf \simeq \mathrm{id}_X$.

Of course in **Top**, homotopy equivalences are precisely the ones we already know. Later we will see that CW complexes are fibrant-cofibrant objects and therefore the following theorem is a generalization of Whiteheads Theorem 2.1. Quillen proved this in Section 1.1 of [Qui06], though a modern approach to proving this can be found in Chapter 3 of [Ste22].

Theorem 5.11. *Let $\mathcal{C}$ be a model category. Then a morphism $f : X \to Y$ between fibrant-cofibrant objects in $\mathcal{C}$ is a weak equivalence if and only if it is a homotopy equivalence.*

The previous lemma and theorem show us that knowing the fibrant and cofibrant objects of a model category is extremely important. Some model categories are nice, in the sense that every object is fibrant, so we only have to look out for cofibrant objects. This is the case for **Top**, since given a topological space X any continuous map $f : D^n \to X$ can be extended to a continuous map $\bar{f} : D^n \times I \to X$ by setting $\bar{f}(x, t) = f(x)$. It then follows that the following square commutes, showing that the unique map $X \to *$ is a Serre fibration.

$$\begin{tikzcd} D^n \arrow[r, "f"] \arrow[d, hook, "(\mathrm{id}_{D^n},0)"'] & X \arrow[d] \\ D^n \times I \arrow[r] \arrow[ur, "\bar{f}"] & * \end{tikzcd}$$

So what are the cofibrant objects in **Top**? It turns out that these are precisely the retracts of CW complexes, but proving this is quite a bit more involved. Quillen discusses this in Chapter II.3 of [Qui06].

In the other two examples of model structures we gave, it is even easier to see that we have similar cases. Since the terminal object in **CDGA** is the trivial algebra 1, every morphism of cdga's $f : A \to 1$ is surjective in every degree and hence a fibration. Again, the cofibrant objects are a bit more difficult, but for our purposes it is enough to know that every minimal model is a cofibrant object. This is proven as Lemma 4.7 in [Hol21].

Since in the Kan-Quillen model structure on $\mathbf{Set}_{\mathbf{\Delta}}$ the cofibrations were the "easy-to-describe" morphisms, we have a dual case, namely that every object in $\mathbf{Set}_{\mathbf{\Delta}}$ is cofibrant. As with the case of **CDGA**, this follows directly from the definition of cofibrations. We have already mentioned that fibrations in $\mathbf{Set}_{\mathbf{\Delta}}$ can be equivalently characterized by saying they satisfy the right lifting property against all horn inclusions. It follows therefore that an object $K \in \mathbf{Set}_{\mathbf{\Delta}}$ is fibrant if and only if there exists a dashed arrow in every diagram of the form

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & K \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

Objects that satisfy this condition are often called *Kan complexes* and besides their obvious importance when talking about model categories, they play a fundamental role in the theory of quasi-categories.

Returning to a general model category $\mathcal{C}$, we note that we can factor the unique map $X \to 1$ from some $X \in \mathcal{C}$ to the terminal object in $\mathcal{C}$, as $X \to X' \to 1$, where the first arrow is a trivial cofibration and the second a fibration. In particular, this means that X' is fibrant and weakly equivalent to X. Because of this, we call X' the *fibrant replacement* of X. Similarly, there is a *cofibrant replacement* of X. For technical reasons, we define the fibrant replacement of a fibrant object to be the object itself and similarly for the cofibrant replacement.

By the functoriality assumption in Definition 5.1 M2, the process of getting these replacements is also functorial. That is, we have a functor $R : \mathcal{C} \to \mathcal{C}$, such that for every $X \in \mathcal{C}$ the object $R(X)$ is fibrant and weakly equivalent to X. Similarly, we write $Q : \mathcal{C} \to \mathcal{C}$ for the functor that send X to its cofibrant replacement.

Since weak equivalences, fibrations and cofibrations are closed under composition, it follows that for any $X \in \mathcal{C}$ the composites $Q(R(X))$ and $R(Q(X))$ are weakly equivalent to X and further fibrant *and* cofibrant. Naturally, we will call this the *fibrant-cofibrant replacement.* Using this, we can finally define the homotopy category of a model category.

Definition 5.12. Let $\mathcal{C}$ be a model category. Then its *homotopy category* $\mathrm{Ho}(\mathcal{C})$ has objects those of $\mathcal{C}$ and for $X, Y \in \mathrm{Ho}(\mathcal{C})$ the hom-set $\mathrm{Ho}(\mathcal{C})(X, Y) = [QRX, QRY]$ is the set of homotopy class of maps from the fibrant-cofibrant replacement of X to the fibrant-cofibrant replacement of Y. Composition of maps $[f] \in [QRX, QRY]$ and $[g] \in [QRY, QRZ]$ is defined as $[g] \circ [f] := [g \circ f]$.[12]

One can equivalently define the homotopy category $\mathrm{Ho}(\mathcal{C})$ to have as objects the fibrant-cofibrant objects of $\mathcal{C}$ and just homotopy classes of maps as the morphisms.

5.2. **Quillen Functors and Derived Functors.** We have not yet talked about functors between model categories. Of course, as model categories are fundamentally just categories, we can just look at normal functors. But this would be like looking at topological spaces and maps of sets between them. In the usual categorical philosophy we want our functors to *preserve* some of the important structure of a model category. One concept that is quite restrictive, but very useful is the following.

Definition 5.13. Let $\mathcal{C}$ and $\mathcal{D}$ be two model categories, then a functor $F : \mathcal{C} \to \mathcal{D}$ is called

(i) *left Quillen* if it is left adjoint and preserves cofibrations, and
(ii) *right Quillen* if it is right adjoint and preserves fibrations.

If there is an adjunction $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ where F is left Quillen and G is right Quillen, we call this adjunction a *Quillen adjunction.*

A natural question then is whether a Quillen functor between model categories somehow induces a functor on their homotopy categories. Given that it would've been quite useless to define them otherwise, the answer is of course yes.

Recall that by R and Q we denote the fibrant and cofibrant replacement functors. Using these we can then define the desired "induced" or *derived* functors.

Proposition 5.14. *Let $\mathcal{C}$ and $\mathcal{D}$ be model categories, $F : \mathcal{C} \to \mathcal{D}$ left and $G : \mathcal{C} \to \mathcal{D}$ right Quillen. Then there are functors $LF : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$, $X \mapsto F(Q(X))$ and $RG : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$, $X \mapsto G(R(X))$.*

We call LF the *left derived functor* of F and RG the *right derived functor* of G. Note that RG is simply the name for the composite of functors $G(R(-))$ and not the composite $R(G(-))$ as the notation may suggest.

An important result about derived functors is that if F and G form a Quillen adjunction, then we get an adjunction on the level of homotopy categories:

$$LF : \mathrm{Ho}(\mathcal{C}) \leftrightarrows \mathrm{Ho}(\mathcal{D}) : RG.$$

If additionally this adjunction is an equivalence of categories, we call the adjunction $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ a *Quillen equivalence.*

[12]Using Lemma 5.10 one can show that this is indeed well defined.

With this we can finally state what we meant by simplicial sets being equivalent to topological spaces. In Chapter 2.III of [Qui06] Quillen showed the following theorem.

Theorem 5.15. *The adjunction*

$$|-|: \mathbf{Set}_{\boldsymbol{\Delta}} \rightleftarrows \mathbf{Top} : \mathrm{Sing}$$

is a Quillen equivalence.

This theorem tells us that if we want to prove certain things about **Top** we may prove them about $\mathbf{Set}_{\boldsymbol{\Delta}}$. This will become fundamental for us in the following chapter.

6. The Quillen-Sullivan Equivalence

A fundamental property of the polynomial de Rham algebra $\Omega(*)$ that Sullivan recognized is the fact that its "bigradedness" allows it to be viewed as an object of **CDGA**, as in Definition 3.5, or as an object of $\mathbf{Set}_{\boldsymbol{\Delta}}$, as in Example 4.4. And further, that when taking the morphism set from some object to $\Omega(*)$ in one category, the result is an object in the other. We will now make this precise.

Definition 6.1. The functor

$$A : \mathbf{Set}_{\boldsymbol{\Delta}}^{\mathrm{op}} \to \mathbf{CDGA},\ Y \mapsto \mathbf{Set}_{\boldsymbol{\Delta}}(Y, \Omega(*))$$

is the *PL de Rham functor* and

$$K : \mathbf{CDGA} \to \mathbf{Set}_{\boldsymbol{\Delta}}^{\mathrm{op}},\ B \mapsto \mathbf{CDGA}(B, \Omega(*))$$

is the *simplicial de Rham functor*.

This definition seems pretty simple at first, but looking more closely one may wonder what precisely we mean and why this is even well defined.

The functor A sends a simplicial set Y to the hom-set between Y and $\Omega(*)$, but as we have noted in Example 4.4, if we view only the n-th piece $\Omega(*)^n$ this also is a simplicial set. So given some n, our definition is meant to say $A(Y)^n = \mathbf{Set}_{\boldsymbol{\Delta}}(Y, \Omega(*)^n)$.

We now want to show that this sequence of sets admits the structure of a cdga. As with maps from a set into a vector space, the algebraic structure on the codomain already induces the same algebraic structure on the set of maps. More concretely, given two simplicial maps $f : Y \to \Omega(*)^n$ and $g : Y \to \Omega(*)^m$, we can form their product $f \cdot g : Y \to \Omega(*)^{n+m}$ degree- and point-wise. That is, for $k \in \mathbb{N}$ and $y \in Y_k$, we define $(f \cdot g)_k(y) := f_k(y) \cdot g_k(y)$, which is in $\Omega(k)^{n+m}$. One can check that this also commutes with face and degeneracy maps, making it a simplicial map. Graded commutativity of

$A(Y)$ follows from our definition and the fact that $\Omega(*)$ is graded commutative. Addition and scalar multiplication can be defined in a similar way, making $A(Y)$ into a commutative graded algebra. The differentials $d_{\Omega(k)}$ on $\Omega(n)$ also induce differentials on $A(Y)$ by post-composing maps $f \in A(Y)$ degree wise, i.e. $d_Y(f)_k = d_{\Omega(k)} \circ f_k$.

On the other hand, the functor K sends a cdga B to the sequence of sets $K(B)_n = \mathbf{CDGA}(B, \Omega(n))$. Since this hom-set is covariant in $\Omega(n)$, the face and degeneracy maps on $\Omega(*)$ induce face and degeneracy maps on $K(B)$. It then follows from the fact that $\Omega(*)$ is a simplicial set that the face and degeneracy maps on $K(B)$ also satisfy the simplicial identities, making $K(B)$ into a simplicial set.

Using the functor A we can define what it means to be a minimal model for a topological space. Namely, a *minimal model* for a topological space X is a minimal model M for the cdga $A(\mathrm{Sing}(X))$.

The two *de Rham* functors further make up an adjunction, but we get even more than that, as the following theorem shows. These results are proven as Theorem 6.2 and Lemma 6.17 in [Hol21] or as Propositions 8.1 and 8.8 of [BG76].

Theorem 6.2. *The PL de Rham functor A and the simplicial de Rham functor K form a Quillen adjunction*

$$K : \mathbf{CDGA} \leftrightarrows \mathbf{Set}^{\mathrm{op}}_{\boldsymbol{\Delta}} : A.$$

This also gives us a first hint as to why a choice of augmentation is similar to a choice of base point. An augmentation of a cdga B is a map $B \to \mathbb{Q}$, applying the functor K this becomes a morphism $K(\mathbb{Q}) = * \to K(B)$, i.e. a choice of base point in the simplicial set $K(B)$. Conversely, we find that a choice of base point becomes an augmentation under A. From this we further conclude that the adjunction from above induces an adjunction on the augmented and pointed versions of these categories.

Corollary 6.3. *The Quillen adjunction from Theorem 6.2 induces a Quillen adjunction*

$$K : \mathbf{CDGA}_{/\mathbb{Q}} \leftrightarrows \mathbf{Set}^{\mathrm{op}}_{\boldsymbol{\Delta}*} : A.$$

Recall that if $B \in \mathbf{CDGA}_{/\mathbb{Q}}$, then there is a morphism $\epsilon : B \to \mathbb{Q}$ such that post-composing it with the unit map yields $\epsilon \circ e_B = \mathrm{id}_{\mathbb{Q}}$. The kernel of the morphism ϵ is called the *augmentation ideal* $\bar{B} = \ker(\epsilon)$. Using this we can define the *indecomposables*, denoted $IB = \bar{B}/(\bar{B} \cdot \bar{B})$. This is again a chain complex with induced differential, so we may take its cohomology, which we denote by $\pi^n(B) = H^n(IB)$. We further write $\mathrm{Hom}_{\mathbb{Q}}(A, B)$ for the set of $\mathbb{Q}$-linear maps between $A, B \in \mathbf{CDGA}_{/\mathbb{Q}}$ and recall that this set inherits the algebraic structure from the codomain. We can then state the following theorem, which can be found as Propositions 8.12 and 8.13 in [BG76].

Theorem 6.4. *Let $B \in \mathbf{CDGA}_{/\mathbb{Q}}$ be cofibrant and homologically connected, then $\pi_0(K(B)) = *$ and for $n \geq 1$ there are natural bijections*

$$\pi_n(K(B)) \cong \mathrm{Hom}_{\mathbb{Q}}(\pi^n(B), \mathbb{Q})$$

which are group isomorphisms for $n \geq 2$.

One can set additional conditions on B to make the bijection of sets $\pi_1(K(B)) \cong \mathrm{Hom}_{\mathbb{Q}}(\pi^1(B), \mathbb{Q})$ into a group isomorphism, but in general this is not given. This theorem tells us that we can compute the homotopy groups of $K(B)$ from B. This will become even more significant later.

It is perhaps not difficult to guess that the Quillen adjunction of Theorem 6.2 will play an important role in the Quillen-Sullivan equivalence. To state this equivalence however, we first need to restrict our categories of interest. For this we make the following definition.

Definition 6.5. Let $X \in \mathbf{Top}$ be a nilpotent rational space and $A \in \mathbf{CDGA}$ a homologically connected cdga, then

(i) X is of *finite $\mathbb{Q}$-type* if all of its homotopy groups are finite dimensional $\mathbb{Q}$-vector spaces,
(ii) A is of *finite $\mathbb{Q}$-type* if its minimal model M is a finite dimensional $\mathbb{Q}$-vector space in each degree.

This is not an overly restrictive condition on our topological spaces, as many examples one cares about are indeed of finite $\mathbb{Q}$-type, in particular, all finite CW complexes.

We now write $\mathrm{Ho}(\mathbf{CDGA}_{/\mathbb{Q}})^{\mathrm{cft}} \subset \mathrm{Ho}(\mathbf{CDGA}_{/\mathbb{Q}})$ for the full subcategory with objects being cofibrant homologically connected augmented cdga's of finite $\mathbb{Q}$-type and $\mathrm{Ho}(\mathbf{Set}^{\mathrm{op}}_{\mathbf{\Delta}*})^{\mathrm{Knft}} \subset \mathrm{Ho}(\mathbf{Set}^{\mathrm{op}}_{\mathbf{\Delta}*})$ for the full subcategory with objects being pointed Kan complexes whose geometric realizations are rational nilpotent spaces of finite $\mathbb{Q}$-type.

We can now state the main theorem of this survey. We formulate it in a similar way to Holstein, who gives a proof in Chapter 9 of [Hol21]. Alternatively, this is also similar to Theorem 9.4 in [BG76], which is proved in Chapter 10.

Theorem 6.6 (The Quillen-Sullivan Equivalence)**.** *The adjunction on homotopy categories $LK : \mathrm{Ho}(\mathbf{CDGA}_{/\mathbb{Q}}) \rightleftarrows \mathrm{Ho}(\mathbf{Set}^{\mathrm{op}}_{\mathbf{\Delta}*}) : RA$, coming from the Quillen adjunction $K : \mathbf{CDGA}_{/\mathbb{Q}} \rightleftarrows \mathbf{Set}^{\mathrm{op}}_{\mathbf{\Delta}*} : A$, descends to an equivalence* *of categories*

$$LK : \mathrm{Ho}(\mathbf{CDGA}_{/\mathbb{Q}})^{\mathrm{cft}} \overset{\simeq}{\rightleftarrows} \mathrm{Ho}(\mathbf{Set}^{\mathrm{op}}_{\mathbf{\Delta}*})^{\mathrm{Knft}} : RA.$$

By the Quillen equivalence of Theorem 5.15 between topological spaces and simplicial sets and by the fact that every homologically connected cdga has a minimal model, the previous theorem implies that we have the following correspondence:

Homotopy classes of rational nilpotent CW complexes of finite $\mathbb{Q}$-type	$\cong$	Isomorphism classes of minimal models of finite $\mathbb{Q}$-type

Now given some path-connected space X, it follows that $A(\mathrm{Sing}(X))$ is actually homologically connected and therefore Theorem 3.7 tells us that there exists a minimal model M for X. If X is now further nilpotent and of finite $\mathbb{Q}$-type, then an important corollary from the proof of Theorem 6.6 is that the homotopy groups of $K(M)$ are the same as those of X. Using Theorem 6.4 it therefore follows that for $n \geq 2$[13], we have group isomorphisms

$$\pi_n(X) \otimes \mathbb{Q} \cong \mathrm{Hom}_{\mathbb{Q}}(\pi^n(M), \mathbb{Q}).$$

This is the big upshot of the Quillen-Sullivan equivalence:

Computing rational homotopy groups is an algebraic problem. This way of determining rational homotopy groups, which are essentially homotopy groups modulo torsion, has been very fruitful. For example in [BJS97] the authors use precisely this approach to compute the rational homotopy type of certain function spaces. Of course, there are also many examples where this is used to compute the rational homotopy groups of classical spaces, such as spheres and projective spaces in Section 2.1 of [Hes06].

References

[BG76] Aldridge Knight Bousfield and Victor KAM Gugenheim. *On PL DeRham theory and rational homotopy type*, volume 179. American Mathematical Soc., 1976.

[BJS97] Edgar Brown Jr and Robert Szczarba. On the rational homotopy type of function spaces. *Transactions of the American Mathematical Society*, 349(12):4931–4951, 1997.

[BK72] Aldridge Knight Bousfield and Daniel Marinus Kan. *Homotopy limits, completions and localizations*, volume 304. Springer Science & Business Media, 1972.

[Dro06] Emmanuel Dror. A generalization of the whitehead theorem. In *Symposium on Algebraic topology*, pages 13–22. Springer, 2006.

[13] It turns out that if $\pi_1(X)$ is abelian we also have an isomorphism for $n = 1$.

[FHT12] Yves Félix, Stephen Halperin, and J-C Thomas. *Rational homotopy theory*, volume 205. Springer Science & Business Media, 2012.

[GTHT00] Antonio Gómez-Tato, Stephen Halperin, and Daniel Tanré. Rational homotopy theory for non-simply connected spaces. *Transactions of the American Mathematical Society*, 352(4):1493–1525, 2000.

[Hat02] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, 2002.

[Hes99] Kathryn Hess. A history of rational homotopy theory. In *History of topology*, pages 757–796. Elsevier, 1999.

[Hes06] Kathryn Hess. Rational homotopy theory: a brief introduction. *arXiv preprint math/0604626*, 2006.

[Hol21] Julian Holstein. Rational homotopy theory. Lecture notes available online at `https://www.math.uni-hamburg.de/home/holstein/lehre/RHTnotes.pdf`, 2021.

[Hov07] Mark Hovey. *Model categories*. American Mathematical Soc., 2007.

[Iva22] Sergei O Ivanov. An overview of rationalization theories of non-simply connected spaces and non-nilpotent groups. *Acta Mathematica Sinica, English Series*, 38(10):1705–1721, 2022.

[May99] J Peter May. *A concise course in algebraic topology*. University of Chicago press, 1999.

[ML98] Saunders Mac Lane. *Categories for the working mathematician*, volume 5. Springer Science & Business Media, 1998.

[Moe17] Ieke Moerdijk. Rational homotopy theory. Lecture notes available online at `https://www.math.ru.nl/~sagave/east2017/moerdijk_rational_homotopy.pdf`, 2017.

[Mor10] Syunji Moriya. Rational homotopy theory and differential graded category. *Journal of Pure and Applied Algebra*, 214(4):422–439, 2010.

[MP12] J Peter May and Kate Ponto. *More concise algebraic topology: localization, completion, and model categories*. University of Chicago Press, 2012.

[Qui69] Daniel Quillen. Rational homotopy theory. *Annals of Mathematics*, 90(2):205–295, 1969.

[Qui06] Daniel G Quillen. *Homotopical algebra*, volume 43. Springer, 2006.

[Rie11] Emily Riehl. A leisurely introduction to simplicial sets. Unpublished expository article available online at `https://emilyriehl.github.io/files/ssets.pdf`, 2011.

[Rie14] Emily Riehl. *Categorical homotopy theory*, volume 24. Cambridge University Press, 2014.

[Ste21] Walker H Stern. An introduction to quasi-categories. Available online at `https://www.walkerstern.com/assets/pdf/QCats_notes.pdf`, 2021.

[Ste22] Walker H Stern. Model categories. Available online at `https://www.walkerstern.com/assets/pdf/Model_Cats.pdf`, 2022.

[Sul77] Dennis Sullivan. Infinitesimal computations in topology. *Publications Mathématiques de l'IHÉS*, 47:269–331, 1977.

[Whi49] John HC Whitehead. Combinatorial homotopy i. *Bull. Amer. Math. Soc*, 55(3):213–245, 1949.